

# ON ENDOMORPHISMS OF HYPERSURFACES

ILYA KARZHEMANOV

ABSTRACT. For any prime  $p \geq 5$ , we show that generic hypersurface  $X_p \subset \mathbb{P}^p$  defined over  $\mathbb{Q}$  admits a non-trivial rational dominant self-map of degree  $> 1$ , defined over  $\overline{\mathbb{Q}}$ . A simple arithmetic application of this fact is also given.

## 1. INTRODUCTION

**1.1.** Let  $X$  be an algebraic variety (smooth, projective, over a field of characteristic 0). Then two groups of symmetries of  $X$ , namely  $\text{Aut}(X)$  (biregular automorphisms) and  $\text{Bir}(X)$  (birational ones), come for free. The question on whether these groups differ is classical and very important. For instance, the property  $\text{Aut}(X) = \text{Bir}(X)$ , signifying *birational (super)rigidity* of  $X$ , is an obstruction for  $X$  to be rational (such  $X$  had been studied in numerous papers including [16], [24], [9], [17] and [18]).

Typically one has  $\text{Aut}(X) = \{\text{id}\}$  however. In this regard, it is natural to consider another (more general) class of symmetries of  $X$ , namely  $\text{End}(X)$ , consisting of *rational dominant* endomorphisms of  $X$ . Then one may ask (what seems to be a folklore) whether  $\text{End}(X) = \text{Bir}(X)$ ? (Note that the latter property is an obstruction for  $X$  to be unirational.) This question gets immediate answer (‘yes’) when  $X$  — with  $\text{Pic}(X) = \mathbb{Z}$  at least — is of general type (as the endomorphisms preserve the spaces  $H^0(X, mK_X)$  for all  $m \in \mathbb{Z}$ ). On the other hand, already in the Calabi-Yau case things are not that straightforward; although still one gets  $\text{End}(X) = \text{Bir}(X)$  when  $X$  is a general K3 surface for example (see [7]).

Presently, we would like to treat rationally connected  $X$ , namely  $X := X_N \subset \mathbb{P}^N$  being a hypersurface of degree  $N$  (see [24], [15], [6] for some other aspects of the geometry of these  $X_N$ ). Recall that according to [10] every  $X_N, N \geq 4$ , is non-rational, having  $\text{Bir}(X_N) = \text{Aut}(X_N)$ .

Our main result is

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*Key words:* hypersurface, rational self-map, incompressibility.

**Theorem 1.2.** *For any prime  $p \geq 5$ , (Zariski) general  $\mathbb{Q}$ -hypersurface  $X_p \subset \mathbb{P}^p$  admits an endomorphism  $f_{X_p}$  of degree  $> 1$ , defined over  $\overline{\mathbb{Q}}$ . More precisely,  $f_{X_p}$  has a field of definition —  $\mathbf{k}$  below, — same for all  $X_p$ .*

Thus the “non-regular” geometry of  $X = X_p$  is pretty much fruitful (compare with results in [2], [4], [23], accounting for *regular* self-maps of  $X$ ). At the same time, the proof of Theorem 1.2 is not effective, and it would be interesting to describe (a part of)  $\text{End}(X)$  explicitly. For example, what is the  $\text{End}(X)$ -action on the universal Chow group of  $X$  (cf. [25], [3])? It is also plausible to get rid of the degree/ground field/dimension assumption in the formulation of our result. Say, can one take *any* Fano manifold  $X$  in place of  $X_p$ , or at least any (composite) integer  $N$  instead of  $p$ ?

**1.3.** In order to prove Theorem 1.2 we first relate  $X_p$  to those hypersurfaces that have lots of endomorphisms. The latter are  $X_d \subset \mathbb{P}^N$ , defined over a given field, that happen to be unirational over this field, provided that  $N \gg d$  is sufficiently large (see **2.1** below for the precise statements).

