SOME PATHOLOGIES OF FANO MANIFOLDS IN POSITIVE CHARACTERISTIC

ILYA KARZHEMANOV

Abstract. We construct examples of Fano manifolds, which are defined over a field of positive characteristic, but not over $\mathbb{C}$.

1. Introduction

1.1. Let X be a Fano variety defined over a ground field $\mathbf{k} = \bar{\mathbf{k}}$ (see [12] or [16, Ch. V] for the basic notions and facts). We will also assume X to be smooth.

The main concern of the present note is the following:

Question. Suppose $p := \mathrm{char}\,\mathbf{k} > 0$. Does there exist X non-liftable to char $= 0$ (hence, in particular, $H^2(X, T_X) \neq 0$), i.e. is it true that there is always some variety X_0, defined over a $\mathbb{Z}$-subalgebra $R \subset \mathbb{C}$ of finite type, so that $X = X_0 \otimes_R \mathbf{k}$ for $\mathbf{k} := R/\mathfrak{p}$ and a prime ideal $\mathfrak{p} \subset R$ with $\mathfrak{p} \cap \mathbb{Z} = (p)$?

The problem is trivial for $\dim X = 1$, a bit trickier when $\dim X = 2$ (but still one does not obtain any new del Pezzo surfaces in positive characteristic due to e.g. [19, Ch. IV,§2]), and does not seem to get much attention in the case of $\dim X \geq 3$.

Remark 1.2. In [5], many (both non- and algebraic) Calabi-Yau 3-folds, defined over $\mathbb{F}_p$, have been constructed. The idea was to take a *rigid* (i.e. with $H^1(T) = 0$) nodal CY variety $\mathcal{X}$, defined over $\mathbb{Z}$, in such a way that its mod p reduction $X_p := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$ acquires an additional node. Then the needed CY 3-fold X is constructed via a small resolution $X \longrightarrow X_p$ (see [5, Theorem 4.3]). We will discuss other (general type) examples in Remark 1.5 below.

Recall that partial classification of Fano 3-folds X, subject to the "numerical" constraints $\mathrm{rk}\,\mathrm{Pic} = 1$ and either $p > 5$ or $D \cdot c_2(X) > 0$ for any ample divisor D on X, was obtained in [30].[1)] As it turns out, all these X do admit a lifting to $\mathbb{C}$, which is insufficient for us (cf. [1], [7], [20], [29]).

Thus, in order to find X with an interesting behavior in positive characteristic one should turn to the "qualitative" properties of Fano varieties, as the following example suggests (compare with Remark 1.2):

Example 1.3. The hypersurface $Y_p := (\sum x_i^p y_i = 0) \subset \mathbb{P}^n \times \mathbb{P}^n$ (see [16, (1.4.3.2)]) is a Fano variety iff $p \leq n$. Furthermore, Y_p is an (obvious) $SL(n+1)$-homogeneous space, which can deform to a non-homogeneous variety. This violates a similar property of homogeneous spaces defined over $\mathbf{k} \subseteq \mathbb{C}$. On the other hand, one

[1)] Actually, in the light of Corollary 1.7 below, for the arguments in [30] to carry on (cf. [30, Theorem 1.4]) one has to *assume* in addition that $H^1(X, \mathcal{O}_X(-D)) = 0$ for any (or at least those considered in [30]) ample divisors D on X.

may consider quotient schemes G/P, where G is a semi-simple algebraic group and $P \subset G$ is its *non-reduced* parabolic subgroup. (These G/P are rational for $p > 3$ due to [35].) Yet, unfortunately, both of the examples lift to characteristic 0, at least when rk Pic $= 1$ (cf. the discussion in [30]).[2)] Finally, let us mention p-coverings X of Fano hypersurfaces in $\mathbb{P}^{n+1}$, for which $\wedge^{n-1}\Omega^1_X$ contains a positive line subbundle (see [15]). But again, even though this property of $\wedge^{n-1}\Omega^1_X$ is specific to positive characteristic (see e.g. [3]), X is (obviously) liftable to $\mathbb{C}$.

