ANALYTICITY OF THE TOTAL ANCESTOR POTENTIAL IN SINGULARITY THEORY

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Abstract. K. Saito's theory of primitive forms gives a natural semi-simple Frobenius manifold structure on the space of miniversal deformations of an isolated singularity. On the other hand, Givental introduced the notion of a total ancestor potential for every semi-simple point of a Frobenius manifold and conjectured that in the settings of singularity theory his definition extends analytically to non-semisimple points as well. In this paper we prove Givental's conjecture by using the Eynard–Orantin recursion.

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1. Introduction

The Gromov–Witten invariants of a compact algebraic manifold V are by definition a virtual count of holomorphic maps from a Riemann surface to V satisfying various incidents constraints. Although the rigorous definition of the Gromov–Witten invariants is very complicated, when it comes to computations, quite a bit of techniques were developed. One of the most exciting achievements is due to Givental who conjectured that under some technical conditions (which amount to saying that V has sufficiently many rational curves) we can reconstruct the higher genus invariants in terms of genus 0 and the higher genus Gromov–Witten invariants of the point. Givental's conjecture was proved recently by Teleman [22] and

2000 Math. Subj. Class. 14D05, 14N35, 17B69.
Key words and phrases: period integrals, Frobenius structure, Goromov–Witten invariants, vertex operators.

its impact on other areas of mathematics, such as integrable systems and the theory of quasi-modular forms is a subject of an ongoing investigation (see [5], [17]).

The higher genus reconstruction formalism of Givental (see [10] or Section 3 bellow) is most naturally formulated in the abstract settings of the so called *semi-simple Frobenius manifolds* (see [7] for some background on Frobenius manifolds). In the case of Gromov–Witten theory, the Frobenius structure is given on the vector space $H^*(V;\mathbb{C})$ and it is induced from the quantum cup product. More precisely, Givental defined the total ancestor potential of a semi-simple Frobenius manifold which in the case of Gromov–Witten theory coincides with a generating function of the so called *ancestor* Gromov–Witten invariants (see [11]).

In this paper we study the total ancestor potential of the semi-simple Frobenius manifold arising in singularity theory. Let $f \in \mathcal{O}_{\mathbb{C}^{2l+1},0}$ be the germ of a holomorphic function with an isolated critical point at 0, i.e., the local algebra $H := \mathcal{O}_{\mathbb{C}^{2l+1},0}/(f_{x_0},\dots,f_{x_{2l}})$ is a finite dimensional vector space (over $\mathbb{C}$). The dimension is called *multiplicity* of the critical point and it will be denoted by N. We fix a miniversal deformation $F(t,x)$, $t \in B$ and a primitive form ω in the sense of K. Saito [19, 21], so that B inherits a Frobenius structure (see [14, 20]). Let B_{ss} be the set of points $t_0 \in B$, such that the critical values $u_1(t),\dots,u_N(t)$ of $F(t,\cdot)$ form a coordinate system for t in a neighborhood of t_0. In such coordinates the product and the residue pairing assume a diagonal form which means that the corresponding Frobenius algebra is semi-simple. Let $\mathbf{t} = \{t_{k,i}\}_{k=0,1,\dots}^{i=1,\dots,N}$ be a sequence of formal variables. For every $t \in B_{\mathrm{ss}}$ we denote by $\mathcal{A}_t(\hbar;\mathbf{t})$ the total ancestor potential of the Frobenius structure (c.f. Section 3.2). It is a formal series in $\mathbb{C}(\!(\hbar)\!)[\![\mathbf{t}]\!]$, whose coefficients are analytic functions in $t \in B_{ss}$. A priori the coefficients could have poles along the caustic $B \setminus B_{\mathrm{ss}}$. Our main result is the following.

Theorem 1.1. *Assume that we have an isolated singularity with an arbitrary primitive form. The corresponding family of total ancestor potentials $\mathcal{A}_t(\hbar;\mathbf{t})$, $t \in B_{\mathrm{ss}}$, determines a formal series in $\mathbb{C}(\!(\hbar)\!)[\![\mathbf{t}]\!]$ whose coefficients are holomorphic in t and extend holomorphically over the caustic to all of B.*

The proof of Theorem 1.1 is based on the local Eynard–Orantin recursion (see [4] and [16]). We follow the approach in [16]. The main advantage of the recursion is that it gives a reconstruction which does not make use of the higher genus theory of the point, but it depends only on the Frobenius structure! Following an idea of Bouchard–Eynard (see [3]) we prove that the local recursion, which apriori is defined only for $t \in B_{\mathrm{ss}}$, extends to generic points $t \in B \setminus B_{\mathrm{ss}}$. Let us point out that at this point we use the fact that for a generic $t \in B \setminus B_{\mathrm{ss}}$ the function $F(t,\cdot)$ has a singularity of type A_2. From here one proves easily by induction that $\mathcal{A}_t(\hbar;\mathbf{q})$ extends analytically for generic $t \in B \setminus B_{\mathrm{ss}}$ provided some initial set of correlators of genus 0 and genus 1 are analytic. While the analyticity of the genus 0 correlators is easy to verify, the analyticity of the genus 1 ones is much more involved. However the computation was already done by C. Hertling (see [14], Theorem 14.6). Therefore, to complete the proof of Theorem 1.1, it remains only to recall the Hartogue's extension theorem.

Remark 1.2. *The fact that we have higher genus reconstruction at non-semisimple points looks quite attractive on its own and it deserves a further investigation. In particular, it will be interesting to find a generalization of Givental's formula at various non-semisimple points and see if similar formulas occur in Gromov–Witten theory as well.*

Remark 1.3. *Theorem 1.1 is very important for the Landau–Ginzburg/Calabi–Yau correspondence (c.f.* [17, 18]*) where it is necessary to restrict the total ancestor potential to* marginal deformations *only and the latter are always non-semisimple.*

For every semi-simple Frobenius structure, one can introduce also the so called *total descendant potential*

$$\mathcal{D}(\hbar;\mathbf{t}) = e^{F^{(1)}(t)}\widehat{S}_t^{-1}\mathcal{A}_t(\hbar;\mathbf{t}). \tag{1}$$

We refer to [11] for the details of this formula. For the discussion that follows it will be important to note only that in singularity theory S_t depends holomorphically on $t\in B$. One can check that the RHS is independent of t in a sense that the derivative with respect to t is 0. However, there is a subtlety here that comes from the fact that the operator $\widehat{S}_t^{-1}$ changes the Fock space. Namely, the total descendant potential is an element of $\mathbb{C}(\!(\hbar)\!)[\![t_0-t,t_1,\dots]\!]$. A good analogy to think about what is going on here is the function $f(q)=1/q$. Take any $t\neq 0$, then the Taylor series expansion of $f(q)$ at $q=t$ gives

$$f(q)=\sum_{n=0}^{\infty}(-1)^n t^{-n-1}(q-t)^n.$$

The derivative of the RHS with respect to t is 0, however we can not quite say that it is independent of t. On the other hand, we can set formally $t=t_0$ in (1), then we get a formal series in $\mathbb{C}(\!(\hbar)\!)[\![t_1,t_2,\dots]\!]$ whose coefficients depend holomorphically on $t_0\in B_{\rm ss}$. So we can interpret formula (1) in the following way. The total descendant potential $\mathcal{D}(\hbar;\mathbf{t})$ is a formal series in $\mathbb{C}(\!(\hbar)\!)[\![t_1,t_2,\dots]\!]$ whose coefficients are holomorphic functions in $t_0\in B_{\rm ss}$ with possible poles along the caustic $B\backslash B_{\rm ss}$. If we pick an arbitrary $t\in B_{\rm ss}$ and take the Taylor series expansion of each coefficient at $t_0=t$, then we get the RHS of formula (1). Theorem 1.1 implies that in singularity theory the coefficients of $\mathcal{D}(\hbar;\mathbf{t})$ are holomorphic in $t_0\in B$. The total descendant potential has the form $e^{\sum_{g=0}^{\infty}\hbar^{g-1}\mathcal{F}^{(g)}(\mathbf{t})}$, where $\mathcal{F}^{(g)}(\mathbf{t})$ is a formal series called the *genus-g descendant potential*. Setting $t_k=0$ for all $k>0$ in $\mathcal{F}^{(g)}(\mathbf{t})$, we get a formal series $F^{(g)}(t_0)$ called *primary* genus-g potential. We have the following corollary.

Corollary 1.4. *The primary potentials $F^{(g)}(t_0)$ are holomorphic on B for all $g\geq 0$.*

2. Frobenius structures in singularity theory

Let us first recall some of the basic settings in singularity theory. For more details we refer the reader to the excellent book [1]. Let $f\colon (\mathbb{C}^{2l+1},0)\to(\mathbb{C},0)$ be the germ of a holomorphic function with an isolated critical point of multiplicity N. Denote by

$$H = \mathcal{O}_{\mathbb{C}^{2l+1},0}/(\partial_{x_0}f,\dots,\partial_{x_{2l}}f)$$

the *local algebra* of the critical point; then $\dim H = N$.

Definition 2.1. *A* miniversal deformation *of f is a germ of a holomorphic function $F\colon (\mathbb{C}^N\times\mathbb{C}^{2l+1},0)\to(\mathbb{C},0)$ satisfying the following two properties:*

(1) F is a deformation of f, i.e., $F(0,x)=f(x)$.
(2) The partial derivatives $\partial F/\partial t^i$ $(1\le i\le N)$ project to a basis in the local algebra

$$\mathcal{O}_{\mathbb{C}^N,0}[[x_0,\dots,x_{2l}]]/\langle\partial_{x_0}F,\dots,\partial_{x_{2l}}F\rangle.$$

Here we denote by $t=(t^1,\dots,t^N)$ and $x=(x_0,\dots,x_{2l})$ the standard coordinates on $\mathbb{C}^N$ and $\mathbb{C}^{2l+1}$ respectively, and $\mathcal{O}_{\mathbb{C}^N,0}$ is the algebra of germs at 0 of holomorphic functions on $\mathbb{C}^N$.