Next we employ the *degree formula* from [26] (cf. [22], [19], [14]). Recall that this formula relates the connective K-theory classes of two algebraic varieties  $X$  and  $Y$  admitting some rational map  $f : X \dashrightarrow Y$ . However, in order to get something fruitful in this way (e.g. to show that  $f$  with  $\deg f = 0$  does not exist) one has to consider  $X, Y$ , etc. to be defined over an algebraically *non-closed* field  $\mathbf{k}$ , and (roughly speaking) to have no points over  $\mathbf{k}$ . This is the reason for the degree (resp. dimension/ground field) restriction in Theorem 1.2.

*Remark 1.4.* One of the basic obstructions for applying our method to arbitrary degree  $d$  (Fano) hypersurfaces  $X_d \subset \mathbb{P}^N$  is that those usually have points over the field of definition (cf. [5]). There may also be no such nice congruence relation as in Corollary 2.5 below. But still, once we find (sufficiently many?) endomorphisms of  $X_p \subset \mathbb{P}^p$  which preserve some projective subspace  $\Gamma \subset X_p$ , one may hope (by projecting from  $\Gamma$ ) to obtain non-trivial endomorphisms of  $X_d$  for some (all?)  $d \leq p - 1$ .

The principal part of our arguments relies on (a part of) the main result in [8] which asserts the existence of  $X = X_p = Y$  as indicated above. It is then not hard to derive Theorem 1.2 for the given  $X_p$  and the general case follows easily (see Section 3 for details). Again we indicate that the initial  $X_p$  is very special. (It is

defined over a field  $\mathbf{k}$  of cohomological dimension  $\leq 1$  and does not contain points over the extensions of  $\mathbf{k}$  whose degrees are coprime with  $p$ .)

**1.5.** The next result was motivated by the paper [1]:

**Corollary 1.6.** *Let  $X_p \subset \mathbb{P}^p$  be as in Theorem 1.2. Then there exists a possibly larger number field  $K \supseteq \mathbb{Q}$  such that Zariski closure of the  $f_{X_p}$ -orbit of the set  $X_p(K)$  of  $K$ -rational points on  $X_p$  has dimension  $\geq 2$ .*

One may consider Corollary 1.6 as a generalization of [13, Theorem 1.4]. Yet, unfortunately, our conclusion is weaker and it would be interesting to establish potential density of the set  $X_p(\mathbb{Q})$  in  $X_p$  (e. g. by refining the arguments of Section 4 below).

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## 2. PRELIMINARIES

**2.1.** Fix some integers  $d$  and  $r \geq 1$ . Recall that every smooth hypersurface  $X_d \subset \mathbb{P}^N$  of degree  $d$  contains a projective subspace  $\Lambda \simeq \mathbb{P}^r$  when  $N \gg 1$  (see [12], [20]). More precisely,  $X_d$  corresponds to (generic) point in the incidence subvariety

$$Z := \{(X_d, \Lambda) \mid \Lambda \subset X_d\} \subset \mathbb{P}^{\binom{N+d}{d}} \times \text{Gr}(r+1, N+1),$$

with *dominant* projections  $Z \longrightarrow \mathbb{P}, \text{Gr}$ .

Further, if  $\Lambda$  is given over an arbitrary field  $\mathbf{k}_0 \subset \mathbb{C}$  by equations  $l_{N-r+1} = \dots = l_N = 0$  for some linear forms  $l_i$ , then  $X_d$  can be chosen to be defined over  $\mathbf{k} := \mathbf{k}_0(\sqrt{-1})$ . Indeed, any  $X_d$  passing through  $\Lambda$  has the defining equation  $\sum_{i=N-r+1}^N \phi_i l_i = 0$ , for some (varying) forms  $\phi_i$  of degree  $d-1$ . Hence, since the set of  $\mathbf{k}$ -points is obviously dense in  $\mathbb{P}^{\binom{N+d-1}{d-1}}$  (w. r. t. the complex analytic

topology), one can approximate  $\phi_i$  by such  $\mathbf{k}$ -forms of degree  $d-1$  that  $X_d$  remains smooth.

