RATIONALITY OF AN S_6-INVARIANT QUARTIC 3-FOLD

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Abstract. We complete the study of rationality problem for hypersurfaces $X_t \subset \mathbb{P}^4$ of degree 4 invariant under the action of the symmetric group S_6.

1. Introduction

1.1. Any quartic 3-fold $X_t \subset \mathbb{P}^4$ with a non-trivial action of the group S_6 can be given by the equations

$$\sum x_i = t \sum x_i^4 - (\sum x_i^2)^2 = 0 \tag{1.2}$$

in $\mathbb{P}^5$. Here the parameter $t \in \mathbb{P}^1$ is allowed to vary.

When $t = 2$ one gets the Burkhardt quartic whose rationality is well-known (see e. g. [10, 5.2.7]). Similarly, $t = 4$ corresponds to the Igusa quartic, which is again rational (see [21, Section 3]). On the other hand, it was shown in [1] that for all other $t \neq 0, 6, 10/7$ the quartic X_t is non-rational.

Example 1.3. Following [4, Section 4], let us blow up an A_6-orbit of 12 lines in $\mathbb{P}^3$ to get a 3-fold that contracts, A_6-equivariantly, onto a quartic threefold with 36 nodes. It follows from *Remark* in [1] that this (Todd) quartic must be $X_{10/7}$. Hence $X_{10/7}$ is rational.

Thus, excluding the trivial case of $t = 0$ it remains to consider only X_6, in order to determine completely the birational type of all S_6-invariant quartics. Here is the result we obtain in this paper:

Theorem 1.4. *The quartic $X := X_6$ is rational.*

Theorem 1.4 is proved in Section 3 by, basically, running the equivariant-MMP-type of arguments as in [22]. (Although the proof also uses some computations carried in Section 2.) Unfortunately, we were not able to apply the results from

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[14], since non-rational X_t all have defect equal 5 (see [1, Lemma 2]), which seems to contradict either [14, 5.2, Lemma 8] or [14, 5.2, Proposition 3] (compare also with [14, Corollary 1] and the list of cases in [14, Main Theorem]).

Conventions. The ground field is $\mathbb{C}$ and X signifies the quartic X_6 in what follows. We will be using freely standard notions and facts from [11] and [16] (but we recall some of them for convenience).

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2. Auxiliary results

2.1. Consider the subspace $\mathbb{P}^3 \subset \mathbb{P}^5$ given by equations

$$x_0 + x_2 + x_5 = x_1 + x_3 + x_4 = 0.$$

We have $X \cap \mathbb{P}^3 = Q_1 + Q_2$, where the quadric $Q_1 \subset \mathbb{P}^3$ is given by

$$x_0^2 + x_0x_2 + x_2^2 + w(x_1^2 + x_1x_3 + x_3^2) = 0,\ w := \sqrt[3]{1},$$

while the equation of $Q_2 \subset \mathbb{P}^3$ is

$$x_0^2 + x_0x_2 + x_2^2 - (w+1)(x_1^2 + x_1x_3 + x_3^2) = 0.$$

Identify the set $\{x_2, x_0, x_4, x_3, x_1\}$ with $\{1, \dots, 5\}$ and consider the corresponding action of the group S_5. Put $\tau := (13524) \in S_4 \subset S_5$ and $o := [1 : 1 : w : w : w^2 : w^2] \in \mathrm{Sing}(X)$ (cf. [1]).[1)] Then the following (evident) assertion holds:

Lemma 2.2. *$\tau^c(Q_i) \ni o$ iff $c = 0$ or 2.*

Consider $h := (23451) \in S_5$. Again a direct computation gives the following:

Lemma 2.3. *$h^a\tau^b(Q_i) \ni o$ iff*

$(a, b) \in \{(0,0), (3,0), (4,0), (0,2), (3,3), (1,2), (4,2), (1,1)\}$. More precisely, we have

[1)] For the set $\{1, \dots, n\}$, any $n \geq 1$, symbol $(i_1 \dots i_n), 1 \leq i_j \leq n$, denotes its permutation $\{i_1, \dots, i_n\}$ (i.e. $1 \mapsto i_1$ and so on). Also, if $i_j = j$ for some j, we will identify (in the obvious way) $(i_1 \dots i_n)$ with permutation of the respective $(n-1)$-element set.

- $\tau^2(Q_i) = h^4(Q_i) \ni o$ *and* $\tau^2(Q_i) \neq Q_i$;
- $h^4\tau^2(Q_i) = h^3(Q_i) \ni o$ *and* $h^4\tau^2(Q_i) \neq Q_i, \tau^2(Q_i)$;
- $h\tau^2(Q_i) = Q_i$;
- $h^3\tau^3(Q_i) = h\tau(Q_i) \ni o$ *and* $h^3\tau^3(Q_i) \neq Q_i, \tau^2(Q_i), h^4\tau^2(Q_i)$.

2.4. Let $G := \langle \tau, h \rangle$ be the group generated by τ and h. Note that the order of G is divisible by 4 and 5. Then from the classification of subgroups in S_5 we deduce that G is the *general affine group* $\mathrm{GA}(1,5)$. Note also that $G = \mathbb{F}_5 \rtimes \mathbb{F}_5^*$ for the field $\mathbb{F}_5$ (here $\mathbb{F}_5, \mathbb{F}_5^*$ are the additive and multiplicative groups, respectively).