We answer our **Question** via the next

Theorem 1.4. *For any $p \geq 3$ and $d, N \gg 1$, there exists a flat family $\mathcal{X} \longrightarrow B_d$ of Fano 3-folds, parameterized by an algebraic variety B_d of dimension $2d$, such that*

- *the fiber $X := \mathcal{X}_b$ is non-liftable to char $= 0$ for generic $b \in B_d$;*
- *the fibers $\mathcal{X}_b$ and $\mathcal{X}_{b'}$ are non-isomorphic for generic $b \in B_d, b' \in B_{d'}$, with $d \neq d'$;*
- *$(-K^3_{\mathcal{X}_b}) \geq N$ for the anticanonical degree of $X = \mathcal{X}_b$.*

Remark 1.5. Examples of manifolds (in arbitrary dimension) of general type violating Kodaira vanishing – more precisely, having $H^1(L^{-1}) \neq 0$ for an ample line bundle L, – have been constructed in [22]. This was a development of the method from [28], where a surface S of general type (with $H^1(S, L^{-1}) \neq 0$) is constructed together with a morphism $S \longrightarrow C$ onto a curve, so that the $\boldsymbol{k}(C)$*-curve* S violates the Mordell-Weill property. We follow these trends starting from **2.6** below, for the 3-fold X we need is constructed by finding a $\mathbb{P}^1$-bundle W over S first, with $-K_W$ nef, and then taking a p-covering $X \longrightarrow W$ à la [9].

Next corollary complements the results mentioned in Remarks 1.2, 1.5:

Corollary 1.6. *For generic X as in Theorem 1.4 and $m \gg 1$, there is a cyclic m-covering $\pi_m : X_m \longrightarrow X$ such that X_m is smooth, K_{X_m} is nef, and X_m is non-liftable to char $= 0$.*

The proof of Theorem 1.4 also yields (compare with [31])

Corollary 1.7. *In the notations from Corollary 1.6, irregularity $q(X) \neq 0$, as well as $q(X_m) \neq 0$.*

(Both of Corollaries 1.6, 1.7 are proved at the end of Section 2.)

1.8. Theorem 1.4 and Corollaries 1.6, 1.7 seem to be the “minimal” illustrations of pathological behavior of Fano varieties in positive characteristic (cf. the discussion after Remark 1.2). Note also that Theorem 1.4 provides an unbounded family of (smooth) Fano 3-folds (cf. [17], [26]). Furthermore, it follows easily from the arguments after Lemma 2.13 below that the cones $\overline{NE}(X_b)$, with varying $b \in B_d$, admit infinitely many “jumps” (compare with [34], [6], [36]).

[2)] Another possible approach to answer **Question** affirmatively is via the *wild* conic bundle structures (for $p = 2$) on Fano manifolds (see [30, Theorem 5.5]). But again one does not get anything new in characteristic 2 due to [21, Theorem 7] (see also [33] for some discussion and results on (birational) geometry of wild conic bundles).

Thus, one can see that essentially every "standard" property of Fano manifolds breaks in positive characteristic, the fact which is due (in our opinion) to the following principle behind (compare with Remark 2.11 below):

every object over $\mathbf{k}$, *char* $\mathbf{k} > 0$, *is defined* only up to *the Frobenius twist.*[3)]

(This was probably first exploited in [32], while the ultimate reading on the subject are [23], [24] and [25] of course.)

Finally, it would be interesting to work out the constructions in **2.8**, taking a (supersingular) Kummer surface in place of S (cf. [4]).

2. The construction

2.1. Fix a smooth projective surface S. We will denote by $\mathcal{E}$ (resp. $\mathcal{F}$) a vector bundle (resp. a coherent sheaf) on S. Let us recall some standard notions and facts about $\mathcal{E}$ and $\mathcal{F}$ (see e.g. [8], [10], [27]).

