GROMOV–WITTEN THEORY OF FANO ORBIFOLD CURVES AND ADE-TODA HIERARCHIES

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Abstract. We construct an integrable hierarchy in the form of Hirota quadratic equations (HQE) that governs the Gromov–Witten invariants of the Fano orbifold projective curve $\mathbb{P}^1_{a_1,a_2,a_3}$ with positive orbifold Euler characteristic. We also identify our HQEs with an appropriate Kac–Wakimoto hierarchy.

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1. Introduction

Since the seminal work of Witten [62], it has been expected that Gromov-Witten (GW) invariants of a target space X should be governed by an integrable hierarchy. Witten's conjecture [62], proven by Kontsevich [45] states that the GW theory of $X = \mathrm{pt}$ is governed by the KdV hierarchy. The Toda conjecture, proven by [27, 52, 49, 50] states that the GW theory of $X = \mathbb{P}^1$ is governed by the extended Toda hierarchy [11]. It is known [51, 40, 12] that the GW theory of $\mathbb{P}^1$-orbifolds with two orbifold points is governed by the extended bigraded Toda hierarchy [10]. It has been conjectured [6] that the GW theory of the resolved conifold is governed by the Ablowitz-Ladik hierarchy [1]. The relationship between topological field theories and integrable hierarchies is studied in other examples, such as [28, 31, 21, 22, 23, 47].

The problem of identifying an integrable hierarchy governing the GW theory of a target space X is a very interesting and very difficult problem. In this paper, we will investigate this problem for Fano orbifold curves, with the help of Givental's higher genus reconstruction in Gromov-Witten theory.

One of the main recent advances in GW theory is the higher-genus reconstruction of the GW invariants of a complex projective manifold X with a semi-simple quantum cohomology. The reconstruction was discovered and proved by Givental in the equivariant settings when X is equipped with a torus action with isolated fixed points [29]. Based on his work [29], Givental conjectured a certain higher genus reconstruction formula for the total ancestor potential of X with semi-simple quantum cohomology. This formula is formulated in terms of a very convenient quantization formalism invented by Givental (see [30]). This formula was proved in various cases in [30, 39, 36, 5], and in full generality by C. Teleman [59]. This formula has many important applications in other areas of mathematics such as integrable systems, representation theory of vertex algebras, and modular forms.

Givental's reconstruction inspires an approach to studying the relation between Gromov-Witten theory and integrable systems. In this approach one aims at constructing an integrable hierarchy in the form of *Hirota quadratic equations* (HQE)[1] and show that the generating function of Gromov-Witten invariants is a tau-function of the hierarchy (i.e. it satisfies the HQEs). This approach has been successfully worked out for Gromov-Witten theory of X when $X = \mathbb{P}^1$ [49, 50] and $X = \mathbb{P}^1_{a,b}$ [51]. See also [28, 31, 24] for instances of this approach in the setting of singularity theory. A more precise description of our results is now in order.

1.1. **Gromov-Witten theory of Fano orbifold curves.** In this paper we solve the problem of constructing HQEs for Gromov-Witten theory of Fano orbifold projective lines with three orbifold points. Let

$$\mathbf{a} = \{a_1, a_2, a_3\}$$

be a triple of positive integers such that $a_1 \leq a_2 \leq a_3$. Let

$$\mathbb{P}^1_{\mathbf{a}}$$

be the orbifold projective line obtained from $\mathbb{P}^1$ by adding[2] $\mathbb{Z}_{a_1}$-, $\mathbb{Z}_{a_2}$-, and $\mathbb{Z}_{a_3}$-orbifold points. The nature of the problem of constructing HQEs depends on the orbifold Euler characteristic of $\mathbb{P}^1_{\mathbf{a}}$:

$$\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1.$$

In this paper we will study the Fano case $\chi > 0$, leaving the other two cases $\chi = 0$ and $\chi < 0$ for a future investigation.

For an Fano orbifold curve $\mathbb{P}^1_{\mathbf{a}}$, we will consider its Chen-Ruan orbifold cohomology (see the explicit definition in Section 2)

$$H := H^*_{\mathrm{orb}}(\mathbb{P}^1_{\mathbf{a}}, \mathbb{C}).$$

It is a graded vector space with a fixed basis $\{\phi_i\}_{i\in\mathfrak{I}}$, where $\mathfrak{I}$ is an index set defined in (4). In orbifold Gromov-Witten theory, one consider the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d)$ of orbifold stable maps f from a domain orbifold curve Σ with genus g and n marked points, to the target orbifold $\mathbb{P}^1_{\mathbf{a}}$, such that the homology class of the image of f is d times the fundamental class of the underlyting curve of $\mathbb{P}^1_{\mathbf{a}}$. See Secton 2.2 for the details. The descendant Gromov-Witten invariants (see (7)) are intersection numbers on the moduli space of stable maps, denoted by

$$\langle \phi_{i_1}\, \psi_1^{k_1}, \ldots, \phi_{i_n} \psi_n^{k_n} \rangle_{g,n,d},$$

where ψ_j are ψ-classes on the moduli space of stable maps.

The object of our main interest is the so-called *total descendant potential*, defined by the following generating series of Gromov–Witten invariants:

$$\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t}) = \exp\Big(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n) \rangle_{g,n,d} \Big), \quad (1)$$

[1]The word "quadratic" in HQE was used by Givental in [28]. The equations are also known as "Hirota bilinear equations."

[2]For example, by root constructions [2], [9].

where Q is a non-zero complex number, $\mathbf{t}(z) := t_0+t_1z+t_2z^2+\cdots$, and $\hbar$, $t_0, t_1, \ldots \in H$ are formal variables. Using the so called *dilaton shift* $q_k = t_k - \delta_{k,1}\mathbf{1}$ we identify $\mathcal{D}_{\mathbf{a}}$ with a vector in the *Fock* space

$$\mathbb{C}_\hbar[\![\mathbf{q}]\!] := \mathbb{C}_\hbar[\![q_0, q_1+\mathbf{1}, q_2, \cdots]\!], \quad \text{where} \quad \mathbb{C}_\hbar = \mathbb{C}(\!(\hbar)\!).$$

The construction of HQEs for the Gromov-Witten theory of $\mathbb{P}^1_{\mathbf{a}}$ in this paper uses the theory of vanishing cycles and period integrals associated to a Landau-Ginzburg mirror of $\mathbb{P}^1_{\mathbf{a}}$. This mirror model of $\mathbb{P}^1_{\mathbf{a}}$ was constructed in [51] in the case $a_1 = 1$ and in general by P. Rossi [53], who managed to compute the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$. For our purposes we need to know how to solve the quantum differential equations in terms of period integrals. This was achieved recently by Ishibashi–Shiraishi–Takahashi [38]. With such a mirror model at hand we can apply the idea of Givental [28], which was further developed in [24, 31, 49, 51].

1.2. **The Kac–Wakimoto hierarchy.** The triplets $\mathbf{a} = \{a_1, a_2, a_3\}$ with $\chi > 0$ are classified by the Dynkin diagrams of type ADE together with a choice of a *branching node*. In the D and E cases there is a unique choice of a branching node, while in the A-case any node can be chosen. By removing the branching node we obtain 3 diagrams of type[3] $A_{a_\mu-1}$, $\mu = 1, 2, 3$. Let us denote by $\mathfrak{h}^{(0)}$ the Cartan subalgebra of the corresponding simple Lie algebra $\mathfrak{g}^{(0)}$ and define (cf. formula (34))

$$\sigma_b = \prod_{\mu=1}^{3} \Big(\cdots s^{(-1)}_{\mu,2} s^{(-1)}_{\mu,1} \Big), \tag{2}$$

where $s^{(-1)}_{\mu,i} : \mathfrak{h}^{(0)} \to \mathfrak{h}^{(0)}$ is the reflection through the hyperplanes orthogonal to $\gamma_{\mu,i}$, which is the i-th simple root on the μ-th component of the Dynkin diagram. The automorphism σ_b can be extended to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Let us denote by κ the order of σ_b. Fix a σ_b-eigenbasis $\{H_i\}_{i\in\mathfrak{I}}$ of $\mathfrak{h}^{(0)}$ satisfying $(H_i|H_{j^*}) = \kappa\,\delta_{ij}$. It turns out that the spectrum of σ_b is given by the degrees of the cohomology classes ϕ_i. More precisely, we can arrange that $\sigma_b(H_i) = e^{-2\pi\sqrt{-1}d_i}H_i$, where $d_i = 1 - \deg(\phi_i)/2 = 1 - i''/a_i$. Put

$$m_{01} := 0, \quad m_{02} := \kappa, \quad m_i := d_{i^*}\,\kappa, \quad i \in \mathfrak{I}_{\rm tw}.$$

The Kac-Wakimoto hierarchy corresponding to the conjugacy class of σ_b in the Weyl group can be described as follows. Let $\mathbb{C}[y]$ be the algebra of polynomials on $y = (y_{i,l})$, $i \in \mathfrak{I}\backslash\{(0,1)\}$ and $l \geq 0$. The vector space[4] $\mathbb{C}[y]^{\mathbb{Z}}$ is equipped with the structure of a module over the algebra of differential operators in e^ω by setting

$$(e^\omega \cdot \tau)_n = \tau_{n-1}, \quad (\partial_\omega \cdot \tau)_n = n\tau_n, \quad \tau = (\tau_n)_{n\in\mathbb{Z}} \in \mathbb{C}[y]^{\mathbb{Z}}.$$

For every root α of $\mathfrak{g}^{(0)}$ we define *vertex operators* acting on $\mathbb{C}[y]^{\mathbb{Z}}$ as follows

$$E^*_\alpha(\zeta) := \exp\Big(\sum_{i,l} (\alpha|H_i) y_{i,l} \zeta^{m_i+l\kappa} \Big) \exp\Big(\sum_{i,l} (\alpha|H_{i^*}) \frac{\partial}{\partial y_{i,l}} \frac{\zeta^{-m_i-l\kappa}}{-m_i-l\kappa} \Big)$$

[3] if $a_\mu = 1$ then the corresponding diagram is empty.

[4] This is a direct product of copies of $\mathbb{C}[y]$ indexed by $n \in \mathbb{Z}$.

and

$$E^0_\alpha(\zeta) = \exp\Big((\omega_b|\alpha)\omega\Big)\exp\Big(\Big((\omega_b|\alpha)\,\chi\,\log\zeta^\kappa\ +2\pi\sqrt{-1}\ (\rho_b|\alpha)\Big)\partial_\omega\Big),$$

where ω_b is the fundamental weight corresponding to the branching node and

$$\rho_b = -\sum_{\mu=1}^{3}\sum_{i=1}^{a_\mu-1}\frac{1}{a_\mu}\omega_{\mu,i}$$

where $\omega_{\mu,i}$ is the fundamental weight corresponding to the i-th node on the μ-th branch of the Dynkin diagram. The HQE of the σ_b-twisted Kac–Wakimoto hierarchy are given by the following bilinear equation for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$:

(3)

$$\mathrm{Res}_{\zeta=0}\frac{d\zeta}{\zeta}\Big(\sum_{\alpha\in\Delta^{(0)}} a_\alpha(\zeta)E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\Big)\tau\otimes\tau = \Big(|\rho_s|^2/\kappa^2+\frac{\chi}{2}(\partial_\omega\otimes 1-1\otimes\partial_\omega)^2+$$
$$+\frac{1}{\kappa}\sum_{i,l}(m_i+l\kappa)(y_{i,l}\otimes 1-1\otimes y_{i,l})(\partial_{y_{i,l}}\otimes 1-1\otimes\partial_{y_{i,l}})\Big)\tau\otimes\tau,$$

where $E_\alpha(\zeta) = E^{(0)}_\alpha(\zeta)E^*_\alpha(\zeta)$, the coefficients

$$a_\alpha(\zeta) = \zeta^{\kappa\,|\alpha_0|^2}\,e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}\kappa^{-2}\prod_{l=1}^{\kappa-1}(1-\eta^l)^{(\sigma_b^l\alpha|\alpha)},$$

and the constant $|\rho_s|^2/\kappa^2 = \frac{1}{12}\sum_{\mu=1}^{3}\Big(a_\mu-\frac{1}{a_\mu}\Big)$.

1.3. **The main result.** Using the change of variables (68)–(69), we can write the Kac–Wakimoto HQE in terms of the descendant variables $\{q_k\}_{k\geq 0}$. Our main result can be stated as follows.

Theorem 1. *Let $\mathcal{D}_{\mathbf{a}}$ (with $\mathbf{a} = \{a_1, a_2, a_3\}$) be the total descendant potential* (1) *of an orbifold projective line $\mathbb{P}^1_{\mathbf{a}}$ with a positive orbifold Euler characteristic. There exists $C = C'Q$, with $C'\neq 0$ a constant independent of the Novikov variable Q, such that the sequence $(\tau_n(\hbar;\mathbf{q}))_{n\in\mathbb{Z}}$ of formal power series defined by*

$$\tau_n(\hbar;\mathbf{q}) = C^{\frac{1}{2}n^2}\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q}+n\sqrt{\hbar}\mathbf{1}),\quad n\in\mathbb{Z}.$$

is a solution to the σ_b-twisted Kac–Wakimoto HQE (3)*, where σ_b is the element* (2) *of the Weyl group of the corresponding finite root system.*

In other words, Theorem 1 shows that the Gromov-Witten theory of $\mathbb{P}^1_{\mathbf{a}}$ is governed by the Kac-Wakimoto hierarchy associated to the triple $\mathbf{a}$.

The proof of Theorem 1 may be outlined as follows. First, the hierarchy (3) is shown to be equivalent (via a Laplace transform) to another hierarchy (77) defined for affine cusp polynomials, see Theorem 33. Then by Proposition 39, the descendant potential $\mathcal{D}_{\mathbf{a}}$ satisfies the hierarchy (77) if and only if the ancestor potential $\mathcal{A}_t$ (see equation (83)) satisfies another hierarchy (95). Finally, it is shown (Theorem 40) that $\mathcal{A}_t$ indeed satisfies (95). Let us point out that although our proof of Theorem 40 follows closely the argument of [31], we managed to simplify one of the crucial steps in [31]. Namely, there is a certain analyticity property (c.f. Section 8.2) of the so called phase factors that was previously established via the theory of finite reflection groups and their relation to Artin groups. This is one of the main obstacles to generalize the result of [31] to other singularities. Our argument now seems to apply in much more general settings, since it relies only on the fact that the Gauss–Manin connection has regular singularities and that the vertex operators are local to each other (in the sense of the theory of vertex operator algebras).

Remark 2. *The constant C' is given explicitly in terms of the root system. However we do not have a closed formula that evaluates C'. Still, C' can be determined from the first few HQE and some easily computable GW invariants of $\mathbb{P}^1_{\mathbf{a}}$. See Section 9 for an example.*

Remark 3. *Let us emphasize that the variables $q_1^{01}, q_2^{01}, \ldots$ appear as parameters in the differential equations for τ. It is natural to expect that the σ_b-twisted Kac–Wakimoto HQE can be extended in order to include differential equations in $q_1^{01}, q_2^{01}, \ldots$ as well. For example, in the case of Dynkin diagram of type A, our hierarchy should agree with a certain reduction of the 2D Toda hierarchy and the required extension was constructed by G. Carlet* [10]*. For the type D and E cases, the extension can be constructed with the same idea as in* [50]*, although a slight modification is necessary. The details will be presented else where.*

Our approach to Theorem 1 systematically explores representation theoretic properties of the Landau-Ginzburg mirror of $\mathbb{P}^1_{\mathbf{a}}$ and realizes these properties in quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ using the period maps. Such an approach should be helpful in identifying integrable hierarchies governing Gromov-Witten theory of more general target spaces.

We suggest to call the σ_b-twisted Kac–Wakimoto hierarchy appearing in Theorem 1 the *ADE-Toda hierarchy*, while the corresponding extension should be called the *Extended ADE-Toda hierarchy.*

Another approach to the relationship between GW theory and integrable hierarchy is due to Dubrovin and Zhang [18], who proposed a class of integrable hierarchies defined for any semi-simple Frobenius manifold. While their construction produces flows that are rational functions on certain jet variables and so in general the integrable system will be ill behaved, it is expected that for the important classes of semi-simple Frobenius manifold, such as quantum cohomology, the flows are in fact polynomial and that the hierarchy can be used to compute uniquely the higher genus invariants. The polynomiality of the flows for a semi-simple Frobenius manifold associated with a cohomological field theory (this includes the case of GW theory) was proved recently by Buryak–Posthuma–Shadrin [7, 8] using the higher genus reconstruction of Givental. The discovery of this new class of integrable hierarchies is a major breakthrough in the theory of integrable systems. It is natural to study further their properties and to look for applications to other areas of Mathematics and even beyond.

It is very interesting also to investigate the relation between the integrable hierarchies obtained by appliying Dubrovin and Zhang's construction [19] to the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ and the integrable hierarchies in Theorem 1. It is natural to expect that the two approaches yield the same integrable hierarchy. We hope to return to this problem in the near future.

The rest of this paper is organized as follows. In Section 2, we recall the orbifold Gromov-Witten theory for Fano projective curves $\mathbb{P}^1_{\mathbf{a}}$. For those orbifolds, we also give an alternative proof for the higher genus reconstruction of total ancestor potential. In Section 3, using the period mapping, we construct an affine root system in the quantum cohomology H of $\mathbb{P}^1_{\mathbf{a}}$ arising from vanishing cycles and prove that the natural weighted-homogeneous basis of H is a Jordan basis for the affine Coxeter transformation. In Section 4 we obtain an explicit description of the leading order terms of the period mapping in terms of the affine root system and the affine Coxeter transformation. In Section 5, using the results from Section 4, we give a Fock-space realization of the basic representations of the affine Lie algebras of ADE type. In Section 6 we construct the Kac-Wakimoto hierarchies and integrable hierarchies for affine cusp polynomials and show that these hierarchies are related by a Laplace transform (Theorem 33). In Section 7 we construct another hierarchy (95) and describe its relation with the hierarchies from previous sections, see Proposition 39. In Section 8 we show that the ancestor potential of $\mathbb{P}^1_{\mathbf{a}}$ satisfies the integrable hierarchy (95) and deduce Theorem 1. In Section 9 we consider the example $\mathbf{a} = \{2, 2, 2\}$.

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2. Orbifold Gromov–Witten theory of $\mathbb{P}^1_{\mathbf{a}}$

2.1. Fano orbifold curves $\mathbb{P}^1_{\mathbf{a}}$ and Chen-Ruan orbifold cohomology.

Fano orbifold curves are close orbifold curves with positive orbifold Euler characteristics. They are classified by triplets of positive integers

$$\mathbf{a} = \{a_1, a_2, a_3\}$$

where $a_1 \le a_2 \le a_3$ and

$$\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 > 0.$$

Each Fano orbifold curve is an orbifold curve with an underlying curve $\mathbb{P}^1$ and has at most three orbifold points p_i $(i = 1, 2, 3)$ with local isotropy groups $\mathbb{Z}_{a_i}$. We denote such an Fano orbifold curve by $\mathbb{P}^1_{\mathbf{a}}$. Note that such notation also includes the smooth curve $\mathbb{P}^1$ with $a_1 = a_2 = a_3 = 1$. It is easy to see that χ is the orbifold Euler characteristic of $\mathbb{P}^1_{\mathbf{a}}$.

For a triplet $\mathbf{a}=\{a_1,a_2,a_3\}$, it is convenient to introduce an index set

$$\mathfrak{I}:=\mathfrak{I}_{\rm tw}\cup\{(01),(02)\}:=\big\{(i',i'')\mid 1\le i'\le 3, 1\le i''\le a_{i'}-1\big\}\cup\{(01),(02)\}. \tag{4}$$

Let $\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}$ be the so called *inertia orbifold* of $\mathbb{P}^1_{\mathbf{a}}$. The Chen-Ruan orbifold cohomology for an Fano orbifold curve $\mathbb{P}^1_{\mathbf{a}}$ is denoted by

$$H:=H^*_{\rm orb}(\mathbb{P}^1_{\mathbf{a}},\mathbb{C})=H^*(\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}};\mathbb{C}),$$

where $H^*(\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}};\mathbb{C})$ is the cohomology of the inertia orbifold $\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}$ with its degree shifted appropriately. We use the index set $\mathfrak{I}$ to label a fixed basis of the Chen-Ruan orbifold cohomology H as follows:

$$\phi_{01}=1,\quad \phi_{02}=P$$

are the unit and the hyperplane class of the underlying $\mathbb{P}^1$ respectively and

$$\phi_i=\phi_{i',i''},\quad i:=(i',i'')\in\mathfrak{I}_{\rm tw}.$$

are the units of the corresponding twisted sectors of $\mathbb{P}^1_{\mathbf{a}}$. The cohomology degree of the classes are as follows:

$$\deg\phi_{01}=0,\quad \deg\phi_{02}=2,\quad \deg\phi_i=\frac{2i''}{i'}, i=(i',i'')\in\mathfrak{I}_{\rm tw}.$$

There is a natural involution $*$ on $\mathfrak{I}$ induced by orbifold Poincaré duality

$$(0,1)^*=(0,2),\quad (i',i'')^*=(i',a_{i'}-i''). \tag{5}$$

The orbifold Poincaré pairing $(-,-)$ on H is non-zero only for the following pairs of cohomology classes

$$(\phi_{01},\phi_{02})=1,\quad (\phi_i,\phi_j)=\frac{1}{a_i}\delta_{i,j^*},$$

where $i,j\in\mathfrak{I}$ correspond to twisted classes, and we set $a_i:=a_{i'}$ for $i=(i',i'')\in\mathfrak{I}_{\rm tw}$.

2.2. **The descendants and the ancestors.** Gromov-Witten theory studies integrals over moduli spaces of stable maps. In this paper, we will use both the descendant invariants and the ancestor invariants. Let us introduce their definitions for Fano orbifold curves $\mathbb{P}^1_{\mathbf{a}}$. Let $d\in\mathrm{Eff}(\mathbb{P}^1_{\mathbf{a}})\subset H_2(\mathbb{P}^1_{\mathbf{a}};\mathbb{Z})\cong\mathbb{Z}$ be an effective curve class. By choosing the homology class $[\mathbb{P}^1_{\mathbf{a}}]$ as a $\mathbb{Z}$-basis of $H_2(\mathbb{P}^1_{\mathbf{a}};\mathbb{Z})$ we may identify d with a non-negative integer. Let

$$\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)$$

be the moduli space of stable orbifold maps f from a genus-g nodal orbifold Riemann surface Σ to $\mathbb{P}^1_{\mathbf{a}}$, such that $f_*[\Sigma]=d$. In addition, Σ is equipped with n marked points $z_1,\dots,z_n$ that are pairwise distinct and not nodal and the orbifold structure of Σ is non-trivial only at the marked points and the nodes. The moduli space $\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)$ has a virtual fundamental cycle $[\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}},d)]^{\rm virt}$. Its homology degree is

$$2\left((3-\dim\mathbb{P}^1_{\mathbf{a}})(g-1)+\chi\cdot d+n\right). \tag{6}$$

The moduli space is naturally equipped with line bundles $\mathcal{L}_i$ formed by the cotangent lines[5] $T^*_{\bar{z}_i}\bar{\Sigma}/\mathrm{Aut}(\Sigma, z_1, \ldots, z_n; f)$ and with evaluation map

$$\mathrm{ev} : \overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d) \to \underbrace{\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}} \times \cdots \times \mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}}_{n},$$

obtained by evaluating f at the (orbifold) marked points $z_1, ..., z_n$ and landing at the connected component of the inertia orbifold $\mathrm{I}\,\mathbb{P}^1_{\mathbf{a}}$ corresponding to the generator of the automorphism group of the orbifold point z_i (c.f. [13]).

The *descendant orbifold Gromov–Witten invariants* of $\mathbb{P}^1_{\mathbf{a}}$ are intersection numbers

$$(7) \qquad \langle \phi_{i_1}\, \psi_1^{k_1}, \ldots, \phi_{i_n} \psi_n^{k_n} \rangle_{g,n,d} := \int_{[\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d)]^{\mathrm{virt}}} \mathrm{ev}^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_n})\, \psi_1^{k_1} \cdots \psi_n^{k_n},$$

where $\phi_{i_s} \in H := H^*_{\mathrm{orb}}(\mathbb{P}^1_{\mathbf{a}}; \mathbb{C})$, $\psi_{i_s} = c_1(\mathcal{L}_{i_s})$. The *total descendant potential* is

$$\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t}) = \exp\Big(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n) \rangle_{g,n,d} \Big),$$

where Q is a non-zero complex number called the *Novikov varaible*, $\hbar$, $t_0, t_1, \ldots \in H$ are formal variables and $\mathbf{t}(z) := t_0 + t_1 z + t_2 z^2 + \cdots$.

Let $\pi : \overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d) \to \overline{\mathcal{M}}_{g,n}$ be the stablization of the forgetful morphism and

$$\Lambda_{g,n,d}\,(\phi_{i_1}, \cdots, \phi_{i_n}) := \pi_* \left([\overline{M}_{g,n}(\mathbb{P}^1_{\mathbf{a}}, d)]^{\mathrm{virt}} \cap \mathrm{ev}^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_n}) \right).$$

The *ancestor orbifold Gromov-Witten invariants* of $\mathbb{P}^1_{\mathbf{a}}$ are intersections numbers over the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ $(2g - 2 + n > 0)$:

$$(8) \qquad \langle \phi_{i_1}\, \bar{\psi}_1^{k_1}, \ldots, \phi_{i_n} \bar{\psi}_n^{k_n} \rangle_{g,n,d} := \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n,d}\,(\phi_{i_1}, \cdots, \phi_{i_n})\; \bar{\psi}_1^{k_1} \cdots \bar{\psi}_n^{k_n},$$

where $\bar{\psi}_{i_s}$ is the i_s-th ψ-class over $\overline{\mathcal{M}}_{g,n}$. We define the total ancestor potential of $\mathbb{P}^1_{\mathbf{a}}$ as follows

$$(9) \qquad \mathcal{A}_{\mathbf{a}}(\hbar; \mathbf{t}) := \exp\Big(\sum_{g,n,d} \hbar^{g-1} \frac{Q^d}{n!} \langle \mathbf{t}(\bar{\psi}_1), \ldots, \mathbf{t}(\bar{\psi}_n) \rangle_{g,n,d} \Big).$$

For each element $t \in H$, it is useful to introduce the double bracket notation:

$$\langle\langle \phi_{i_1}\, \bar{\psi}_1^{k_1}, \ldots, \phi_{i_n} \bar{\psi}_n^{k_n} \rangle\rangle_{g,n}(t) := \sum_{k,d} \frac{Q^d}{k!} \langle \phi_{i_1}\, \bar{\psi}_1^{k_1}, \ldots, \phi_{i_n} \bar{\psi}_n^{k_n}, t, \ldots, t \rangle_{g,n+k,d}$$

We define a total ancestor potential that depends on the choice of t,

$$(10) \qquad \mathcal{A}_t(\hbar; \mathbf{t}) = \exp\Big(\sum_{g,n} \hbar^{g-1} \frac{1}{n!} \langle\langle \mathbf{t}(\bar{\psi}_1), \ldots, \mathbf{t}(\bar{\psi}_n) \rangle\rangle_{g,n}(t) \Big).$$

The total ancestor potential $\mathcal{A}_t(\hbar; \mathbf{t})$ and the total descendant potential $\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{t})$ are related by the quantization of a calibration operator $S_t(z)$ in Section 3.4. We will explain the details of the quantization in Section 7.

[5]Here $\bar{\Sigma}$ is the nodal Riemann surface underlying Σ and $\bar{z}_i \in \bar{\Sigma}$ is the i-th marked point on $\bar{\Sigma}$.

For more details on orbifold Gromov–Witten theory we refer to [13] for the analytic approach and to [2] for the algebraic geometry approach.

2.3. **Quantum cohomology and Teleman's theorem.** By definition, the *quantum cup product* is a family of associative commutative multiplications $\bullet_t$ in H defined for each $t \in H$ via the correlators

$$(\phi_i \bullet_t \phi_j, \phi_k) = \langle\langle \phi_i, \phi_j, \phi_k \rangle\rangle(t).$$

Let t_i, $i \in \mathfrak{I}$ be the corresponding coordinates of ϕ_i. The quantum cup product induces on H a Frobenius structure of conformal dimension 1 with respect to the *Euler* vector field

$$E = \sum_{i \in \mathfrak{I}} d_i t_i \frac{\partial}{\partial t_i} + \chi \frac{\partial}{\partial t_{02}}$$

where $d_i = 1 - \deg(\phi_i)/2$, i.e.,

$$d_{01} = 1, \quad d_{02} = 0,$$

and

$$d_i = 1 - \frac{i''}{a_{i'}}, \quad i = (i', i'') \text{ twisted class (i.e. not } (0,1),(0,2)).$$

A Frobenius manifold is called semisimple if the multiplication has a semisimple basis. The Frobenius manifold $(H, (\ ,\ ), \bullet_t, \phi_{01}, E)$ is isomorphic to the Frobenius manifold constructed from the mirror model of $\mathbb{P}^1_{\mathbf{a}}$ [51, 53, 38]. Using the mirror model, it is quite easy to see that $\bullet_t$ is semisimple for generic t.

For any semisimple Frobenius manifold, Givental has a higher genus reconstruction formula [29] and conjectured that the higher genus Gromov-Witten ancestor invariants are uniquely determined from its semisimple quantum cohomology. Teleman [59] has proved this conjecture. More explicitly,

Theorem 4 ([59])**.** *The Gromov-Witten ancestor invariants for a complex projective manifold X are determined by a recursive relation from the quantum cohomology $\bullet_t$ at a single semisimple point $t \in H^{\mathrm{ev}}(X)$ in the even cohomology, and from the Euler vector field.*

This reconstruction works for orbifolds as well. Combine with Theorem 9 in Section 3, it allows us to identify the total ancestor potential defined in (10) and in (83). Hence we can also identify the total descendant potentials defined in (1) and in (82).

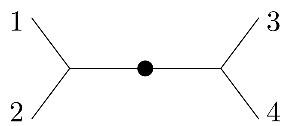
Remark 5. *We briefly explain how a Givental-style mirror theorem for $\mathbb{P}^1_{\mathbf{a}}$ can be established. Let $\mathbb{P}^2_{\mathbf{a}}$ be an orbifold $\mathbb{P}^2$ obtained from $\mathbb{P}^2$ by adding $\mathbb{Z}_{a_1}$-, $\mathbb{Z}_{a_2}$-, and $\mathbb{Z}_{a_3}$-orbifold structures along the three toric prime divisors of $\mathbb{P}^2$. Then $\mathbb{P}^1_{\mathbf{a}}$ is a hyperplane section of $\mathbb{P}^2_{\mathbf{a}}$. The orbifold $\mathbb{P}^2_{\mathbf{a}}$ is a Fano toric orbifold, and its J-function can be computed by the mirror theorem of* [15]*. $\mathbb{P}^1_{\mathbf{a}}$ is the zero locus of a generic section of a convex line bundle on $\mathbb{P}^2_{\mathbf{a}}$. Hence the J-function of $\mathbb{P}^1_{\mathbf{a}}$ can be computed from that of $\mathbb{P}^2_{\mathbf{a}}$ using the quantum Lefschetz theorem of* [14]*. The computation of the J-function* *of $\mathbb{P}^1_{\mathbf{a}}$ can be used to derive an identification between the quantum cohomology D-module of $\mathbb{P}^1_{\mathbf{a}}$ and the D-module defined by $f_{\mathbf{a}}(x)$. Since we do not use these results in this paper, we omit the details.*

2.4. **An alternative proof of higher genus reconstruction.** In this subsection, we use the degree of virtual fundamental cycle and tautological relations to give a simple proof for Teleman's higher genus reconstruction theorem for the target $\mathbb{P}^1_{\mathbf{a}}$, see Proposition 7 below. This proof does not require the semisimple assumption.

