Representation theory for systems of projectors and discrete Laplace operators

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1 Introduction

In this paper, we introduce a class of algebras $B(\Gamma)$ related to a simply laced graphs Γ and study their representation theory. Algebras $B(\Gamma)$ depend on a system of parameters s_{ij} corresponding to edges of the graph. They are generated by idempotents x_i, numbered by the vertices of the graph, which subject to relations:

- $x_i^2 = x_i$, for every i in $V(\Gamma)$,
- $x_ix_jx_i = s_{ij}^2x_i$, $x_jx_ix_j = s_{ij}^2x_j$, if i and j are adjacent in Γ,
- $x_ix_j = x_jx_i = 0$, if i and j are not adjacent in Γ.

Our original motivation to consider algebras $B(\Gamma)$ was the problem on classifying systems of m Cartan subalgebras in Lie algebra $sl(n,\mathbb{C})$, pairwise orthogonal with respect to Killing form. This problem is related to representation theory of algebras $B(\Gamma_m(n))$, where $\Gamma_m(n)$ is a graph with vertices arranged in m rows, by n vertices in each row, such that vertices in different rows are connected by edges and those in the same row are not. The problem on orthogonal Cartan subalgebras can be interpreted as the problem on finding system of projectors satisfying the conditions of unbiasedness. These conditions imply that projectors define a representation of $B(\Gamma_m(n))$ for particular value of parameters $s_{ij} = \frac{1}{n}$.

The representation theory of algebras $B(\Gamma)$ for other graphs are related to many classical and modern problems in algebraic geometry: Poncelet porism and its generalizations, etc.

There is a unitary version for representation theory of $B(\Gamma)$, when we consider representations in a Hermitian (or Euclidian) space and require that the generating idempotents in $B(\Gamma)$ are represented by Hermitian (resp. orthogonal) projectors. Classification of such representations includes the problem of classifying systems of lines in a a Hermitian vector space with given angles between them. The above algebraic version of the problem can be considered as the complexification of the Hermitian problem.

The Hermitian representations for $B(\Gamma_m(n))$ give *mutually unbiased bases*. The problem of classifying such bases has recently attracted a lot of attention in the quantum information theory due to its relevance to quantum encoding, decoding and quantum tomography. We shortly described an instance of how mutually unbiased bases appear in this context in 3.3.

Algebra $B(\Gamma)$ is a quotient of Temperley-Lieb algebra and of Hecke algebra of the graph (see 4.2). Thus, the study of its representation theory can be viewed as the first step in studying representation theory of Hecke algebras of complicated graphs. Note that there is an ample representation theory of Hecke algebras of Dynkin and extended Dynkin graphs, but virtually nothing is known about representations of Hecke algebras for graphs with say noncommutative fundamental group.

The representation theory for $B(\Gamma)$ is closely related to harmonic analysis of local systems on the graph. The crucial fact is that the algebra $B(\Gamma)$ can be obtained from Poincare groupoid of the graph (see 4.3) by changing the multiplication via a (generalized) Laplacian of the graph (see 4.4, 4.5). This construction implies a pair of homomorphism of $B(\Gamma)$ into the algebra of the Poincare groupoid of the graph. Representations of $B(\Gamma)$ can be understood from representations of the Poincare groupoid by push-forward and pull-back functors along these homomorphisms. Since the representations of Poincare groupoid are identified with local systems on the graph, the relevance of the harmonic analysis becomes clear. The Hermitian (as well as Euclidian) version of the problem is related to the positivity of Laplace operator.

We consider general construction of modifying multiplication in an algebra A by means of its element Δ. The modified multiplication is given by:

$$a \cdot b = a\Delta b$$

Since the new algebra might not have a unit, we adjoin the unit to it.

We formalize suitable conditions on the element Δ which imply good relation between representation theory of the new algebra and that of A. Such elements we call *well-tempered*. (Generalized) Laplacians are examples of well-tempered elements. We show that well-temperedness is the property of the double cosets with respect to the action of the group of invertible elements in A. We obtain an upper bound for Hochschild and global dimension for the algebra with modified multiplication in terms of the same dimensions for A.

Note that the Poincare groupoid is isomorphic to the matrix algebra over the group algebras of the fundamental group of the graph. Thus, for simply connected graph it is the matrix algebra over a field. All nonzero elements in this algebra are well-tempered. It is interesting to address the problem of classifying well-tempered elements in matrix algebras over other group algebras.

In section 6 we study the representation theory of algebras $B(\Gamma)$. First, we describe properties of functors and natural transformations between the categories of representations of algebra A and algebra B constructed from A via a well-tempered element. Second, we address the problem of coherence for $B(\Gamma)$. We actually start with coherence of Poincare groupoid. We prove theorem 47, which has an independent interest, that an algebra which is quasi-free relatively over a commutative noetherian ring $\mathbb{K}$ is coherent. This result is easy when the ring $\mathbb{K}$ is a field but requires more techniques in general case. Our proof is based on the Chase criterion for coherence. As a consequence of this result, Poincare groupoid over a noetherian ring of any finite graph is coherent. Then we prove theorem 48 which claims that if A is coherent and Δ a well-tempered element then B is coherent too. This allows us to consider a reasonable *abelian* category of representations for $B(\Gamma)$ with appropriate finiteness conditions, the category of finitely represented modules.

We also consider the derived categories and show that the derived category of representations for B, $D^b(B - \text{mod})$, has a semiorthogonal decomposition into categories $D^b(A - \text{mod})$ and $D^b(\mathbb{K} - \text{mod})$ and determine the gluing functor $D^b(A - \text{mod}) \to D^b(\mathbb{K} - \text{mod})$. The abelian category $B - \text{mod}$ is the heart of the t-structure which is obtained by gluing the standard t-structures on the two semiorthogonal components.

We are particularly interested in the representations of $B(\Gamma)$ that have minimal possible (non-zero) dimension. If $B(\Gamma) = B(\Gamma_m(n))$, then the minimal dimension is n, and this is exactly the case that corresponds to orthogonal Cartan subalgebras. Another good example is when Γ is a cyclic graph with n vertices. Then the minimal possible representation of $B(\Gamma)$ has dimension $n - 2$ for suitable choice of parameters s_{ij}. The category $B - \text{mod}$ for this case is equivalent to the category of perverse sheaves on the complex rational curve with a double point.

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2 Orthogonal Cartan subalgebras and algebraically unbiased projectors

2.1 Orthogonal Cartan subalgebras

Consider a simple Lie algebra L over an algebraically closed field of characteristic zero. Let K be the Killing form on L. In 1960, J.G.Thompson, in course of constructing integer quadratic lattices with interesting properties, introduced the following definitions.

Definition. Two Cartan subalgebras H_1 and H_2 in L are said to be *orthogonal* if $K(h_1, h_2) = 0$ for all $h_1 \in H_1, h_2 \in H_2$.

Definition. Decomposition of L into the direct sum of Cartan subalgebras $L = \oplus_{i=1}^{h+1} H_i$ is said to be *orthogonal* if H_i is orthogonal to H_j, for all $i \neq j$.

Intensive study of orthogonal decompositions has been undertaken since then (see the book [KT] and references therein). For Lie algebra $sl(n)$, A.I. Kostrikin et all arrived to the following conjecture, called *Winnie-the-Pooh Conjecture* (cf. *ibid.* where, in particular, the name of the conjecture is explained by a wordplay in the Milne's book in Russian translation).

Conjecture 1. *Lie algebra $sl(n)$ has an orthogonal decomposition if and only if $n = p^m$, for a prime number p.*

The conjecture has proved to be notoriously difficult. Even the non-existence of an orthogonal decomposition for $sl(6)$, when $n = 6$ is the first number which is not a prime power is still open. It is also important to find the maximal number of pairwise orthogonal Cartan subalgebras in $sl(n)$ for any given n as well as to classify them up to obvious symmetries.

We recall an interpretation and generalization of the problem in terms of systems of minimal projectors and its relation to representation theory of Temperley-Lieb algebras and Hecke algebras of some graphs. This was discovered by the first author about 25 years ago (cf. *ibid.*).

2.2 Algebraically unbiased projectors

Let V be a n-dimensional space over a field of characteristic zero.

Two minimal (i.e. rank 1) projectors p and q in V are said to be *algebraically unbiased* if

$$tr(pq) = \frac{1}{n} \tag{1}$$

Equivalently, this reads as one of the two (equivalent) algebraic relations:

$$pqp = \frac{1}{n}p, \tag{2}$$

$$qpq = \frac{1}{n}q. \tag{3}$$

We will also consider *orthogonal* projectors. Orthogonality of p and q is algebraically expressed as

$$pq = qp = 0 \tag{4}$$

Two maximal (i.e. of cardinality n) sets of minimal orthogonal projectors $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ are said to be *algebraically unbiased* if p_i and q_j are algebraically unbiased for all pairs (i, j).

Let $sl(V)$ be the Lie algebra of traceless operators in V. Killing form is given by the trace of product of operators. A Cartan subalgebra H in V defines a unique maximal set of minimal orthogonal projectors in V. Indeed, H can be extended to the Cartan subalgebra H' in $gl(V)$ spanned by H and the identity operator E. Rank 1 projectors in H' are pairwise orthogonal and comprise the required set. We say that these projectors are *associated* to H.

If p is a minimal projector in H', then trace of p is 1, hence, $p - \frac{1}{n}E$ is in H. If projectors p and q are associated to orthogonal Cartan subalgebras, then

$$\mathrm{Tr}(p - \frac{1}{n}E)(q - \frac{1}{n}E) = 0,$$

which is equivalent to p and q to be algebraically unbiased.

Therefore, an orthogonal pair of Cartan subalgebras is in one-to-one correspondence with two algebraically unbiased maximal sets of minimal orthogonal projectors. Similarly, orthogonal decompositions of $sl(n)$ correspond to $n + 1$ of pairwise algebraically unbiased sets of minimal orthogonal projectors. In the analysis of the problem, it is worthwhile to consider not only maximal sets of orthogonal projectors, but also study mutual unbiasedness for various subsets of maximal sets. Thus, we come to the problem of studying the sets of projectors where every pair satisfies either conditions (2-3) or (4). This will lead us to the representation theory of reduced Temperley-Lieb algebras which we study in the next section.

More explicitly, algebraic unbiasedness can be expressed as follows. Let projectors p and q be given as

$$p = e \otimes x, \quad q = f \otimes y,$$

where e and f are in V and x and y are in V^*. The equations $p^2 = p$ and $q^2 = q$ imply:

$$(e, x) = 1, \quad (f, y) = 1, \tag{5}$$

where $(-,-)$ stands for the pairing between vectors and covectors. Then the algebraic unbiasedness of p and q reads:

$$(x,f)(y,e) = \frac{1}{n}. \tag{6}$$

Orthogonality conditions (4) reads:

$$(x,f) = 0, \quad (y,e) = 0. \tag{7}$$

2.3 Formulation of the problem in terms of graphs

The above discussion suggests to consider the following general problem on systems of projectors.

Assume we are given a simply laced graph Γ with a finite number of vertices. Consider a finite dimensional vector space V. Assign a rank 1 projector in V to every vertex of the graph. If two vertices are related by an edge, then we require the corresponding two projectors to be algebraically unbiased. If there is no edge between the vertices, then we require the projectors to be orthogonal. The problem is to classify, for a given graph, all possible systems of projectors satisfying the required conditions modulo automorphisms of V.

The problem enjoys a *duality* which exchanges the roles of the space V and its dual V^*. Indeed, equations (2), (3), (4) are invariant under the operation of assigning to the vertices the adjoint projectors in the dual space V^*:

$$p \mapsto p^*. \tag{8}$$

Thus, having a configuration in the space V, we obtain a dual configuration in V^* and *vice versa.*

Rank 1 projectors are parameterized by the variety $\mathbb{P}(V) \times \mathbb{P}(V^*) \setminus D$, where D is the incidence divisor. Indeed, every projector p is given by $p = e \otimes x$, for e a vector and x a covector satisfying $(e,x) = 1$. The k^*-action $e \mapsto \lambda e$, $x \mapsto \lambda^{-1}x$, for $\lambda \in k^*$, identifies points corresponding to the same projector. Note that this is an instance of symplectic reduction. Indeed, $V \times V^*$ is endowed with the standard symplectic form. The equation $(e,x) = 1$ fixes the value of the moment map for the above k^*-action. The induced symplectic structure on the space of projectors coincides with Kostant-Kirillov-Lie bracket on the coadjoint orbit of Lie algebra $gl(V)$.

We consider the moduli space of solutions to our problem. To be more precise the fine moduli is clearly a stack, because every configuration of projectors has obvious automorphism subgroup of k^*, the dilations. Under suitable conditions, there is no other automorphism for a configuration.

Proposition 2. *Let Γ be a connected graph. Assume that either the configuration of projectors is such that the images of the projectors span the space V or the intersection of kernels of projectors is trivial. Then the only automorphisms of the configuration are elements from k^* (scalar operators).*

Proof. Assume that the images of projectors span the space V. An automorphism of the configuration must preserve the image of every projector. If projectors p and q are algebraically unbiased, then for a vector v in the image of p we have:

$$pqv = \frac{1}{n}v,$$

which shows, in particular, that qv is a nonzero vector in the image of q. By applying to this equality any operator that commutes with p and q, we see that it acts on v and qv via multiplication with the same scalar. Hence, it acts on the images of p and q by multiplication with the same scalar. Since the graph is connected, the scalar is the same for images of all projectors. As images of projectors span V, the statement follows. For the case when the intersection of the kernels of the projectors is trivial, the statement follows due to duality. □

We say that a configuration is *minimal* if the images of projectors span vector space V and the intersection of kernels of all projectors is zero. Let $\tilde{\mathcal{M}}^n(\Gamma)$ be the moduli stack of minimal configurations in $\mathbb{C}^n$. It follows from the proposition that it is a k^*-gerb. We denote by $\mathcal{M}^n(\Gamma)$ the coarse moduli space of this stack. Clearly, it is the quotient of a subvariety in the cartesian product of copies of $\mathbb{P}(V) \times \mathbb{P}(V^*) \setminus D$, one copy for every vertex of the graph, modulo the action of $GL(V)$. Note that this presentation is not particularly convenient for calculation.

To every configuration, we can assign in a canonical way a minimal configuration at the price of reducing the dimension of the vector space. To this end, take the subspace generated by the images of all projectors. There is a configuration subordinated to the same graph in this subspace. If the intersection of the kernels for all projectors in the new configuration is nontrivial, then mode out this subspace. The resulting configuration is minimal.

We say that two configurations are *S-equivalent* if the corresponding minimal configurations are isomorphic. Consider the set $\mathcal{M}(\Gamma)$ of isomorphism classes of all minimal configurations for all possible dimensions n. We will endow it with the structure of an affine variety. The variety has a stratification by subvarieties, such that each stratum is isomorphic to $\mathcal{M}^n(\Gamma)$, i.e. it parameterizes minimal configurations of given dimension n:

$$\mathcal{M}(\Gamma) = \bigcup_n \mathcal{M}^n(\Gamma) \tag{9}$$

Equivalently, this variety parameterizes S-equivalence classes of configurations in the vector space of dimension equal to the number of vertices in the graph. All this has a natural interpretation in the representation theoretic context which we discuss below.

2.4 The moduli space of configurations as a torus

We will show that $\mathcal{M}(\Gamma)$ is in fact a k^*-torus. Let us describe a generating set of functions on it. Consider a cyclic path γ of length s in the graph. Choose any orientation of the path and take the product $p_\gamma = p_1 \dots p_s$ of projectors corresponding to the vertices of the path taken in any full order compatible with the cyclic order defined by the orientation. The trace T_γ of the resulting operator does not depend on the choice of full order. It gives a set, parameterized by homotopy classes of cyclic paths, of regular function on the moduli space of solutions.

Since the operator $p_\gamma p_1$, if nonzero, has the same kernel and image as p_1 has, it is proportional to p_1. By checking the trace, we obtain:

$$p_\gamma p_1 = T_\gamma p_1.$$

Let $\hat{\gamma}$ be the cyclic path γ with inverse orientation and $p_{\hat{\gamma}} = p_s \dots p_1$, the operator which corresponds to the full order inverse to the one chosen for γ. Then we have as above:

$$p_1 p_{\hat{\gamma}} = T_{\hat{\gamma}} p_1.$$

Using this equalities, we obtain:

$$T_\gamma T_{\hat{\gamma}} p_1 = p_\gamma p_1 p_{\hat{\gamma}} = \frac{1}{n^s} p_1. \tag{10}$$

The latter equality is seen by iteratively applying (2) and (3). Therefore:

$$T_\gamma T_{\hat{\gamma}} = \frac{1}{n^s} \tag{11}$$

It is convenient to introduce normalized functions S_γ:

$$S_\gamma = n^{\frac{1}{2}|\gamma|} T_\gamma, \tag{12}$$

where $|\gamma|$ is the length of the cyclic path γ, i.e. the number of edges in γ. Equation (11) reads as a cancelation law:

$$S_\gamma S_{\hat{\gamma}} = 1. \tag{13}$$

Let us consider Γ as a topological space, a 1-dimensional CW-complex.

Theorem 3. *(i) S_γ depends only on the homotopy class of the free loop γ;*

(ii) The assignment $\gamma \mapsto S_\gamma$ extends to a homomorphism $\mathrm{H}_1(\Gamma, \mathbb{Z}) \to k^*$;

(iii) S_γ does not depend on choice of representation in the S-equivalence class.

(iv) The map $\mathcal{M}(\Gamma) \to \mathrm{H}^1(\Gamma, k^*)$ defined by S_γ's is bijective.

Proof. Given a cyclic path γ, there is a minimal free loop γ_0 in the graph which represents the homotopy class of γ. The cyclic path γ can be contracted to γ_0 by elementary contraction, i.e. contractions where only paths along an edge in one direction and then immediately back, is contracted. Equations (2) and (3) imply that S_γ does not change under such contractions. This proves(i).