First of all, one defines the Chern classes $c_i := c_i(\mathcal{E}) \in A^i(S)$ for $\mathcal{E}$ (and similarly for $\mathcal{F}$, using a locally free resolution), with $c_1 = \det \mathcal{E}$. In fact, letting $W := \mathbb{P}(\mathcal{E})$ there is a natural inclusion $A^*(S) \hookrightarrow A^*(W)$ of groups of cycles induced by the projection $\pi : W \longrightarrow S$, and the following identity (called the *Hirsch formula*) holds:

$$H^2 + H \cdot c_1 + c_2 = 0.$$

Here $H := \mathcal{O}(1) \in A^1(W)$ is the Serre line bundle on W and c_i are identified with $\pi^*(c_i)$ (cf. Remark 2.2). In particular, for $r := \mathrm{rk}\ \mathcal{E} = 2$ we have

$$H^3 = -c_1^2 - c_2,$$

which together with the Euler's formula

$$K_W = -rH + \pi^*(K_S + c_1)$$

gives

$$(-K_W^3) = 6K_S^2 + 10c_1^2 + 24K_S \cdot c_1 - 8c_2.$$

Remark 2.2. When $\mathbf{k} \subseteq \mathbb{C}$ (so that $c_i \in H^{2i}(S, \mathbb{Z})$) and the structure group of $\mathcal{E}$ is not $SU(2)$ (or $H^1(S, \mathbb{Z}) \neq 0$), the previous formulae (except for the Euler's one and with suitably corrected $(-K_W^3)$) should be read with all the c_i up to $\pm$. In fact, when $\mathcal{E} \simeq \mathcal{E}^*$ (the dual of $\mathcal{E}$) and $\simeq$ is non-canonical, there is no preference in choosing $\pi^*(c_i)$ or $-\pi^*(c_i)$ as one may change the orientation in the fibers of $\mathcal{E}$. In particular, the Hirsch formula turns into $H^2 \pm H \cdot c_1 \pm c_2 = 0$, with both c_i having a definite sign at once.

Further, let $Z \subset S$ be a 0-dimensional subscheme supported at a finite number of points $p_1, \ldots, p_m$. Put $\ell_i := \dim\ \mathcal{O}_{S,p_i}/I_{Z,p_i}$ and $\ell(Z) := \sum_i \ell_i$ (the *length* of Z) for the defining ideal I_Z of Z. One can show that $c_2(j_*\mathcal{O}_Z) = -\ell(Z) \in A^2(S)$, where $j : Z \longrightarrow X$ is the inclusion map. In particular, if $\mathcal{E}$ admits a splitting

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \otimes I_Z \to 0 \tag{2.3}$$

for some line bundles $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(S)$, we obtain (using the Whitney's formula)

$$c_1 = c_1(\mathcal{L}) + c_1(\mathcal{L}'), \qquad c_2 = c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') + \ell(Z). \tag{2.4}$$

[3)] As, for example, the hyperelliptic curve $y^2 = x^p - a, a \in \mathbf{k}$, of genus $(p-1)/2$, covered (after Frobenius twist) by a rational curve (see [32, Corollary 1]).

Example 2.5. Let $s \in H^0(S, \mathcal{E})$ be a section. Assume for simplicity that the zero locus $(s)_0 \subset S$ of s has codimension ≥ 2. Then one gets an exact sequence of sheaves $0 \to \mathcal{O}_S \xrightarrow{s} \mathcal{E} \xrightarrow{\wedge s} \bigwedge^2 \mathcal{E} \otimes I_{(s)_0} \to 0$ (cf. (2.3)).

Suppose now that $r = 2$. Let $\mathcal{L} \subset \mathcal{E}$ be a line subbundle such that the sheaf $\mathcal{E}/\mathcal{L}$ is torsion-free. Then by working locally it is easy to obtain an exact sequence (2.3). All such sequences (*extensions* of $\mathcal{E}$) are classified by the group $\mathrm{Ext}^1(\mathcal{L}' \otimes I_Z, \mathcal{L})$, which in the case when Z is a locally complete intersection coincides with $\mathcal{O}_Z$. (More precisely, since $\mathcal{L}, \mathcal{L}'$, etc. are defined up to the $\mathbf{k}^*$-action, the classifying space for the extensions is the projectivization $\mathbb{P}(\mathrm{Ext}^1(\mathcal{L}' \otimes I_Z, \mathcal{L})) = \mathbb{P}(\mathcal{O}_Z)$.) Note also that (2.3) provides a *locally free* extension when $H^2(S, \mathcal{L}'^{-1} \otimes \mathcal{L}) = 0$ (Serre's criterion).