Consider the divisor $D := \sum_{\gamma \in G} \gamma(Q_1)$ and the local class group $\mathrm{Cl}_{o,X}$ at o. Note that both Q_i are smooth because they are projectively equivalent to $x_0^2 + x_2^2 + x_1^2 + x_3^2 = 0$. In particular, blowing up $\mathbb{P}^4 \supset X$ at Q_1 yields a small resolution of the singularity $o \in X$. Then by the standard properties of (small) extremal contractions we may identify Q_1 with the generator $1 \in \mathrm{Cl}_{o,X} = \mathbb{Z}$.

With all this set-up we get the following:

Proposition 2.5. $\operatorname{rk} \mathrm{Cl}^G X > 1$ *for* $D \in \mathrm{Cl}_{o,X}$ *being equal to either* 4 *or* 8.

Proof. Let us recall the construction of the group $\mathrm{Cl}_{o,X}$. One identifies $X = \operatorname{Spec} \mathcal{O}_{o,X}$ and considers various morphisms $\mu : X \longrightarrow X'$. Here X' is any (not necessarily normal) variety. Then $\mathrm{Cl}_{o,X}$ is generated by the sheaves $\mathcal{O}_X(Q_1)$ and $\mu^*\mathcal{O}_{X'}(H)$ for all Cartier divisors H on X' (note that $\mu^*\mathcal{O}_{X'}(H)$ may no longer be a divisorial sheaf for non-flat μ). The group operation "+" on $\mathrm{Cl}_{o,X}$ is induced by the usual product of $\mathcal{O}_X$-modules.

Further, by construction of τ, h (cf. Lemmas 2.2, 2.3) we have

$$D = \sum_{(a,b) \in \{(0,0),\dots,(1,1)\}} \tau^a h^b(Q_1) = 2h^4(Q_1) + 2h^3(Q_1) + 2Q_1 + 2h\tau(Q_1) \tag{2.6}$$

in $\mathrm{Cl}_{o,X}$ (we have identified $\mathcal{O}_X(Q_1)$ with Q_1). Now, since $h^3(Q_1), h^4(Q_1) \ni o$, both h^3, h^4 act on $\mathrm{Cl}_{o,X} = \mathbb{Z}$. Indeed, $h^3(Q_1)$ and $h^4(Q_1)$ differ from (a power of) Q_1 by some suitable μ^*H as above.

For $h^3 = (h^4)^2$ we get $h^3(Q_1) = 1 = (h^3)^3(Q_1) = h^4(Q_1)$ and hence $D = 4$ or 8. This means in particular that the product of $\mathcal{O}_X$-modules

$$\mathcal{I} := \prod_{\gamma \in G} \mathcal{O}_X(-\gamma(Q_1)),$$

identified with D as an element in $\mathrm{Cl}_{o,X}$, is not invertible (otherwise D will be zero).

Take a G-equivariant resolution $r : W \longrightarrow X$. Then the sheaf $r^*\mathcal{I}$ becomes invertible and the corresponding (effective) divisor is *not* of the form [relatively trivial part] + [r-exceptional part]. Indeed, otherwise $\mathcal{I}$ will be equal to $\mu^*\mathcal{O}_{X'}(H)$, with some X' and H as earlier, which is impossible for $D \neq 0$ in $\mathrm{Cl}_{o,X}$.

Applying relative G-equivariant MMP to W (cf. [25, 9.1]) yields a small G-equivariant contraction $Y \longrightarrow X$ and a relatively non-trivial G-invariant Cartier divisor on Y (note that according to [15, Lemma 5.1] "Cartier = $\mathbb{Q}$-Cartier" in this case). This shows that $\operatorname{rk} \mathrm{Cl}^G X > 1$ and completes the proof of Proposition 2.5.[2)]

□

2.7. Fix some terminal $G\mathbb{Q}$-factorial modification $\phi : Y \longrightarrow X$. Here ϕ is a G-equivariant birational morphism with 1-dimensional exceptional locus (see Proposition 2.5). Let also $\psi : Y \longrightarrow Z$ be a K_Y-negative G-extremal contraction.

Lemma 2.8. 3-*fold* Y *is Gorenstein.*

Proof. This follows from the relation $\phi^*\omega_X = \omega_Y$, the fact that ϕ is small, and the freeness of $|-K_X|$. □

Recall that the singular locus of X consists of two S_6-orbits, of length 30 and 10, respectively, where the first orbit contains the point o, while the second one contains $o' := [-1 : -1 : -1 : 1 : 1 : 1]$ (see *Remark* in [1]).

For an appropriate Y we get the following:

Lemma 2.9. $\operatorname{Sing} Y = \emptyset$ *or* $G \cdot o'$.

Proof. Indeed, the divisor D from **2.4** contains o and the morphism ϕ makes X $G\mathbb{Q}$-factorial near o, which means that one may take ϕ to resolve the singularities in $G \cdot o \subset D$ (run the G-equivariant $\mathbb{Q}$-factorialization procedure from the proof of Proposition 2.5).

The complement $\Sigma :=$ [the longest S_6-orbit in $\operatorname{Sing} X$] $\setminus G \cdot o$ is also a G-orbit (of length 10). Furthermore, we have $s(o) \neq o \in \Sigma$ for $s := (21) \in S_5$ (see **2.1**), and

[2)] Present definition of $\mathrm{Cl}_{o,X}$ differs from the usual (algebraic) one that is via the direct limit of groups $\mathrm{Cl}\, U/\mathrm{Pic}\, U$ over all Zariski opens $U \ni o$ on X. A priori there is no natural isomorphism of the latter with $\mathrm{Cl}_{o,X}$. At the same time, we have used the fact that $0 \neq D \in \mathrm{Cl}_{o,X}$ in order to construct Y as above, thus proving the existence of *some* G-invariant non-Cartier divisor on X.

so the arguments in the proof of Proposition 2.5, with $s(Q_1) = Q_1$, apply to show that X not $G\mathbb{Q}$-factorial near Σ as well. Hence we may assume that ϕ resolves the singularities in Σ as well.