It is more convenient for us to postpone the proof for the other statements until we develop a more conceptional approach via Poincare groupoid (see **??**).

□

Let $\Gamma_m(n)$ be the graph with m rows and n vertices in each row, such that any two vertices from different rows are connected by an edge and any two vertices in the same row are disconnected.

In view of what was explained in section 2.2, a configuration of projectors subordinated to the graph $\Gamma_m(n)$ in the vector space V of dimension n gives a configuration of m pairwise orthogonal Cartan subalgebras in $sl(n)$. The restriction on the dimension is crucial here. In the above picture, this corresponds to distinguishing the stratum of the minimal dimension on the torus. Moduli of configurations of orthogonal Cartan subalgebras is the quotient of $\mathcal{M}^n(\Gamma_m(n))$ by the action of the product of m symmetric groups S_n that permute the vertices in the rows of the graph.

2.5 Generalizations of the problem

It is natural to put the problem on projectors into a broader context. We can substitute the constant $\frac{1}{n}$ at the right hand side of (2) and (3) by arbitrary non-zero constant $r \in k^*$. We say that two rank 1 projectors are *r-unbiased* if:

$$pqp = rp, \tag{14}$$

or, equivalently,

$$qpq = rq. \tag{15}$$

Further, when considering the problem on system of projectors subordinated to a graph, we can make r to be dependent on the edge. Then the initial data is a graph Γ with labels $r_{ij} \in k^*$ assigned to edges (ij) in the graph. The relations are:

$$p_i p_j p_i = r_{ij} p_i, \tag{16}$$

$$p_j p_i p_j = r_{ij} p_j, \tag{17}$$

if there is an edge connecting i with j. We keep also the condition that p_i and p_j are orthogonal projectors if there is no edge.

All what was said in the previous subsection holds true for this generalized version. Formula (12) for S_γ then reads:

$$S_\gamma = \frac{1}{\sqrt{\prod r_{ii+1}}} T_\gamma, \tag{18}$$

where the product under the square root is taken over all edges in γ and T_γ is the trace of the product of projectors along cyclic path γ. The rest goes *mutatis mutandis*.

The generalized version of Theorem (3) holds true with the same wording.

We can also generalize the problem by considering projectors of higher rank. Equations (16) and (17) are not equivalent conditions for this case. We should pose them both. It follows immediately from these equations that p_i and p_j have the same rank. Thus all projectors from the system are of the same rank if the graph is connected. We will reformulate this problem in terms of the representation theory of algebras $B_r(\Gamma)$, which we consider in the next section. Similar to the homological interpretation for the case of rank 1 projectors, the higher rank case is related to local systems of higher rank on Γ (regarded as a topological space).

2.6 Generalized Hadamard matrices.

In this subsection, we will show how generalized Hadamard matrices are related to orthogonal decompositions of $sl(n)$.

Let $\mathcal{M}$ be the set of $n \times n$ matrices with non-zero entries. A matrix $A = \{a_{ij}\}$ from $\mathcal{M}$ is said to be a *generalized Hadamard* matrix if

$$\sum_{j=1}^{n} \frac{a_{ij}}{a_{kj}} = 0. \tag{19}$$

for all $i \neq k$.

This condition can be recast by means of *Hadamard involution* $h : \mathcal{M} \to \mathcal{M}$ defined by

$$h : a_{ij} \mapsto \frac{1}{n a_{ji}}. \tag{20}$$

Proposition 4. *A is a generalized Hadamard matrix if and only if A is invertible and $h(A) = A^{-1}$.*

Proof. Indeed, (19) is equivalent to $A \cdot h(A) = 1$. □

For any two Cartan subalgebras in a simple Lie algebra, one is known to be always a conjugate for the other by an automorphism of the Lie algebra. For the case of $sl(n)$, Cartan subalgebras are conjugate by an element of $GL_n(k)$, i.e. if (H, H') is a pair of Cartan subalgebras, then we have an element $A \in GL_n(k)$ such that $H' = AHA^{-1}$. The transition matrix A is uniquely defined when we fix basis $\{e_i\}$ and $\{f_i\}$ such that H consists of diagonal matrices for the first basis and H' does so for the second basis. The freedom of choice for one basis is given by the normalizer in $GL_n(k)$ for one Cartan subalgebra, i.e. the group of monomial matrices. Therefore, the transition matrix A is defined up to transformations

$$A' = M_1 A M_2, \tag{21}$$

where M_1 and M_2 are invertible monomial matrices.

Proposition 5. *[KT] Two Cartan subalgebras H and AHA^{-1} form an orthogonal pair of Cartan subalgebras in $sl(n)$ if and only if A is a generalized Hadamard matrix.*

Recall that the problem on orthogonal pairs of Cartan subalgebras is governed by the graph $\Gamma_2(n)$. Hence, the moduli space of configurations of rank 1 projectors in an n-dimensional space subordinated to this graph is closely related to the variety of Hadamard matrices. According to theorem 3, the moduli of configurations in all dimensions is a k^*-torus of dimension $(n-1)^2$ (rank of homology for graph $\Gamma_2(n)$). This torus is identified with the quotient of $\mathcal{M}$, the torus of matrices with invertible entries, by the left and right actions of the torus of diagonal matrices. Equations (19) are invariant under these left and right actions. They define the minimal stratum of this torus, which corresponds to the case of n-dimensional configurations.

3 The Hermitian case

3.1 Mutually unbiased bases and configurations of lines in a Hermitian space

The terminology of unbiased bases first appeared in physics. It is a unitary version of the algebraic unbiasedness introduced above. We will define it and explain on an example how unbiased bases show up in quantum information theory.

Let V ba an n dimensional complex space with a fixed Hermitian metric $\langle\ ,\ \rangle$. Two orthonormal Hermitian bases $\{e_i\}$ and $\{f_j\}$ in V are *mutually unbiased* if, for all (i, j),

$$|\langle e_i, f_j\rangle|^2 = \frac{1}{n} \tag{22}$$

Consider the orthogonal projectors p_i and q_j, corresponding to these bases, defined by:

$$p_i(-) = e_i \otimes \langle -, e_i\rangle, \quad q_j(-) = f_j \otimes \langle -, f_j\rangle.$$

Then, the condition (6) is satisfied for them, hence they are algebraically unbiased. Note that these operators are rank 1 Hermitian projectors, and, being such, are defined by non-zero vectors in their images. We say that two rank 1 projectors are *unbiased* if they are algebraically unbiased Hermitian projectors.

We can formulate a general problem about unbiased Hermitian projectors for graphs similar to that for algebraically unbiased projectors. Since rank 1 Hermitian projectors are defined by a line in V, the image of the projector, the problem concerns collections of points in $\mathbb{P}V$, but we will formulate it in terms of representing vectors of length 1 in V.

We again fix a simply laced graph Γ with a finite number of vertices and a finite dimensional vector space V of dimension n, now endowed with a Hermitian form. Assign a length 1 vector in V to every vertex in Γ. If two vertices are connected by an edge, then we put condition

$$|\langle e, f\rangle|^2 = \frac{1}{n} \tag{23}$$

on vectors e and f corresponding to the vertices. If the vertices are disconnected, then we require the vectors to be perpendicular. The problem is to classify all systems of vectors modulo linear automorphisms of V and the products of $U(1)$'s corresponding to change of phases of all vectors.

Similar to the case of the algebraic problem we have a generalization of the problem to the case of Hermitian projectors. To this end, we consider a *full* graph Γ together with labels r_{ij}, $0 \leq r_{ij} \leq 1$, on edges (ij). The problem is to find all, up to linear automorphisms of V, configurations of lines labeled by vertices of the graph and satisfying

$$|\langle e_i, e_j\rangle|^2 = r_{ij}, \tag{24}$$

for any choice of length 1 vectors e_i and e_j in the lines corresponding to vertices i and j of the graph. Vanishing $r_{ij} = 0$ corresponds to the absence of edge between i and j in the graph in the algebraic version of the problem (in which case the condition was given by two equations (7), instead of one in the Hermitian case).

Note that all our conditions are just to fix particular angles between lines in the configuration. More generally, we can consider subspaces of fixed dimension in the Hermitian space V and fix angles between them. This corresponds to considering higher rank representations of the algebra $B_r(\Gamma)$ (see below).

3.2 Moduli of configurations of lines

We can regard algebraic unbiasedness as the complexification of unbiasedness. To be more precise, consider the space $\mathcal{M}(\Gamma)$ of all configurations of projectors satisfying the problem for graph Γ on systems of algebraically unbiased projectors in a space V. It is an algebraic variety in cartesian product of copies of $\mathbb{P}(V) \times \mathbb{P}(V^*) \setminus D$, one copy for every vertex of Γ.

Now fix a Hermitian form on V. The Hermitian involution gives a new duality on the set of algebraic configurations:

$$p_i \mapsto p_i^\dagger \tag{25}$$

The duality induces an involution on $\mathcal{M}(\Gamma)$. The involution takes S_γ into $\bar{S}_{\hat{\gamma}}$. Since $\mathcal{M}(\Gamma) = \mathrm{H}^1(\Gamma, \mathbb{C}^*)$, this is an anti-holomorphic involution on $\mathcal{M}(\Gamma)$. Clearly, the involution takes a minimal configuration into a minimal one. Hence, it preserves all strata $\mathcal{M}^n(\Gamma)$. It follows that For a Hermitian configuration, $p_i = \bar{p}_i$. Hence

on $\mathcal{S}$ which takes every projector to its Hermitian conjugate. Clearly, this involution preserves $\mathcal{S}$.

Since mutually unbiased bases are algebraically unbiased, they are related to orthogonal Cartan subalgebras in $sl(n)$. Given m pairwise mutually unbiased bases $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m$ in a Hermitian space V, we obtain m Cartan subalgebras $H_1, H_2, ..., H_m$ in $sl(n)$ which are pairwise orthogonal with respect to the Killing form. In particular, a collection of $n+1$ mutually unbiased bases in a Hermitian vector space of dimension n gives rise to an orthogonal decomposition of $sl(n)$. This fact was noticed by P.Oscar Boykin, Pham Huu Tiep, Meera Sitharam and Pawel Wocjan in [BTSW].

Let $\mathcal{B}$ an orthonormal basis in $\mathbb{C}^n$. Matrix $A = (a_{ij})$ is said to be *complex Hadamard* if bases $\mathcal{B}$ and $A(\mathcal{B})$ are mutually unbiased. Let A and C be a complex Hadamard matrices. We will say that A is equivalent to C if $A = M_1 C M_2$ for some unitary monomial matrices M_1, M_2.

There exists the following relation between complex Hadamard matrices and generalized Hadamard ones: A is a complex Hadamard if and only if A is a generalized Hadamard and $|a_{ij}| = 1$.

3.3 Protocols of quantum information transition

Consider a finite dimensional quantum mechanical system for which V is the space of states. *States* of the system are vectors in V (considered up to multiplication by a non-zero constant), while *observables* are Hermitian operators. Choose an observable which is diagonal in the basis of e_i's and has pairwise distinct real eigenvalues λ_i's. Due to the basic quantum mechanical principles, the *measurement* of this observable when the system is in the state f, normalized to be of Hermitian norm one, returns us one of the eigenvalue of the Hermitian operator and the probability to have eigenvalue λ_i as the result of measuring the observable is $|\langle e_i, f\rangle|^2$. The state of the system after the measurement is the eigenvector e_i.

The condition (22) frequently appears in the Quantum Information Theory problems. Here is one of possible ways how this happens.

Let Alice transmits quantum information to Bob using a quantum system in the space V. A protocol for transmitting quantum information is a set of pairwise mutually unbiased bases in V. Alice and Bob has arranged the sequence of pairwise mutually unbiased bases which they will use in their communication, say $\{e_i\}$, $\{f_j\}$, $\{g_k\}$, etc. Alice sends states of the systems: firstly, one of the vector of the basis $\{e_i\}$, then a vector of the basis $\{f_j\}$, etc. Possible transmitted string might look as $(e_2, f_5, g_3, \dots)$. The actual information is contained in the indices $2, 5, 3, \dots$. Assume also the presence of an eavesdropper, Eve. Since the protocol is open information, Eve knows the set of bases but she does not know the sequence that Alice and Bob arranged among themselves. Eve will get no information even probabilistically if she uses a wrong basis from the protocol to determine a transmitted state. Indeed, if Alice sends one of the e_i's and Eve measures an observable which, as a Hermitian operator, is diagonal in the basis $\{f_j\}$, then she will get one of the eigenvalues of this observable with equal probability $\frac{1}{n}$. Clearly, the more is the number of mutually unbiased bases, the more secure is the protocol, whence the relevance of the Winnie-the-Pooh problem to Quantum Information Theory.

There are other instances when mutually unbiased bases appear in physics, see [?], [?].

4 Algebras $B(\Gamma)$, Hecke algebras and Poincare groupoids of graphs.

4.1 Algebra $B(\Gamma)$.

The above discussion of the problem on configurations of projectors motivates the study of representation theory for algebras $B(\Gamma)$, which we introduce here. Under some specialization of parameters, these algebras become quotients of more familiar Temperley-Lieb algebras of graphs. The latter are, in their turn, quotients of Hecke algebras of graphs.

Let Γ be a simply laced graph with no loop (i.e. no edge with coinciding ends). Denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of the graph. Let $\mathbb{K}$ be a commutative ring and $\{s_{ij}\}$, where (ij) runs over the set of edges of the graph (i.e. $s_{ij} = s_{ji}$), a set of invertible elements in $\mathbb{K}$. For example, one can take the universal ring $\mathbb{K} = k[\{s_{ij}\}, \{s_{ij}^{-1}\}]$, where k is a field of characteristic zero. We define algebra $B(\Gamma)$ as a unital algebra over Generators x_i of $B(\Gamma)$, except for 1, are numbered by all vertices i of Γ. They subject relations:

- $x_i^2 = x_i$, for every i in $V(\Gamma)$,
- $x_i x_j x_i = s_{ij}^2 x_i$, $x_j x_i x_j = s_{ij}^2 x_j$, if i and j are adjacent in Γ,
- $x_i x_j = x_j x_i = 0$, if there is no edge connecting i and j in Γ.

We define $B^+(\Gamma)$ as the augmentation ideal in $B(\Gamma)$ generated by all x_i's. Note that automorphisms of the graph induce automorphisms of $B(\Gamma)$.

A configuration of projectors p_i's considered in section 2 can be understood as a representation of $B(\Gamma)$:

$$x_i \mapsto p_i.$$

Clearly, we have

$$r_{ij} = s_{ij}^2.$$

It will be convenient for us to keep the square roots of r_{ij} as basic parameters.

A path in the graph is a sequence of vertices

$$\gamma = (i_0, \dots, i_t),$$

where $i_l = i_{l+1}$ or $(i_l i_{l+1}) \in E(\Gamma)$, for $0 \le l \le t-1$. To such a path, we assign an element x_γ in $B(\Gamma)$:

$$x_\gamma = x_{i_0} \dots x_{i_t}.$$

We assign 1 to the empty path.

Path γ is said to be *contracted* if $i_l \neq i_{l+1}$, for all $0 \le l \le t-1$, and $i_l \neq i_{l+2}$, for all $0 \le l \le t-2$, or if it is an empty path. If γ is not contracted, i.e. $i_l = i_{l+1}$ or $i_l = i_{l+2}$ for some l, then we define its *elementary contraction* as a new path with vertex i_{l+1} or, respectively, two vertices i_{l+1} and i_{l+2} removed from γ. We can obtain other contractions of the path by iterating minimal contractions. Note that all contractions of γ are homotopic to γ in the class of pats with the same start and end point as γ. Every homotopy class of paths with fixed start and end point contains a unique contracted path, called *minimal contraction* for paths in the homotopy class. The contraction for a given path can be achieved by a sequence of elementary contractions.

Proposition 6. *The set of elements x_γ where γ runs over the set of all contracted paths in Γ, is a $\mathbb{K}$-basis of $B(\Gamma)$.*

Proof. The defining relations for $B(\Gamma)$ imply that x_γ remains the same up to invertible multiplier after an elementary contraction and that the $\mathbb{K}$-linear span of x_γ's is $B(\Gamma)$. Hence, x_γ's, where γ runs over all contracted paths, span $B(\Gamma)$.

It remains to show that these elements are linearly independent. To this end, note that any element in the ideal of relations in $B(\Gamma)$ among x_i's is a $\mathbb{K}$-linear combination of two classes of relations:

- $x_{i_1} \dots x_{i_t} = 0$ when, for some l, the pair (i_l, i_{l+1}) is not an edge of the graph.
- $x_\gamma = \lambda_{\gamma\gamma'} x_{\gamma'}$ where γ' is an elementary contraction of γ and $\lambda_{\gamma\gamma'} \in \mathbb{K}^*$;

The first class shows that it is enough to consider only monomials corresponding to paths in the graph. The second class of relations are divided into groups, where each group consists of relations among monomials x_γ where γ is in a homotopy class of paths with fixed starting and ending points. Hence, it is enough to check that monomials from every homotopy class of paths span a 1-dimensional space in $B(\Gamma)$. Denote by q_{ij} the difference between multiplicities of edge (ij) in γ and its contraction γ'.Then one can easily check by induction on the number of elementary contractions that lead from γ to γ' the formula:

$$x_\gamma = \prod s_{ij}^{q_{ij}} x_{\gamma'}.$$

Taking $\gamma' = \gamma_o$, the minimal contraction in the homotopy class of γ, we get that all monomials have an expression as the monomial of the contracted path with a unique invertible multiplier. Thus, the space spanned by the monomials from a homotopy class is indeed 1-dimensional. □

When we need to specify the values of s_{ij} or r_{ij}, we will write $B_s(\Gamma)$ or $B_{(r)}(\Gamma)$ for the algebra with specified parameters $s = \{s_{ij}\}$ and $r = \{r_{ij}\}$. We shall consider the representation theory of these algebras in the next section.