2.6. Let Y be a smooth projective variety with a line bundle $L \in \mathrm{Pic}(Y)$. Frobenius $F : Y \longrightarrow Y$ induces a homomorphism

$$F^* : H^1(Y, L^{-1}) \to H^1(Y, L^{-p})$$

and one has

$$\mathrm{Ker}\, F^* \simeq \{df \in \Omega_{\mathbf{k}(Y)} \mid f \in \mathbf{k}(Y), (df) \geq pD\}$$

once $L = \mathcal{O}_Y(D)$ for $D \geq 0$ (see [22, Theorem 1]). We may assume that $\mathrm{Ker}\, F^* \neq 0$ (see [22, Theorem 2]), $H^1(Y, \mathcal{O}_Y(-pD)) = 0$ for an ample D (cf. the proof of [30, Theorem 1.4]), which yields a non-split extension

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{E}_{L,Y} \to \mathcal{O}_Y \to 0$$

such that the corresponding extension for $F^*\mathcal{E}_{L,Y}$ splits. This defines a $\mathbb{G}_a$-torsor over Y (w.r.t. the line bundle L^{-1}) and a subscheme $X \subset \mathbb{P}(\mathcal{E}_{L,Y})$ which projects "p-to-1" onto Y. More precisely, the morphism $\phi : X \longrightarrow Y$ is purely inseparable of degree p, so that X is regular and $K_X = \phi^*(K_Y - (p-1)D)$ (see [22, Proposition 1.7, (16)]).

Example 2.7. Let $P(y)$ be a polynomial of degree e and $C \subset \mathbb{A}^2$ a plane curve given by the equation $P(y^p) - y = z^{pe-1}$ (see [22, Example 1.3]). One easily checks that $(dz) = pe(pe-3)(\infty)$ for $C \subset \mathbb{P}^2$ identified with its projectivization and $\infty \in C$. Then, taking $y^p z$ (with $d(y^p z) = y^p dz$) and $e \gg 1$, we obtain $\dim \mathrm{Ker}\, F^* = h^1(C, (3 - pe)(\infty)) \geq 2$ for $D := (pe-3)(\infty)$. Moreover, [22, Propositions 2.3, 2.6] shows that instead of C we may take Y as above, with $h^1(Y, L^{-1}) \geq 2$, ample K_Y (for $p \geq 3$) and arbitrary $\dim Y \geq 2$.

2.8. Now let $S := Y$ be the *Raynaud surface*. Recall that S can be realized as a double cover of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_{L,C})$ over the curve C from Example 2.7, ramified along a section and the p-cover X of C as above (run the constructions in [22, 2.1] with $k := 2$). Let $\psi : S \longrightarrow C$ be the induced rational curve fibration.

Proposition 2.9. *In the previous setting, there exists a vector bundle $\mathcal{E}$ on S such that*

- $r = 2, c_2 = 0$ *and* $c_1 = -nK_S$ *for an arbitrary* $n \gg 1$;
- *the divisor* $-K_W$ *is nef on* $W = \mathbb{P}(\mathcal{E})$ *(cf.* **2.1***).*

Proof. In the exact sequence (2.3), take $\mathcal{L}$ an arbitrary ample, $\mathcal{L}' := -\mathcal{L} - nK_S$ and Z any finite with $\ell(Z) = -c_1(\mathcal{L}) \cdot c_1(\mathcal{L}')$. Then from (2.4) we get $c_1 = -nK_S, c_2 = 0$, and the Euler's formula gives

$$-K_W = 2H + \pi^*(n-1)K_S.^{4)}$$

We will deduce that this $-K_W$ is nef from the two forthcoming lemmas.

Consider $l \subset S$, a general fiber of ψ, and put $\Sigma_l := \pi^{-1} l = \mathbb{P}(\mathcal{E}\big|_l)$.

