Coherence of relatively quasi-free algebras

Alexey Bondal, Ilya Zhdanovskiy

1 Introduction

An important problem in representation theory of associative algebras is to prove coherence for various noncommutative algebras. Left coherence implies that the category of finitely presented left modules is abelian. This category might then be considered as being analogous to the category of coherent sheaves on an affine commutative variety. Thus, coherence is the initial point for developing noncommutative geometry in representation theory of algebras in the style similar to the theory of coherent sheaves.

Despite its crucial importance the problem of coherence is basically *terra incognita*: we don't know so far about many interesting classes of noncommutative algebras with nice homological properties whether they are coherent or not.

There is a class of algebras closed to free associative algebras for which coherence is known. These are so-called quasi-free algebras over fields in the sense of Kuntz and Quillen [CQ1]. We reproduce the proof in the main body of the text.

In this paper, we are interested in algebras that are relatively quasi-free over a commutative algebra, say $\mathbb{K}$. By definition, such an algebra A is central over $\mathbb{K}$ and A-bimodule of noncommutative 1-forms $\Omega^1_{A/\mathbb{K}}$ is projective. Similar to the case of algebras that are quasi-free over a field one can show that relatively quasi-free algebras are exactly those which satisfy the lifting property for nilpotent $\mathbb{K}$-central extensions (proposition 1).

We prove a theorem that an algebra which is quasi-free relatively over a commutative *noetherian* ring $\mathbb{K}$ is left and right coherent.

This result requires a different and more involved techniques in comparison to the case when the ring $\mathbb{K}$ is a field. Our proof is based on the Chase criterion for coherence, which states that a ring is left coherent if and only if the product of any number of flat *right* modules is flat. In fact the criterion is enough to check for the product of sufficiently many copies of the rank one free modules over the algebra. We use also a criterion of similar style for noetherian algebras.

It would be interesting to check whether the method of this paper is applicable to proving coherence for other important classes of algebras. In [BZ], we explore an algebra B constructed via algebra A and an element Δ in it. The algebra is constructed by modifying multiplication in the vector space A via the rule:

$$a \cdot_B b = a\Delta b$$

and adjoining a unit to the resulting algebra. If element Δ verifies some conditions, in which case it is called well-tempered in [BZ], we prove that if the original algebra was coherent then so is the new one, thus providing an iterative way of constructing coherent algebras.

We are grateful to Dmitry Piontkovsky for useful references. This work was done during authors visit to Kavli IPMU and was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

2 Coherence of algebras and categories of finitely presented modules

A left module M of an algebra is said to be *coherent* if it is finitely generated and for every morphism $\varphi : P \to M$ with free module P of finite rank the kernel of φ is finitely generated. An algebra is *(left) coherent* if it is coherent as a left module over itself. If algebra is coherent, then finitely presented modules are the same as coherent modules and the category of finitely presented modules is abelian [Bou].

Let A be an algebra over a commutative ring $\mathbb{K}$. The definition due to Cuntz and Quillen [CQ1] of a quasi-free algebra, which we adopt to the relative case (i.e. for algebras over a commutative ring $\mathbb{K}$ rather than over a field), is that algebra A over a field k is *quasi-free* if the bimodule of noncommutative differential 1-forms $\Omega^1 A$ (see definition below) is a projective $A \otimes_k A^{opp}$-module.

Every quasi-free algebra is *hereditary*, i.e. has global dimension ≤ 1. Indeed, for any two left A-modules M and N we have:

$$\mathrm{Ext}^i_A(M, N) = \mathrm{Ext}^i_{A-A}(A, \mathrm{Hom}_k(M, N)).$$

Since A has a projective bimodule resolution of length 1, it follows that $\mathrm{Ext}^i_A(M, N) = 0$, for $i \geq 2$.

Hereditary algebras are coherent. Indeed, a submodule of a projective module is projective for such algebras. Thus, given a morphism $\varphi : P_1 \to P_2$ between finitely generated projective modules, the image I of φ is a submodule of a projective module, hence projective too. Then short exact sequence induced by φ:

$$0 \to K \to P_1 \to I \to 0,$$

where K is the kernel of φ, splits. Hence, we have an epimorphism $P_1 \to K$, which proves that K is finitely generated.

This proves that every quasi-free algebra is left (and, similarly, right) coherent. Note that the number of generators for the kernel K as a left A-module is bounded by the number of generators of P_1, which is not true in general even for coherent commutative algebras. For that reason this proof of coherence is not applicable to the case of relatively quasi-free algebras that we consider below. We will need a substantially different argument.

Let us develop the theory for the case of relatively quasi-free algebras. A $\mathbb{K}$-algebra A is said to be *central* over $\mathbb{K}$ if the image of $\mathbb{K} \to A$ is in the center of A. An A-*bimodule* is said to be $\mathbb{K}$-*central* if left and right $\mathbb{K}$ actions coincide. Such A-bimodules are identified with left $A \otimes_{\mathbb{K}} A^{opp}$-modules.