4.2 Temperley-Lieb algebras and Hecke algebras of graphs

Consider the special case when $s_{ij} = s$ for all edges (ij) of the graph. To describe the relation to Hecke algebra, note that the standard Temperley-Lieb algebra $TL(\Gamma)$ is defined similarly with the last relation replaced by $x_i x_j = x_j x_i$ when there is no edge connecting i and j. Hence, algebra $B(\Gamma)$ for the case $s_{ij} = s$ is a quotient of $TL(\Gamma)$.

Recall that Hecke algebra $H(\Gamma)$ of graph Γ is the unital algebra over $k[q, q^{-1}]$ generated by elements $T_i, i \in V(\Gamma)$ subject the following relations:

- $T_i T_j T_i = T_j T_i T_j$, if $(i, j) \in E(\Gamma)$,
- $T_i T_j = T_j T_i$, otherwise,
- $(T_i + 1)(T_i - q) = 0$ for any i.

We obtain $H_q(\Gamma)$, an algebra over k, by specializing q to a non-zero value. For the case Γ is Dynkin graph of type A_n $H(\Gamma)$ is known to be a q-deformation of the group algebra for S_{n+1}. Its representation theory is useful in constructing polynomial invariants of knots (cf. [?]). If q is not a root of unity, $H(\Gamma)$ is isomorphic to the group algebra of for S_{n+1}.

There is a homomorphism $H(\Gamma) \to B(\Gamma)[q]$, where $B(\Gamma)$ is extended by the central element q, a root of the equation:

$$q + 2 + q^{-1} = s^{-2}.$$

The homomorphism is defined by mapping:

$$T_i \mapsto (q + 1)x_i - 1.$$

One can easily see that via this homomorphism $B(\Gamma)$ becomes isomorphic to the quotient of $H(\Gamma)$ by relations:

- $-1 + T_i + T_j - T_i T_j - T_j T_i + T_i T_j T_i = 0$, if (i, j) is an edge;
- $-1 + T_i + T_j - T_i T_j = 0$, otherwise.

To give the idea about how much we quotient out when passing from $H(\Gamma)$ to $B(\Gamma)$, let us note that if Γ is a graph of type A_n and q is not a root of unity, algebra $B(\Gamma)$, as a quotient of $H(\Gamma)$ is a direct sum of the matrix algebra, the operators in the standard n-dimensional representation of the permutation group S_{n+1}, and the trivial 1-dimensional representation k. Thus the representation theory of $B(\Gamma)$ is very simple for this graph. The reason is that the homology of the graph is trivial. Real complications appear for graphs which are not trees.

4.3 Poincare groupoids of graphs

It turns out that the representation theory of $B(\Gamma)$ is closely related to that of Poincare groupoid of the graph.

Let again Γ be a simply-laced graph with no loop. Consider it as a topological space. Let $\mathcal{P}(\Gamma)$ be the Poincare groupoid of graph Γ, i.e. a category with objects vertices of the graph and morphisms homotopic classes of paths. Composition of morphisms is given by concatenation of paths.

Let $\mathbb{K}$ be a commutative ring. Denote by $\mathbb{K}\Gamma$ the algebra over $\mathbb{K}$ with a free $\mathbb{K}$-basis numbered by morphisms in $\mathcal{P}(\Gamma)$ and multiplication induced by concatenation of paths (when it makes sense, and zero when it does not). Let e_i be the element of $\mathbb{K}\Gamma$ which is the constant path in vertex i. Any oriented edge (ij) can be interpreted as a morphism in $\mathcal{P}(\Gamma)$, hence it gives an element l_{ij} in $\mathbb{K}\Gamma$. Defining relations are:

- $e_ie_j = \delta_{ij}e_i, \quad e_il_{jk} = \delta_{ij}l_{ik}, \quad l_{jk}e_i = \delta_{ki}l_{jk};$
- $l_{ij}l_{ji} = e_i, \quad l_{ji}l_{ij} = e_j, \quad l_{ij}l_{km} = 0$, if $j \neq k$.

We consider $\mathbb{K}\Gamma$ as an algebra with unit:

$$1 = \sum_{i \in V(\Gamma)} e_i.$$

Let $\gamma \in \mathcal{P}(\Gamma)$ be a path in Γ. Denote by l_γ the element in $\mathbb{K}\Gamma$ corresponding to this path. There is an involutive anti-isomorphism $\sigma : \mathbb{K}\Gamma \to \mathbb{K}\Gamma^{opp}$ defined by

$$\sigma(l_\gamma) = l_{\hat{\gamma}}, \tag{26}$$

where $\hat{\gamma}$ is the path inverse to γ. It implies a duality, i.e. an involutive anti-equivalence, $D : \mathbb{K}\Gamma - \mathrm{mod}_{fd} \simeq \mathbb{K}\Gamma - \mathrm{mod}_{fd}^{opp}$, on the category $\mathbb{K}\Gamma - \mathrm{mod}_{fd}$ of finite dimensional $\mathbb{K}\Gamma$-representations, if $\mathbb{K} = k$ is a field. If $\rho : k\Gamma \to \mathrm{End}(V)$ is a representation, then the dual representation $D(\rho) : k\Gamma \to \mathrm{End}(V^*)$ is defined, for $l \in k\Gamma$, by:

$$D(\rho)(l) = \rho(\sigma(l))^*. \tag{27}$$

Let Γ be in addition a connected graph. Then the category of representations for Poincare groupoid and that for the fundamental group of the graph are equivalent. To see this, fix $t \in V(\Gamma)$. Denote by $\mathbb{K}[\pi(\Gamma, t)]$ the group algebra of the fundamental group $\pi(\Gamma, t)$. Consider projective $\mathbb{K}\Gamma$ - module $P_t = \mathbb{K}\Gamma e_t$. Clearly, P_t is a $\mathbb{K}\Gamma$ - $\mathbb{K}[\pi(\Gamma, t)]$ - bimodule. Note that P_t are isomorphic as left $\mathbb{K}\Gamma$-modules for all choices of the vertex t. Indeed, the right multiplication by an element corresponding to a path starting at t_1 and ending at t_2 would give an isomorphism of P_{t_1} with P_{t_2}.

Proposition 7. *Bimodule P_t induces a Morita equivalence between $\mathbb{K}\Gamma$ and $\mathbb{K}[\pi(\Gamma,t)]$. Thus, the categories $\mathbb{K}\Gamma - \mathrm{mod}$ and $\mathbb{K}[\pi(\Gamma,t)] - \mathrm{mod}$ are equivalent. Moreover, algebra $\mathbb{K}\Gamma$ is isomorphic to the matrix algebra over $\mathbb{K}[\pi(\Gamma,t)]$, with size of (square) matrices equal to $|V(\Gamma)|$.*

The functors that induce equivalence between categories $\mathbb{K}\Gamma - \mathrm{mod}$ and $\mathbb{K}[\pi(\Gamma,t)] - \mathrm{mod}$ are:

$$V \mapsto P_t \otimes_{\mathbb{K}[\pi(\Gamma,t)]} V, \quad W \mapsto \mathrm{Hom}_{\mathbb{K}\Gamma}(P_t, W). \tag{28}$$

In order to define an isomorphism $\mathbb{K}\Gamma \to \mathrm{Mat}_n(\mathbb{K}[\pi(\Gamma,t)])$, fix a system of paths $\{\gamma_i\}$ connecting the vertex t with every vertex i. For any element $\pi \in \mathbb{K}[\pi(\Gamma,t)]$ consider an element $\gamma_i^{-1}\pi\gamma_j$ in $\mathbb{K}\Gamma$. The homomorphism is defined be the assignment:

$$\gamma_i^{-1}\pi\gamma_j \mapsto \pi \cdot E_{ij},$$

where E_{ij} stands for the elementary matrix with the only nontrivial entry 1 at (ij)-th place. This is clearly a well-defined ring isomorphism.

Let $\mathbb{K} = k$ be again a field. Since the fundamental group $\pi(\Gamma,t)$ is free, the equivalence implies that homological dimension of category $k\Gamma - \mathrm{mod}$ is 0 if graph is a tree and 1 otherwise.

The equivalence takes the duality functor (27) for $k\Gamma - \mathrm{mod}$ into the standard duality $W \mapsto W^*$ for representations of the group $\pi(\Gamma,t)$.

4.4 A formal construction of a new algebra by an element in an algebra

We shall use the following general algebraic construction. Let A be an algebra over a commutative ring $\mathbb{K}$ and Δ an element in A. Consider the non-unital algebra, A_Δ, with the same space as A but with new multiplication defined by:

$$a \cdot_\Delta b = a\Delta b. \tag{29}$$

By adjoining the identity element to algebra A_Δ, we get a unital algebra $\widehat{A}_\Delta = \mathbb{K} \cdot 1 \oplus A_\Delta$. Define two homomorphisms:

$$\psi_1, \psi_2 : A_\Delta \to A, \tag{30}$$

by assigning:

$$\psi_1 : a \mapsto a\Delta, \quad \psi_2 : a \mapsto \Delta a. \tag{31}$$

Note that

$$\mathrm{Im}\psi_1 = A\Delta, \quad \mathrm{Im}\psi_2 = \Delta A, \tag{32}$$

i.e. the images of the non-unital algebra A_Δ are the one-sided ideals in A generated by Δ. We use the same notation ψ_1 and ψ_2 for the extensions to homomorphisms of unital algebras:

$$\psi_1, \psi_2 : \widehat{A}_\Delta \to A,$$

Note that A_Δ is the augmentation ideal, and hence $\widehat{A}_\Delta$ - bimodule. Left multiplication in A commutes with right multiplication in A_Δ and vice versa. We use left and right multiplication in A to endow A_Δ with the structure of A-bimodule. Clearly, as a right or left A-module, A_Δ is free of rank 1. The left $\widehat{A}_\Delta$-module structure on A_Δ coincides with the structure obtained by pull back of the left A-module structure along morphism ψ_1. Similarly, the right $\widehat{A}_\Delta$-module structure coincides with structure obtained by pull back of the right A-module structure along morphism ψ_2.

Denote by ${}_{\psi_i}A_{\psi_j}$ the B - bimodule structure on A with left module structure defined by ψ_i and right module structure defined by ψ_j. The identification of B^+ with A gives an isomorphism of B-bimodules :

$$B^+ \cong {}_{\psi_1}A_{\psi_2} \tag{33}$$

For that reason, we shall consider A to be endowed always with left B-module structure coming from ψ_1 and right B-module structure coming from ψ_2.

If Δ is an invertible element in A, then ψ_1 and ψ_2 are isomorphisms. Thus, A_Δ is a unital algebra with the unit Δ^{-1}. $\widehat{A}_\Delta$ is obtained from a unital algebra, isomorphic to A, by adjoining a new unit 1_B. In this case, algebra $\widehat{A}_\Delta$ is isomorphic to the direct sum of algebras $A_\Delta \simeq A$ and $\mathbb{K}$. Algebra A_Δ is already embedded into $\widehat{A}_\Delta$, and $\mathbb{K}$ is embedded into $\widehat{A}_\Delta$ via $\mathbb{K}(1_B - \Delta^{-1})$(in order to annihilate A_Δ). So the construction is interesting only when Δ is not invertible.

4.5 Laplace operator on the graph and construction of $B(\Gamma)$ via Poincare groupoid.

Let $\mathbb{K}$ be a ring with a set of invertible elements $\{s_{ij}\}$. A universal choice is $\mathbb{K} = k[\{s_{ij}\}, \{s_{ij}^{-1}\}]$. For our purposes we need an element Δ in algebra $\mathbb{K}\Gamma$:

$$\Delta = 1 + \sum s_{ij} l_{ij}, \tag{34}$$

where sum is taken over all oriented edges. We call it (generalized) Laplace operator of the graph.

Consider algebra $\mathbb{K}\Gamma_\Delta$ obtained from algebra $\mathbb{K}\Gamma$ of Poincare groupoid and the element Δ via the above construction. Denote by x_i's the elements in $\mathbb{K}\Gamma_\Delta$ that correspond to e_i's in $\mathbb{K}\Gamma$. An important relation between algebras $B(\Gamma)$ and $\mathbb{K}\Gamma$ is established by the following

Theorem 8. *There is a unique isomorphism of non-unital algebras:*

$$B^+(\Gamma) \cong \mathbb{K}\Gamma_\Delta, \tag{35}$$

and hence, an isomorphism of unital algebras:

$$B(\Gamma) \cong \widehat{\mathbb{K}\Gamma_\Delta} \tag{36}$$

that takes x_i into e_i.

Proof. Firstly,

$$x_i^2 = e_i \cdot_\Delta e_i = e_i \Delta e_i = e_i = x_i.$$

Further, we have:

$$x_i x_j = e_i \cdot_\Delta e_j = e_i \Delta e_j = s_{ij} l_{ij},$$

for $(ij) \in E(\Gamma)$ and

$$x_i x_j = 0, \tag{37}$$

when (ij) is not an edge of Γ.

We also have:

$$l_{ij} \cdot_\Delta l_{jk} = l_{ij} l_{jk} \tag{38}$$

whenever $(ij) \in E(\Gamma)$ and $(jk) \in E(\Gamma)$. Hence,

$$x_i x_j x_i = s_{ij}^2 l_{ij} l_{ji} = s_{ij}^2 x_i. \tag{39}$$

Thus, we have checked the defining relations for $B(\Gamma)$. In other words, we constructed a homomorphism

$$B(\Gamma) \to \widehat{\mathbb{K}\Gamma_\Delta},$$

which we need to show to be an isomorphism. The above equations imply that for every path $\gamma = (i_1 \dots i_l)$, $l \geq 2$, in the graph we have

$$x_{i_1} \dots x_{i_l} = x_{i_1} x_{i_2}^2 \dots x_{i_{l-1}}^2 x_{i_l} = s_{i_1 i_2} \dots s_{i_{l-1} i_l} l_{i_1 i_2} \cdot_\Delta \dots \cdot_\Delta l_{i_{l-1} i_l} = s_{i_1 i_2} \dots s_{i_{l-1} i_l} l_{i_1 i_2} \dots l_{i_{l-1} i_l}. \tag{40}$$

where we consider multiplication in $\mathbb{K}\Gamma_\Delta$ at the left hand side and that in $\mathbb{K}\Gamma$ at the right hand side. To get the last equality we use iteratively (38) and the fact that left multiplication in $\mathbb{K}\Gamma_\Delta$ commutes with right multiplication in $\mathbb{K}\Gamma$ (or the explicit formula for multiplication).

Since the set of elements $l_{i_1 i_2} \dots l_{i_{l-1} i_l}$ where $(i_1, \dots, i_l)$ run over all contracted paths in Γ is a $\mathbb{K}$-basis in $\mathbb{K}\Gamma$, bijectivity of the homomorphism follows from proposition 6. □

A tail is a vertex in the graph with valency 1.

Proposition 9. *Let Γ be a non-empty connected graph with no tail and $\mathbb{K}$ a commutative ring with no zero divisor. Then Δ is neither left nor right zero divisor in $\mathbb{K}\Gamma$.*

Proof. Assume that $z \in \mathbb{K}\Gamma$ is such that $z\Delta = 0$. Since $\mathbb{K}\Gamma$ is a direct sum of right modules P_t, where t runs over all vertices of the graph, every component z_t of z in this direct sum decomposition satisfies $z_t \Delta = 0$. Thus, we reduced the problem to the case when $z \in P_t$ for some $t \in V(\Gamma)$. Consider the universal covering graph $\tilde{\Gamma}$ with a lift $\tilde{t} \in V(\tilde{\Gamma})$ of the vertex t. Then we have a unique lift $\tilde{z} \in \mathbb{K}\tilde{\Gamma}$ of z in the groupoid algebra of the universal cover. Clearly, $\tilde{z}\tilde{\Delta} = 0$, where $\tilde{\Delta}$ is the Laplace operator for the universal cover. Note that $\mathrm{H}^1(\Gamma, \mathbb{Z}) \neq 0$, because the graph with zero homology is a tree and tree always has a tail. This implies that $\tilde{\Gamma}$ is infinite. Thus the sum in (34) for $\tilde{\Delta}$ is infinite, but $\tilde{z}\tilde{\Delta}$ is a finite element, because $\tilde{z}$ has an expression as a finite sum

$$\tilde{z} = \sum \lambda_\gamma l_\gamma, \tag{41}$$

where γ runs over some finite set of contracted paths in $\tilde{\Gamma}$ (with start at $\tilde{t}$), and $\lambda_\gamma \neq 0$. Let γ_o is a contracted path in this sum with maximal length, and i is the ending vertex of this path. Because the vertex is not a tail, there is an edge l_{ij} incident to this vertex, which is not contained in γ_o. Decomposition of $\tilde{z}\tilde{\Delta}$ into linear combination of contracted paths will contain a non-zero summand $\lambda_{\gamma_o} s_{ij} l_{\gamma_o} l_{ij}$ (no other terms than $\lambda_{\gamma_o} l_{\gamma_o}$ from (41) can contribute to it). Since $\widetilde{z\Delta} = \tilde{z}\tilde{\Delta}$, we got a contradiction. The absence of right zero divisors for Δ is proven similarly.

□

One can easily find graphs with tails for which Δ is a zero divisor.

Consider the homomorphisms: $\psi_i : B(\Gamma) \to \mathbb{K}\Gamma, i = 1, 2$. They map the generators of $B(\Gamma)$ as follows:

$$\psi_1 : x_i \mapsto e_i + \sum_j s_{ij} l_{ij}, \tag{42}$$

$$\psi_2 : x_i \mapsto e_i + \sum_j s_{ji} l_{ji}. \tag{43}$$

Corollary 10. *Let Γ be a non-empty connected graph with no tail and $\mathbb{K}$ a commutative ring with no zero divisor. Then homomorphisms $\psi_i : B(\Gamma) \to \mathbb{K}\Gamma$, for $i = 1, 2$, are injective.*

The last remark we would like to make is about the parameters s_{ij} which enter the definition of Laplace operator. We can consider a symmetric square matrix $S = \{s_{ij}\}$ with invertible elements at the diagonal and define Laplace operator by:

$$\Delta = \sum s_{ii} e_i + \sum s_{ij} l_{ij}.$$

The corresponding algebra $\mathbb{K}\Gamma_\Delta$ will have generators x_i, corresponding to point paths in the groupoid, which are not projectors. But they can be normalized by a nonzero scalar to be projectors. Multiplication of x_i by λ corresponds to multiplication of the elements in i-th line and i-th row of the matrix S by λ. These multiplications allow to reduce to the case when diagonal entries of the matrix are ones, which we considered. The combinatorial structure of the graph encodes the places with nonzero entries in the matrix S.