We first recall the g-reduction property introduced in [21], which is a consequence of results by Ionel [35], and by Faber and Pandharipande [20]:

Lemma 6 ([35, 20])**.** *If $M(\psi,\kappa)$ is a polynomial of ψ-classes and κ-classes with $\deg M \geq g$ for $g \geq 1$ or $\deg M \geq 1$ for $g = 0$, then $M(\psi,\kappa)$ can be presented as a linear combination of dual graphs on the boundary of $\overline{\mathcal{M}}_{g,n}$.*

Our second tool is the Getzler's relation in [26]. It is a linear relation between codimension two cycles in $H_*(\overline{\mathcal{M}}_{1,4},\mathbb{Q})$. Here we briefly introduce this relation for our purpose. Consider the dual graph,

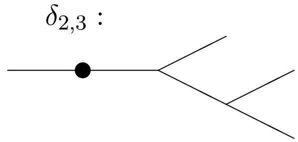
$\Delta_{12,34} =$ 

This graph represents a codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$: A filled circle represents a genus-1 component, other vertices represent genus-0 components. An edge connecting two vertices represents a node, a tail (or half-edge) represents a marked point on the component of the corresponding vertex. $\Delta_{2,2}$ is defined to be the S_4-invariant of the codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$,

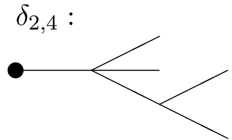
$$\Delta_{2,2} = \Delta_{12,34} + \Delta_{13,24} + \Delta_{14,23}.$$

We denote $\delta_{2,2} = [\Delta_{2,2}]$ the corresponding cycle in $H_4(\overline{\mathcal{M}}_{1,4},\mathbb{Q})$. We list the corresponding unordered dual graph for other strata below, see [26] for more details.

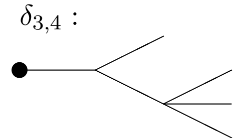
$\delta_{2,3}$:



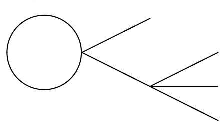
$\delta_{2,4}$:



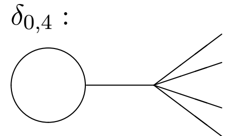
$\delta_{3,4}$:



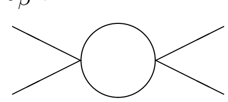
$\delta_{0,3}$:



$\delta_{0,4}$:



δ_{β} :



In [26], Getzler found the following identity:

$$12\delta_{2,2} + 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_{\beta} = 0 \in H_4(\overline{\mathcal{M}}_{1,4},\mathbb{Q}). \tag{11}$$

Now we prove the following higher genus reconstruction result.

Proposition 7. *The total ancestor potential $\mathcal{A}_{\mathbf{a}}(\hbar;\mathbf{t})$ is uniquely determined by the quantum cohomology of $\mathbb{P}^1_{\mathbf{a}}$ when $\mathbf{a}\neq\{1,1,1\}$ and $\chi>0$.*

Proof. We consider the ancestor correlator $\langle\phi_{i_1}\,\bar\psi_1^{k_1},\ldots,\phi_{i_n}\bar\psi_n^{k_n}\rangle_{g,n,d}$ in (8). According to the degree formula (6), if the correlator is nonzero, then

$$\frac{1}{2}\sum_{j=1}^{n}\deg\phi_{i_j}+\sum_{j=1}^{n}k_j=(3-\frac{1}{2}\dim\mathbb{P}^1_{\mathbf{a}})(g-1)+\chi\cdot d+n. \tag{12}$$

Now if $\sum_{j=1}^{n}k_j\geq g$ for $g\geq 1$ or $\sum_{j=1}^{n}k_j\geq 1$ for $g=0$, then we can apply Lemma 6 of g-reduction to rewrite the ancestor correlator as a linear combination of intersection numbers over the corresponding homology cycles of some dual graphs, each of the dual graph lives on the boundary of $\overline{\mathcal{M}}_{g,n}$. The splitting axiom in Gromov-Witten theory allows us to reconstruct the ancestor correlator in (8) using intersection numbers over each component of the boundaries. We can keep doing this process until on each component, the g-reduction property does not hold. In another words, all the ancestor correlators are determined completed by those (8) which satisfies $\sum_{j=1}^{n}k_j\leq g-1$ for $g\geq 1$ or $\sum_{j=1}^{n}k_j=0$ for $g=0$. On the other hand, since $\deg\phi_{i_j}\leq 2$, $\chi>0$ and $\dim\mathbb{P}^1_{\mathbf{a}}=1$, the formula (12) implies such intersection numbers must vanish unless $g=0$ and all $k_j=0$, or $g=1,d=0$, all $k_j=0$ and all $\deg\phi_{i_j}=2$.

In order to finish the proof, it only remains to consider genus 1 correlator $\langle P\rangle_{1,1,0}$. If $\mathbf{a}\neq\{1,1,1\}$, then according to Rossi's computation [53], we can always find a twisted sector $\phi_i\in H$, such that

$$\langle\phi_i,\phi_i,\phi_{i^*},\phi_{i^*}\rangle_{0,4,0}\neq 0. \tag{13}$$

We consider the integration of the cohomology cycle $\Lambda_{1,4,0}(\phi_i,\phi_i,\phi_{i^*},\phi_{i^*})$ over the Getzler's relation (11), with four fixed insertions $\phi_i,\phi_i,\phi_{i^*},\phi_{i^*}$. Using the splitting axiom in Gromov-Witten theory, it is not hard to see that the integration vanishes on those homology classes with a genus-1 component except that

$$\int_{\delta_{3,4}}\Lambda_{1,4,0}(\phi_i,\phi_i,\phi_{i^*},\phi_{i^*})$$

is a multiplication by a nonzero scalar and $\langle P\rangle_{1,1,0}$, because of (13). Thus the equality (11) implies $\langle P\rangle_{1,1,0}$ is reconstructed from genus-0 correlators.

□

Remark 8. *The technique above only uses properties of cohomology field theories and tautological relations over the moduli space of stable curves. So it also works for the reconstruction of the ancestor potential in* (83)*. It also works for elliptic orbifold projective curves $\mathbb{P}^1_{\mathbf{a}}$, where $\chi=0$, see* [46]*. The genus-1 correlator $\langle P\rangle_{1,1,0}$ in Gromov-Witten theory can be calculated directly using virtual cycle or virtual localization, see* [60]*.*

3. Quantum cohomology and root systems

3.1. **Mirror symmetry for the quantum cohomology.** The Frobenius structure on H arising from quantum cohomology can be identified with the Frobenius structure on a certain deformation space of the *affine cusp polynomial*

$$f_{\mathbf{a}}(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Q} x_1 x_2 x_3, \quad x = (x_1, x_2, x_3). \tag{14}$$

where $Q \in \mathbb{C}^*$ is the Novikov variable. The isomorphism in the case $a_1 = 1$ was established in [51] and the general case can be found in [53]. According to Ishibashi–Shiraishi–Takahashi (see [38]), the Frobenius structure can be described also in the general framework of K. Saito's theory of primitive forms. This is precisely the point of view suitable for our purposes.

Let

$$\mu = a_1 + a_2 + a_3 - 1$$

be the Milnor number of $f_{\mathbf{a}}$, i.e., the number of critical points of a Morsification of $f_{\mathbf{a}}$. Let

$$M = \mathbb{C}^\mu$$

be the space of a miniversal deformation of the polynomial $f_{\mathbf{a}}$. Note that the cardinality of the set $\mathfrak{I}$ is μ, so we can enumerate the coordinates on M via $s = (s_i)_{i \in \mathfrak{I}}$. Given $s \in M$, we put

$$F(x, s) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - \frac{1}{Qe^{s_{02}}}\, x_1 x_2 x_3 + s_{01} + \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} s_i\, x_{i'}^{i''}\,.$$

Here we put $\mathfrak{I}_{\mathrm{tw}} := \mathfrak{I} \setminus \{(0,1),(0,2)\}$. Let $C \subset M \times \mathbb{C}^3$ be the analytic subvariety with structure sheaf

$$\mathcal{O}_C = \mathcal{O}_{M \times \mathbb{C}^3} / (\partial_{x_1} F, \partial_{x_2} F, \partial_{x_3} F);$$

then the *Kodaira-Spencer map*

$$\mathcal{T}_M \to p_* \mathcal{O}_C, \quad \frac{\partial}{\partial s_i} \mapsto \frac{\partial F}{\partial s_i} \bmod (\partial_{x_1} F, \partial_{x_2} F, \partial_{x_3} F), \tag{15}$$

where $p : M \times \mathbb{C}^3 \to M$ is the projection onto the first factor, is an isomorphism which allows us to define an associative, commutative multiplication $\bullet$ on $\mathcal{T}_M$. The main result in [38] is that

$$\omega = \frac{\sqrt{-1}}{Qe^{s_{02}}}\, dx_1 \wedge dx_2 \wedge dx_3$$

is a *primitive form* in the sense of K. Saito (see [54]), which allows us to construct a Frobenius structure on M (see [55]). More precisely, the form ω gives rise to a residue pairing on $\mathcal{O}_C$

$$(\phi_1, \phi_2) = -\frac{1}{Q^2 e^{2\, s_{02}}} \operatorname{Res}_{M \times \mathbb{C}^3 / M} \frac{\phi_1 \phi_2\, dx_1 \wedge dx_2 \wedge dx_3}{\partial_{x_1} F \partial_{x_2} F \partial_{x_3} F}\,,$$

which via the Kodaira–Spencer isomorphism (15) induces a non-degenerate bilinear form on $\mathcal{T}_M$. Let us form the following family of connections on $\mathcal{T}_M$

$$\nabla = \nabla^{\mathrm{L.C.}} - \frac{1}{z} \sum_{i \in \mathfrak{I}} (\partial_{s_i} \bullet)\, ds_i,$$

where $\nabla^{\mathrm{L.C.}}$ is the Levi-Cevita connection associated with the residue pairing and $\partial_{s_i}\bullet$ is the operator of multiplication by the vector field $\partial/\partial s_i$. Let us also introduce the so called *oscillatory integrals*

$$J_{\mathcal{A}}(s,z) = (-2\pi z)^{-3/2}\, zd \int_{\mathcal{A}_{s,z}} e^{F(x,s)/z}\,\omega \in T_s^*M,$$

where d is the de Rham differential on M, and $\mathcal{A}$ is a flat section of the bundle on $M\times\mathbb{C}^*$, whose fiber over a point (s,z) is given by the space of semi-infinite homology cycles

$$H_3(\mathbb{C}^3, \{x|\mathrm{Re}(F(x,s)/z) \ll 0\};\mathbb{C}) \cong \mathbb{C}^\mu.$$

The fact that ω is primitive means that the connection ∇ is flat for all $z\neq 0$ and that after identifying $\mathcal{T}_M\cong\mathcal{T}_M^*$ via the residue pairing, the oscillatory integrals $J_{\mathcal{A}}$ give rise to flat sections of ∇. Moreover, since the oscillatory integrals are weighted-homogeneous functions if one assigns weights d_i $(i\in\mathfrak{I})$, $1/a_j$ $(1\leq j\leq 3)$, and χ to s_i, x_j, and Q respectively, they satisfy an additional differential equation with respect to z. Let $E\in\mathcal{T}_M$ be the *Euler* vector field

$$E = \sum_{i\in\mathfrak{I}} d_i s_i \frac{\partial}{\partial s_i} + \chi\frac{\partial}{\partial s_{02}}.$$

Note that under the Kodaira–Spencer isomorphism E corresponds to the equivalence class of F in $p_*\mathcal{O}_C$. The oscillatory integrals satisfy the following differential equation:

$$(z\partial_z + E)\, J_{\mathcal{A}}(t,z) = \theta\, J_{\mathcal{A}}(t,z), \tag{16}$$

where $\theta:\mathcal{T}_M\to\mathcal{T}_M$ is defined via

$$\theta(X) = \nabla_X^{L.C.}(E) - \frac{1}{2}X$$

where the constant $\frac{1}{2}$ is chosen in such a way that θ is *anti-symmetric* with respect to the residue pairing: $(\theta(X),Y) = -(X,\theta(Y))$.

The quantum cohomology computed at $t=0$ is isomorphic as a Frobenius algebra with T_0M (see [38, 53]). The identification has the following form

$$\phi_i = x_{i'}^{i''} + \cdots, \quad \phi_{01} = 1, \quad \phi_{02} = \frac{1}{Q}x_1x_2x_3 + \cdots.$$

where $i=(i',i'')$ is the index of a twisted class and the dots stand for some polynomials that involve higher-order powers of Q. More precisely, using the Kodaira-Spencer isomorphism we have

$$\phi_i = \partial_{s_i} + \cdots, \quad \phi_{01} = \partial_{s_{01}}, \quad \phi_{02} = \partial_{s_{02}} + \cdots,$$

where the dots stand for some vector fields depending holomorphically on Q near $Q=0$ and vanishing at $Q=0$. These additional terms are uniquely fixed by the requirement that the vector fields ϕ_i $(i\in\mathfrak{I})$ are flat, i.e., the residue pairing is constant independent of Q. On the other hand the flatness of ∇ implies that the residue pairing is flat, therefore we can extend uniquely the isomorphism $H \cong T_0M$ to an isomorphism

$$TH \cong TM$$

such that the residue pairing coincides with the Poincaré pairing. In other words, the linear coordinates t_i, $i \in \mathfrak{I}$ on H are functions on M such that $t_i(0) = 0$, the vector field $\partial/\partial t_i$ is flat with respect to the Levi–Cevita connection, and at $s = 0$ it coincides with ϕ_i. The mirror symmetry for quantum cohomology can be stated as follows.

Theorem 9 ([38], Theorem 4.1)**.** *The isomorphism $M \cong H$, $s \mapsto t(s)$ is an isomorphism of Frobenius manifolds, i.e., $T_sM \cong T_{t(s)}H$ as Frobenius algebras.*

From now on we will make use of the residue pairing to identify $T^*M \cong TM$. Also the flat Levi–Cevita connection $\nabla^{\mathrm{L.C.}}$ allows us to construct a trivialization

$$TM \cong M \times T_0M,$$

and finally, the Kodaira–Spencer map (15) together with the mirror symmetry isomorphism gives $T_0M \cong H$. In other words, we have natural trivializations

$$T^*M \cong TM \cong M \times H. \tag{17}$$

3.2. **The period integrals.** Givental noticed that certain period integrals (c.f. formula (18) bellow) in singularity theory play a crucial role in the theory of integrable systems. In this section, we recall Givental's construction as well as some of its basic properties. For more details we refer to [28].

Put $X = M \times \mathbb{C}^3$ and let

$$\varphi : X \to M \times \mathbb{C}, \quad (s, x) \mapsto (s, F(x, s)).$$

Let

$$X_{s,\lambda} = \varphi^{-1}(s, \lambda)$$

be the fibers of φ. The set of all $(s, \lambda) \in M \times \mathbb{C}$ such that the fiber $X_{s,\lambda}$ is singular is an analytic hypersurface, called *discriminant*. Its complement in $M \times \mathbb{C}$ will be denoted by

$$(M \times \mathbb{C})'.$$

The homology and cohomology groups $H_2(X_{s,\lambda}; \mathbb{C})$ and $H^2(X_{s,\lambda}; \mathbb{C})$, $(s, \lambda) \in (M \times \mathbb{C})'$ form vector bundles over the base $(M \times \mathbb{C})'$. Moreover, the integral structure in the fibers allows us to define a flat connection known as the *Gauss–Manin* connection.

Let us fix the point $(0, 1) \in (M \times \mathbb{C})'$ (this is true for $Q \ll 1$) to be our reference point. The vector space

$$\mathfrak{h} = H_2(X_{0,1}; \mathbb{C})$$

has a very rich structure, which we would like to recall. Let

$$\Delta \subset \mathfrak{h}$$

be the set of *vanishing cycles*, and $(\cdot|\cdot)$ be the *negative* of the intersection pairing. The negative sign is chosen so that $(\alpha|\alpha) = 2$ for all $\alpha \in \Delta$. The parallel transport with respect to the Gauss–Manin connection induces a monodromy representation

$$\pi_1((M \times \mathbb{C})') \to \mathrm{GL}(\mathfrak{h}).$$

The image

$$W \subset \mathrm{GL}(\mathfrak{h})$$

of the fundamental group under this representation is a subgroup of the group of linear transformations of $\mathfrak{h}$ that preserve the intersection form. The Picard–Lefschetz theory can be applied in our setting as well and W is in fact a reflection group generated by the reflections

$$s_\alpha(x) = x - (\alpha|x)\alpha, \quad \alpha \in \Delta.$$

The reflection s_α is the monodromy transformation along a simple loop that goes around a generic point on the discriminant over which the cycle α vanishes. Finally, recall that the *classical* monodromy $\sigma \in W$ is the monodromy transformation along a big loop around the discriminant. For more details on vanishing homology and cohomology and the Picard–Lefschetz theory we refer to the book [3].

Proposition 10.

(1) *The set of vanishing cycles Δ is an affine root system of type $X_N^{(1)}$, where $N = \mu - 1 = a_1 + a_2 + a_3 - 2$ and*

$$X = \begin{cases} A & \text{if } a_1 = 1, \\ D & \text{if } a_1 = a_2 = 2, \\ E & \text{otherwise.} \end{cases}$$

(2) *There exists a basis of simple roots such that the classical monodromy σ is an affine Coxeter transformation.*

Part (1) of Proposition 10 is due to A. Takahashi (see [58]). The proof is based on a standard method developed by Gusein-Zade and A'Campo. Part (2) is not hard to verify as well. For the reader's convenience we outlined the main steps of the proof in Appendix A.

The main objects in our construction are the following multi-valued analytic functions:

$$I^{(n)}_\alpha(t,\lambda) = -\frac{1}{2\pi}\,\partial_\lambda^{n+1}\, d_M \int_{\alpha_{t,\lambda}} d^{-1}\omega, \tag{18}$$

where the value of the RHS depends on the choice of a path avoiding the discriminant, connecting the reference point with (t,λ). The cycle $\alpha_{t,\lambda}$ is obtained from $\alpha \in \mathfrak{h}$ via a parallel transport (along the chosen path), $d^{-1}\omega$ is any holomorphic 2-form η on $\mathbb{C}^3$ such that $\omega = d\eta$, and d_M is the de Rham differential on M. The RHS in (18) defines naturally a cotangent vector in T_t^*M, which via the trivialization (17) is identified with a vector in H.

The period vectors (18) are uniquely defined for all $n \geq -1$. For $n \leq -2$ there is an ambiguity in choosing integration constants, which however can be removed by means of the following differential equations:

$$\partial_{t_i} I^{(n)}_\alpha(t,\lambda) = -\phi_i \bullet I^{(n+1)}_\alpha(t,\lambda), \quad i \in \mathfrak{I}, \tag{19}$$

$$\partial_\lambda I^{(n)}_\alpha(t,\lambda) = I^{(n+1)}_\alpha(t,\lambda), \tag{20}$$

$$(\lambda - E\bullet)\partial_\lambda I^{(n)}_\alpha(t,\lambda) = \Big(\theta - n - 1/2\Big) I^{(n)}_\alpha(t,\lambda). \tag{21}$$

The oscillatory integrals are related to the period integrals via a Laplace transform along an appropriately chosen path:

$$J_{\mathcal{A}}(t,z) = (-2\pi z)^{-1/2}\int_{u_i}^{\infty} e^{\lambda/z} I^{(0)}_{\alpha}(t,\lambda)d\lambda, \tag{22}$$

where $u_i(t)$ is such that $(t, u_i(t))$ is a point on the discriminant over which the cycle α vanishes. The differential equations (19) are the Laplace transform of $\nabla J_{\mathcal{A}} = 0$, while the equation (21) is the Laplace transform of the differential equation (16). Using equations (20) and (21) we can express $I^{(n)}$ in terms of $I^{(n+1)}$ as long as the operator $\theta - n - 1/2$ is invertible. This is the case for $n \leq -2$, which allows us to extend the definition of $I^{(n)}$ to all $n \in \mathbb{Z}$.

3.3. **Stationary phase asymptotic.** Let $u_i(t)$, $1 \leq i \leq \mu$ be the critical values of $F(x,t)$. The set

$$M_{\rm ss} \subset M$$

of all points $t \in M$ such that the critical values $u_i(t)$ form locally near t a coordinate system is open and dense. Let us fix some $t_0 \in M_{\rm ss}$; then in a neighborhood of t_0 the critical values give rise to a coordinate system in which the pairing and the product $\bullet$ are diagonal, i.e.,

$$\partial/\partial u_i \bullet \partial/\partial u_j = \delta_{i,j}\partial/\partial u_j, \quad (\partial/\partial u_i, \partial/\partial u_j) = \delta_{i,j}/\Delta_j,$$

where Δ_j are some multi-valued analytic functions on $M_{\rm ss}$. Following Dubrovin's terminology (see [17]), we refer to u_i as *canonical coordinates.*

Remark 11. *It is easy to see that the critical variety C of the function F is non-singular, i.e., it is a manifold. It can be proved that the projection map $p : C \subset M \times \mathbb{C}^3 \to M$ is a finite branched covering of degree μ. The branching points are precisely $M \setminus M_{ss}$.*

Using the canonical coordinates we can construct a trivialization of the tangent bundle

$$\Psi : M_0 \times \mathbb{C}^{\mu} \cong TM_0, \quad e_i \mapsto \sqrt{\Delta_i}\,\frac{\partial}{\partial u_i},$$

where $M_0 \subset M_{\rm ss}$ is an open contractible neighborhood of t_0 and $\{e_i\}$ is the standard basis of $\mathbb{C}^{\mu}$. According to Givental (see [29]), there exists a unique formal asymptotic series $\Psi_t R_t(z) e^{U_t/z}$, where

$$R_t(z) = 1 + R_1(t)z + R_2(t)z^2 + \cdots, \tag{23}$$

and $R_k(t)$ are linear operators on $\mathbb{C}^N$, that satisfies the same differential equations as the oscillatory integrals $J_{\mathcal{A}}$.

We will make use of the following formal series

$$\mathbf{f}_{\alpha}(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k)}_{\alpha}(t,\lambda)\,(-z)^k\,, \alpha \in \mathfrak{h}. \tag{24}$$

Example 12. Note that for A_1-singularity $F(t,x) = x^2/2+t$ we have $u := u_1(t) = t$. Up to a sign there is a unique vanishing cycle. The series (24) will be denoted simply by $\mathbf{f}_{A_1}(t,\lambda;z)$. The corresponding period vectors can be computed explicitly and they are given by the following formulas:

$$I^{(k)}_{A_1}(u,\lambda) = (-1)^k\,\frac{(2k-1)!!}{2^{k-1/2}}\,(\lambda-u)^{-k-1/2},\quad k\geq 0$$

$$I^{(-k-1)}_{A_1}(u,\lambda) = 2\,\frac{2^{k+1/2}}{(2k+1)!!}\,(\lambda-u)^{k+1/2},\quad k\geq 0.$$

The key lemma (see [28]) is the following.

Lemma 13. *Let $t \in M_{\rm ss}$ and β be a vanishing cycle vanishing over the point $(t, u_i(t))$. Then for all λ near $u_i := u_i(t)$, we have*

$$\mathbf{f}_\beta(t,\lambda;z) = \Psi_t R_t(z)\, e_i\, \mathbf{f}_{A_1}(u_i,\lambda;z)\,.$$

An important corollary of Lemma 13 is the following remarkable formula due to K. Saito (see [54]):

$$(\alpha|\beta) = (I^{(0)}_\alpha(t,\lambda), (\lambda - E\bullet) I^{(0)}_\beta(t,\lambda)). \tag{25}$$

To prove this formula, first note that the the differential equations (19)–(21) imply that the RHS is independent of t and λ. In order to compute the RHS, let us fix $t\in M_{\rm ss}$ and let λ approach one of the critical values $u_i(t)$ in such a way that the cycle β vanishes over $(t, u_i(t))$. According to Lemma 13 we have

$$I^{(0)}_\beta(t,\lambda) = 2(2(\lambda-u_i))^{-1/2} e_i + O((\lambda-u_i)^{1/2}).$$

Similarly, decomposing $\alpha = \alpha' + (\alpha|\beta)\beta/2$, where α' is invariant with respect to the local monodromy, we get

$$I^{(0)}_\alpha(t,\lambda) = (\alpha|\beta)\,(2(\lambda-u_i))^{-1/2} e_i + O((\lambda-u_i)^{1/2}).$$

It is well known (see [17]) that in canonical coordinates the Euler vector field has the form $E = \sum u_i \partial_{u_i}$. Now it is easy to see that the RHS of (25), up to higher order terms in $(\lambda - u_i)$ is $(\alpha|\beta)$ and since the latter must be independent of λ the higher-order terms must vanish.

3.4. **The calibration operator.** The *calibration* of the Frobenius structure on H is by definition a gauge transformation S of the form

$$S_t(z) = 1 + \sum_{k=1}^\infty S_k(t) z^{-k},\quad S_k(t)\in \mathrm{End}(H), \tag{26}$$

such that $\nabla = S d S^{-1}$. In Gromov–Witten theory there is a canonical choice of calibration given by genus-0 descendant invariants as follows (see [30]):

$$(S_t(z)\phi_i, \phi_j) = (\phi_i,\phi_j) + \sum_{k=0}^\infty \langle \phi_i\psi^k, \phi_j\rangle_{0,2}(t) z^{-k-1}.$$

Here

$$\langle \phi_i\psi^k, \phi_j\rangle_{0,2}(t) = \sum_{m,d} \frac{Q^d}{m!} \langle \phi_i\psi^k, \phi_j, t, ..., t\rangle_{0,2+m,d}.$$

It is a general fact in GW theory (see [30]) that

$$S_t(z)^{-1}\Big(\partial_z - z^{-1}\theta + z^{-2}E\bullet\Big)S_t(z) = \partial_z - z^{-1}\theta + z^{-2}\rho, \tag{27}$$

where $\rho = c_1(T\mathbb{P}^1_{\mathbf{a}})\cup = \chi\, P\cup$, where for a cohomology class $A \in H$ we denote by $A\cup$ the operator of orbifold cup product multiplication by A.

We define a new series

$$\widetilde{\mathbf{f}}_\alpha(\lambda; z) := S_t(z)^{-1}\, \mathbf{f}_\alpha(t, \lambda; z). \tag{28}$$

Note that the RHS is independent of t. Put

$$\widetilde{\mathbf{f}}_\alpha(\lambda; z) = \sum_{n\in\mathbb{Z}} \widetilde{I}^{(n)}_\alpha(\lambda)\,(-z)^n. \tag{29}$$

We will refer to $\widetilde{I}^{(n)}_\alpha(\lambda)$ as the *calibrated* limit of the period vector $I^{(n)}_\alpha(t,\lambda)$.

In our general set up the Novikov variable Q is a fixed non-zero constant. However, it will be useful also to allow Q to vary in a small contractible neighborhood and to study the dependence of the periods and their calibrated limits on Q. By definition $I^{(n)}_\alpha(t,\lambda)$ depend on $Qe^{t_{02}}$, so we simply have

$$Q\partial_Q\, I^{(n)}_\alpha(t,\lambda) = \partial_{t_{02}}\, I^{(n)}_\alpha(t,\lambda).$$

Using the *divisor equation* in Gromov–Witten theory, it is easy to prove (c.f. [30]) that the gauge transformation $S_t(z)$ satisfies the following differential equation:

$$zQ\partial_Q\, S_t(z) = z\partial_{t_{02}} S_t(z) - S_t(z)\,(P\cup).$$

Finally, the gauge identity $\nabla = SdS^{-1}$ and the differential equations (19)–(21) imply that the calibrated limit of the period vectors satisfy the following system of differential equations:

$$Q\partial_Q \widetilde{I}^{(n)}_\alpha(\lambda) = -P\cup \widetilde{I}^{(n+1)}_\alpha(\lambda) \tag{30}$$

$$\partial_\lambda \widetilde{I}^{(n)}_\alpha(\lambda) = \widetilde{I}^{(n+1)}_\alpha(\lambda), \tag{31}$$

$$(\lambda - \rho\,\cup)\partial_\lambda \widetilde{I}^{(n)}_\alpha(\lambda) = \Big(\theta - n - 1/2\Big)\widetilde{I}^{(n)}_\alpha(\lambda). \tag{32}$$

Lemma 14. *The following formula holds*

$$\widetilde{I}^{(-1)}_\alpha(\lambda) = \langle B_{01}, \alpha\rangle\Big(\lambda + (\chi\log\lambda - \log Q)\,P\Big) + \langle B_{02},\alpha\rangle\,P + \sum_{i\in\mathfrak{I}_{\rm tw}} \langle B_i,\alpha\rangle\,\lambda^{d_i}\,\phi_i,$$

where $\{B_i\}_{i\in\mathfrak{I}}$ *is a basis of* $\mathfrak{h}^\vee := H^2(X_{0,1};\mathbb{C})$.

Proof. By definition the operator ρ acts on H as follows

$$\rho(\phi_{01}) = \chi\phi_{02}, \quad \rho(\phi_i) = 0, \quad \text{for } i\neq(0,1),$$

while the Hodge grading operator has the form

$$\theta(\phi_i) = (d_i - 1/2)\,\phi_i, \quad i\in\mathfrak{I}.$$

Note that the H-valued functions that follow the pairings $\langle B_i, \alpha\rangle$ are solutions to the system (30)–(32) with $n = -1$. These solutions are linearly independent, therefore they must give a basis in the space of all solutions. The lemma follows. $\square$

An immediate corollary of Lemma 14 is that the periods $\widetilde{I}^{(n)}_\alpha(\lambda)$ are multi-valued analytic functions on $\mathbb{C} \setminus \{0\}$. In order to keep track of the multi-valuedness let us fix a ray in $\mathbb{C}$ starting at $\lambda = 1$. For every $t \in M$ and a point λ_0 on the ray, we construct a path in $M \times \mathbb{C}$ connecting the reference point $(0, 1)$ with (t, λ_0) consisting of two straight line segments: one from $(0, 1)$ to $(0, \lambda_0)$ and another one from $(0, \lambda_0)$ to (t, λ_0). If $|\lambda_0| \gg 1$, then the path does not intersect the discriminant and the values of the period vectors $I^{(n)}_\alpha(t, \lambda)$ are uniquely fixed for all λ sufficiently close to λ_0. Note that the series

$$I^{(n)}_\alpha(t,\lambda) = \widetilde{I}^{(n)}_\alpha(\lambda) + \sum_{k=1}^{\infty} (-1)^k S_k(t) \widetilde{I}^{(n+k)}_\alpha(\lambda)$$

must be convergent for $|\lambda| \gg 1$, because it is a solution to the differential equation (21), which has a regular singular point at $\lambda = \infty$. In other words, the monodromy of the functions $\widetilde{I}^{(n)}_\alpha(\lambda)$ when analytically continued along a loop around $\lambda = 0$ coincides with the monodromy transformation $\sigma \in W$ of $\mathfrak{h}$ corresponding to a loop that goes around the discriminant, i.e., σ is the classical monodromy. Let $\sigma^* : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$ be the induced transformation on the dual space $\mathfrak{h}^\vee := H^2(X_{0,1}; \mathbb{C})$, i.e.,

$$\langle \sigma^* B, \beta\rangle = \langle B, \sigma(\beta)\rangle.$$

Note that the induced action of W on $\mathfrak{h}^\vee$ is a *right* action.

