THE SKEW GROWTH FUNCTIONS $N_{M,\deg}(t)$ FOR THE MONOID OF TYPE $\mathrm{B_{ii}}$ AND OTHERS.

TADASHI ISHIBE

Abstract. Let M be a positive homogeneously finitely presented monoid $\langle L \mid R\rangle_{mo}$ that satisfies the cancellation condition and is equipped with the degree map $\deg: M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words, and let $P_{M,\deg}(t) := \sum_{u \in M} t^{\deg(u)}$ be its generating series, called the (*spherical*) *growth function*. If M satisfies the condition that any subset J of I_0 (:= the image of the set L in M) admits either the least right common multiple Δ_J or no common multiple in M, then the inversion function $P_{M,\deg}(t)^{-1}$ is given by the polynomial $\sum_{J \subset I_0} (-1)^{\#J} t^{\deg(\Delta_J)}$, where the summation index J runs over all subsets of I_0 whose least right common multiple exists. Since a monoid M, in general, may not admit the least right common multiple Δ_J for a given subset J of it, if we attempt to generalize the formula, the consideration to obtain the above formula is invalid. In order to resolve this obstruction, we will examine the set $\mathrm{mcm}(J)$ of minimal common right multiples of J. Then, we need to introduce a concept of a *tower of minimal common multiples* of elements of M and denote the set of all the towers in M by $\mathrm{Tmcm}(M)$. By considering the structure of the set $\mathrm{Tmcm}(M)$ of all the towers, K. Saito has proved the *inversion formula*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1,$$

where the second factor in LHS is a suitably signed generating series

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in \mathrm{mcm}(J_n)} t^{\deg(\Delta)},$$

called the *skew growth function*.

In this article, we will present several explicit calculations of examples of the skew growth functions for the monoid of type $\mathrm{B_{ii}}$ and others whose towers do not stop on the first stage J_1. For monoids of this kind, on one hand, we generally find difficulty in calculating the skew growth functions, since, for any tower $T = (I_0, J_1, J_2, \cdots, J_n)$, we need to calculate the set $\mathrm{mcm}(J_i)$ explicitly for each J_i. On the other hand, we find difficulty in showing the cancellativity of them, since the pre-existing technique is far from perfect. By improving the technique, we will show the cancellativity of the above examples successfully.

1. Introduction

Let M be a positive homogeneously finitely presented monoid $\langle L \mid R\rangle_{mo}$ that satisfies the cancellation condition (i.e. $axb = ayb$ implies $x = y$). Due to the homogeneity, we naturally define a map $\deg: M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words. By analogy to the (spherical) growth function for a group, the (*spherical*) *growth function* for a monoid M is defined as $P_{M,\deg}(t) := \sum_{u \in M} t^{\deg(u)}$ and has been discussed and calculated by several authors ([A-N][B][Bro][De][I1][S2][S3][S4][S5][Xu]). In these studies except the [I1], the monoid M satisfies the **condition** $\mathcal{L}$ that any subset J of I_0 (:= the image of the set L in M) admits either the least right common multiple Δ_J or no common multiple in M. Then, the inversion function $P_{M,\deg}(t)^{-1}$ is given by the following function, called the *skew growth function*,

$$N_{M,\deg}(t) := \sum_{J \subset I_0} (-1)^{\#J} t^{\deg(\Delta_J)},$$

where the summation index J runs over all subsets of I_0 whose least right common multiple exists. Thus, the growth function $P_{M,\deg}(t)$ can be calculated through the skew growth function $N_{M,\deg}(t)$. We remark that, in this case, $N_{M,\deg}(t)$ is a polynomial.

In general, a monoid M may not admit the least right common multiple Δ_J for a given subset J of it. Indeed, the monoid, called the type $\mathrm{B_{ii}}$, that was discussed in [I1], does not satisfy the **condition** $\mathcal{L}$. Nevertheless, the author has made a success in calculating the spherical growth function for it. The spherical growth function can be expressed as a rational function $\frac{1-t+t^2}{(1-t)^4}$. Since the form of the numerator polynomial $1 - t + t^2$ is suggestive, a generalization of the inversion formula to a wider class of monoids which do not always satisfy the **condition** $\mathcal{L}$ is naturally expected. Instead of considering the least common multiple Δ_J, we study the set $\mathrm{mcm}(J)$ of *minimal common right multiples* of J. However, the datum $\{\mathrm{mcm}(J)\}_{J \subset I_0}$ is still not sufficient to recover the inversion formula, since a subset J' of $\mathrm{mcm}(J)$ in general may have common right multiples. Thus, we need to consider the set $\mathrm{mcm}(J')$ for a subset J' of $\mathrm{mcm}(J)$. Then, we may again need to consider $\mathrm{mcm}(J'')$ for a subset J'' of $\mathrm{mcm}(J')$, and so on. Repeating this process, we are naturally led to consider a notion of *tower*: a finite sequence $I_0 \supset J_1, J_2, \ldots, J_n$ of subsets of M such that $J_2 \subset \mathrm{mcm}(J_1), \ldots, J_n \subset \mathrm{mcm}(J_{n-1})$.

In [S1], K. Saito has made a success in generalizing the inversion formula for a rather wider class of monoids. Namely, for a cancellative monoid M equipped with a discrete degree map $\deg : M \to \mathbb{R}_{\geq 0}$ (see [S1]§4), he defined the (spherical) growth function $P_{M,\deg}(t)$ for M with respect to deg, and, by considering the set $\mathrm{Tmcm}(M)$ of all towers $T = (I_0, J_1, J_2, \cdots, J_n)$ in M (we do not explain here what the set I_0 is about, cf. [S1]), he defined the skew growth function

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in \mathrm{mcm}(J_n)} t^{\deg(\Delta)}.$$

Then, he has shown the inversion formula for M with respect to deg

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

Thus, when we put $h(M, \deg) := \max\{n \mid T = (I_0, J_1, J_2, \cdots, J_n) \in \mathrm{Tmcm}(M)\}$, the inversion formula covers all the cases $\infty \geq h(M, \deg) \geq 0$.

In this article, for a positive homogeneously finitely presented cancellative monoid $M = \langle L \mid R \rangle_{mo}$ that do not satisfy the **condition** $\mathcal{L}$, we will present several explicit calculations of examples of the skew growth functions. For a non-abelian monoid $M = \langle L \mid R \rangle_{mo}$ whose $h(M, \deg)$ is equal to ∞, one may think that calculations of the skew growth functions are not practicable. However, in §5, we succeed in the non-trivial calculation for the monoid of type $\mathrm{B_{ii}}$, partially because, for any tower $T = (I_0, J_1, J_2, \cdots, J_n)$, we can calculate the set $\mathrm{mcm}(J_i)$ explicitly for each J_i due to the Lemma 5.5. For the same reason, for the monoids G_n^+ and H_n^+ whose $h(M, \deg)$ is equal to 2 and the abelian monoid M_{abel} whose $h(M, \deg)$ is equal to ∞, we can calculate the skew growth functions in §5. As far as we know, for non-abelian monoids that do not satisfy the **condition** $\mathcal{L}$, there are few examples for which the cancellativity of them has been shown, since the pre-existing technique to show the cancellativity is far from perfect ([G][B-S][Deh1][Deh2]). In [Deh1], [Deh2], if presentation of a positive homogeneously presented monoid satisfies some condition, called completeness, the cancellativity of it can be trivially checked. When the presentation is not complete, in order to obtain a complete presentaion, some procedure, called completion, is carried out. From our experience, for most of non-abelian monoids that do not satisfy the **condition** $\mathcal{L}$, these procedures do not finish in finite steps. For monoids of this kind, nothing is discussed in [Deh1], [Deh2]. On the other hand, the presentations of the examples $G^+_{\mathrm{B_{ii}}}$ ([I1]), $G^+_{m,n}$ ([I2]), G^+_n and H^+_n are not complete and the procedures do not finish in finite steps. Nevertheless, by improving the technique, we show the cancellativity of them successfully. In §4, we will show the cancellativity of the monoids G^+_n and H^+_n by the improved technique. In [S2][S3][S4][S5][K-T-Y], the distribution of the zeroes of the denominator polynomials of the growth functions are investigated from the viewpoint of limit functions ([S4]). In §6, we will explore another viewpoint from an analysis of the three examples.

2. Positive homogeneous presentation

In this section, we first recall from [S-I], [B-S] some basic definitions and notations. Secondly, for a positive homogeneously finitely presented group

$$G = \langle L \,|\, R \,\rangle,$$

we associate a monoid defined by it. We will prepare basic definitions in a positive homogeneously presented monoid. Lastly, we define two operations on the set of subsets of a monoid.

First, we recall from [S-I] basic definitions on a monoid M.

Definition 2.1. *1. A monoid M is called* cancellative, *if a relation $AXB = AYB$ for $A, B, X, Y \in M$ implies $X = Y$.*

2. For two elements u, v in M, we denote

$$u|_l v$$

if there exists an element x in M such that $v = ux$. We say that u divides v from the left, or, v is a right-multiple of u.

3. We say that G^+ is conical, *if 1 is the only invertible element in G^+.*

Next, we recall from [B-S] some terminologies and concepts. Let L be a finite set. Let $F(L)$ be the free group generated by L, and let L^* be the free monoid generated by L inside $F(L)$. We call the elements of $F(L)$ *words* and the elements of L^* *positive words*. The empty word ε is the identity element of L^*. If two words A, B are identical letter by letter, we write $A \equiv B$. Let $G = \langle L \,|\, R \,\rangle$ be a positive homogeneously presented group (i.e. the set R of relations consists of those of the form $R_i = S_i$ where R_i and S_i are positive words of the same length), where R is the set of relations. We often denote the images of the letters and words under the quotient homomorphism

$$F(L) \longrightarrow G$$

by the same symbols and the equivalence relation on elements A and B in G is denoted by $A = B$.