4.6 Properties of Poincare groupoid as a module over $B(\Gamma)$

We shall prove several useful facts about $\mathbb{K}\Gamma$ which reflect its properties as a (bi)module over $B(\Gamma)$.

Proposition 11. *The isomorphism in theorem 8 identifies $B(\Gamma)x_i$ with ${}_{\psi_1}P_i = {}_{\psi_1}(\mathbb{K}\Gamma e_i)$ (i.e. with P_i with left $B(\Gamma)$-module structure induced by ψ_1) and $x_i B(\Gamma)$ with $(e_i\mathbb{K}\Gamma)_{\psi_2}$ (i.e. with $e_i\mathbb{K}\Gamma$ with right $B(\Gamma)$-module structure induced by ψ_2). We have an isomorphism of left $B(\Gamma)$-modules:*

$$B^+(\Gamma) = \oplus_i \, B(\Gamma)x_i \tag{44}$$

and an isomorphism of right $B(\Gamma)$-modules

$$B^+(\Gamma) = \oplus_i \, x_i B(\Gamma) \tag{45}$$

Proof. Formulas (37) and (40) show that $B(\Gamma)x_i$ coincides with $\mathbb{K}\Gamma e_i$ and $x_i B(\Gamma)$ does with $e_i\mathbb{K}\Gamma$. Since the left $B(\Gamma)$-module structure on $\mathbb{K}\Gamma$ comes via ψ_1 and the right module structure does via ψ_2 the identifications follow. The decompositions

$$\mathbb{K}\Gamma = \oplus_i\ \mathbb{K}\Gamma e_i, \quad \mathbb{K}\Gamma = \oplus_i\ e_i\mathbb{K}\Gamma$$

imply the decompositions for $B^+(\Gamma)$. □

Since $\mathbb{K}\Gamma$ is isomorphic to $B^+(\Gamma)$, this proposition is applicable to $\mathbb{K}\Gamma$ when we consider it as a left or right module over $B(\Gamma)$.

Corollary 12. *$B^+(\Gamma)$ (hence, $\mathbb{K}\Gamma$) is projective as left and right $B(\Gamma)$-module.*

Proof. Since, x_i is an idempotent, we have a decomposition:

$$B(\Gamma) = B(\Gamma)x_i \oplus B(\Gamma)(1 - x_i).$$

Being a direct summand of a free module, module $B(\Gamma)x_i$ is projective. Hence $B^+(\Gamma)$ is so. □

In the framework of algebra A with a fixed element Δ, the multiplication map defines an A-bimodule homomorphism $A \otimes_{\widehat{A}_\Delta} A \to A$.

Proposition 13. *Multiplication map in $B^+(\Gamma)$ defines an isomorphism of $\mathbb{K}\Gamma$-bimodules:*

$$\mathbb{K}\Gamma \otimes_{B(\Gamma)} \mathbb{K}\Gamma = \mathbb{K}\Gamma. \tag{46}$$

Proof. $\mathbb{K}\Gamma$ is identified with $B^+(\Gamma)$ as a left and right $B(\Gamma)$-module. Thus we need to show that $B^+(\Gamma) \otimes_{B(\Gamma)} B^+(\Gamma) = B^+(\Gamma)$. Every element in $B^+(\Gamma)$ has the form

$$b = \sum x_i b_i,$$

for some $b_i \in B^+$, in particular, for $b = x_i$, we have $x_i = x_i x_i$. This implies that b is the image of $x_i \otimes b_i$ under multiplication map, i.e. the map is surjective. Also, this implies that every element $p \in B^+(\Gamma) \otimes_{B(\Gamma)} B^+(\Gamma)$ can be presented in the form

$$p = \sum x_i \otimes c_i.$$

If the image of p under multiplication is zero, then

$$\sum x_i c_i = 0$$

By proposition 11 this implies that $x_i c_i = 0$ for all i. Then, $x_i \otimes c_i = x_i^2 \otimes c_i = x_i \otimes x_i c_i = 0$, i.e. $p = 0$. Hence, multiplication map is also injective. □

5 $\widehat{A}_\Delta$ for well-tempered Δ

Given an algebra A over a commutative ring $\mathbb{K}$ and a fixed element Δ, we formalize suitable conditions on this element which imply properties of $B(\Gamma)$ described in the previous subsection. We call elements satisfying these conditions well-tempered. They provide a good framework for studying representation theory of a class of algebras whom $B(\Gamma)$ belongs to.

5.1 Well-tempered elements

When considering algebras over a commutative ring $\mathbb{K}$, we always assume them to be free as modules over $\mathbb{K}$. By default, tensor products are assumed to be over $\mathbb{K}$.

Recall a bit of noncommutative differential geometry. Let B be a unital algebra over $\mathbb{K}$. Then B-bimodule Ω^1_B of noncommutative 1-forms is defined as the kernel of the multiplication map: $B \otimes B \to B$. Thus, we have a short exact sequence:

$$0 \to \Omega^1_B \to B \otimes B \to B \to 0. \tag{47}$$

The universal derivation $B \to \Omega^1_B$ is defined by the formula:

$$\mathrm{d}b = b \otimes 1 - 1 \otimes b,$$

for $b \in B$.

Let M be a left module over algebra B. We say that $\nabla : M \to \Omega^1_B \otimes_B M$ is a connection on M if it satisfies the condition:

$$\nabla(bm) = (\mathrm{d}b)m + b\nabla(m),$$

for all $b \in B$ and $m \in M$.

Lemma 14. *[CQ1] Module M is projective if and only if there exists a connection upon it.*

Proof. Taking tensor product over B of sequence (47) with M gives a short exact sequence:

$$0 \to \Omega^1_B \otimes_B M \to B \otimes M \to M \to 0,$$

because $\mathrm{Tor}^B_1(B, M) = 0$. Since $B \otimes M$ is a free B-module, projectiveness of M is equivalent to splitting of this sequence, i.e. existence of a B-module homomorphism $B \otimes M \to \Omega^1_B \otimes_B M$ which is a retraction on $\Omega^1_B \otimes_B M$. When composed with the embedding $M \to B \otimes M$, where $m \mapsto 1 \otimes m$ this retraction is nothing but a connection on M. □

Note that Ω^1_B is projective as both left and right B-module. The splitting of the sequence (47) as of left (respectively, right) B-modules is given by map $B \to B \otimes B$ defined by $b \mapsto b \otimes 1_B$ (respectively, $b \mapsto 1_B \otimes b$).

Let B be an augmented unital algebra with augmentation ideal B^+. We shall use the short exact sequence given by the augmentation:

$$0 \to B^+ \to B \to \mathbb{K} \to 0. \tag{48}$$

Lemma 15. *We have an isomorphism of left B-modules:*

$$\Omega^1_B = B \otimes B^+,$$

and an isomorphism of right B-modules:

$$\Omega^1_B = B^+ \otimes B,$$

Proof. By definition of Ω^1_B, we have an embedding $\Omega^1_B \to B \otimes B$. Composite of this map with the projection $B \otimes B \to B \otimes B^+$ that kills $B \otimes 1_B$ gives the required isomorphism for left B-modules. The projection $B \otimes B \to B^+ \otimes B$ that kills $1_B \otimes B$ does the case for right B-modules. □

Now we consider $M = B^+$ as a left or right B-module.

Proposition 16. *Assume that multiplication map $B^+ \otimes B^+ \to B^+$ is epimorphic. Then the following are equivalent:*

(i) B^+ is projective as a left (respectively, right) B-module,

(ii) there is a left (respectively, right) B-module homomorphism which is a section to the map $B^+ \otimes B^+ \to B^+$ given by multiplication in B^+,

(iii) left B-submodule $\Omega^1_B B^+$ in Ω^1_B is a direct summand (respectively, right B-submodule $B^+\Omega^1_B$ in Ω^1_B is a direct summand).

Proof. Left projectiveness of B^+ is equivalent to the existence of a left B-module homomorphism $s : B^+ \to B \otimes B^+$, a section to the multiplication map $B \otimes B^+ \to B^+$. By assumption, every element $b \in B^+$ has a decomposition $b = \sum b_i c_i$ with $b_i, c_i \in B^+$. Since s is a B-module homomorphism, we got that $s(b) = \sum b_i s(c_i) \in B^+ \otimes B^+$, i.e. the image of s lies in $B^+ \otimes B^+$. Thus, we can consider s as a section required in (ii). This proves equivalence of (i) and (ii).

Apply functor $\Omega^1_B \otimes_B (-)$ to the augmentation sequence (48). Since Ω^1_B is projective as a right B-module, it is also flat. Thus we get a short exact sequence:

$$0 \to \Omega^1_B \otimes_B B^+ \to \Omega^1_B \to \Omega^1_B \otimes_B \mathbb{K} \to 0.$$

The image of the first homomorphism is $\Omega^1_B B^+$. In view of lemma 15, we have isomorphisms of left B-modules $\Omega^1_B = B \otimes B^+$. Applying functor $(-) \otimes_B \mathbb{K}$ to (47), we get that $\Omega^1_B \otimes_B \mathbb{K} = B^+$ as a left B-module. Easy calculation shows that, under this identifications, the homomorphism $\Omega^1_B \to \Omega^1_B \otimes_B \mathbb{K}$ in the above short exact sequence coincides up to sign with the morphism $B \otimes B^+ \to B^+$ given by multiplication. Thus the kernel of the multiplication map $B \otimes B^+ \to B^+$ is isomorphic to $\Omega^1_B B^+$. This map has a section if and only if B^+ is a projective left B-module. Therefore $\Omega^1_B B^+$ is a direct summand in Ω^1_B exactly when B^+ is projective. □

Given an element Δ in an algebra A, let $B = \widehat{A}_\Delta$ and $B^+ = A_\Delta$ the augmentation ideal in B.

Consider complex K_Δ:

$$\cdots \to A^{\otimes n} \to \cdots \to A^{\otimes 3} \to A^{\otimes 2} \to A \to 0 \tag{49}$$

with differential $\mathrm{d} : A^{\otimes n+1} \to A^{\otimes n}$ defined by:

$$d(a_0 \otimes \cdots \otimes a_n) = \sum (-1)^i a_0 \otimes \cdots \otimes a_i \Delta a_{i+1} \otimes \cdots \otimes a_n.$$

This is the bar-complex for B^+ expressed in terms of A. It might not be exact in general, because B^+ is not a unital algebra.

Lemma 17. *Complex of A-bimodules K_Δ is isomorphic to $K_{\Delta'}$ with*

$$\Delta' = c\Delta d,$$

where c and d are any invertible elements in A.

Proof. The isomorphism of complexes $K_{\Delta'} \simeq K_\Delta$ is given by:

$$a_0 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n \mapsto a_0 c \otimes \cdots \otimes d a_i c \otimes \cdots \otimes d a_n. \tag{50}$$

□

Definition. An element Δ in algebra A is said to be *well-tempered* if the multiplication map $B^+ \otimes B^+ \to B^+$ is an epimorphism and B^+ is projective as left and right B-module.

By the previous subsection, Laplace operators are well-tempered elements in $\mathbb{K}\Gamma$.

Lemma 18. *If algebra A is a commutative, then Δ is well-tempered if and only if it is invertible.*

Proof. Note that the image of the multiplication map $B^+ \otimes B^+ \to B^+$, in terms of A, is the two-sided ideal $A\Delta A$. Therefore, if A is a commutative and Δ is well-tempered, then $\Delta A = A$, i.e. Δ is invertible. Conversely, if Δ is invertible, then the multiplication map is epimorphic, and we can define the left (respectively, right) B-module section to it by mapping $b \mapsto b \otimes \Delta^{-1}$ (respectively, $\Delta^{-1} \otimes b$). □

As we have seen, this implies that there is an isomorphism of algebras $B \simeq A \oplus \mathbb{K}$ for the commutative case.

According to proposition 16, Δ is well-tempered if and only if the most right differential $A \otimes A \to A$ in the complex K_Δ is epimorphic and allows sections which are left and right B-module homomorphism.

Lemma 19. *Let c and d be any invertible elements in A. Element Δ is well-tempered if and only if $\Delta' = c\Delta d$ is so.*

Proof. Left B-module structure on K_Δ is the pull-back along ψ_1 of the left A-module structure and right B-module structure on K_Δ is the pull-back along ψ_2 of the right A-module structure. Hence isomorphism (50) of complexes K_Δ and $K_{\Delta'}$ is a morphism of B-bimodules. Thus, the property to have a left or right B-module section for the most right differential is preserved by the isomorphism of complexes. □

Proposition 20. *If Δ is well-tempered, then complex K_Δ is exact.*

Proof. As we already mentioned, complex K_Δ coincides with bar-complex for B^+:

$$\cdots \to (B^+)^{\otimes i} \to \cdots \to B^+ \otimes B^+ \to B^+.$$

Let $h : B^+ \to B^+ \otimes B^+$ be a homomorphism of right B-modules and a section to the most right differential in the complex. It exists by proposition 16. Consider homotopy in this complex defined by :

$$h(b_1 \otimes b_2 \otimes b_2 \otimes \cdots \otimes b_n) = h(b_1) \otimes b_2 \otimes \cdots \otimes b_n.$$

One can easily check that it satisfies the equation:

$$\mathrm{d}h + h\mathrm{d} = \mathrm{id}$$

Hence the complex is exact. □

Note that we used only one-sided projectiveness of B^+ in the proof of this proposition.

Corollary 21. *If Δ is well-tempered, then multiplication in B^+ gives an isomorphism of A-bimodules:*

$$B^+ \otimes_B B^+ = B^+. \tag{51}$$

Proof. Indeed, the quotient of $(B^+)^{\otimes 2}$ by the image of the differential $(B^+)^{\otimes 3} \to (B^+)^{\otimes 2}$ is isomorphic to $B^+ \otimes_B B^+$. Since there is no homology of K_Δ in the terms $B^+ \otimes B^+$ and B^+, then this quotient is identified with B^+. □

Consider the homomorphism of A-bimodules

$$A \to \mathrm{Hom}_B(A, A) \tag{52}$$

that takes $a \in A$ to the homomorphism $A \to A$ of left B-modules defined by right multiplication with a in A. To make it compatible with multiplication in the endomorphisms ring of A we need to take the opposite multiplication in the algebra A at the left hand side.

Proposition 22. *Let $\Delta \in A$ is well-tempered, and $B = \widehat{A}_\Delta$. Then map (52) defines an isomorphism of algebras and of A-bimodules:*

$$A^{opp} = \mathrm{Hom}_B(A, A) \tag{53}$$

Proof. Recall that $B^+ = A$ as an A-bimodule. It is a straightforward check that (52) defines a homomorphism of algebras and of A-bimodules. Apply functor $\mathrm{Hom}(-, A)$ to isomorphism (51). It gives:

$$A = \mathrm{Hom}_A(A, A) = \mathrm{Hom}_A(A \otimes_B A, A) = \mathrm{Hom}_B(A, A).$$

Again, it is straightforward to check that this isomorphism coincides with the one we consider. □

Since Laplace operator is a well-tempered element in $\mathbb{K}\Gamma$, this proposition implies

Corollary 23. *We have an isomorphism of algebras and of $\mathbb{K}\Gamma$-bimodules:*

$$\mathbb{K}\Gamma^{opp} = \mathrm{Hom}_{B(\Gamma)}(\mathbb{K}\Gamma, \mathbb{K}\Gamma) \tag{54}$$

There is a morphism of left A-modules

$$A \to \mathrm{Hom}_B(A, B) \tag{55}$$

that takes an element $a \in A$ to the composite of the operator of right multiplication by a, $R_a : A \to A = B^+$, with embedding $B^+ \to B$.

Proposition 24. *Let $\Delta \in A$ is well-tempered, and $B = \widehat{A}_\Delta$. Morphism (55) is an isomorphism of left A-modules:*

$$A = \mathrm{Hom}_B(A, B)$$

Proof. Apply functor $\mathrm{Hom}_B(A, -)$ to the short exact sequence (48). We obtain:

$$0 \to \mathrm{Hom}_B(A, B^+) \to \mathrm{Hom}_B(A, B) \to \mathrm{Hom}_B(A, \mathbb{K}) \to \tag{56}$$

The first term in this sequence is isomorphic to A by the previous proposition. The last term is zero, because any homomorphism from $A = B^+$ to the trivial B-module $\mathbb{K}$ must annihilate the image of the multiplication map $B^+ \otimes B^+ \to B^+$, which is epimorphic by the definition of well-tempered elements.

□

Corollary 25. *We have an isomorphism of left $\mathbb{K}\Gamma$-modules:*

$$\mathbb{K}\Gamma = \mathrm{Hom}_{B(\Gamma)}(\mathbb{K}\Gamma, B(\Gamma))$$

The finite generation for B^+ as a B-module that we have seen to hold for $B(\Gamma)$ is also true in general for algebras B constructed from a well-tempered elements. More precisely, we have

Lemma 26. *Let $B = \widehat{A}_\Delta$ is such that the multiplication map $B^+ \otimes B^+ \to B^+$ is epimorphic. Then B^+ is finitely generated as a right and left B-module.*

Proof. The multiplication map for B^+ is epimorphic if two-sided ideal $A\Delta A$ is A. Therefore, we have a decomposition for the unit in A:

$$1_A = \sum x_i \Delta y_i.$$

It implies that any $b \in B^+ = A$ has a decomposition:

$$b = \sum b x_i \Delta y_i = \sum (b x_i) \cdot_B y_i,$$

where $\cdot_B$ is multiplication in B. Therefore, B^+ is left generated by the finite set $\{y_i\}$. Similarly, it is right generated by the set $\{x_i\}$. □

5.2 Hochschild and global dimension of $\widehat{A}_\Delta$

Hochschild dimension of a $\mathbb{K}$-algebra B is defined as projective dimension of B as a B-bimodule, i.e. as a module over $B \otimes B^{opp}$ (note that the tensor product is taken over $\mathbb{K}$). We denote it by $\mathrm{Hdim}B$. It is invariant under Morita equivalences. In more general approach, when B is a DG-algebra, it is called *smooth* if B is a perfect $B \otimes B^{opp}$-bimodule. Smoothness is a derived Morita invariant property, but Hochschild dimension might not be preserved under derived equivalences. For a genuine algebra B, smoothness is equivalent to finiteness of Hochschild dimension.