Thus, since one can always find generic  $\Lambda \in \text{Gr}(r, N)$  to be defined over  $\mathbf{k}$ , the above discussion provides a smooth hypersurface  $X_d$  and a projective subspace  $\Lambda \subset X_d$ , both over  $\mathbf{k}$ , for any given  $d, r$  and sufficiently large  $N = N(d, r)$ . From [12, Corollary 3.7] we obtain

**Theorem 2.2** (Harris, Mazur, Pandharipande).  *$X_d$  is  $\mathbf{k}$ -unirational.*

**2.3.** Let  $X, Y$  be smooth projective geometrically irreducible  $\mathbf{k}_0$ -varieties of dimension  $d = \dim X = \dim Y$ . Assume that there is a rational  $\mathbf{k}_0$ -map  $f : X \dashrightarrow Y$ . The degree  $\deg f$  equals 0 if  $f$  is non-dominant; otherwise  $\deg f := [\mathbf{k}_0(X) : f^*\mathbf{k}_0(Y)]$ .

We recall the next result from [26]:

**Theorem 2.4** (Zainoulline).  $\chi(\mathcal{O}_X) \cdot \tau_{d-1} \equiv \deg f \cdot \chi(\mathcal{O}_Y) \cdot \tau_{d-1} \pmod{n_Y}$ , where  $\chi(\cdot)$  is the Euler characteristic,

$$\tau_{d-1} := \prod_{p \text{ prime}} p^{\lfloor \frac{d-1}{p-1} \rfloor}$$

is the  $(d-1)$ -st Todd number, and  $n_Y$  is the g. c. d. of degrees of all closed points on  $Y$ .

An immediate consequence of Theorem 2.4 is

**Corollary 2.5** (cf. [26, Example 6.4]). *In the previous notation, if  $n_X = p$ ,  $X = Y = X_p \subset \mathbb{P}^p$  (i. e.  $d = p-1$  and  $N = p$ ) for some prime  $p \geq 3$ , and (more generally)  $f$  is a  $\mathbf{k}$ -map, then  $\deg f > 0$ . (Such  $X$  is called incompressible over  $\mathbf{k}$ .)<sup>1)</sup>*

*Proof.* Regard  $X$  as a hypersurface over  $\mathbf{k}$ . Then, since  $[\mathbf{k} : \mathbf{k}_0] \leq 2$ , we get  $n_X = p$  again. The claim now follows from Theorem 2.4 (with  $\mathbf{k}_0$  replaced by  $\mathbf{k}$ ) and the fact that  $\chi(\mathcal{O}_X) = 1 = (\tau_{p-2}, p)$ .  $\square$

Existence of  $X = X_p$  as in Corollary 2.5 is guaranteed by the following result (see [8, Theorem 8]):

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<sup>1)</sup> Existence of similar hypersurfaces  $X$  of general type, as suggested by [26, Example 6.4], is not clear in the current setting because the condition  $n_X = p$  need not be satisfied (compare with Proposition 2.6 below).

**Proposition 2.6** (Colliot-Thélène). *For every prime  $p \geq 5$ , there is a smooth (over  $\overline{\mathbb{Q}}$ ) hypersurface  $X \subset \mathbb{P}^p$ , given (over  $\mathbb{Q}$ ) by the equation*

$$x_1^p + lx_2^p + \dots + l^{p-1}x_p^p - \alpha x_0^p = 0$$

for some integers  $l, \alpha$ , so that  $n_X = p$ .

*Remark 2.7.* Let  $t$  be a transcendental parameter,  $X_d \subset \mathbb{P}^N$  some  $\mathbb{Q}(t)$ -hypersurface, and  $d = N = p$ . One may assume (e.g. by identifying  $X_d$  with an appropriate pencil of degree  $d = p$  hypersurfaces) that this  $X_d$  specializes via  $t \mapsto 0$  to  $X$  from Proposition 2.6. It follows then that generic such  $X_d$  also satisfies  $n_{X_d} = p$ . (Here “generic hypersurface” means a point in a Zariski open subset of the  $\mathbb{Q}(t)$ -variety  $\mathbb{P}^{\binom{N+d}{d}}$  parameterizing all hypersurfaces of degree  $d$  in  $\mathbb{P}^N$  defined over the field  $\mathbb{Q}(t)$ .) Indeed, by definition of  $n_Y$  (see Theorem 2.4) and specialization  $t \mapsto 0$  one finds that  $n_{X_d}$  divides  $p = n_X$ , and once  $n_{X_d} = 1$  we get (by the same reasoning) that  $n_X = 1$  as well, a contradiction.