Finally, ϕ either resolves or not the singularities in $G \cdot o'$, depending on whether there is a G-invariant non-Cartier divisor passing through o' or there is no such. $\square$

We will assume from now on that Y is as in Lemma 2.9.

Proposition 2.10. *If ψ is birational, with exceptional locus E, then $\psi(E)$ is a curve.*

Proof. Firstly, recall that Y is terminal, $G\mathbb{Q}$-factorial (but not necessarily $\mathbb{Q}$-factorial) and Gorenstein (see Lemma 2.8).

Lemma 2.11. *Y is $\mathbb{Q}$-factorial with* $\operatorname{rk}\operatorname{Pic} Y = 11$.

Proof. Note that $\mathbb{F}_5 = \langle h \rangle$ is the unique normal subgroup in $G = \mathbb{F}_5 \rtimes \mathbb{F}_5^*$. Then we have $Q_i \not\sim h(Q_i)$. Indeed, otherwise $D \sim 5 \sum_{\gamma \in \langle \tau \rangle} \gamma(Q_i)$, where D is as in **2.4**. But in this case $D = 5(Q_1 + \tau^2(Q_1))$ in $\mathrm{Cl}_{o,X}$ (see Lemma 2.3), which is either 0 or 10, thus contradicting Proposition 2.5.

Further, since D is a G-orbit of Q_1, all of its components are linearly independent in $\operatorname{Cl} X \otimes \mathbb{R}$. Indeed, otherwise we get $\sum \gamma(Q_1) = 0$, which is an absurd. This, together with computation of the defect in [1], yields $\operatorname{rk}\operatorname{Cl} X = 11$ for $\operatorname{Cl} X$ being generated by K_X, a G-invariant class of some Weil divisor D_o and by the components of D (the number of these components is 10 because $Q_1 \not\sim h(Q_1)$).

Similarly, we find that $\operatorname{Cl} Y$ is generated by K_Y, $\phi_*^{-1} D_o$ and by the components of $\phi_*^{-1} D$, all being Cartier according to Lemma 2.9 and the fact that $D \not\ni o'$. Thus $\operatorname{Cl} Y = \operatorname{Pic} Y$ and the claim follows. $\square$

Now let E_i be the irreducible 2-dimensional components of E. Suppose that $\dim \psi(E) = 0$. Then we get the following:

Lemma 2.12. *E is a disjoint union of E_i.*

Proof. Since the divisor $-K_Y$ is nef and big, it follows from Lemma 2.11 and [24] that the Mori cone $\overline{NE}(Y)$ is polyhedral, spanned by extremal rays, so that every extremal ray on Y is contractible. This implies that some (at least 1-dimensional) family of curves in every E_i generates an extremal ray because there are no small K_Y-negative extremal contractions on Y (see [5] and Lemmas 2.8, 2.11). In particular, E_i do not intersect, since $\dim \psi(E) = 0$ by assumption. $\square$

Note that $\operatorname{Cl} X \simeq \operatorname{Cl} Y$ as G-modules. This induces a natural G-action on the cone $\overline{NE}(Y)$. Consider the G-extremal ray in $\overline{NE}(Y)$ corresponding to ψ. By Lemma 2.12 this is a G-orbit of some K_Y-negative contractible extremal rays R_i corresponding to E_i.

It remains to exclude the cases $E_i = \mathbb{P}^1 \times \mathbb{P}^1$ or quadratic cone, and $E_i = \mathbb{P}^2$, both for $\dim \psi(E) = 0$ (cf. [5]). Suppose one of these possibilities does occur. Then we get

Lemma 2.13. *Every surface E_i is not preserved by the subgroup $\langle h \rangle \subset G$.*

Proof. Assume the contrary. Then all R_i are invariant with respect to $\langle h \rangle$ and there is a subspace $\mathbb{P}^3 \subset \mathbb{P}^4 \supset X$ (with $\phi(E_i) \subseteq X \cap \mathbb{P}^3$) invariant under $\mathbb{F}_5 = \langle h \rangle$. Recall that $h = (23451)$ permutes x_0, x_2, x_1, x_3, x_4. Thus the equation of $\mathbb{P}^3$ is $\sum_{i=0}^{4} x_i = 0$. This implies that $X \cap \mathbb{P}^3 \cap \operatorname{Sing} X = \emptyset$ and so $\phi(E_i)$ is Cartier. But the latter is impossible for otherwise $\phi(E_i)$ would intersect all the curves on X negatively. $\square$

It follows from Lemma 2.13 that all E_i are linearly independent in $\operatorname{Pic} Y \otimes \mathbb{R}$ and together with K_Y they generate $\operatorname{Pic} Y$ (argue exactly as in the proof of Lemma 2.11). Note also that $E_i \cdot C \geq 0$ for all i and any K_Y-trivial curve $C \subset Y$ because otherwise the class of C belongs to R_i (recall that by our assumption $\psi(E_i)$ is a point). In particular, there is such C that any other K_Y-trivial curve $\neq C$ on Y is numerically equivalent to $C + \sum a_i R_i$ for all $a_i \geq 0$, and so there is just one C. This implies that every surface $\phi(E_i) \subseteq X \cap \mathbb{P}^3$ (of degree $(K_Y)^2 \cdot E_i \leq 2$) contains a G-orbit of length at least 30 (see Lemma 2.9). Hence $\phi(E_i)$ together with E_i are all $\langle h \rangle$-invariant.[3)] The latter contradicts Lemma 2.13 and Proposition 2.10 is completely proved. $\square$

We conclude by the following simple, although useful in what follows, observation:

Lemma 2.14. $G \not\subset \mathrm{GL}(3, \mathbb{C})$.