Lemma 27. *Let Δ be a well-tempered element in algebra A and $B = \widehat{A}_\Delta$. If P is a projective A-bimodule, then ${}_{\psi_1}P_{\psi_2}$ is a projective B-bimodule.*

Proof. Since B^+ is projective as a left and right B-module, then $B^+ \otimes B^{+opp}$ is projective as a $B \otimes B^{opp}$-module. Projective $A \otimes A^{opp}$-module P is a direct summand in a free $A \otimes A^{opp}$-module $A \otimes U \otimes A^{opp}$, where U is a free $\mathbb{K}$-module. Therefore, ${}_{\psi_1}P_{\psi_2}$ is a direct summand in ${}_{\psi_1}A \otimes U \otimes A^{opp}_{\psi_2} = B^+ \otimes U \otimes B^{+opp}$, which is a projective $B \otimes B^{opp}$-module. Therefore, it is projective itself. □

Theorem 28. *Let Δ be a well-tempered element in algebra A and $B = \widehat{A}_\Delta$. Then* $\mathrm{Hdim} B$ *is less than or equal to* $\max(\mathrm{Hdim} A, 2)$.

Proof. Since A has a projective resolution of length $\mathrm{Hdim} A$ as $A \otimes A^{opp}$-bimodule, then B^+, which is isomorphic to ${}_{\psi_1}A_{\psi_2}$, has a projective resolution of the same length as $B \otimes B^{opp}$-bimodule, by the above lemma.

The trivial module $\mathbb{K}$ has the following resolution of length 2 by projective $B \otimes B^{opp}$-modules:

$$0 \to B^+ \otimes B^+ \to (B \otimes B^+) \oplus (B^+ \otimes B) \to B \otimes B \to \mathbb{K} \to 0.$$

The augmentation exact sequence (48) implies that projective $B \otimes B^{opp}$-dimension of B is not greater than maximum of projective dimensions of B^+ and of $\mathbb{K}$. □

Thus, we see that the smoothness of A in the DG-sense implies smoothness of B.

Corollary 29. *Hochschild dimension of $B(\Gamma)$ is less than or equal to 2.*

Proof. By proposition 7, algebra $\mathbb{K}\Gamma$ is Morita equivalent to a matrix algebra over $\mathbb{K}[\pi(\Gamma), t]$. Since the fundamental group of Γ is free, algebra $\mathbb{K}[\pi(\Gamma), t]$ is quasi-free (relatively over $\mathbb{K}$) in the sense of Cuntz and Quillen [CQ1], hence Hochschild dimension of $\mathbb{K}\Gamma$ is ≤ 1. As Hochschild dimension is Morita invariant, the above theorem gives the required upper bound for Hochschild dimension of $B(\Gamma)$. □

Recall that left (respectively, right) global dimension of an algebra is the maximum of projective dimensions of left (respectively, right) modules over the algebra. For an algebra B, we denote its left global dimension by $\mathrm{gldim}_l B$ and right global dimension by $\mathrm{gldim}_r B$.

As a consequence of the above theorem, we obtain

Theorem 30. *Let Δ be a well-tempered element in algebra A and $B = \widehat{A}_\Delta$. Then we have inequalities for left and right global dimensions of B:*

$$\mathrm{gldim}_l B \leq \max(\mathrm{Hdim} A, 2) + \mathrm{gldim}_l \mathbb{K},$$

$$\mathrm{gldim}_r B \leq \max(\mathrm{Hdim} A, 2) + \mathrm{gldim}_r \mathbb{K}.$$

Proof. Let M and N be any two left (or right) A-modules. We have spectral sequence with the sheet E_2:

$$\mathrm{E}_2^{ij} = \mathrm{Ext}^i_{A-A}(A, \mathrm{Ext}^j_{\mathbb{K}}(M, N)) \tag{57}$$

that converges to $\mathrm{Ext}^{i+j}_A(M, N)$. Thus, we get an upper bound for (left or right) global dimension of B from the upper bound on Hochschild dimension obtained in the previous theorem and from the upper bound on j for non-zero $\mathrm{Ext}^j_{\mathbb{K}}$ by global dimension of $\mathbb{K}$. □

Corollary 31. *If $\mathbb{K} = k$ is a field, then global dimension of the category of $B(\Gamma)$-modules is not higher than 2.*

6 Representation theory for $B(\Gamma)$.

6.1 Functors between categories $A - \mathrm{mod}$ and $\widehat{A}_\Delta - \mathrm{mod}$.

Assume again that we have a unital algebra A, a fixed element Δ in A , a non-unital algebra $B^+ = A_\Delta$ and its unital extension, algebra $B = \widehat{A}_\Delta$. We do not assume in this subsection that Δ is well-tempered.

By restricting the module structure along ψ_1 and ψ_2, we define push-forward functors on the categories of left modules:

$$\psi_{1*} : A - \mathrm{mod} \to B - \mathrm{mod}, \quad \psi_{2*} : A - \mathrm{mod} \to B - \mathrm{mod}.$$

We use notations for these functors that are compatible with the viewpoint of Noncommutative Algebraic Geometry, where homomorphisms ψ_1 and ψ_2 are assumed to define geometric maps between noncommutative affine spectra of unital algebras: $\mathrm{Spec}A \to \mathrm{Spec}B$. The noncommutative affine spectrum of an algebra is understood as an object of the category opposite to the category of associative algebras. Modules are understood as sheaves on the affine spectra.

There is a natural transformation of functors:

$$\lambda : \psi_{1*} \to \psi_{2*}, \tag{58}$$

For a representation $\rho : A \to \mathrm{End}V$ and $v \in V$, it is defined by

$$\lambda(v) = \rho(\Delta)v. \tag{59}$$

It is a straightforward check that this formula defines a natural transformation.

We say that a B-module is *B^+-trivial*, if B^+ acts by zero on it.

Lemma 32. *Let W be an A-module. Then the kernel and cokernel of λ_W are B^+-trivial modules. Moreover, we have an exact sequence with the middle morphism λ_W:*

$$0 \to (\psi_{1*}W)^{B^+} \to \psi_{1*}W \to \psi_{2*}W \to \psi_{2*}W/(B^+\psi_{2*}W) \to 0. \tag{60}$$

Proof. The map λ_W is given by the action of Δ on the representation space. This implies that the action of $\psi_1(b) = b\Delta$ on the kernel of λ_W is zero for any $b \in B^+$. Taking $b = 1_A$, we see that the kernel is exactly the submodule in $\psi_{1*}W$ which contains all elements on which B^+ acts trivially. Hence it is $(\psi_{1*}W)^{B^+}$.

The image of λ_W is the image of the action of Δ. It contains the image of the action of $\psi_2(b) = \Delta b$, for any $b \in B^+$, and, in fact, coincides with $B^+ \cdot \psi_{2*}W$ (again, consider $b = 1_A$). Thus the quotient of $\psi_{2*}W$ by the image is the space of co-invariants for B^+-action.

□

Functors ψ_{1*} and ψ_{2*} have right adjoints $\psi_1^!,\ \psi_2^! : B-\text{mod} \to A-\text{mod}$ defined by:

$$\psi_1^!,\ \psi_2^! : V \mapsto \text{Hom}_B(A,V), \tag{61}$$

where A is endowed with the structure of left B-module via ψ_1 or ψ_2 respectively.

Also functors ψ_{1*} and ψ_{2*} have left adjoints $\psi_1^*,\ \psi_2^* : B-\text{mod} \to A-\text{mod}$ defined by:

$$\psi_1^*,\ \psi_2^* : V \mapsto A \otimes_B V, \tag{62}$$

where A is endowed with right B-module structure via ψ_1 or ψ_2 respectively.

In order to distinguish the multiplication in A from that in B^+, we denote it by $\cdot_A$. Let $\rho : B \to \text{End}(V)$ be a representation of B. Consider map

$$\mu_V : A \otimes_B V \to \text{Hom}_B(A,V)$$

defined by:

$$a \otimes v \mapsto \phi_{a\otimes v} \in \text{Hom}_B(A,V), \tag{63}$$

where

$$\phi_{a\otimes v}(a') = \rho(a' \cdot_A a)v \tag{64}$$

Note that we used ψ_2 to define $A \otimes_B V$ and ψ_1 to define $\text{Hom}_B(A,V)$.

Lemma 33. *Formulas (63) and (64) define a natural transformation of functors:*

$$\mu : \psi_2^* \to \psi_1^!. \tag{65}$$

Proof. There are several things that we need to check. First, morphism ϕ is indeed a homomorphism of left B-modules. Second, morphism μ_V is well-defined on $A\otimes_B V$. Third, morphism μ_V is compatible with left A-module structures. Forth, morphisms μ_V are functorial with respect V. All this is a straightforward check, which we leave to the reader. □

By adjunction, we have a natural transformation:

$$\chi : \psi_{1*}\psi_2^* \to \text{id}.$$

Let V be a B-module with action of B given by $\rho : B \to \text{End}V$. Then $\psi_{1*}\psi_2^* V = A \otimes V$ and

$$\chi(a \otimes v) = \rho(a)v,$$

where a is interpreted as an element in B^+.

Denote the following adjunction morphisms by ϵ and δ:

$$\epsilon : \psi_{1*}\psi_1^! \to \text{id},$$

$$\delta : \text{id} \to \psi_{2*}\psi_2^*.$$

For a B-module V and $\varphi \in \text{Hom}_B(A,V) = \psi_{1*}\psi_1^! V$, we have

$$\epsilon_V(\varphi) = \varphi(1).$$

For $v \in V$, we have

$$\delta_V(v) = 1 \otimes v$$

as an element in $A \otimes_B V = \psi_{2*}\psi_2^* V$.

Lemma 34. *The kernel and cokernel of both ϵ and δ evaluated on any B-module V are B^+-trivial modules.*

Proof. We denote by ρ the action of B on V. Let $\varphi \in \mathrm{Hom}_B(A, V) = \psi_{1*}\psi_1^! V$. If φ belongs to the kernel of ϵ_V, then $\varphi(1) = 0$ and

$$(b \cdot \varphi)(a) = \varphi(a \cdot_A b \cdot_A \Delta) = \rho(a \cdot_A b)\varphi(1) = 0,$$

for any $a \in A$ and $b \in B^+$. Hence the kernel of ϵ is a B^+-trivial B-module.

If $v = \rho(b)u$ for some $b \in B^+$ and $u \in V$, then define $\varphi \in \mathrm{Hom}_B(A, V)$ by $\varphi(a) = \rho(a \cdot_A b)u$. This is indeed a homomorphism of B-modules and $\varphi(1) = v$. Hence the action by elements from B^+ on V has values in the image of ϵ, i.e. the cokernel of ϵ is a B^+-trivial B-module.

Let $a \otimes v$ be in $A \otimes_B V = \psi_{2*}\psi_2^* V$ and $b \in B^+$. Then

$$b \cdot (a \otimes v) = (\Delta \cdot_A b \cdot_A a) \otimes v = 1 \otimes \rho(b \cdot_A a)v.$$

Thus cokernel of δ_V is a B^+-trivial B-module.

If $v \in V$ is such that $\delta_V(v) = 1 \otimes v = 0$ in $A \otimes_B V = \psi_{2*}\psi_2^* V$. Using the fact that $A \otimes_B V$ is a left A-module, we get $a \otimes v = a(1 \otimes v) = 0$, for any $a \in A$. Now let us use the fact that $A \otimes_B V$ is the underlying vector space for $\psi_{1*}\psi_2^* V$, though with a different B-module structure, and apply $\chi_V : A \otimes_B V \to V$. We get:

$$\chi_V(a \otimes v) = \rho(a)v = 0,$$

where a is now interpreted as an arbitrary element in B^+. Therefore, the kernel of δ_V is a B^+-trivial B-module. □

Lemma 35. *Let V be a B-module. The composite of natural transformations:*

$$\psi_{1*}\psi_1^! V \xrightarrow{\epsilon_V} V \xrightarrow{\delta_V} \psi_{2*}\psi_2^* V \xrightarrow{\psi_{2*}(\mu_V)} \psi_{2*}\psi_1^! V \tag{66}$$

coincides with $\lambda_{\psi_1^! V}$.

Proof. This is a straightforward check. □

Example. If A is a commutative algebra, then $\psi_1 = \psi_2$. Let $A = k[t]$ be the algebra of polynomial in one variable and $\Delta = t^2$. Algebra $B = \widehat{A}_\Delta$ is the algebra of an affine curve with a cusp:

$$B = k[x, y]/(x^3 - y^2).$$

The homomorphism $\psi_1 = \psi_2 : B \to A$ is the normalization map for the cusp curve given by:

$$x = t^2, \quad y = t^3.$$

Take $V = A$ as a B-module. Then easy calculation shows that

$$A \otimes_B A \simeq k[t] \oplus k[t]/t^2,$$

and

$$\mathrm{Hom}_B(A, A) \simeq k[t].$$

Thus μ_A is not an isomorphism.

6.2 Calculations in the well-tempered case

Lemma 36. *Let Δ be a well-tempered element in A. Then functors $\psi_1^!$ and ψ_2^* are exact.*

Proof. Since B^+ is projective as left B-module, then $\psi_1^!$ is exact. Since B^+ is projective, hence flat, as right B-module, then ψ_2^* is exact. □

Lemma 37. *Let Δ be well-tempered, W an A-module and V a B^+-trivial B-module. Then*

(i) $\mathbb{R}\psi_1^! V = \mathbb{R}\mathrm{Hom}_B(A, V) = 0$;

(ii) $\mathbb{L}\psi_2^* V = A \otimes_B^{\mathbb{L}} V = 0$;

(iii) $\mathrm{Ext}_B^\bullet(\psi_{1*}W, V) = 0$;

(iv) $\mathrm{Ext}_B^\bullet(V, \psi_{2*}W) = 0$.

Proof. First,

$$\mathbb{R}\mathrm{Hom}_B(A, V) = \mathbb{R}\mathrm{Hom}_B(B^+, V) = \mathrm{Hom}_B(B^+, V),$$

because B^+ is a projective left B-module. Further any homomorphism from $A = B^+$ to a B^+-trivial B-module must annihilate the image of the multiplication map $B^+ \otimes B^+ \to B^+$. This map is epimorphic by the definition of well-tempered elements. This implies (i).

We have by adjunction:

$$\mathrm{Ext}_B^\bullet(\psi_{1*}W, V) = \mathrm{Ext}_A^\bullet(W, \mathbb{R}\psi_1^! V) = 0.$$

which proves (iii).

Similarly,

$$A \otimes_B^{\mathbb{L}} V = B^+ \otimes_B^{\mathbb{L}} V = B^+ \otimes_B V = 0,$$

because every element $b \in B^+$ has the form $b = \sum a_i \cdot_B b_i$, where $a_i, b_i \in B^+$. Hence $b \otimes v = \sum a_i \otimes b_i v = 0$ for any $v \in V$. This proves (ii) and implies

$$\mathrm{Ext}_B^\bullet(V, \psi_{2*}W) = \mathrm{Ext}_A^\bullet(\mathbb{L}\psi_2^* V, W) = \mathrm{Ext}_A^\bullet(A \otimes_B^{\mathbb{L}} V, W) = 0,$$

which proves (iv). □

Proposition 38. *The adjunction morphisms*

$$\psi_2^*\psi_{2*} \to \mathrm{id}, \quad \mathrm{id} \to \psi_1^!\psi_{1*}$$

are isomorphisms.

Proof. Let W be a left A-module. Since A is isomorphic B^+, isomorphism (51) reads as $A \otimes_B A = A$. Applying tensor product $(-) \otimes_A W$ to it gives: $A \otimes_B W = W$, which implies that the morphism $\eta_W : \psi_2^*\psi_{1*}W \to W$ obtained from λ_W by adjunction is an isomorphism. Now apply ψ_2^* to λ_W. By lemmas 32, 36 and 37(ii) we get that $\psi_2^*(\lambda_W) : \psi_2^*\psi_{1*}W \to \psi_2^*\psi_{2*}W$ is an isomorphism. Since η_W is the composite of $\psi_2^*(\lambda_W)$ and the adjunction morphism $\psi_2^*\psi_{2*}W \to W$, we get that $\psi_2^*\psi_{2*} \to \mathrm{id}$ is an isomorphism.