### 3. PROOF OF THEOREM 1.2

We keep on with notation of Section 2.

**3.1.** Let  $X$  be as in Proposition 2.6. Consider a cone  $\widehat{X} \subset \mathbb{P}^N$  over  $X$  of sufficiently large dimension and a family  $\mathcal{X}$  of degree  $p$  hypersurfaces ( $\subset \mathbb{P}^N$ ) over  $\mathbb{Q}$  that smooths out the singularities of  $\widehat{X}$ . We may regard (the general fiber of)  $\mathcal{X}$  as a smooth  $\mathbf{k}_0$ -hypersurface of degree  $p$  in  $\mathbb{P}^N$  for some purely transcendental field  $\mathbf{k}_0 \supset \mathbb{Q}$ .

Next, using the (dominant) projections  $Z \rightarrow \mathbb{P}, \text{Gr}$  one can see by the same argument as in **2.1** that the set of all  $\mathbf{k}$ -points  $(X_p, \Lambda) \in Z$  is dense on  $Z$  in the complex analytic topology, and so  $\mathcal{X}$  can be approximated by  $\mathbf{k}$ -hypersurfaces  $X_p$  (for  $\mathcal{X}, X_p$  treated as points on  $\mathbb{P}$ , with  $\mathbf{k} \subset \mathbb{C}$ ). The hypersurfaces  $\mathcal{X}$  and  $X_p$  can actually be put on an affine line  $\mathbb{A}_{\mathbf{k}}^1 \subset \mathbb{P}$  in such a way that the preimage  $\widetilde{\mathbb{A}}_{\mathbf{k}}^1 \subset Z$  of  $\mathbb{A}_{\mathbf{k}}^1$  has all fibers = some projective spaces and generic fiber isomorphic to  $\Lambda$  (apply the reasoning from **2.1** to this family over  $\mathbb{A}_{\mathbf{k}}^1$  considered as a degree  $p$  hypersurface over  $\mathbf{k}_0(t)$ ).

**Lemma 3.2.** *In the previous setting, the hypersurface  $\mathcal{X}$  contains a projective subspace  $\subset \mathbb{P}^N$ , isomorphic to  $\Lambda$  and defined over  $\mathbf{k}$ .*

*Proof.* It suffices to show that there exists a  $\mathbf{k}$ -point on  $Z$  whose projection to  $\mathbb{P}$  coincides with  $\mathcal{X}$ . Indeed, the preceding natural projection  $\widetilde{\mathbb{A}}_{\mathbf{k}}^1 \rightarrow \mathbb{A}_{\mathbf{k}}^1$  is  $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ -equivariant, hence its fiber over  $\mathcal{X}$  is a projective  $\mathbf{k}$ -space.  $\square$

Without loss of generality we will assume that  $\mathbf{k}_0 = \mathbb{Q}(t)$  in what follows.

**Lemma 3.3.** *There exists a  $\mathbf{k}$ -endomorphism  $f : \mathcal{X} \dashrightarrow \mathcal{X}$  such that  $\deg f > 1$ .*

*Proof.* It follows from Lemma 3.2 and Theorem 2.2 (applied to  $X_d := \mathcal{X}$ ) that  $\mathcal{X}$  is  $\mathbf{k}$ -unirational. This yields two rational dominant  $\mathbf{k}$ -maps  $\phi : \mathbb{P}^{N-1} \dashrightarrow \mathcal{X}$  and  $\psi : \mathcal{X} \dashrightarrow \mathbb{P}^{N-1}$ . We may assume one of  $\phi, \psi$  to have degree  $> 1$ . Then it remains to take  $f := \phi \circ \psi$ .  $\square$