Lemma 15. *The vectors $\{B_i\}_{i\in\mathfrak{I}}$ provide a Jordan basis for the classical monodromy*

$$\begin{aligned} \sigma^*(B_{01}) &= B_{01}, \\ \sigma^*(B_{02}) &= B_{02} + 2\pi\sqrt{-1}\chi B_{01}, \\ \sigma^*(B_i) &= e^{2\pi\sqrt{-1}\, d_i} B_i, \quad i \in \mathfrak{I}_{\rm tw}. \end{aligned}$$

Proof. The analytical continuation of $\widetilde{I}^{(-1)}_\alpha(\lambda)$ around $\lambda = 0$ in a counter-clockwise direction is $\widetilde{I}^{(-1)}_{\sigma(\alpha)}(\lambda)$. The lemma follows from Lemma 14. $\square$

4. The calibrated periods

In this section we will use the period mapping to embed the root system $X_N^{(1)}$ of vanishing cycles in the quantum cohomology. The cohomology classes ϕ_i provide a Jordan basis of the classical monodromy. Following ideas of Steinberg, we will express the classical monodromy in terms of an automorphism of the finite Weyl group of type X_N. This alows us to express the period vectors $\widetilde{I}^{(n)}_\alpha(\lambda)$ (we call them *calibrated periods*) in terms of the finite root system X_N.

4.1. **The period maps.** Recall the period vectors $\widetilde{I}^{(n)}_\alpha(\lambda)$ introduced in equation (29) in Section 3.4. We will be interested in the two maps from the sequence

$$\widetilde{I}^{(n)}(1) : \mathfrak{h} \to H, \quad \alpha \mapsto \widetilde{I}^{(n)}_\alpha(1) \tag{33}$$

corresponding to $n = -1$ and $n = 0$. According to Lemma 14 we have

$$\widetilde{I}^{(-1)}(1) = B_{01}\,(1 - \log Q\,P) + B_{02}\,P + \sum_{i\in\mathfrak{I}_{\rm tw}} B_i\,\phi_i.$$

In other words, the map for $n = -1$ is an isomorphism and the pre-images of vectors

$$b_{01} = 1 - \log Q\,P, \quad b_{02} = P, \quad b_i = \phi_i,$$

form a basis of $\mathfrak{h}$ dual to the Jordan basis $\{B_i\}$ of σ^*. In particular, $\{b_i\}$ give a Jordan basis for the classical monodromy σ. Using the isomorphism $\widetilde{I}^{(-1)}(1)$ we equip H with an intersection pairing $(\cdot|\cdot)$ and a root system

$$\Delta^{(-1)} \subset H.$$

Lemma 16. *In the basis $\{b_i\} \subset H$ the classical monodromy takes the form*

$$\sigma(b_{01}) = b_{01} - 2\pi\sqrt{-1}\,\chi\,b_{02}, \quad \sigma(b_{02}) = b_{02}, \quad \sigma(b_i) = e^{-2\pi\sqrt{-1}\,d_i} b_i.$$

The intersection form becomes

$$(b_{01}|b_{01}) = \chi, \quad b_{02} \in \operatorname{Ker}(\cdot|\cdot), \quad (b_i|b_j) = d_i d_{i^*}\delta_{i,j^*}/a_i.$$

Here $*$ is the involution on $\mathfrak{I}$ introduced in equation (5).

Proof. The first part of the lemma is just the dual statement from Lemma 15. The second one is a consequence of Saito's formula (25). □

The period map (33) with $n = 0$ has a 1-dimensional kernel. In fact, using (32) we get

$$\widetilde{I}^{(0)}(1) = (1-\rho)^{-1}(\theta + 1/2)\widetilde{I}^{(-1)}(1) = (1+\rho)(\theta+1/2)\widetilde{I}^{(-1)}(1).$$

We denote the image of $\widetilde{I}^{(0)}(1)$ by $H^{(0)}$. Let

$$\Delta^{(0)} \subset H^{(0)}$$

be the image of the root system. Let us denote by $r : H \to H^{(0)}$ the map defined by $\widetilde{I}^{(0)}(1) = r \circ \widetilde{I}^{(-1)}(1)$, i.e.,

$$r(b) = (1+\rho)(\theta + 1/2)(b).$$

Note that the intersection pairing on H takes the form

$$(a_1|a_2) = (r(a_1), (1-\rho)r(a_2)), \quad a_1, a_2 \in H.$$

It follows that we can pushforward the intersection form to a non-degenerate bilinear pairing on $H^{(0)}$, which we denote again by $(\cdot|\cdot)$. Moreover, r maps the affine root system $\Delta^{(-1)}$ to a finite root system, i.e., $\Delta^{(0)}$ is the finite root system obtained as the quotient of $\Delta^{(-1)}$ by the imaginary root translations.

4.2. **Spliting of the affine root system.** We will restrict only to the case when $\Delta^{(-1)}$ is of type $X_N^{(1)}$, $X = DE$. In that case the Dynkin diagram is a tree and the Coxeter number h of the corresponding finite root system X_N is an even number.

Let

$$\gamma_i^{(-1)}, \quad 0 \le i \le N$$

be a basis of simple roots in $\Delta^{(-1)}$ such that the Dynkin diagram is $X_N^{(1)}$. We will assume that $\gamma_0^{(-1)}$ is the *affine vertex*, i.e., the extra node that we have to attach to X_N in order to obtain $X_N^{(1)}$. Vectors $\gamma_i^{(0)} = r(\gamma_i^{(-1)})$, $1 \le i \le N$, form a basis of simple roots of $\Delta^{(0)}$. Let $W^{(0)}$ be the reflection group generated by $\gamma_i^{(0)}$. It is well known that the map

$$s_i^{(0)} := s_{\gamma_i^{(0)}} \mapsto s_i^{(-1)} := s_{\gamma_i^{(-1)}}, \quad 1 \le i \le N$$

induces a group embedding $W^{(0)} \to W$. Furthermore, for every $\alpha \in \Delta^{(0)}$, let us define a *lift* $\alpha^{(-1)} \in \Delta^{(-1)}$ as follows

$$\alpha = \sum_{i=1}^N a_i \gamma_i^{(0)} \quad \mapsto \quad \alpha^{(-1)} := \sum_{i=1}^N a_i \gamma_i^{(-1)}.$$

Then the root system $\Delta^{(-1)}$ coincides with the set

$$\left\{ \alpha^{(-1)} + n\,\delta \mid \alpha \in \Delta^{(0)},\ n \in \mathbb{Z} \right\},$$

where $\delta = \gamma_0^{(-1)} + \theta^{(-1)}$ and $\theta \in \Delta^{(0)}$ is the highest root with respect to the basis $\{\gamma_i^{(0)}\}_{i=1}^N$ (see [41]). Following Kac, we will refer to $n\,\delta$ $(n \in \mathbb{Z})$ as *imaginary roots*. Finally, let us denote by

$$\Lambda^{(-1)} := H_2(X_{0,1}; \mathbb{Z})$$

the root lattice of $\Delta^{(-1)}$. Given $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 := (\alpha|\alpha) \neq 0$, recall that the reflection with respect to α is defined by

$$s_\alpha(x) = x - 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\alpha.$$

We also define the following translation:

$$t_\alpha(x) := s_{\alpha+\delta} s_\alpha(x) = x + 2\frac{(\alpha|x)}{(\alpha|\alpha)}\,\delta.$$

This definition induces a group embedding $t : \Lambda^{(0)} \to W$. Recall that $w\,s_\alpha\,w^{-1} = s_{w(\alpha)}$ for all $w \in W$ and $\alpha \in \Lambda^{(-1)}$ such that $|\alpha|^2 \neq 0$. Therefore, $\Lambda^{(0)}$ is a normal subgroup of W and we have an isomorphism

$$W \cong \Lambda^{(0)} \rtimes W^{(0)}.$$

Let us emphasize that the above isomorphism is not canonical – it depends on the choice of a basis of simple roots of $\Delta^{(-1)}$.

We would like to choose a basis of simple roots in $\Delta^{(-1)}$ such that the classical monodromy σ takes a very elegant form. The Dynkin diagram X_N $(X = DE)$ has a unique branching node $\gamma_b^{(-1)}$. After deleting $\gamma_b^{(-1)}$ and all edges incident with $\gamma_b^{(-1)}$ we get 3 Dynkin diagrams of type $A_{a_\mu - 1}$, $\mu = 1, 2, 3$. Let us relabel the nodes of X_N via $\gamma_{\mu,i}^{(-1)}$, where $\mu = 1, 2, 3$ enumerates the 3 branches and i enumerates the nodes of the corresponding branch starting from the node closest to the branching node (see Figure 1). Put

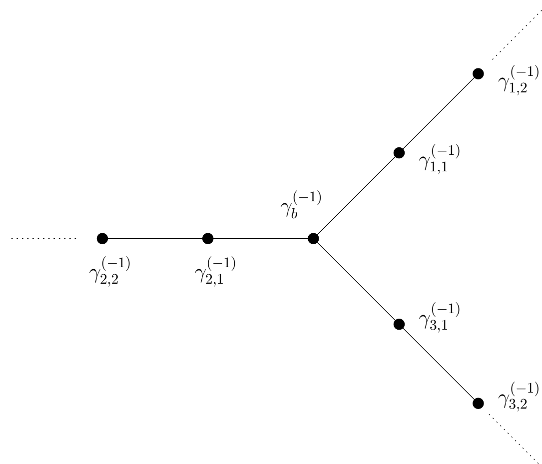


FIGURE 1. The branching node

$$\sigma_b = \prod_{\mu=1}^{3} \Big(\cdots s_{\mu,2}^{(-1)} s_{\mu,1}^{(-1)} \Big), \tag{34}$$

where the order of the reflections that enter each factor of the above product is important, but the order in which the 3 factors are arranged is irrelevant since any two reflections associated with different branches of X_N commute. Following the ideas of Steinberg [57], we have

Proposition 17. *There exists a basis $\{\gamma_i^{(-1)}\}_{i=0}^N$ of simple roots such that $\sigma = t_{\gamma_b^{(-1)}}\, \sigma_b$.*

For the reader's convenience, we put a proof of this proposition in Appendix B.

4.3. **The defect of σ.** From now on we assume that $\{\gamma_i^{(-1)}\}_{i=0}^N$ is a basis of simple roots of $\Delta^{(-1)}$ so that the classical monodromy is given by $\sigma = t_b\, \sigma_b$, where for brevity we put $t_b = t_{\gamma_b^{(-1)}} = t_{\gamma_b^{(0)}}$ and σ_b is given by (34). Let

$$\omega_i^{(-1)} \in H^\vee, \quad 0 \le i \le N$$

be the fundamental weights, i.e.,

$$\Big\langle \omega_i^{(-1)}, \gamma_j^{(-1)} \Big\rangle = \delta_{i,j}, \quad 0 \le i, j \le N.$$

We identify the dual space of $H^{(0)}$ with $H^{(0)}$ via the intersection pairing. Let $\omega_i^{(0)} \in H^{(0)}$ be the fundamental weights of the finite root system corresponding to the simple roots $\gamma_i^{(0)}, 1 \leq i \leq N$, i.e.,

$$\left(\omega_i^{(0)}|\gamma_j^{(0)}\right) = \delta_{i,j}, \quad 1 \leq i, j \leq N.$$

Recall the Jordan basis $\{b_i\}$ of σ from Section 4.1 (see Lemma 16). Then we can write every $\alpha \in H$ as

$$\alpha = \sum_{i \in \mathfrak{I}} \langle b^i, \alpha \rangle \, b_i, \tag{35}$$

where $\{b^i\}_{i\in\mathfrak{I}}$ is a basis of $H^\vee$ dual to $\{b_i\}_{i\in\mathfrak{I}}$. Using Lemma 16 we get

Lemma 18.

$$\begin{aligned} \langle b^{01}, \alpha \rangle &= (r(\alpha)|h_1)/\chi, \\ \langle b^{i}, \alpha \rangle &= (r(\alpha)|h_{i^*})a_i/(d_i d_{i^*}), \quad i \in \mathfrak{I}_{\rm tw} \end{aligned}$$

where

$$h_i := r(b_i), \quad i \in \mathfrak{I} \setminus \{(0,2)\}.$$

Let $\gamma_b^{(-1)}$ be the branching node and let k_i $(1 \leq i \leq N)$ be the so called *Kac labels*, i.e., the positive integers that appear in the decomposition of the highest root

$$\theta = k_1\gamma_1^{(0)} + \cdots + k_N\gamma_N^{(0)}.$$

For $\alpha \in H^{(0)}$ let us denote by $r^*\alpha \in H^\vee$ the linear functional

$$\langle r^*\alpha, x\rangle := (\alpha|r(x)), \quad x \in H.$$

Lemma 19. *In the above notation we have $h_1 = \pm\chi\,\omega_b^{(0)}$ and*

$$\sigma^*(\omega_b^{(-1)}) = \omega_b^{(-1)} + k_b\, r^*\sigma_b^*(\gamma_b^{(0)}).$$

Proof. Since $\{h_i\}_{i\in\mathfrak{I}\setminus\{(0,2)\}}$ form an eigenbasis for σ_b we see that the subspace of fixed points of σ_b is 1-dimensional. On the other hand, $\omega_b^{(0)}$ is a fixed point of σ_b, so $\omega_b^{(0)}$ must be proportional to h_1. It remains only to compare the length squares

$$(h_1|h_1) = \chi, \quad (\omega_b^{(0)}|\omega_b^{(0)}) = 1/\chi,$$

where the second identity was workout in [18], or it can be verified on a case by case basis directly from the Dynkin diagrams. The second statement of the Lemma follows from a direct computation, using that $\sigma^* = \sigma_b^*\, t_b^*$ and $\delta = \gamma_0^{(-1)} + \sum k_i\gamma_i^{(-1)}$. □

In the next proposition we will prove a formula for the *defect* of σ, which is the linear form Δ that appears in the identity

$$(1-\sigma^m)(x) = \langle \Delta, x\rangle\, \delta,$$

where m is the order of the image of $\sigma \in W$ in $W^{(0)}$ under the quotient map $H \to H^{(0)} \cong H/\langle\delta\rangle$. The importance of the notion of a defect was discovered by Dlab and Ringel in the representation theory of quivers (see [16]).

Proposition 20. *Let $m=|\sigma_b|$ be the order of σ_b. Then for $\alpha\in H$, we have*

$$(1-\sigma^m)(\alpha)=\pm m\chi\,(\omega_b^{(0)}|r(\alpha))\,2\pi\sqrt{-1}\,P. \tag{36}$$

The imaginary root $\delta=\mp 2\pi\sqrt{-1}\,P$. So the defect is $-r^*(\omega_b^{(0)})$.

Proof. The first part of the proposition is easy to prove: decompose α into a sum of eigenvectors as in (35), recall Lemma 16 and the formula $h_1=\pm\chi\,\omega_b^{(0)}$ (see Lemma 19). The difficult part is to compute the defect of σ. Using Lemma 19 we get

$$(\sigma^m)^*\omega_b^{(-1)}=\omega_b^{(-1)}+k_b r^*(\sigma_b+\cdots+\sigma_b^m)^*(\gamma_b^{(0)}).$$

By expressing $\gamma_b^{(0)}$ using the eigenbasis $\{h_i\}$ we see that only the h_1 component contributes to the RHS. On the other hand, the h_1-component is

$$\langle b^{01},\gamma_b^{(-1)}\rangle=\pm(\gamma_b^{(0)}|\omega_b^{(0)})=\pm 1,$$

where we used that $h_1=\pm\chi\omega_b^{(0)}$. In other words,

$$(1-\sigma^m)^*(\omega_b^{(-1)})=-(m\chi)k_b\,r^*(\omega_b^{(0)}), \tag{37}$$

where we used again that $h_1=\pm\chi\omega_b^{(0)}$. Let us compare formulas (36) and (37) by pairing them with $\omega_b^{(-1)}$ and α respectively:

$$\pm m\chi\,(\omega_b^{(0)}|r(\alpha))\,2\pi\sqrt{-1}\,\langle\omega_b^{(-1)},P\rangle=-(m\chi)k_b\,(\omega_b^{(0)}|r(\alpha)).$$

Since P and δ are in the 1-dimensional kernel of the intersection form, they must be proportional. It remains only to notice that

$$\pm\langle\omega_b^{(-1)},2\pi\sqrt{-1}\,P\rangle=-k_b=-\langle\omega_b^{(-1)},\delta\rangle.$$

□

4.4. **The toroidal cycle.** Let $\Gamma_\varepsilon\subset\mathbb{C}^3$ be the torus

$$\Gamma_\epsilon:=\{|x_1|=|x_2|=1,|x_3|=\varepsilon\}.$$

If ε is sufficiently big, Γ_ε does not intersect the Milnor fiber $X_{0,1}$. Hence we have a well-defined cycle

$$[\Gamma_\varepsilon]\in H_3(\mathbb{C}^3\setminus X_{0,1};\mathbb{Z})\cong H_2(X_{0,1};\mathbb{Z}),$$

where the isomorphism is given by the so called *tube mapping* (for more details see [32]). Let us denote by φ the image of $[\Gamma_\varepsilon]$ under the above isomorphism.

Proposition 21. *We have $I_\varphi^{(-1)}(t,\lambda)=2\pi\sqrt{-1}\,P$.*

Proof. Increasing ε does not change the homology class $[\Gamma_\varepsilon]$, therefore by choosing $\varepsilon\gg 0$ we may arrange that Γ_ε does not intersect the Milnor fiber $X_{t,\lambda}$ for all (t,λ) sufficiently close to $(0,1)$. In particular, the cycle $\varphi_{t,\lambda}$ obtained from φ via a parallel transport with respect to the Gauss–Manin connection coincides with the image of $[\Gamma_\varepsilon]$ via the tube mapping. We have (c.f. [32])

$$I(t,\lambda,Q):=\int_{[\Gamma_\varepsilon]}\frac{\omega}{F(t,x)-\lambda}=2\pi\sqrt{-1}\int_{\varphi_{t,\lambda}}\frac{\omega}{dF}=2\pi\sqrt{-1}\,\partial_\lambda\int d^{-1}\omega. \tag{38}$$

Comparing with the definition (18) we get

$$I(t,\lambda,Q) = -(2\pi)^2\sqrt{-1}\,(I^{(-1)}_\varphi(t,\lambda),1).$$

Using the differential equation (21), we get

$$(\lambda\partial_\lambda + E)I(t,\lambda,Q) = 0. \tag{39}$$

The integral $I(t,\lambda,Q)$ is analytic at $(t,\lambda,Q)=(0,0,0)$ because it has the form

$$\sqrt{-1}\int_{[\Gamma_\varepsilon]} \frac{dx_1dx_2dx_3}{Qe^{t_{02}}\,(G(t,x)-\lambda) - x_1x_2x_3}\,,$$

where $G(t,x)$ is a holomorphic function in t and x. However, equation (39) means that $I(t,\lambda,Q)$ is homogeneous of degree 0 and since the weights of all variables are positive, the integral must be a constant. In particular, we may set $t=Q=\lambda=0$, which gives

$$I(t,\lambda,Q) = -\sqrt{-1}\int_{[\Gamma_\varepsilon]}\frac{dx_1dx_2dx_3}{x_1x_2x_3} = (2\pi)^3.$$

Note that equation (38) implies that $I^{(0)}_\varphi(t,\lambda)=0$. Recalling again the differential equation (21), we get

$$I^{(-1)}_\varphi(t,\lambda) = (I^{(-1)}_\varphi(t,\lambda),1)\,P = (2\pi)^{-2}\sqrt{-1}\,I(t,\lambda,Q)\,P = 2\pi\sqrt{-1}\,P.$$

□

4.5. **The linear form b^{02}.** The linear forms $b^i\in H^\vee$, $i\in\mathfrak{I}$ are the images of $B_i\in\mathfrak{h}^\vee$ (see Lemma 14) under the period isomorphism

$$\widetilde{I}^{(-1)}:\mathfrak{h}\to H.$$

In Section 4.3 we have expressed b^i, $i\in\mathfrak{I}\setminus\{(0,2)\}$ in terms of the finite root system X_N, see Lemma 18. We need to do something similar with b^{02} as well. Recall the notation $\gamma^{(-1)}_{\mu,i}$ for the simple roots associated with the branches of the Dynkin diagram X_N (c.f. Figure 1).

Proposition 22. *The following formulas hold*

$$\langle\, b^{02}\,,\delta\rangle = \pm 2\pi\sqrt{-1},\quad \langle\, b^{02}\,,\gamma^{(-1)}_{\mu,i}\,\rangle = \pm 2\pi\sqrt{-1}\Big(\delta_{i,1}-\frac{1}{a_\mu}\Big).$$

Proof. We will use that if $x=(1-\sigma)y$, then according to Lemma 15 we have

$$\langle\, b^{02}\,,x\rangle = -2\pi\sqrt{-1}\,\chi\,\langle\, b^{01}\,,y\rangle = \mp 2\pi\sqrt{-1}\,\chi(\omega^{(0)}_b|r(y)). \tag{40}$$

We now analyze the image of $1-\sigma$. By definition

$$\sigma(x) = t_b(\sigma_b(x)) = \sigma_b(x) + (\gamma^{(-1)}_b|\sigma_b(x))\delta.$$

By definition $\sigma_b = \prod_{\mu=1}^3 \sigma_\mu^{(0)}$. The action of $\sigma_\mu^{(0)}$ on the subspace with basis $\{\gamma_{\mu,1}^{(-1)}, \dots, \gamma_{\mu,a_\mu-1}^{(-1)}\}$ is represented by the following matrix

$$\sigma_\mu^{(0)} = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{41}$$

We also have

$$\sigma_b(b_{01}) = b_{01}, \quad \sigma_b^{-1}(\gamma_b^{(-1)}) = \gamma_b^{(-1)} + \gamma_{1,1}^{(-1)} + \gamma_{2,1}^{(-1)} + \gamma_{3,1}^{(-1)}.$$

The map $1-\sigma$ induces a linear map between the two subspaces of H with bases respectively

$$\{\gamma_{\mu,i}^{(-1)}\,,\, b_{01} | 1 \le \mu \le 3, 1 \le i \le a_\mu - 1\}$$

and

$$\{\gamma_{\mu,i}^{(-1)}\,,\, \delta | 1 \le \mu \le 3, 1 \le i \le a_\mu - 1\}.$$

The corresponding matrix (in block form) is

$$\begin{bmatrix} 1-\sigma_1^{(0)} & 0 & 0 & 0 \\ 0 & 1-\sigma_2^{(0)} & 0 & 0 \\ 0 & 0 & 1-\sigma_3^{(0)} & 0 \\ c_1 & c_2 & c_3 & \mp\chi \end{bmatrix}$$

where the subrows c_μ are $(-1,1,0,\dots,0)$. The inverse of the above matrix has last row of the form $(c_1', c_2', c_3', \mp 1/\chi)$, where

$$\mp\chi\, c_\mu' = (1-1/a_\mu, -1/a_\mu, \dots, -1/a_\mu).$$

To prove the Proposition, it remains only to apply formula (40) and notice that since $\langle \omega_b^{(0)}, \gamma_{\mu,i}^{(0)} \rangle = 0$, only the last row of the inverse matrix contributes to the RHS. □

4.6. **The calibrated periods.** Let $\beta \in \Delta$ be an arbitrary vanishing cycle. Put

$$\beta^{(-1)} := \widetilde{I}_\beta^{(-1)}(1) \in \Delta^{(-1)} \quad \text{and} \quad \alpha := \pm r(\beta^{(-1)}) = \pm \widetilde{I}_\beta^{(0)}(1) \in \Delta^{(0)},$$

where the sign is the same as in Lemma 19, i.e.,

$$\omega_b^{(0)} = \pm h_1/\chi = \pm\Big(\frac{1}{\chi} + P\Big).$$

Let κ be a positive constant whose value will be specified later on. Put

$$H_{01} := H_{02} := \pm(\kappa/\chi)^{\frac12}\, h_1, \quad H_i := \pm(\kappa\, a_i)^{\frac12}\, h_i/d_i, \quad i \in \mathfrak{I}_{\rm tw}.$$

Note that $\{H_i\}_{i\in\mathfrak{I}}$ is a σ_b-eigenbasis of $H^{(0)}$ with $\sigma_b(H_i) = e^{-2\pi\sqrt{-1}d_i} H_i$ in which the intersection form takes the form

$$(H_i|H_j) = \kappa\, \delta_{i,j^*}, \quad i,j \in \mathfrak{I}. \tag{42}$$

By definition

$$\beta^{(-1)} = \langle \omega_0^{(-1)}, \beta^{(-1)} \rangle\, \delta \pm \sum_{i=1}^{N} (\omega_i^{(0)} | \alpha)\, \gamma_i^{(-1)}.$$

We get the following identities, using Lemma 18 and Proposition 22:

$$\langle B_{01}, \beta \rangle = \langle b^{01}, \beta^{(-1)} \rangle = (\omega_b^{(0)} | \alpha),$$

$$\langle B_i, \beta \rangle = \langle b^i, \beta^{(-1)} \rangle = \pm (h_{i^*} | \alpha) a_i / (d_i d_{i^*}) = (H_{i^*} | \alpha)\, (a_i/\kappa)^{\frac{1}{2}}\, \frac{1}{d_i},$$

and

$$\langle B_{02}, \beta \rangle = \langle b^{02}, \beta^{(-1)} \rangle = (\omega_b^{(0)} | \alpha)\, c + 2\pi\sqrt{-1}\Big((\rho_b | \alpha) + n \Big),$$

where $c = \pm \langle b^{02}, \gamma_b^{(-1)} \rangle$,

$$n = \pm \langle \omega_0^{(-1)}, \beta^{(-1)} \rangle + \sum_{\mu=1}^{3} (\omega_{\mu,1}^{(0)} | \alpha) \quad \in \quad \mathbb{Z},$$

and

$$\rho_b = -\sum_{\mu,i} \frac{1}{a_\mu} \omega_{\mu,i}^{(0)}.$$

Clearly, the map $\beta \mapsto (\alpha, n)$ is a one-to-one correspondence between the set Δ of vanishing cycles and $\Delta^{(0)} \times \mathbb{Z}$, so we can use the latter set to parameterize vanishing cycles. Recalling Lemma 14 we get

$$\begin{aligned} \widetilde{I}_\beta^{(-1)}(\lambda) \;=\; & (\alpha | \omega_b^{(0)})\, \lambda + (\alpha | \omega_b^{(0)}) \chi (\log \lambda - C_0)\, P + 2\pi\sqrt{-1}(n + (\rho_b | \alpha))\, P + \\ & \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} (\alpha | H_{i^*})\, \sqrt{a_i/\kappa}\, \frac{\lambda^{d_i}}{d_i}\, \phi_i, \end{aligned}$$

where $C_0 = \frac{1}{\chi}(-c + \log Q)$, with c some constant independent of Q. From here we find, using equations (19), (20), (21), that the remaining periods are:

$$\widetilde{I}_\beta^{(l)}(\lambda) = (-1)^l l! (\alpha | \omega_b^{(0)}) \chi \lambda^{-l-1} P + \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} (\alpha | H_{i^*}) (d_i - 1) \cdots (d_i - l) \lambda^{d_i - l - 1} \sqrt{a_i/\kappa} \phi_i,$$

$$\widetilde{I}_\beta^{(0)}(\lambda) = (\alpha | \omega_b^{(0)}) + (\alpha | \omega_b^{(0)}) \chi \lambda^{-1} P + \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} (\alpha | H_{i^*}) \lambda^{d_i - 1} \sqrt{a_i/\kappa} \phi_i,$$

$$\begin{aligned} \widetilde{I}_\beta^{(-1-l)}(\lambda) = & (\alpha | \omega_b^{(0)}) \frac{\lambda^{l+1}}{(l+1)!} + \Big((\alpha | \omega_b^{(0)}) \chi \frac{\lambda^l}{l!} (\log \lambda - C_l) + 2\pi\sqrt{-1}(n + (\rho_b | \alpha)) \frac{\lambda^l}{l!} \Big) P + \\ & \sum_{i \in \mathfrak{I}_{\mathrm{tw}}} (\alpha | H_{i^*})\, \sqrt{a_i/\kappa}\, \frac{\lambda^{d_i + l}}{d_i (d_i + 1) \cdots (d_i + l)}\, \phi_i, \end{aligned}$$

where $l \geq 1$ and C_l ($l \geq 1$) are constants defined recursively by $C_l = C_{l-1} + \frac{1}{l}$.

Remark 23. *Unfortunately, we do not have a closed formula for the constant c. Nevertheless, Theorem 1 and the small quantum cohomology of* $\mathbb{P}^1_{\mathbf{a}}$ *allow us to compute c on a case by case basis.*

Remark 24. *It is natural to speculate that the root systems in the quantum cohomology of* $\mathbb{P}^1_{\mathbf{a}}$ *studied here are related to the integral structure in quantum cohomology introduced in* [37] *and* [44]*. We plan to study this in the near future.*

5. Realization of the basic representation via periods

Let $\mathfrak{g}^{(0)}$ be a simple Lie algebra of type ADE with an invariant bilinear form $(\,|\,)$, normalized in such a way that all roots have length $\sqrt{2}$. By definition, the affine Kac–Moody algebra corresponding to $\mathfrak{g}$ is the vector space

$$\mathfrak{g} := \mathfrak{g}^{(0)}[t,t^{-1}] \oplus \mathbb{C}\,K \oplus \mathbb{C}\,d$$

equipped with a Lie bracket defined by the following relations:

$$[X\ t^n, Y\ t^m] := [X,Y]\ t^{n+m} + n\delta_{n,-m}(X\,|\,Y)K,$$
$$[d, X\ t^n] := n(X\ t^n), \quad [K, \mathfrak{g}^{(0)}] := 0,$$

where $X, Y \in \mathfrak{g}^{(0)}$.

We fix a Cartan subalgebra $\mathfrak{h}^{(0)} \subset \mathfrak{g}^{(0)}$ and let $\Delta^{(0)}$ be the root system of $\mathfrak{g}^{(0)}$, i.e.,

$$\mathfrak{g}^{(0)} = \oplus_{\alpha\in\Delta^{(0)}} \mathfrak{g}^{(0)}_\alpha.$$

Let us define σ_b to be the authomorphism given in equation (34). If the root system is of type A; then we choose any of the nodes to be a branching node and we have 2 instead of 3 branches.