Lemma 2.10. *We have $(\Omega^2_{\Sigma_l})^{**} \simeq \omega_{\Sigma_l/l} \otimes \mathcal{L}^{-1} \otimes \mathcal{L}'$ for the double dual of $\Omega^2_{\Sigma_l}$.*

Proof. Note that l is a rational curve with unique singular point $o := \mathrm{Sing}(l)$ such that l is given by the equation $y^2 = z^p$ (affine) locally near o. This shows that the sheaf Ω^1_l has torsion, with support in o, for $2ydy = pz^{p-1}dz = 0$. In particular, we get $(\Omega^1_l)^{**} = \mathcal{O}_l \cdot dz$ near o, with dy being the second (torsion) generator of Ω^1_l. This easily yields

$$h^0(l, (\Omega^1_l)^{**}) \ge \frac{(p-1)(p-2)}{2} - \frac{(p-2)(p-3)}{2} - 1 \ge 0.$$

Indeed, if $\tilde{l} \subset \mathbb{P}^2$ is the closure of the curve $(y^2 = z^p)$, then $\mathrm{Sing}(\tilde{l}) = \{o, \infty\}$ and there is a natural morphism $l \longrightarrow \tilde{l}$. The above estimate $h^0(l, (\Omega^1_l)^{**}) \ge 0$ now follows because $l \longrightarrow \tilde{l}$ is the normalization near ∞ and $\mathrm{mult}_\infty(\tilde{l}) = p - 2$.

Further, we have

$$\mathbb{P}(\mathcal{E}\big|_l) = \mathbb{P}(\mathcal{L}\big|_l \oplus \mathcal{L}'\big|_l)$$

with a local fiber coordinate t, so that (Kähler) dt induces a section τ of $(\Omega^2_{\Sigma_l})^{**}$. Namely, given the indicated properties of $\mathcal{E}\big|_l$ and Ω^1_l, we obtain

$$dt \wedge \xi = \tau\big|_{U \cap U'} = f dt \wedge \xi\big|_{U' \cap U}$$

for some $\xi \in H^0(l, (\Omega^1_l)^{**})$ and $f \in \mathcal{O}(U \cap U' \cap l)$ generating a cycle in $H^1(l, \mathcal{O}^*_l)$. (Here $U, U' \subset \Sigma_l$ are arbitrary open charts pulled back from l, and the latter is identified with the zero-section of restrictions $\mathcal{E}\big|_U, \mathcal{E}\big|_{U'}$.)

More precisely, we get that τ lifts to a section of $(\mathcal{L}^{-1} \otimes \mathcal{L}')\big|_l$, which gives $(\Omega^2_{\Sigma_l})^{**} \simeq \omega_{\Sigma_l/l} \otimes \mathcal{L}^{-1} \otimes \mathcal{L}'$. $\square$

Remark 2.11. The proof of Lemma 2.10 illustrates the following phenomenon typical to char > 0: the sheaf Ω^1_X may be rank 1 torsion, thus not equal to $\mathrm{Hom}(T_X, \mathcal{O}_X)$, where $\dim T_{X,x} \ge 2$ for some $x \in \mathrm{Sing}(X)$. Let us also point out that the interrelation between $\Omega^2_{\Sigma_l}$ and canonical bundle of the normalization of Σ_l is not clear (meaning the conductor cycle need not be effective) because Σ_l violates the S_2-property (for the curve l obviously does, – the rational function $y^{1/p} : l \longrightarrow \mathbb{P}^1$ is defined at every point near o, but is not *regular* at o).