**3.4.** Let  $f$  be as in Lemma 3.3. Note that  $f$  is induced by a rational self-map of  $\mathbb{P}^N \supset \mathcal{X}, \widehat{X}$ . Indeed, since  $p \geq 5$  and thus  $\text{Pic}(\mathcal{X}) = \mathbb{Z}$  (Lefschetz), the map  $f$  is given by such a linear system on  $\mathcal{X}$  that is obtained via restriction of a linear system from  $\mathbb{P}^N$ .

Put  $f_0$  to be the specialization of  $f$  at the fiber  $\widehat{X}$  of the family  $\mathcal{X}$  (recall that  $\mathbf{k}_0 = \mathbb{Q}(t)$ ). More precisely, if  $\mathcal{L}_t$  is a (movable) linear system defining  $f$ , then its specialization  $\mathcal{L}_0$  to  $\widehat{X}$  may acquire some divisorial components in the base locus.<sup>2)</sup> Subtracting all these we arrive at a linear system which we set to define  $f_0$ .

**Lemma 3.5.**  *$f_0$  is not induced by a self-map of  $\text{Sing}(\widehat{X})$  (for an appropriate  $f$ ). In other words, if  $x_0 = \dots = x_p = 0$  are the equations of the singular locus  $\text{Sing}(\widehat{X}) \subset \widehat{X}$ , then  $f_0 \neq \text{id}, 0$  on the subspace  $\mathbb{P}^p \subset \mathbb{P}^N$  complementary to  $\text{Sing}(\widehat{X})$ .*

*Proof.* Let the notation be as in the proof of Lemma 3.3. One can choose a  $\mathbf{k}$ -point  $o \in \mathcal{X}$  such that  $\psi$  (resp.  $\phi$ ) is unramified at  $o$  (resp.  $\psi(o)$ ).

Take  $f := \phi \circ \sigma \circ \psi$  for some  $\sigma \in \text{PGL}(N, \mathbf{k})$  (to be specified further) in such a way that  $f(o) = o$ . Namely, since  $\psi, \phi$  induce isomorphisms of the tangent spaces  $T_o = T_{\phi(\psi(o))}, T_{\psi(o)}$ , and (adjoint of)  $\sigma$  can act transitively on the  $N$ -tuples of  $\mathbf{k}$ -vectors in  $T_{\psi(o)}$ , we arrive at such  $f$  whose Jacobian  $\text{Jac}_o(f)$  is a  $\mathbf{k}$ -matrix with pairwise distinct eigen values and all defined over  $\mathbb{Q}$ . Then, considering  $f_0$  as a rational self-map of  $\mathbb{P}^N$  (cf. the discussion at the beginning of **3.4**), we obtain that all non-zero eigen values of the matrix  $\text{Jac}_{o'}(f_0)$  are pairwise distinct as well. Here  $o' \in \widehat{X}$  is the specialization of  $o$ .<sup>3)</sup>

<sup>2)</sup> The latter stems from the fact that  $f$  may not be defined (in codimension 2) on  $\widehat{X}$ .

<sup>3)</sup> Note that  $o'$  is not contained in the indeterminacy locus of  $f_0$  for the indicated  $o$  and  $\sigma$ . Thus  $f_0$  is defined at  $o'$  and satisfies  $f_0(o') = o'$ .

Now suppose that  $f_0$  is induced by some rational endomorphism of  $\text{Sing}(\widehat{X})$ . Again we regard  $f_0$  as a self-map of  $\mathbb{P}^N$ . Then  $f_0$  is regular at some point  $o' \in \mathbb{P}^N$ ,  $f_0(o') = o'$  and the non-zero eigen values of  $\text{Jac}_{o'}(f_0)$  are all different. On the other hand,  $f_0 = \text{id}$  on the subspace  $\mathbb{P}^p \subset \mathbb{P}^N$  complementary to  $\text{Sing}(\widehat{X})$ , a contradiction.