We fix a representative of the holomorphic germ F, which we denote again by F, with a domain X constructed as follows. Let

$$B^{2l+1}_\rho\subset\mathbb{C}^{2l+1}\,,\qquad B=B^N_\eta\subset\mathbb{C}^N\,,\qquad B^1_\delta\subset\mathbb{C}$$

be balls with centers at 0 and radii ρ,η, and δ, respectively. We set

$$S=B\times B^1_\delta\subset\mathbb{C}^N\times\mathbb{C}\,,\quad X=(B\times B^{2l+1}_\rho)\cap\phi^{-1}(S)\subset\mathbb{C}^N\times\mathbb{C}^{2l+1}\,,$$

where

$$\phi\colon B\times B^{2l+1}_\rho\to B\times\mathbb{C}\,,\qquad (t,x)\mapsto(t,F(t,x))\,.$$

This map induces a map $\phi\colon X\to S$ and we denote by X_s or $X_{t,\lambda}$ the fiber

$$X_s=X_{t,\lambda}=\{(t,x)\in X\,|\,F(t,x)=\lambda\}\,,\qquad s=(t,\lambda)\in S.$$

The number ρ is chosen so small that for all r, $0<r\le\rho$, the fiber $X_{0,0}$ intersects transversely the boundary ∂B^{2l+1}_r of the ball with radius r. Then we choose the numbers η and δ small enough so that for all $s\in S$ the fiber X_s intersects transversely the boundary ∂B^{2l+1}_ρ. Finally, decreasing further η and ρ if necessary we may assume that the critical values of F are contained in a disk $B^1_{\delta''}$ with radius $\delta''<\delta$.

Let Σ be the *discriminant* of the map ϕ, i.e., the set of all points $s\in S$ such that the fiber X_s is singular. Put

$$S'=S\setminus\Sigma\subset\mathbb{C}^N\times\mathbb{C}\,,\qquad X'=\phi^{-1}(S')\subset X\subset\mathbb{C}^N\times\mathbb{C}^{2l+1}\,.$$

Then the map $\phi\colon X'\to S'$ is a smooth fibration, called the *Milnor fibration*. Let δ' be a real number, s.t., $\delta''<\delta'<\delta$, then $(0,\delta')\in S'$ and all smooth fibers are diffeomorphic to $X_{0,\delta'}$. To avoid cumbersome notation let us assume that $\delta'=1$.

The middle homology group of the smooth fiber, equipped with the bilinear form $(\cdot|\cdot)$ equal to $(-1)^l$ times the intersection form, is known as the *Milnor lattice* $Q = H_{2l}(X_{0,1};\mathbb{Z})$. For a generic point $s \in \Sigma$, the singularity of the fiber X_s is Morse. Thus, every choice of a path from $(0,1)$ to s avoiding Σ leads to a group homomorphism $Q \to H_{2l}(X_s;\mathbb{Z})$. The kernel of this homomorphism is a free $\mathbb{Z}$-module of rank 1. A generator $\alpha \in Q$ of the kernel is called a *vanishing cycle* if $(\alpha|\alpha) = 2$.

2.1. **Frobenius structure.** Let $\mathcal{T}_B$ be the sheaf of holomorphic vector fields on B. Condition (2) in Definition 2.1 implies that the map

$$\partial/\partial t^i \mapsto \partial F/\partial t^i \quad \mathrm{mod}\ \langle \partial_{x_0}F, \dots, \partial_{x_{2l}}F\rangle \qquad (1 \le i \le N)$$

induces an isomorphism between $\mathcal{T}_B$ and $p_*\mathcal{O}_C$, where $p\colon X \to B$ is the natural projection $(t,x) \mapsto t$ and

$$\mathcal{O}_C := \mathcal{O}_X/\langle \partial_{x_0}F, \dots, \partial_{x_{2l}}F\rangle$$

is the structure sheaf of the critical set of F. In particular, since $\mathcal{O}_C$ is an algebra, the sheaf $\mathcal{T}_B$ is equipped with an associative commutative multiplication, which will be denoted by $\bullet$. It induces a product $\bullet_t$ on the tangent space of every point $t \in B$. The class of the function F in $\mathcal{O}_C$ defines a vector field $E \in \mathcal{T}_B$, called the *Euler vector field*.

Given a holomorphic volume form ω on $(\mathbb{C}^{2l+1}, 0)$, possibly depending on $t \in B$, we can equip $p_*\mathcal{O}_C$ with the so-called *residue pairing*

$$(\psi_1(t,x), \psi_2(t,x)) := \Big(\frac{1}{2\pi i}\Big)^{2l+1} \int_{\Gamma_\epsilon} \frac{\psi_1(t,y)\,\psi_2(t,y)}{\partial_{y_0}F\cdots\partial_{y_{2l}}F}\,\omega\,,$$

where $y = (y_0, \dots, y_{2l})$ is a ω-unimodular coordinate system (i.e. $\omega = dy_0\wedge\cdots\wedge dy_{2l}$) and the integration cycle Γ_ϵ is supported on $|\partial_{y_0}F| = \cdots = |\partial_{y_{2l}}F| = \epsilon$. Using that $\mathcal{T}_B \cong p_*\mathcal{O}_C$, we get a non-degenerate complex bilinear form $(\ ,\ )$ on $\mathcal{T}_B$, which we still call residue pairing.

For $t \in B$ and $z \in \mathbb{C}^*$ put

$$X_t = \{x \in B_\rho^{2l+1} \ :\ |F(t,x)| < \delta\}$$

and

$$X_{t,z}^- = \{x \in X_t \cap \partial B_\rho^{2l+1} \ :\ \mathrm{Re}(z^{-1}F(t,x)) < 0\}.$$

The homology groups

$$H_{2l+1}(X_t, X_{t,z}^-;\mathbb{C}) \cong \mathbb{C}^N$$

form a vector bundle on $B \times \mathbb{C}^*$ equipped naturally with a Gauss–Manin connection. For every flat section $\mathcal{B} = \mathcal{B}_{t,z}$ let us denote by $J_{\mathcal{B}}(t,z)$ the stationary phase asymptotic (see [1] for more details) as $z \to 0$ of the oscillatory integral

$$(2\pi z)^{-l-\frac{1}{2}}\ (z d^B) \int_{\mathcal{B}_{t,z}} e^{z^{-1}F(t,x)}\omega \in \mathcal{T}_B^*,$$

where d^B is the de Rham differential on B. According to K. Saito's theory of *primitive forms* [14, 19, 21] there exists a form ω, called primitive, such that $J_{\mathcal{B}}(t,z)$ are horizontal sections for the following connection:

$$\nabla_{\partial/\partial t^i} = \nabla^{\mathrm{L.C.}}_{\partial/\partial t^i} - z^{-1}(\partial_{t^i}\bullet_t), \qquad 1 \le i \le N \tag{2}$$

$$\nabla_{\partial/\partial z} = \partial_z - z^{-1}\theta + z^{-2}E\bullet_t\ . \tag{3}$$

Here $\nabla^{\mathrm{L.C.}}$ is the Levi–Civita connection associated with the residue pairing and

$$\theta := \nabla^{\mathrm{L.C.}}E - \Big(1-\frac{d}{2}\Big)\mathrm{Id},$$

where d is some complex number. In particular, this means that the residue pairing and the multiplication $\bullet$ form a *Frobenius structure* on B of conformal dimension d with identity 1 and Euler vector field E. For the definition of a Frobenius structure we refer to [7] .

Assume that a primitive form ω is chosen. Note that the flatness of the Gauss–Manin connection implies that the residue pairing is flat. Denote by $(\tau_1,\ldots,\tau_N)$ a coordinate system on B that is flat with respect to the residue pairing, and write ∂_i for the vector field $\partial/\partial\tau_i$. We can further modify the flat coordinate system so that the Euler field is the sum of a constant and linear fields:

$$E = \sum_{i=1}^N (1-d_i)\tau_i\partial_i + \sum_{i=1}^N \rho_i\partial_i\,.$$

The constant part represents the class of f in H, and the spectrum of degrees $d_1,\ldots,d_N$ ranges from 0 to d. Note that in the flat coordinates τ_i the operator θ (called sometimes the *Hodge grading operator*) assumes diagonal form:

$$\theta(\partial_i) = \Big(\frac{d}{2}-d_i\Big)\partial_i\,, \qquad 1\le i\le N\,.$$

Finally, let us trivialize the tangent and the cotangent bundle. We have the following identifications:

$$T^*B \cong TB \cong B\times T_0B \cong B\times H,$$

where H is the Jacobi algebra of f, the first isomorphism is given by the residue pairing, the second by the Levi–Cevita connection of the flat residue pairing, and the last one is the Kodaira–Spencer isomorphism

$$T_0B \cong H, \quad \partial/\partial t_i \mapsto \partial_{t_i}F\Big|_{t=0} \ \mathrm{mod}\ (f_{x_0},\ldots,f_{x_{2l}}). \tag{4}$$

Let $v_i \in H$ be the images of the flat vector fields ∂_i via the Kodaira–Spencer isomorphism (4). We assume that $v_N = 1$ is the unity of the algebra H.