Define the $A \otimes_{\mathbb{K}} A^{opp}$-module of relative forms by an exact sequence

$$0 \to \Omega^1_{A/\mathbb{K}} \to A \otimes_{\mathbb{K}} A \to A \to 0 \tag{1}$$

Definition Let $\mathbb{K}$ be a commutative ring. An algebra A central over $\mathbb{K}$ is said to be quasi-free over $\mathbb{K}$ if $\Omega^1_{A/\mathbb{K}}$ is a projective $A \otimes_{\mathbb{K}} A^{opp}$-module.

Let A be a relatively quasifree algebra over $\mathbb{K}$. By applying functor $\mathrm{Ext}^i(-, M)$ to the exact sequence (1) for an arbitrary $\mathbb{K}$-central A-bimodule M, we obtain:

$$\mathrm{Ext}^2_{A\otimes_{\mathbb{K}} A^{opp}}(A, M) = \mathrm{Ext}^1_{A\otimes_{\mathbb{K}} A^{opp}}(\Omega^1_{A/\mathbb{K}}, M) = 0. \tag{2}$$

A generalization of Cuntz-Quillen criterion for quasi-free algebras [CQ1] holds in the relative case.

Proposition 1. *Algebra A is quasi-free over $\mathbb{K}$ if and only if for any $\tilde{R}$, a nilpotent extension of the algebra R by square zero ideal in the category of algebras central over $\mathbb{K}$, and any homomorphism $A \to R$ there exists its lifting to a homomorphism $A \to \tilde{R}$.*

Proof. Let A be a quasi-free algebra. Consider a square-zero extension $\tilde{R} \to R$ and a homomorphism $A \to R$. Denote by I the kernel of $\tilde{R} \to R$. By assumptions, it is a square zero ideal in $\tilde{R}$, hence it has a natural structure of $\mathbb{K}$-central R-bimodule. Let $\tilde{A}$ be the fibred product over R of A and $\tilde{R}$. This is an algebra central over $\mathbb{K}$ too. It is a square zero extension of A by I, where I is endowed with a $\mathbb{K}$-central A-bimodule structure which is the pull-back of R-bimodule structure. Such extensions are classified by $\mathrm{Ext}^2_{A\otimes_{\mathbb{K}} A^{opp}}(A, I)$. This group is trivial for quasi-free algebras by (2). Hence, we have a splitting homomorphism $A \to \tilde{A}$. When combined with the map $\tilde{A} \to \tilde{R}$ it gives the required lifting.

Conversely, if all square zero extensions allow liftings, then, by taking $R = A$ and I arbitrary $\mathbb{K}$-central A-bimodule, we see that $\mathrm{Ext}^2_{A\otimes_{\mathbb{K}} A^{opp}}(A, I) = \mathrm{Ext}^1_{A\otimes_{\mathbb{K}} A^{opp}}(\Omega^1_{A/\mathbb{K}}, I) = 0$, i.e. $\Omega^1_{A/\mathbb{K}}$ is a projective $A \otimes_{\mathbb{K}} A^{opp}$-bimodule.

□

Recall the following criterion of coherence due to Chase [Chase].

Lemma 2. *For any ring A the following are equivalent:*

- *A is left coherent,*
- *For any family of* right *flat modules F_i, $i \in I$, the product $\prod_{i\in I} F_i$ is right flat,*
- *For the family $F_i \cong A$ of free modules with* card $I =$ card A, *the product $\prod_{i\in I} F_i$ is right flat.*

We will use also a criterion of the same style for noetherianess (cf. [Ab]).

Lemma 3. *For any ring A the following are equivalent:*

- *A is left noetherian,*

- *For any left A-module M and any family of flat* right *modules F_i, $i \in I$, the morphism $(\prod_{i\in I} F_i) \otimes_A M \to \prod_{i\in I}(F_i \otimes_A M)$ is mono,*
- *For any left A-module M and the family of rank 1 free* right *modules $F_i \cong A$, $i \in I$,* $\operatorname{card} I = \operatorname{card} A$*, the morphism $(\prod_{i\in I} F_i) \otimes_A M \to \prod_{i\in I}(F_i \otimes_A M)$ is mono.*

Theorem 4. *Let $\mathbb{K}$ be a commutative noetherian ring. Assume that A is an algebra quasi-free relatively over $\mathbb{K}$ and A is flat as a $\mathbb{K}$-module. Then A is a left and right coherent algebra.*

Proof. Consider left A-module M and a family of rank 1 free right A-modules $F_i \cong A$. We shall consider all left (respectively, right) A-modules to be always endowed with the right (respectively, left) $\mathbb{K}$-module structure identical to the left (respectively, right) $\mathbb{K}$-module structure.