Same argument shows that  $f_0$  has non-trivial components on  $\mathbb{P}^p$ , i. e.  $f_0 \neq 0$  there, which concludes the proof.  $\square$

Note that generic subspace  $\mathbb{P}^p \subset \mathbb{P}^N$  cuts out a subvariety on  $\widehat{X}$  isomorphic  $X$  (cf. the beginning of **3.1**). Identify  $X$  with such a section  $\mathbb{P}^p \cap \widehat{X}$  so that the restriction  $f_0|_X$  is a well-defined rational map. Let  $f_X : X \dashrightarrow X$  be the composition of  $f_0|_X$  and the linear projection  $\widehat{X} \dashrightarrow X$  from  $\text{Sing}(\widehat{X})$ .

**Lemma 3.6.**  $f_X \neq \text{id}$  and is dominant.

*Proof.* Indeed,  $f_0$  induces a non-identical map on  $X$  by Lemma 3.5, and it remains to apply Corollary 2.5 to the  $\mathbb{Q}(\sqrt{-1})$ -map  $f_X$ .  $\square$

*Remark 3.7.* For the last part of Lemma 3.6, observe that the fact  $\deg f_X \neq 0$  is not immediate from the proof of Lemma 3.5, and hence one needs an additional argument (results of the second half of Section 2 for instance) in order to proceed.

Theorem 1.2 (for the given  $X$ ) now follows from

**Proposition 3.8.**  $\deg f_X > 1$ .

*Proof.* Regard  $\mathcal{X}$  as a (flat) family of hypersurfaces  $\mathcal{X}_t \subset \mathbb{P}^N \times t \subset \mathbb{P}^N \times \mathbb{P}^1$  (with  $\mathcal{X}_0 := \widehat{X}$ ). Let  $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$  be the natural projection (so that  $\pi^{-1}(t) = \mathcal{X}_t$ ). Let also  $\mathcal{L}_t$  be as in the second paragraph of **3.4**.

**Lemma 3.9.** The linear system  $\mathcal{L}_0$  is non-trivial on  $\widehat{X}$  (unless  $\deg f_X > 1$ ).

*Proof.* Suppose the contrary (i. e.  $\mathcal{L}_0 = \{0\}$ ). Then the  $\pi$ -map  $f : \mathcal{X} \dashrightarrow \mathcal{X}$  has indeterminacies along  $\widehat{X}$ . Resolve these by some  $\pi$ -blow-up  $\sigma : \mathcal{Y} \rightarrow \mathcal{X}$ . Let  $\mathcal{Y} \xrightarrow{a} \mathcal{Z} \xrightarrow{b} \mathcal{X}$  be the Stein factorization of the resolved  $f$ . Here  $a, b$  are some  $\pi$ -morphisms, with  $a$  inducing a birational isomorphism between  $\mathcal{Y}$  and  $\mathcal{Z}$ . Moreover, since  $f$  is not defined on  $\widehat{X}$ , the proper transform  $\sigma_*^{-1}\widehat{X}$  of  $\widehat{X}$  on  $\mathcal{Y}$  belongs to the exceptional locus of  $a$ . Composing further with  $b$  yields a rational self-map  $\mu : \widehat{X} \dashrightarrow \widehat{X}$  of degree 0.

Notice that  $\mu$  does not coincide with the projection  $\widehat{X} \dashrightarrow X$  from  $\text{Sing}(\widehat{X})$  because  $\deg b > 1$  (cf. Lemma 3.3) and  $\mathcal{X}_0 = \widehat{X}$  is a non-multiple fiber of  $\pi$  (so that

$\widehat{X}$  is not a branching divisor of  $b$ ). Thus one may assume  $X = \mathbb{P}^p \cap \widehat{X}$  intersects generic fiber of  $\mu$  at  $> 1$  points. Composing  $\mu|_X$  with  $\widehat{X} \dashrightarrow X$  either gives a self-map  $X \dashrightarrow X$  of degree  $> 1$  (which we take for  $f_X$ ), or that  $X$  is not incompressible, in contradiction with Corollary 2.5.  $\square$