5.1. Twisted realization of the affine Lie algebra.

The Lie algebra $\mathfrak{g}^{(0)}$ can be constructed in terms of the root system via the so-called *Frenkel–Kac* construction [25]. Let $\Lambda^{(0)} \subset \mathfrak{h}^{(0)}$ be the root lattice. There exists a bimultiplicative function

$$\epsilon : \Lambda^{(0)} \times \Lambda^{(0)} \to \{\pm 1\}$$

satisfying

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}, \quad \epsilon(\alpha,\alpha) = (-1)^{|\alpha|^2/2},$$

where $|\alpha|^2 := (\alpha|\alpha)$. The map $(\alpha,\beta) \mapsto \varepsilon(\sigma_b(\alpha), \sigma_b(\beta))$ is another bimultiplicative function satisfying the above properties. It is known that all bi-multiplicative functions of the above form are equivalent (see [42], Corollary 5.5). Hence there exists a function $\upsilon : \Lambda^{(0)} \to \{\pm 1\}$ such that

$$\upsilon(\alpha)\upsilon(\beta)\varepsilon(\alpha,\beta) = \upsilon(\alpha+\beta)\varepsilon(\sigma_b\alpha, \sigma_b\beta). \tag{43}$$

There exists a set of root vectors

$$A_\alpha \in \mathfrak{g}^{(0)}_\alpha \tag{44}$$

such that

$$\begin{aligned}[A_\alpha, A_{-\alpha}] &= \epsilon(\alpha,-\alpha)\alpha \\ [A_\alpha, A_\beta] &= \epsilon(\alpha,\beta)A_{\alpha+\beta}, \quad \text{if} \quad (\alpha|\beta) = -1 \\ [A_\alpha, A_\beta] &= 0, \quad \text{if} \quad (\alpha|\beta) \geq 0.\end{aligned}$$

We can extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$ as follows

$$\sigma_b(A_\alpha) = \upsilon(\alpha)^{-1} A_{\sigma_b(\alpha)}, \quad \alpha \in \Delta^{(0)}.$$

Let us denote by κ the order of the extended automorphism $\sigma_b : \mathfrak{g}^{(0)} \to \mathfrak{g}^{(0)}$. Clearly we have $\kappa = |\sigma_b|$ or $2|\sigma_b|$. Since $(\ |\ )$ is both $\mathfrak{g}^{(0)}$-invariant (with respect to the adjoint representation) and $W^{(0)}$-invariant, we have

$$(A_\alpha|A_{-\alpha}) := \epsilon(\alpha,-\alpha), \quad (A_\alpha|A_\beta) := (A_\alpha|H) = 0, \quad \forall \beta \neq -\alpha,\ H \in \mathfrak{h}^{(0)}.$$

Put $\eta = e^{2\pi\sqrt{-1}/\kappa}$. We extend the action of σ_b to the affine Lie algebra $\mathfrak{g}$ by

$$\sigma_b \cdot (X \otimes t^n) = \sigma_b(X) \otimes (\eta^{-1}t)^n, \quad \sigma_b \cdot K = K, \quad \sigma_b \cdot d = d.$$

Let

$$\mathfrak{g}^{\sigma_b} \subset \mathfrak{g}$$

be the Lie subalgebra of σ_b-fixed points. According to Kac (see [41], Theorem 8.6.) $\mathfrak{g}^{\sigma_b} \cong \mathfrak{g}$. Let us recall the isomorphism. The fixed points subspace $(\mathfrak{g}^{(0)})^{\sigma_b}$ contains a Cartan subalgebra $\widetilde{\mathfrak{h}}^{(0)}$. We have a corresponding decomposition into root subspaces

$$\mathfrak{g}^{(0)} = \widetilde{\mathfrak{h}}^{(0)} \oplus \bigoplus_{\widetilde{\alpha} \in \widetilde{\Delta}^{(0)}} \mathfrak{g}^{(0)}_{\widetilde{\alpha}},$$

where $\widetilde{\Delta}^{(0)} \subset \widetilde{\mathfrak{h}}^{(0)}$ are the corresponding roots. Note that since the root subspaces are 1 dimensional, they must be eigen-subspaces of σ_b. Therefore, by choosing a set of simple roots $\widetilde{\alpha}_i$, $i = 1, 2, \ldots, N$ in $\widetilde{\Delta}^{(0)}$ we can uniquely define an integral vector $s = (s_1, \ldots, s_N)$, $0 \leq s_i < \kappa$ such that the eigenvalue of the eigensubspace $\mathfrak{g}^{(0)}_{\widetilde{\alpha}_i}$ is η^{s_i}. Put

$$\rho_s : \widetilde{\mathfrak{h}}^{(0)} \to \widetilde{\mathfrak{h}}^{(0)}, \quad \rho_s = \sum_{i=1}^N s_i \widetilde{\omega}_i,$$

where $\widetilde{\omega}_i \in \widetilde{\mathfrak{h}}^{(0)}\, (1 \leq i \leq N)$ are the fundamental weights corresponding to the simple roots $\widetilde{\alpha}_i\, (1 \leq i \leq N)$, i.e., $(\widetilde{\omega}_i|\widetilde{\alpha}_j) = \delta_{ij}$. The isomorphism

$$\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^{\sigma_b}$$

is defined as follows

$$\Phi(Xt^n) = t^{n\kappa + \mathrm{ad}_{\rho_s}} X + \delta_{n,0}\, (\rho_s|X) K \tag{45}$$

$$\Phi(K) = \kappa\, K$$

$$\Phi(d) = \kappa^{-1}\Big(d - \rho_s - \frac{1}{2}(\rho_s|\rho_s)\, K\Big), \tag{46}$$

where

$$t^{\mathrm{ad}_{\rho_s}} X = \exp\Big(\log t\ \mathrm{ad}_{\rho_s}\Big) X.$$

Note that the RHS is single-valued in t and σ_b-invariant in X, because

$$\exp\Big(2\pi\sqrt{-1}\mathrm{ad}_{\rho_s/\kappa}\Big) = \sigma_b.$$

5.2. **The Kac–Peterson construction.** Following [43], we would like to recall the realization of the basic level 1 representation of the affine Lie algebra $\mathfrak{g}$ corresponding to the automorphism σ_b. The idea is to construct a representation of the Lie algebra $\mathfrak{g}^{\sigma_b}$ on a Fock space, which induces via the isomorphism Φ the basic level-1 representation.

Consider again the situation studied in Sections 3 and 4. Fix an eigenbasis $\{H_i\}_{i\in\mathfrak{I}}$ of σ_b satisfying $(H_i|H_{j^*}) = \kappa\,\delta_{ij}$ (compare with (42)). It is convenient to put

$$m_{01} := 0, \quad m_{02} := \kappa, \quad m_i := d_{i^*}\,\kappa, \quad i \in \mathfrak{I}_{\mathrm{tw}},$$

where $e^{-2\pi\sqrt{-1}\,d_i} = \eta^{m_i}$ is the eigenvalue of H_i. The elements $H_{i,l} := H_i t^{m_i + l\kappa}$ ($i \in \mathfrak{I}, l \in \mathbb{Z}$) generate a *Heisenberg* Lie subalgebra

$$\mathfrak{s} \subset \mathfrak{g}^{\sigma_b},$$

i.e., the following commutation relations hold

$$[H_{i,l}, H_{j,m}] = (m_i + l\kappa)\,\delta_{i,j^*}\,\delta_{l+m,-1}\,\kappa\,K.$$

Let us also fix a $\mathbb{C}$-linear basis of $\mathfrak{s}$

$$H_0 := H_{01}, \quad H_{i,l}, \quad H_{i^*,-l-1}\ (l \geq 0, i \in \mathfrak{I}\backslash\{(0,1)\}), \quad K. \tag{47}$$

Let $\mathfrak{S}$ be the subgroup of the affine Kac–Moody Lie group generated by the lifts of the following loops:

$$h_{\alpha,\beta} = \exp\Big(\alpha \log t^{\kappa} + 2\pi\sqrt{-1}\ \beta\Big), \tag{48}$$

where $\alpha, \beta \in \mathfrak{h}^{(0)}$ are such that

$$\sigma_b\alpha = \alpha, \quad \sigma_b(\beta) - \beta + \alpha \in \Lambda^{(0)}.$$

Let us point out that under the analytical continuation around $t = 0$, the loop $h_{\alpha,\beta}$ gains the factor $e^{2\pi\sqrt{-1}\kappa\alpha}$. The latter must be 1 because

$$\kappa\alpha = (\alpha + \sigma_b(\beta) - \beta) + \sigma_b(\alpha + (\sigma_b(\beta) - \beta)) + \cdots + {\sigma_b}^{\kappa-1}(\alpha + (\sigma_b(\beta) - \beta)) \in \Lambda^{(0)}.$$

It follows that $h_{\alpha,\beta}$ is single-valued and σ_b-invariant, i.e., it defines an element of the affine Kac–Moody loop group acting on $\mathfrak{g}^{\sigma_b}$ by conjugation.

The main result of Kac and Peterson [43] is the following: *the basic representation of $\mathfrak{g}^{\sigma_b}$ remains irreducible when restricted to the pair $(\mathfrak{s}, \mathfrak{S})$*. Let us recall the construction of the representation. Put

$$X_\alpha(\zeta) = \sum_{n\in\mathbb{Z}} A_{\alpha,n}\,\zeta^{-n} = \frac{1}{\kappa}\sum_{l=1}^{\kappa}\sum_{n\in\mathbb{Z}} \eta^{-nl}({\sigma_b}^l(A_\alpha)t^n)\zeta^{-n}, \quad \alpha \in \Delta^{(0)},$$

where A_α appears in (44), and

$$H_i(\zeta) = \sum_{l\in\mathbb{Z}} H_{i,l}\,\zeta^{-m_i-l\kappa}, \quad i\in\mathfrak{I}\backslash\{(0,1)\},$$

where $H_{i,l}$ appears in (47). Let us denote by

$$\pi_0 : \mathfrak{h}^{(0)} \to \mathfrak{h}_0^{(0)} \text{ and } \pi_* : \mathfrak{h}^{(0)} \to (\mathfrak{h}_0^{(0)})^\perp$$

the orthogonal projections of $\mathfrak{h}^{(0)}$ onto $\mathfrak{h}_0^{(0)} := \mathbb{C}\,H_0$ and $(\mathfrak{h}_0^{(0)})^\perp$ respectively. Given $x\in\mathfrak{h}^{(0)}$, we will sometimes use the notation x_0 and x_* for $\pi_0(x)$ and $\pi_*(x)$ respectively. Let $E^*_\alpha(\zeta)$ be the *vertex operator*

$$E^*_\alpha(\zeta) = \exp\Big(\sum_{i,l}(\alpha|H_i)H_{i^*,-l-1}\frac{\zeta^{m_i+l\kappa}}{m_i+l\kappa}\Big)\exp\Big(\sum_{i,l}(\alpha|H_{i^*})H_{i,l}\frac{\zeta^{-m_i-l\kappa}}{-m_i-l\kappa}\Big), \tag{49}$$

where both sums are over all $i\in\mathfrak{I}\setminus\{(0,1)\}$ and all $l\in\mathbb{Z}_{\geq 0}$.

Lemma 25. *There are operators C_α, $\alpha\in\Delta^{(0)}$, independent of ζ, that commute with all basis vectors (47) of $\mathfrak{s}$ different from H_0, such that*

$$X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta),$$

where

$$X^0_\alpha(\zeta) = \zeta^{\kappa\,|\alpha_0|^2/2}C_\alpha\zeta^{\kappa\,\alpha_0}, \tag{50}$$

and $\alpha_0 = \pi_0(\alpha)$.

Proof. After a direct computation we get

$$[H_{i,l}, X_\alpha(\zeta)] = (\alpha|H_i)\zeta^{m_i+l\kappa}X_\alpha(\zeta).$$

It follows that $X_\alpha(\zeta) = X^0_\alpha(\zeta)E^*_\alpha(\zeta)$, where $X^0_\alpha(\zeta)$ is an operator commuting with all $H_{i,l}\neq H_0$.

After a direct computation we get the following commutation relations:

$$\begin{aligned} h_{\alpha,\beta}\ (-d)\ h^{-1}_{\alpha,\beta} &= -d+\kappa\,\alpha+\frac{1}{2}|\alpha|^2\,\kappa^2K, \\ h_{\alpha,\beta}\ A_{\gamma,n}\ h^{-1}_{\alpha,\beta} &= e^{2\pi\sqrt{-1}\,(\beta|\gamma)}A_{\gamma,n+\kappa(\alpha|\gamma)}+\delta_{n,0}(\alpha|A_\gamma)\kappa\,K, \end{aligned}$$

and $h_{\alpha,\beta}$ commute with the Heisenberg algebra $\mathfrak{s}$ except for:

$$h_{\alpha,\beta}\ H_0\ h^{-1}_{\alpha,\beta} = H_0+(\alpha|H_0)\kappa K.$$

Here $h_{\alpha,\beta}$ are given in (48). In order to determine the dependence on ζ of $X^0_\alpha(\zeta)$ we first have to notice that

$$-d = \frac{1}{2}|\rho_s|^2K+\frac{1}{2}H_0^2+\sum_{i,l}H_{i^*,-l-1}H_{i,l}, \tag{51}$$

where $H_0 = H_{01} = H_{02}$. Indeed, if we decompose the basic representation into a direct sum of weight subspaces of $\mathfrak{s}$, then using the above commutation relations, we get that the LHS of (51) is an operator that preserves these weight subspaces while the difference of the LHS and the RHS commutes with $\mathfrak{s}$ and $\mathfrak{S}$. The formula follows up to the constant term $\frac{1}{2}|\rho_s|^2K$, which is fixed by examining the action of the operator $d \in \mathfrak{g}$ on the vacuum vector. Using formula (45) for $Xt^n = \rho_s$ we get that ρ_s (viewed as an element of $\mathfrak{g}^{\sigma_b}$) acts on the vacuum by the scalar $-|\rho_s|^2/\kappa$; then since the RHS of formula (46) acts by 0 on the vacuum, we get that $d \in \mathfrak{g}^{\sigma_b}$ acts by the scalar

$$-|\rho_s|^2/\kappa + \frac{1}{2}|\rho_s|^2(1/\kappa) = -\frac{1}{2\kappa}|\rho_s|^2.$$

Since we have

$$[d, X_\alpha(\zeta)] = -\zeta\partial_\zeta X_\alpha(\zeta), \quad [d, E^*_\alpha(\zeta)] = -\zeta\partial_\zeta E^*_\alpha(z),$$

we easily get $-\zeta\partial_\zeta X^0_\alpha = [d, X^0_\alpha]$. On the other hand, $X^0_\alpha(\zeta)$ commutes with $H_{i,l}$ for all i, l, except

$$[H_0, X^0_\alpha(\zeta)] = (\alpha|H_0)X^0_\alpha. \tag{52}$$

It follows that

$$\zeta\partial_\zeta X^0_\alpha = \kappa\Big(X^0_\alpha \alpha_0 + \frac{|\alpha_0|^2}{2}X^0_\alpha\Big).$$

Solving the above equation we get formula (50). □

Lemma 26. *The operators C_α in* (50) *satisfy the following commutation relation*

$$C_\alpha C_\beta = \epsilon(\alpha,\beta)B^{-1}_{\alpha,\beta}C_{\alpha+\beta}, \tag{53}$$

where

$$B_{\alpha,\beta} = \kappa^{-(\alpha|\beta)}\prod_{l=1}^{\kappa-1}(1-\eta^l)^{({\sigma_b}^l(\alpha)|\beta)}.$$

Proof. Let us assume first that $\alpha \neq -\beta$ are two roots. After a direct computation we get that the commutator $[X_\alpha(\zeta), X_\beta(w)]$ is given by the following formula:

$$\frac{1}{\kappa}\sum_{l=1}^{\kappa}\left(\prod_{i=1}^{l-1}\upsilon^{-1}({\sigma_b}^i(\beta))\right)\epsilon(\alpha,{\sigma_b}^l(\beta))\,\delta(\eta^{-l}\zeta, w)\,w\,X_{\alpha+{\sigma_b}^l(\beta)}(\zeta),$$

where $\delta(x,y) := \sum_{n\in\mathbb{Z}} x^n y^{-n-1}$ is the formal delta function. On the other hand,

$$E^*_\alpha(\zeta)E^*_\beta(w) = \prod_{l=1}^{\kappa}\Big(1-\eta^l\frac{w}{\zeta}\Big)^{({\sigma_b}^l(\alpha)|\beta)} : E^*_\alpha(\zeta)E^*_\beta(w):,$$

where : : is the standard normal ordering in the Heisenberg group - all annihilation operators $H_{i,l}$ must be moved to the right. Substituting in the above commutator $X_\gamma(\zeta) = X^0_\gamma(\zeta)E^*_\gamma(\zeta)$ we get that the following two expressions are equal:

$$\prod_{l=1}^{\kappa}\Big(1-\eta^l\frac{w}{\zeta}\Big)^{({\sigma_b}^l(\alpha)|\beta)}X^0_\alpha(\zeta)X^0_\beta(w) - \prod_{l=1}^{\kappa}\Big(1-\eta^l\frac{\zeta}{w}\Big)^{({\sigma_b}^l(\beta)|\alpha)}X^0_\beta(w)X^0_\alpha(\zeta) \tag{54}$$

and

$$\frac{1}{\kappa}\sum_{l=1}^{\kappa}\left(\prod_{i=1}^{l-1}\upsilon^{-1}({\sigma_b}^i(\beta))\right)\epsilon(\alpha,{\sigma_b}^l(\beta))\delta(\eta^{-l}\zeta,w)\,w\,X^0_{\alpha+{\sigma_b}^l(\beta)}(\zeta). \tag{55}$$

Both formulas have the form $i_{\zeta,w}P_1(\zeta,w) - i_{\zeta,w}P_2(\zeta,w)$, where P_1 and P_2 are some rational functions and $i_{\zeta,w}$ (resp. $i_{w,\zeta}$) means the Laurent series expansion in the region $|\zeta| > |w|$ (resp. $|w| < |\zeta|$). Since $P_1 = P_2$ for the second expression, the same must be true for the first one, i.e.,

$$\prod_{l=1}^{\kappa}\Big(1-\eta^l\frac{w}{\zeta}\Big)^{(\sigma_b{}^l(\alpha)|\beta)} X^0_\alpha(\zeta)X^0_\beta(w) = \prod_{l=1}^{\kappa}\Big(1-\eta^l\frac{\zeta}{w}\Big)^{(\sigma_b{}^l(\beta)|\alpha)} X^0_\beta(w)X^0_\alpha(\zeta).$$

Recalling formula (50) and (52), the above equality implies:

$$C_\alpha C_\beta = \prod_{l=1}^{\kappa}(-\eta^l)^{(\alpha|\sigma_b{}^l(\beta))}\; C_\beta C_\alpha. \tag{56}$$

Using this equality we can easily write (54) as a sum of formal delta functions. Comparing with (55) we get (53). □

Using formula (53) we define C_α for all α in the root lattice $\Lambda^{(0)}$; then formula (56) still holds. Finally, a similar argument gives us that

$$C_\alpha C_{-\alpha} = \epsilon(\alpha,-\alpha)\, B^{-1}_{\alpha,-\alpha}, \text{i.e., } C_0 = 1. \tag{57}$$

Let $\gamma_i := \gamma_i^{(0)}$ be simple roots, γ_b be the branching node, and $\gamma_{\mu,i}$ be the enumeration of the non-branching simple roots that we used before (see Figure 1). Let

$$\mathfrak{s}_- \subset \mathfrak{s}$$

be the Lie subalgebra of $\mathfrak{s}$ spanned by the vectors

$$H_{i^*,-l-1}, \quad i \in \mathfrak{I}\backslash\{(0,1)\}, l \geq 0.$$

The basic representation can be realized on the following vector space:

$$V_x = S^*(\mathfrak{s}_-) \otimes \mathbb{C}[e^\omega]e^{x\omega}, \tag{58}$$

where x is a complex number and $\omega := \pi_0(\gamma_b)$. The first factor of the tensor product in (58) is the symmetric algebra on $\mathfrak{s}_-$, and the second one is isomorphic to the group algebra of the lattice $\pi_0(\Lambda^{(0)}) = \mathbb{Z}\,\pi_0(\gamma_b)$. We will refer to $|0\rangle := 1 \otimes e^{x\omega}$ as the *vacuum vector*. Slightly abusing the notation we define the operator

$$\partial_\omega := \frac{\partial}{\partial\omega} - x,$$

acting on V_x, so that $\partial_\omega\,|0\rangle = 0$.

Lemma 27. *The following formulas hold*

$$(\omega_{\mu,i}|\chi\omega_b) = \Big(1-\frac{i}{a_\mu}\Big), \quad \pi_0(\gamma_b) = \chi\omega_b, \quad \pi_*(\gamma_b) = -\sum_{\mu,i}\Big(1-\frac{i}{a_\mu}\Big)\gamma_{\mu,i}.$$

Proof. Let $\{\varepsilon_i\}_{i=1}^a$ be the standard basis of $\mathbb{C}^a$. The root system of type A_{a-1} is given by $\{\varepsilon_i - \varepsilon_j\}$ and the standard choice of simple roots is $\gamma_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq a-1$. Note that the fundamental weights corresponding to the basis of simple roots are

$$\widetilde{\omega}_i = \Big(1-\frac{i}{a}\Big)(\varepsilon_1+\cdots+\varepsilon_i) - \frac{i}{a}(\varepsilon_{i+1}+\cdots+\varepsilon_a).$$

It follows that the pairing between the fundamental weights is

$$(\widetilde{\omega}_i|\widetilde{\omega}_j) = \min(i,j) - ij/a.$$

In particular, we have

$$\widetilde{\omega}_i = \Big(1-\frac{i}{a}\Big)\gamma_1 + \cdots , \tag{59}$$

where the remaining terms involve only $\gamma_2,\dots,\gamma_{a-1}$.

In our settings, for any fixed $\mu=1,2,3$, the roots $\{\gamma_{\mu,i}\}_{i=1}^{a_\mu-1}$ give rise to a subroot system of type $A_{a_\mu-1}$. Let us denote by $\widetilde{\omega}_{\mu,i}$ the corresponding fundamental weights. Note that

$$\omega_{\mu,i} = \widetilde{\omega}_{\mu,i} - (\widetilde{\omega}_{\mu,i}|\gamma_b)\,\omega_b,$$

so the first formula of the Lemma follows from (59) and

$$(\gamma_b|\gamma_{\mu,i}) = -\delta_{i,1}, \quad (\widetilde{\omega}_{\mu,i}|\omega_b) = 0.$$

The other two identities follow easily from the first one. □

Lemma 28. *Let c_α ($\alpha\in\Lambda^{(0)}$) be operators defined by*

$$C_\alpha = c_\alpha \exp\Big((\omega_b|\alpha)\omega\Big)\exp\Big(2\pi\sqrt{-1}\,(\rho_b|\alpha)\partial_\omega\Big). \tag{60}$$

Then $[c_\alpha, c_\beta]=0$.

Proof. To begin with, note that by definition, the commutator $C_\alpha C_\beta C_\alpha^{-1}C_\beta^{-1}$ is given by the following formula:

$$\prod_{l=1}^{\kappa}(-\eta^l)^{(\alpha|{\sigma_b}^l(\beta))} = e^{\pi\sqrt{-1}\,(\alpha_0|\beta)}e^{2\pi\sqrt{-1}\,((1-\sigma_b)^{-1}\alpha_*|\beta)}.$$

On the other hand, using (60), the commutator becomes

$$c_\alpha c_\beta c_\alpha^{-1}c_\beta^{-1}\exp 2\pi\sqrt{-1}\,\Big((\rho_b|\alpha)(\omega_b|\beta)-(\rho_b|\beta)(\omega_b|\alpha)\Big). \tag{61}$$

Recall that σ_b is a composition of 3 matrices $\sigma_\mu^{(0)}, \mu=1,2,3$ whose action on the subspace with basis $\{\gamma_{\mu,1},\dots,\gamma_{\mu,a_\mu-1}\}$ is represented by the matrix (41). It is easy to check that the (i,j)-th entry

$$\Big[(1-\sigma_\mu^{(0)})^{-1}\Big]_{ij} = \frac{i}{a_\mu} - \varepsilon_{ij}, \quad \varepsilon_{ij} = \begin{cases} 0 & \text{if } i\le j,\\ 1 & \text{if } i>j.\end{cases} \tag{62}$$

A straightforward computation using formula (62) and Lemma 27 yields

$$\begin{aligned}
((1-\sigma_\mu^{(0)})^{-1}\gamma_{\mu,i}|\gamma_b) &= -\frac{1}{a_\mu},\\
((1-\sigma_\mu^{(0)})^{-1}\gamma_{\mu,i}|\gamma_{\mu,j}) &= \delta_{i,j}-\delta_{i+1,j},\\
((1-\sigma_b)^{-1}(\gamma_b)_*|\gamma_{\mu,i}) &= \frac{1}{a_\mu} \pmod{\mathbb{Z}},\\
((1-\sigma_b)^{-1}(\gamma_b)_*|\gamma_b) &= 1-\frac{1}{2}\chi\,.
\end{aligned}$$

Using the above formulas we get

$$((1-\sigma_b)^{-1}\pi_*(\alpha)|\beta) = (\rho_b|\alpha)\,(\omega_b|\beta) - (\rho_b|\beta)\,(\omega_b|\alpha) - \frac{1}{2}(\alpha_0|\beta_0) \pmod{\mathbb{Z}}.$$

For the commutator we get

$$C_\alpha C_\beta C_\alpha^{-1} C_\beta^{-1} = \exp\Big(2\pi\sqrt{-1}\Big((\rho_b|\alpha)\,(\omega_b|\beta) - (\rho_b|\beta)\,(\omega_b|\alpha)\Big)\Big).$$

Comparing with (61) we get $c_\alpha c_\beta c_\alpha^{-1} c_\beta^{-1} = 1$. □

Lemma 28 implies that the operators c_α can be represented by scalars, i.e., we can find complex numbers c_α, $\alpha \in \Lambda^{(0)}$ such that

$$c_\alpha c_\beta = \epsilon(\alpha,\beta)\, B_{\alpha,\beta}^{-1}\, e^{-2\pi\sqrt{-1}(\rho_b|\beta)(\omega_b|\alpha)}\; c_{\alpha+\beta}. \tag{63}$$

For example we can choose c_{α_i} arbitrarily for the simple roots α_i and then use formula (63) to define the remaining constants.

The level 1 basic representation can be realized on V_x as follows. Let us represent the Heisenberg algebra $\mathfrak{s}$ on $\mathbb{C}[e^\omega]e^{x\omega}$ by letting all generators act trivially, except for $H_0 \mapsto (H_0|\gamma_b)\,\partial_\omega$. The latter is forced by the commutation relation

$$[H_0, C_\alpha] = (\alpha|H_0)\, C_\alpha = (\omega_b|\alpha)\,(H_0|\gamma_b)\, C_\alpha.$$

In this way V_x naturally becomes an $\mathfrak{s}$-module. Furthermore, put

$$E^0_\alpha(\zeta) = \exp\Big((\omega_b|\alpha)\omega\Big) \exp\Big(\Big((\omega_b|\alpha)\,\chi\,\log\zeta^\kappa\; + 2\pi\sqrt{-1}\;(\rho_b|\alpha)\Big)\partial_\omega\Big) \tag{64}$$

and $E_\alpha(\zeta) = E^0_\alpha(\zeta) E^*_\alpha(\zeta)$, where $E^*_\alpha(\zeta)$ is defined by formula (49). We get that the representation of the Heisenberg algebra $\mathfrak{s}$ on V_x can be lifted to a representation of the affine Lie algebra $\mathfrak{g}^{\sigma_b}$ as follows:

$$\begin{aligned} X_\alpha(\zeta) &\mapsto c_\alpha \zeta^{\kappa|\alpha_0|^2/2} E_\alpha(\zeta), \quad \alpha \in \Delta^{(0)} \\ K &\mapsto 1/\kappa, \\ d &\mapsto -\frac{1}{2\kappa}|\rho_s|^2 - \frac{1}{2}H_0^2 - \sum_{i,l} H_{i^*,-l-1} H_{i,l}. \end{aligned}$$

Finally, let us finish this section by making a remark on κ. There is no a canonical way to extend σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Therefore, the value of κ depends on our choice of the cocycle $\epsilon(\alpha,\beta)$ and the corresponding sign-function $\upsilon(\alpha)$. We will see however that replacing κ by $m\kappa$, where m is a positive integer, does not change the HQEs, so we may assume that κ is a sufficiently big integer, s.t., $\sigma_b^\kappa = 1$. For the sake of completeness, let us fix an extension that seems natural for our purposes. Put $\omega_{\mu,0} = \omega_b$ and $\omega_{\mu,a_\mu} = 0$ and define

$$\mathrm{SF}(\alpha,\beta) = \sum_{\mu=1}^{3} \sum_{i=0}^{a_\mu-1} (\omega_{\mu,i}|\alpha)(\omega_{\mu,i} - \omega_{\mu,i+1}|\beta). \tag{65}$$

Since $\mathrm{SF}(\alpha,\beta) + \mathrm{SF}(\beta,\alpha) = (\alpha|\beta)$, the bi-multiplicative function $\epsilon(\ ,\ ) = (-1)^{\mathrm{SF}(\ ,\ )}$ is an acceptable choice for the Frenkel–Kac construction. Note that

$$\upsilon(\alpha) = (-1)^{\sum_{\mu=1}^{3}(\omega_b|\alpha)(\omega_{\mu,1}|\alpha)} \tag{66}$$

satisfies formula (43), so we get an exlicit formula for an extension of σ_b to a Lie algebra automorphism of $\mathfrak{g}^{(0)}$. Moreover, since

$$\prod_{l=1}^{|\sigma_b|} \upsilon(\sigma_b^l(\alpha)) = (-1)^{\chi|\sigma_b|},$$

we get that $\kappa = |\sigma_b|$ if $\chi|\sigma_b|$ is even and $\kappa = 2|\sigma_b|$ if $\chi|\sigma_b|$ is odd. Notice that $|\sigma_b| = \mathrm{lcm}(a_1, a_2, a_3)$, the least common multiple of a_1, a_2, a_3.