2.2. **Period integrals.** Given a middle homology class $\alpha \in H_{2l}(X_{0,1};\mathbb{C})$, we denote by $\alpha_{t,\lambda}$ its parallel transport to the Milnor fiber $X_{t,\lambda}$. Let $d^{-1}\omega$ be any $2l$-form whose differential is ω. We can integrate $d^{-1}\omega$ over $\alpha_{t,\lambda}$ and obtain multivalued functions of λ and t ramified around the discriminant in S (over which the Milnor fibers become singular). To $\alpha \in H_{2l}(X_{0,1};\mathbb{C})$, we associate the *period vectors* $I^{(k)}_\alpha(t,\lambda) \in H$ ($k \in \mathbb{Z}$) defined by

$$(I^{(k)}_\alpha(t,\lambda), v_i) := -(2\pi)^{-l}\partial_\lambda^{l+k}\partial_i \int_{\alpha_{t,\lambda}} d^{-1}\omega\,, \qquad 1 \le i \le N\,. \tag{5}$$

Note that this definition is consistent with the operation of stabilization of singularities. Namely, adding the squares of two new variables does not change the right-hand side, since it is offset by an extra differentiation $(2\pi)^{-1}\partial_\lambda$. In particular, this defines the period vector for a negative value of $k \ge -l$ with l as large as one wishes. Note that, by definition, we have

$$\partial_\lambda I^{(k)}_\alpha(t,\lambda) = I^{(k+1)}_\alpha(t,\lambda)\,, \qquad k \in \mathbb{Z}\,.$$

The following lemma is a consequence of the definition of a primitive form.

Lemma 2.2. *The period vectors* (5) *satisfy the differential equations*

$$\partial_i I^{(k)}_\alpha = -v_i \bullet_t (\partial_\lambda I^{(k)}_\alpha)\,, \qquad 1 \le i \le N\,, \tag{6}$$

$$(\lambda - E\bullet_t)\partial_\lambda I^{(k)}_\alpha = \Big(\theta - k - \frac{1}{2}\Big) I^{(k)}_\alpha\,. \tag{7}$$

The connection corresponding to the differential equations (6)–(7) is a Laplace transform of the connection (2)–(3). In particular, since the oscillatory integrals are related to the period vectors via the Laplace transform, Lemma 2.2 follows from the fact that the oscillator integrals are horizontal sections for the connection (2)–(3).

Using equation (7), we analytically extend the period vectors to all $|\lambda| > \delta$. It follows from (6) that the period vectors have the symmetry

$$I^{(k)}_\alpha(t,\lambda) \;=\; I^{(k)}_\alpha(t - \lambda\mathbf{1}, 0)\,, \tag{8}$$

where $t \mapsto t - \lambda\mathbf{1}$ denotes the time-λ translation in the direction of the flat vector field $\mathbf{1}$ obtained from $1 \in H$. (The latter represents identity elements for all the products $\bullet_t$.)

Let $t \in B_{ss}$ be a semi-simple point; then the period vector $I^{(0)}_\alpha(t,\lambda)$ could have singularities only at the critical values $u_i(t)$. Moreover, using equation (3) it is easy to see that the order of the pole at a given singular point $\lambda = u_i(t)$ is at most $\frac{1}{2}$. A simple corollary of this observation, which will be used repeatedly, is that if the cycle α is invariant with respect to the local monodromy around $\lambda = u_i(t)$; then the corresponding period vectors $I^{(n)}_\alpha(t,\lambda)$ must be analytic in a neighborhood of $\lambda = u_i(t)$.

2.3. **Stationary phase asymptotic.** Let $u_i(t)$ $(1 \le i \le N)$ be the critical values of $F(t,\cdot)$. For a generic t, they form a local coordinate system on B in which the Frobenius multiplication and the residue pairing are diagonal. Namely,

$$\partial/\partial u_i \bullet_t \partial/\partial u_j = \delta_{ij}\partial/\partial u_j\,, \quad \Big(\partial/\partial u_i, \partial/\partial u_j\Big) = \delta_{ij}/\Delta_i\,,$$

where Δ_i is the Hessian of F with respect to the volume form ω at the critical point corresponding to the critical value u_i. Therefore, the Frobenius structure is *semi-simple*. We denote by Ψ_t the following linear isomorphism

$$\Psi_t\colon \mathbb{C}^N \to T_tB\,, \qquad e_i \mapsto \sqrt{\Delta_i}\partial/\partial u_i\,,$$

where $\{e_1,\dots,e_N\}$ is the standard basis for $\mathbb{C}^N$.

Let U_t be the diagonal matrix with entries $u_1(t),\dots,u_N(t)$. According to Givental [10], the system of differential equations (cf. (2), (3))

$$z\partial_i J(t,z) = v_i \bullet_t J(t,z)\,, \qquad 1 \le i \le N\,, \tag{9}$$

$$z\partial_z J(t,z) = (\theta - z^{-1}E\bullet_t)J(t,z) \tag{10}$$

has a unique formal asymptotic solution of the form $\Psi_t R_t(z) e^{U_t/z}$, where

$$R_t(z) = 1 + R_1(t)z + R_2(t)z^2 + \cdots\,,$$

and $R_k(t)$ are linear operators on $\mathbb{C}^N$ uniquely determined from the differential equations (9) and (10).

We will make use of the following formal series

$$\mathbf{f}_\alpha(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k)}_\alpha(t,\lambda)\,(-z)^k\,, \tag{11}$$

and

$$\phi_\alpha(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k+1)}_\alpha(t,\lambda)\,d\lambda\,(-z)^k\,. \tag{12}$$

Note that for A_1-singularity $F(t,x) = x^2/2 + t$ we have $u := u_1(t) = t$. Up to a sign there is a unique vanishing cycle. The corresponding series (11) and (12) will be denoted simply by $\mathbf{f}_{A_1}(t,\lambda;z)$ and $\phi_{A_1}(t,\lambda;z)$. The period vectors can be computed explicitly and they are given by the following formulas:

$$I^{(k)}_{A_1}(u,\lambda) = (-1)^k\,\frac{(2k-1)!!}{2^{k-1/2}}\,(\lambda-u)^{-k-1/2}, \quad k \ge 0$$

$$I^{(-k-1)}_{A_1}(u,\lambda) = 2\,\frac{2^{k+1/2}}{(2k+1)!!}\,(\lambda-u)^{k+1/2}, \quad k \ge 0.$$

The key lemma (see [12]) is the following.

Lemma 2.3. *Let $t \in B$ be generic and β be a vanishing cycle vanishing over the point $(t,u_i(t)) \in \Sigma$. Then for all λ near $u_i := u_i(t)$, we have*

$$\mathbf{f}_\beta(t,\lambda;z) = \Psi_t R_t(z)\,e_i\,\mathbf{f}_{A_1}(u_i,\lambda;z)\,.$$

3. Symplectic loop space formalism

The goal of this section is to introduce Givental's quantization formalism (see [11]) and use it to define the higher genus potentials in singularity theory.

3.1. **Symplectic structure and quantization.**

The space $\mathcal{H} := H(\!(z^{-1})\!)$ of formal Laurent series in z^{-1} with coefficients in H is equipped with the following *symplectic form*:

$$\Omega(\phi_1, \phi_2) := \mathrm{Res}_z \,(\phi_1(-z), \phi_2(z))\ , \qquad \phi_1, \phi_2 \in \mathcal{H}\,,$$

where, as before, $(,)$ denotes the residue pairing on H and the formal residue Res_z gives the coefficient in front of z^{-1}.

Let $\{v_i\}_{i=1}^N$ and $\{v^i\}_{i=1}^N$ be dual bases of H with respect to the residue pairing. Then

$$\Omega(v^i(-z)^{-k-1}, v_j z^l) = \delta_{ij}\delta_{kl}\,.$$

Hence, a Darboux coordinate system is provided by the linear functions q_k^i, $p_{k,i}$ on $\mathcal{H}$ given by:

$$q_k^i = \Omega(v^i(-z)^{-k-1}, \cdot)\,, \qquad p_{k,i} = \Omega(\cdot, v_i z^k)\,.$$

In other words,

$$\phi(z) = \sum_{k=0}^{\infty}\sum_{i=1}^{N} q_k^i(\phi) v_i z^k + \sum_{k=0}^{\infty}\sum_{i=1}^{N} p_{k,i}(\phi) v^i(-z)^{-k-1}\,, \qquad \phi \in \mathcal{H}\,.$$

The first of the above sums will be denoted $\phi^+(z)$ and the second $\phi^-(z)$.

The *quantization* of linear functions on $\mathcal{H}$ is given by the rules:

$$\widehat{q_k^i} = \hbar^{-1/2} q_k^i\,, \qquad \widehat{p_{k,i}} = \hbar^{1/2}\frac{\partial}{\partial q_k^i}\,.$$

Here and further, $\hbar$ is a formal variable. We will denote by $\mathbb{C}_\hbar$ the field $\mathbb{C}(\!(\hbar^{1/2})\!)$.

Every $\phi(z) \in \mathcal{H}$ gives rise to the linear function $\Omega(\phi, \cdot)$ on $\mathcal{H}$, so we can define the quantization $\widehat{\phi}$. Explicitly,

$$\widehat{\phi} = -\hbar^{1/2}\sum_{k=0}^{\infty}\sum_{i=1}^{N} q_k^i(\phi)\frac{\partial}{\partial q_k^i} + \hbar^{-1/2}\sum_{k=0}^{\infty}\sum_{i=1}^{N} p_{k,i}(\phi) q_k^i\,. \tag{13}$$

The above formula makes sense also for $\phi(z) \in H[[z, z^{-1}]]$ if we interpret $\widehat{\phi}$ as a formal differential operator in the variables q_k^i with coefficients in $\mathbb{C}_\hbar$.