Let  $\mathcal{Y}, \sigma, a, \dots$  be as above. It follows from Lemma 3.9 that  $\widehat{X} = \mathcal{X}_0$  and the scheme  $\mathcal{Z}_0 := b^{-1}(0)$  are birationally isomorphic via  $a \circ \sigma^{-1}$  (recall that  $b$  is *finite*). In particular,  $\mathcal{Z}_0$  is not a ramification divisor of  $b$ , which implies that  $\deg b|_{\mathcal{Z}_t} = \deg b|_{\mathcal{Z}_0}$  for all  $t \in \mathbb{C}$  close to 0.

Now, since  $\deg b|_{\mathcal{Z}_t} > 1$  by construction, we deduce that  $\deg f_0 = \deg b|_{\mathcal{Z}_0} > 1$  as well. The latter also gives  $\deg f_X > 1$ . Indeed, otherwise restricting  $\mathcal{L}_0$  to generic  $X = \mathbb{P}^p \cap \widehat{X}$ , we get  $f_X \in \mathbf{Aut}(X)$  according to Lemma 3.6 and [10]. But then, since  $f_X$  is composed of  $f_0|_X$  and projection  $\widehat{X} \dashrightarrow X$  to the base of the cone, we obtain that  $f_0$  must be induced by some projective transformation of  $\mathbb{P}^N \supset \widehat{X}$ . The latter obviously contradicts  $\deg f_0 > 1$  and the proof of Proposition 3.8 is finished.  $\square$

To complete the proof of Theorem 1.2 we simply apply Remark 2.7 and the fact that all the preceding arguments go verbatim with  $\mathbb{Q}$  replaced by  $\mathbb{Q}(t)$ . It remains to set  $t := t_0 \in \mathbb{Q}$  — a given general parameter value — to obtain hypersurfaces as in the statement of Theorem 1.2. Furthermore, in the generic setting one can replace the argument with  $f_X \in \mathbf{Aut}(X)$  at the end of the proof of Proposition 3.8 by that with  $f_X = \text{id}$  (cf. [21, Theorem 5]), thus getting a contradiction with Lemma 3.6.<sup>4)</sup>

#### 4. PROOF OF COROLLARY 1.6

**4.1.** We proceed with applying the constructions of Section 3 to study the arithmetics of hypersurfaces  $X_p \subset \mathbb{P}^p$  (the notation is as earlier).<sup>5)</sup>

Let again  $X$  be as in Proposition 2.6.

**Lemma 4.2.**  *$f_X$  is non-periodic.*

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<sup>4)</sup> This argument may be considered as another way to prove Theorem 1.2.

<sup>5)</sup> Note that one can not specialize the (abundance of)  $\mathbf{k}$ -points on  $\mathcal{X}$  to “many”  $\mathbb{Q}$ -points on the cone  $\widehat{X}$  (as we did with (some of) endomorphisms) because  $n_X = p$  and so all the points on  $\widehat{X}$  we obtain this way are concentrated on  $\text{Sing}(\widehat{X})$ .

*Proof.* Indeed, otherwise we have  $f_X^k = \text{id}$  for some  $k$ , so that both  $f_X, f_X^{k-1}$  are invertible. But this contradicts Proposition 3.8.  $\square$

Fix  $f_X$  as in Lemma 4.2. Then after possibly replacing  $f_X$  by  $f_X^k, k \gg 1$ , we obtain a point  $o \in X(K)$  such that  $f_X(o) = o$  and  $f_X$  is defined at  $o$  (see [11]). We also have  $\det \text{Jac}_o(f_X) \neq 0$  because  $f_X$  is dominant.