Proof. The group G has only one 4-dimensional and four 1-dimensional irreducible representations. The claim follows by decomposing $\mathbb{C}^3$ into the direct sum of irreducible G-modules. $\square$

[3)] As there are no G-invariant curves in $\mathbb{P}^3 \cap S_1 \cap S_2$ for two different surfaces S_i of degree ≤ 2 containing common G-orbit of length 30 (cf. Lemma 2.14).

3. Proof of Theorem 1.4

3.1. We retain the notation and results of Section 2. Consider some G-extremal contraction $\psi : Y \longrightarrow Z$. Let us assume for a moment that ψ is birational with exceptional locus E. Recall that E is a union of (generically) ruled surfaces E_i contracted by ψ onto some curves (see Proposition 2.10).

Lemma 3.2. *$E \cap \operatorname{Sing} Y = \emptyset$.*

Proof. Over the general point of $\psi(E_i)$ morphism ψ coincides with the blow-up of a curve (see [5]). Then for any ruling $C \subset E_i$ contracted by ψ we have $K_Y \cdot C = -1$. Hence the surfaces $\phi(E_i) \subset X$ are swept out by the lines $\phi(C)$.

Note further that C corresponds to a contractible extremal face of $\overline{NE}(Y)$ (cf. the proof of Lemma 2.12). In particular, one may assume that C generates a K_Y-negative extremal ray, which shows that C is Cartier on E_i because all scheme fibers of $\psi\big|_{E_i}$ are smooth (lines) and C varies in a flat family.

Recall that all divisors E_i are Cartier (cf. Lemmas 2.8, 2.11 and [15, Lemma 5.1]). Now, if $E_i \cap \operatorname{Sing} Y \neq \emptyset$, then $\phi(C)$ is a singular curve for some C as above, which is impossible. Hence $E_i \cap \operatorname{Sing} Y = \emptyset$. But then $E \cap \operatorname{Sing} Y = \emptyset$ as well because $E_j \cap \operatorname{Sing} Y \subset E_i$ for all surfaces E_j from the corresponding extremal face. □

Remark 3.3. We have $h^{1,2} = 0$ for a resolution of Y according to *Remark* in [1]. Then it follows from [5] and Lemma 3.2 that $\psi(E_i) = \mathbb{P}^1$ for all i.

Lemma 3.4. *We have $K_Y = \psi^* K_Z + E$ (hence Z is Gorenstein), $K_Y \cdot C = -1$ for any ruling $C \subset E_i$ contracted by ψ, and Z is smooth near $\psi(E)$.*

Proof. One obtains the first two identities by exactly the same argument as in the proof of Lemma 3.2. Further, since the linear system $|-K_Y|$ is basepoint-free (with $K_Y = \phi^* K_X$), generic surface $S \in |-K_Y|$ passing through a given point on Y is smooth. Then, for $S \cdot C = 1$, we find that the surface $\psi(S) \in |-K_Z|$ is smooth as well, so that Z is smooth near $\psi(E)$. □

Now let ψ be the result of running a G-MMP on Y.

Lemma 3.5. *In the above setting, ψ is a birational contraction that maps its exceptional loci onto 1-dimensional centers, so that the corresponding 3-folds are smooth near these centers. In particular, all these 3-folds are $\mathbb{Q}$-factorial Gorenstein and terminal, with nef and big $-K$, and ψ is composed of blow-downs onto smooth rational curves.*

Proof. It follows from Lemmas 2.8, 2.11, 3.2, 3.4 and [23, Corollary 4.9] that each step of ψ produces a $\mathbb{Q}$-factorial Gorenstein terminal 3-fold, with a G-action and nef and big $-K$, unless all exceptional $E_i = \mathbb{P}^2$ on this step. One can easily see the proper transform of such E_i on X will be a plane. Moreover, arguing as at the end of the proof of Proposition 2.10 we find that this plane will be $\langle h \rangle$-invariant, which contradicts Lemma 2.13.

Further, arguing as in the proof of Corollary 3.9 below one computes that whenever $E_i =$ quadric or $\mathbb{P}^2$, contracted to a point in both cases, its proper transform on Y (hence on X as well) will also have degree ≤ 2 w. r. t. $-K$. This leads to contradiction as earlier.

Thus on each step ψ can contract E_i to curves only. Applying the same arguments as in the proof of Proposition 2.10 to each step of ψ gives the claim (the last assertion of lemma follows from [5]). □

Let, as above, E be the ψ-exceptional locus. Note that Y contains the G-orbit of 20 curves C_j contracted by ϕ (see Lemma 2.9). In particular, G induces a non-trivial action on the set of these C_j, which leads to the next

Lemma 3.6. *E can not consist of only one (connected) surface.*

Proof. Indeed, otherwise we have $(E = E_i) \cap C_j \neq \emptyset$ for all j, which yields a faithful G-action on the base of the ruled surface E. Hence we get $G \subset \mathrm{PGL}(2, \mathbb{C})$. On the other hand, we have $G \not\subset A_5, S_4$ (see Lemma 2.14), a contradiction. □

Proposition 3.7. *$E \neq \emptyset$ unless Y is rational.*

Proof. Let $E = \emptyset$. Then we get $\mathrm{rk}\,\mathrm{Pic}^G Y = 2$ and $\overline{NE}(Y)$ is generated by (G-orbits of) the classes of C_j and an extremal ray corresponding to some G-Mori fibration $\varphi : Y \longrightarrow S$ ($\dim S > 0$).