Remark 29. *The notation* SF *is motivated from the notion of a Seifert form in singularity theory (cf.* [3, 4]*). We do not claim that* (65) *is a Seifert form, although it would be interesting to investigate whether definition* (65) *can be interpreted as a linking number between* α *and* β*.*

6. Integrable hierarchies

6.1. **The Kac–Wakimoto hierarchy.** Following Kac–Wakimoto (see [41]), we can define an integrable hierarchy in the Hirota form whose solutions are parametrized by the orbit of the vacuum vector $|0\rangle$ of the affine Kac–Moody group. A vector $\tau \in V_x$ belongs to the orbit if and only if $\Omega_x\,(\tau \otimes \tau) = 0$, where Ω_x is the operator representing the following bi-linear Casimir operator:

$$\sum_{\alpha\in\Delta^{(0)}} \sum_n \frac{1}{(A_\alpha|A_{-\alpha})} A_{\alpha,n} \otimes A_{-\alpha,-n} + K\otimes d + d\otimes K + \frac{1}{\kappa} H_0 \otimes H_0 + \\ + \frac{1}{\kappa} \sum_{i,l} \Big(H_{i,l} \otimes H_{i^*,-l-1} + H_{i^*,-l-1} \otimes H_{i,l} \Big),$$

where the second sum is over all $i \in \mathfrak{I}\backslash\{(0,1)\}$ and all $l \geq 0$. On the other hand, we have

$$\sum_n \frac{1}{(A_\alpha|A_{-\alpha})} A_{\alpha,n} \otimes A_{-\alpha,-n} = \mathrm{Res}_{\zeta=0}\, \frac{d\zeta}{\zeta} a_\alpha(\zeta) E_\alpha(\zeta) \otimes E_{-\alpha}(\zeta),$$

where the coefficients a_α can be computed explicitly thanks to formula (63), i.e.,

$$a_\alpha(\zeta) = B_{\alpha,\alpha}\, \zeta^{\kappa\,|\alpha_0|^2}\, e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}. \tag{67}$$

We identify the symmetric algebra $S^*(\mathfrak{s}_-)$ with the Fock space $\mathbb{C}[y]$, where $y = (y_{i,l})$ is a sequence of formal variables indexed by $i \in \mathfrak{I}\backslash\{(0,1)\}$ and $l \geq 0$, by identifying $H_{i^*,-l-1} = (m_i + l\kappa) y_{i,l}$. Then (note that $(H_0|\gamma_b) = (\kappa\chi)^{1/2}$)

$$H_{i,l} = \frac{\partial}{\partial y_{i,l}}, \quad H_0 = (\kappa\chi)^{1/2}\, \partial_\omega, \quad K = 1/\kappa,$$

and

$$d = -\frac{|\rho_s|^2}{2\kappa} - \frac{\kappa\chi}{2}\partial_\omega^2 - \sum_{i,l} (m_i + l\kappa) y_{i,l} \partial_{y_{i,l}}.$$

The elements of the representation space can also be thought as sequences of polynomials in the following way:

$$V_x \cong \mathbb{C}[y]^{\mathbb{Z}}, \quad \sum_{n\in\mathbb{Z}} \tau_n(y)e^{(n+x)\omega} \mapsto \tau := (\tau_n(y))_{n\in\mathbb{Z}}.$$

The above isomorphism turns $\mathbb{C}[y]^{\mathbb{Z}}$ into a module over the algebra of differential operators in e^ω:

$$(e^\omega \cdot \tau)_n = \tau_{n-1}, \quad (\partial_\omega \cdot \tau)_n = n\tau_n.$$

The Hirota equations then assume the form (3) stated in Section 1.2. It remains only to verify the value of the constant $|\rho_s|^2/\kappa^2$.

Lemma 30. *We have*

$$|\rho_s|^2/\kappa^2 = \frac{1}{2}\,\mathrm{tr}\Big(\frac{1}{4} + \theta\,\theta^{\mathrm{T}}\Big),$$

where θ is the Hodge grading operator (see equation (16)).

Proof. Since $\tau = |0\rangle$ is a solution to the hierarchy, we must have

$$|\rho_s|^2/\kappa^2 = \sum_{\alpha:\ (\omega_b|\alpha)=0} a_\alpha(\zeta).$$

Let $\alpha \in \Delta^{(0)}$ be such that $(\omega_b|\alpha) = 0$, then formula (67) reduces simply to

$$a_\alpha(\zeta) = B_{\alpha,\alpha} = \kappa^{-2}\prod_{l=1}^{\kappa-1}(1-\eta^l)^{(\sigma_b^l\alpha|\alpha)}.$$

Recall the notation in the proof of Lemma 27. We claim that α must belong to one of the root subsystems $\Delta_\mu^{(0)}$ of type $A_{a_\mu-1}$ corresponding to the legs of the Dynkin diagram for some μ. Indeed, let us write α as a linear combination $\sum_{\mu,i} c_{\mu,i}\gamma_{\mu,i}$ for some integers $c_{\mu,i}$. If this linear combination involves a simple root $\gamma_{\mu,i}$ for some μ, then using reflections $s_{\mu,i}$ with $i > 1$ we can transform α to a cycles α' such that the decomposition of α' as a sum of simple roots will involve $\gamma_{\mu,1}$. Moreover, we still have $(\omega_b|\alpha') = 0$. In other words, we may assume that $c_{\mu,1} \neq 0$ as long as $c_{\mu,i} \neq 0$ for some i. However, since $(\alpha|\gamma_b) = -\sum_\mu c_{\mu,1}$ and the coefficients $c_{\mu,i}$ have the same sign (depending on whether α is a positive or a negative root) we get that there is precisely one μ for which $c_{\mu,1} \neq 0$.

Assume that $\alpha \in \Delta_\mu^{(0)}$, then since σ_b is a product of the Coxeter transformations $\sigma_{\mu'} = \cdots s_{\mu',2}s_{\mu',1}$, in the above formula for a_α only σ_μ contributes and since the order of σ_μ is a_μ, after a short computation we get

$$a_\alpha(\zeta) = a_\mu^{-2}\prod_{l=1}^{a_\mu-1}(1-\eta_\mu^l)^{(\sigma_\mu^l\alpha|\alpha)}, \quad \eta_\mu = e^{2\pi\sqrt{-1}/a_\mu}.$$

These are precisely the coefficients of the principal Kac-Wakimoto hierarchy of type $A_{a_\mu-1}$. According to [24] the sum

$$\sum_{\alpha\in\Delta_\mu^{(0)}} a_\alpha(\zeta) = |\rho_\mu|^2/a_\mu^2.$$

where ρ_μ is the sum of the fundamental weights of $\Delta_\mu^{(0)}$. It is well known that $|\rho_\mu|^2=(a_\mu-1)a_\mu(a_\mu+1)/12$. We get

$$|\rho_s|^2/\kappa^2 = \frac{1}{12}\sum_{\mu=1}^3\Big(a_\mu-\frac{1}{a_\mu}\Big).$$

It remains only to notice (using $\theta^{\rm T}=-\theta$) that

$$\begin{aligned}\frac12\,{\rm tr}\Big(\frac14+\theta\,\theta^{\rm T}\Big) &= \frac12\,{\rm tr}\Big(\frac12+\theta\Big)\Big(\frac12-\theta\Big)=\frac12\sum_i d_i(1-d_i)=\frac12\sum_{\mu=1}^3\sum_{i=1}^{a_\mu-1}\frac{i(a_\mu-i)}{a_\mu^2}\\ &= \frac{1}{12}\sum_{\mu=1}^3\Big(a_\mu-\frac{1}{a_\mu}\Big).\end{aligned}$$

□

6.2. **Formal discrete Laplace transform.** Let $\alpha\in\Delta^{(0)}$ and $\beta\in\Delta$ be as in Section 4.6. We would like to compare the vertex operators[6] $E_\alpha(\zeta)$ and $\widetilde{\Gamma}^\beta(\lambda):=e^{(\widetilde{\mathbf{f}}_\beta(\lambda;z))\widehat{}}$, where

$$\widetilde{\mathbf{f}}_\beta(\lambda;z)=\sum_{n\in\mathbb{Z}}\widetilde{I}^{(n)}_\beta(\lambda)\,(-z)^n,$$

see (29). Using the formulas for the calibrated periods from Section 4.6 we get

$$\widetilde{\Gamma}^\beta(\lambda)=U_\beta(\lambda)\,\widetilde{\Gamma}^\beta_0(\lambda)\,\widetilde{\Gamma}^\beta_*(\zeta),$$

where (we dropped the superscript and set $\omega_b:=\omega_b^{(0)}$)

$$U_\beta(\lambda)=\exp\Big(\sum_{l=1}^\infty\Big((\omega_b|\alpha)\chi(\log\lambda-C_l)+2\pi\sqrt{-1}(n+(\rho_b|\alpha))\Big)\frac{\lambda^l}{l!}\,q_l^{01}/\sqrt{\hbar}\Big),$$

$$\begin{aligned}\widetilde{\Gamma}^\beta_0(\lambda) &= \exp\Big(\Big((\omega_b|\alpha)\,\chi\,(\log\,\lambda-C_0)+2\pi\sqrt{-1}(n+(\rho_b|\alpha))\Big)\frac{q_0^{01}}{\sqrt{\hbar}}\Big)\times\\ &\quad\exp\Big(-(\omega_b|\alpha)\,\sqrt{\hbar}\frac{\partial}{\partial q_0^{01}}\Big),\end{aligned}$$

and

$$\widetilde{\Gamma}^\beta_*(\lambda)=\exp\Big(\sum_{i,l}(\alpha|H_i)\,\zeta^{m_i+l\kappa}\,y_{i,l}\Big)\ \exp\Big(\sum_{i,l}(\alpha|H_{i^*})\,\frac{\zeta^{-m_i-l\kappa}}{-m_i-l\kappa}\,\frac{\partial}{\partial y_{i,l}}\Big),$$

[6]See Section 7.1 for the definition of the quantization operation $(-)\widehat{}$.

where the sums are over all $i \in \mathfrak{I}\backslash\{(0,1)\}$ and $l \geq 0$, $\lambda = \zeta^\kappa/\kappa$, and we use the change of variables ($l \geq 0$)

$$y_{02,l} = \frac{1}{\sqrt{\hbar}} \frac{\kappa^{d_{02}}}{\sqrt{\kappa\chi}} \frac{q_l^{02}}{m_{02}(m_{02}+\kappa)\cdots(m_{02}+l\kappa)}, \tag{68}$$

$$y_{i,l} = \frac{1}{\sqrt{\hbar}} \frac{\kappa^{d_i}}{\sqrt{\kappa a_i}} \frac{q_l^{i}}{m_i(m_i+\kappa)\cdots(m_i+l\kappa)}. \tag{69}$$

Comparing with (49) and (64) we get that $\widetilde{\Gamma}^\beta_* = E^*_\beta$ and that $\widetilde{\Gamma}^\beta_0$ is a Laplace transform of E^0_β. We make the last statement precise as follows. Put

$$\widehat{V} := \mathbb{C}_\hbar[\![y, x, q_1^{01}+1, q_2^{01}, \dots]\!]^{\mathbb{Z}}.$$

The space $\widehat{V}$ contains a completion of the basic representation V_x. It has also some aditional variables q_l^{01}, $l \geq 1$ which will be treated as parameters. Just like before, we identify the elements of $\widehat{V}$ with formal Fourier series

$$f = (f_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} f_n\, e^{(n+x)\omega}.$$

Given $f(\hbar;\mathbf{q}) \in \mathbb{C}_\hbar[\![\mathbf{q}]\!]$ satisfying the condition

$$f(\hbar;\mathbf{q})\Big|_{q_0^{01}=x\sqrt{\hbar}} \in \mathbb{C}_\hbar[\![\mathbf{q}]\!] \quad \forall x\in\mathbb{C}, \tag{70}$$

define the formal Laplace transform of f depending on a parameter C ($C \neq 0$)

$$\mathcal{F}_C(f(q_0^{01},\dots)) := \sum_{n\in\mathbb{Z}} f((x+n)\sqrt{\hbar},\dots)\, e^{(n+x)\omega}\, C^{\frac{1}{2}n^2} \quad \in \widehat{V},$$

where the dots stand for the remaining $\mathbf{q}$-variables on which f depends. It is easy to check that

$$\mathcal{F}_C \circ q_0^{01}/\sqrt{\hbar} = \frac{\partial}{\partial\omega} \circ \mathcal{F}_C \tag{71}$$

and

$$\mathcal{F}_C \circ e^{-m\sqrt{\hbar}\partial/\partial q_0^{01}} = e^{m\omega}\, C^{\frac{1}{2}m^2+m\partial_\omega} \circ \mathcal{F}_C, \tag{72}$$

where recall that $\partial_\omega = \frac{\partial}{\partial\omega} - x$.

Lemma 31. *Let $C = \kappa^\chi\, e^{\chi C_0}$, then*

$$E^0_\alpha(\zeta)\ \mathcal{F}_C = \mathcal{F}_C e^{-AB-\frac{1}{2}B^2\log C}\, e^{Ax}\, \widetilde{\Gamma}^\beta_0,$$

where

$$A = (\omega_b|\alpha)\,\chi\,(\log\lambda - C_0) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha)), \quad B = (\omega_b|\alpha).$$

Proof. Using (71) and (72) we get that the vertex operators in q_0^{01} transform as follows:

$$\mathcal{F}_C\, e^{A\, q_0^{01}/\sqrt{\hbar}}\, e^{-B\sqrt{\hbar}\partial/\partial q_0^{01}} = e^{AB+\frac{1}{2}B^2\log C}\, e^{Ax}\, e^{B\omega} e^{(A+B\log C)\partial_\omega}\, \mathcal{F}_C.$$

On the other hand, after a straightforward computation, we get

$$e^{AB+\frac{1}{2}B^2\log C} = \zeta^{\kappa|\alpha_0|^2} e^{-\frac{|\alpha_0|^2}{2\chi}\,(2\chi(C_0+\log\kappa)-\log C)} e^{2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}$$

and

$$A+B\log C = (\omega_b|\alpha)\Big(\chi\log\zeta^\kappa + \log C - \chi(C_0+\log\kappa)\Big) + 2\pi\sqrt{-1}(n+(\rho_b|\alpha)). \tag{73}$$

Furthermore, note that since the operator $e^{2\pi\sqrt{-1}\partial_\omega}$ acts as the identity on $\widehat{V}$, the integer n in (73) may be set to 0. Finally, it remains only to compare with (64) and to recall our assumption

$$\log C = \chi(C_0+\log\kappa). \tag{74}$$

□

6.3. **Integrable hierarchies for the affine cusp polynomials.** For every root $\alpha\in\Delta^{(0)}\subset H^{(0)}$ we fix an arbitrary lift $\beta\in\Delta\subset\mathfrak{h}$ (cf. Section 4.6). The subset of affine roots obtained in this way will be denoted by Δ'. Following the construction of Givental and Milanov in [31] we introduce the following Casimir-like operator

$$\begin{aligned}\widetilde{\Omega}_{\Delta'}(\lambda) \quad = \quad & -\frac{\lambda^2}{2}\Big(\sum_{i=1}^N :(\widetilde{\phi}_i(\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\widetilde{\phi}_i(\lambda))(\widetilde{\phi}^i(\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\widetilde{\phi}^i(\lambda)):\Big) + \\ & + \sum_{\beta\in\Delta'} \tilde{b}_\beta(\lambda)\widetilde{\Gamma}^\beta(\lambda)\otimes_{\mathfrak{a}}\widetilde{\Gamma}^{-\beta}(\lambda) - \frac{1}{2}\operatorname{tr}\Big(\frac{1}{4}+\theta\,\theta^{\mathrm{T}}\Big),\end{aligned}$$

where the notation is as follows. Let $\{\beta_i\}_{i=1}^N$ and $\{\beta^i\}_{i=1}^N$ be two sets of vectors in $\mathfrak{h}$ such that under the projection $\widetilde{I}^{(0)}(1):\mathfrak{h}\to H^{(0)}$ they project to bases dual with respect to the intersection form $(\cdot|\cdot)$, i.e., $(\beta_i|\beta^j)=\delta_{ij}$. Then

$$\widetilde{\phi}_i(\lambda) = (\partial_\lambda\widetilde{\mathbf{f}}_{\beta_i}(\lambda;z))\widehat{\ }, \quad \widetilde{\phi}^i(\lambda) = (\partial_\lambda\widetilde{\mathbf{f}}_{\beta^i}(\lambda;z))\widehat{\ }, \quad 1\le i\le N.$$

The tensor product is over the polynomial algebra $\mathfrak{a}:=\mathbb{C}_\hbar[q_1^{01},q_2^{01},\dots]$, which in particular means that almost all terms that involve $\log\lambda$ cancel.

The first sum in the definition of $\widetilde{\Omega}_{\Delta'}$ is monodromy invariant around $\lambda=\infty$ and hence it expands in only integral powers of λ. In fact one can check that the corresponding coefficients give rise to a representation of the Virasoro algebra, which can be identified with an instance of the so called *coset Virasoro construction*[7]. After a straightforward computation using the formulas for the periods from Section 4.6, we get the following formula for the coefficient in front of λ^{-2} (i.e., the L_0-Virasoro operator)

$$\frac{\chi}{2\hbar}(q_0^{01}\otimes_{\mathfrak{a}}1 - 1\otimes_{\mathfrak{a}} q_0^{01})^2 + \sum_{i,l}\Big(\frac{m_i}{\kappa}+l\Big)(q_l^i\otimes_{\mathfrak{a}}1 - 1\otimes_{\mathfrak{a}} q_l^i)(\partial_{q_l^i}\otimes_{\mathfrak{a}}1 - 1\otimes_{\mathfrak{a}}\partial_{q_l^i}).$$

The coefficient $\tilde{b}_\beta$ are defined in terms of the vertex operators $\widetilde{\Gamma}^\beta(\lambda)$ as follows

$$\tilde{b}_\beta^{-1}(\lambda) = \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^2\widetilde{B}_{\beta,-\beta}(\lambda,\mu), \tag{75}$$

[7]We are thankful to B. Bakalov for this remark.

where $\widetilde{B}_{\alpha,\beta}(\lambda,\mu)$ is the phase factor from the composition of the following two vertex operators:

$$\widetilde{\Gamma}^{\alpha}(\lambda)\widetilde{\Gamma}^{\beta}(\mu)=\widetilde{B}_{\alpha,\beta}(\lambda,\mu)\ :\widetilde{\Gamma}^{\alpha}(\lambda)\widetilde{\Gamma}^{\beta}(\mu):\ .$$

After a straightforward computation as in Section 5.2, we get

$$\widetilde{B}_{\alpha,\beta}(\lambda,\mu)=\mu^{-(\alpha_0|\beta_0)}e^{C_0(\alpha_0|\beta_0)-2\pi\sqrt{-1}(\omega_b|\alpha)\,(\rho_b|\beta)}\prod_{l=1}^{\kappa}\Big(1-\eta^l(\mu/\lambda)^{1/\kappa}\Big)^{(\sigma_b^l(\alpha)|\beta)}, \tag{76}$$

where we are slightly abusing the notation on the RHS by using α and β to denote the image of α and β in $\mathfrak{h}^{(0)}$.

We are interested in the following system of Hirota quadratic equations: for every integer $n\in\mathbb{Z}$

$$\mathrm{Res}_{\lambda=\infty}\ \frac{d\lambda}{\lambda}\Big(\widetilde{\Omega}_{\Delta'}(\lambda)\,(\tau\otimes_{\mathfrak{a}}\tau)\Big)\Big|_{q_0^{01}\otimes 1-1\otimes q_0^{01}=n\sqrt{\hbar}}=0 \tag{77}$$

where $\tau\in\mathbb{C}_\hbar[\![q_0,q_1+1,q_2\dots]\!]$. The operator $\widetilde{\Omega}_{\Delta'}(\lambda)$ is multivalued near $\lambda=\infty$: the analytic continuation around $\lambda=\infty$ corresponds to a monodromy transformation of each cycles $\beta\in\Delta'$ of the type $\beta\mapsto\sigma_b(\beta)+n_\beta\varphi$, where $n_\beta\in\mathbb{Z}$. Using Proposition 21 we get that the analytical continuation transforms $\widetilde{\Omega}_{\Delta'}(\lambda)$ by permuting the cycles β and multiplying each vertex operator term by $e^{2\pi\sqrt{-1}n_\beta(q_0^{01}\otimes 1-1\otimes q_0^{01})}$. Therefore the 1-form in (77) is invariant with respect to the analytic continuation near $\lambda=\infty$. Moreover, for the same reason the equations (77) are independent of the choice of a lift Δ' of $\Delta^{(0)}$.

Remark 32. *The Hirota quadratic equations* (77) *are a straightforward generalization of the construction of Givental and Milanov* [31] *(see also* [24], *where the coefficients* $\tilde{b}_\beta$ *were interpreted in terms of the vertex operators) of integrable hierarchies for simple singularities.*

The following is the main result of this Section.

Theorem 33. *There exists a constant* C *such tha if* τ *is a solution to the Hirota quadratic equations* (77)*, then* $\mathcal{F}_C(\tau)$ *is a tau-function of the* σ_b*-twisted Kac–Wakimoto hierarchy.*

Proof. Let us fix the constant C as in (74). We just have to find the Laplace transform of the Hirota quadratic equations (3) of the Kac–Wakimoto hierarchy. Let $\alpha\in\Delta^{(0)}$ and $\beta\in\Delta$ be as in Section 4.6. Using Lemma 31 we get

$$\Big(a_\alpha(\zeta)\,E_\alpha(\zeta)\otimes E_{-\alpha}(\zeta)\Big)\ \Big(\mathcal{F}_C\otimes\mathcal{F}_C\Big)=\Big(\mathcal{F}_C\otimes\mathcal{F}_C\Big)\ \Big(b_\beta(\lambda)\,\widetilde{\Gamma}^{\beta}(\lambda)\otimes_{\mathfrak{a}}\widetilde{\Gamma}^{-\beta}(\lambda)\Big),$$

where the coefficient b_β is given by

$$a_\alpha(\zeta)\,\zeta^{-2\kappa|\alpha_0|^2}e^{\frac{|\alpha_0|^2}{\chi}\ \log C}e^{-4\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}.$$

Recalling formula (67) and $\lambda=\zeta^\kappa/\kappa$ we get

$$b_\beta(\lambda)=B_{\alpha,\alpha}\,\lambda^{-|\alpha_0|^2}\,e^{|\alpha_0|^2\,C_0}\,e^{-2\pi\sqrt{-1}(\omega_b|\alpha)(\rho_b|\alpha)}. \tag{78}$$

Using (75) and (76), it is not hard to verify that $b_\beta(\lambda) = \tilde{b}_\beta(\lambda)$.

In other words, $\mathcal{F}_C(\tau)$ is a solution to the Kac–Wakimoto hierarchy if τ satisfies the following equations:

$$\mathrm{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda}\Big((\mathcal{F}_C \otimes \mathcal{F}_C)\widetilde{\Omega}_{\Delta'}(\lambda)\, (\tau \otimes_{\mathfrak{a}} \tau)\Big) = 0.$$

Comparing the coefficients in front of $e^{(n'+x)\omega} \otimes e^{(n''+x)\omega}$ we get (77) with $n = n' - n''$. □

7. The symplectic loop space formalism

The symplectict loop space formalism in Gromov–Witten theory was introduced by Givental [30]. We apply this natural framework to describe and investigate further the Hirota quadratic equations (77).

7.1. Canonical quantization. The space

$$\mathcal{H} := H(\!(z^{-1})\!)$$

of formal Laurent series in z^{-1} with coefficients in H is equipped with the following *symplectic form*:

$$\Omega(\phi_1, \phi_2) := \mathrm{Res}_z \left(\phi_1(-z), \phi_2(z)\right) , \qquad \phi_1, \phi_2 \in \mathcal{H}\, ,$$

where, as before, $(,)$ denotes the residue pairing on H and the formal residue Res_z gives the coefficient in front of z^{-1}.

Let $\{\phi_i\}_{i\in\mathfrak{I}}$ and $\{\phi^i\}_{i\in\mathfrak{I}}$ be dual bases of H with respect to the residue pairing. Then

$$\Omega(\phi^i(-z)^{-k-1}, \phi_j z^l) = \delta_{ij}\delta_{kl}\, .$$

Hence, a Darboux coordinate system is provided by the linear functions q_k^i, $p_{k,i}$ on $\mathcal{H}$ given by:

$$q_k^i = \Omega(\phi^i(-z)^{-k-1}, \cdot)\, , \qquad p_{k,i} = \Omega(\cdot, \phi_i z^k)\, .$$

In other words,

$$\phi(z) = \sum_{k=0}^{\infty}\sum_{i\in\mathfrak{I}} q_k^i(\phi)\phi_i z^k + \sum_{k=0}^{\infty}\sum_{i\in\mathfrak{I}} p_{k,i}(\phi)\phi^i(-z)^{-k-1}\, , \qquad \phi \in \mathcal{H}\, .$$

The first of the above sums will be denoted by $\phi(z)_+$ and the second by $\phi(z)_-$.

The *quantization* of linear functions on $\mathcal{H}$ is given by the rules

$$\widehat{q}_k^i = \hbar^{-1/2} q_k^i\, , \qquad \widehat{p}_{k,i} = \hbar^{1/2}\frac{\partial}{\partial q_k^i}\, ,$$

where the RHSs of the above definitions are operators acting on the Fock space

$$\mathbb{C}_\hbar[\![\mathbf{q}]\!] := \mathbb{C}_\hbar[\![q_0, q_1 + \mathbf{1}, q_2, \cdots]\!], \quad \text{where} \quad \mathbb{C}_\hbar = \mathbb{C}(\!(\hbar)\!). \tag{79}$$

Every $\phi(z) \in \mathcal{H}$ gives rise to the linear function $\Omega(\phi, \cdot)$ on $\mathcal{H}$, so we can define the quantization $\widehat{\phi}$. Explicitly,

$$(\phi_i z^k)\widehat{\ } = -\hbar^{1/2}\frac{\partial}{\partial q_k^i}, \quad (\phi^i(-z)^{-k-1})\widehat{\ } = \hbar^{-1/2}\, q_k^i. \tag{80}$$

The quantization also makes sense for $\phi(z) \in H[[z, z^{-1}]]$ if we interpret $\widehat{\phi}$ as a formal differential operator in the variables q_k^i with coefficients in $\mathbb{C}_{\hbar,Q}$.

Lemma 34. *For all $\phi_1, \phi_2 \in \mathcal{H}$, we have $[\widehat{\phi}_1, \widehat{\phi}_2] = \Omega(\phi_1, \phi_2)$.*

Proof. It is enough to check this for the basis vectors $\phi^i(-z)^{-k-1}$, $\phi_i z^k$, in which case it is true by definition. □

7.2. **Quantization of symplectic transformations.** It is known that both series $S_t(z)$ and $R_t(z)$ described in Sections 3.3 and 3.4 are symplectic transformations on $(\mathcal{H}, \Omega)$. Moreover, they both have the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation.

A linear operator $A(z)$ on $\mathcal{H} := H(\!(z^{-1})\!)$ is infinitesimal symplectic if and only if the map $\mathcal{H} \ni \phi \mapsto A\phi \in \mathcal{H}$ is a Hamiltonian vector field with a Hamiltonian given by the quadratic function

$$h_A(\phi) = \frac{1}{2}\Omega(A\phi, \phi).$$

By definition, the *quantization* of $e^{A(z)}$ is given by the differential operator $e^{\widehat{h}_A}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k,i}p_{l,j})\widehat{\ } = \hbar \frac{\partial^2}{\partial q_k^i \partial q_l^j}, \quad (p_{k,i}q_l^j)\widehat{\ } = (q_l^j p_{k,i})\widehat{\ } = q_l^j \frac{\partial}{\partial q_k^i}, \quad (q_k^i q_l^j)\widehat{\ } = \frac{1}{\hbar} q_k^i q_l^j .$$

In the case of the orbifold $\mathbb{P}^1_{\mathbf{a}}$, Givental's higher genus reconstruction formula [30], proved by C. Teleman [59], can be stated as follows. If $t \in H$ is a semi-simple point, so that there exists a canonical coordinate system at t (see Section 3.3), then

$$\mathcal{D}_{\mathbf{a}}(\hbar; \mathbf{q}(z)) = e^{F^{(1)}(t)}\, \widehat{S}_t^{-1}\, \widehat{\Psi}_t\, \widehat{R}_t\, e^{\widehat{U_t/z}} \prod_{i=1}^{\mu} \mathcal{D}_{\rm pt}(\hbar\, \Delta_i; \sqrt{\Delta_i} Q^i(z)) , \tag{81}$$

where $\mathcal{D}_{\rm pt}$ is the total descendant potential of a point and the factor

$$F^{(1)}(t) = \sum_{d,n=0}^{\infty} \frac{Q^d}{n!} \langle t, \ldots, t \rangle_{1,n,d}$$

is the genus-1 primary (i.e. no descendants) potential. Let us examine more carefully the quantized action of the operators in formula (81).

7.2.1. *The action of the asymptotical operator.* The operator $\widehat{U_t/z}$ is known to annihilate the Witten–Kontsevich tau-function. Therefore, $e^{\widehat{U_t/z}}$ is redundant and it can be dropped from the formula. By definition, $\widehat{\Psi}_t$ is the following change of variables:

$$\mathbf{q}(z) = \Psi_t \sum_{i=1}^{\mu} Q^i(z) e_i , \quad \text{i.e.,} \quad \sqrt{\Delta_i} Q_k^i = \sum_{j \in \mathfrak{I}} (\partial_j u_i)\, q_k^j .$$

Put $\widehat{\mathcal{R}}_t = \widehat{\Psi}_t \widehat{R}_t \widehat{\Psi}_t^{-1}$ and

$${}^i\mathbf{q}(z) = \sum_{k=0}^{\infty} \sum_{j \in \mathfrak{I}} q_k^j (\partial_j u_i) z^k .$$

Then the total descendant potential assumes the form:

$$\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q}(z)) = e^{F^{(1)}(t)}\widehat{S}_t^{-1}\mathcal{A}_t(\hbar;\mathbf{q}(z))\,, \tag{82}$$

where

$$\mathcal{A}_t(\hbar;\mathbf{q}(z)) = \widehat{\mathcal{R}}_t\prod_{i=1}^{\mu}\mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i;{}^i\mathbf{q}(z)) \in \mathbb{C}_{\hbar,Q}[[q_0,q_1+\mathbf{1},q_2\dots]] \tag{83}$$

is the so-called *total ancestor potential* of $\mathbb{P}^1_{\mathbf{a}}$.

The action of the operator $\widehat{R}_t$ on formal functions, whenever it makes sense, is given as follows.

Lemma 35 (Givental [29])**.** *We have*

$$\widehat{R}_t^{-1}\,F(\mathbf{q}) = \Big(e^{\frac{\hbar}{2}V_t(\partial,\partial)}F(\mathbf{q})\Big)\Big|_{\mathbf{q}\mapsto R_t\mathbf{q}}\,,$$

where $V_t(\partial,\partial)$ is the quadratic differential operator

$$V_t(\partial,\partial) = \sum_{k,l=0}^{\infty}\sum_{i,j\in\mathfrak{I}}(\phi^i,V_{kl}(t)\phi^j)\,\frac{\partial^2}{\partial q_k^i\partial q_l^j}$$

whose coefficients $V_{kl}(t)$ are given by

$$\sum_{k,l=0}^{\infty}V_{kl}(t)z^kw^l = \frac{1-R_t(z)({}^TR_t(w))}{z+w}$$

and ${}^TR_t(w)$ denotes the transpose of $R_t(w)$ with respect to the Poincare pairing.