Lemma 2.12. *The linear system $|-K_W\big|_{\Sigma_l}|$ is basepoint-free.*

Proof. It follows from Lemma 2.10 that the line bundle $(\Omega^2_{\Sigma_l})^*$ satisfies

$$H^0(\Sigma_l, (\Omega^2_{\Sigma_l})^*) \supseteq H^0(\Sigma_l, \mathcal{L} \otimes \mathcal{L}'^{-1}).$$

Indeed, since $\mathbb{P}^1$ normalizes l, we get $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \cdot dz$ near (the preimage of) o (cf. the proof of Lemma 2.10). Hence the sheaf $\omega_{\Sigma_l/l}$ pulls back to $\omega_{\Sigma/\mathbb{P}^1}$ for the

4) Note also that $H^2(S, \mathcal{L}'^{-1} \otimes \mathcal{L}) = H^2(nK_S) = 0$ and so $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{L}' \otimes I_Z, \mathcal{L})$ is locally free.

ruled surface $\Sigma := \mathbb{P}(\mathcal{E}\big|_{\mathbb{P}^1})$. It remains to notice that (the class of) $(\omega_{\Sigma/\mathbb{P}^1})^* = 2[\text{minimal section on } \Sigma] + 2[\text{fiber}]$ is ample.

On the other hand, for $s \in H^0(\Sigma_l, (\Omega^2_{\Sigma_l})^*)$ we get

$$-K_W\big|_{\Sigma_l} = (s = 0) + \Sigma_l\big|_{\Sigma_l} = (s = 0)$$

by adjunction, and the claim follows. □

Lemma 2.13. *$H^1(W, -K_W - \Sigma_l) = 0$.*

Proof. Notice that

$$H^1\big(W, \pi_*(2H + \pi^*(n-1)K_S - \Sigma_l)\big) = H^1\big(S, S^2\mathcal{E} \otimes \mathcal{O}_S((n-1)K_S - l)\big) = 0$$

by Serre vanishing (for K_S is ample). Furthermore, we have

$$R^1\pi_*(2H + \pi^*(n-1)K_S - \Sigma_l) = 0$$

by Grothendieck comparison, since the divisor $2H + \pi^*(n-1)K_S - \Sigma_l$ is π-ample. The assertion now follows from the Leray spectral sequence. □

From the defining exact sequence for Σ_l and Lemma 2.13 we get a surjection

$$H^0(W, -K_W) \twoheadrightarrow H^0(\Sigma_l, -K_W\big|_{\Sigma_l}).$$

Then by Lemma 2.12 the base locus of $|-K_W|$ is contained in the fibers of π. This shows that $-K_W$ is nef and finishes the proof of Proposition 2.9. □

Recall that $H^1(S, L^{-1}) \neq 0$ for $L = \mathcal{O}_S(D)$ (cf. the beginning of **2.8**). One can also observe that $H^1(W, \pi^*L^{-1}) \simeq H^1(S, L^{-1})$. Hence, setting $Y := W$ (resp. $L := \pi^*L$) in the notations of **2.6**, we can pass to the corresponding p-cover $\phi : X = \mathbb{P}(\mathcal{E}_{\pi^*L,W}) \longrightarrow W$. The divisor $-K_X$ is ample by Proposition 2.9 and Kleiman's criterion (see [11, Ch. I, Theorem 8.1]).

Finally, since $W = \mathbb{P}(\mathcal{E})$ for $\mathcal{E}$ constructed as an extension (cf. **2.1**), to complete the proof of Theorem 1.4 it suffices to vary $\mathcal{L}$ and Z as above in such a way that both $c_1(\mathcal{E}) = -nK_S, c_2(\mathcal{E}) = 0$ remain fixed, while $\ell(Z) \to \infty$ (cf. (2.4) and the properties of $\mathcal{E}$ in Proposition 2.9). The assertion then follows from the formula for $(-K_W^3)$ in **2.1** and the fact that the cone $\overline{NE}(X)$ is finite polyhedral (see e.g. [14], [13]). More precisely, we have obtained that $X = \mathcal{X}_b$ surjects onto $W = \mathbb{P}(\mathcal{E})$, with $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{L}' \otimes I_Z, \mathcal{L})$ varying together with Z (or $b \in B_d$). Then it follows from the discussion after Example 2.5 that for different $\mathcal{E}$ the 3-folds W are non-isomorphic. This gives $\mathcal{X}_b \not\simeq \mathcal{X}_{b'}$ for generic $b' \in B_{d'}, d' \neq d$, and proves the second claim of Theorem 1.4, whereas the other two are evident from the construction of X.