Lemma 3.1. *For all $\phi_1, \phi_2 \in \mathcal{H}$, we have $[\widehat{\phi_1}, \widehat{\phi_2}] = \Omega(\phi_1, \phi_2)$.*

Proof. It is enough to check this for the basis vectors $v^i(-z)^{-k-1}$, $v_i z^k$, in which case it is true by definition. □

It is known that the operator series $\mathcal{R}_t(z) := \Psi_t R_t(z)\Psi_t^{-1}$ is a symplectic transformation. Moreover, it has the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation. A linear operator $A(z)$ on $\mathcal{H} := H(\!(z^{-1})\!)$ is infinitesimal symplectic if and only if the map $\phi \in \mathcal{H} \mapsto A\phi \in \mathcal{H}$ is a Hamiltonian vector field with a Hamiltonian given by the quadratic function $h_A(\phi) = \frac{1}{2}\Omega(A\phi, \phi)$. By definition, the *quantization* of $e^{A(z)}$ is given by the differential operator $e^{\widehat{h_A}}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k,i}p_{l,j})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_k^i \partial q_l^j}\,, \quad (p_{k,i}q_l^j)\widehat{\ } = (q_l^j p_{k,i})\widehat{\ } = q_l^j \frac{\partial}{\partial q_k^i}\,, \quad (q_k^i q_l^j)\widehat{\ } = \frac{1}{\hbar} q_k^i q_l^j\,.$$

3.2. **The total ancestor potential.** Let us make the following convention. Given a vector

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} q_k z^k \in H[z]\,, \qquad q_k = \sum_{i=1}^{N} q_k^i v_i \in H\,,$$

its coefficients give rise to a vector sequence $q_0, q_1, \dots$. By definition, a *formal function* on $H[z]$, defined in the formal neighborhood of a given point $c(z) \in H[z]$, is a formal power series in $q_0 - c_0, q_1 - c_1, \dots$. Note that every operator acting on $H[z]$ continuously in the appropriate formal sense induces an operator acting on formal functions.

The *Witten–Kontsevich tau-function* is the following generating series:

$$\mathcal{D}_{\mathrm{pt}}(\hbar; Q(z)) = \exp\Big(\sum_{g,n} \frac{1}{n!}\hbar^{g-1} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} (Q(\psi_i) + \psi_i)\Big), \tag{14}$$

where $Q_0, Q_1, \dots$ are formal variables, and ψ_i ($1 \le i \le n$) are the first Chern classes of the cotangent line bundles on $\overline{\mathcal{M}}_{g,n}$ (see [23, 15]). It is interpreted as a formal function of $Q(z) = \sum_{k=0}^{\infty} Q_k z^k \in \mathbb{C}[z]$, defined in the formal neighborhood of $-z$. In other words, $\mathcal{D}_{\mathrm{pt}}$ is a formal power series in $Q_0, Q_1 + 1, Q_2, Q_3, \dots$ with coefficients in $\mathbb{C}(\!(\hbar)\!)$.

Let $t \in B$ be a *semi-simple* point, so that the critical values $u_i(t)$ ($1 \le i \le N$) of $F(t, \cdot)$ form a coordinate system. Recall also the flat coordinates $\tau = (\tau_1(t), \dots, \tau_N(t))$ of t. The *total ancestor potential* of the singularity is defined as follows

$$\mathcal{A}_t(\hbar; \mathbf{q}(z)) = \widehat{\mathcal{R}}_t \prod_{i=1}^{N} \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i; {}^i\mathbf{q}(z)) \in \mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2 \dots]], \tag{15}$$

where $\mathcal{R}_t(z) := \Psi_t R_t(z)\Psi_t^{-1}$ and

$${}^i\mathbf{q}(z) = \sum_{k=0}^{\infty} \sum_{a=1}^{N} \frac{\partial u_i}{\partial \tau_a}\, q_k^a\, z^k\,.$$

It will be convenient also to consider another set $\mathbf{t} = \{t_k^i\}$ of formal variables related to $\mathbf{q}$ via the so called *dilaton shift:*

$$t_k^i = \begin{cases} q_k^i & \text{if } (k,i) \neq (1,N) \\ q_1^N + 1 & \text{otherwise,} \end{cases} \tag{16}$$

where recall that $v_N = 1 \in H$ is the unit for the Frobenius multiplication.

3.3. **The correlator functions.** In order to motivate our definition of correlators, let us first recall the definition in the geometric settings, following [11]. For a given projective manifold V, let us denote by $\overline{\mathcal{M}}_{g,n}(V,d)$ the moduli space of degree-d stable maps from a genus-g nodal Riemann surface, equipped with n marked points, to V. The ancestor correlator functions are defined by the following intersection numbers:

$$\langle v_{i_1}\overline{\psi}_1^{k_1}, \dots, v_{i_n}\overline{\psi}_n^{k_n}\rangle_{g,n}(t) := \sum_{m=0}^{\infty}\sum_d \frac{Q^d}{m!}\int_{[\overline{\mathcal{M}}_{g,n+m}(V,d)]^{\mathrm{virt}}} \mathrm{ev}^*(v_{i_1}\otimes\cdots\otimes v_{i_n}\otimes t^{\otimes m})\prod_{a=1}^n \overline{\psi}_a^{k_a},$$

where the notation is as follows. The classes $\{v_{i_s}\}_{s=1}^n$ and t are cohomology classes on V, the 2-nd sum is over all effective curve classes $d \in H_2(V;\mathbb{Z})$ and Q^d is an element of the Novikov ring. Furthermore, evaluating the stable map at the marked points gives rise to the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n+m}(V,d) \to V^{n+m},$$

while the operation forgetting the last m marked points, the stable map, and stabilizing (i.e. contracting the unstable components) gives a map $\mathrm{ft} : \overline{\mathcal{M}}_{g,n+m}(V,d) \to \overline{\mathcal{M}}_{g,n}$. The cohomology classes $\overline{\psi}_s := \mathrm{ft}^*(\psi_s)$ $(1 \leq s \leq n)$. Finally, $[\overline{\mathcal{M}}_{g,n+m}(V,d)]^{\mathrm{virt}}$ is the virtual fundamental cycle. Let us point out that if $\overline{\mathcal{M}}_{g,n}$ is empty, i.e., $2g-2+n \leq 0$, then the ancestor correlator is by definition 0. The total ancestor potential of V has the form

$$\mathcal{A}_t(\hbar;\mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty}\hbar^{g-1}\mathcal{F}_t^{(g)}(\mathbf{t})\Big), \tag{17}$$

where $t \in H := H^*(V;\mathbb{C})$, $\mathbf{t} = \{t_k^i\}$ is a set of formal variables and

$$\mathcal{F}_t^{(g)}(\mathbf{t}) = \sum_{n=0}^{\infty}\frac{1}{n!}\langle \mathbf{t}(\overline{\psi}_1), \dots, \mathbf{t}(\overline{\psi}_n)\rangle(t)_{g,n}$$

is the so called genus-g ancestor potential, where $\mathbf{t}(z) = \sum_{k,i} t_k^i v_i z^k$ and the definition of the correlator is extended mult-linearly.

Let us return to the settings of singularity theory. It can be proved that the ancestor potential (15) still has the form (17). Motivated by Gromov–Witten theory we would like to define the analogues of the ancestor correlator functions, so that the ancestor potential can be written in the same way. Put

$$\langle v_{i_1}\psi^{k_1},\dots,v_{i_n}\psi^{k_n}\rangle_{g,n}(t;\mathbf{t}) := \partial_{t^{i_1}_{k_1}}\cdots\partial_{t^{i_n}_{k_n}}\,\mathcal{F}^{(g)}_t(\hbar;\mathbf{t}), \tag{18}$$

then by the Taylor's formula we have

$$\mathcal{A}_t(\hbar;\mathbf{t}) = \exp\Big(\sum_{g,n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\,\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle_{g,n}(t;0)\Big),$$

where by extending multi-linearly the definition (18) we allow the insertions of the correlator to be any formal power series from $H[\![\psi]\!]$. Our main interest is in the correlators (18) with $\mathbf{t}=0$. Using the definition (15), it is easy to check that each of these correlators (with $\mathbf{t}=0$) is an analytic function in $t\in B_{\rm ss}$. The statement of Theorem 1.1 is that each correlator is in fact analytic on the entire space B.

4. The ancestors for generic non-semisimple points

Let $t_0\in B$ be a point, such that the function $F(t_0,\cdot): B^{2l+1}_\rho\to\mathbb{C}$ has $N-2$ Morse critical points $\xi^0_i (1\leq i\leq N-2)$ and a critical point ξ^0_{N-1} of type A_2, i.e., we can choose local coordinate system $y=(y_0,y_1,\dots,y_{2l})$ centered at the critical point ξ^0_{N-1} such that

$$F(t_0,y) = u^0_{N-1} + y_0^3 + \sum_{i=1}^{2l} y_i^2,\quad u^0_{N-1} := F(t_0,\xi^0_{N-1}).$$

Let us assume that the critical values $u^0_i := F(t_0,\xi^0_i)$, $1\leq i\leq N-1$, are pairwise distinct. Note that $t_0\in B\setminus B_{ss}$ and that all other points in $B\setminus B_{ss}$ form an analytic subvariety in B of codimension at least 2.