Lemma 3.8. *Let $\dim S = 1$. Then Y is minimal over S unless it is rational.*

Proof. Suppose there is a surface Ξ which is exceptional for some (relative) K_Y-negative extremal contraction on Y/S. Then Ξ necessarily contains one of C_j. Indeed, otherwise Ξ intersects all curves on Y non-negatively by the structure of $\overline{NE}(Y)$, which is impossible. In particular, we find that Ξ must be a minimal ruled surface (same argument as in the proof of Lemma 2.12), with the negative section equal some C_j.

We may assume $K^2_{Y_\eta} \leq 4$ for generic fiber Y_η of φ – otherwise Y is rational (see [9], [18]). Moreover, we have $K^2_{Y_\eta} \neq 1$, since otherwise the group $G \subseteq \mathrm{Aut}(Y_\eta)$ must act faithfully on elliptic curves from $|-K_\eta|$, which is impossible (cf. Lemma 2.14). One also has $K^2_{Y_\eta} \neq 2$ because the order of the group of automorphisms of del Pezzo surfaces of degree 2 is not divisible by 5 (see e. g. [7, Table 8.9]).

Further, if $K^2_{Y_\eta} = 4$, then contracting Ξ we arrive at a del Pezzo fibration of degree 5, so that Y is rational.

Now, if $K^2_{Y_\eta} = 3$, then all smooth fibers of φ are isomorphic and have $\mathrm{Aut}\, Y_\eta = S_5$ (see [7, Table 9.6]). Away from the singular fibers φ defines a locally trivial (in analytic topology) fibration on smooth cubic surfaces Y_η. Two charts, $Y_\eta \times S'$ and $Y_\eta \times S''$, say (for some analytic subsets $S', S'' \subseteq S$), are glued together via an automorphism $t \in \mathrm{Aut}\, Y_\eta$, which preserves the elements in the G-orbit of Ξ and satisfies $tGt^{-1} = G$. Since G is not a normal subgroup in S_5, one gets $t \in G$, and the letter is impossible, once $t \neq 1$, by the way G acts on Ξ (a. k. a. on C_j). Thus $t = 1$ and φ induces a locally trivial fibration in the Zariski topology, so that Y is rational, and the proof is complete. □

Note further that the subgroup $\langle h \rangle \subset G$ must act faithfully on $\mathrm{Pic}\, Y$. Indeed, otherwise $Q_i \sim h^a(Q_i)$ for all a, i, which implies that Q_i contains the orbit $\langle h \rangle \cdot o$, a contradiction. In particular, if $\dim S = 1$, then from Lemma 3.8 we deduce that either $\mathrm{Pic}\, Y = \mathbb{Z}^2$ (this contradicts Lemma 2.11), or φ contains a fiber with ≥ 5 irreducible components (interchanged by $\langle h \rangle$). In the latter case, we get $K^2_{Y_\eta} \geq 5$ for generic fiber Y_η of φ, and rationality of Y follows from [9], [18].

Finally, one excludes the case when φ is a G-conic bundle exactly as in the proof of Lemma 3.12 below, and Proposition 3.7 is completely proved. □

Here is a refinement of Lemma 3.6 and Proposition 3.7:

Corollary 3.9. *E is a disjoint union of G-orbits (length ≥ 2), corresponding to extremal faces of $\overline{NE}(Y)$, unless Y is rational.*

Proof. Let $E, \tilde{E}$ be two ψ-exceptional orbits in question. Choose some connected components $E_j \subset E, \tilde{E}_j \subset \tilde{E}$ and suppose they intersect. One may assume both $E_j, \tilde{E}_j$ to be ruled surfaces that can be contracted by the blow-downs, one for each surface (cf. Lemma 3.5 and the proof of Lemma 3.2).

Let $\psi_j : Y \longrightarrow Y_j$ be the contraction of E_j. Then, given that $E_j \cap \tilde{E}_j \neq \emptyset$, there is a ψ-exceptional curve $C \subset \tilde{E}_j$ such that $E_j \cdot C \geq 0$. On the other hand, we

have $K_Y = \psi_j^* K_{Y_j} + E_j$ and $K_{Y_j} \cdot \psi_j(C) = -1$ (for ψ blows down $\psi_j(\tilde{E}_j)$), which gives either $K_Y \cdot C = -1$ or $K_Y \cdot C = 0$ (recall that $-K_Y$ is nef). The latter case is an absurd by construction of ψ. In the former case, we get $E_j \cdot C = 0$ and so $\psi_*(E_j \cap \tilde{E}_j) = \psi_* C = 0$, which is impossible for the ruled surfaces $E_j \neq \tilde{E}_j$, since then $0 = E_i \cdot C = (C^2) < 0$ on E_i, a contradiction. □

3.10. We will assume from now on that $E \neq \emptyset$ is as in Corollary 3.9. It follows from Lemma 3.5 that Z is $\mathbb{Q}$-factorial Gorenstein and terminal. Note also that $-K_Z$ is nef and big by [23, Corollary 4.9].