The substitution $\mathbf{q}\mapsto R_t\mathbf{q}$ can be written more explicitly as follows:

$$q_0\mapsto q_0,\quad q_1\mapsto R_1(t)q_0+q_1,\quad q_2\mapsto R_2(t)q_0+R_1(t)q_1+q_2\,,\dots.$$

The above substitution is not a well-defined operation on the space of formal functions. This complication, however, is offset by a certain property of the Witten–Kontsevich tau-function, which we now explain. By definition, an *asymptotical function* is a formal function of the type:

$$\mathcal{A}(\mathbf{q}) = \exp\Big(\sum_{g=0}^{\infty}F^{(g)}(\mathbf{q})\hbar^{g-1}\Big)\,.$$

Such a function is called *tame* if the following $(3g-3+r)$*-jet constraints* are satisfied:

$$\left.\frac{\partial^r F^{(g)}}{\partial q_{k_1}^{i_1}\cdots\partial q_{k_r}^{i_r}}\right|_{\mathbf{q}=0} = 0 \quad\text{if}\quad k_1+\dots+k_r > 3g-3+r\,.$$

The Witten–Kontsevich tau-function (up to the shift $q_1\mapsto q_1+1$) is tame for dimensional reasons: $\dim\overline{\mathcal{M}}_{g,r}=3g-3+r$. The total ancestor potential $\mathcal{A}_t$ is also tame, as it can be seen from its geometric definition (cf. [30]) or by using the fact that the action of the operator $\widehat{R}_t$ on tame functions is well defined and it preserves the tameness property ([28]).

7.2.2. *The action of the calibration.* The quantized symplectic transformation $\widehat{S}_t^{-1}$ acts on formal functions as follows.

Lemma 36 (Givental [29]). *We have*

$$\widehat{S}_t^{-1}\, F(\mathbf{q}) = e^{\frac{1}{2\hbar}W_t\mathbf{q}^2} F\big((S_t\mathbf{q})_+\big)\,, \tag{84}$$

where $W_t\mathbf{q}^2$ is the quadratic form

$$W_t\mathbf{q}^2 = \sum_{k,l=0}^{\infty} (W_{kl}(t)q_l, q_k)$$

whose coefficients are defined by

$$\sum_{k,l=0}^{\infty} W_{kl}(t)z^{-k}w^{-l} = \frac{{}^TS_t(z)S_t(w)-1}{z^{-1}+w^{-1}}\,.$$

The subscript + in (84) means truncation of all negative powers of z, i.e., in $F(\mathbf{q})$ we have to substitute (cf. (26)):

$$q_k \mapsto q_k + S_1(t)q_{k+1} + S_2(t)q_{k+2} + \cdots\,, \quad k = 0, 1, 2, \dots\,.$$

This operation is well-defined on the space of formal power series.

7.3. **Vertex operators.** Recall the series (24). We are interested in the vertex operators

$$\Gamma^\alpha(t,\lambda) =: e^{\widehat{\mathbf{f}}^\alpha(t,\lambda)} :, \quad \alpha \in \Delta, \tag{85}$$

and their *phase factors* $B_{\alpha,\beta}(t,\lambda,\mu)$ defined by

$$\Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu) = B_{\alpha,\beta}(t,\lambda,\mu) : \Gamma^\alpha(t,\lambda)\Gamma^\beta(t,\mu) :,$$

where $:\cdot:$ is the usual normal ordering – move all differentiation operators to the right of the multiplication operators. Note that

$$B_{\alpha,\beta}(t,\lambda,\mu) := e^{\Omega(\mathbf{f}_\alpha(t,\lambda;z)_+,\mathbf{f}_\beta(t,\mu;z)_-)}. \tag{86}$$

The action of the vertex operators on the Fock space is not well defined in general. We would like to recall the conjugation laws from [28] and to make sense of the vertex operator action on the Fock space.

7.3.1. *Vertex operators at infinity.* Let us fix $t \in M$ and expand the vertex operators $\Gamma^\alpha(t,\lambda)$ in a neighborhood of $\lambda = \infty$. By definition (see (28)) we have $f_\alpha(t,\lambda;z) = S_t\widetilde{\mathbf{f}}_\alpha(\lambda;z)$. Using formula (84), it is easy to prove that

$$\widetilde{\Gamma}^\alpha(\lambda)\,\widehat{S}_t^{-1} = e^{\frac{1}{2}W(\widetilde{\mathbf{f}}_\alpha(\lambda)_+,\widetilde{\mathbf{f}}_\alpha(\lambda)_+)}\widehat{S}_t^{-1}\,\Gamma^\alpha(t,\lambda). \tag{87}$$

In particular, using the formal λ^{-1}-adic topology we get that the vertex operator $\Gamma^\alpha(t,\lambda)$ defines a linear map

$$\mathbb{C}_\hbar[\![\mathbf{q}]\!] \to K_\hbar[\![\mathbf{q}]\!],$$

where K is an appropriate field extension of the field $\mathbb{C}(\!(\lambda^{-1})\!)$.

Let us explain the relation between the phase factors. Recall formula (76), the RHS is interpreted as an element in $\mathbb{C}(\!(\lambda^{-1/\kappa})\!)(\!(\mu^{-1/\kappa})\!)$ by taking the Laurent series expansion with respect to λ at $\lambda = \infty$.

Proposition 37. *The following formula holds:*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \widetilde{B}_{\alpha,\beta}(\mu,\lambda) e^{W_t(\widetilde{\mathbf{f}}_\alpha(\mu)_+,\widetilde{\mathbf{f}}_\beta(\lambda)_+)}.$$

Proof. Conjugating the identity $\widetilde{\Gamma}^\alpha(\lambda)\widetilde{\Gamma}^\beta(\mu) = \widetilde{B}_{\alpha,\beta}(\lambda,\mu) :\widetilde{\Gamma}^\alpha(\lambda)\widetilde{\Gamma}^\beta(\mu):$ by $\widehat{S}$ and using formula (87) we get that

$$e^{\frac{1}{2}\left(W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+,\widetilde{\mathbf{f}}_\alpha(\lambda)_+)+W_t(\widetilde{\mathbf{f}}_\beta(\mu)_+,\widetilde{\mathbf{f}}_\beta(\mu)_+)\right)} B_{\alpha,\beta}(t,\lambda,\mu)$$

coincides with

$$e^{\frac{1}{2}W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_++\widetilde{\mathbf{f}}_\beta(\mu)_+,\widetilde{\mathbf{f}}_\alpha(\lambda)_++\widetilde{\mathbf{f}}_\beta(\mu)_+)}\widetilde{B}_{\alpha,\beta}(\lambda,\mu).$$

The quadratic form W is symmetric, so comparing the above identities yields the desired formula. □

7.3.2. *Vertex operators at a critical value.* Let us assume now that λ is near one of the critical values $u_i(t)$ and that β is a cycle vanishing over $\lambda = u_i(t)$. According to Lemma 13 we have $\mathbf{f}_\beta(t,\lambda;z) = \Psi_t R_t(z)\mathbf{f}_{A_1}(u_i,\lambda;z)$. Using Lemma 35 it is easy to prove (see [28], Section 7) that

$$\Gamma^\beta(t,\lambda)\,\widehat{\Psi}_t\widehat{R}_t = e^{\frac{1}{2}V_t(\mathbf{f}_\beta(t,\lambda)_-,\mathbf{f}_\beta(t,\lambda)_-)}\widehat{\Psi}_t\widehat{R}_t\,\Gamma^\pm_{A_1}(u_i,\lambda), \tag{88}$$

where $\Gamma^\pm_{A_1}(u_i,\lambda) =: e^{\pm\widehat{\mathbf{f}}_{A_1}(u_i,\lambda)}:$ is the vertex operator of the A_1-singularity, V_t is the second order differential operator defined in Lemma 35, and

$$V_t(\mathbf{f}_\beta(t,\lambda)_-,\mathbf{f}_\beta(t,\lambda)_-) = \sum_{k,l=0}^\infty (I_\beta^{(-k)}(t,\lambda), V_{kl}\, I_\beta^{(-l)}(t,\lambda)).$$

In this case, the action of the vertex operators is well-defined on the subspace spanned by the tame asymptotical functions and it yields a linear map

$$\Gamma^\beta(t,\lambda) : \mathbb{C}_\hbar[\![\mathbf{q}]\!]_{\text{tame}} \to K_\hbar[\![\mathbf{q}]\!],$$

where $K = \mathbb{C}(\!((\lambda-u_i)^{1/2})\!)$. Furthermore, the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is well defined if β is a vanishing cycle, since it can be interpreted as an element in $\mathbb{C}(\!((\mu-u_i)^{1/2})\!)(\!((\lambda-u_i)^{1/2})\!)$. Finally, similarly to Proposition 37, we have

$$B_{\beta,\beta}(t,\lambda,\mu) = B_{A_1}(u_i,\lambda,\mu)e^{-V_t(\mathbf{f}_\beta(t,\lambda)_-,\mathbf{f}_\beta(t,\mu)_-)}, \tag{89}$$

where $B_{A_1}(u_i,\lambda,\mu)$ is the phase factor of the product $\Gamma^\pm_{A_1}(u_i,\lambda)\Gamma^\pm_{A_1}(u_i,\mu)$. A straightforward computation gives

$$B_{A_1}(u_i,\lambda,\mu) = \left(\frac{\sqrt{\lambda-u_i}-\sqrt{\mu-u_i}}{\sqrt{\lambda-u_i}+\sqrt{\mu-u_i}}\right)^2, \tag{90}$$

where the RHS should be expanded into a Laurent series with respect to μ at $\mu = u_i$.

7.4. **From descendants to ancestors.** Following our construction of the HQEs from Section 6.3 we would like to introduce an integrable hierarchy for the ancestor potential $\mathcal{A}_t$. Let us introduce the Heisenberg fields

$$\phi_\beta(t,\lambda) = \partial_\lambda \widehat{\mathbf{f}}^\beta(t,\lambda),$$

and the corresponding Casimir operator

$$\begin{aligned}\Omega_{\Delta'}(t,\lambda) &= -\frac{\lambda^2}{2}\Big(\sum_{i=1}^N :(\phi_i(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi_i(t,\lambda))(\phi^i(t,\lambda)\otimes_{\mathfrak{a}} 1 - 1\otimes_{\mathfrak{a}}\phi^i(t,\lambda)): \Big) + \\ &\quad + \sum_{\beta\in\Delta'} b_\beta(t,\lambda)\Gamma^\beta(t,\lambda)\otimes_{\mathfrak{a}}\Gamma^{-\beta}(t,\lambda) - \frac{1}{2}\operatorname{tr}\Big(\frac{1}{4}+\theta\,\theta^{\mathrm{T}}\Big),\end{aligned}$$

where $\phi_i := \phi_{\beta_i}$, $\phi^i := \phi_{\beta^i}$ (with $\{\beta_i\}$ and $\{\beta^i\}$ chosen as in Section 6.3), and the coefficient $b_\beta(t,\lambda)$ are defined by

$$b_\beta(t,\lambda)^{-1} = \lim_{\mu\to\lambda}\ \Big(1-\frac{\mu}{\lambda}\Big)^2 B_{\beta,-\beta}(t,\lambda,\mu). \tag{91}$$

Finally, we need also to discretize the HQEs corresponding to the above Casimir operator so that we offset the problem of multivaluedness. Note that, for the toroidal cycle φ in Section 4.4, according to Proposition 21 the vector $\mathbf{f}^\varphi(t,\lambda;z)$ has only negative powers of z, so the quantization $\widehat{\mathbf{f}}^\varphi(t,\lambda)$ is a linear function in $\mathbf{q}$.

Lemma 38. *Let φ be the toroidal cycle. Then the equation*

$$\widehat{\mathbf{f}}^\varphi(t,\lambda)\otimes 1 - 1\otimes\widehat{\mathbf{f}}^\varphi(t,\lambda) = 2\pi\sqrt{-1}\,n \tag{92}$$

is equivalent to

$$[S_t^{-1}\mathbf{q}(z)]_{0,01}\otimes 1 - 1\otimes [S_t^{-1}\mathbf{q}(z)]_{0,01} = n\sqrt{\hbar} \tag{93}$$

$$[S_t^{-1}\mathbf{q}(z)]_{l,01}\otimes 1 - 1\otimes [S_t^{-1}\mathbf{q}(z)]_{l,01} = 0, \quad \forall l\geq 1, \tag{94}$$

where $[f(z)]_{l,i}$ denotes the coefficient in front of $\phi_i z^l$.

Proof. Note that

$$\widetilde{\mathbf{f}}^\varphi(\lambda;z) = 2\pi\sqrt{-1}\ \sum_{l=0}^\infty \frac{\lambda^l}{l!}\,\phi_{02}\,(-z)^{-l-1}.$$

The equations (93)–(94) can be written equivalently as

$$\Omega(\widetilde{\mathbf{f}}^\varphi(\lambda;z), S_t^{-1}\mathbf{q}(z)) = 2\pi\sqrt{-1}\,n\sqrt{\hbar}.$$

It remains only to recall that S_t is a symplectic transformation and that

$$\mathbf{f}^\varphi(t,\lambda;z) = S_t\widetilde{\mathbf{f}}^\varphi(\lambda;z).$$

□

We will be interested in the following HQEs: for every integer $n\in\mathbb{Z}$

$$\operatorname{Res}_{\lambda=\infty}\frac{d\lambda}{\lambda}\Big(\Omega_{\Delta'}(t,\lambda)\,(\tau\otimes\tau)\Big)\Big|_{\widehat{\mathbf{f}}^\varphi(t,\lambda)\otimes 1 - 1\otimes\widehat{\mathbf{f}}^\varphi(t,\lambda) = 2\pi\sqrt{-1}\,n} = 0, \tag{95}$$

where τ belongs to an appropriate Fock space and we have to require also that the discretization is well defined. For our purposes the HQEs (95) will be on the Fock space $\mathbb{C}_\hbar[\![q_0+t, q_1+1, q_2, \dots]\!]$. On the other hand the operator $\widehat{S}_t^{-1}$ gives rise to an isomorphism

$$\widehat{S}_t^{-1} : \mathbb{C}_\hbar[\![q_0+t, q_1+\mathbf{1}, q_2, \dots]\!] \to \mathbb{C}_\hbar[\![q_0, q_1+1, q_2, \dots]\!].$$

which allows us to identify the HQEs (77) and (95).

Proposition 39. *A function τ is a solution to the HQEs* (95) *iff $\widehat{S}_t^{-1}\tau$ is a solution to the HQEs* (77).

Proof. Using Proposition 37 we get that

$$\widetilde{\Omega}_{\Delta'}(\lambda)\,(\widehat{S}_t^{-1}\otimes\widehat{S}_t^{-1}) = (\widehat{S}_t^{-1}\otimes\widehat{S}_t^{-1})\,\Omega_{\Delta'}(t,\lambda).$$

It remains only to notice that the discretizations in both HQEs are compatible with the action of $\widehat{S}_t$, which follows easily from Lemma 36 and Lemma 38. □

8. The phase factors and the ancestor hierarchy

In this section we prove

Theorem 40. *The total ancestor potential $\mathcal{A}_t(\hbar;\mathbf{q})$ is a solution to the HQEs* (95).

Given this, Proposition 39 implies that the total descendant potential $\mathcal{D}_{\mathbf{a}}(\hbar;\mathbf{q})$ is a solution to the HQEs (77). Finally, recalling Theorem 33, we obtain a proof of our main result Theorem 1.

8.1. The integrable hierarchy for A_1-singularity.

It was conjectured by Witten [62] and first proved by Kontsevich [45] that the total descendant potential of a point is a tau-function of the KdV hierarchy. The latter can be written in two different ways: via the Kac-Wakimoto construction and as a reduction of the KP hierarchy. We will need both realizations, so let us recall them.

8.1.1. *The Kac–Wakimoto construction of KdV.* The Casimir operator (cf. Section 7.4) for the A_1-singularity $f(x) = x^2/2 + u$ takes the form

$$\begin{aligned}\Omega_{A_1}(u,\lambda) \;=\; & -\frac{\lambda^2}{4} : \phi_\beta^{V\otimes V}(u,\lambda)\phi_\beta^{V\otimes V}(u,\lambda) : + \\ & + b_\beta(u,\lambda)\Big(\Gamma^{\beta}_{A_1}(u,\lambda)\otimes\Gamma^{-\beta}_{A_1}(u,\lambda) + \Gamma^{-\beta}_{A_1}(u,\lambda)\otimes\Gamma^{\beta}_{A_1}(u,\lambda)\Big) - \frac{1}{8},\end{aligned}$$

where the coefficient

$$b_\beta(u,\lambda) = \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2} B_{\beta,\beta}(u,\mu,\lambda) = \frac{\lambda^2}{16(\lambda-u)^2}.$$

We denoted by V the Fock space $\mathbb{C}_\hbar[\![\mathbf{q}]\!]$, and

$$\phi_\beta^{V\otimes V}(u,\lambda) := \phi_\beta(u,\lambda)\otimes 1 - 1\otimes\phi_\beta(u,\lambda).$$

Witten's conjecture (Kontsevich's theorem) can be stated as follows:

$$\operatorname{Res}_{\lambda=\infty}\Omega_{A_1}(0,\lambda)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0. \tag{96}$$

To compare the above equation with the principal Kac-Wakimoto hierarchy of type A_1, note that

$$\Gamma^{\beta}_{A_1}(u,\lambda) = \exp\Big(2\sum_{n=0}^{\infty}\frac{(2(\lambda-u))^{n+1/2}}{(2n+1)!!}\,\frac{q_n}{\sqrt{\hbar}}\Big)\exp\Big(-2\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2(\lambda-u))^{n+1/2}}\sqrt{\hbar}\partial_n\Big),$$

and that the coefficient in front of λ^{-2} in $\frac{1}{4}:\phi_{\beta}^{V\otimes V}(0,\lambda)\phi_{\beta}^{V\otimes V}(0,\lambda):$ is precisely

$$\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)(q_n\otimes 1 - 1\otimes q_n)(\partial_n\otimes 1 - 1\otimes\partial_n),$$

where $\partial_n := \partial/\partial q_n$. It follows that the above equations coincide with the Kac–Wakimoto form of the KdV hierarchy up to the rescaling $q_n = t_{2n+1}(2n+1)!!$.

On the other hand, the total descendant potential $\mathcal{D}_{\rm pt}$ satisfies the string equation, which can be stated as follows (see [30]): $e^{(u/z)^{\widehat{}}}\mathcal{D}_{\rm pt} = \mathcal{D}_{\rm pt}$. Using that

$$\Omega_{A_1}(0,\lambda)\left(e^{(u/z)^{\widehat{}}}\otimes e^{(u/z)^{\widehat{}}}\right) = \left(e^{(u/z)^{\widehat{}}}\otimes e^{(u/z)^{\widehat{}}}\right)\Omega_{A_1}(u,\lambda)$$

we get that $\mathcal{D}_{\rm pt}$ satisfies also the following HQEs:

$$\mathrm{Res}_{\lambda=\infty}\,\Omega_{A_1}(u,\lambda)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0. \tag{97}$$

8.1.2. *The KdV hierarchy as a reduction of KP.* According to Givental [28] the KdV hiearachy (96) can be written also as

$$\mathrm{Res}_{\lambda=0}\,\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda}}\,\Gamma^{\pm\beta/2}_{A_1}(0,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(0,\lambda)\Big)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0.$$

Using again the string equation and Proposition 37 we get that $\mathcal{D}_{\rm pt}$ satisfies also the following HQEs:

$$\mathrm{Res}_{\lambda=u}\,\Big(\sum_{\pm}\frac{d\lambda}{\pm\sqrt{\lambda-u}}\,\Gamma^{\pm\beta/2}_{A_1}(u,\lambda)\otimes\Gamma^{\mp\beta/2}_{A_1}(u,\lambda)\Big)\,(\mathcal{D}_{\rm pt}\otimes\mathcal{D}_{\rm pt}) = 0. \tag{98}$$

8.2. **The phase factors.** Recall that the phase factor (86) has the form $B_{\alpha,\beta} = \exp\Omega_{\alpha,\beta}$ with

$$\Omega_{\alpha,\beta}(t,\lambda,\mu) := \sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_{\alpha}(t,\lambda), I^{(-n-1)}_{\beta}(t,\mu)), \tag{99}$$

where each term on the RHS is expanded into a Laurent series near $\lambda=\infty$ and $\mu=\infty$. On the other hand if λ and μ are close to some critical value $u_i(t)$ and β vanishes over $\mu = u_i(t)$, then the Laurent series expansion of the RHS of (99) in $\lambda - u_i(t)$ and $\mu - u_i(t)$ makes sense and we obtain yet another formal Laurent series. We would like to prove that these two different formal interpretations of (99) are related to each other via analytic continuation. This is the key to proving Theorem 40, because it allows us to reduce the HQE to several copies of HQE for A_1-singularity.

The differential of (99) with respect to t is the 1-form

$$I^{(0)}_\alpha(t,\lambda)\bullet I^{(0)}_\beta(t,\mu)=\sum_{i\in\mathfrak{I}}(I^{(0)}_\alpha(t,\lambda),\partial_i\bullet I^{(0)}_\beta(t,\mu))\,dt_i.$$

This suggests that the analytic continuation that we are looking for can be constructed in terms of integrals along the path of the following family of 1-forms:

$$\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)=I^{(0)}_\alpha(t,\lambda)\bullet_t I^{(0)}_\beta(t,\mu),$$

where λ and μ are viewed as parameters. Sometimes it will be convenient to use also the 1-form $\mathcal{W}_{\alpha,\beta}(\xi):=\widetilde{\mathcal{W}}_{\alpha,\beta}(\xi,0)$, which is also known as the *phase form* (see [28]). Let us introduce the open subset

$$D=\{(t,\lambda,\mu)\in M\times\mathbb{C}^2\ :\ |\mu-\lambda|<\max(|\lambda-u_i(t)|,|\mu-u_i(t)|)\quad\forall i\},$$

where $\{u_i(t)\}_i$ is the set of critical values of $F(x,t)$. Since we will be dealing with multivalued analytic functions on D that have a pole along the diagonal $\mu=\lambda$, let us fix a reference point $(t_*,\lambda_*,\mu_*)\in D$, with $\mu_*\neq\lambda_*$. In order to specify the value of a multivalued analytic function at any other point $(t,\lambda,\mu)\in D$ we always select a path consisting of the composition of a straight segment $[(t_*,\lambda_*,\mu_*),(t_*,\lambda_*,\lambda_*+\varepsilon)]$, a path from $(t_*,\lambda_*,\lambda_*+\varepsilon)$ to $(t,\lambda,\lambda+\varepsilon)$, and a straight segment $[(t,\lambda,\lambda+\varepsilon),(t,\lambda,\mu)]$, where $|\varepsilon|\ll 1$ and the second path consists of points (t',λ',μ') sufficiently close to the diagonal, e.g., $0<|\lambda'-\mu'|<\varepsilon$. Furthermore, for each $t\in M$, put $r(t)=\max_i|u_i(t)|$, i.e., this is the radius of the smallest disk (with center at 0) that contains all critical values. Let

$$D_\infty=\{(t,\lambda,\mu)\in D\ :\ |\lambda|>|\mu|>r(t),\ |\lambda-\mu|<\max(|\lambda|,|\mu|)\}.$$

Proposition 41. *The series* (99) *is convergent for all* $(t,\lambda,\mu)\in D_\infty$.

Proof. Using Proposition 37 we can write (99) as a sum of two formal series

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)=\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)+W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+,\widetilde{\mathbf{f}}_\beta(\mu)_+),$$

where $\widetilde{\Omega}_{\alpha,\beta}:=\log\widetilde{B}_{\alpha,\beta}$ is expanded into a Laurent series in the domain $|\lambda|>|\mu|$. It is enough to prove the proposition for the second series on the RHS of the above equality. Recalling the definition of W_t and using the fact that modulo Q the series $S_t(z)=e^{t\cup/z}$, where $t\cup$ means the classical orbifold cup product multiplication by t, we get that

$$\lim_{\mathrm{Re}(t_{02})\to-\infty}\ \lim_{'t\to 0}\ (W_t-t_{02}\,P)=0,$$

where $'t$ is the point with coordinates t_i, $i\in\mathfrak{I}\setminus\{02\}$. On the other hand, since

$$dW_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+,\widetilde{\mathbf{f}}_\beta(\mu)_+)=d\Omega_{\alpha,\beta}(t,\lambda,\mu)=I^{(0)}_\alpha(t,\lambda)\bullet I^{(0)}_\beta(t,\mu),$$

the series

$$W_t(\widetilde{\mathbf{f}}_\alpha(\lambda)_+,\widetilde{\mathbf{f}}_\beta(\mu)_+)-t_{02}(\alpha_0|\beta_0)/\chi,\tag{100}$$

viewed as a formal Laurent series in λ^{-1} and μ^{-1} can be identified with the improper integral

$$\lim_{\varepsilon\to\infty}\int_\varepsilon^t \Big(I^{(0)}_\alpha(t',\lambda)\bullet I^{(0)}_\beta(t',\mu) - dt'_{02}(\alpha_0|\beta_0)/\chi\Big), \tag{101}$$

where $\varepsilon\in M$ and the limit is taken along a straight segment, s.t., $\varepsilon_i\to 0$ for $i\neq 02$ and $\mathrm{Re}(\varepsilon_{02})\to-\infty$. More precisely, if we take the Laurent series expansion of the integrand at $\lambda=\infty$ and $\mu=\infty$ and integrate termwise, we get (100).

In order to prove the convergence, it is enough to choose an intgeration path, such that the Laurent series expansions of $I^{(0)}_\alpha(t',\lambda)$ and $I^{(0)}_\beta(t',\mu)$ are convergent for $|\lambda|>|\mu|>r(t)$ and such that the termwise integration preserves convergence. The period $I^{(0)}_\alpha(t',\lambda)$ (resp. $I^{(0)}_\beta(t',\mu)$) is a solution to an ordinary differential equation in λ that has a regular singular point at $\lambda=\infty$ and all other singular points are at the critical values $u_j(t)$ (which are also regular). Therefore, the Laurent series expansion of $I^{(0)}_\alpha(t',\lambda)$ (resp. $I^{(0)}_\beta(t',\mu)$) is convergent for $|\lambda|>r(t')$. (resp. $|\mu|>r(t')$). Let us pick the integration path C (e.g. a trajectory of the Euler vector field E), s.t., $r(t')$ is decreasing as t' varies along C from t to ∞; then the Laurent series expansion of the integrand is convergent for $|\lambda|>r(t),|\mu|>r(t)$. Moreover, after changing the integration variables $q'_i=t'_i$ for $i\neq 02$ and $q'_{02}=e^{t'_{02}}$ and setting $\lambda=\zeta^\kappa$, $\mu=w^\kappa$, we get that the integrand depends holomorphically on (q',ζ,w) for $|\zeta|>r(t)^{1/\kappa},|w|>r(t)^{1/\kappa}$ and $q'\in C$. Hence, the Laurent series expansion in λ^{-1} and μ^{-1} is uniformly convergent in $q'\in C$, which implies that the termwise integration preserves the convergence. □

The proof of Proposition 41 yields slightly more. Namely, it tells us how to extend analytically the phase factors from D_∞ to D. Let us give the precise statements. Put

$$D^c_\infty=\{(t,\lambda,\mu)\in D\ :\ |\lambda|>r(t),\ |\mu|>r(t),\ |\lambda-\mu|<\max(|\lambda|,|\mu|)\}.$$

Note that $D_\infty\subset D^c_\infty\subset D$ and that the phase factor $\widetilde{B}_{\alpha,\beta}(\lambda,\mu)$ is a multivalued analytic function on D^c_∞ with poles along $\lambda=\mu$.

Corollary 42. *The Laurent series expansion of $\widetilde{B}^{-1}_{\alpha,\beta}(\lambda,\mu)B_{\alpha,\beta}(t,\lambda,\mu)$ in λ^{-1} and μ^{-1} is convergent $\forall(t,\lambda,\mu)\in D^c_\infty$.*

In other words, the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ extends analytically to D^c_∞ except for a possible pole along the diagonal $\lambda=\mu$.

Corollary 43. *Let $\lambda\neq\mu$ be fixed numbers and C be a path in D of the form $C'\times\{\lambda\}\times\{\mu\}$ (C' is a path in M) connecting $(t,\lambda,\mu)\in D$ with a point $(t_0,\lambda,\mu)\in D^c_\infty$, then the phase factor $B_{\alpha,\beta}(t_0,\lambda,\mu)$ extends analytically along C. Moreover, the analytic extension is given by*

$$B_{\alpha,\beta}(t,\lambda,\mu)=B_{\alpha,\beta}(t_0,\lambda,\mu)\,e^{\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)}.$$

Let $t_0\in M$ be a generic point, so that all critical points of $F(x,t_0)$ are of type A_1 and the corresponding critical values are pairwise distinct. Let $u_i(t_0)$ be a critical value of $F(x,t_0)$ with a maximal absolute value, i.e., $|u_i(t_0)|=r(t_0)$. Since the period vectors $I^{(0)}_\alpha(t_0,\lambda)$ satisfy a Fuchsian differential equation in λ with singularities at the critical values of $F(x,t_0)$ we get that the Laurent series expansion of $I^{(0)}_\alpha(t_0,\lambda)$ at $\lambda=u_i(t_0)$ is convergent for all λ, s.t., $0<|\lambda-u_i(t_0)|<r_i(t_0)$, where $r_i(t_0)$ is the distance from $u_i(t_0)$ to the closest other singular point. Furthermore, let $(t,\lambda,\mu)\in D_\infty$ be an arbitrary point satisfying $|\lambda-\mu|<\frac{1}{2}r_i(t_0)$. Using the triangle inequality we have (for $j\neq i$)

$$|\lambda-\mu|<|u_j(t_0)-u_i(t_0)|-|\lambda-\mu|\leq|\lambda-(u_j(t_0)+\mu-u_i(t_0))|,$$

which proves that the point $(t_0+(\mu-u_i(t_0))\mathbf{1},\lambda,\mu)$ is on the boundary of D_∞. Every path C in D_∞ from (t,λ,μ) approaching the boundary point $(t_0+(\mu-u_i(t_0))\mathbf{1},\lambda,\mu)$ determines a path in $(M\times\mathbb{C})'$ from (t,μ) approaching the discriminant at $(t_0,u_i(t_0))$, so it makes sense to say that a cycle $\beta\in H_2(X_{t,\mu};\mathbb{Z})$ vanishes along C. Finally, note that the unit vector $\mathbf{1}\in H\cong M$ has coordinates $t_{01}=1$, $t_i=0$ for $i\neq(0,1)$ and that the period vectors have the following translation symmetry:

$$I^{(n)}_\alpha(t,\lambda)=I^{(n)}_\alpha(t-\lambda\mathbf{1},0),\quad\forall n\in\mathbb{Z},\quad\forall\alpha\in\mathfrak{h}.$$

Lemma 44. *Let $t_0\in M$, $(t,\lambda,\mu)\in D_\infty$ with $|\lambda-\mu|<\frac{1}{2}r_i(t_0)$, and $\beta\in H_2(X_{t,\mu};\mathbb{Z})$ be a cycle vanishing over $(t_0,u_i(t_0))$ along some path $C\subset D_\infty$, then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)=\lim_{\varepsilon\to0}\int_{t_0+(\varepsilon+\mu-u_i(t_0))\mathbf{1}}^{t}\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu),\tag{102}$$

where the integration is along the path C and the limit is taken along a straight segment.