Next, we recall from [S-I], [I1] some basic concepts on positive homogeneously presented monoid.

Definition 2.2. *Let $G = \langle L \,|\, R \,\rangle$ be a positive homogeneously finitely presented group, where L is the set of generators (called alphabet) and R is the set of relations. Then we associate a monoid $G^{+} = \langle L \mid R \,\rangle_{mo}$ defined as the quotient of the free monoid L^{*} generated by L by the equivalence relation defined as follows:*

i) *two words U and V in L^{*} are called* elementarily equivalent *if either $U \equiv V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),*

ii) *two words U and V in L^{*} are called* equivalent, *denoted by $U \doteq V$, if there exists a sequence $U \equiv W_0, W_1, \ldots, W_n \equiv V$ of words in L^{*} for $n \in \mathbb{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \ldots, n$.*

Due to the homogeneity of the relations, we define a homomorphism:

$$\deg \;:\; G^{+} \;\longrightarrow\; \mathbb{Z}_{\geq 0}$$

by assigning to each equivalence class of words the length of the words.

Remark 1. *For a positive homogeneously presented group $G = \langle L \,|\, R \,\rangle$, the associated monoid $G^{+} = \langle L \mid R \,\rangle_{mo}$ is conical.*

Remark 2. *In* [S1]*, for a monoid M, the quotient set $M/\sim$ is considered, where the equivalence relation $\sim$ on M is defined by putting $u \sim v \Leftrightarrow_{def.} u|_l v \,\&\, v|_l u$. Due to the conicity, if $M = \langle L \mid R \,\rangle_{mo}$, then we say that $M/\sim\, = M$.*

Lastly, we consider two operations on the set of subsets of a monoid M. For a subset J of M, we put

$$\mathrm{cm}_r(J) := \{u \in M \mid j \,|_l\, u,\ \forall j \in J\},$$

$$\min_r(J) := \{u \in J \mid \exists v \in J \text{ s.t. } v \,|_l\, u \Rightarrow v = u\},$$

and their composition: the set of *minimal common multiples* of the set J by

$$\mathrm{mcm}(J) := \min_r(\mathrm{cm}_r(J)).$$

3. Generating functions $P_{M,\deg}$ and $N_{M,\deg}$

In this section, for a positive homogeneous presented cancellative monoid

$$M = \langle L \mid R \,\rangle_{mo},$$

we define a growth function $P_{M,\deg}$ and a skew growth function $N_{M,\deg}$. Next, we recall from [S1] the inversion formula for the growth function of M.

First, we introduce a concept of towers of minimal common multiples in M.

Definition 3.1. *A* tower *of M of height $n \in \mathbb{Z}_{\geq 0}$ is a sequence*

$$T := (I_0, J_1, J_2, \cdots, J_n)$$

of subsets of M satisfying the followings.

i) *$I_0 :=$ the image of the set L in M.*
ii) *$mcm(J_k) \neq \emptyset$ and we put $I_k := \mathrm{mcm}(J_k)$ for $k = 1, \ldots, n$.*
iii) *$J_k \subset I_{k-1}$ such that $1 < \# J_k < \infty$ for $k = 1, \ldots, n$.*

Here, we call I_0, J_k and I_k, the ground, the kth stage and the set of minimal common multiples on the kth stage of the tower T, respectively. In particular, the set of minimal common multiples on the top stage is denoted by $|T| := I_n$.

The set of all towers of M shall be denote by $\mathrm{Tmcm}(M)$. *We put* $h(M, \deg) := \max\{\text{height of } T \in \mathrm{Tmcm}(M)\}$.

Remark 3. i) *It is clear that M is a free monoid if and only if $h(M, \deg) = 0$.*
ii) All of the monoids discussed in [A-N], [B-S], [S2], [S3] have $h(M, \deg) \leq 1$.
iii) For the following cancellative monoid

$$G^+_{\mathrm{B_{ii}}} := \left\langle a, b, c \;\middle|\; \begin{array}{c} cbb = bba, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo},$$

we have $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$ (§5).
iv) For the following cancellative monoids

$$G^+_n := \left\langle a, b, c \;\middle|\; \begin{array}{c} cb^n = b^n a, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo} (n = 3, 4, \ldots),$$

$$H^+_n := \left\langle a, b, c \;\middle|\; \begin{array}{c} b(ab)^n ba = cb(ab)^n b, \\ ab = bc, \\ ac = ca \end{array} \right\rangle_{mo} (n = 1, 2, \ldots),$$

we have $h(G^+_n, \deg) = 2$ and $h(H^+_n, \deg) = 2$ (§5).
v) For the following abelian cancellative monoid

$$M_{\mathrm{abel},m} := \left\langle a, b \;\middle|\; \begin{array}{c} a^m = b^m, \\ ab = ba \end{array} \right\rangle_{mo} (m = 2, 3, \ldots),$$

we have $h(M_{\mathrm{abel},m}, \deg) = \infty$ (§5).

Secondly, we define a growth function $P_{M,\deg}$ and a skew growth function $N_{M,\deg}$. In the previous section, we have fixed a degree map deg on M. Then, we define the *growth function of the monoid* $(M, \deg)$ by

$$P_{M,\deg} := \sum_{u \in M} t^{\deg(u)} = \sum_{d \in \mathbb{Z}_{\geq 0}} \#(M_d)\ t^d,$$

where we put $M_d := \{u \in M \mid \deg(u) = d\}$. And we define the *skew growth function of the monoid* $(M, \deg)$ by

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in |T|} t^{\deg(\Delta)}. \tag{3.1}$$

Remark 4. *In the definition* (3.1)*, we can write down the coefficient of the term t directly. Namely, we write*

$$N_{M,\deg}(t) = 1 - \#(I_0)t + \sum_{\text{height of } T \geq 1} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in |T|} t^{\deg(\Delta)}.$$

Remark 5. Therefore, if M is a free monoid of rank n, then we have $N_{M,\deg}(t) = 1 - nt$.

Lastly, we recall from [S1] the inversion formula for the growth function of $(M, \deg)$.

Theorem 3.2. *We have the inversion formula*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$

4. Cancellativity of G_n^+ and H_n^+

In this section, for a preparation for calculations of the skew growth functions for the monoids G_n^+ and H_n^+ in §5, we prove the cancellativity of them.

First, we show the cancellativity of G_n^+.

Theorem 4.1. *The monoid G_n^+ is a cancellative monoid.*

Proof. First, we remark the following.

Proposition 4.2. *The left cancellativity on G_n^+ implies the right cancellativity.*

Proof. Consider a map $\varphi : G_n^+ \to G_n^+$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where σ is a permutation $\binom{a,b,c}{c,b,a}$ and $rev(W)$ is the reverse of the word $W = x_1x_2\cdots x_k$ (x_i is a letter) given by the word $x_k \cdots x_2x_1$. In view of the defining relation of G_n^+, φ is well-defined and is an anti-isomorphism. If $\beta\alpha \doteqdot \gamma\alpha$, then $\varphi(\beta\alpha) \doteqdot \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) \doteqdot \varphi(\alpha)\varphi(\gamma)$. Using the left cancellativity, we obtain $\varphi(\beta) \doteqdot \varphi(\gamma)$ and, hence, $\beta \doteqdot \gamma$.

□

The following is sufficient to show the left cancellativity on G_n^+.

Proposition 4.3. *Let Y be a positive word in G_n^+ of length $r \in \mathbb{Z}_{\geq 0}$ and let $X^{(h)}$ be a positive word in G_n^+ of length $r - h \in \{ r - n + 1, \ldots, r \}$.*
(i) *If $vX^{(0)} \doteqdot vY$ for some $v \in \{a, b, c\}$, then $X^{(0)} \doteqdot Y$.*
(ii) *If $aX^{(0)} \doteqdot bY$, then $X^{(0)} \doteqdot bZ$, $Y \doteqdot cZ$ for some positive word Z.*
(iii) *If $aX^{(0)} \doteqdot cY$, then $X^{(0)} \doteqdot cZ$, $Y \doteqdot aZ$ for some positive word Z.*
(iv - 0) *If $bX^{(0)} \doteqdot cY$, then there exist an integer k $(0 \leq k \leq r - n)$ and a positive word Z such that $X^{(0)} \doteqdot c^kb^{n-1}a \cdot Z$ and $Y \doteqdot a^kb^n \cdot Z$.*
(iv - 1 - a) *There does not exist words $X^{(1)}$ and Y that satisfy an equation $ba \cdot X^{(1)} \doteqdot cY$.*
(iv - 1 - b) *If $bb \cdot X^{(1)} \doteqdot cY$, then $X^{(1)} \doteqdot b^{n-2}a \cdot Z$ and $Y \doteqdot b^n \cdot Z$ for some positive word Z.*
(iv - 1 - c) *If $bc \cdot X^{(1)} \doteqdot cY$, then there exist an integer k $(0 \leq k \leq r - n - 1)$ and a positive word Z such that $X^{(1)} \doteqdot c^kb^{n-1}a \cdot Z$ and $Y \doteqdot a^{k-1}b^n \cdot Z$.*
If $n \geq 4$, then, for $2 \leq h \leq n - 2$, we have to prepare the following propositions (iv - h - a) (iv - h - b) *and* (iv - h - c)
(iv - h - a) *There does not exist positive words $X^{(h)}$ and Y that satisfy an equation $b^ha \cdot X^{(h)} \doteqdot cY$.*
(iv - h - b) *If $b^{h+1} \cdot X^{(h)} \doteqdot cY$, then $X^{(h)} \doteqdot b^{n-h-1}a \cdot Z$ and $Y \doteqdot b^n \cdot Z$ for some positive word Z.*
(iv - h - c) *There does not exist positive words $X^{(h)}$ and Y that satisfy an equation $b^hc \cdot X^{(h)} \doteqdot cY$.*
(iv - $(n-1)$ - a) *If $b^{n-1}a \cdot X^{(n-1)} \doteqdot cY$, then $X^{(n-1)} \doteqdot ba \cdot Z$ and $Y \doteqdot b^nc \cdot Z$ for some positive word Z.*
(iv - $(n-1)$ - b) *If $b^n \cdot X^{(n-1)} \doteqdot cY$, then $X^{(n-1)} \doteqdot aZ$ and $Y \doteqdot b^n \cdot Z$ for some positive word Z.*