Lemma 3.11. *We have $\phi_*^{-1} Q_j \not\subset E$ for some j.*

Proof. Note that $\psi_* K_Y = K_Z$ because Z has rational singularities. This gives the claim as $-K_Y = \phi_*^{-1} Q_1 + \phi_*^{-1} Q_2$. □

Let us treat the case when Z admits a G-Mori fibration.

Lemma 3.12. *Z is not a G-conic bundle.*

Proof. Suppose we are given a G-conic bundle structure on Z with generic fiber $C = \mathbb{P}^1$. Then if $\phi_*^{-1} Q_1 \not\subset E$, say (see Lemma 3.11), it follows from the definition of Q_i and G in **2.1** that the G-orbit of Q_1 (hence also of $\phi_*^{-1} Q_1$) has length ≥ 10 (cf. the proof of Lemma 2.11). This yields a faithful G-action on C which in turn contradicts Lemma 2.14. □

Lemma 3.13. *Z is not a G-del Pezzo fibration unless Z is rational.*

Proof. Argue exactly as in the del Pezzo case from the proof of Proposition 3.7. □

3.14. We will assume from now on that Z is a $G\mathbb{Q}$-Fano (cf. Lemmas 3.12 and 3.13). Note that any two components of exceptional locus of ψ can intersect only along the fibers. Then it follows from Remark 3.3, Lemma 3.4 and Corollary 3.9 that either

$$-K_Z^3 = 4 + 2k(-K_Z \cdot \mathbb{P}^1 + 1) \tag{3.15}$$

for some even $k \leq 10$ or

$$-K_Z^3 = 24 - 20 K_Z \cdot \mathbb{P}^1 - 10k' \tag{3.16}$$

for some $k' \leq -K_Z \cdot \mathbb{P}^1$ (recall that $\operatorname{rk} \operatorname{Pic} Y = 11$ by Lemma 2.11 and the subgroup $\langle h \rangle \subset G$ acts faithfully on $\operatorname{Pic} Y$).

Lemma 3.17. *The linear system $|-K_Z|$ is basepoint-free.*

Proof. Assume the contrary. Then it follows from [12] that Z is a G-equivariant double cover of the cone over a ruled surface (note that $-K_Z^3 \geq 12$ is divisible by 4). This easily gives $G \subset \mathrm{PGL}(2, \mathbb{C})$ and contradiction with Lemma 2.14. □

Lemma 3.18. *The morphism defined by $|-K_Z|$ is an embedding.*

Proof. Assume the contrary. Then it follows from [3, Theorem 1.5] that Z is a G-equivariant double cover of either a rational scroll or the cone over a ruled surface. In both cases, arguing similarly as in the proof of Lemma 3.12, one gets contradiction. □

Lemmas 3.17 and 3.18 allow one to identify Z with its anticanonical model $Z_{2g-2} \subset \mathbb{P}^{g+1}$ (here $g := -K_X^3/2 + 1$ is the genus of Z).

Lemma 3.19. *Z is singular unless it is rational.*

Proof. Suppose that Z is smooth. Then rationality of Z follows from the fact that $h^{1,2}(Z) = 0$ (see Remark 3.3) and [11, §§12.2 – 12.6]. □

According to Lemmas 3.19, 2.9 and 3.5 we may reduce to the case when $|\mathrm{Sing}\, Z| = |\mathrm{Sing}\, Y| = 10$, with the locus $\mathrm{Sing}\, Z$ being some G-orbit.

Proposition 3.20. *$g \leq 9$.*

Proof. Let $g > 9$. Note that the linear span of any G-orbit in $\mathrm{Sing}\, Z$ has dimension ≤ 9. Hence we can consider a G-invariant hyperplane section $S \in |-K_Z|$ (satisfying $S \cap \mathrm{Sing}\, Z \neq \emptyset$).

Further, since $G \not\subset \mathrm{GL}(3, \mathbb{C})$, the group G acts on Z without smooth fixed points. On the other hand, since Z is G-isomorphic to X near $\mathrm{Sing}\, Z$ by construction, we obtain that G does not have fixed points on Z at all.

Lemma 3.21. *There are no G-invariant smooth rational curves on Z.*

Proof. Indeed, otherwise the action $G \circlearrowleft \mathbb{P}^1 \subset Z$ is cyclic, which gives a G-fixed point $\in \mathbb{P}^1$, a contradiction. □

Lemma 3.22. *The pair (Z, S) is plt.*

Proof. Lemma 3.21 and the proof of [22, Lemma 4.6] show that the pair (Z, S) is log canonical. Moreover, if (Z, S) is not plt, the same argument as in [22] reduces

the claim to the case when S is a ruled surface over an elliptic curve, say B. On the other hand, since $|S \cap \operatorname{Sing} Z| = 10$, we get either $G \subset \mathrm{PGL}(2, \mathbb{C})$ or a faithful G-action on B, a contradiction. $\square$

It follows from Lemma 3.22 and [25, Corollary 3.8] that S is either normal or reducible. But in the latter case, $-K_Z \sim$ [disconnected surface], which is impossible.

Thus the surface S is normal with at most canonical singularities. Let us identify S with its (G-equivariant) minimal resolution. In particular, we may assume that S contains a G-invariant collection of disjoint (-2)-curves $C_i, 1 \leq i \leq 10$.