Proof. According to Corollary 43, it is enough to prove the lemma for a single point $(t,\lambda,\mu)\in D_\infty$. Let us choose a point $(t_0',\lambda,\mu)\in D_\infty$ sufficiently close to the boundary point $(t_0+(\mu-u_i(t_0))\mathbf{1},\lambda,\mu)$, s.t., $t_0'=t_0+x_0\mathbf{1}$, $x_0\in\mathbb{C}$. By definition $\Omega_{\alpha,\beta}(t_0',\lambda,\mu)$ is the Laurent series expansion near $\lambda=\infty$ of the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}(I^{(n)}_\alpha(t_0',\lambda),I^{(-n-1)}_\beta(t_0',\mu)).\tag{103}$$

Let us point out that (103) might be divergent, although its Laurent series expansion $\Omega_{\alpha,\beta}(t_0',\lambda,\mu)$ is convergent. We may assume that our integration path is a straight segment, which allows us to write the RHS of (102) as

$$\lim_{\varepsilon\to0}\int_{\varepsilon+\mu-u_i}^{x_0}(I^{(0)}_\alpha(t_0,\lambda-x),I^{(0)}_\beta(t_0,\mu-x))dx.\tag{104}$$

Using integration by parts $(n+1)$ times and the fact that the periods $I^{(-p-1)}_\beta(t_0,\mu-x)$ vanish at $x=\mu-u_i$, we get that the integral (104) coincides with

$$\sum_{p=0}^{n}(-1)^{p+1}(I^{(p)}_\alpha(t_0',\lambda),I^{(-p-1)}_\beta(t_0',\mu))+$$
$$\lim_{\varepsilon\to0}(-1)^{n+1}\int_{\varepsilon+\mu-u_i}^{x_0}(I^{(n+1)}_\alpha(t_0,\lambda-x),I^{(-n-1)}_\beta(t_0,\mu-x))dx.\tag{105}$$

The Laurent series expansion of $I^{(n+1)}_\alpha(t_0,\lambda-x) = I^{(n+1)}_\alpha(t_0+x\mathbf{1},\lambda)$ in λ^{-1} is uniformly convergent for all x that vary along a compact subset of the open subset in $\mathbb{C}$ defined by the inequality

$$\{x\in\mathbb{C} \ : \ |\lambda| > |u_i(t_0)+x|\}.$$

Since $|\lambda| > |\mu| > |u_i(t_0)|$, by choosing t_0' sufficiently close to $t_0+(\mu-u_i(t_0))\mathbf{1}$, we may arrange that the integration path is entirely contained in the above open subset. Hence the integral (105) has a convergent Laurent series in λ^{-1}. Moreover, the leading order term of the expansion is λ^{-e} for some rational number $e>n$. This proves that the integral (104) has a Laurent series expansion in λ^{-1} that coincides with the Laurent series expansion of the series (103). □

So far we have managed to prove that the phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ extends to a multivalued analytic function on D except for a possible pole along the diagonal $\lambda=\mu$. Our next goal is to prove that the analytic continuation is compatible with the monodromy representation. Let us make this statement precise. The projection

$$\pi : D\to (M\times\mathbb{C})', \quad (t,\lambda,\mu)\mapsto (t,\lambda)$$

is surjective and the fiber over (t,λ) is an open disk with radius $\min_i |\lambda-u_i(t)|$, so π is a deformation retract. In particular, we see that π induces an isomorphism of the fundamental groups, so we have a well defined monodromy representation (cf. Section 3.2)

$$\rho : \pi_1(D)\to \mathrm{GL}(\mathfrak{h}).$$

Let $U\subset D$ be an open subdomain and $f_{\alpha,\beta}(t,\lambda,\mu)$ be a function depending bilinearly on $(\alpha,\beta)\in\mathfrak{h}\times\mathfrak{h}$ and analytic in a neighborhoood of some point $(t_0,\lambda_0,\mu_0)\in U$. We say that $f_{\alpha,\beta}$ is *multi-valued analytic* on U if it can be extended analytically along any path in U. Furthermore, we say that $f_{\alpha,\beta}$ is *compatible* with the monodromy representation ρ, if for every closed loop C in U, the analytic continuation of $f_{\alpha,\beta}(t,\lambda,\mu)$ along C coincides with $f_{w(\alpha),w(\beta)}(t,\lambda,\mu)$, where $w=\rho(C)$ is the corresponding monodromy transformation.

Lemma 45. *Let α and β be cycles in the vanishing cohomology, s.t., $(\alpha|\beta)=0$ then*

$$\Omega_{\alpha,\beta}(t,\lambda,\mu)-\Omega^{\mathrm{an}}_{\beta,\alpha}(t,\mu,\lambda) = 2\pi\sqrt{-1}\,\mathrm{SF}(\alpha,\beta) \quad \forall (t,\lambda,\mu)\in D_\infty,$$

where SF *is the bi-linear form* (65) *and $\Omega^{\mathrm{an}}_{\beta,\alpha}(t,\mu,\lambda)$ is the analytic continuation of $\Omega_{\beta,\alpha}(t,\lambda,\mu)$ along the straight segment $[(t,\lambda,\mu),(t,\mu,\lambda)]$.*

Proof. According to Corollary 42 the phase factor $\Omega_{\beta,\alpha}(t,\lambda',\mu')$ extends analytically to a multi-valued analytic function $\Omega^{\mathrm{an}}_{\beta,\alpha}(t,\lambda',\mu')$ defined for all $(t,\lambda',\mu')\in D^{\mathrm{c}}_\infty$, s.t., $\lambda'\neq\mu'$. Moreover, the difference

$$\Omega_{\beta,\alpha}(t,\lambda',\mu')-\widetilde{\Omega}_{\beta,\alpha}(\lambda',\mu'), \quad \text{where} \quad \widetilde{\Omega}_{\beta,\alpha}(\lambda',\mu') := \log\widetilde{B}_{\beta,\alpha}(\lambda',\mu'),$$

has a convergent Laurent series expansion in D^{c}_∞ and it is invariant under switching $(\beta,\lambda')\leftrightarrow(\alpha,\mu')$. Therefore, it is enough to prove the statement for $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ where (λ,μ) is a point in the open subdomain of $\mathbb{C}^2$ defined by

$$|\lambda-\mu| < \max(|\lambda|,|\mu|).$$

Recalling formula (76), the rest of the proof is a streightforward computation (see also the proof of Lemma 28, where some of the computations were already done). □

Remark 46. *If we omit the condition $(\alpha|\beta)=0$ in Lemma 45, then the identity is true only up to an integer multiple of $2\pi\sqrt{-1}(\alpha|\beta)$. The ambiguity comes from the fact that the phase factor $\widetilde{\Omega}_{\alpha,\beta}(\lambda,\mu)$ has a logarithmic singularity along $\lambda=\mu$ of the type $(\alpha|\beta)\log(\lambda-\mu)$.*

Proposition 47. *The phase factor $B_{\alpha,\beta}(t,\lambda,\mu)$ is compatible with the monodromy representation in the domain D.*

Proof. According to Corollary 43, we have to prove that if $C=C'\times\{(\lambda,\mu)\}\subset D$ is an arbitrary loop based at (t,λ,μ), then

$$B_{w(\alpha),w(\beta)}(t,\lambda,\mu)=B_{\alpha,\beta}(t,\lambda,\mu)\,e^{\int_C \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)},$$

where $w=\rho(C)$. We may assume that $(t,\lambda,\mu)\in D_\infty$, then the above equality is equivalent to

$$\Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu)=\Omega_{\alpha,\beta}(t,\lambda,\mu)+\int_C \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\pmod{2\pi\sqrt{-1}\,\mathbb{Z}}. \tag{106}$$

We first prove a special case of the above formula. Namely, let us choose a generic point $t_0\in M$, s.t., if $u_i(t_0)$ is a critical value with maximal absolute value then $|\lambda-\mu|\ll r_i(t_0)$ (see the notation in Lemma 44). We will assume that $t=t_0+x_0\mathbf{1}$ is sufficiently close to $t_0+(\mu-u_i(t_0))\mathbf{1}$ and that C is a closed loop of the type $t_0+x\mathbf{1}$, where the parameter x varies along a small closed loop based at $x_0\in\mathbb{C}$ going around $\mu-u_i(t_0)$, so that the line segment $[\lambda-x,\mu-x]$ moves around u_i. Let us denote by $\gamma\in H_2(X_{t,\lambda};\mathbb{Z})$ the vanishing cycle vanishing over $(t_0,u_i(t_0))$, then we have the following decompositions:

$$\alpha=\alpha'+\frac{(\alpha|\gamma)}{2}\,\gamma,\quad \beta=\beta'+\frac{(\beta|\gamma)}{2}\,\gamma,$$

where α' and β' are cycles invariant w.r.t. the local monodromy around the point $(t_0,u_i(t_0))$. After a straightforward computation we get

$$\Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu)-\Omega_{\alpha,\beta}(t,\lambda,\mu)=-(\alpha|\gamma)\Omega_{\gamma,\beta'}(t,\lambda,\mu)-(\beta|\gamma)\Omega_{\alpha',\gamma}(t,\lambda,\mu),$$

while $\int_C\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ is

$$\frac{1}{2}(\beta|\gamma)\int_C\widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu)+\frac{1}{2}(\alpha|\gamma)\int_C\widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu)+\frac{1}{4}\,(\alpha|\gamma)\,(\beta|\gamma)\int_C\widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu), \tag{107}$$

where we used that $\int_C\widetilde{\mathcal{W}}_{\alpha',\beta'}(\lambda,\mu)=0$, because the periods $I^{(0)}_{\alpha'}(t_0,\lambda-x)$ and $I^{(0)}_{\beta'}(t_0,\mu-x)$ are holomorphic respectively at $x=\lambda-u_i$ and $x=\mu-u_i$, which means that the phase form is holomorphic inside the loop C. The last integral in the above formula is easy to compute because only the singular terms of $I^{(0)}_\gamma(t_0,\lambda-x)$ and $I^{(0)}_\gamma(t_0,\mu-x)$ contribute, i.e.,

$$\int_C\widetilde{\mathcal{W}}_{\gamma,\gamma}(\lambda,\mu)=2\oint\frac{dx}{\sqrt{(\lambda-u_i(t_0)-x)(\mu-u_i(t_0)-x)}}=4\pi\sqrt{-1}.$$

According to Lemma 44

$$\Omega_{\alpha',\gamma}(t,\lambda,\mu) = \int_{t_0+(\mu-u_i(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu)$$

and the integral on the RHS has a convergent Laurent series expansion in $\lambda - u_i(t)$ and $(\mu - u_i(t))^{1/2}$, which allows us to evaluate the integral

$$\int_C \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) = -2\int_{t_0+(\mu-u_i(t_0))\mathbf{1}}^{t} \widetilde{\mathcal{W}}_{\alpha',\gamma}(\lambda,\mu) = -2\Omega_{\alpha',\gamma}(t,\lambda,\mu).$$

It remains only to evaluate the 2-nd integral in (107). We have

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = \int_C \widetilde{\mathcal{W}}_{\beta',\gamma}(\mu,\lambda) = -2\Omega^{\mathrm{an}}_{\beta',\gamma}(t,\mu,\lambda),$$

where the 2-nd identity is derived just like above. Recalling Lemma 45, we get

$$\Omega^{\mathrm{an}}_{\beta',\gamma}(t,\mu,\lambda) = \Omega_{\gamma,\beta'}(t,\lambda,\mu) + 2\pi\sqrt{-1}\,\mathrm{SF}(\beta',\gamma).$$

Using that $\beta' = \beta - (\beta|\gamma)\gamma/2$ and that $\mathrm{SF}(\gamma,\gamma) = 1$, we finally get

$$\int_C \widetilde{\mathcal{W}}_{\gamma,\beta'}(\lambda,\mu) = -2\Omega_{\gamma,\beta'}(t,\lambda,\mu) - 4\pi\sqrt{-1}\,\mathrm{SF}(\beta,\gamma) + 2\pi\sqrt{-1}(\beta|\gamma).$$

Since $\mathrm{SF}(\beta,\gamma) \in \mathbb{Z}$, the proof of formula (106) in the special case is complete.

The general case follows easily, because the fundamental group $\pi_1((M\times\mathbb{C})')$ is generated by loops like the above one. Indeed, we already know that the affine cusp polynomial $f(x)$ has a real Morsification $F(x,t_0')$, i.e., all critical points of $F(x,t_0')$ are real and the corresponding critical values are real as well. In particular, we can find a small deformation $F(x,t_0)$ of the real Morsification, s.t., the critical values u_i are vertices of a convex polygon. The fundamental group $\pi_1((M\times\mathbb{C})')$ is generated by simple loops in $\{t_0\}\times\mathbb{C}$ that go around the vertices of the polygon. Let us pick one of these loops and let $(t_0,u_i(t_0))$ be the corresponding vertex of the polygon. Since the translations of the type $t_0\mapsto t_0+c\mathbf{1}$, $c\in\mathbb{C}$, do not change the homotopy class of the loop, we can find a representative (namely, pick c, s.t., the $|u_i(t_0)+c| > |u_j(t_0)+c|$ for all other vertices $(t_0,u_j(t_0))$) of the homotopy class, which has the special form from above. □

Proposition 48. *There exists a generic point $t_0\in M$ (i.e. $F(x,t_0)$ is a Morse function) and a critical value $u_i(t_0)$, s.t.,*

$$B_{\alpha,\beta}(t,\lambda,\mu) = \lim_{\varepsilon\to 0}\exp\Big(-\int_t^{t_0+(\varepsilon+\mu-u_i(t_0))\mathbf{1}} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)\Big), \tag{108}$$

where the integration is along any path of the form $C\times\{\lambda\}\times\{\mu\}\subset D$, s.t., the cycle $\beta\in H_2(X_{t,\mu},\mathbb{Z})$ vanishes along it.

Proof. Let us assume that t_0 is a generic point and that $u_i(t_0)$ is the critical value with maximal absolute value. It is enough to prove the statement for an arbitrary point $(t,\lambda,\mu)\in D_\infty$, because, according to Corollary 43, the value of $B_{\alpha,\beta}(t',\lambda,\mu)$ at any other point $(t',\lambda,\mu)\in D$ differs by an integral of $\widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu)$ along a path connecting t and t', while the RHS of (108) clearly has the same property. Furthermore, the integration path is homotopic to $C'' \circ C'$, where $C' \times \{\lambda\} \times \{\mu\} \subset D$ is a closed loop based at (t, λ, μ) and $C'' \times \{\lambda\} \times \{\mu\} \subset D_\infty$ is a path along which the cycle $w(\beta)$ vanishes, where $w = \rho(C')$ is the monodromy transformation along C'. According to Lemma 44, formula (108) holds for C'' and $B_{w(\alpha),w(\beta)}$. Therefore, we need to prove that

$$(109) \qquad -\int_{C'} \widetilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\mu) = \Omega_{\alpha,\beta}(t,\lambda,\mu) - \Omega_{w(\alpha),w(\beta)}(t,\lambda,\mu) \pmod{2\pi\sqrt{-1}\mathbb{Z}},$$

which follows immediately from Proposition 47. □

8.3. **The ancestor solution.** Now we are in a position to prove that the total ancestor potential $\mathcal{A}_t$ is a solution to the HQEs (95), i.e. Theorem 40. To begin with, put $\mathbf{q}' = \mathbf{q} \otimes 1$, $\mathbf{q}'' = 1 \otimes \mathbf{q}$, and let us assume that the discretization condition (92) is satisfied for some integer n. The tameness of $\mathcal{A}(\hbar; \mathbf{q})$ implies that the LHS of (95) (for $\tau = \mathcal{A}(\hbar; \mathbf{q})$) is a formal series in $\mathbf{q}'$ and $\mathbf{q}''$ with coefficients formal Laurent series in $\sqrt{\hbar}$, whose coefficients are polynomial expressions of the period vectors $I^{(n)}_\alpha(t, \lambda)$. In particular, the residue in (95) can be computed via the residue theorem, i.e., we have to compute the residues at all critical points and at $\lambda = 0$ and prove that their sum is 0.

Let $u_i(t)$ be one of the critical points of F, where $t \in M$ is a generic point such that all critical values are pairwise different. Furthermore, we assume that λ is near $u_i(t)$ and that a path in $(M \times \mathbb{C})'$ from the reference point $(0, 1)$ to (t, λ) is fixed in such a way that the vanishing cycle β, vanishing over $\lambda = u_i(t)$, belongs to the subset Δ' of affine roots defined in Section 6.3.

8.3.1. *The Virasoro term.* Let us compute

$$(110) \qquad -\operatorname{Res}_{\lambda=u_i(t)} \frac{\lambda}{2} d\lambda \sum_{i=1}^N :\phi^{V\otimes V}_{\beta_i}(t,\lambda)\phi^{V\otimes V}_{\beta^i}(t,\lambda):\ \mathcal{A}_t^{\otimes 2},$$

where $\phi^{V\otimes V}_\alpha := \phi_\alpha \otimes 1 - 1 \otimes \phi_\alpha$. Put $\beta_i = \alpha_i + (\beta_i|\beta)\beta/2$ and $\beta^i = \alpha^i + (\beta^i|\beta)\beta/2$, where $(\alpha_i|\beta) = (\alpha^i|\beta) = 0$. The above operator can be written as the sum of

$$\sum_{i=1}^N :\phi^{V\otimes V}_{\alpha_i}(t,\lambda)\phi^{V\otimes V}_{\alpha^i}(t,\lambda): + \Big(\sum_{i=1}^N (\beta_i|\beta)(\beta^i|\beta)\Big)\frac{1}{4} :\phi^{V\otimes V}_\beta(t,\lambda)\phi^{V\otimes V}_\beta(t,\lambda):$$

and

$$(111) \qquad \sum_{i=1}^N \frac{1}{2}\Big((\beta_i|\beta) :\phi^{V\otimes V}_\beta(t,\lambda)\phi^{V\otimes V}_{\alpha^i}(t,\lambda): + (\beta^i|\beta) :\phi^{V\otimes V}_\beta(t,\lambda)\phi^{V\otimes V}_{\alpha_i}(t,\lambda):\Big)$$

The Picard-Lefschetz formula implies that the periods $I^{(n)}_{\alpha_i}(t,\lambda)$ and $I^{(n)}_{\alpha^i}(t,\lambda)$ are invariant with respect to the local monodromy around $\lambda = u_i(t)$, so they must be holomorphic in a neighborhood of $\lambda = u_i(t)$. The operator $\phi^{V\otimes V}_\varphi(t,\lambda)$, where φ is the toroidal cycle, vanishes after we impose the discretization condition (92). On the other hand, since $\sum_i (\beta_i|\beta)(\beta^i|\alpha) = (\beta|\alpha)$, the cycles

$$-\beta + \sum_{i=1}^{N} (\beta_i|\beta)\beta^i \quad \text{and} \quad -\beta + \sum_{i=1}^{N} (\beta^i|\beta)\beta_i$$

are in the kernel of the intersection form, so they must be proportional to φ. Hence the operator (111) vasnishs after the discretization condition (92) is imposed. The residue (110) turns into

$$-\operatorname{Res}_{\lambda=u_i(t)} \frac{\lambda}{4} d\lambda \, :\phi_\beta^{V\otimes V}(t,\lambda)\phi_\beta^{V\otimes V}(t,\lambda):\; \mathcal{A}_t(\hbar;\mathbf{q}')\mathcal{A}_t(\hbar;\mathbf{q}'').$$

To compute the above residue, note that the expression

$$:\phi_\beta^{V\otimes V}(t,\lambda)\phi_\beta^{V\otimes V}(t,\lambda):(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}$$

can be written as

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2} :\phi_{A_1}^{V\otimes V}(u_i,\lambda)\phi_{A_1}^{V\otimes V}(u_i,\lambda): +2V_t(\phi_\beta(t,\lambda)_-,\phi_\beta(t,\lambda)_-).$$

Let us compute

$$-\operatorname{Res}_{\lambda=u_i(t)} \frac{\lambda}{4} d\lambda\, 2V_t(\phi_\beta(t,\lambda)_-,\phi_\beta(t,\lambda)_-) = -\operatorname{Res}_{\lambda=u_i(t)} \frac{\lambda}{2} d\lambda\, (V_{00}(t)I_\beta^{(0)}(t,\lambda), I_\beta^{(0)}(t,\lambda)),$$

where we used the fact that only the leading term (w.r.t. z) of $\phi_\beta(t,\lambda;z)_- = -I_\beta^{(0)}(t,\lambda)z^{-1} + \cdots$ will contribute because the remaining ones have a zero at $\lambda = u_i(t)$ of order at least $\frac{1}{2}$. Furthermore, the Laurent series expansion of $I_\beta^{(0)}$ at $\lambda = u_i(t)$ has the form

$$I_\beta^{(0)}(t,\lambda) = 2(2(\lambda-u_i))^{-1/2}e_i + \cdots, \quad e_i = du_i/\sqrt{\Delta_i},$$

where the dots stand for terms that have at $\lambda = u_i$ a zero of order at least $\frac{1}{2}$. These terms do not contribute to the residue, so we get

$$-\operatorname{Res}_{\lambda=u_i(t)} \frac{\lambda}{2} d\lambda\, (V_{00}(t)e_i,e_i)\frac{2}{\lambda-u_i(t)} = u_i(t)\,(R_1(t)e_i,e_i).$$

We get the following formula for the residue (110):

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\Big(u_iR_1^{ii} - \operatorname{Res}_{\lambda=u_i}\frac{\lambda}{4}d\lambda\, :\phi_{A_1}^{V\otimes V}(u_i,\lambda)\phi_{A_1}^{V\otimes V}(u_i,\lambda):\Big)\prod_{j=1}^{\mu}\mathcal{D}_{\rm pt}(\hbar\Delta_j;{}^j\mathbf{q})^{\otimes 2},$$

where $R_1^{ii} = (R_1e_i,e_i)$ is the i-th diagonal entry of R_1.

8.3.2. *The A_1-subroot system.* The vanishing cycles $\{-\beta,\beta\}$ form a subroot system of type A_1. Let us compute the residue of the corresponding vertex operator terms, i.e.,

$$\operatorname{Res}_{\lambda=u_i(t)} \frac{d\lambda}{\lambda}\Big(\sum_{\pm} b_{\pm\beta}(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)\Big)\mathcal{A}_t^{\otimes 2}. \tag{112}$$

We have $b_\beta(t,\lambda)=b_{-\beta}(t,\lambda)$ and

$$b_\beta(t,\lambda)\Gamma^{\pm\beta}(t,\lambda)\otimes\Gamma^{\mp\beta}(t,\lambda)(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}=(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}b_{A_1}(u_i,\lambda)\Gamma^{\pm\beta}_{A_1}(u_i,\lambda)\otimes\Gamma^{\mp\beta}_{A_1}(u_i,\lambda),$$

where we used formula (88) together with the identity

$$b_\beta(t,\lambda)e^{V_t(\mathbf{f}_\beta(t,\lambda)_-,\mathbf{f}_\beta(t,\lambda)_-)}=b_{A_1}(u_i,\lambda),$$

which follows immediately from (89). Using that $\mathcal{A}_t=\widehat{\Psi}_t\widehat{R}_t\prod_j\mathcal{D}^{(j)}_{\rm pt}$, where the factors $\mathcal{D}^{(j)}_{\rm pt}=\mathcal{D}_{\rm pt}(\hbar\Delta_j;{}^j\mathbf{q})$ are solutions to KdV, we can compute the residue (112) via the Kac-Wakimoto form of the KdV hierarchy (97). After a short computation we get that the residue (112) is

$$(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\Big(\frac{1}{8}+\mathrm{Res}_{\lambda=u_i}\frac{\lambda}{4}d\lambda\ :\phi^{V\otimes V}_{A_1}(u_i,\lambda)\phi^{V\otimes V}_{A_1}(u_i,\lambda):\Big)\prod_{j=1}^{\mu}\mathcal{D}_{\rm pt}(\hbar\Delta_j;{}^j\mathbf{q})^{\otimes 2}.$$

8.3.3. *The A_2 subroot subsystem.* Let $\alpha\in\Delta'$ be a cycle such that $(\alpha|\beta)=1$. We claim that the expression

$$\Big(b_\alpha(t,\lambda)\Gamma^{\alpha}(t,\lambda)\otimes\Gamma^{-\alpha}(t,\lambda)+b_{\alpha-\beta}(t,\lambda)\Gamma^{\alpha-\beta}(t,\lambda)\otimes\Gamma^{-\alpha+\beta}(t,\lambda)\Big)\mathcal{A}_t^{\otimes 2}\tag{113}$$

is analytic near $\lambda=u_i$. Using the decompositions

$$\alpha=\alpha'+\beta/2,\quad \alpha-\beta=\alpha'-\beta/2,$$

where $(\alpha'|\beta)=0$, the above expression can be written as

$$\Gamma^{\alpha'}\otimes\Gamma^{-\alpha'}\Big(a'\Gamma^{\beta/2}\otimes\Gamma^{-\beta/2}+a''\Gamma^{-\beta/2}\otimes\Gamma^{\beta/2}\Big)\ \mathcal{A}_t^{\otimes 2},$$

where the coefficients a' and a'' are given by

$$\begin{aligned} a'(t,\lambda) &= \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2}B_{\alpha,\alpha}(t,\lambda,\mu)B_{\alpha',-\beta}(t,\lambda,\mu),\\ a''(t,\lambda) &= \lim_{\mu\to\lambda}\Big(1-\frac{\mu}{\lambda}\Big)^{-2}B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B_{\alpha',\beta}(t,\lambda,\mu).\end{aligned}$$

On the other hand we have

$$\Gamma^{\pm\beta/2}\otimes\Gamma^{\mp\beta/2}(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}=(\widehat{\Psi}_t\widehat{R}_t)^{\otimes 2}\,e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)}\Gamma^{\pm\beta/2}_{A_1}\otimes\Gamma^{\mp\beta/2}_{A_1}.$$

The exponential factor can be expressed in terms of the phase factors as follows (cf. Section 7.3.2):

$$e^{V_t(\mathbf{f}_{\beta/2}(t,\lambda)_-,\mathbf{f}_{\beta/2}(t,\lambda)_-)}=\frac{1}{2\sqrt{\lambda-u_i}}\lim_{\mu\to\lambda}(\lambda-\mu)^{1/2}B_{\beta/2,-\beta/2}(t,\lambda,\mu),$$

where the limit is taken in the region $|\lambda|>|\mu|$. Recalling the KP-reduction HQEs of KdV (98) we get that if the coefficients

$$c'(t,\lambda)=\lambda^2\lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha,\alpha}(t,\lambda,\mu)B_{\alpha',-\beta}(t,\lambda,\mu)B_{\beta/2,-\beta/2}(t,\lambda,\mu)$$

and

$$c''(t,\lambda)=\lambda^2\lim_{\mu\to\lambda}(\lambda-\mu)^{-3/2}B_{\alpha-\beta,\alpha-\beta}(t,\lambda,\mu)B_{\alpha',\beta}(t,\lambda,\mu)B_{\beta/2,-\beta/2}(t,\lambda,\mu)$$

are analytic near $\lambda = u_i$, and $c'/c'' = -1$, then the expression (113) is analytic near $\lambda = u_i$.

Let us prove the analyticity of c'. The argument for c'' is similar. Let us choose a small $\varepsilon \in \mathbb{C}$ and a generic point $t_0 \in M$ on the discriminant. Furthermore, we fix 3 paths $C_\varepsilon, C'_\varepsilon$, and C''_ε in $M' = M \setminus \{\mathrm{discr}\}$ from $t_0 + (\mu - \lambda + \varepsilon)\mathbf{1}$ to $t - \lambda\mathbf{1}$ such that the parallel transport along the 3 paths transforms the cycle vanishing over t_0 respectively into β, α, and $\alpha - \beta$. The phase factors in the definition of c' can be written in terms of integrals along the path as follows

$$\begin{aligned} B_{\alpha,\alpha}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0} \exp\Big(\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda)\Big), \\ B_{\alpha',-\beta}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0} \exp\Big(\int_{C_\varepsilon} \mathcal{W}_{\alpha',-\beta}(\mu-\lambda)\Big), \\ B_{\beta/2,-\beta/2}(t,\lambda,\mu) &= \lim_{\varepsilon\to 0} \exp\Big(\int_{C_\varepsilon} \mathcal{W}_{\beta/2,-\beta/2}(\mu-\lambda)\Big). \end{aligned}$$

Using these formulas, we can express the coefficient $c'(t,\lambda)$ as the limit $\varepsilon \to 0$ of the following expression:

$$\lambda^2 \lim_{\mu\to\lambda} (\lambda-\mu)^{-3/2} \exp\Big(\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda) - \int_{C_\varepsilon} \mathcal{W}_{\alpha,\alpha}(\mu-\lambda) + \int_{C_\varepsilon} \mathcal{W}_{\alpha',\alpha'}(\mu-\lambda)\Big).$$

Let us examine the dependence on the parameters t, λ, and $\xi := \mu - \lambda$. The difference of the first two integrals in the above formula does not depend on t and λ, because the paths C'_ε and C_ε have the same starting and ending points. After passing to the limit the difference contributes a constant independent of t, λ, and μ. The last integral is analytic near $\lambda = u_i$, because the cycle α' is invariant with respect to the local monodromy, which means that the period vector $I^{(0)}_{\alpha'}(t',\xi)$ and respectively the phase form $\mathcal{W}_{\alpha',\alpha'}(\xi)$ are analytic for t' sufficiently close to $t - u_i\mathbf{1}$ and $|\xi| \ll 1$. This proves the analyticity of c'.