(iv - $(n-1)$ - c) *There does not exist positive words $X^{(n-1)}$ and Y that satisfy an equation $b^{n-1}c \cdot X^{(n-1)} \doteq cY$.*

Proof. We will show the general theorem, by refering to the triple induction (see [I2] for instance). The theorem for a positive word Y of word-length r and $X^{(h)}$ of word-length $r-h \in \{r-n+1,\dots,r\}$ will be refered to as $\mathrm{H}_{r,h}$. It is easy to show that, for $r=0,1$, $\mathrm{H}_{r,h}$ is true. If a positive word U_1 is transformed into U_2 by using t single applications of the defining relations of G_n^+, then the whole transformation will be said to be of *chain-length t*. For induction hypothesis, we assume
(A) $\mathrm{H}_{s,h}$ is true for $s=0,\dots,r$ and arbitrary h for transformations of all chain-lengths,
and
(B) $\mathrm{H}_{r+1,h}$ is true for $0 \le h \le n-1$ for all chain-lengths $\le t$.
We will show the theorem $\mathrm{H}_{r+1,h}$ for chain-lengths $t+1$. For the sake of simplicity, we devide the proof into two steps.
Step 1. $\mathrm{H}_{r+1,h}$ for $h=0$
Let X, Y be of word-length $r+1$, and let

$$v_1X \doteq v_2W_2 \doteq \cdots \doteq v_{t+1}W_{t+1} \doteq v_{t+2}Y$$

be a sequence of single transformations of $t+1$ steps, where $v_1,\dots,v_{t+2} \in \{a,b,c\}$ and $W_2,\dots,W_{t+1}$ are positive words of length $r+1$. By the assumption $t>1$, there exists an index $\tau \in \{2,\dots,t+1\}$ such that we can decompose the sequence into two steps

$$v_1X \doteq v_\tau W_\tau \doteq v_{t+2}Y,$$

in which each step satisfies the induction hypothesis (B).

If there exists τ such that v_τ is equal to either to v_1 or v_{t+2}, then by induction hypothesis, W_τ is equivalent either to X or to Y. Hence, we obtain the statement for the $v_1X \doteq v_{t+2}Y$. Thus, we assume from now on $v_\tau \neq v_1, v_{t+2}$ for $1<\tau\le t+1$.

Suppose $v_1 = v_{t+2}$. If there exists τ such that $(v_1 = v_{t+2}, v_\tau) \neq (b,c),(c,b)$, then each of the equivalences says the existence of $\alpha,\beta \in \{a,b,c\}$ and positive words Z_1, Z_2 such that $X \doteq \alpha Z_1$, $W_\tau \doteq \beta Z_1 \doteq \beta Z_2$ and $Y \doteq \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \doteq \beta Z_2$, we get $Z_1 \doteq Z_2$. Hence, we obtain the statement $X \doteq \alpha Z_1 \doteq \alpha Z_2 \doteq Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $(v_1 = v_{t+2}, v_\tau) = (b,c)$. However, because of the above consideration, we say $v_2 = \cdots = v_{t+1} = c$. Hence, we consider the following case

$$bX \doteq cW_1 \doteq \cdots \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis (B) to each step, we say that there exist positive words Z_3 and Z_4 such that

$$X \doteq b^{n-1}a \cdot Z_3,\ W_1 \doteq b^n \cdot Z_3,$$

$$W_{t+1} \doteq b^n \cdot Z_4,\ Y \doteq b^{n-1}a \cdot Z_4.$$

Since an equation $W_1 \doteq W_{t+1}$ holds, we say that

$$b^n \cdot Z_3 \doteq b^n \cdot Z_4.$$

By induction hypothesis, we have $X \doteq Y$.
In the case of $(v_1 = v_{t+2}, v_\tau) = (c,b)$, we can prove the statement in a similar

manner.
Suppose $v_1 \neq v_{t+2}$. We consider the following three cases.
Case 1: $(v_1, v_\tau, v_{t+2}) = (a, b, c)$
Because of the above consideration, we consider the case $\tau = t+1$, namely

$$aX \doteq bW_{t+1} \doteq cY.$$

Applying the induction hypothesis to each step, we say that there exist positive words Z_1 and Z_2 such that

$$X \doteq bZ_1,\ W_{t+1} \doteq cZ_1,$$

$$W_{t+1} \doteq b^{n-1}a \cdot Z_2,\ Y \doteq b^n \cdot Z_2.$$

Thus, we say that $c \cdot Z_1 \doteq b^{n-1}a \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word Z_3 such that

$$Z_1 \doteq b^n c \cdot Z_3,\ Z_2 \doteq ba \cdot Z_3.$$

Hence, we have $X \doteq cb^{n+1} \cdot Z_3$ and $Y \doteq ab^{n+1} \cdot Z_3$.
Case 2: $(v_1, v_\tau, v_{t+2}) = (a, c, b)$
We consider the case $\tau = t+1$, namely

$$aX \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis to each step, we say that there exist positive words Z_1 and Z_2 such that

$$X \doteq cZ_1,\ W_{t+1} \doteq aZ_1,$$

$$W_{t+1} \doteq b^n \cdot Z_2,\ Y \doteq b^{n-1}a \cdot Z_2.$$

Thus, we say that $aZ_1 \doteq b^n \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word Z_3 such that

$$Z_1 \doteq b^{n+1} \cdot Z_3,\ Z_2 \doteq ba \cdot Z_3.$$

Hence, we have $X \doteq b \cdot b^n c \cdot Z_3$ and $Y \doteq c \cdot b^n c \cdot Z_3$.
Case 3: $(v_1, v_\tau, v_{t+2}) = (b, a, c)$
Then, we consider the following case

$$bX \doteq aW_\tau \doteq cY.$$

Applying the induction hypothesis to each step, we say that there exist positive words Z_1 and Z_2 such that

$$X \doteq cZ_1,\ W_\tau \doteq bZ_1,$$

$$W_\tau \doteq cZ_2,\ Y \doteq aZ_2.$$

Moreover, we say that there exist a positive word Z_3 and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$Z_1 \doteq c^k b^{n-1}a \cdot Z_3,\ Z_2 \doteq a^k b^n \cdot Z_3.$$

Thus, we have

$$X \doteq c^{k+1}b^{n-1}a \cdot Z_3,\ Y \doteq a^{k+1}b^n \cdot Z_3.$$

Step 2. $\mathrm{H}_{r+1,h}$ for $0 \leq h \leq n-1$
We will show the general theorem $\mathrm{H}_{r+1,h}$ by induction on h. The case $h = 0$ is proved in Step 1. First, we show the case $h = 1$. Let $X^{(1)}$ be of word-length r, and

let Y be of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(1)} \doteq \cdots \doteq cY,$$

where V is a positive word of length 2. We discuss three cases.

Case $1: V = ba$.

We consider the following case

$$ba \cdot X^{(1)} \doteq \cdots \doteq cY. \tag{4.1}$$

By the result of Step 1, we say that there exist a positive word Z_1 and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$aX^{(1)} \doteq c^k b^{n-1} a \cdot Z_1,\ Y \doteq a^k b^n \cdot Z_1.$$

Applying the induction hypothesis (A), we say that there exists a positive word Z_2 such that

$$X^{(1)} \doteq c^k \cdot Z_2,\ b^{n-1} a \cdot Z_1 \doteq aZ_2.$$

Moreover, we say that there exists a positive word Z_3 such that

$$b^{n-2} a \cdot Z_1 \doteq cZ_3,\ Z_2 \doteq bZ_3.$$

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(1)}$ and Y that satisfy the equation (4.1).

Case $2: V = bb$.

We consider the following case

$$bb \cdot X^{(1)} \doteq V_2 \cdot W_2 \doteq \cdots \doteq V_{t+1} \cdot W_{t+1} \doteq cY,$$

where V_2 and V_{t+1} are positive words. It is enough to discuss the case $(V_2, V_{t+1}) = (bcb^n, ac)$. Applying the induction hypothesis (A) to the equation

$$bcb^n \cdot W_2 \doteq ac \cdot W_{t+1}, \tag{4.2}$$

we say that there exists a positive word Z_1 such that $cW_{t+1} \doteq bZ_1$. Applying the induction hypothesis, we say that there exist a positive word Z_2 and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$W_{t+1} \doteq a^k b^n \cdot Z_2,\ Z_1 \doteq c^k b^{n-1} a \cdot Z_2. \tag{4.3}$$

Applying (4.3) to the equation (4.2), we have

$$bcb^n \cdot W_2 \doteq ac \cdot a^k b^n \cdot Z_2.$$

Moreover, we say

$$b^n \cdot W_2 \doteq c^k b^{n-1} a \cdot Z_2. \tag{4.4}$$

We consider the following two cases.