Now, applying tensor product $(-) \otimes_A W$ to isomorphism (53) implies that $\mathrm{id} \to \psi_1^!\psi_{1*}$ is an isomorphism. □

Proposition 39. *Let Δ be a well-tempered element in algebra A and $B = \widehat{A}_\Delta$. Then natural transformation μ from (65) gives an isomorphism $\psi_2^* \simeq \psi_1^!$ on the category $B-\mathrm{mod}$.*

Proof. Functor ψ_2^* takes rank 1 free B-module B to A and so does the functor $\psi_1^!$ due to proposition 24. Functors ψ_2^* and $\psi_1^!$ are exact and commute with infinite direct sum. The latter does, because A is a finitely generated left B-module due to proposition 26. Hence, a presentation for a B-module V as a cokernel of a homomorphism of free B-modules implies similar presentation for $\psi_2^* V$ and $\psi_1^! V$ as cokernels of a homomorphisms of free A-modules, while μ induces an isomorphism of these presentations. □

Proposition 40. *Let Δ be a well-tempered. For any two B^+-trivial B-modules U and V and any $i \in \mathbb{Z}$, we have:*

$$\mathrm{Ext}^i_B(U,V) = \mathrm{Ext}^i_{\mathbb{K}}(U,V).$$

In particular, $\mathbb{K}$ is an exceptional object, i.e. $\mathrm{Hom}_B(\mathbb{K},\mathbb{K}) = \mathbb{K}$ *and* $\mathrm{Ext}^i_B(\mathbb{K},\mathbb{K}) = 0$, *for $i \neq 0$.*

Proof. Clearly, $\mathrm{Hom}_B(U,V) = \mathrm{Hom}_{\mathbb{K}}(U,V)$. Let F be a free $\mathbb{K}$-module. Let us show that $\mathrm{Ext}^{>0}_B(F,V) = 0$, for any B^+-module V. Applying functor $(-)\otimes F$ to the augmentation exact sequence (48) gives a short exact sequence of B-modules:

$$0 \to B^+ \otimes F \to B \otimes F \to F \to 0$$

Apply functor $\mathrm{Ext}^i_B(-,V)$ to it. Since B^+ and B are projective B-modules, we get $\mathrm{Ext}^{>1}(F,V) = 0$ and an exact sequence:

$$0 \to \mathrm{Hom}_B(F,V) \to \mathrm{Hom}_B(B\otimes F,V) \to \mathrm{Hom}_B(B^+ \otimes F,V) \to \mathrm{Ext}^1_B(F,V) \to 0$$

Since $B^+ = \psi_{1*}A$, then $\mathrm{Hom}_B(B^+,V) = 0$ by lemma 37(iii). Since F is free, we have: $\mathrm{Hom}_B(B^+ \otimes F,V) = 0$ and, in view of the exact sequence, $\mathrm{Ext}^1_B(F,V) = 0$, too. Calculation of $\mathrm{Ext}^i_B(U,V)$ via free $\mathbb{K}$-resolutions imply the required isomorphism wit $\mathrm{Ext}^i_{\mathbb{K}}(U,V)$.

□

Proposition 41. *Let Δ be well-tempered and W an A-module. Then we have an isomorphisms of B-modules:*

$$\mathrm{Hom}_B(B^+,\psi_{1*}W) = \psi_{2*}W. \tag{67}$$

Further, there is a quasi-isomorphism of complexes:

$$\mathbb{R}\mathrm{Hom}_B(\mathbb{K},\psi_{1*}W) = \{\, 0 \longrightarrow \psi_{1*}W \xrightarrow{\lambda_W} \psi_{2*}W \longrightarrow 0 \,\}, \tag{68}$$

with two non-trivial components in degree 0 and 1.

Proof. We have an isomorphism of A-modules $\mathrm{Hom}_B(A,\psi_{1*}W) = \psi_1^!\psi_{1*}W = W$ by proposition 38. $\mathrm{Hom}_B(B^+,\psi_{1*}W)$ is the same vector space with the left action of B coming from its right action on A, defined by ψ_2. This proves (67).

Using augmentation sequence (48) as a projective resolution for left B-module $\mathbb{K}$ and the above identification for $\mathrm{Hom}_B(B^+,\psi_{1*}W)$, we obtain (68). □

Lemma 42. *Let Δ be well-tempered and W an A-module. Then:* $\mathrm{Hom}_B(\mathbb{K}, \psi_{1*}W) = (\psi_{1*}W)^{B^+}$, $\mathrm{Ext}^1_B(\mathbb{K}, \psi_{1*}W) = \psi_{2*}W/B^+\psi_{2*}W$, *and* $\mathrm{Ext}^{>1}_B(\mathbb{K}, \psi_{1*}W) = 0$. *In particular,* $\mathrm{Hom}_B(\mathbb{K}, B^+) = \{a \in A | \Delta a = 0\}$, $\mathrm{Ext}^1_B(\mathbb{K}, B^+) = A/\Delta A$ *and* $\mathrm{Ext}^{>1}_B(\mathbb{K}, B^+) = 0$.

Proof. This follows from (68) and 60. The particular case is when we take $W = A$. □

Proposition 43. *Let Δ be well-tempered, V a B-module and U a B^+-trivial B-module. Then* $\mathrm{Ext}^i_B(V, U) = 0$*, for* $i > \mathrm{gldim}\mathbb{K} + 1$.

Proof. By lemma 34 we have an exact sequence

$$0 \to K \to \psi_{1*}\psi_1^! V \to V \to Q \to 0 \qquad (69)$$

with K and Q B^+-trivial B-modules. Consider the spectral sequence that calculates functor $\mathrm{Ext}^\bullet_B(-, U)$ on this exact sequence. Its E_1-term has Ext-groups on top of terms of the sequence and it converges to zero, because the sequence is exact. We have by adjunction:

$$\mathrm{Ext}^\bullet_B(\psi_{1*}\psi_1^! V, U) = \mathrm{Ext}^\bullet_A(\psi_1^! V, \psi_1^! U),$$

because $\psi_1^!$ is an exact functor by lemma 36. By lemma 37, $\psi_1^! U = 0$. Therefore, $\mathrm{Ext}^\bullet_B(\psi_{1*}\psi_1^! V, U) = 0$. Moreover, since K and Q are trivial modules, $\mathrm{Ext}^i_B(K, U) = 0$ and $\mathrm{Ext}^i_B(Q, U) = 0$, for $i > \mathrm{gldim}\mathbb{K}$, by proposition 40. The spectral sequence immediately implies that $\mathrm{Ext}^i_B(V, U) = 0$, for $i > \mathrm{gldim}\mathbb{K} + 1$. □

6.3 Coherence of algebras and categories of finitely presented modules

Let $A = \mathbb{K}\Gamma$. Then we know that $B(\Gamma) = \widehat{A}_\Delta$. According to our original problem, we are interested in the categories of representations for $B(\Gamma)$, which are of finite rank over $\mathbb{K}$, but general theory dictates that we should start with the category of finitely presented modules. First we need to show that this category is abelian.

A left module M of an algebra is said to be *coherent* if it is finitely generated and for every morphism $\varphi : P \to M$ with free module P of finite rank the kernel of φ is finitely generated. An algebra is *(left) coherent* if it is coherent as a left module over itself. If algebra is coherent, then finitely presented modules are the same as coherent modules and the category of finitely presented modules is abelian [**?**].

By proposition 7, algebra $\mathbb{K}\Gamma$ is isomorphic to the matrix algebra over $\mathbb{K}[\pi(\Gamma, t)]$. Hence, it is quasi-free relatively over $\mathbb{K}$. The definitions due to Cuntz and Quillen [CQ1] of a quasi-free algebra, which we adopt to the relative case (i.e. for algebras over a a commutative ring $\mathbb{K}$ rather than over a field), is that algebra A is *quasi-free* if the bimodule of noncommutative differential 1-forms $\Omega^1 A$ is a projective $A \otimes A^{opp}$-module.

If $\mathbb{K} = k$ is a field, then every quasi-free algebra is *hereditary*, i.e. has global dimension ≤ 1. Indeed, for any two A-modules M and N we have:

$$\mathrm{Ext}^i_A(M, N) = \mathrm{Ext}^i_{A-A}(A, \mathrm{Hom}_k(M, N)).$$

Since A has a projective bimodule resolution of length 1, it follows that $\mathrm{Ext}^i_A(M,N)=0$, for $i\geq 2$.

Hereditary algebras are coherent. Indeed, a submodule of a projective module is projective for such algebras. Thus, given a morphism $\varphi : P_1 \to P_2$ between finitely generated projective modules, the image I of φ is a submodule of a projective module, hence projective too. Then short exact sequence induced by φ:

$$0 \to K \to P_1 \to I \to 0,$$

where K is the kernel of φ, splits. Hence, we have an epimorphism $P_1 \to K$, which proves that K is finitely generated. Since $k\Gamma$ is quasi-free, hence hereditary, we got that the category $k\Gamma - \mathrm{mod}_{fp}$ of finitely presented $k\Gamma$-modules is abelian.

The above argument works for quasi-free algebras when the base ring is a field only. We develop the theory for the case of relatively quasi-free algebras. A $\mathbb{K}$-algebra A is said to be *central over* $\mathbb{K}$ if the image of $\mathbb{K} \to A$ is in the center of A. Define the $A \otimes_{\mathbb{K}} A^{opp}$-module of relative forms by an exact sequence

$$0 \to \Omega^1_{A/\mathbb{K}} \to A \otimes_{\mathbb{K}} A \to A \to 0 \tag{70}$$

Definition Let $\mathbb{K}$ be a commutative ring. An algebra A central over $\mathbb{K}$ is said to be quasi-free over $\mathbb{K}$ if $\Omega^1_{A/\mathbb{K}}$ is a projective $A \otimes_{\mathbb{K}} A^{opp}$-module.

A generalization of Cuntz-Quillen criterion for quasi-freeness [CQ1] holds in the relative case.

Proposition 44. *Algebra A is quasi-free over $\mathbb{K}$ if and only if for any $\tilde{R}$, a nilpotent extension of the algebra R by square zero ideal I in the category of algebras central over $\mathbb{K}$, and any homomorphism $A \to R$ there exists its lifting to a homomorphism $A \to \tilde{R}$.*

Proof. Consider such an extension and such a homomorphism. Let $\tilde{A}$ be the fibred product over R of A and $\tilde{R}$. This is an algebra central over $\mathbb{K}$ too. It is a square zero extension of A by means of I endowed with an A-bimodule structure pulled back from R-bimodule structure. Note that I is is $\mathbb{K}$-central bimodule (i.e. left $\mathbb{K}$ action coincides with the right one). Such extensions are classified by $\mathrm{Ext}^2_{A\otimes A^{opp}}(A, I)$ which are trivial for quasi-free algebras. Hence, we have a splitting homomorphism $A \to \tilde{A}$. When combined with the map $\tilde{A} \to \tilde{R}$ it gives the required lifting. Conversely, if all square zero extensions allow liftings, then, by taking $R = A$ and I arbitrary $\mathbb{K}$-central A-bimodule, we see that $\mathrm{Ext}^2_{A\otimes A^{opp}}(A, I) = \mathrm{Ext}^1_{A\otimes A^{opp}}(\Omega^1_{A/\mathbb{K}}, I) = 0$, i.e. $\Omega^1_{A/\mathbb{K}}$ is a projective $A \otimes A^{opp}$-bimodule.

□

The following criterion of coherence is due to Chase [Chase].

Lemma 45. *For any ring A the following are equivalent:*

- *A is left coherent*
- *For any family of* right *flat modules F_i, $i \in I$, the product $\prod_{i\in I} F_i$ is right flat.*

- *For the family* $F_i \cong A$ *of free modules with* $\operatorname{card} I = \operatorname{card} A$*, the product* $\prod_{i \in I} F_i$ *is right flat.*

We will use also a similar criterion for noetherianess (cf. [Ab]).

Lemma 46. *For any ring A the following are equivalent:*

- *A is left noetherian*
- *For any left A-module M and any family of flat* right *modules* F_i*,* $i \in I$*, the morphism* $(\prod_{i \in I} F_i) \otimes_A M \to \prod_{i \in I}(F_i \otimes_A M)$ *is mono.*
- *For any left A-module M and the family of rank 1 free* right *modules* $F_i \cong A$*,* $i \in I$*,* $\operatorname{card} I = \operatorname{card} A$*, the morphism* $(\prod_{i \in I} F_i) \otimes_A M \to \prod_{i \in I}(F_i \otimes_A M)$ *is mono.*

Theorem 47. *Let* $\mathbb{K}$ *be a commutative noetherian ring. Assume that A is an algebra quasi-free relatively over* $\mathbb{K}$ *and A is flat as a* $\mathbb{K}$*-module. Then A is a left and right coherent algebra.*

Proof. Consider left A-module M and a family of rank 1 free right A-modules $F_i \cong A$. We shall consider all left (respectively, right) A-modules to be always endowed with the right (respectively, left) $\mathbb{K}$-module structure identical to the left (respectively, right) $\mathbb{K}$-module structure.

By applying functors $(-) \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i)$ and $\prod_i((-) \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i))$ and their derived functors to the exact sequence (70), we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 \longrightarrow & \operatorname{Tor}_1^{A \otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} \prod_i F_i) & \longrightarrow & \Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & \prod_i \operatorname{Tor}_1^{A \otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} F_i) & \longrightarrow & \prod_i(\Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) \end{array} \tag{71}$$

For any left A-module M and right A-module N, we have an isomorphism of objects in the derived categories:

$$A \otimes^{\mathbb{L}}_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes^{\mathbb{L}}_{\mathbb{K}} N) = M \otimes^{\mathbb{L}}_A N,$$

which implies a spectral sequence:

$$\operatorname{Tor}_i^{A \otimes_{\mathbb{K}} A^{opp}}(A, \operatorname{Tor}_j^{\mathbb{K}}(M, N)) \Longrightarrow \operatorname{Tor}^A_{i+j}(M, N)$$

For a flat $\mathbb{K}$-module N, it implies that

$$\operatorname{Tor}_1^{A \otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} N) = \operatorname{Tor}_1^A(M, N). \tag{72}$$

Since F_i are rank 1 free as A-modules and A is a flat $\mathbb{K}$-module, F_i are also flat $\mathbb{K}$-modules. Hence $\operatorname{Tor}_1^A(M, F_i) = 0$. Since $\mathbb{K}$ is noetherian, it is coherent, hence by Chase criterion, lemma 45, $\prod_i F_i$ is also flat. In view of (72), diagram (71) reads:

$$\begin{array}{ccccc} 0 \longrightarrow & \operatorname{Tor}_1^A(M, \prod_i F_i) & \longrightarrow & \Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \\ & \downarrow & & \downarrow \\ & 0 & \longrightarrow & \prod_i(\Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) \end{array} \tag{73}$$

Let us show that $\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \to \prod_i (\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes A^{opp}} (M \otimes_{\mathbb{K}} F_i))$ is an embedding. By criterion of noetherianess, lemma 46, for $\mathbb{K}$, we have that $M \otimes_{\mathbb{K}} \prod_i F_i \to \prod_i (M \otimes_{\mathbb{K}} F_i)$ is an embedding.

Let $D = \oplus_j (A \otimes_{\mathbb{K}} A^{opp})$ be a free $A \otimes_{\mathbb{K}} A^{opp}$-module, then

$$D \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) = \oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i)$$

and

$$\prod_i (D \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) = \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i)).$$

The morphism

$$D \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \to \prod_i (D \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i))$$

is the composite of two morphism:

$$\oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i) \to \oplus_j \prod_i (M \otimes_{\mathbb{K}} F_i) \to \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i)).$$

Both morphism are readily embeddings, hence so is the composite.

Since $\Omega^1_{A/\mathbb{K}}$ is a projective bimodule, we have an imbedding

$$\Omega^1_{A/\mathbb{K}} \to \oplus_j A \otimes_{\mathbb{K}} A^{opp}$$

as a direct summand. This implies a diagram:

$$\begin{array}{ccc} \Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) & \longrightarrow & \oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i) \\ \downarrow & & \downarrow \\ \prod_i (\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) & \longrightarrow & \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i)) \end{array} \tag{74}$$

The upper horizontal arrow is an embedding because it obtained by tensoring up an embedding of a direct summand with a module $M \otimes_{\mathbb{K}} \prod_i F_i$. We have shown above that the right vertical arrow is an embedding too. Therefore, the left vertical arrow is an embedding. Then, diagram (73) implies that $\mathrm{Tor}_1^A(M, \prod_i F_i) = 0$, i.e $\prod_i F_i$ is a right flat module. By Chase criterion, lemma 45, algebra A is left coherent.

Right coherence follows similarly.

□

Theorem 48. *Let $\Delta \in A$ be well-tempered and $B = \widehat{A}_{\Delta}$. Assume that A is a coherent algebra over a coherent ring $\mathbb{K}$. Then B is coherent too.*

Proof. Consider a homomorphism $M \to N$ of free B-modules of finite rank. Embed it into a four term exact sequence

$$0 \to K \to M \to N \to Q \to 0. \tag{75}$$

We need to show that K is finitely generated. Apply functor $\psi_{1*}\psi_1^!$ to it. Since both functors ψ_{1*} and $\psi_1^!$ are exact (lemma 36), we get an exact sequence:

$$0 \to \psi_{1*}\psi_1^! K \to \psi_{1*}\psi_1^! M \to \psi_{1*}\psi_1^! N \to \psi_{1*}\psi_1^! Q \to 0$$

By proposition 24, $\psi_1^! M$ and $\psi_1^! N$ are free A-modules of finite rank. Since A is a coherent algebra, this implies that $\psi_1^! K$ is a finitely generated A-module. Further, $\psi_{1*}A$ is finitely generated as a B-module by lemma 26. Therefore, ψ_{1*} of a finitely generated A-module is a finitely generated B-module. Hence, B-module $\psi_{1*}\psi_1^! K$ is finitely generated too.

Now consider the exact sequence induced by the adjunction morphism:

$$\psi_{1*}\psi_1^! K \to K \to S \to 0$$

As $\psi_{1*}\psi_1^! K$ is finitely generated, it is remained to show that S is so. By lemma 34, S is a B^+-trivial B-module. Therefore, S is a quotient of K/B^+K. Let us check that the latter is finitely generated.