Let us choose a small disc D_i with center the critical value u^0_i for each $i=1,2,\dots,N-1$. We are going to let t vary in a small open neighborhood U of t_0, so that the critical values $u_i(t)$, $1\leq i\leq N$ of $F(t,\cdot)$ satisfy the conditions

$$u_i(t)\in D_i,\quad 1\leq i\leq N-2,\quad u_{N-1}(t),u_N(t)\in D_{N-1},$$

and

$$u_i(t_0)=u^0_i,\quad 1\leq i\leq N-2,\quad u_{N-1}(t_0)=u_N(t_0)=u^0_{N-1}.$$

Let us fix an arbitrary $t\in U\cap B_{ss}$.

4.1. **Twisted representations of the local Heisenberg algebras.** We fix a reference point p_i in the complement D^*_i to the critical values in D_i and denote by

$$\Delta_i\subset H_{2l}(X_{t,p_i};\mathbb{Z}),\quad 1\leq i\leq N-1$$

the cycles vanishing over the critical values contained in D_i. Note that together with the intersection pairing $(\cdot|\cdot)$ the sets $\Delta_i = \{\pm\beta_i\}$, $1 \le i \le N-2$ are root systems of type A_1, while

$$\Delta_{N-1} = \{\alpha := \beta_{N-1}, \beta := \beta_N, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta\}$$

is a root system of type A_2.

Let us fix $\Delta = \Delta_i$, $D := D_i$, and $D^* = D_i^*$ for some $i = 1, 2, \ldots, N-1$. We denote by Q_Δ the corresponding root lattice and put $\mathfrak{h}_\Delta = \mathbb{C} \otimes_{\mathbb{Z}} Q_\Delta$. The vector space $\widehat{\mathfrak{h}_\Delta} = \mathfrak{h}_\Delta[t, t^{-1}] \oplus \mathbb{C}\, K$ has a natural structure of a Heisenberg Lie algebra with Lie bracket given by

$$[\alpha\, t^m, \beta\, t^n] = m\delta_{m+n,0}\, (\alpha|\beta)\, K\ .$$

We denote by $\mathcal{F}_\Delta = \mathrm{Sym}(\mathfrak{h}_\Delta[t^{-1}]t^{-1})$ the Fock space of $\widehat{\mathfrak{h}_\Delta}$, i.e., the unique irreducible highest weight representation of $\widehat{\mathfrak{h}_\Delta}$, such that the center K acts by 1 and $\widehat{\mathfrak{h}}_\Delta^+ := \mathfrak{h}_\Delta[t]$ annihilates the *vacuum* 1. The notation t that appears here has nothing to do with the deformation parameters that we introduced before. In order to avoid confusion, from now on we put $a_m := a\, t^m$, $a \in \mathfrak{h}_\Delta$, $m \in \mathbb{Z}$.

Following [2], we define *bosonic fields*

$$X_t(\alpha, \lambda) = \partial_\lambda\, \widehat{\mathbf{f}_\alpha}(t, \lambda), \tag{19}$$

and *propagators*

$$P_{\alpha,\beta}(t, \lambda; \mu-\lambda) = \partial_\lambda\partial_\mu \lim_{\epsilon\to 0} \int_{t-(u_i(t)+\epsilon)\mathbf{1}}^{t-\lambda\,\mathbf{1}} I_\alpha^{(0)}(t', \mu-\lambda) \bullet I_\beta^{(0)}(t', 0) \tag{20}$$

where $\alpha, \beta \in \Delta$, for each $\lambda \in D^*$ we pick $\mu \in D^*$ sufficiently close to λ, and the integration is along a path such that $\beta_{t',0} \in H_{2l}(X_{t',0}; \mathbb{Z})$ vanishes at the end point $t' = t - u_i(t)\mathbf{1}$. The integrand is a 1-form obtained as follows: each period vector is by definition a co-vector in $T_{t'}^*B$; we identify vectors and co-vectors via the residue pairing and hence the Frobenius multiplication in $T_{t'}B$ induces a multiplication on T_t^*B. Finally, we can extend the definition of the propagator bi-linearly to all $\alpha, \beta \in \mathfrak{h}_\Delta$.

The Laurent series expansion of the propagator (20) at $\mu = \lambda$ (see [2] Lemma 7.5) has the following form:

$$P_{\alpha,\beta}(t, \lambda; \mu-\lambda) = \frac{(\alpha|\beta)}{(\lambda-\mu)^2} + \sum_{k=0}^{\infty} P_{\alpha,\beta}^k(t, \lambda)\, (\mu-\lambda)^k.$$

The above series has a non-zero radius of convergence and the coefficients $P_{\alpha,\beta}^k(t, \lambda)$ are multi-valued analytic functions on D^*, i.e., the analytic continuation in λ along any path in D^* is compatible with the monodromy action on α and β. The latter statement follows from Lemmas 7.1–7.3 in [2], which can be applied in our settings as well because the root system Δ is of type A.

For $a \in \mathcal{F}_\Delta$ of the form

$$a = \alpha^1_{(-k_1-1)} \cdots \alpha^r_{(-k_r-1)} 1 \,, \qquad r \geq 1\,,\ \alpha^i \in \mathfrak{h}_\Delta\,,\ k_i \geq 0\,,$$

we define

$$X_t(a,\lambda) = \sum_J \Big(\prod_{(i,j)\in J} \partial_\lambda^{(k_j)} P^{k_i}_{\alpha^i,\alpha^j}(t,\lambda) \Big) :\!\Big(\prod_{l\in J'} \partial_\lambda^{(k_l)} X_t(\alpha^l,\lambda) \Big)\!:, \tag{21}$$

where the sum is over all collections J of disjoint ordered pairs $(i_1, j_1), \ldots, (i_s, j_s) \subset \{1, \ldots, r\}^2$ such that $i_1 < \cdots < i_s$ and $i_l < j_l$ for all l, $J' = \{1, \ldots, r\}\backslash\{i_1, \ldots, i_s, j_1, \ldots, j_s\}$, and the normally ordered product : : means the usual composition of differential operators, except that we apply the differentiation operations $\frac{\partial}{\partial q^i_k}$ before the multiplication ones q^j_l. The main property of the above operators is that their Laurent series expansions at the critical values contained in D form a twisted representation of $\mathcal{F}_\Delta$ (see [2] Section 6 for more precise statement).

4.2. **The local Eynard–Orantin recursion.** Let $\Delta = \Delta_i$ be one of the root systems and let $a = \alpha^1_{(-k_1-1)} \cdots \alpha^r_{(-k_r-1)} 1 \in \mathcal{F}_\Delta$ be any vector. We define a multi-valued analytic symmetric r-form

$$\Omega^a_g(t,\lambda;\mathbf{t}) = f^a_g(t,\lambda;\mathbf{t})\ \underbrace{d\lambda\cdots d\lambda}_{r \text{ times}}$$

as follows

$$X_t(a,\lambda)\, \mathcal{A}_t(\hbar;\mathbf{q}) = \Big(\sum_{g=0}^{\infty} f^a_g(t,\lambda;\mathbf{q})\, \hbar^{g-1} \Big)\, \mathcal{A}_t(\hbar;\mathbf{q}). \tag{22}$$

Note that in the definition of $\Omega^a_g(t,\lambda;\mathbf{t})$ we replaced $\mathbf{q}$ by $\mathbf{t}$, so we did not use the dilaton-shift identification (16). If $a = \alpha^1_{(-1)} \cdots \alpha^r_{(-1)} 1$; then we will write $\Omega^{\alpha^1,\ldots,\alpha^r}_g(t,\lambda;\mathbf{t})$ instead of $\Omega^a_g(t,\lambda;\mathbf{t})$.

Let us point out that in definition (22) we are using in an essential way that the total ancestor potential $\mathcal{A}_t(\hbar;\mathbf{t})$ is *tame*. The latter by definition means that the correlator functions (18) vanish for $\mathbf{t} = 0$ and $k_1 + \cdots + k_n > 3g-3+n$. The tameness guarantees that inserting formal power series from $H[\![\psi]\!]$ in the correlators (18) does not produce divergent series.

According to [16], the ancestor correlators $\langle \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n}(t;0)$ satisfy the local Eynard–Orantin recursion, which can be written in the following way. If $(g, n+1)$ is in the stable range, i.e., $2g-2+n+1 > 0$, then

$$\Big\langle v_a \psi_1^m, \mathbf{t}, \ldots, \mathbf{t} \Big\rangle_{g,n+1} = \tag{23}$$

$$-\frac{1}{4} \sum_{j=1}^{N} \mathrm{Res}_{\lambda=u_j} \frac{[\widehat{v_a z^m}, \widehat{\mathbf{f}^{\beta_j}}_-(t,\lambda)]}{(I^{(-1)}_{\beta_j}(t,\lambda), \mathbf{1})\, d\lambda} \times \Bigg(\Big\langle \phi^+_{\beta_j}(t,\lambda;\psi_1), \phi^+_{\beta_j}(t,\lambda;\psi_2), \mathbf{t}, \ldots, \mathbf{t} \Big\rangle_{g-1,n+2} + \tag{24}$$

$$\sum_{\substack{g'+g''=g\\ n'+n''=n}} \binom{n}{n'} \Big\langle \phi^+_{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\dots,\mathbf{t}\Big\rangle_{g',n'+1} \Big\langle \phi^+_{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\dots,\mathbf{t}\Big\rangle_{g'',n''+1}\Bigg), \tag{25}$$

where β_j is a cycle vanishing over $\lambda = u_j$, we are allowing *unstable* correlators (see [16] for the definition) on the RHS, and we suppressed the argument $(t;0)$ of the correlators. In order to see that the above formula defines a recursion let us order all correlators lexicographically according to the pair (g,n) formed by the genus g and the number of insertions n of the correlator. The LHS is a single correlator of type $(g,n+1)$, while the RHS involves only correlators that are lexicographically smaller, because if $(g',n') = (0,0)$ or $(g'',n'') = (0,0)$, then the corresponding term in the sum (25) involves unstable correlators that vanish.