Case $2-1: k = 0$

There exists a positive word Z_3 such that

$$W_2 \doteq cZ_3,\ Z_2 \doteq bZ_3.$$

Thus, we have

$$X^{(1)} \doteq b^{n-1} a \cdot cZ_3 \doteq b^{n-2} a \cdot ba \cdot Z_3,$$

$$Y \doteq ab^n b \cdot Z_3 \doteq b^n \cdot baZ_3.$$

Case $2-2: k \geq 1$
Applying the induction hypothesis to the equation (4.4), we say that there exists a positive word Z_3 such that

$$W_2 \doteq a^k \cdot Z_3.$$

Thus, we consider the equation $b^n \cdot Z_3 \doteq b^{n-1}a \cdot Z_2$. We say that there exists a positive word Z_4 such that

$$Z_2 \doteq bZ_4, \ Z_3 \doteq cZ_4.$$

Thus, we have

$$X^{(1)} \doteq b^{n-1}a \cdot a^k c \cdot Z_3 \doteq b^{n-2}a \cdot ba^{k+1}Z_3,$$
$$Y \doteq aa^k b^n b \cdot Z_3 \doteq b^n \cdot ba^{k+1}Z_3.$$

Case $3: V = bc$.
Then, we consider the following case

$$bc \cdot X^{(1)} \doteq \cdots \doteq cY.$$

By the induction hypothesis, we say that there exist a positive word Z_1 and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$cX^{(1)} \doteq c^k b^{n-1}a \cdot Z_1, \ Y \doteq a^k b^n \cdot Z_1.$$

We consider the following two cases.
Case $3-1: k = 0$
By the induction hypothesis, we say that there exists a positive word Z_2 such that

$$X^{(1)} \doteq b^n c \cdot Z_2, \ Z_1 \doteq ba \cdot Z_2.$$

Thus, we have

$$X^{(1)} \doteq b^{n-1}a \cdot bZ_2, \ Y \doteq b^n ba \cdot Z_2 \doteq ab^n \cdot bZ_2.$$

Case $3-2: k \geq 1$
Then, we have

$$X^{(1)} \doteq c^{k-1}b^{n-1}a \cdot Z_1, \ Y \doteq a^k b^n \cdot Z_1.$$

Second, when $n \geq 4$, we show the theorem $\mathrm{H}_{r+1,h}$ $(2 \leq h \leq n-2)$ by induction on h. We assume $h = 1, 2, \ldots, j$ $(j \leq n-3)$. The case $h = 1$ has been proved. Let $X^{(j+1)}$ be of word-length $r - j$, and let Y be of word-length $r + 1$. We consider a sequence of single transformations of $t + 1$ steps

$$V \cdot X^{(j+1)} \doteq \cdots \doteq cY, \tag{4.5}$$

where V is a positive word of length $j + 2$. We discuss the following three cases.
Case $1: V \doteq bb^j a$.
Applying the induction hypothesis, we say that there exists a positive word Z_1 such that

$$aX^{(j+1)} \doteq b^{n-j-1}a \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

By the induction hypothesis, we say that there exists a positive word Z_2 such that

$$X^{(j+1)} \doteq bZ_2, \ b^{n-j-2}a \cdot Z_1 \doteq cZ_2.$$

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(j+1)}$ and Y that satisfy the equation (4.5).

Case $2: V \doteq bb^{j+1}$.
Applying the induction hypothesis, we say that there exists a positive word Z_1 such that

$$bX^{(j+1)} \doteq b^{n-j-1}a \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

Thus, we have $X^{(j+1)} \doteq b^{n-j-2}a \cdot Z_1$.
Case $3: V \doteq bb^j c$.
Applying the induction hypothesis, we say that there exists a positive word Z_1 such that

$$cX^{(j+1)} \doteq b^{n-j-1}a \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(j+1)}$ and Y that satisfy the equation (4.5).

Lastly, we show the theorem $\mathrm{H}_{r+1,n-1}$. Let $X^{(n-1)}$ be of word-length $r-n+2$, and let Y be of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(n-1)} \doteq \cdots \doteq cY, \tag{4.6}$$

where V is a positive word of length n. We discuss the following three cases.
Case $1: V \doteq b^{n-1}a$.
By the above result, we say that there exists a positive word Z_1 such that

$$aX^{(n-1)} \doteq ba \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

By the induction hypothesis, we say that there exists a positive word Z_2 such that

$$X^{(n-1)} \doteq ba \cdot Z_2, \ Z_1 \doteq cZ_2.$$

Thus, we have $Y \doteq b^n c \cdot Z_2$.
Case $2: V \doteq b^{n-1}b$.
By the above result, we say that there exists a positive word Z_1 such that

$$bX^{(n-1)} \doteq ba \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

Thus, we have $X^{(n-1)} \doteq aZ_1$.
Case $3: V \doteq b^{n-1}c$.
By the above result, we say that there exists a positive word Z_1 such that

$$cX^{(n-1)} \doteq ba \cdot Z_1, \ Y \doteq b^n \cdot Z_1.$$

We have a contradiction. Hence, there does not exist positive words $X^{(n-1)}$ and Y that satisfy the equation (4.6).

□

This completes the proof of Theorem 4.1. □

Secondly, we show the cancellativity of H_n^+.

Theorem 4.4. *The monoid H_n^+ is a cancellative monoid.*

Proof. First, we remark the following.

Proposition 4.5. *The left cancellativity on H_n^+ implies the right cancellativity.*

Proof. Consider a map $\varphi : H_n^+ \to H_n^+$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where σ is a permutation $\binom{a,b,c}{c,b,a}$. By following the proof in Proposition 4.2, we can show the statement. □

The following is sufficient to show the left cancellativity of the monoid H_n^+.

Proposition 4.6. *Let Y be a positive word in H_n^+ of length $r \in \mathbb{Z}_{\geq 0}$ and let $X^{(h)}$ be a positive word in H_n^+ of length $r-h \in \{\,2n,\dots,r\,\}$.*
(i) *If $vX^{(0)} \underset{\cdot}{=} vY$ for some $v \in \{a,b,c\}$, then $X^{(0)} \underset{\cdot}{=} Y$.*
(ii) *If $aX^{(0)} \underset{\cdot}{=} bY$, then $X^{(0)} \underset{\cdot}{=} bZ$, $Y \underset{\cdot}{=} cZ$ for some positive word Z.*
(iii) *If $aX^{(0)} \underset{\cdot}{=} cY$, then $X^{(0)} \underset{\cdot}{=} cZ$, $Y \underset{\cdot}{=} aZ$ for some positive word Z.*
(iv) *If $bX^{(0)} \underset{\cdot}{=} cY$, then there exist an integer k $(0 \leq k \leq r-2n-2)$ and a positive word Z such that $X^{(0)} \underset{\cdot}{=} c^k(ab)^n ba \cdot Z$ and $Y \underset{\cdot}{=} a^k b(ab)^n b \cdot Z$.*
(v) *If $bb \cdot X^{(1)} \underset{\cdot}{=} cY$, then $X^{(1)} \underset{\cdot}{=} c(ab)^{n-1}ba \cdot Z$, $Y \underset{\cdot}{=} b(ab)^n b \cdot Z$ for some positive word Z.*
For $2 \leq h \leq r-2n$, we prepare the following propositions.
(vi-h) *If $c^{h-1}bb \cdot X^{(h)} \underset{\cdot}{=} bY$, then $X^{(h)} \underset{\cdot}{=} c(ab)^{n-1}b \cdot Z$ and $Y \underset{\cdot}{=} (ab)^n ba^{h-1} \cdot Z$ for some positive word Z.*

Proof. By refering to the double induction (see [G], [B-S] for instance), we show the general theorem. The theorem for a positive word Y of word-length r and $X^{(h)}$ of word-length $r-h \in \{\,r-2n,\dots,r\,\}$ will be refered to as $\mathrm{H}_{r,h}$. It is easy to show that, for $r = 0,1$, $\mathrm{H}_{r,h}$ is true. For induction hypothesis, we assume
(A) $\mathrm{H}_{s,h}$ is true for $s = 0,\dots,r$ and arbitrary h for transformations of all chain-lengths,
and
(B) $\mathrm{H}_{r+1,h}$ is true for $0 \leq h \leq \max\{0, r+1-2n\}$ for all chain-lengths $\leq t$.
We will show the theorem $\mathrm{H}_{r+1,h}$ for chain-lengths $t+1$. For the sake of simplicity, we devide the proof into two steps.
Step 1. $\mathrm{H}_{r+1,h}$ for $h = 0$
Let X, Y be of word-length $r+1$, and let

$$v_1X \underset{\cdot}{=} v_2W_2 \underset{\cdot}{=} \cdots \underset{\cdot}{=} v_{t+1}W_{t+1} \underset{\cdot}{=} v_{t+2}Y$$

be a sequence of single transformations of $t+1$ steps, where $v_1,\dots,v_{t+2} \in \{\,a,b,c\,\}$ and $W_2,\dots,W_{t+1}$ are positive words of length $r+1$. By the assumption $t > 1$, there exists an index $\tau \in \{\,2,\dots,t+1\,\}$ such that we can decompose the sequence into two steps

$$v_1X \underset{\cdot}{=} v_\tau W_\tau \underset{\cdot}{=} v_{t+2}Y,$$

in which each step satisfies the induction hypothesis (B).

If there exists τ such that v_τ is equal to either to v_1 or v_{t+2}, then by induction hypothesis, W_τ is equivalent either to X or to Y. Hence, we obtain the statement for the $v_1X \underset{\cdot}{=} v_{t+2}Y$. Thus, we assume from now on $v_\tau \neq v_1, v_{t+2}$ for $1 < \tau \leq t+1$.