By applying functors $(-) \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i)$ and $\prod_i((-) \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i))$ and their derived functors to the exact sequence (1), we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{Tor}_1^{A\otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} \prod_i F_i) & \longrightarrow & \Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_i \operatorname{Tor}_1^{A\otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} F_i) & \longrightarrow & \prod_i(\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) \end{array} \tag{3}$$

For any left A-module M and right A-module N, we have an isomorphism of objects in the derived categories:

$$A \otimes^{\mathbb{L}}_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes^{\mathbb{L}}_{\mathbb{K}} N) = M \otimes^{\mathbb{L}}_A N,$$

which implies a spectral sequence:

$$\operatorname{Tor}_i^{A\otimes_{\mathbb{K}} A^{opp}}(A, \operatorname{Tor}_j^{\mathbb{K}}(M, N)) \Longrightarrow \operatorname{Tor}^A_{i+j}(M, N)$$

For a flat $\mathbb{K}$-module N, it implies that

$$\operatorname{Tor}_1^{A\otimes_{\mathbb{K}} A^{opp}}(A, M \otimes_{\mathbb{K}} N) = \operatorname{Tor}_1^A(M, N). \tag{4}$$

Since F_i are rank 1 free as A-modules and A is a flat $\mathbb{K}$-module, F_i are also flat $\mathbb{K}$-modules. Hence $\operatorname{Tor}_1^A(M, F_i) = 0$. Since $\mathbb{K}$ is noetherian, it is coherent, hence by Chase criterion, lemma 2, $\prod_i F_i$ is also flat. In view of (4), diagram (3) reads:

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{Tor}_1^A(M, \prod_i F_i) & \longrightarrow & \Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \prod_i(\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) \end{array} \tag{5}$$

Let us show that $\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \to \prod_i(\Omega^1_{A/\mathbb{K}} \otimes_{A\otimes A^{opp}} (M \otimes_{\mathbb{K}} F_i))$ is an embedding. By criterion of noetherianess, lemma 3, for $\mathbb{K}$, we have that $M \otimes_{\mathbb{K}} \prod_i F_i \to \prod_i(M \otimes_{\mathbb{K}} F_i)$ is an embedding.

Let $D = \oplus_j (A \otimes_{\mathbb{K}} A^{opp})$ be a free $A \otimes_{\mathbb{K}} A^{opp}$-module, then

$$D \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) = \oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i)$$

and

$$\prod_i (D \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) = \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i)).$$

The morphism

$$D \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \to \prod_i (D \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i))$$

is the composite of two morphism:

$$\oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i) \to \oplus_j \prod_i (M \otimes_{\mathbb{K}} F_i) \to \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i)).$$

Both morphism are readily embeddings, hence so is the composite.

Since $\Omega^1_{A/\mathbb{K}}$ is a projective bimodule, we have an imbedding

$$\Omega^1_{A/\mathbb{K}} \to \oplus_j A \otimes_{\mathbb{K}} A^{opp}$$

as a direct summand. This implies a diagram:

$$\begin{tikzcd}
\Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} \prod_i F_i) \arrow[r] \arrow[d] & \oplus_j (M \otimes_{\mathbb{K}} \prod_i F_i) \arrow[d] \\
\prod_i (\Omega^1_{A/\mathbb{K}} \otimes_{A \otimes_{\mathbb{K}} A^{opp}} (M \otimes_{\mathbb{K}} F_i)) \arrow[r] & \prod_i (\oplus_j (M \otimes_{\mathbb{K}} F_i))
\end{tikzcd} \tag{6}$$

The upper horizontal arrow is an embedding because it obtained by tensoring up an embedding of a direct summand with a module $M \otimes_{\mathbb{K}} \prod_i F_i$. We have shown above that the right vertical arrow is an embedding too. Therefore, the left vertical arrow is an embedding. Then, diagram (5) implies that $\mathrm{Tor}^A_1(M, \prod_i F_i) = 0$, i.e $\prod_i F_i$ is a right flat module. By Chase criterion, lemma 2, algebra A is left coherent.

Right coherence follows similarly.

$\square$

References

[Ab] H. Aberg, *Coherence of Amalgamations*, Journal of Algebra **78** (1982), 372–385.

[Bou] N. Bourbaki, *Algebre Homologique*, Masson, Paris, New-York, Barcelone, Milan, 1980.

[BZ] A. Bondal, I. Zhdanovskiy, *Representation theory for systems of projectors and discrete Laplace operators*, IPMU13-0001, IPMU, Kashiwa, Japan, 2013, 48 ., http://db.ipmu.jp/ipmu/sysimg/ipmu/1012.pdf

[Chase] S. Chase, *Direct Products of Modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.

[CQ1] J. Cuntz, and D. Quillen, *Algebra extensions and nonsingularity*, J. of Amer. Math. Soc. **8** (1995), no. 2, 251–289.