The above recursion can be written also in the following way

$$\langle v_a\,\psi^m\rangle_{g,1}(t;\mathbf{t}) = -\frac{1}{4}\sum_{i=1}^N \mathrm{Res}_{\lambda=u_i} \frac{\Omega(v_a\,z^m,\mathbf{f}^-_{\beta_i}(t,\lambda;z))}{y_{\beta_i}(t,\lambda)}\,\Omega_g^{\beta_i,\beta_i}(t,\lambda;\mathbf{t}), \tag{26}$$

where β_i is a cycle vanishing over $\lambda = u_i$ and

$$y_\beta(t,\lambda) := (I^{(-1)}_\beta(t,\lambda),1)\,d\lambda.$$

If we compare the degree-n terms with respect to $\mathbf{t}$ in (26), then we get precisely (23)–(25). Let us point out that while in (25) one has to give a special definition of the unstable correlator $\langle\phi^+_{\beta_j}(t,\lambda;\psi_1),\mathbf{t}\rangle_{0,2}$ and $\langle\phi^+_{\beta_j}(t,\lambda;\psi_1)\rangle_{0,1}$, in (26) this definition is automatically incorporated in the notation $\Omega_g^{\beta_j,\beta_j}(t,\lambda;\mathbf{t})$.

4.3. **Extending the recursion.** Let $\Delta = \Delta_{N-1}$ be the root system of type A_2. Put

$$\chi_1 = \frac{2}{3}\,\alpha + \frac{1}{3}\beta,\quad \chi_2 = -\frac{1}{3}\,\alpha + \frac{1}{3}\beta,\quad \chi_3 = -\frac{1}{3}\,\alpha - \frac{2}{3}\beta \quad\in\quad \mathfrak{h}_\Delta.$$

We refer to these as 1-point cycles. Note that the root system Δ consists of all differences $\chi_i-\chi_j$ for $i\neq j$. Motivated by the construction of Bouchard–Eynard [3] we introduce the following integral

$$-\frac{1}{2\pi\sqrt{-1}}\oint\sum_{c_1,\dots,c_r}\frac{1}{(r-1)!}\,\frac{\Omega(v_a\,z^m,\mathbf{f}^-_{c_1}(t,\lambda;z))}{\prod_{k=2}^r y_{c_k-c_1}(t,\lambda)}\,\Omega_g^{c_1,\dots,c_r}(t,\lambda;\mathbf{t}), \tag{27}$$

where the integral is along a closed loop in D^*_{N-1} that goes once counterclockwise around the critical values $u_{N-1}(t)$ and $u_N(t)$ and the sum is over all $r=2,3$ and all $c_1,\dots,c_r\in\{\chi_1,\chi_2,\chi_3\}$ such that $c_i\neq c_j$ for $i\neq j$. The monodromy group is the Weyl group of the root system and it acts on the 1-point cycles via permutations. In other words the integrand is monodromy invariant, hence a single valued analytic 1-form in D^*_{N-1}, so the integral makes sense.

Theorem 4.1. *For $g \neq 1$, the integral* (27) *coincides with the sum of the last two residues (at $\lambda = u_{N-1}, u_N$) in the sum* (26)*. For $g = 1$ the same identification holds up to terms independent of* $\mathbf{t}$.

The proof of Theorem 4.1 relies on a certain identity that we would like to present first. Let u_i and u_j ($1 \leq i, j \leq N$) be two of the critical values, $\beta := \beta_j$ be the cycle vanishing over u_j, and $a \in \mathfrak{h}_{\Delta_i}$ (we assume that $\Delta_N = \Delta_{N-1}$). Let us fix some Laurent series

$$f(\lambda,\mu) \in (\lambda - u_i)^{1/2}\mathbb{C}(\!(\lambda - u_i, \mu - u_j)\!) + \mathbb{C}(\!(\lambda - u_i, \mu - u_j)\!)$$

where $\mathbb{C}(\!(\lambda - u_i, \mu - u_j)\!)$ denotes the space of formal Laurent series. We have to evaluate residues of the following form:

$$\operatorname{Res}_{\lambda=u_i} \operatorname{Res}_{\mu=u_j} \sum_{\text{all branches}} \frac{\Omega(\phi_a^+(t,\lambda;z), \mathbf{f}_\beta^-(t,\mu;z))}{y_\beta(t,\mu)}\, f(\lambda,\mu)\, d\mu, \tag{28}$$

where ϕ_a is the formal series (12) and the sum is over all branches (2 of them) of the multivalued function that follows.

Lemma 4.2. *If $f(\lambda,\mu)$ does not have a pole at $\lambda = u_i$; then the residue* (28) *is non-zero only if $i = j$ and in the latter case it equals to*

$$(a|\beta) \operatorname{Res}_{\lambda=u_i} \sum_{\text{all branches}} \frac{f(\lambda,\lambda)}{y_\beta(t,\lambda)}\, d\lambda^2.$$

Proof. Put $a = a' + (a|\beta_i)\beta_i/2$; then a' is invariant with respect to the monodromy around $\lambda = u_i$. From this we get that $\phi_{a'}^+(t,\lambda;z)$ is analytic at $\lambda = u_i$, so it does not contribute to the residue. In other words, it is enough to prove the lemma only for $a = \beta_i$. Let us assume that $a = \beta_i$. We have

$$\Omega(\phi_a^+(t,\lambda;z), \mathbf{f}_\beta^-(t,\mu;z)) = \Omega(\mathbf{f}_\beta^+(t,\mu;z), \phi_a^-(t,\lambda;z)) + \Omega(\phi_a(t,\lambda;z), \mathbf{f}_\beta(t,\mu;z)).$$

The first symplectic pairing on the RHS does not contribute to the residue, because $\phi_a^-(t,\lambda;z)$ has a pole of order at most $\frac{1}{2}$ so after taking the sum over all branches, the poles of fractional degrees cancel out and hence the 1-form at hands is analytic at $\lambda = u_i$. For the second symplectic pairing we recall Lemma 2.3 and after a straightforward computation we get

$$\Omega(\phi_{A_1}(u_i,\lambda;z)e_i, \mathbf{f}_{A_1}(u_j,\mu;z)e_j) = 2\delta_{i,j}\, \frac{(\mu - u_j)^{\frac{1}{2}}}{(\lambda - u_i)^{\frac{1}{2}}}\, \delta(\lambda - u_i, \mu - u_j)\, d\lambda,$$

where

$$\delta(x,y) = \sum_{n\in\mathbb{Z}} x^n y^{-n-1}$$

is the formal δ-function. It is an easy exercise to check that for every $f(y) \in \mathbb{C}(\!(y)\!)$ we have

$$\mathrm{Res}_{y=0}\, \delta(x,y)\, f(y) = f(x).$$

The lemma follows. □

4.4. **Proof of Theorem 4.1.** The integral (27) can be written as a sum of two residues: $\mathrm{Res}_{\lambda=u_{N-1}}$ and $\mathrm{Res}_{\lambda=u_N}$. We claim that each of these residues can be reduced to the corresponding residue in the sum (26). Let us present the argument for $\lambda = u_{N-1}$. The other case is completely analogous.

Recall that we denoted by $\alpha = \beta_{N-1}$ the cycle vanishing over u_{N-1}. The summands in (27) for which $r = 2$ and $c_1, c_2 \in \{\chi_1, \chi_2\}$ give precisely

$$\mathrm{Res}_{\lambda=u_{N-1}} \frac{\Omega(v_a\, z^m, \mathbf{f}^-_{\chi_1-\chi_2}(t,\lambda;z))}{y_{\chi_1-\chi_2}(t,\lambda)}\, \Omega_g^{\chi_1,\chi_2}(t,\lambda;\mathbf{t}).$$

On the other hand, using that $\alpha = \chi_1 - \chi_2$ we get

$$\Omega_g^{\chi_1,\chi_2}(t,\lambda;\mathbf{t}) = -\frac{1}{4}\,\Omega_g^{\alpha,\alpha}(t,\lambda;\mathbf{t}) + \frac{1}{4}\,\Omega_g^{\chi_1+\chi_2,\chi_1+\chi_2}(t,\lambda;\mathbf{t})$$

Since $(\chi_1+\chi_2|\alpha) = 0$, the form $\Omega_g^{\chi_1+\chi_2,\chi_1+\chi_2}(t,\lambda;\mathbf{t})$ is analytic at $\lambda = u_{N-1}$, so it does not contribute to the residue. Therefore we obtain precisely the $(N-1)$-st residue in (26). It remain only to see that the remaining summands with $r = 2$ cancel out with the summand with $r = 3$.