Suppose $v_1 = v_{t+2}$. If there exists τ such that $(\,v_1 = v_{t+2}, v_\tau\,) \neq (\,b,c\,),(\,c,b\,)$, then each of the equivalences says the existence of $\alpha,\beta \in \{\,a,b,c\,\}$ and positive words Z_1, Z_2 such that $X \underset{\cdot}{=} \alpha Z_1$, $W_\tau \underset{\cdot}{=} \beta Z_1 \underset{\cdot}{=} \beta Z_2$ and $Y \underset{\cdot}{=} \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \underset{\cdot}{=} \beta Z_2$, we get $Z_1 \underset{\cdot}{=} Z_2$. Hence, we obtain the statement $X \underset{\cdot}{=} \alpha Z_1 \underset{\cdot}{=} \alpha Z_2 \underset{\cdot}{=} Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $(\,v_1 = v_{t+2}, v_\tau\,) = (b,c)$. However, because of the above consideration, we say $v_2 = \cdots = v_{t+1} = c$. Hence, we consider the following case

$$bX \underset{\cdot}{=} cW_1 \underset{\cdot}{=} \cdots \underset{\cdot}{=} cW_{t+1} \underset{\cdot}{=} bY.$$

Applying the induction hypothesis (B) to each step, we say that there exist positive words Z_3 and Z_4 such that

$$X \doteq (ab)^n ba \cdot Z_3,\ W_1 \doteq b(ab)^n b \cdot Z_3,$$

$$W_{t+1} \doteq b(ab)^n b \cdot Z_4,\ Y \doteq (ab)^n ba \cdot Z_4.$$

Since an equation $W_1 \doteq W_{t+1}$ holds, we say that $X \doteq Y$.
In the case of $(v_1 = v_{t+2}, v_\tau) = (c, b)$, we can prove the statement in a similar manner.
Suppose $v_1 \neq v_{t+2}$. It suffices to consider the following two cases.
Case 1 : $(v_1, v_\tau, v_{t+2}) = (a, b, c)$
Because of the above consideration, we consider the case $\tau = t + 1$, namely

$$aX \doteq bW_{t+1} \doteq cY.$$

Applying the induction hypothesis to each step, we say that there exist positive words Z_1 and Z_2 such that

$$X \doteq bZ_1,\ W_{t+1} \doteq cZ_1,$$

$$W_{t+1} \doteq (ab)^n ba \cdot Z_2,\ Y \doteq b(ab)^n b \cdot Z_2.$$

Thus, we say that $cZ_1 \doteq (ab)^n ba \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word Z_3 such that

$$Z_1 \doteq aZ_3,\ b(ab)^{n-1} ba \cdot Z_2 \doteq cZ_3.$$

Hence, we have $bbc(ab)^{n-2}ba \cdot Z_2 \doteq cZ_3$. Applying the induction hypothesis (A) to this equation, there exists a positive word Z_4 such that

$$c(ab)^{n-2}ba \cdot Z_2 \doteq c(ab)^{n-1}ba \cdot Z_4,\ Z_3 \doteq b(ab)^n b \cdot Z_4.$$

Hence, we have $ba \cdot Z_2 \doteq abba \cdot Z_4$. Moreover, we say that there exists a positive word Z_5 such that

$$Z_2 \doteq cba \cdot Z_5,\ Z_4 \doteq cZ_5.$$

Thus, we have

$$X \doteq bab(ab)^n bc \cdot Z_5 \doteq c \cdot b(ab)^n bcb \cdot Z_5,$$

$$Y \doteq b(ab)^n bcba \cdot Z_5 \doteq a \cdot b(ab)^n bcb \cdot Z_5.$$

Case 2 : $(v_1, v_\tau, v_{t+2}) = (a, c, b)$
We consider the case $\tau = t + 1$, namely

$$aX \doteq cW_{t+1} \doteq bY.$$

Applying the induction hypothesis to each step, we say that there exist positive words Z_1 and Z_2 such that

$$X \doteq cZ_1,\ W_{t+1} \doteq aZ_1,$$

$$W_{t+1} \doteq b(ab)^n b \cdot Z_2,\ Y \doteq (ab)^n ba \cdot Z_2.$$

Thus, we say that $aZ_1 \doteq b(ab)^n b \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word Z_3 such that

$$Z_1 \doteq bZ_3,\ (ab)^n b \cdot Z_2 \doteq cZ_3.$$

Hence, there exists a positive word Z_4 such that

$$b(ab)^{n-1} b \cdot Z_2 \doteq cZ_4,\ Z_3 \doteq aZ_4.$$

We have $bbc(ab)^{n-2}b \cdot Z_2 \doteq cZ_4$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word Z_5 such that

$$c(ab)^{n-2}b \cdot Z_2 \doteq c(ab)^{n-1}ba \cdot Z_5,\ Z_4 \doteq b(ab)^n b \cdot Z_5.$$

Hence, we have $Z_2 \doteq cba \cdot Z_5$. Thus, we have

$$X \doteq cbab(ab)^n b \cdot Z_5 \doteq b(ab)^n bacb \cdot Z_5,$$

$$Y \doteq (ab)^n bacba \cdot Z_5 \doteq c(ab)^n bacb \cdot Z_5.$$

Step 2. $\mathrm{H}_{r+1,h}$ for $1 \leq h \leq r+1-2n$
We will show the general theorem $\mathrm{H}_{r+1,h}$. First, we show the case $h=1$. Then, we consider the following case

$$bb \cdot X^{(1)} \doteq \cdots \doteq cY.$$

By the result of Step 1, we say that there exist a positive word Z_1 and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$bX^{(1)} \doteq c^k(ab)^n ba \cdot Z_1,\ Y \doteq a^k b(ab)^n b \cdot Z_1.$$

Thus, we have $bX^{(1)} \doteq ac^k b(ab)^{n-1}ba \cdot Z_1$. Applying the induction hypothesis (A), we say that there exists a positive word Z_2 such that

$$X^{(1)} \doteq cZ_2,\ bZ_2 \doteq c^k b(ab)^{n-1}ba \cdot Z_1 \doteq c^k bbc(ab)^{n-2}ba \cdot Z_1.$$

We consider the case $k \geq 1$. By the induction hypothesis, we say that there exists a positive word Z_3 such that

$$Z_2 \doteq (ab)^n ba^k \cdot Z_3,\ c(ab)^{n-2}ba \cdot Z_1 \doteq c(ab)^{n-1}b \cdot Z_3.$$

Hence, we have $ba \cdot Z_1 \doteq abb \cdot Z_3$. Then, we have $aZ_1 \doteq cb \cdot Z_3$. By the induction hypothesis, there exists a positive word Z_4 such that

$$Z_1 \doteq cb \cdot Z_4,\ Z_3 \doteq cZ_4.$$

Thus, we have

$$X^{(1)} \doteq c(ab)^n ba^k c \cdot Z_4 \doteq c(ab)^{n-1}ba \cdot cba^k \cdot Z_4,$$

$$Y \doteq a^k b(ab)^n bcb \cdot Z_4 \doteq b(ab)^n b \cdot cba^k \cdot Z_4.$$

Next, we consider the case $2 \leq k \leq r+1-2n$. We consider the following case

$$c^{h-1}bb \cdot X^{(h)} \doteq \cdots \doteq bY. \tag{4.7}$$

By the result of Step 1, we say that there exist a positive word Z_1 and an integer $k_1 \in \mathbb{Z}_{\geq 0}$ such that

$$c^{h-2}bb \cdot X^{(h)} \doteq a^{k_1}b(ab)^n b \cdot Z_1,\ Y \doteq c^{k_1}(ab)^n ba \cdot Z_1.$$

By repeating the same process $h-1$ times, there exist integers $k_2, \ldots, k_{h-1} \in \mathbb{Z}_{\geq 0}$ and positive word Z_{h-1} such that

$$bb \cdot X^{(h)} \doteq a^{k_{h-1}} \cdot b(ab)^n b \cdot Z_{h-1}.$$

Then, we have $b \cdot X^{(h)} \doteq c^{k_{h-1}} \cdot (ab)^n b \cdot Z_{h-1} \doteq ac^{k_{h-1}} \cdot b(ab)^{n-1}b \cdot Z_{h-1}$. By the induction hypothesis, there exists a positive word Z_h such that

$$X^{(h)} \doteq cZ_h,\ c^{k_{h-1}} \cdot b(ab)^{n-1}b \cdot Z_{h-1} \doteq bZ_h.$$

Hence, we have $bZ_h \doteq c^{k_{h-1}} \cdot bbc(ab)^{n-2}b \cdot Z_{h-1}$. By the induction hypothesis, there exists a positive word Z_0 such that

$$c(ab)^{n-2}b \cdot Z_{h-1} \doteq c(ab)^{n-1}b \cdot Z_0,\ Z_h \doteq (ab)^n ba^{k_{h-1}} \cdot Z_0.$$

Thus, we have $bZ_{h-1} \doteq abb \cdot Z_0$. We have $Z_{h-1} \doteq cb \cdot Z_0$. Then, we have

$$X^{(h)} \doteq c(ab)^n ba^{k_{h-1}} \cdot Z_0 \doteq c(ab)^{n-1}b \cdot cba^{k_{h-1}} \cdot Z_0.$$

Applying this result to (4.7), we have

$$bY \doteq c^{h-1}bb \cdot c(ab)^{n-1}b \cdot cba^{k_{h-1}} \cdot Z_0 \doteq b(ab)^n ba^{h-1} \cdot cba^{k_{h-1}} \cdot Z_0.$$

Hence, we have $Y \doteq (ab)^n ba^{h-1} \cdot cba^{k_{h-1}} \cdot Z_0$.

□

This completes the proof of Theorem 4.4. □

5. Calculations of the skew growth functions

In this section, we will calculate the skew growth functions for the monoids $G^+_{\mathrm{B_{ii}}}$, G^+_n, H^+_n and $M_{\mathrm{abel},m}$.