Consider the spectral sequence which is obtained by applying functor $\mathbb{K} \otimes_B (-)$ to the sequence (75). The spectral sequence converges to zero, because the original sequence is exact. Taking into account that M and N are projective, hence, flat, we see that the contribution to $\mathbb{K} \otimes_B K = K/B^+K$ come from the kernel of $\mathbb{K} \otimes_B M \to \mathbb{K} \otimes_B N$ and $\mathrm{Tor}_2^B(\mathbb{K}, Q)$. The former is the kernel of a morphism between finite rank free $\mathbb{K}$-module, hence, in view of coherence of $\mathbb{K}$, is finitely generated, the latter is zero because augmentation sequence (48) provides a flat B-resolution of length 2 for $\mathbb{K}$. □

Corollary 49. *Let $\mathbb{K}$ be a noetherian ring. Then Poincare groupoid $\mathbb{K}\Gamma$ and algebra $B(\Gamma)$ are coherent.*

Proof. Since the fundamental group $\pi(\Gamma)$ is free, the group ring $\mathbb{K}[\pi(\Gamma)]$ is quasi-free relatively over $\mathbb{K}$ (cf. [CQ1]). Indeed, the free algebra $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ is relatively quasi-free over $\mathbb{K}$ because it obviously satisfies the lifting criterion of proposition 44. The group ring $\mathbb{K}[\pi(\Gamma)]$ is a localization of of the free algebra, hence satisfies the lifting criterion too. By theorem 47 it is coherent. By proposition 7, algebra $\mathbb{K}\Gamma$ is Morita equivalent to the group ring of $\mathbb{K}[\pi(\Gamma)]$. Therefore, it is coherent too. Coherence of $B(\Gamma)$ follows by theorem 48. □

This corollary implies that the category of finitely presented left modules over $\mathbb{K}\Gamma$ and over $B(\Gamma)$ are abelian. We indicate the categories of finitely presented (left) modules by subscript fp.

Proposition 50. *Let Δ be a well-tempered element in algebra A and $B = \widehat{A}_\Delta$. Then functors ψ_{1*}, ψ_2^*, $\psi_1^!$ and ψ_1^* take finitely presented left modules over corresponding algebras to finitely presented modules.*

Proof. A finitely presented A-module W has a presentation by finitely generated free A-modules. By applying exact functor ψ_{1*} to this presentation, we obtain a presentation for $\psi_{1*}W$ by projective B-modules which are finite sums of copies of B^+. Since B^+ is finitely generated B-module by lemma 26, $\psi_{1*}W$ is finitely presented.

Functors ψ_1^* and ψ_2^* take finitely generated free B-modules to finitely generated free A-modules. As the both are right exact, it follows that they take finitely presented B-modules to finitely presented A-modules.

Functor $\psi_1^!$ is isomorphic to ψ_2^* by proposition 39.

□

Lemma 51. *Let $\mathbb{K}$ be a noetherian ring. Then any finitely generated $\mathbb{K}$-module is finitely presented as a B-module.*

Proof. Since $\mathbb{K}$ is noetherian every finitely generated module has a presentation by finitely generated free modules over $\mathbb{K}$. Finitely generated free modules over $\mathbb{K}$ have finite projective resolution over B, because $\mathbb{K}$ has a projective B-resolution (48). The fact follows. □

Proposition 52. *Functor ψ_{2*} takes finitely presented modules to finitely presented if and only if $\{a \in A | \Delta a = 0\}$ and $A/\Delta A$ are finitely generated $\mathbb{K}$-modules.*

Proof. Assume that ψ_{2*} takes finitely presented modules to finitely presented. Consider exact sequence (60) applied to module $W = A$. Since $\psi_{1*}W$ is finitely presented and the category of finitely presented modules is abelian, the kernel and cokernel of λ_A must be finitely presented modules. But they are B^+-trivial modules with the kernel being isomorphic to $\{a \in A | \Delta a = 0\}$ and cokernel to $A/\Delta A$. Hence they must be finitely generated $\mathbb{K}$-modules.

Conversely, assume that these $\mathbb{K}$-modules are finitely presented. By lemma 51 it is enough to show that the kernel and cokernel of λ_W are finitely generated $\mathbb{K}$-modules. For $W = A$ it is clear from (60).

Consider the 4-term exact sequence that come from this finite presentation:

$$0 \to Z \to A^k \to A^l \to W \to 0$$

First, since functor $\psi_{2*}(-)/B^+\psi_{2*}(-)$ is right exact and its value on A^l is a finitely generated $\mathbb{K}$-module then so is its value on W. Thus the cokernel of λ_W is a finitely generated $\mathbb{K}$-module. Second, by applying exact functor ψ_{1*} to the exact sequence, we get a 4-terms exact sequence:

$$0 \to \psi_{1*}(Z) \to (B^+)^k \to (B^+)^l \to \psi_{1*}(W) \to 0$$

By applying functor $\mathrm{Hom}_B(\mathbb{K}, -)$ to it, we get a spectral sequence that converges to 0. It shows that the contribution to $(\psi_{1*}W)^{B^+} = \mathrm{Hom}_B(\mathbb{K}, \psi_{1*}W)$ might come from the cokernel of $\mathrm{Hom}_B(\mathbb{K}, (B^+)^k) \to \mathrm{Hom}_B(\mathbb{K}, (B^+)^l)$, from $\mathrm{Ext}^1_B(\mathbb{K}, (B^+)^k)$ and from $\mathrm{Ext}^2_B(\mathbb{K}, \psi_{1*}Z)$. By lemma 42 contributions from the first two are finitely generated $\mathbb{K}$-modules, while the third module is zero. Therefore, the kernel of λ_W is finitely generated over $\mathbb{K}$. □

For an A-module W, consider the map $\lambda_W : \psi_{1*}W \to \psi_{2*}W$ defined by natural transformation (58). Define the B-module W_{min} as the image of λ_W.

Lemma 53. *Let W be an A-module and V a B^+-trivial B-module. Then $B(\Gamma)$-module W_{min} satisfies* $\mathrm{Hom}_B(V, W_{min}) = 0$ *and* $\mathrm{Hom}_B(W_{min}, V) = 0$.

Proof. Module W_{min} is a submodule in $\psi_{2*}W$, hence $\mathrm{Hom}_{B(\Gamma)}(V, W_{min}) = 0$ by lemma 37(iv). Also it is a quotient of $\psi_{1*}W$, hence $\mathrm{Hom}_{B(\Gamma)}(W_{min}, V) = 0$ by lemma 37(iii). □

If V is a B-module, then define the *minimal shadow* of V by $V_{min} := (\psi_1^! V)_{min} = (\psi_2^* V)_{min}$.

Proposition 54. *Given B-module V we have a commutative diagram:*

$$\begin{array}{ccc} \psi_{1*}\psi_1^! V & \xrightarrow{\alpha} & V_{min} \\ {\scriptstyle \epsilon_V}\downarrow & & \downarrow{\scriptstyle \beta} \\ V & \xrightarrow[\delta_V]{} & \psi_{2*}\psi_2^* V \end{array}$$

with an epimorphism α and monomorphism β.

Proof. This follows from lemma 35 and proposition 39. □

For a left $B(\Gamma)$-module V and a vertex $i \in V(\Gamma)$, define $\mathbb{K}$-modules

$$V_i = \{v \in V | \; \rho(x_i)v = v\}.$$

If Γ is a connected graph, then V_i are isomorphic for all $i \in V(\Gamma)$. Indeed, if $(ij) \in E(\Gamma)$ and $v \in V_i$, then $s_{ij}^{-1}\rho(x_j)v \in V_j$. Moreover, $s_{ij}^{-1}\rho(x_i)(s_{ij}^{-1}\rho(x_j)v) = s_{ij}^{-2}\rho(x_i x_j x_i)v = v$. Thus, $s_{ij}^{-1}\rho(x_j)$ and $s_{ij}^{-1}\rho(x_i)$ are mutually inverse transformations between V_i and V_j. In general, the isomorphism depends on the choice of path connecting vertices i and j.

Proposition 55. *Let V be a left $B(\Gamma)$-module. Then there is a natural decomposition:*

$$\psi_2^* V = \oplus_i \; V_i \tag{76}$$

Proof. Since $\mathbb{K}\Gamma$ is isomorphic to $B^+(\Gamma)$ as a right $B(\Gamma)$-module, we can apply (45) to calculation of $\mathbb{K}\Gamma \otimes_{B(\Gamma)} V$. Since x_i is an idempotent we have a short exact sequence of right modules (actually, a direct sum decomposition):

$$0 \to (1 - x_i)B(\Gamma) \to B(\Gamma) \to x_i B(\Gamma) \to 0 \tag{77}$$

Taking tensor product of (77) with V over $B(\Gamma)$ shows that the

$$x_i B \otimes_B V = V/\mathrm{Im}(\mathrm{id}_V - \rho(x_i)) = V_i.$$

□

We say that a finite dimensional $B(\Gamma)$-module is *of rank* r, and write rank $V = r$, if $\mathrm{dim} V_i = r$.

6.4 Duality

Assume that we have an involutive anti-automorphism $\sigma = \sigma_A$ on A that preserves Δ:

$$\sigma(\Delta) = \Delta.$$

Then, σ induces an anti-automorphism on B^+, hence on B, which we denote by σ_B when we need to distinguish it from σ_A. Indeed,

$$\sigma(a \cdot_\Delta b) = \sigma(a\Delta b) = \sigma(b)\sigma(\Delta)\sigma(a) = \sigma(b) \cdot_\Delta \sigma(a).$$

Further, the anti-involutions on A and B satisfy a relation with respect to ψ_1 and ψ_2:

$$\sigma_A \circ \psi_1 = \psi_2 \circ \sigma_B. \tag{78}$$

Remember that we have an anti-involution (26) on the algebra $\mathbb{K}\Gamma$. It preserves Laplace operator Δ, hence it induces an anti-involution $\sigma_B : B(\Gamma) \to B(\Gamma)^{opp}$, which is defined by:

$$\sigma_B(x_i) = x_i.$$

Any anti-isomorphism of an algebra induces an equivalence between the categories of left and right modules of the algebra. If $\mathbb{K} = k$ is field and a left B-module is finite dimensional over k, then the dual vector space is a right B-module. The composition of the equivalence induced by an anti-isomorphism with "taking dual" gives a duality, i.e. an involutive anti-equivalence, on the category of finite dimensional right B-modules. Anti-involution σ_B induces a duality $D : B(\Gamma) - \text{mod} \simeq B(\Gamma) - \text{mod}^{opp}$. For the representation $\rho : B(\Gamma) \to \text{End}(V)$, the dual representation $D(\rho) : B(\Gamma) \to \text{End}(V^*)$ in V^* is defined by

$$D(\rho)(b) = \rho(\sigma_B(b))^*.$$

On generators x_i's of $B^+(\Gamma)$ the duality acts by $x_i \mapsto x_i^*$. Thus, the duality is an algebraic version of the duality discussed in the beginning of section 2.3.

An interesting problem is to study self-dual representations of $B^+(\Gamma)$, i.e representations V that allow an isomorphism with the dual representation: $V \simeq V^*$.

6.5 Morita equivalence

Consider a pair of elements Δ, Δ' in an algebra A. Here we address the problem when the corresponding algebras $B = \widehat{A}_\Delta$ and $B' = \widehat{A}_{\Delta'}$ are Morita equivalent, i.e. when there exists a $B' - B$-bimodule N such that the functor $\Phi_N : \text{mod} - B \to \text{mod} - B'$ given by

$$\Phi_N : V \mapsto N \otimes_B V$$

induces an equivalence on categories of modules.

We use superscript $'$ for notation of homomorphisms related to Δ', to distinguish them from similar homomorphisms related to Δ.

Fix elements $c, d \in A$ which satisfy the equation:

$$c \cdot_A \Delta = \Delta' \cdot_A d. \tag{79}$$

Let us construct bimodule $N = N_{cd}$ as follows. We want N to be an extension of a trivial bimodule $\mathbb{K}$ by A, the latter is considered as a $B' - B$-bimodule with left B'-module structure coming via ψ_1' and with right B-module structure coming from ψ_2:

$$0 \to A \to N_{cd} \to \mathbb{K}\langle z \rangle \to 0 \tag{80}$$

with z being a generator in $\mathbb{K} = \mathbb{K}\langle z \rangle$. Take $N = \mathbb{K}\langle z \rangle \oplus A$ as a vector space. Define left action of $b' \in B'^+$ on z by:

$$b'z = b' \cdot_A c.$$

This is an element in $A \subset N_{cd}$. Similarly, define right action of $b \in B^+$ on z by:

$$zb = d \cdot_A b.$$

The action of the unit is, of course, by identity.

Lemma 56. *This endows N_{cd} with a $B' - B$-bimodule structure.*

Proof. Indeed, we have for the left B'-module structure:

$$b_1'(b'z) = b_1' \cdot_{B'} b' \cdot_A c = (b_1'b')z$$

and similar for the right B-module structure. Also, the left and the right module structures commute:

$$(b'z) \cdot_B b = (b' \cdot_A c) \cdot_B b = b' \cdot_A c \cdot_A \Delta \cdot_A b = b' \cdot_A \Delta' \cdot_A d \cdot_A b = b' \cdot_{B'} (d \cdot_A b) = b' \cdot_{B'} (zb).$$

□

Let $\Delta' = \Delta$, then $B' = B$. Let $(c, d) = (1_A, 1_A)$ be two copies of the unit in A. Then $N_{11} = B$ as a B-bimodule. Hence the functor $\Phi_{N_{11}}$ is the identity functor in $\mathrm{mod} - B$.

Composition of functors of type Φ_N is compatible with products of bimodules:

$$\Phi_L \circ \Phi_N = \Phi_{L \otimes_{B'} N}$$

for a $B' - B$-bimodule L and a $B - B''$-bimodule N.

Proposition 57. *Let Δ be a well-tempered element in A. Let N_{cd} be a $B' - B$-bimodule and N_{uv} be a $B - B''$-bimodule. Then*

$$N_{cd} \otimes_B N_{uv} = N_{c \cdot_A u, d \cdot_A v}$$

as a $B' - B''$-module.

Proof. Let $\Delta, \Delta', \Delta''$ be the defining elements for B, B' and B'' respectively. By definition of bimodules N_{cd} and N_{uv}, we have:

$$c \cdot_A \Delta = \Delta' \cdot_A d, \quad u \cdot_A \Delta'' = \Delta \cdot_A v,$$

which implies that

$$cu \cdot_A \Delta'' = \Delta' \cdot_A dv.$$

That is, the module $N_{cu,dv}$ does exist.

We know by (51) that $A \otimes_B A = A$ and by lemma 37 (ii) that $A \otimes^{\mathbb{L}}_B \mathbb{K} = 0$. By taking tensor product over B of sequences (80) for N_{cd} and for N_{uv}, we obtain that $N_{cd} \otimes N_{uv}$ fits in a similar short exact sequence

$$0 \to A \to N_{cd} \otimes_B N_{uv} \to \mathbb{K}\langle z \otimes w \rangle \to 0,$$

where z and w stand for generators in the components $\mathbb{K}$ for N_{cd} and for N_{uv}, respectively. Thus, we need to check the left action of B' and right action of B'' on $z \otimes w$. Let b' be in B'. Then

$$b'(z \otimes w) = b' \cdot_A c \otimes w.$$

Now $b' \cdot_A c$ is an element in $A \simeq B^+$. According to (51), we can decompose it into

$$b' \cdot_A c = \sum p_i \cdot_B q_i,$$

where p_i and q_i are some elements in B^+. Further,

$$b' \cdot_A c \otimes w = \sum p_i \cdot_B q_i \otimes w = \sum p_i \otimes q_i w = \sum p_i \otimes q_i \cdot_A u.$$

The identification $B^+ \otimes_B B^+ = B^+$ in (51) is given by multiplication in B^+. Hence under this identification, we have

$$b'(z \otimes w) = \sum p_i \otimes q_i \cdot_A u = \sum p_i \cdot_B q_i \cdot_A u = b' \cdot_A c \cdot_A u.$$

Thus the left action of B' on $z \otimes w$ is 'via' $c \cdot_A u$. Similarly for the right action of B''. □

Corollary 58. *If $c, d \in A$ is a pair of invertible elements in A satisfying (79), then bimodule N_{cd} provides a Morita-equivalence.*

Proof. For $B' - B$-bimodule N_{cd}, the inverse $B' - B$-bimodule is $N_{c^{-1},d^{-1}}$. □

Let G be the group of invertible elements in A.

Corollary 59. *If Δ' is in the double orbit $G\Delta G$ of Δ, then categories $B - \mathrm{mod}$ and $B' - \mathrm{mod}$ are equivalent.*

Take tensor product over B of the sequence (80) with the trivial left B-module k. We know that $k\Gamma \otimes^{\mathbb{L}}_B k = 0$, therefore $\Phi_{N_{cd}}(k) = k$.

Proposition 60. *Functors ψ_{1*} and ψ'_{1*} are compatible with equivalences $\Phi_{N_{cd}}$, i.e.*

$$\Phi_{N_{cd}} \circ \psi_{1*} = \psi'_{1*}.$$

The same is true for ψ_{2*} and ψ'_{2*}

Proof. □

If c and d are invertible, then $\Phi_{N_{cd}}$ is an equivalence for both abelian and derived categories. We got that it is identical on the two pieces of semiorthogonal decomposition for derived categories.

6.6 Example: matrix algebra over a field

Let A be a matrix algebra over a field $\mathbb{K} = k$, We identify A with the algebra of operators in a vector space V over k of dimension n.

Proposition 61. *All non-zero elements in A are well-tempered. If A is an operator of corank s, then B is Morita equivalent to the algebra of the quiver with two vertices, $[0]$ and $[1]$, having s arrows*
$alpha_i$ from $[0]$ to $[1]$ and s arrows β_j from $[1]$ to $[0]$ and relations $\beta_j\alpha_i = 0$, for all $1 \leq i, j \leq s$.

Proof. First, A is simple, hence, for any non-zero element Δ, the two sided ideal $A\Delta A$ coincides with A. Let Δ be an operator of rank k. We know that well-temperateness is a property of a double coset. By multiplying Δ from both sides by invertible operators, we can make it to become a projector P of rank k. Let U and W be the image and the kernel of P. Then $V = U \oplus W$. □

6.7 Derived categories

We assume here that $\Delta \in A$ is well-tempered and $B = \widehat{A}_\Delta$. Consider bounded derived categories $D^b(A - \text{mod})$ and $D^b(B - \text{mod})$. Functors ψ_{1*} and ψ_{2*} and functors $\psi_2^* \simeq \psi_1^!$ are exact. We denote by the same symbols the corresponding derived functors. We show that they provide semi-orthogonal decompositions for $D^b(B - \text{mod})$. Natural transformations (58) and (65) and the fact that μ is an isomorphism of functors carry on to the derived category context.