There are two types of quadratic summands: $c_1, c_2 \in \{\chi_1, \chi_3\}$ and $c_1, c_2 \in \{\chi_2, \chi_3\}$. They add up respectively to

$$\frac{\Omega(v_a\, z^m, \mathbf{f}^-_{\chi_1-\chi_3}(t,\lambda;z))}{y_{\chi_1-\chi_3}(t,\lambda)}\, \Omega_g^{\chi_1,\chi_3}(t,\lambda;\mathbf{t}) \tag{29}$$

and

$$\frac{\Omega(v_a\, z^m, \mathbf{f}^-_{\chi_2-\chi_3}(t,\lambda;z))}{y_{\chi_2-\chi_3}(t,\lambda)}\, \Omega_g^{\chi_2,\chi_3}(t,\lambda;\mathbf{t}). \tag{30}$$

By definition

$$\sum_{g=0}^{\infty} \hbar^{g-1}\, \Omega_g^{\chi_i,\chi_3}(t,\lambda;\mathbf{t})\mathcal{A}_t = \Big(:\widehat{\phi}_{\chi_i}(t,\lambda)\widehat{\phi}_{\chi_3}(t,\lambda): + P^0_{\chi_i,\chi_3}(t,\lambda)\, d\lambda^2 \Big)\, \mathcal{A}_t. \tag{31}$$

The term $P^0_{\chi_i,\chi_3}$ contributes only to genus 1 and the contribution is independent of $\mathbf{t}$, so we may ignore this term. The normal product on the RHS is by definition

$$\widehat{\phi}_{\chi_3}(t,\lambda)\, \widehat{\phi}^+_{\chi_i}(t,\lambda) + \widehat{\phi}^-_{\chi_i}(t,\lambda)\widehat{\phi}_{\chi_3}(t,\lambda). \tag{32}$$

Since $(\chi_3|\alpha) = 0$ the field $\widehat{\phi}_{\chi_3}(t,\lambda)$ is analytic at $\lambda = u_i$. In addition $\widehat{\phi}^-_{\chi_i}(t,\lambda)$ has a pole of order at most $\frac{1}{2}$ at $\lambda = u_i$. It follows that the second summand in (32) does not contribute to the residue and therefore it can be ignored as well. For the RHS of (31) we get

$$\sum_{g=0}^{\infty} \hbar^{g-1}\, \widehat{\phi}_{\chi_3}(t,\lambda)\, \langle \phi^+_{\chi_i}(t,\lambda;\psi)\rangle_{g,1}(t;\mathbf{t})\, \mathcal{A}_t.$$

Recalling the local recursion (26) we get

$$-\frac{1}{4}\sum_{j=1}^{N} \mathrm{Res}_{\mu=u_j}\, \frac{\Omega(\phi^+_{\chi_i}(t,\lambda;z), \mathbf{f}^-_{\beta_j}(t,\mu;z))}{y_{\beta_j}(t,\mu)}\, \widehat{\phi}_{\chi_3}(t,\lambda)\, Y_t^{u_j}(\beta_j^2,\mu)\, d\mu^2\, \mathcal{A}_t,$$

where $Y_t^{u_j}(a,\mu)$ is the Laurent series expansion of $X_t(a,\mu)$ in $(\mu-u_j)$. Therefore we need to compute the residues $\mathrm{Res}_{\lambda=u_{N-1}}\, \mathrm{Res}_{\mu=u_j}$ of the following expressions

$$-\frac{1}{4}\sum_{i=1,2} \frac{\Omega(v_a\, z^m, \mathbf{f}^-_{\chi_i-\chi_3}(t,\lambda;z))}{y_{\chi_i-\chi_3}(t,\lambda)}\, \frac{\Omega(\phi^+_{\chi_i}(t,\lambda;z), \mathbf{f}^-_{\beta_j}(t,\mu;z))}{y_{\beta_j}(t,\mu)}\, \widehat{\phi}_{\chi_3}(t,\lambda)\, Y_t^{u_j}(\beta_j^2,\mu)\ d\mu^2\, \mathcal{A}_t.$$

The operator $\widehat{\phi}_{\chi_3}(t,\lambda)\, Y_t^{u_j}(\beta_j^2,\mu)\, d\mu^2$ can be written as

$$:\widehat{\phi}_{\beta_j}(t,\mu)^2\, \widehat{\phi}_{\chi_3}(t,\lambda): +2[\widehat{\phi}^+_{\chi_3}(t,\lambda), \widehat{\phi}^-_{\beta_j}(t,\mu)]\, \widehat{\phi}_{\beta_j}(t,\mu) + P^0_{\beta_j,\beta_j}(t,\mu)\widehat{\phi}_{\chi_3}(t,\lambda)\, d\mu^2\,. \quad (33)$$

Since $(\chi_3|\alpha) = 0$ the operator $\widehat{\phi}^+_{\chi_3}(t,\lambda)$ is regular at $\lambda = u_i$. It follows that the commutator

$$[\widehat{\phi}^+_{\chi_3}(t,\lambda), \widehat{\phi}^-_{\beta_j}(t,\mu)] \in \mathbb{C}(\!(\lambda-u_{N-1}, \mu-u_j)\!)$$

and therefore we may recall Lemma 4.2. The above residue is non-zero only if $j = N-1$. In the latter case we get

$$-\frac{1}{4}\mathrm{Res}_{\lambda=u_{N-1}} \sum_{i=1,2} (\chi_i|\alpha)\, \frac{\Omega(v_a\, z^m, \mathbf{f}^-_{\chi_i-\chi_3}(t,\lambda;z))}{y_{\chi_i-\chi_3}(t,\lambda)\, y_\alpha(t,\lambda)}\, \widehat{\phi}_{\chi_3}(t,\lambda)\, Y_t^{u_{N-1}}(\alpha_{-1}^2,\lambda)\, \mathcal{A}_t. \quad (34)$$

Note that (c.f. [2], Section 7)

$$[\widehat{\phi}^+_{\chi_3}(t,\lambda), \widehat{\phi}^-_{\beta_j}(t,\mu)] = \iota_{\lambda-u_{N-1}}\, \iota_{\mu-u_{N-1}}\ P_{\chi_3,\beta_j}(t,\lambda;\mu-\lambda),$$

where $\iota_{\lambda-u_{N-1}}$ is the Laurent series expansion at $\lambda = u_{N-1}$. Hence

$$\widehat{\phi}_{\chi_3}(t,\lambda)\, Y_t^{u_{N-1}}(\alpha_{-1}^2,\lambda) = \iota_{\lambda-u_{N-1}} X_t((\chi_3)_{-1}\alpha_{-1}^2,\lambda).$$

By definition

$$-\frac{1}{4}\,\alpha_{-1}^2 = (\chi_1)_{-1}\,(\chi_2)_{-1} - \frac{1}{4}\,(\chi_3)_{-1}^2$$

and since χ_3 is invariant with respect to the local monodromy around $\lambda = u_{N-1}$, the field $X_t((\chi_3)_{-1}^3,\lambda)$ does not contribute to the residue. We get the following formula for the residue (34):

$$\operatorname{Res}_{\lambda=u_{N-1}} \sum_{i=1,2} (\chi_i|\alpha) \frac{\Omega(v_a z^m, \mathbf{f}^-_{\chi_i-\chi_3}(t,\lambda;z))}{y_{\chi_i-\chi_3}(t,\lambda)\, y_\alpha(t,\lambda)} Y_t^{u_{N-1}}((\chi_1)_{-1}\,(\chi_2)_{-1}\,(\chi_3)_{-1},\lambda)\,\mathcal{A}_t.$$

Using that $\alpha = \chi_1 - \chi_2$, $(\chi_1|\alpha) = 1$, and $(\chi_2|\alpha) = -1$ we get

$$\operatorname{Res}_{\lambda=u_{N-1}} \Bigg(\frac{\Omega(v_a z^m, \mathbf{f}^-_{\chi_1}(t,\lambda;z))}{y_{\chi_2-\chi_1}(t,\lambda)\, y_{\chi_3-\chi_1}(t,\lambda)} + \frac{\Omega(v_a z^m, \mathbf{f}^-_{\chi_2}(t,\lambda;z))}{y_{\chi_1-\chi_2}(t,\lambda)\, y_{\chi_3-\chi_2}(t,\lambda)} + \\ + \frac{\Omega(v_a z^m, \mathbf{f}^-_{\chi_3}(t,\lambda;z))}{y_{\chi_1-\chi_3}(t,\lambda)\, y_{\chi_2-\chi_3}(t,\lambda)} \Bigg) \times \sum_{g=0}^\infty \hbar^{g-1} \Omega_g^{\chi_1,\chi_2,\chi_3}(t,\lambda;\mathbf{t})\,\mathcal{A}_t.$$

This sum cancels out the contribution to the residue at $\lambda = u_{N-1}$ of the cubic terms (i.e. the terms with $r = 3$) of the integral (27). □

4.5. **Proof of Theorem 1.1.** The most difficult part of the proof is already completed. We just need to take care of several initial cases. Let t_0 be a generic point in $B \setminus B_{ss}$. Let us write each 1-point correlator as a sum of terms homogeneous in $\mathbf{t}$

$$\langle v_a\psi^m\rangle_{g,1}(t;\mathbf{t}) = \sum_{n=0}^\infty \frac{1}{n!} \langle v_a\psi^m, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi)\rangle_{g,n+1}(t;0).$$

We claim that each summand is analytic at $t = t_0$. In order to see this let us put a lexicographical order on the summands according to (g, n) – genus and degree. Note that the local Eynard–Orantin recursion tells us how to find a correlator of a fixed genus and degree in terms of correlators of lower lexicographical order. Assuming that the lower order correlators are analytic at $t = t_0$ and that we can apply Theorem 4.1, we get the analyticity of the next correlator, because the integral (27) is analytic at $t = t_0$. Therefore, the proof would be completed by induction if we establish the initial cases

$$\langle v_a\psi^m, \mathbf{t}(\psi), \mathbf{t}(\psi)\rangle_{0,3}(t;0) \quad \text{and} \quad \langle v_a\psi^m\rangle_{1,1}(t;0).$$

Note that the above genus-1 correlators are the only ones for which Theorem 4.1 can not be applied. Hence if we verify the analyticity of the above correlators; then the proof of the analyticity of the 1-point correlators will be completed.