Remark 2.14. Alternatively, one could show that already W is Fano, by applying the results in [2], [37] (and the formula for $(-K_W^3)$ of course).

We now pass on to the proof of Corollaries:

Proof of Corollary 1.6. X_m is obtained by the cyclic covering of X with ramification at a smooth surface from $|-mK_X|$ (see e.g. [18]). This gives $\mu_m \in \mathrm{Aut}(X_m)$ (with $X = X_m/\mu_m$) and $(m, p) = 1$. We may assume the μ_m-action on X_m comes from a cyclic projective action on some $\mathbb{P}^N \supset X_m$.

Now, if X_m were liftable to $\mathbb{C}$, the μ_m-action on $X_m \subset \mathbb{P}^N$ must also lift. Then we would get that $X = X_m/\mu_m$ is liftable to $\mathbb{C}$, a contradiction. □

Proof of Corollary 1.7. Suppose that $q(X) = 0$. We are going to find an ample line bundle $\mathfrak{L}$ on X such that $H^1(X_m, \pi_m^*\mathfrak{L}^{-1}) \neq 0$ and $(p-1)\pi_m^*\mathfrak{L} - K_{X_m}$ is ample. This will contradict [16, Ch. II, Corollary 6.3]

The description of $\operatorname{Ker} F^*$ in **2.6** and the fact that $h^1(W, \pi^*L^{-1}) = h^1(S, L^{-1}) \geq 2$ (cf. Example 2.7) yield $h^1(X, (\phi \circ \pi)^*L^{-1}) \geq 2$ – for not every element in $H^1(W, \pi^*L^{-1})$ is a p-power (when lifted to X). Furthermore, the same argument shows that $H^1(X_m, \pi_m^*\mathfrak{L}^{-1}) \neq 0$, provided we have found $\mathfrak{L}$ on X as needed.

Now set $\mathfrak{L} := \mathcal{O}_X(-K_X + (\pi \circ \phi)^*D)$. This is obviously ample. Moreover, by the vanishing in [30, Proposition 6.1] and results from [30, §§3,4] we may assume generic surface $S_X \in |-K_X|$ to be smooth (hence *K3-like*), which leads to an exact sequence

$$0 \to H^1(X, -S_X - (\pi \circ \phi)^*D) \to H^1(X, -(\pi \circ \phi)^*D) \to H^1(S, -(\pi \circ \phi)^*D\big|_S).$$

Here $H^1(S, -(\pi \circ \phi)^*D\big|_S) = 0$ by [22, Proposition 3.4], $H^1(X, -(\pi \circ \phi)^*D) \neq 0$ by the previous considerations, $\mathcal{O}_X(K_X - (\pi \circ \phi)^*D) = \mathfrak{L}^{-1}$, and thus we are done.[5)]

Hence we get $q(X) \neq 0$. Then also $q(X_m) \neq 0$ for the flat morphism π_m. $\square$

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References

[1] E. Ballico, On Fano threefolds in characteristic p, Rend. Sem. Mat. Univ. Politec. Torino **47** (1989), no. 1, 57–70 (1991).

[2] C. Birkar, Existence of flips and minimal models for 3-folds in char p, Preprint arXiv:1311.3098 (to appear in Annales scientifiques de l'ENS).

[3] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective manifolds, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), no. 6, 1227–1287, 1439.

[4] F. Bogomolov and Y. Tschinkel, Rational curves and points on $K3$ surfaces, Amer. J. Math. **127** (2005), no. 4, 825–835.

[5] S. Cynk and D. van Straten, Small resolutions and non-liftable Calabi-Yau threefolds, Manuscripta Math. **130** (2009), no. 2, 233–249.

[6] T. de Fernex and C. D. Hacon, Rigidity properties of Fano varieties, in *Current developments in algebraic geometry*, 113–127, Math. Sci. Res. Inst. Publ., 59, Cambridge Univ. Press, Cambridge.

[7] A. J. de Jong and M. Starr, A note on Fano manifolds whose second Chern character is positive, Preprint arXiv:math/0602644.

[8] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford Univ. Press, New York, 1990.