It remains only to prove that $c'/c'' = -1$. Using the above path integrals, we can write $\log(c'/c'')$ in the following way:

$$\int_{C'_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \int_{C_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \int_{C''_\varepsilon} \mathcal{W}_{\alpha-\beta,\alpha-\beta} + \int_{C_\varepsilon} \mathcal{W}_{\alpha-\beta,\alpha-\beta} + \int_\gamma \mathcal{W}_{\alpha,\alpha} - \int_\gamma \mathcal{W}_{\alpha,\alpha},$$

where γ is a small loop in M' based at $t - \lambda\mathbf{1}$ that goes counterclockwise around the discriminant. The above expression coincides with

$$\oint_{(C''_\varepsilon)^{-1}\circ\gamma\circ C'_\varepsilon} \mathcal{W}_{\alpha,\alpha} - \oint_{C_\varepsilon^{-1}\circ\gamma\circ C_\varepsilon} \mathcal{W}_{\alpha,\alpha},$$

where the branch of the phase form is determined by its value at the point $t - \lambda\mathbf{1}$ (which belongs to all integration paths involved). By definition the cycle α is invariant along the loop $(C''_\varepsilon)^{-1} \circ \gamma \circ C'_\varepsilon$, so the first integral is an integer multiple of $2\pi\sqrt{-1}$. Let α_0 and β_0 be the cycles over $t_0 + (\xi+\varepsilon)\mathbf{1}$ obtained respectively from α and β via the parallel transport along C_ε. By definition β_0 is the cycle vanishing over t_0. Let us take a small loop γ_ε based at $t_0 + (\xi+\varepsilon)\mathbf{1}$ that goes counterclockwise around the discriminant. Then the parallel transport of α_0 along γ_ε is $\alpha_0-\beta_0$ which coincides with the parallel transport of α_0 along $C_\varepsilon^{-1}\circ\gamma\circ C_\varepsilon$. Hence

$$\oint_{C_\varepsilon^{-1}\circ\gamma\circ C_\varepsilon}\mathcal{W}_{\alpha,\alpha}-\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha_0,\alpha_0}=\oint_{C_\varepsilon^{-1}\circ\gamma\circ C_\varepsilon}\mathcal{W}_{\alpha_0,\alpha_0}-\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha_0,\alpha_0}$$

is an integer multiple of $2\pi\sqrt{-1}$. In other words, we get

$$c'/c''=\lim_{\xi\to 0}\lim_{\varepsilon\to 0}\exp\Big(-\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha_0,\alpha_0}(\xi)\Big).$$

The limit here is easy to compute because the integral involves only local information. Namely, let $\alpha_0=\alpha_0'+\beta_0/2$, where α_0' is invariant with respect to the local monodromy. There exists a function $u(t')$ analytic in a neighborhood of $t'=t_0$ such that the discriminant is given locally by the equation $u=0$ ($u(t')$ is a critical value of $F(x,t')$). Using Lemma 13 we get

$$I^{(0)}_{\beta_0}(t',\xi)=2(2(\xi-u))^{-1/2}\,\frac{du}{\sqrt{\Delta}}+\cdots,$$

where the dots stand for higher order terms. On the other hand, the period vector $I^{(0)}_{\alpha'}(t',\xi)$ is analytic for (t',ξ) sufficiently close $(t_0,0)$. Expanding the phase form into a Laurent series about $\xi=u$ we get

$$\lim_{\varepsilon\to 0}\oint_{\gamma_\varepsilon}\mathcal{W}_{\alpha_0,\alpha_0}(\xi)=\frac{1}{4}\oint_{\gamma_\varepsilon}\mathcal{W}_{\beta_0,\beta_0}(\xi)=\frac{1}{4}\oint\frac{2du}{\sqrt{(-u)(\xi-u)}}=\pi\sqrt{-1},$$

i.e., $c'/c''=-1$.

8.3.4. *Proof of Theorem 40.* The 1-form

$$\frac{d\lambda}{\lambda}\Omega_{\Delta'}(t,\lambda)\,\mathcal{A}_t(\hbar;\mathbf{q}')\mathcal{A}_t(\hbar;\mathbf{q}'')$$

has poles only at $\lambda=0,\infty$, and the critical values u_i, $1\le i\le\mu$. Let u_i be one of the critical values and β be the cycle vanishing over $\lambda=u_i$. Note that non-trivial contributions to the residue at $\lambda=u_i$ come only from vertex operator terms corresponding to vanishing cycles that have non-zero intersection with β. Recalling our computations in Sections 8.3.1, 8.3.2, and 8.3.3, we get that the residue at $\lambda=u_i$ is $(1/8+u_iR_1^{ii})\mathcal{A}_t^{\otimes 2}$, while the residue at $\lambda=0$ is $-\frac{1}{2}\,\mathrm{tr}\left(\frac{1}{4}+\theta\theta^T\right)\mathcal{A}_t^{\otimes 2}$. In order to prove that the residue at $\lambda=\infty$ is 0, we just need to check that

$$\sum_{i=1}^{\mu}u_iR_1^{ii}=\frac{1}{2}\,\mathrm{tr}\left(\theta\theta^T\right).$$

The above identity is well-known from the theory of Frobenius manifolds (see [31, 33]). Hence the ancestor potential $\mathcal{A}_t(\hbar;\mathbf{q})$ is a solution to the HQEs (95). Theorem 40 is thus proved.

8.3.5. *Proof of Theorem 1.* It is easy to see that Theorem 1 follows from Theorem 40, Proposition 39 and Theorem 33.

9. Example: $\mathbf{a} = \{2,2,2\}$

In this Section we consider the example $\mathbf{a} = \{2,2,2\}$, namely $\mathbb{P}^1_{\mathbf{a}} = \mathbb{P}^1_{2,2,2}$. In this case $\Delta^{(0)}$ is the root system of type D_4. It is convenient to denote the indexes in the index set $\mathfrak{I}_{\rm tw} = \{(1,1),(2,1),(3,1)\}$ simply by $1,2,3$. There are 12 positive roots

$$\begin{aligned}&\gamma_i\ (1\leq i\leq 3),\quad \gamma_b,\quad \gamma_b+\gamma_i\ (1\leq i\leq 3),\quad \gamma_b+\gamma_i+\gamma_j\ (1\leq i<j\leq 3),\\ &\gamma_b+\gamma_1+\gamma_2+\gamma_3,\quad 2\gamma_b+\gamma_1+\gamma_2+\gamma_3,\end{aligned}$$

where γ_b is the simple root corresponding to the branching node of the Dynkin diagram and $\gamma_i\,(1\leq i\leq 3)$ are the remaining simple roots. The fundamental weight is $\omega_b = 2\gamma_b+\gamma_1+\gamma_2+\gamma_3$. It is easy to find that an eigenbasis for σ_b is given by

$$H_i := (\kappa/2)^{1/2}\gamma_i\quad (1\leq i\leq 3),\quad H_0 := (\kappa/2)^{1/2}\omega_b,$$

and we have $m_i = \frac{\kappa}{2}$, $d_i = \frac{1}{2}$, $1\leq i\leq 3$, where $\kappa = 4$.

Let us write the HQEs for $\tau = (\tau_n(y))_{n\in\mathbb{Z}}$. We have

$$a_\alpha(\zeta) = \frac{1}{4}2^{(\sigma_b(\alpha)|\alpha)}\zeta^{\kappa|\alpha_0|^2}e^{2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}$$

and

$$\Big(E_\alpha(\zeta)\tau\Big)_0 = \zeta^{-\kappa|\alpha_0|^2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}E^*_\alpha(\zeta)\tau_{-(\omega_b|\alpha)},$$

where the subscript 0 on the LHS means the 0-th component of the corresponding vector in our Fock space. Recall that the HQE give rise to a system of PDE in the following way. First we make a substitution

$$\mathbf{y}' := \mathbf{y}\otimes 1 = \mathbf{x}+\mathbf{t},\quad \mathbf{y}'' := 1\otimes\mathbf{y} = \mathbf{x}-\mathbf{t},$$

which implies that

$$\mathbf{y}'-\mathbf{y}'' = 2\mathbf{t},\quad \frac{\partial}{\partial\mathbf{y}'}-\frac{\partial}{\partial\mathbf{y}''} = \frac{\partial}{\partial\mathbf{t}},$$

and that

$$\mathrm{Res}_{\zeta=0}\Big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\Big)_{0,0}$$

is the coefficient in front of ζ^0 in the following expression

$$\begin{aligned}&2^{(\sigma_b(\alpha)|\alpha)-2}e^{-2\pi\sqrt{-1}(\rho_b|\alpha)(\omega_b|\alpha)}\Big(\zeta^{-\kappa|\alpha_0|^2}e^{\sum_{i,l}2(\alpha|H_i)\zeta^{m_i+l\kappa}t_{i,l}}\Big)\\ &\Big(e^{-\sum_{i,l}(\alpha|H_{i^*})\frac{\zeta^{-m_i-l\kappa}}{m_i+l\kappa}\partial_{x_{i,l}}}\tau_{-(\omega_b|\alpha)}(\mathbf{x}+\mathbf{t})\Big)\Big(e^{\sum_{i,l}(\alpha|H_{i^*})\frac{\zeta^{-m_i-l\kappa}}{m_i+l\kappa}\partial_{x_{i,l}}}\tau_{(\omega_b|\alpha)}(\mathbf{x}-\mathbf{t})\Big).\end{aligned}$$

By definition the HQE are

$$\begin{aligned}&\mathrm{Res}_{\zeta=0}\sum_{\alpha\in\Delta^{(0)}}\Big(a_\alpha(\zeta)E_\alpha(\zeta)\tau\otimes E_{-\alpha}(\zeta)\tau\Big)_{m,n} =\\ &\Big(\frac{3}{8}+\frac{1}{4}(m-n)^2+2\sum_{i,l}(d_{i^*}+l)t_{i,l}\partial_{t_{i,l}}\Big)\tau_m(\mathbf{x}+\mathbf{t})\tau_n(\mathbf{x}-\mathbf{t}).\end{aligned}$$

Comparing the coefficients in front of the various monomials in $\mathbf{t}$ we obtain a system of PDE whose equations are some quadratic polynomials in the partial derivatives of τ. Let us specialize to the case $m = n = 0$. In order to get non-trivial equations we have to compare coefficients in front of monomials that are invariant under the involution $\mathbf{t} \mapsto -\mathbf{t}$. The simplest case is $\mathbf{t}^0$, which corresponds to the identity

$$\sum_{\alpha\in\Delta^{(0)}:(\omega_b|\alpha)=0} 2^{(\sigma_b(\alpha)|\alpha)-2} = \frac{3}{8}.$$

Comparing the coefficients in front of the monomial $t_{02,0}^2$, we get

$$4\frac{\partial^2}{\partial x_{02,0}^2}\log\tau(\mathbf{x}) = 8\kappa\frac{\tau_{-2}(\mathbf{x})\tau_2(\mathbf{x})}{\tau^2(\mathbf{x})} + 4(2/\kappa)^{1/2}\frac{\partial^3}{\partial t_{1,0}\partial t_{2,0}\partial t_{3,0}}\Big(\frac{\tau_{-1}(\mathbf{x}+\mathbf{t})\tau_1(\mathbf{x}-\mathbf{t})}{\tau^2(\mathbf{x})}\Big)\Big|_{\mathbf{t}=0}.$$

Recalling the substitution (68)–(69), which in this case is

$$\begin{aligned} y_{02,0} &= \frac{1}{\sqrt{\hbar}}\frac{\sqrt{2}}{\kappa\sqrt{\kappa}}q_0^{02}, \\ y_{i,0} &= \frac{1}{\sqrt{\hbar}}\frac{\sqrt{2}}{\kappa}q_0^i, \quad 1\le i\le 3, \end{aligned}$$

we get

$$\hbar\frac{\partial^2}{\partial(q_0^{02})^2}\log\tau(\mathbf{q}) = \frac{4}{\kappa^2}\frac{\tau_{-2}(\mathbf{q})\tau_2(\mathbf{q})}{\tau^2(\mathbf{q})} + \frac{\hbar^{3/2}}{\kappa^{1/2}}\,\partial_1\partial_2\partial_3\Big(\frac{\tau_{-1}(\mathbf{q}+\mathbf{t})\tau_1(\mathbf{q}-\mathbf{t})}{\tau^2(\mathbf{q})}\Big)\Big|_{\mathbf{t}=0},$$

where for brevity we put $\partial_i := \partial/\partial t_0^i$. To get a differential equation for the total descendant potential we just have to substitute

$$\tau_{\pm 2}(\mathbf{q}) = C^2\mathcal{D}(\hbar;\mathbf{q}\pm 2\sqrt{\hbar}),\quad \tau_{\pm 1}(\mathbf{q}) = C^{1/2}\mathcal{D}(\hbar;\mathbf{q}\pm\sqrt{\hbar}).$$

In order to find the constant $C = C'Q$, let us restrict only to primary invariants, i.e., set $q_k^i = 0$, $\forall k>0$, and comapre the leading terms of the genus expansion. We get the following PDE for the primary genus-0 potential F:

$$F_{02,02} = 4\frac{C^4}{\kappa^2}e^{4F_{01,01}} + \frac{-C}{\sqrt{\kappa}}e^{F_{01,01}}\Big(8F_{01,1}F_{01,2}F_{01,3} + 4(F_{01,1}F_{2,3}+F_{01,2}F_{1,3}+F_{01,3}F_{1,2})\Big),$$

where $F_{i,j} := \partial^2 F/\partial q_0^i\partial q_0^j$. To simplify the notation, let us put $t_i := q_0^i$. Using the string equation we get

$$F_{01,01} = t_{02},\quad F_{01,i} = \frac{1}{2}t_i,$$

so from the above equation we get the following relation

$$F_{02,02} = 4\frac{C^4}{\kappa^2}e^{4t_{02}} + \frac{-C}{\sqrt{\kappa}}e^{t_{02}}\Big(t_1t_2t_3 + 2(t_1F_{2,3}+t_2F_{1,3}+t_3F_{1,2})\Big). \tag{114}$$

Note that the degree-0 term of $F_{i,j}$ for $i\neq j$ must be 0, therefore comparing the degree 1 terms in the above equation, and using the divisor equation we get

$$\frac{-C'}{\sqrt{\kappa}} = \langle\phi_1,\phi_2,\phi_3\rangle_{g=0,3,d=1} = 1.$$

In other words, $C = C'Q = -\kappa^{1/2}Q$.

In fact, equation (114) allows us to compute the potential F recursively, by the degree of the Novikov variable Q. Indeed, it is easy to see that up to degree-1 terms, F is given by

$$\frac{1}{2}t_{01}^2t_{02}+\frac{1}{4}t_{01}(t_1^2+t_2^2+t_3^2)+\frac{1}{96}(t_1^4+t_2^4+t_3^4)+Qe^{t_{02}}t_1t_2t_3.$$

Comparing the degree-2 terms in (114) we get that the dgree-2 term of F must be $\frac{1}{2}(t_1^2+t_2^2+t_3^2)Q^2e^{2t_{02}}$. Arguing in the same way we get that F does not have degree-3 terms, while the degree 4 term must be $\frac{1}{4}Q^4e^{4t_{02}}$. The potential F takes the form

$$\begin{aligned} F(t) &= \frac{1}{2}t_{01}^2t_{02}+\frac{1}{4}t_{01}(t_1^2+t_2^2+t_3^2)+\frac{1}{96}(t_1^4+t_2^4+t_3^4)+Qe^{t_{02}}t_1t_2t_3+ \\ &+\frac{1}{2}Q^2e^{2t_{02}}(t_1^2+t_2^2+t_3^2)+\frac{1}{4}Q^4e^{4t_{02}}. \end{aligned}$$

The above formula agrees with the computation of P. Rossi (see [53], Example 3.2) based on Symplectic Field Theory.

Remark 49. *In many cases (e.g. the Example above), the choice of the eigenvectors H_i normalized by $(H_i|H_j)=\kappa\delta_{i,j^*}$ is not unique. Choosing a different basis amounts to a homogeneous change of the flat coordinates on M, so when applying our result to GW theory, one should keep in mind that the descendant potential in GW theory satisfies the HQEs provided we choose the correct basis $\{\phi_i\}_{i\in\mathfrak{I}}$ of the orbifold cohomology.*

Appendix A. The intersection form for the affine cusp polynomials

Let us outline the main steps in the proof of Proposition 10 for the cases with $a_1\geq 2$, otherwise we are in the settings of [51] and the proposition can be proved by explicit computations. Note that a_1 must be 2 in order to have $\chi>0$. Therefore we can write the deformation $F(x,s)$ as follows:

$$\Big(x_1+\frac{1}{2}\Big(s_{1,1}-x_2x_3/(Qe^{s_{02}})\Big)\Big)^2+G(x,s),$$

where the function $G(x,s)$ depends only on x_2 and x_3. In other words F is *stably* equivalent to G. In these settings one can use a method developed by Gusein-Zade and A'Campo to compute the Dynkin diagram of F (see [3] and the references there in). The main statement is the following. Let us fix a parameter $s\in M$ such that the function $G(\cdot,s):\mathbb{C}^2\to\mathbb{C}$ satisfies the following conditions:

1. The function takes real values on $\mathbb{R}^2\subset\mathbb{C}^2$.
2. All critical points are real and non-degenerate.
3. The critical values at the saddle points are 0.
4. The zero level $\{G(x,s)=0\}\subset\mathbb{R}^2$ is a plane curve with only nodal singularities, such that

$$\mu=2\delta-r+1, \tag{115}$$

where r is the number of connected components of the plane curve and δ is the number of nodes.

If we find such a deformation then we draw the zero level of G in the plane $\mathbb{R}^2$ and we associate 2 types of vanishing cycles: cycles corresponding to the nodes of the curve and cycles corresponding to the compact connected components of the complement of the curve (in $\mathbb{R}^2$). We call them respectively *nodal* and *compact* vanishing cycles. The compact ones can be split further into positive and negative ones depending on whether G is positive or negative inside the corresponding compact domain. Note that equation (115) implies that the total number of vanishing cycles constructed in the above scheme is μ.

The intersection form $(\alpha|\beta)$ is determined by the following rules which fall into 3 different cases:

(1) First, if α and β are nodal type, then the intersection is 0 for $\alpha \neq \beta$ and 2 otherwise.
(2) Next, if α and β are compact type, then the intersection is the number of common edges of the corresponding compact domains (each compact domain is a curved polygon with edges the nodes).
(3) Finally, if α is compact and β is nodal; then the intersection is $-n$, where if we put a small disk at the node corresponding to β then n is the number of connected components of the intersection of the disk with the compact domain corresponding to α (note that n could be 0, 1, or 2).

The classical monodromy σ can be written as $\sigma_-\sigma_0\sigma_+$, where $\sigma_\pm$ is a product of reflections corresponding to positive/negative compact type cycles and σ_0 is the product of the reflections corresponding to the nodal type cycles. Therefore, the proof of Proposition 10 is reduced to finding an appropriate deformation satisfying properties (1)–(4), finding an appropriate basis of vanishing cycles such that the Dynkin diagram obtained from the corresponding plane curve is transformed into an affine Dynkin diagram, and verifying that σ is an affine Coxeter transformation.

Case 1. (Type D) If $a_1 = a_2 = 2$; then one can construct a deformation in terms of the *Chebyshev polynomials* $T_n(x)$. Recall that the latter are defined by the trigonometric identity $T_n(\cos(x)) = 2^{1-n}\cos(nx), \forall x \in \mathbb{R}$, e.g.,

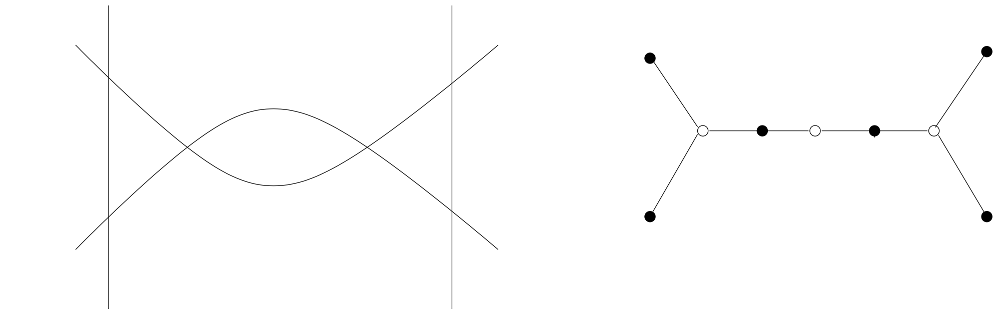
$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = -\frac{1}{2} + x^2, \quad \dots .$$

Put $m = a_3 - 2$ for brevity; then a deformation satisfying the required properties can be chosen in the form

$$G(x,s) = (x_3 - \varepsilon_1)(x_3 + \varepsilon_1)\Big(T_m(x_3) - \varepsilon_2^2\, T_2(x_2/\varepsilon_2)\Big),$$

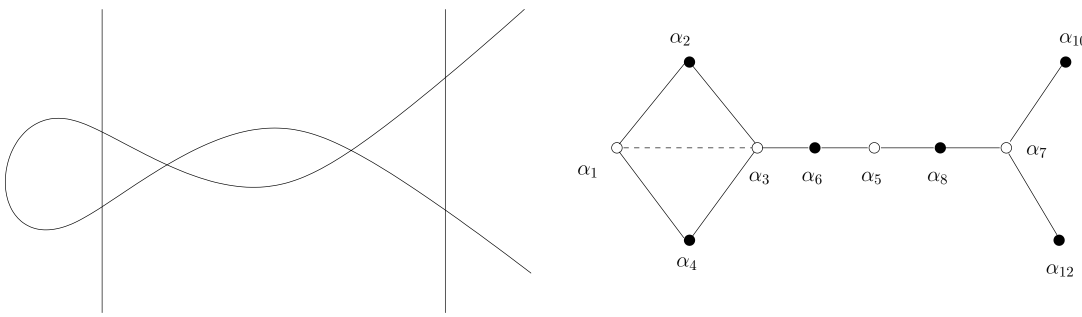
where ε_1 and ε_2 are appropriately chosen. In fact, it is not hard to see that ε_2 must be $2^{(2-m)/2}$.

The 0-level curves and the corresponding Dynkin diagrams are given on Figure 2 if $N = \mu - 1$ is even and on Figure 3 if $N = \mu - 1$ is odd. The black and white nodes correspond to vanishing cycles of nodal and compact types respectively. In the even case there is no need of additional transformations. In the odd case however, we have to find an appropriate basis of vanishing cycles. Let us label the vanishing cycles (see Figure 3) of compact type by $\alpha_1, \alpha_3, \dots$ (odd indices) and the other cycles by



$r = 4, \quad \delta = 6, \quad \mu = 9$

FIGURE 2. Dynkin diagram $D_N^{(1)}$ (N even)



$r = 3, \quad \delta = 6, \quad \mu = 10$

FIGURE 3. Dynkin diagram equivalent to $D_N^{(1)}$ (N odd)

$\alpha_2, \alpha_4, \ldots$ (even indices). Put

$$\alpha_1' := s_1(\alpha_1) = -\alpha_1, \quad \alpha_2' = s_1(\alpha_2) = \alpha_2 + \alpha_1, \quad \alpha_4' = s_1(\alpha_4) = \alpha_4 + \alpha_1,$$

where $s_i := s_{\alpha_i}$. We replace α_i with α_i', for $i = 1, 2, 4$. A straightforward computation yields

$$(\alpha_1'|\alpha_3) = -1, \quad (\alpha_2'|\alpha_3) = (\alpha_4'|\alpha_6) = 0,$$

so in the new basis the Dynkin diagram is $D_N^{(1)}$. For the classical monodromy we have

$$\sigma = s_1(s_2 s_4 \cdots)(s_3 s_5 \cdots) = (s_{\alpha_2'} s_{\alpha_4'} \cdots)(s_{\alpha_1'} s_3 s_5 \cdots).$$

The RHS is clearly an affine Coxeter transformation.

$r = 2, \quad \delta = 4, \quad \mu = 7$

FIGURE 4. Dynkin diagram equivalent to $E_6^{(1)}$

Case 2. (Type E) If $a_1 = 2, a_2 = 3$, and $a_3 = 3$. Then the deformation can be taken of the form

$$G(x,s) = (x_2^2 + x_3 - 3)(x_3^2 + x_2 - 2).$$

The 0-level curve and the corresponding Dynkin diagram are given on Figure 4. We enumerate the nodes of the Dynkin diagram as shown on Figure 4. Note that the classical monodromy is given by

$$\sigma = s_1 s_5 (s_2 s_4 s_6 s_8) s_3.$$

We leave it to the reader to check that the substitution

$$\begin{aligned} \alpha_2' &= \alpha_2 \\ \alpha_6' &= \alpha_6 \\ \alpha_8' &= \alpha_8 \\ \alpha_4' &= s_4(\alpha_4) = -\alpha_4 \\ \alpha_5' &= s_4(\alpha_5) \\ \alpha_1' &= s_4(\alpha_1) \\ \alpha_3' &= s_1 s_5(\alpha_3) \end{aligned}$$

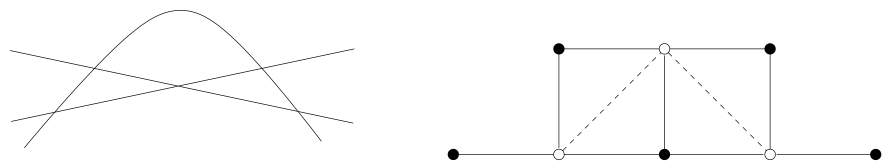
transforms the Dynkin diagram on Figure 4 into the affine Dynkin diagram $E_6^{(1)}$. The classical monodromy takes the form

$$\sigma = s_1 s_5 (s_2' s_6' s_8' s_1' s_5' s_4' s_3') s_1 s_5 \,.$$

Passing to a new basis of vanishing cycles $\alpha_i'' := s_1 s_5(\alpha_i')$ we get (recall that $w s_\alpha w^{-1} = s_{w(\alpha)}$) that σ is an affine Coxeter transformation. This completes the proof of Case 2. The argument for the remaining two cases (see Figures 5 and 6) $a_1 = 2$, $a_2 = 3$, and $a_3 = 4$, or 5, is similar and it will be omitted.

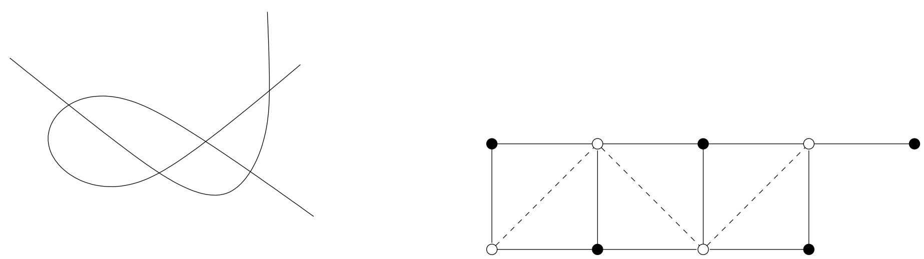
APPENDIX B. A PROOF OF PROPOSITION 17

Proposition 17 follows the ideas of Steinberg [57] (see also [56] for some more backgraound on the affine Coxeter transformation). According to Proposition 10



$r = 3, \quad \delta = 5, \quad \mu = 8$

FIGURE 5. Dynkin diagram equivalent to $E_7^{(1)}$



$r = 2, \quad \delta = 5, \quad \mu = 9$

FIGURE 6. Dynkin diagram equivalent to $E_8^{(1)}$

we can find a basis of simple roots so that σ is a product of all simple reflections in some order. On the other hand, if the affine Dynkin diagram is a tree; then all (affine) Coxeter transformations are conjugate (see [34], Section 3.16). Therefore we may assume that we have a basis of simple roots, such that σ is a bicolored Coxeter transformation. Recall that a bicolored Coxeter transformation is defined as follows. We decompose the nodes Π of the Dynkin diagram into two disjoint parts Π_1 (odd) and Π_2 (even),

$$\Pi = \Pi_1 \cup \Pi_2,$$

such that in each part the corresponding roots are orthogonal. A bicolored Coxeter transformation is the composition of all reflections corresponding to the odd nodes (i.e. from Π_1) followed by the composition of the reflections corresponding to the even nodes (i.e. from Π_2). We may further assume that the affine vertex $\gamma_0^{(-1)}$ is an even node. For notational convenience we set $\Pi_g = \Pi_1$ if g is odd and $\Pi_g = \Pi_2$ if g is even.

Let $\theta \in \Delta^{(0)}$ be the highest root. We denote by σ_1 and σ_2 the product of all reflections corresponding respectively to odd and even nodes of $\Delta^{(0)}$. It is a theorem of Steinberg [57] that the branching node of X_N is

$$\gamma_b^{(-1)} = \begin{cases} (\sigma_2\sigma_1)^k(\theta) & \text{if the Coexter number } h = 4k+2; \\ \sigma_1(\sigma_2\sigma_1)^{k-1} & \text{if } h = 4k. \end{cases}$$

Moreover, the branching node is in $\Pi_{h/2}$, i.e., it is odd in the case $h = 4k+2$, while in the other case when $h = 4k$ it is even.

The affine Coxeter transformation is

$$\sigma = s_0^{(-1)}\,\sigma_2\sigma_1 = s_{-\theta+\delta}s_{-\theta}s_\theta\sigma_2\sigma_1 = t_{-\theta}\,s_\theta\sigma_2\sigma_1,$$

where we are using the splitting of the affine root system $\Delta^{(-1)} = \Delta^{(0)} + \mathbb{Z}\,\delta$ to embed $\Delta^{(0)}$ and the Weyl group $W^{(0)}$ into $\Delta^{(-1)}$ and W respectively.

If $h = 4k+2$; then we put $w = \sigma_1(\sigma_2\sigma_1)^k$ and we get

$$w\cdot\sigma\cdot w^{-1} = t_{-w(\theta)}s_{w(\theta)}\sigma_1\sigma_2. \tag{116}$$

On the other hand, using Steinberg's theorem we get $w(\theta) = \sigma_1(\gamma_b^{(-1)})$ and that $\gamma_b^{(-1)}$ is an odd node. Since σ_1 is the product of all odd reflections and they pairwise commute, we get easily that $\sigma_1(\gamma_b^{(-1)}) = -\gamma_b^{(-1)}$. In other words (116) is precisely $t_{\gamma_b}s_b^{(-1)}\sigma_1\sigma_2$. Note that since the node $\gamma_{\mu,1}^{(-1)}$ (see Figure 1) is even, the product $s_b^{(-1)}\sigma_1\sigma_2$ coincides with

$$\prod_{\mu=1}^{3}\Big(\Big(\cdots s_{\mu,4}^{(-1)}s_{\mu,2}^{(-1)}\Big)\Big(\cdots s_{\mu,3}^{(-1)}s_{\mu,1}^{(-1)}\Big)\Big). \tag{117}$$

If $h = 4k$; then we put $w = (\sigma_2\sigma_1)^k$ and a similar analysis shows that $\gamma_b^{(-1)}$ is an even node and that

$$w\cdot\sigma\cdot w^{-1} = t_{\gamma_b}s_b^{(-1)}\sigma_2\sigma_1.$$

The product $s_b^{(-1)}\sigma_2\sigma_1$ is given by the same formula (117).

To finish the proof we need just to find an element $w\in W$ that fixes the branching node and it permutes the reflections in (117) in such a way that we get (34). Clearly we can work within each branch of the Dynkin diagram X_N. For each $\mu = 1,2,3$, the reflection group generated by $\{s_{\mu,i}^{(-1)}, 1\le i\le a_\mu - 1\}$ is the permutation group on $a := a_\mu$ letters $\varepsilon_1,\dots,\varepsilon_a$, where we can take ε_i to be the standard basis of $\mathbb{C}^a$, so that the simple roots $\gamma_{\mu,i}^{(-1)} = \varepsilon_i - \varepsilon_{i+1}$. Then we have

$$s_{\mu,i}^{(-1)} = (i,i+1),\quad i = 1,2,\dots,a-1.$$

It follows that the μ-th factor in (117) is the cycle

$$(1,3,5,\dots,2a'+1,2a'',2a''-2,\dots,2), \tag{118}$$

where $a'' = a'$ if $a = 2a'+1$ is odd and $a'' = a'+1$ if $a = 2a'+2$ is even. On the other hand the μ-th factor in the product (34) is the cycle

$$(a,a-1,\dots,2,1) = (1,a,a-1,\dots,3,2). \tag{119}$$

Comparing (118) and (119) we get that there exists a permutation w_μ permuting only $\{\varepsilon_3, \varepsilon_4, \dots, \varepsilon_a\}$ that transforms the cycle (118) into the cycle (119). Clearly w_μ belongs to the subgroup generated by the reflections $\{s_{\mu,2}^{(-1)}, s_{\mu,3}^{(-1)}, \dots\}$ and hence $\gamma_b^{(-1)}$ is fixed by w_μ, so it remains only to set $w = w_1 w_2 w_3$. □

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