First, we present an explicit calculation of the skew growth function for the monoid $G^+_{\mathrm{B_{ii}}}$. In [I1], we have made a success in calculating the growth function $P_{G^+_{\mathrm{B_{ii}}},\deg}(t)$ by using the normal form for the monoid $G^+_{\mathrm{B_{ii}}}$. By the inversion formula, we can calculate the skew growth function $N_{G^+_{\mathrm{B_{ii}}},\deg}(t)$. Nevertheless, we present an explicit calculation, because, in spite of the fact that the monoid is non-abelian and the height of it is infinite, we succeed in the non-trivial calculation.

Example. 1. First of all, we recall a fact from [S-I] §7.

Lemma 5.1. *Let X and Y be positive words in $G^+_{\mathrm{B_{ii}}}$ of length $r \in \mathbb{Z}_{\geq 0}$.*
(i) *If $vX \doteq vY$ for some $v \in \{a, b, c\}$, then $X \doteq Y$.*
(ii) *If $aX \doteq bY$, then $X \doteq bZ$, $Y \doteq cZ$ for some positive word Z.*
(iii) *If $aX \doteq cY$, then $X \doteq cZ$, $Y \doteq aZ$ for some positive word Z.*
(iv) *If $bX \doteq cY$, then there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word Z such that $X \doteq c^k ba \cdot Z$, $Y \doteq a^k bb \cdot Z$.*

Thanks to the Lemma 5.1, we have proved the cancellativity in [S-I]. And we prove the following Lemma.

Lemma 5.2. *If an equation $bb \cdot X \doteq cY$ in $G^+_{\mathrm{B_{ii}}}$ holds, then $X \doteq aZ$, $Y \doteq bb \cdot Z$ for some positive word Z.*

Proof. Due to the Lemma 5.1, we say that there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word Z_0 such that

$$bX \doteq c^k ba \cdot Z_0,\ Y \doteq a^k bb \cdot Z_0. \tag{5.1}$$

We consider the case $k \geq 1$. Due to the Lemma 5.1, we say that there exist an integer $i_1 \in \mathbb{Z}_{\geq 0}$ and a positive word Z_1 such that

$$X \doteq c^{i_1} ba \cdot Z_1,\ c^{k-1}ba \cdot Z_0 \doteq a^{i_1} bb \cdot Z_1.$$

Moreover, we say that there exists a positive word $Z_0^{(1)}$ such that

$$Z_0 \doteq c^{i_1} \cdot Z_0^{(1)},\ c^{k-1}ba \cdot Z_0^{(1)} \doteq bb \cdot Z_1.$$

Repeating the same process k-times, there exist integers $i_2, \dots, i_k \in \mathbb{Z}_{\geq 0}$ and positive words $Z_0^{(k)}$ and Z_k such that

$$Z_0 \underset{\cdot}{=} c^{i_1+i_2+\cdots+i_k} \cdot Z_0^{(k)},\ ba \cdot Z_0^{(k)} \underset{\cdot}{=} bb \cdot Z_k.$$

Moreover, we say that there exists a positive word Z' such that

$$Z_0^{(k)} \underset{\cdot}{=} bZ',\ Z_k \underset{\cdot}{=} cZ'.$$

Applying this result to (5.1), we have

$$\begin{aligned} bX &\underset{\cdot}{=} c^k bac^{i_1+i_2+\cdots+i_k} b \cdot Z' \underset{\cdot}{=} bac^{i_1+i_2+\cdots+i_k} ba^k \cdot Z', \\ Y &\underset{\cdot}{=} a^k bbc^{i_1+i_2+\cdots+i_k} b \cdot Z' \underset{\cdot}{=} bb \cdot c^{i_1+i_2+\cdots+i_k} ba^k \cdot Z'. \end{aligned}$$

Thus, we have $X \underset{\cdot}{=} a \cdot c^{i_1+i_2+\cdots+i_k} ba^k \cdot Z'$. □

As a consequence of Lemma 5.2, we obtain the followings.

Corollary 5.3. *If an equation $bb \cdot X \underset{\cdot}{=} c^l \cdot Y$ in $G^+_{\mathrm{B_{ii}}}$ holds for some positive integer l, then $X \underset{\cdot}{=} a^l \cdot Z$, $Y \underset{\cdot}{=} bb \cdot Z$ for some positive word Z.*

Due to the Corollary 5.3, we can solve the following equation.

Proposition 5.4. *If, for $0 \leq i < j$, an equation $c^i b \cdot X \underset{\cdot}{=} c^j b \cdot Y$ in $G^+_{\mathrm{B_{ii}}}$ holds, then there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word Z such that*

$$X \underset{\cdot}{=} c^k ba^{j-i} \cdot Z,\ Y \underset{\cdot}{=} c^k b \cdot Z.$$

Proof. Due to the cancellativity, we show $c^i b \cdot X \underset{\cdot}{=} c^j b \cdot Y \Leftrightarrow bX \underset{\cdot}{=} c^{j-i} b \cdot Y$. Thanks to the Lemma 5.1, we say that there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word Z_1 such that

$$X \underset{\cdot}{=} c^k ba \cdot Z_1,\ c^{j-i-1} b \cdot Y \underset{\cdot}{=} a^k bb \cdot Z_1.$$

Moreover, we say that there exist Y'

$$Y \underset{\cdot}{=} c^k \cdot Y',\ c^{j-i-1} b \cdot Y' \underset{\cdot}{=} bb \cdot Z_1.$$

Due to the Corollary 5.3, there exists a positive word Z_2 such that

$$bY' \underset{\cdot}{=} bb \cdot Z_2,\ Z_1 \underset{\cdot}{=} a^{j-1-1} \cdot Z_2.$$

Thus, we have

$$X \underset{\cdot}{=} c^k ba^{j-i} \cdot Z_2,\ Y \underset{\cdot}{=} c^k b \cdot Z_2.$$

□

As a corollary of the Proposition 5.4, we show the following lemma.

Lemma 5.5. *For $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m$,*

$$\mathrm{mcm}(\{\, c^{\kappa_1} b, c^{\kappa_2} b, \dots, c^{\kappa_m} b \,\}) = \{\, c^{\kappa_m} b \cdot c^k b \mid k = 0, 1, \dots \}$$

By using the Lemma 5.5, we easily show the following.

Proposition 5.6. *We have $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$.*

Proof. Due to the Proposition 5.1, we show

$$\mathrm{mcm}(\{b, c\}) = \{\, cb \cdot c^k b \mid k = 0, 1, \dots \}.$$

Due to the Lemma 5.1, for $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m$, we say

$$\mathrm{mcm}(\{\, cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \dots, cb \cdot c^{\kappa_m} b \,\}) = \{\, cb \cdot c^{\kappa_m} b \cdot c^k b \mid k = 0, 1, \dots \}.$$

By using the Lemma 5.5 repeatedly, we show $h(G^+_{\mathrm{B_{ii}}}, \deg) = \infty$. □

By using the Lemma 5.5, we calculate the skew growth function. We have to consider four cases $J_1 = \{a,b\},\{a,c\},\{b,c\},\{a,b,c\}$. The set $\mathrm{Tmcm}(G^+_{\mathrm{B_{ii}}}, J_1)$ denotes the set of all the towers starting from a fixed J_1. If $J_1 = \{a,b\},\{a,c\}$, due to the Lemma 5.1, then $\mathrm{mcm}(\{a,b\})$ and $\mathrm{mcm}(\{a,c\})$ consist of only one element, respectively. Next, we consider the case $J_1 = \{b,c\}$. For a fixed tower T, if there exists an element $\Delta \in |T|$ such that $\deg(\Delta) = l+2$, then, from the Lemma 5.5, we say the uniqueness. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{l+2} which is denoted by a_l, by counting all the signs $(-1)^{\#J_1+\cdots+\#J_n-n+1}$ in the definition (3.1) associated with the towers $T = (I_0, J_1, J_2, \cdots, J_n)$ for which $\deg(\Delta)$ can take a value $l+2$. In order to calculate the a_l, we consider the set

$$\mathcal{T}^l_{G^+_{\mathrm{B_{ii}}}} := \{\, T \in \mathrm{Tmcm}(G^+_{\mathrm{B_{ii}}}, J_1) \mid \Delta \in |T| \text{ s.t.} \deg(\Delta) = l+2 \,\}.$$

By using the Lemma 5.5 repeatedly, we show

$$\max\{\text{height of } T \in \mathcal{T}^l_{G^+_{\mathrm{B_{ii}}}}\} = [(l+1)/2].$$

For $u \in \{\, 1, \ldots, [(l+1)/2] \,\}$, we define the set

$$\mathcal{T}^l_{G^+_{\mathrm{B_{ii}}},u} := \{\, T \in \mathrm{Tmcm}(G^+_{\mathrm{B_{ii}}}, J_1) \mid \text{height of } T = u, \Delta \in |T| \text{ s.t.} \deg(\Delta) = l+2 \,\}.$$

From here, we write $\mathcal{T}^l_{G^+_{\mathrm{B_{ii}}}}$ (resp. $\mathcal{T}^l_{G^+_{\mathrm{B_{ii}}},u}$) simply by $\mathcal{T}^l$ (resp. $\mathcal{T}^l_u$). Thus, we have the decomposition:

$$\mathcal{T}^l = \bigsqcup_u \mathcal{T}^l_u. \tag{5.2}$$

Claim 1. For any u, we show the following equality

$$(-1)^{u-1}{}_{l-u}C_{u-1} = \sum_{T \in \mathcal{T}^l_u} (-1)^{\#J_1+\cdots+\#J_u-u+1}.$$

Proof. For the case of $u = 1$, the equality holds. For the case of $u = 2$, we calculate the sum $\sum_{T\in\mathcal{T}^l_2}(-1)^{\#J_2-1}$. By indices $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_m$, the set J_2 is generally written by $\{\, cb\cdot c^{\kappa_1}b, cb\cdot c^{\kappa_2}b, \ldots, cb\cdot c^{\kappa_m}b \,\}$. Due to the Lemma 5.5, we show that the maximum index κ_m can range from 1 to $l-2$. For a fixed index $\kappa_m = \kappa \in \{1, \ldots, l-2\}$, we easily show

$$\sum_{T\in\mathcal{T}^l_2, \kappa_m=\kappa} (-1)^{\#J_2-1} = -1.$$

Therefore, we show that the sum $\sum_{T\in\mathcal{T}^l_2}(-1)^{\#J_2-1} = -(l-2) = -{}_{l-2}C_{2-1}$.