From $G \subseteq \operatorname{Aut} S$ one obtains a G-action on the space $H^{2,0}(S) = \mathbb{C}[\omega_S]$. In particular, the subgroup $\langle \tau^2 \rangle \subset G$ preserves the 2-form ω_S, which implies that the quotient $S_\tau := S/\langle \tau^2 \rangle$ has at worst canonical singularities. Note also that $\tau^2(C_i) = C_i$ and $h(C_i) \neq C_i$ for all i.

Let $\tilde{C}_i$ be the image of C_i on S_τ.

Lemma 3.23. $|\tilde{C}_i \cap \operatorname{Sing} S_\tau| = 2$ *for all* i.

Proof. This follows from the fact that $(\tilde{C}_i^2) = -1$ by the projection formula. $\square$

Let S'_τ be the minimal resolution of S_τ. From Lemma 3.23 we obtain that S'_τ contains ≥ 20 disjoint (-2)-curves. This contradicts $h^{1,1}(S'_\tau) = 20$ and finishes the proof of Proposition 3.20. $\square$

According to Proposition 3.20 and (3.15), (3.16) we may assume that $-K_Z^3 \in \{12, 16\}$.[4)]

Remark 3.24. Actually, since $Z = Z_{16} \subset \mathbb{P}^{10}$ and the projective G-action is induced from the linear one on $\mathbb{C}^{11} = H^0(Z, -K_Z)$, one gets a pencil on Z consisting of G-invariant hyperplane sections. In particular, there is such S intersecting $\operatorname{Sing} Z$, so that the arguments in the proof of Proposition 3.20 apply and exclude the case $-K_Z^3 = 16$.

Proposition 3.25. $\operatorname{rk} \operatorname{Pic} Z \neq 2$.

Proof. Suppose that $\operatorname{rk} \operatorname{Pic} Z = 2$ and consider a 1-parameter family $s : \mathcal{Z} \longrightarrow \Delta$ over a small disk $\Delta \subset \mathbb{C}$ of smooth Fano 3-folds $Z_t, t \neq 0$, deforming to $Z_0 = Z$ (see Lemma 3.5 and [20]). Since $H^i(Z_t, nK_{Z_t}) = 0$ for all $n \leq 0, i > 1$ and t, we deduce that the sheaf $s_*(-K_{\mathcal{Z}})$ is locally free.

[4)] Note that the case $k = 10$ yields $\operatorname{rk} \operatorname{Pic} Z = 1$ and can be excluded exactly as in the proof of Proposition 3.27 below.

Similarly to Y, the cone $\overline{NE}(Z)$ is polyhedral, with contractible extremal rays (cf. the proof of Lemma 2.12). Let H be a nef divisor on Z that determines one of these contractions. Then [13] and [17, Proposition 1.4.13] imply that H varies in the family H_t of nef divisors on Z_t. It follows from the condition $\operatorname{rk}\operatorname{Pic}^G Z = 1$ that both of the extremal contractions on each Z_t must be either birational or Mori fibrations. Now [11, §12.3] (cf. Remark 3.24) shows that Z can only be a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$.

Lemma 3.26. *Z is smooth.*

Proof. Let x_i (resp. y_i) be coordinates on the first (resp. second) $\mathbb{P}^2$-factor of $\mathbb{P}^2 \times \mathbb{P}^2$. Let also $f(x, y) = 0$ be the equation of Z (so that it defines a conic in $\mathbb{P}^2$ whenever $x := [x_0 : x_1 : x_2]$ or y is fixed).

Note that projections to the $\mathbb{P}^2$-factors induce conic bundle structures on Z. These are interchanged by G (because of $\operatorname{rk}\operatorname{Pic}^G Z = 1$) and are $\langle h, \tau^2\rangle$-invariant.

One may assume that $\operatorname{Sing} Z$ belongs to the affine chart $x_0 = y_0 = 1$ on $\mathbb{P}^2 \times \mathbb{P}^2$. Then, after a coordinate change, we obtain that $f(x, y) = x_1x_2y_1y_2+x_1x_2+y_1y_2+1$ in this chart, for h acting diagonally on x_i and y_i.

Now, differentiating $f(x, y)$ by x_1, x_2 we get $x_i = -y_1y_2$, and similarly $y_i = -x_1x_2$. This gives $x_1 = x_2, y_1 = y_2 \in \{-1, -w\}$, which contradicts $f(x, y) = 0$. □

Lemma 3.26 contradicts $|\operatorname{Sing} Z| = 10$ and Proposition 3.25 follows. □

Proposition 3.27. $\operatorname{rk}\operatorname{Pic} Z \neq 1$.

Proof. Let $\operatorname{rk}\operatorname{Pic} Z = 1$. Then we have $Z_t \subset \mathbb{P}^8$ (in the notation from the proof of Proposition 3.25) are Fano 3-folds of the principal series.

Note that there is a G-invariant surface $S \in |-K_Z|$, since $\mathbb{P}^8 = \mathbb{P}(\mathbb{C}^9) \supset Z$, similarly as in Remark 3.24.

Lemma 3.28. *The pair (Z, S) is plt.*

Proof. As in the proof of Lemma 3.22, it suffices to exclude the case when (the normalization of) the surface S is ruled, over a base curve B of genus ≤ 1.