**Lemma 4.3.** *There exists a  $\mathbf{k}$ -cube  $\square_o \subset X$  (i. e.  $\square_o$  is given by linear inequalities with coefficients in  $\mathbf{k}$ ) centered at  $o$  and invariant under  $f_X$ .*

*Proof.* Let  $\square \subset X$  be some cube containing  $o$  and defined over  $\mathbb{Q}$ . Then the set  $\bigcup_{k>0} X \setminus f_X^k(\square)$  is not everywhere dense in  $X$ . Indeed, otherwise there would exist a subsequence  $f_X^{k_i}(\square) \rightarrow o$  for  $i \rightarrow \infty$ , which implies that  $o \in X(\mathbb{Q})$  and contradicts  $n_X = p$ . It remains to take  $f_X$ -invariant  $\square_o \subseteq \bigcap_{k>0} f_X^k(\square)$ .  $\square$

It follows from Lemma 4.3 (and the implicit function theorem) that the eigenvalues  $\lambda_i$  of the matrix  $\text{Jac}_o(f_X)$  are algebraic numbers (from  $K$ ), all having norms  $|\lambda_i| = 1, 1 \leq i \leq p-1$ . Furthermore, making if necessary a coordinate change on  $\mathbb{P}^p \supset X$  of the form  $x_j \mapsto \alpha_j x_j, 0 \leq j \leq p$  (i. e. rescaling the metric on  $X$ ), for some  $\alpha_j \in \mathbb{Q}^*$ , we may assume  $\lambda_i$  to be algebraic integers (from  $O_K$ ).

**Lemma 4.4.** *The matrix  $\text{Jac}_o(f_X)$  is semi-simple. Moreover, there is  $j < p-2$  such that  $\lambda_1 = \dots = \lambda_j = \pm 1$ , while  $\lambda_{j+1}, \dots, \lambda_{p-1}$  are multiplicatively independent.*

*Proof.* The first claim follows from the fact that  $f_X$  linearizes on  $\square_o$ . Now, if  $\lambda_i = \pm 1$  (or equivalently  $\lambda_i \in \mathbb{R}$ ) for all  $i$ , then  $f_X^2 = \text{id}$ , a contradiction.

Further, Dirichlet's unit theorem yields  $\lambda_{j+1} \dots \lambda_{p-1} = |\lambda_{j+1} \dots \lambda_{p-1}| = 1, \lambda_j = \lambda_k^{-1}$  as the only relations between  $\lambda_{j+1}, \dots, \lambda_{p-1} \in O_K^*$ . In particular, since  $j < p-1, \lambda_{j+1}, \dots, \lambda_{p-1}$  generate a free subgroup  $\subseteq \mathbb{Z}^{p-j-1} \subset O_K^*$ .

Suppose that  $j = p-2$ . Then  $\lambda_{j+1}, \dots, \lambda_{p-1}$  are all equal to  $\lambda_{j+1}^{\pm 1}$  say. On the other hand, characteristic polynomial of  $\text{Jac}_o(f_X)$  is defined over  $\mathbf{k}$  (by construction of  $\square_o$  in Lemma 4.3), and hence the minimal polynomial of  $\lambda_{j+1}$  divides it. All together, this implies that  $\lambda_i^2 = \pm 1$  for all  $i \geq j+1$ , a contradiction.

Thus we get  $j < p-2$  and the claim follows.  $\square$

The arguments in [1, Section 2] and Lemma 4.4 imply that Zariski closure of the  $f_X$ -orbit of the locus  $X(K)$  has dimension  $\geq p-j-1 > 1$ .

Finally, to complete the proof of Corollary 1.6 one replaces  $\mathbb{Q}$  by  $\mathbb{Q}(t)$ , as at the end of Section 3, and repeats the previous arguments.

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KAVLI IPMU (WPI), THE UNIVERSITY OF TOKYO, 5 - 1 - 5 KASHIWANOHA,  
KASHIWA, 277 - 8583, JAPAN

MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, 9 INSTITUTSKIY PER.,  
DOLGOPRUDNY, MOSCOW REGION, 141701, RUSSIA

*E-mail address:* ILYA.KARZHEMANOV@IPMU.JP