We show the case for $3 \le u \le [(l+1)/2]$ by induction on u. We assume the case $u = j$. For the case of $u = j+1$, we focus our attention to the set J_2. If we write the set J_2 by $\{\, cb\cdot c^{\kappa_1}b, cb\cdot c^{\kappa_2}b, \ldots, cb\cdot c^{\kappa_m}b \,\}$, due to the the Lemma 5.5, we show that the maximum index κ_m can range from 1 to $l-2j$. By induction hypothesis, it suffices to show the following equality

$$\sum_{k=1}^{l-2j} {}_{l-j-k-1}C_{j-1} = {}_{l-j-1}C_j.$$

Therefore, we have shown the case $u = j+1$. This completes the proof. $\square$

By the decomposition (5.2), we show the following equality.

Claim 2. $a_l = \sum_{k=0}^{[(l-1)/2]}(-1)^k {}_{l-k-1}C_k$.

Then, we easily show the following.

Claim 3. $a_{l+2} - a_{l+1} + a_l = 0$.

Proof. Since an equality ${}_{n+1}C_k - {}_nC_k = {}_nC_{k-1}$ holds, we can show our statement. □

We easily show $a_1 = a_2 = 1$. Hence, the sequence $\{a_l\}_{l=1}^{\infty}$ has a period 6. Lastly, we consider the case $J_1 = \{a, b, c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{l+3} which is denoted by b_l. Since $\mathrm{mcm}(\{a, b, c\}) = \{\, cb \cdot c^k b \mid k = 1, 2, \dots \}$, we can reuse the Lemma 5.5. In a similar manner, we have the following conclusion.

Claim 4. $b_{l+2} - b_{l+1} + b_l = 0$.

Since $b_1 = b_2 = 1$, we also show that the sequence $\{b_l\}_{l=1}^{\infty}$ has a period 6. After all, we can calculate the skew growth function for the monoid $G^+_{\mathrm{B_{ii}}}$:

$$N_{G^+_{\mathrm{B_{ii}}},\deg}(t) = 1 - 3t + 2t^2 + \frac{t^3}{1-t+t^2} - \frac{t^4}{1-t+t^2} = \frac{(1-t)^4}{1-t+t^2}.$$

Secondly, we present an explicit calculation of the skew growth function for the monoid G^+_n.

Example. 2. First of all, we show the following proposition.

Proposition 5.7. *If, for $0 \le i < j$, an equation $c^i b^{n-1} \cdot X \doteq c^j b^{n-1} \cdot Y$ in G^+_n holds, then there exists a positive word Z such that*

$$X \doteq ba^{j-i} \cdot Z,\ Y \doteq bZ.$$

Proof. Since we have shown the cancellativity in §4, we show $c^i b^{n-1} \cdot X \doteq c^j b^{n-1} \cdot Y \Leftrightarrow b^{n-1} \cdot X \doteq c^{j-i} b^{n-1} \cdot Y$. Thanks to the Proposition 4.3 (iv - $(n-2)$ - b), we say that there exists a positive word Z such that

$$X \doteq ba^{j-i} \cdot Z,\ Y \doteq bZ.$$

□

As a corollary of the Proposition 5.7, we show the following lemma.

Lemma 5.8. *For $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_m$,*

$$\mathrm{mcm}(\{\, c^{\kappa_1} b^{n-1}, c^{\kappa_2} b^{n-1}, \dots, c^{\kappa_m} b^{n-1} \,\}) = \{c^{\kappa_m} b^n\}$$

Thus, we obtain the following proposition.

Proposition 5.9. *We have $h(G^+_n, \deg) = 2$.*

By using the Lemma 5.8, we calculate the skew growth function. We have to consider four cases $J_1 = \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. The set $\mathrm{Tmcm}(G^+_n, J_1)$ denotes the set of all the towers starting from a fixed J_1. If $J_1 = \{a, b\}, \{a, c\}$, due to the Proposition 4.3, then $\mathrm{mcm}(\{a, b\})$ and $\mathrm{mcm}(\{a, c\})$ consist of only one element, respectively. Next, we consider the case $J_1 = \{b, c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{n+l} which is denoted by c_l. In order to calculate the c_l, we consider the set

$$\mathcal{T}^l_{G^+_n} := \{\, T \in \mathrm{Tmcm}(G^+_n, J_1) \mid \Delta \in |T| \text{ s.t.} \deg(\Delta) = n + l \,\}.$$

For $u \in \{1,2\}$, we define the set

$$\mathcal{T}^{l}_{G_n^+,u} := \{\, T \in \mathrm{Tmcm}(G_n^+, J_1) \mid \text{height of } T = u, \Delta \in |T| \text{ s.t.} \deg(\Delta) = n + l \,\}.$$

Since $\mathrm{mcm}(\{b,c\}) = \{\, cb \cdot c^k b^{n-1} \mid k = 0,1,\ldots \}$, we easily show $c_1 = c_2 = 1$. Moreover, we show the following.

Proposition 5.10. *We have $c_l = 0$ $(l = 3,4,\ldots)$.*

Proof. From the consideration in Claim 1 of Example 1, for $u = 2$, we also show

$$\sum_{T \in \mathcal{T}^{l}_{G_n^+,u}} (-1)^{\#J_1 + \cdots + \#J_u - u + 1} = -1.$$

Thus, we have $c_l = 0$ $(l = 3,4,\ldots)$. □

Lastly, we consider the case $J_1 = \{a,b,c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{n+l+1} which is denoted by d_l. In a similar way, we show $d_1 = d_2 = 1$ and $d_l = 0$ $(l = 3,4,\ldots)$. After all, we calculate the skew growth function for the monoid G_n^+:

$$N_{G_n^+,\deg}(t) = 1 - 3t + 2t^2 + (t^{n+1} + t^{n+2}) - (t^{n+2} + t^{n+3}) = (1-t)(t^{n+2} + t^{n+1} - 2t + 1).$$

Remark 6. *By the inversion formula, we can calculate the growth function $P_{G_n^+,\deg}(t)$. As far as we know, it is difficult to calculate $P_{G_n^+,\deg}(t)$ directly.*

Thirdly, we present an explicit calculation of the skew growth function for the monoid H_n^+.

Example. 3. First of all, we show the following proposition.

Proposition 5.11. *If, for $0 \le i < j$, an equation $c^i b(ab)^{n-1} ba \cdot X \doteq c^j b(ab)^{n-1} ba \cdot Y$ in H_n^+ holds, then there exists a positive word Z such that*

$$X \doteq cba^{j-i} \cdot Z,\ Y \doteq cb \cdot Z.$$

Proof. Since we have shown the cancellativity in §4, we show $c^i b(ab)^{n-1} ba \cdot X \doteq c^j b(ab)^{n-1} ba \cdot Y \Leftrightarrow b(ab)^{n-1} ba \cdot X \doteq c^{j-i} b(ab)^{n-1} ba \cdot Y$. Thanks to the Proposition 4.6 (vi - h), we say that there exists a positive word Z_1 such that

$$(ab)^{n-1} ba \cdot X \doteq (ab)^n ba^{j-i} \cdot Z_1,\ c(ab)^{n-2} ba \cdot Y \doteq c(ab)^{n-1} b \cdot Z_1.$$

Therefore, we say that there exists a positive word Z_2 such that

$$X \doteq cba^{j-i} \cdot Z_2,\ Y \doteq cb \cdot Z_2.$$

□

As a corollary of the Proposition 5.11, we show the following lemma.

Lemma 5.12. *For $0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_m$,*

$$\mathrm{mcm}(\{\, c^{\kappa_1} b(ab)^{n-1} ba, c^{\kappa_2} b(ab)^{n-1} ba, \ldots, c^{\kappa_m} b(ab)^{n-1} ba \,\}) = \{ c^{\kappa_m} b(ab)^{n-1} bacb \}$$

Thus, we obtain the following proposition.

Proposition 5.13. *We have $h(H_n^+, \deg) = 2$.*

Thanks to the Lemma 5.12, we can calculate the skew growth function. We have to consider four cases $J_1 = \{a,b\},\{a,c\},\{b,c\},\{a,b,c\}$. The set $\mathrm{Tmcm}(H_n^+, J_1)$ denotes the set of all the towers starting from a fixed J_1. If $J_1 = \{a,b\},\{a,c\}$, due to the Proposition 4.6, then $\mathrm{mcm}(\{a,b\})$ and $\mathrm{mcm}(\{a,c\})$ consist of only one element, respectively. Next, we consider the case $J_1 = \{b,c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{2n+3+l} which is denoted by e_l. In order to calculate the e_l, we consider the set

$$\mathcal{T}^l_{H_n^+} := \{\, T \in \mathrm{Tmcm}(H_n^+, J_1) \mid \Delta \in |T| \text{ s.t.} \deg(\Delta) = 2n+3+l \,\}.$$

For $u \in \{1,2\}$, we define the set

$$\mathcal{T}^l_{H_n^+,u} := \{\, T \in \mathrm{Tmcm}(H_n^+, J_1) \mid \text{height of } T = u, \Delta \in |T| \text{ s.t.} \deg(\Delta) = 2n+3+l \,\}.$$

Since $\mathrm{mcm}(\{b,c\}) = \{\, bc^k(ab)^n ba \mid k = 0,1,\ldots \,\}$, we easily show $e_1 = e_2 = e_3 = 1$. Moreover, we show the following.