[9] T. Ekedahl, Canonical models of surfaces of general type in positive characteristic, Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 97–144.

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5) Note that $(p-1)\pi_m^*\mathfrak{L} - K_{X_m} \equiv \pi_m^* \; -(p-1+\dfrac{m+1}{m})K_X + (\pi \circ \phi)^*D$ is ample.

[10] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext, Springer, New York, 1998.

[11] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, Vol. 156, Springer, Berlin, 1970.

[12] V. A. Iskovskikh and Yu. G. Prokhorov, Fano varieties, in *Algebraic geometry, V*, 1–247, Encyclopaedia Math. Sci., 47, Springer, Berlin.

[13] S. Keel, Basepoint freeness for nef and big line bundles in positive characteristic, Ann. of Math. (2) **149** (1999), no. 1, 253–286.

[14] J. Kollár, Extremal rays on smooth threefolds, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 3, 339–361.

[15] J. Kollár, Nonrational hypersurfaces, J. Amer. Math. Soc. **8** (1995), no. 1, 241–249.

[16] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32, Springer, Berlin, 1996.

[17] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom. **36** (1992), no. 3, 765–779.

[18] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, translated from the 1998 Japanese original, Cambridge Tracts in Mathematics, 134, Cambridge Univ. Press, Cambridge, 1998.

[19] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, translated from the Russian by M. Hazewinkel, North-Holland, Amsterdam, 1974.

[20] G. Megyesi, Fano threefolds in positive characteristic, J. Algebraic Geom. **7** (1998), no. 2, 207–218.

[21] S. Mori and N. Saito, Fano threefolds with wild conic bundle structures, Proc. Japan Acad. Ser. A Math. Sci. **79** (2003), no. 6, 111–114.

[22] S. Mukai, Counterexamples to Kodaira's vanishing and Yau's inequality in positive characteristics, Kyoto J. Math. **53** (2013), no. 2, 515–532.

[23] D. Mumford, Pathologies of modular algebraic surfaces, Amer. J. Math. **83** (1961), 339–342.

[24] D. Mumford, Further pathologies in algebraic geometry, Amer. J. Math. **84** (1962), 642–648.

[25] D. Mumford, Pathologies. III, Amer. J. Math. **89** (1967), 94–104.

[26] T. Okada, On the birational unboundedness of higher dimensional $\mathbb{Q}$-Fano varieties, Math. Ann. **345** (2009), no. 1, 195–212.

[27] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, corrected reprint of the 1988 edition, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2011.

[28] M. Raynaud, Contre-exemple au "vanishing theorem" en caractéristique $p > 0$, in *C. P. Ramanujam—a tribute*, 273–278, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin.

[29] N. Saito, Fano threefolds with Picard number 2 in positive characteristic, Kodai Math. J. **26** (2003), no. 2, 147–166.

[30] N. I. Shepherd-Barron, Fano threefolds in positive characteristic, Compositio Math. **105** (1997), no. 3, 237–265.

[31] Y. Takayama, Raynaud-Mukai construction and Calabi-Yau threefolds in positive characteristic, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4063–4074.

[32] J. Tate, Genus change in inseparable extensions of function fields, Proc. Amer. Math. Soc. **3** (1952), 400–406.

[33] B. Totaro, Birational geometry of quadrics in characteristic 2, J. Algebraic Geom. **17** (2008), no. 3, 577–597.

[34] B. Totaro, Jumping of the nef cone for Fano varieties, J. Algebraic Geom. **21** (2012), no. 2, 375–396.

[35] C. Wenzel, Rationality of G/P for a nonreduced parabolic subgroup-scheme P, Proc. Amer. Math. Soc. **117** (1993), no. 4, 899–904.

[36] J. A. Wiśniewski, Rigidity of the Mori cone for Fano manifolds, Bull. Lond. Math. Soc. **41** (2009), no. 5, 779–781.

[37] C. Xu, On the base-point-free theorem of 3-folds in positive characteristic, J. Inst. Math. Jussieu **14** (2015), no. 3, 577–588.

Kavli IPMU (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan
E-mail address: ilya.karzhemanov@ipmu.jp