We will compute the above correlators via the local recursion. Using Lemma 2.3 we can express the Laurent series expansion

$$\iota_{\lambda-u_i}\,\iota_{\mu-u_i}\, P_{\beta_i,\beta_i}(t,\lambda;\mu-\lambda)$$

in terms of the Givental's higher-genus reconstruction operator R. After a straightforward computation we get

$$\frac{\mu+\lambda-2u_i}{(\lambda-\mu)^2(\lambda-u_i)^{1/2}(\mu-u_i)^{1/2}} + \sum_{k,l=0}^\infty 2^{k+l+1}(e_i, V_{kl}e_i)\,\frac{(\mu-u_i)^{k-\frac12}}{(2k-1)!!}\,\frac{(\lambda-u_i)^{l-\frac12}}{(2l-1)!!},$$

where the matrices $V_{kl} \in \mathrm{End}(\mathbb{C}^N)$ are defined as follows:

$$\sum_{k,l=0}^{\infty} V_{kl}\, w^k z^l = \frac{1 - {}^T R(-w) R(-z)}{w+z}.$$

The Laurent series expansion of the propagator at $\mu = \lambda$ becomes

$$\frac{2}{(\lambda-\mu)^2} + P^0_{\beta_i,\beta_i}(t,\lambda) + \cdots ,$$

where the dots stand for higher order terms in $(\mu - \lambda)$ and

$$P^0_{\beta_i,\beta_i}(t,\lambda) = \frac{1}{4}(\lambda - u_i)^{-2} + 2(e_i, R_1 e_i)\,(\lambda - u_i)^{-1}.$$

Similarly, we can find the Laurent series expansion of the period vectors

$$I^{(-1-m)}_{\beta_i}(t,\lambda) = 2\frac{(2(\lambda-u_i))^{m+\frac{1}{2}}}{(2m+1)!!}\Big(e_i - \sum_{k=1}^{N} \frac{R_1^{ki}}{2m+3}\, e_k\, 2(\lambda - u_i) + \cdots\Big),$$

where R_1^{ki} is the (k,i)-th entry of the matrix R_1 and slightly abusing the notation we put $e_i = du_i/\sqrt{\Delta_i}$.

Let us begin with the genus-0 case. The quadratic part of the form $\Omega_0^{\beta_i,\beta_i}(\lambda;\mathbf{t})$ is

$$\sum_{k,l=0}^{\infty}\sum_{a,b=1}^{N} (I^{(-k)}_{\beta_i}(t,\lambda), v_a)\,(I^{(-l)}_{\beta_i}(t,\lambda), v_b)\, d\lambda^2\, t^a_k\, t^b_l.$$

Applying the local recursion and leaving out the terms with $k > 0$ or $l > 0$ (since they do not contribute to the residue) we get that $\langle v_c\psi^m, \mathbf{t}(\psi), \mathbf{t}(\psi)\rangle_{0,3}(t;0)$ is

$$\frac{1}{4}\sum_{i=1}^{N} \mathrm{Res}_{\lambda=u_i} \sum_{a,b=1}^{N} \frac{(I^{(-1-m)}_{\beta_i}(t,\lambda), v_c)}{(I^{(-1)}_{\beta_i}(t,\lambda), 1)}\,(I^{(0)}_{\beta_i}(t,\lambda), v_a)\,(I^{(0)}_{\beta_i}(t,\lambda), v_b)\, d\lambda\, t^a_0\, t^b_0$$

The above residue is non-zero only for $m = 0$. Using the Laurent series expansion of the periods we get

$$\sum_{a,b=1}^{N} \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\Delta_i} \frac{\partial u_i}{\partial \tau_a}\frac{\partial u_i}{\partial \tau_b}\frac{\partial u_i}{\partial \tau_c}\, t^a_0\, t^b_0 = \sum_{a,b=1}^{N} \frac{1}{2}\,(v_a \bullet v_b, v_c)\, t^a_0\, t^b_0.$$

The coefficients of the above quadratic form are precisely the structure constants of the Frobenius multiplication, so they are analytic.

Let us continue with the genus-1 case. Now the local recursion takes the form

$$\langle v_a\psi^m\rangle_{1,1}(t;0) = \frac{1}{4}\sum_{i=1}^{N} \mathrm{Res}_{\lambda=u_i} \frac{(I^{(-1-m)}_{\beta_i}(t,\lambda), v_c)}{(I^{(-1)}_{\beta_i}(t,\lambda), 1)}\, P^0_{\beta_i,\beta_i}(t,\lambda).$$

The above residue is non-zero only if $m = 0$ or $m = 1$, because $P^0_{\beta_i,\beta_i}$ has a pole of order at most 2. In the case when $m = 1$, after substituting the Laurent series expansions of the propagator and of the periods we get

$$\langle v_a\psi\rangle_{1,1}(t;0) = \frac{1}{24}\sum_{i=1}^N \frac{\partial u_i}{\partial \tau_a} = \frac{1}{24}\frac{\partial}{\partial \tau_a}\,\mathrm{Tr}(E\bullet) = \frac{1}{24}\,\mathrm{Tr}(v_a\bullet),$$

where we used that in canonical coordinates the Euler vector field takes the form $\sum_{i=1}^N u_i\partial_{u_i}$ and hence $\sum_i u_i = \mathrm{Tr}(E\bullet)$.

For $m = 0$ after a straightforward computation, using also that $R_1^{ki} = R_1^{ik}$, we get

$$\langle v_a\rangle_{1,1}(t;0) = \frac{1}{2}\sum_{i=1}^N R_1^{ii}\frac{\partial u_i}{\partial \tau_a} + \frac{1}{24}\sum_{k,i=1}^N \frac{\sqrt{\Delta_k}}{\sqrt{\Delta_i}}R_1^{ki}\Big(\frac{\partial u_k}{\partial \tau_a} - \frac{\partial u_i}{\partial \tau_a}\Big).$$

The differential equations (9)–(10) imply the following relation:

$$[\partial_a U_t, R_1] = \Psi_t^{-1}\,\partial_a \Psi_t.$$

From this equation, using that the entries of the matrix Ψ and Ψ^{-1} are respectively

$$\Psi^{bi} = \sqrt{\Delta_i}\,\frac{\partial \tau_b}{\partial u_i} \quad \text{and} \quad (\Psi^{-1})^{kb} = \frac{1}{\sqrt{\Delta_k}}\frac{\partial u_k}{\partial \tau_b}$$

we get

$$\frac{\sqrt{\Delta_k}}{\sqrt{\Delta_i}}R_1^{ki}\Big(\frac{\partial u_k}{\partial \tau_a} - \frac{\partial u_i}{\partial \tau_a}\Big) = \frac{\partial u_k}{\partial \tau_b}\partial_a\Big(\sqrt{\Delta_i}\,\frac{\partial \tau_b}{\partial u_i}\Big)\frac{1}{\sqrt{\Delta_i}} = \delta_{ki}\,\frac{1}{2}\,\partial_a\,\log\Delta_i + \frac{\partial u_k}{\partial \tau_b}\frac{\partial}{\partial \tau_a}\Big(\frac{\partial \tau_b}{\partial u_i}\Big).$$

If we sum the above expression over all $i = 1, 2, \ldots, N$, since $\sum_i \partial_{u_i} = \partial_N$, we get simply $\frac{1}{2}\partial_a \log\Delta_k$. Hence the 1-point genus-1 correlator becomes

$$\langle v_a\rangle_{1,1}(t;0) = \frac{1}{2}\sum_{i=1}^N R_1^{ii}\frac{\partial u_i}{\partial \tau_a} + \frac{1}{48}\sum_{k=1}^N \partial_a\,\log\Delta_k.$$

The RHS is a well known expression, i.e., it is $\partial_a F^{(1)}(t)$, where $F^{(1)}(t)$ is the genus-1 potential of the Frobenius structure (see [13]), also known as the *G-function* (see [8, 9]). According to Hertling [14], Theorem 14.6, the function $F^{(1)}(t)$ is analytic. This completes the proof of the analyticity of all correlators that have at least 1 insertion.

To finish the proof of Theorem 4.1 we still have to prove that the correlators with no insertions $\langle\ \rangle_{g,0}(t;0)$ are analytic. Such correlators are identically 0 for $g = 0$, due to the tameness property of the ancestor potential. In genus 1, in the settings of Gromov–Witten theory, the correlator is 0 because the moduli space $\overline{\mathcal{M}}_{1,0}$ is empty. It is not hard to check (using the differential equation (9)) that in the abstract settings of semis-simple Frobenius manifolds this correlator still vanishes. For higher genera, using the differential equation (9) one can check easily that

$$\partial_a \langle\,\rangle_{g,0}(t;\mathbf{t}) = \frac{\partial}{\partial t_0^a}\langle\,\rangle_{g,0}(t;\mathbf{t}) = \langle v_a\rangle_{g,1}(t;\mathbf{t}).$$

In other words, the differential of the correlator $\langle\,\rangle_{g,0}(t;0)$ is an analytic 1-form on B and since B is simply connected the correlator must be analytic as well. □

Acknowledgements

I am thankful to B. Bakalov and Y. Ruan for many stimulating discussions. This work is supported by Grant-In-Aid and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

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