Proposition 5.14. *We have* $e_l = 0$ $(l = 4,5,\ldots)$.

Proof. From the consideration in Claim 1 of Example 1, for $u = 2$, we also show

$$\sum_{T \in \mathcal{T}^l_{H_n^+,u}} (-1)^{\#J_1 + \cdots + \#J_u - u + 1} = -1.$$

Thus, we have $e_l = 0$ $(l = 4,5,\ldots)$. □

Lastly, we consider the case $J_1 = \{a,b,c\}$. For any fixed $l \in \mathbb{Z}_{>0}$, we calculate the coefficient of the term t^{2n+4+l} which is denoted by f_l. In a similar way, we show $f_1 = f_2 = f_3 = 1$ and $f_l = 0$ $(l = 4,5,\ldots)$. After all, we calculate the skew growth function for the monoid H_n^+:

$$\begin{aligned} N_{H_n^+,\deg}(t) &= 1 - 3t + 2t^2 + (t^{2n+3} + t^{2n+4} + t^{2n+5}) - (t^{2n+4} + t^{2n+5} + t^{2n+6}) \\ &= (1-t)(t^{2n+5} + t^{2n+4} + t^{2n+3} - 2t + 1). \end{aligned}$$

Remark 7. *By the inversion formula, we can calculate the growth function* $P_{H_n^+,\deg}(t)$. *As far as we know, it is difficult to calculate* $P_{H_n^+,\deg}(t)$ *directly.*

Lastly, we calculate the skew growth function for the monoid $M_{\mathrm{abel},m}$.

Example. 4. First of all, we easily show the following proposition.

Proposition 5.15. *Let* X *and* Y *be positive words in* $M_{\mathrm{abel},m}$ *of length* $r \in \mathbb{Z}_{\geq 0}$.
(i) *If* $vX \doteq vY$ *for some* $v \in \{a,b\}$, *then* $X \doteq Y$.
(ii) *If* $aX \doteq bY$, *then either* $X \doteq a^{m-1} \cdot Z_1$ *and* $Y \doteq b^{m-1} \cdot Z_1$ *for some positive word* Z_1 *or* $X \doteq bZ_2$ *and* $Y \doteq aZ_2$ *for some positive word* Z_2.

Lemma 5.16. *There exists a unique tower* $T_n = (I_0, J_1, J_2, \cdots, J_n)$ *of height* $n \in \mathbb{Z}_{>0}$ *with the ground set* $I_0 = \{a,b\}$ *such that*

$$\begin{aligned} J_{2k-1} &= \{a^{(k-1)m+1}, a^{(k-1)m}b\} \ (k = 1,\ldots,[(n+1)/2]), \\ J_{2k} &= \{a^{km}, a^{(k-1)m+1}b\} \ (k = 1,\ldots,[n/2]). \end{aligned}$$

Proof. We easily show $J_1 = \{a,b\}$ and $J_2 = \{a^m, ab\}$. Thanks to the Proposition 5.15, we show our statement by induction on k. □

Therefore, we immediately show $h(M_{\mathrm{abel},m},\deg)=\infty$. And, from the definition (3.1), we can calculate the skew growth function

$$N_{M_{\mathrm{abel},m},\deg}(t)=(1-2t+t^2)(1+t^m+t^{2m}+\cdots)=\frac{(1-t)^2}{1-t^m}.$$

6. Appendix

In this section, we present three examples that suggest the ralationship between the form of the spherical growth function for a monoid $\langle L \mid R\rangle_{mo}$ and properties of the corresponding group $\langle L \mid R\rangle$. For the three examples, by observing the distribution of the zeroes of the denominator polynomials of the growth functions for them, we conjecture that the corresponding groups contain free abelian subgroups of finite index.

Example. 1. We recall an example, the monoid $G^+_{\mathrm{B_{ii}}}$, from [I1]. By using the normal form of the monoid $G^+_{\mathrm{B_{ii}}}$, the author has calculated the spherical growth function $P_{G^+_{\mathrm{B_{ii}}},\deg}(t)$ for the monoid. The spherical growth function $P_{G^+_{\mathrm{B_{ii}}},\deg}(t)$ can be expressed as a rational function $\frac{1-t+t^2}{(1-t)^4}$. Since the zeroes of the denominator polynomial of $P_{G^+_{\mathrm{B_{ii}}},\deg}(t)$ only consists of 1 with multiplicity 4, it is conjectured that the corresponding group $G_{\mathrm{B_{ii}}}$ contains a free abelian subgroup of rank 4 of finite index. Indeed, the author has shown the following.

Proposition 6.1. *The followig* (i),(ii) *and* (iii) *hold.*

(i) *The group $G_{\mathrm{B_{ii}}}$ contains a subgroup of index three isomorphic to $\mathbb{Z}^4$.*

(ii) *The group $G_{\mathrm{B_{ii}}}$ has a polynomial growth rate.*

(iii) *The group $G_{\mathrm{B_{ii}}}$ is a solvable group.*

Example. 2. We consider the following example

$$\left\langle a,b,c \;\middle|\; \begin{array}{c} cb=ba, \\ ab=bc, \\ ac=ca \end{array} \right\rangle_{mo}.$$

We easily show the cancellativity of it by refering to the double induction (see [G]). The spherical growth function can be expressed as a rational function $\frac{1}{(1-t)^3}$. It is also conjectured that the corresponding group contains a free abelian subgroup of rank 3 of finite index. Indeed, we can show that corresponding group contains a subgroup of index two isomorphic to $\mathbb{Z}^3$. Moreover, we show that the group has a polynomial growth rate and is a solvable group.

Example. 3. We consider the following example

$$\left\langle a,b,c,d \;\middle|\; \begin{array}{c} ab=bc, ac=ca, \\ cb=ba, bd=db, \\ ad=dc, cd=da \end{array} \right\rangle_{mo}.$$

We easily show the cancellativity of it by refering to the double induction (see [G]). The spherical growth function can be expressed as a rational function $\frac{1}{(1-t)^4}$. It is also conjectured that the corresponding group contains a free abelian subgroup of rank 4 of finite index. Indeed, we can show that corresponding group contains a subgroup of index three isomorphic to $\mathbb{Z}^4$. Moreover, we show that the group has a polynomial growth rate and is a solvable group.

Acknowledgement. The author is grateful to Kyoji Saito for very interesting discussions and encouragement. This research is supported by JSPS Fellowships for Young Scientists ($23 \cdot 10023$). This researsh is also supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

References

[A-N] M. Albenque and N. Philippe: Growth function for a class of monoids, FPSAC 2009, DMTCS proc. 25-38.

[B] M. Brazila: Growth functions for some one-relation monoids, communications in Algebra vol.**21**, (1993) 3135-3146.

[Bro] A. Bronfman: Growth functions of a class of monoids, preprint, 2001.

[B-S] E. Brieskorn and K. Saito: Artin-Gruppen und Coxeter-Gruppen, *Inventiones Math.* **17** (1972) 245-271, English translation by C. Coleman, R. Corran, J. Crisp, D. Easdown, R. Howlett, D. Jackson and A. Ram at the University of Sydney, 1996.

[Deh1] P. Dehornoy: Complete positive group presentations, J. Algebra **268** (2003) 156-197.

[Deh2] P. Dehornoy: The subword reversing method, Intern. J. Alg. and Comput. **21** (2011) 71-118.

[De] P. Deligne: Les immeubles des tresses généralizé, *Inventiones Math.* **17** (1972) 273-302.

[G] F.A. Garside: The braid groups and other groups, Quart. J. Math. Oxford, **20** (1969), 235-254.

[I1] T. Ishibe: On the monoid in the fundamental group of type $\mathrm{B_{ii}}$, Hiroshima Mathematical Journal Vol.42 No.1(March) 2012.

[I2] T. Ishibe: Infinite examples of cancellative monoids that do not always have least common multiple, submitted.

[I3] T. Ishibe: Infinite examples of cancellative monoids that do not always have least common multiple II, in preparation.

[K-T-Y] K. Kobayashi, S. Tsuchioka and S. Yasuda: Partial theta and growth series of Artin monoids of finite type, in preparation.

[S1] K. Saito: Inversion formula for the growth function of a cancellative monoid, submitted.

[S2] K. Saito: Growth functions associated with Artin monoids of finite type, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), no.10, 179-183.

[S3] K. Saito: Growth functions for Artin monoids, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no.7, 84-88.

[S4] K. Saito: Limit elements in the Configuration Algebra for a Cancellative Monoid, Publ. RIMS Kyoto Univ. **46** (2010), 37-113.

[S5] K. Saito: Growth partition functions for cancellative infinite monoids, preprint RIMS-1705 (2010).

[S-I] K. Saito and T. Ishibe: Monoids in the fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$, Journal of Algebra 344 (2011), 137-160.

[Xu] P. Xu: Growth of the positive braid semigroups, Journal of Pure and Applied Algebra **80** (1992), no.2, 197-215.

Kavli IPMU,
University of Tokyo,
Kashiwa, Chiba 277-8583 Japan
e-mail address : tishibe@ms.u-tokyo.ac.jp