Note that any line L passing through two points from $\operatorname{Sing} Z$ is contained in Z (as Z is an intersection of quadrics). In particular, we have $S \cdot L > 0$ for > 10 of such L, which yields either $G \subset \operatorname{PGL}(2, \mathbb{C})$ or a faithful G-action on B, a contradiction. □

It follows from Lemma 3.28 that S is normal and connected. Further, we have $k \leq 2$ and $-K_Z \cdot \mathbb{P}^1 \leq 2$ in (3.15), which means (cf. Lemma 3.21) that the

exceptional locus of $\psi : Y \longrightarrow Z$ consists of two disjoint surfaces, say E_1, E_2, so that $L_i := \psi(E_i)$ are two lines on Z. In particular, there is a G-invariant subspace $\mathbb{P}^3 \subset \mathbb{P}^8$, with $Z \cap \mathbb{P}^3 = L_1 \cup L_2$, such that X is obtained from Z via the linear projection from $\mathbb{P}^3$ (recall that both X and Z are anticanonically embedded).

We may assume that $Z \cap \mathbb{P}^3 \subset S$ (otherwise there is a pencil as in Remark 3.24). Hence S contains the (-2)-curve L_1 (we have identified S with its minimal resolution). Note that L_1 is preserved by the group $\langle h \rangle$.

Consider the quotient $S_h := S/\langle h \rangle$. Then the image of L_1 on S_h has self-intersection $= -2/5$ by projection formula. On the other hand, this self-intersection $\in \mathbb{Z}[0.5]$ (for S_h has at most canonical singularities due to $h^*(\omega_{S_h}) = \omega_{S_h}$), a contradiction.

Proposition 3.27 is completely proved. □

It follows from Propositions 3.25, 3.27, Remark 3.24, (3.15), (3.16) and [20], [13], [11, §§12.4 – 12.6] that Z is a deformation of either $\mathbb{P}^1 \times$ [del Pezzo surface of degree 2] or of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, ramified along a divisor of tridegree $(2, 2, 2)$. In both cases, Z is hyperelliptic (cf. the beginning of the proof of Proposition 3.25), which contradicts Lemma 3.18.

The proof of Theorem 1.4 is finished.

4. Concluding discussion

4.1. Equations (1.2) and the results of [6] show that any S_6-invariant quartic X_t is not $\mathbb{Q}$-factorial. In turn, as we saw in Section 2, it is indispensable to compute the group $\operatorname{Cl} X_t = H_4(X_t, \mathbb{Z})$ (e. g. for the arguments of Section 3 to carry on).

This amazing interrelation between topology and (birational) geometry of X_t provides one with a hint for studying the birational type of X_t by "topological" means. In this regard, let us give a sketch of an argument, showing that X_t is unirational for generic $t \in \mathbb{R}$, hence for (again generic) $t \in \mathbb{C}$ (cf. [8, Proposition 2.3]).

Namely, differentiating (1.2) one interprets this system of equations as the graph of a Morse function $F : \mathbb{RP}^4 \longrightarrow \mathbb{R}$, so that $X_t^{\mathbb{R}} = F^{-1}(t)$ are smooth level sets for $t \notin \{\infty, 0, 10/7, 2, 4, 6\}$, while the rest of $t \notin \{0, 4\}$ correspond to critical level sets of (maximal) index 3 (here $X_t^{\mathbb{R}}$ denotes the real locus of X_t).

We may replace $\mathbb{RP}^4$ by its double cover S^4. Then F lifts to a Morse function on S^4 and thus all smooth $X_t^{\mathbb{R}}$ are homotopy $\mathbb{RP}^3$. In fact general $X_t^{\mathbb{R}}$ is *diffeomorphic* to $\mathbb{RP}^3$ (note that this $X_t^{\mathbb{R}}$ is smooth and connected).

Further, $X_t^{\mathbb{R}}$ is contained in an affine space $\mathbb{R}^N$, some N, because $\sum x_i^4 \neq 0$ over $\mathbb{R}$. Then the function $F_p := \mathrm{dist}(\cdot, p)$ defines a Morse function on $X_t^{\mathbb{R}}$ for very general points $p \in \mathbb{R}^N$. (Here $\mathrm{dist}(x, y) := \|x - y\|^2$ is the standard Euclidean distance.)

The layers of F_p yield a vector field on $X_t^{\mathbb{R}}$, which is non-degenerate and normal to these layers outside two points, where this field vanishes. We thus obtain a (Hopf) fibration on $X_t^{\mathbb{R}}$ with a section $F_p^{-1}(o) \setminus \{2 \text{ points } o_1, o_2\} = \mathbb{RP}^2$ such that $F_p^{-1}(o) \subset X_t^{\mathbb{R}}$ as an algebraic subset. It remains to apply a diffeomorphism over $F_p^{-1}(o) \setminus \{o_1, o_2\}$ which makes $X_t^{\mathbb{R}} \setminus \{F_p^{-1}(o_1), F_p^{-1}(o_2)\} = \mathbb{RP}^1 \times F_p^{-1}(o) \setminus \{o_1, o_2\}$ as algebraic varieties.

The upshot of the above discussion is that $X_t^{\mathbb{R}}$ (hence X_t) admits *many cancellations* in the sense of [2]. This implies that X_t is unirational.

4.2. We conclude with the following questions:

- What is the Fano 3-fold which the quartic X_6 is G-birationally isomorphic to (cf. Section 3)?
- Are there non-trivial G-birational modifications of X_6 for other subgroups $G \subset S_6$?
- Is X_t unirational over a number field field? [5)]
- Does the set of $\mathbb{Q}$-points on X_t satisfy the potential density property?
- Does X_t carry a pencil of (birationally) Abelian surfaces? [6)]

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[5)] Note that all rational quartics are $\mathbb{Q}$-rational.

[6)] Again this holds for rational X_t.

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