Denote by $D^b_0(B - \text{mod})$ the full subcategory in $D^b(B - \text{mod})$ of complexes with cohomology B^+-trivial modules. In view of proposition 40, it is equivalent to $D^b(\mathbb{K} - \text{mod})$. Denote by $i_* : D^b(\mathbb{K} - \text{mod}) \to D^b(B - \text{mod})$ the corresponding embedding functor.

Recall some definitions from [Bon1]. A triangulated subcategory is said to be *right (resp., left) admissible* if it has right (resp. left) adjoint to the embedding functor.

Proposition 62. *Subcategory $D^b_0(B - \text{mod})$ in $D^b(B - \text{mod})$ is left and right admissible. Functors ψ_{1*} and ψ_{2*} are fully faithful and identify category $D^b(A - \text{mod})$ with, respectively, left and right orthogonal to the subcategory $D^b_0(B - \text{mod})$ in $D^b(B - \text{mod})$.*

Proof. Since ψ_{1*} and ψ_{2*} are exact functors between triangulated categories (i.e. the one that takes exact triangles into exact ones), they are fully faithful if the adjunction morphisms are isomorphisms: $\mathrm{id} \simeq \psi_2^*\psi_{2*}$, $\mathrm{id} \simeq \psi_1^!\psi_{1*}$. This follows from proposition 38, because functors $\psi_2^* = \psi_1^!$, ψ_{2*} and ψ_{1*} are exact as functors between abelian categories.

Let W be an A-module and V_0 a B^+-trivial B-module. We already know by lemma 37 that $\mathrm{Hom}^\bullet_{B(\Gamma)}(\psi_{1*}W, V_0) = 0$ and $\mathrm{Hom}^\bullet_{B(\Gamma)}(V_0, \psi_{2*}W) = 0$. To prove that $D^b_0(B - \mathrm{mod})$ is left admissible and, simultaneously, that the image of functor ψ_{1*} is indeed the whole left orthogonal to $D^b_0(B - \mathrm{mod})$, it is enough to check [Bon1] that every object, say V, in $D^b(B - \mathrm{mod})$ has a decomposition into exact triangle $U \to V \to W$ with U in the image of ψ_{1*} and $W \in D^b_0(B - \mathrm{mod})$. Consider the adjunction morphism $\psi_{1*}\psi_1^! V \to V$. By lemma 34 its cone is in $D^b_0(B - \mathrm{mod})$ if V is a pure B-module. This gives a triangle with required properties for such V. For general V? this follows from exactness of functors ψ_{1*} and $\psi_1^!$. For ψ_{2*}, the proof is similar. $\square$

In accordance with ideology of [Bon1], this proposition is interpreted as existence of decompositions into semiorthogonal pairs:

$$D^b(B-\mathrm{mod}) = \langle i_*D^b(\mathbb{K}-\mathrm{mod}), \psi_{1*}D^b(A-\mathrm{mod})\rangle = \langle \psi_{2*}D^b(A-\mathrm{mod}), i_*D^b(\mathbb{K}-\mathrm{mod})\rangle. \quad (81)$$

Given an admissible subcategory $\mathcal{B}$, there is an equivalence between the left and right orthogonals to it, ${}^\perp\mathcal{B}$ and $\mathcal{B}^\perp$. The mutually inverse 'mutation' functors $L_\mathcal{B} : {}^\perp\mathcal{B} \to \mathcal{B}^\perp$ and $R_\mathcal{B} : \mathcal{B}^\perp \to {}^\perp\mathcal{B}$ are given by restricting to ${}^\perp\mathcal{B}$ the left adjoint to the embedding functor $\mathcal{B}^\perp \to \mathcal{B}$ and by restricting to $\mathcal{B}^\perp$ the right adjoint functor to the embedding ${}^\perp\mathcal{B} \to \mathcal{B}$.

For the case of our interest subcategories $D^b_0(B - \mathrm{mod})^\perp$ and ${}^\perp D^b_0(B - \mathrm{mod})$ are both equivalent to $D^b(A - \mathrm{mod})$ via functors ψ_{1*} and ψ_{2*}, with equality

$$\psi_{2*} = i_{\mathcal{B}^\perp} \circ L_\mathcal{B} \circ \psi_{1*},$$

where $i_{\mathcal{B}^\perp}$ is the embedding functor for the subcategory $\mathcal{B}^\perp$. Since $L_\mathcal{B}$ is basically the adjoint to $i_{\mathcal{B}^\perp}$ we have the adjunction morphism:

$$\lambda : \mathrm{id}_{{}^\perp\mathcal{B}} \to i_{\mathcal{B}^\perp} \circ L_\mathcal{B}. \quad (82)$$

It implies the functorial morphism $\psi_{1*} \to \psi_{2*}$, which coincides with the natural transformation λ in (58) when extended to a transformation between the derived functors.

The embedding functor i_* has the left adjoint $i^*, i^! : D^b(B - \mathrm{mod}) \to D^b(\mathbb{K} - \mathrm{mod})$ defined by

$$i^*(V) = \mathbb{R}\mathrm{Hom}_B(V, \mathbb{K})^*$$

and the right adjoint $i^!$ defined by

$$i^!(V) = \mathbb{R}\mathrm{Hom}_B(\mathbb{K}, V).$$

All together, functors $\psi_{1*}, \psi_{2*}, \psi_1^!, i_*, i^*, i^!$ fit Grothendieck formalism of six functors.

Given an admissible subcategory $\mathcal{B}$ in a triangulated category one can consider the ambient category as being 'glued' from the subcategories $\mathcal{B}$ and its orthogonal ${}^\perp\mathcal{B}$. The necessary extra data to define gluing is the functor $^{\perp}\mathcal{B} \to \mathcal{B}$ which takes an object from $^{\perp}\mathcal{B}$ to the cone of the natural transformation (82) applied to this object. The resulting object lies in $\mathcal{B}$.

To be more precise, this approach works only in the framework of pre-triangulated DG-categories [BK2], rather than for ordinary triangulated categories. Thus we should consider suitable DG-categories, *enhancements* for $\mathcal{B}$ and for its orthogonal $^{\perp}\mathcal{B}$, and a gluing DG-functor (or DG-bimodule) between them. Given this data, one can construct in an essentially unique way a new DG-category, such that its homotopy category has a semiorthogonal decomposition into a pair $\langle \mathcal{B},\ ^{\perp}\mathcal{B} \rangle$ (cf. [?]).

When applied to the our categories, this means that category $D^b(B-\mathrm{mod})$ is glued from the categories $D^b(A-\mathrm{mod})$ and $D^b(\mathbb{K}-\mathrm{mod})$ by means of a functor which is given by the cone of the natural transformation λ in (58).

The first semiorthogonal decomposition from (81) restricts to the category $D^b_{fp}(B-\mathrm{mod})$, but the second one restricts only when ψ_{2*} takes finitely presented modules to finitely presented. The conditions for this is given in proposition 52.

6.8 Example: matrix algebra over a commutative ring

Example. Assume that graph Γ is a tree. Then homology of Γ is trivial and $\mathbb{K}\Gamma$ is isomorphic to the algebra of $\mathbb{K}$-valued matrices of size $n \times n$ with n equal to the number of vertices in the graph. The group G of invertible elements is just $GL(n, \mathbb{K})$. The category $\mathbb{K}\Gamma - \mathrm{mod}$ is equivalent to the category of $\mathbb{K}$-modules. Functor $F : D^b(\mathbb{K}\Gamma - \mathrm{mod}) \to D^b(\mathbb{K} - \mathrm{mod})$ is fully defined by its values on the standard representation of the matrix algebra $\mathbb{K}\Gamma - \mathrm{mod}$ in the n dimensional vector space V. The value of the functor F on this representation is the cone λ_V, which is nothing but the Laplace operator Δ understood as an element of the matrix algebra. Since the cone consists of the kernel and cokernel of Δ, the corank of Δ only matters. This is in accordance with the fact that double orbits of Δ with respect to $GL(n)$ action produce Morita equivalent algebras.

If Δ is nondegenerate, then $F = 0$ and category $D^b(B(\Gamma - \mathrm{mod}_{fd}))$ is equivalent to the direct sum of two categories $D^b(k-\mathrm{mod})$. In general, if Δ has corank s, then $D^b(B(\Gamma - \mathrm{mod}_{\mathrm{fd}}))$ has a full exceptional collection with two elements (E_1, E_2) and extensions: $\mathrm{Ext}^{0,1}(E_1, E_2) = k^s$ and $\mathrm{Ext}^{\neq 0,1}(E_1, E_2) = 0$.

Example. Let $\mathrm{H}^1(\Gamma) = \mathbb{Z}$. Then the group ring $k[\pi_1(\Gamma)]$ is the algebra $k[x, x^{-1}]$ of Laurent polynomials in one variable. In view of Morita equivalence between $k\Gamma$ and $k[\pi_1(\Gamma)]$, the category $k\Gamma - \mathrm{mod}_{fp}$ is equivalent to the category of coherent sheaves on a punctured affine line $\mathbb{A}^1 \setminus 0$. The category $k\Gamma - \mathrm{mod}_{fd}$ is equivalent to the subcategory of artinian sheaves. Since $k\Gamma$ is isomorphic to the matrix algebra over $k[\pi_1(\Gamma)]$, elements of $k\Gamma$ can be understood as homomorphisms of free sheaves of $\mathcal{O}$-modules on $\mathbb{A}^1 \setminus 0$ of rank $n = |V(\Gamma)|$. The Laplace operator gives a homomorphism $\phi_\Delta : \mathcal{O}^n \to \mathcal{O}^n$ given by the matrix:

If Γ is a cyclic graph with n vertices and n edges $l_{i,i+1}$, where $i \in \mathbb{Z}/n$, then

$$\Delta = 1 + \sum_{i \in \mathbb{Z}/n} s_{i,i+1} l_{i,i+1}.$$

Then m Matrix of ϕ_Δ in a suitable basis of the representation has the form:

$$\begin{pmatrix} 1 & s_{1,2} & 0 & ... & 0 & s_{n,1}x^{-1} \\ s_{1,2} & 1 & s_{2,3} & 0 & ... & 0 \\ 0 & s_{2,3} & 1 & s_{3,4} & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & s_{n-2,n-1} & 1 & s_{n-1,n} \\ s_{n,1}x & 0 & ... & 0 & s_{n-1,n} & 1 \end{pmatrix} \tag{83}$$

Proposition 63. *Let Δ and Δ' be two Laplace operators. Categories $B(\Gamma) - \mathrm{mod}_{fp}$ and $B'(\Gamma) - \mathrm{mod}_{fp}$ are equivalent if and only if the cokernel of ϕ_Δ and $\phi_{\Delta'}$ are isomorphic.*

Proof. If determinant of Δ is not tautologically zero, then homomorphism $\Delta : \mathcal{O}^n \to \mathcal{O}^n$ has trivial kernel. The cokernel $\mathrm{coker}\phi_\Delta$ is an artinian sheaf with support at zeros of $\det\phi_\Delta$. .... □

Let the field be algebraically closed. Then an artinian sheaf is decomposed into direct sum of sheaves with support in closed points. All indecomposable artinian sheaves with support in a given closed point are isomorphic to the structure sheaf of a k-th infinitesimal neighborhood of the point, for some k. The support is given by zeros of $\det\phi_\Delta$. The multiplicity of a closed point corresponds to the length of the part of the $\mathrm{coker}\phi_\Delta$ with support in the point. Thus the isomorphism class of an artinian sheaf is fully defined by partitions of multiplicities in $\mathbb{A}^1 \setminus 0$ of all zeros of $\det\phi_\Delta$. To find the partition of multiplicity at a given point x_0 one has to look at the dimension d_s of the kernel of action of $(x - x_0)^s$ on $\mathrm{coker}\phi_\Delta$, for all s. The number of entries of size $\geq s$ in the partition of multiplicity at point x_0 is equal to d_s. We come to the following corollary which explicitly describes Laplace operators with equivalent categories of finitely presented modules.

Corollary 64. *Let Δ and Δ' be two Laplace operators. Categories $B(\Gamma) - \mathrm{mod}_{fp}$ and $B'(\Gamma) - \mathrm{mod}_{fp}$ are equivalent if and only if the partitions of multiplicities at every closed point in $\mathbb{A}^1 \setminus 0$ for Δ and Δ' coincide.*

Note that the above discussion concerned description of the category of finitely presented modules. Since the equivalences are Morita equivalences, they induce equivalences of relevant categories of finite dimensional modules too. But, if we interested in equivalences of categories of finite dimensional representations only, then more equivalences are possible. Indeed, the category $k\Gamma - \mathrm{mod}_{fp}$ is equivalent to the category of artinian sheaves which is the direct sum of subcategories of sheaves with support in closed points of $\mathbb{A}^1 \setminus 0$.

6.9 Recollement

Given an admissible subcategory in a triangulated category and two t-structures, one in a subcategory and another one in its orthogonal, on can define a t-structure in the ambient category by the procedure known as *recollement* [BBD]. In this subsection, we show that the standard t-structure in $D^b(B - \mathrm{mod})$ is obtained by recollement from the standard t-structure on $D^b(A - \mathrm{mod})$ and $D^b(\mathbb{K} - \mathrm{mod})$.

The initial data for recollement are 3 triangulated categories D, D_U and D_F and six exact functors between them. Two of the functors are:

$$i_* : D_F \to D, \quad j^* : D \to D_U.$$

The other functors $i^*, i^!, j_*, j_!$ are the left and right adjoints to these two. They must satisfy a number of constraints which are equivalent to saying that D_F is an admissible subcategory in D and D_U is the quotient of D by D_F, with i_* the embedding functor and j^* the quotient functor.

Given a t-structure $(D_U^{\leq 0}, D_U^{\geq 0})$ in D_U and a t-structure $(D_F^{\leq 0}, D_F^{\geq 0})$ in D_F, the glued t-structure is defined by:

$$D^{\leq 0} := \{K \in D |\ j^*K \in D_U^{\leq 0} \text{ and } i^*K \in D_F^{\leq 0}\},$$

$$D^{\geq 0} := \{K \in D |\ j^*K \in D_U^{\geq 0} \text{ and } i^!K \in D_F^{\geq 0}\}.$$

According to theorem 1.4.10 in [BBD], this pair of categories indeed defines a t-structure.

As we got in the previous subsection, category $D^b(B - \text{mod})$ has an admissible subcategory $D_F = D^b(\mathbb{K} - \text{mod})$, with the quotient category (orthogonal) being the category equivalent to $D_U = D^b(A - \text{mod})$. We denote by $(D_F^{\leq 0}, D_F^{\geq 0})$ and by $(D_U^{\leq 0}, D_U^{\geq 0})$ the standard t-structure in these categories.

The dictionary between our functors and the standard notations for six functors is as follows:

$$\psi_{1*} \longleftrightarrow j_!,$$

$$\psi_{2*} \longleftrightarrow j_*,$$

$$\psi_1^! = \psi_2^* \longleftrightarrow j^*,$$

while the notation for $i_*, i^!, i^*$ coincide.

Theorem 65. *The standard t-structure in* $D^b(B - \text{mod})$ *coincides with the one obtained by recollement of the standard t-structures in* $D^b(\mathbb{K} - \text{mod})$ *and in* $D^b(A - \text{mod})$*.*

Proof. Denote by $(D^{\leq 0}, D^{\geq 0})$ the standard t-structure in $D^b(B - \text{mod})$ and by $D_{gl}^{\leq 0} \cap D_{gl}^{\geq 0}$ the glued t-structure. The embedding $D^{\leq 0} \subset D_{gl}^{\leq 0}$ and $D^{\geq 0} \subset D_{gl}^{\geq 0}$ follows from t-exactness (in the standard t-structures) of $\psi_1^!$ and from right t-exactness of i^* and left t-exactness of $i^!$. The inverse inclusions follow from the fact that both $(D^{\leq 0}, D^{\geq 0})$ and $D_{gl}^{\leq 0} \cap D_{gl}^{\geq 0}$ are t-structures, in particular, $D^{\leq 0}$ is the left orthogonal to $D^{\geq 0}$. □

Let V be a B-module. We say that it is *minimal* if

(i) $\text{Hom}_{B(\Gamma)}(V, k) = 0$,

(ii) $\text{Hom}_{B(\Gamma)}(k, V) = 0$.

This definition is in compliance with the definition of minimal configurations of projectors given in subsection 2.3. An object $V \in D^b(B(\Gamma)-\text{mod})$ is said to be an *extension* of an object $W \in D^b(k\Gamma-\text{mod})$, if $\psi_1^! V \simeq W$. Given a $k\Gamma$-module W, there is a unique up to isomorphism *minimal extension* of W. It can be defined as the image:

$$W_{min} := \text{Im } \lambda_W : \psi_{1*}W \to \psi_{2*}W.$$

Given an object $V \in D^b(B(\Gamma)-\text{mod})$, we define its *minimal shadow* as the minimal extension for $\psi_1^! V$:

$$V_{min} = (\psi_1^! V)_{min} = \text{Im } \lambda_{\psi_1^! V} : \psi_{1*}\psi_1^! V \to \psi_{2*}\psi_1^! V.$$

Clearly, the assignment $V \mapsto V_{min}$ produce a functor. Note that $\lambda_{\psi_1^! V}$ is identified with the composite of adjunction maps:

$$\psi_{1*}\psi_1^! V \to V \to \psi_{2*}\psi_2^* V$$

Come back to our example of the cyclic graph.

Theorem 66. *Assume that graph Γ is cyclic and parameters s_{ij} are such that corank of Laplace operator for $x = 1$ is 2 (this is in fact the maximal possible corank for all x). Consider a singular complex rational curve with one double point stratified by the singular point and the smooth complement to it. The category of $B(\Gamma)$-modules for this choice of parameters s_{ij} is equivalent to the category of perverse sheaves locally constant on the strata.*

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