MODULAR OPERADS OF EMBEDDED CURVES

SATOSHI KONDO, CHARLES SIEGEL, AND JESSE WOLFSON

Abstract. For each $k \geq 6$, we construct a modular operad $\overline{\mathcal{E}}^k$ of "k-log-canonically embedded" curves.

1. Introduction

Fix a natural number $k \geq 6$. For natural numbers g and n, we define

$$N^k_{g,n} := (2k-1)(g-1) + nk - 1.^1$$

Definition 1.1. Let S be a scheme and $k \geq 6$. We define a *k-log-canonically embedded* stable marked curve $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ over S, of genus g, to be a stable marked curve $(\mathcal{C}, \{\sigma_i\}_{i=1}^n)$ over S, along with:

(1) a projective embedding $\eta\colon \mathcal{C} \longrightarrow \mathbb{P}_S^{N^k_{g,n}}$ over S by a complete linear system, and
(2) an isomorphism $\varphi\colon \eta^*\mathcal{O}_{\mathbb{P}_S^{N^k_{g,n}}}(1) \xrightarrow{\cong} \omega_{\mathcal{C}/S}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}$.

Isomorphisms of k-log-canonically embedded stable marked curves are defined in the natural manner.

A pair of stable marked curves $(\mathcal{C}_1, \{\sigma_i\}_{i=1}^n)$ and $(\mathcal{C}_2, \{\tau_j\}_{j=1}^m)$ can be glued together to obtain a third such curve $(\mathcal{C}_1 \cup_{\sigma_k \sim \tau_\ell} \mathcal{C}_2, \{\sigma_i, \tau_j\}_{i\neq k, j\neq \ell})$, for any choice of k and ℓ. Similarly, two points σ_k and σ_ℓ on the same curve $(\mathcal{C}, \{\sigma_i\}_{i=1}^n)$ can be glued together to obtain a new curve $(\mathcal{C}/\sigma_k \sim \sigma_\ell, \{\sigma_i\}_{i\neq k,\ell})$. In this article, we construct analogous gluings for k-log-canonically embedded curves. More conceptually, denote by $\overline{\mathcal{E}}^k_{g,n}$ the moduli of k-log-canonically embedded stable curves of genus g with n marked points (see Definition 4.1). For $k \geq 6$, we construct maps

$$\overline{\mathcal{E}}^k_{g_1,n_1+1} \times \overline{\mathcal{E}}^k_{g_2,n_2+1} \longrightarrow \overline{\mathcal{E}}^k_{g_1+g_2,n_1+n_2} \tag{1.1}$$

encoding the gluing of two embedded curves, as well as maps

$$\overline{\mathcal{E}}^k_{g,n+2} \longrightarrow \overline{\mathcal{E}}^k_{g+1,n} \tag{1.2}$$

which encode gluing two points together on the same embedded curve. Our main result is now the following.

The first author was partially supported by Grant-in-Aid for Young Scientists (B) 30372577. The first and second authors were partially supported by World Premier International Research Center Initiative (WPI Initiative), MEXT Japan.

[1] cf. Lemma 2.5.

Theorem 1.2. *For each $k \geq 6$, the maps* (1.1) *and* (1.2) *endow the collection $\{\overline{\mathcal{E}}^k_{g,n}\}$ with the structure of a modular operad (in schemes) which we denote $\overline{\mathcal{E}}^k$. Further, the maps*

$$\overline{\mathcal{E}}^k_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

given by forgetting the embedding determine a map of modular operads (in DM stacks)

$$\overline{\mathcal{E}}^k \longrightarrow \overline{\mathcal{M}}.$$

We refer to the operad $\overline{\mathcal{E}}^k$ as the *modular operad of k-log-canonically embedded curves*. We were led to this operad by the analogy between $\overline{\mathcal{M}}$ and the topological modular operad $\mathcal{M}_{top}$ of smooth, connected, oriented surfaces with boundary. Because the space of embeddings of a manifold M in $\mathbb{R}^\infty$ is contractible, the operad $\mathcal{M}_{top}$ is equivalent to the (homotopy coherent) operad $\mathcal{E}^\infty_{top}$ of smooth, connected, oriented surfaces with boundary inside $\mathbb{R}^\infty$. This equivalence provides the starting point for many results on moduli of topological surfaces (e.g. Madsen and Weiss's proof of the Mumford conjecture [MW07]). It is natural to ask whether one can similarly obtain information about the moduli of stable marked curves by studying moduli of embedded curves, and the modular operad of embedded curves provides a tool with which to begin investigating this question.

Acknowledgements. We are very grateful to Kyoji Saito and Kavli IPMU for making this collaboration possible by hosting the third author during the 2013–2014 academic year.

2. Preliminaries on Curves

Definition 2.1 (cf. [Knu83])**.** Let S be a scheme, and let g and n be non-negative integers such that $n \geq 3-2g$. A *stable marked curve of genus g* over S, $(\mathcal{C}, \{\sigma_i\}_{i=1}^n)$, is a flat, projective morphism

$$\pi\colon \mathcal{C} \longrightarrow S,$$

of relative dimension 1, along with pairwise disjoint sections

$$\sigma_i\colon S \longrightarrow \mathcal{C}$$

for $i = 1, \dots, n$. We require that, for all geometric points s of S,

(1) the fibers $\mathcal{C}_s$ are reduced, connected curves with at most nodal singularities,
(2) the points $\sigma_i(s)$ lie in the smooth locus of $\mathcal{C}_s$ for all i,
(3) $h^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = g$, and
(4) the normalization $\mathcal{C}^\nu_{a,s}$ of each irreducible component of $\mathcal{C}_s$ contains at least $3-2g_{a,s}$ special points, where $g_{a,s}$ is the arithmetic genus of $\mathcal{C}^\nu_{a,s}$ and where a point is *special* if it is either a point of the form $\sigma_i(s)$ or the pre-image of a node.

Now let $\mathcal{C} \longrightarrow S$ be a curve, i.e. a flat, projective morphism of relative dimension 1, not necessarily connected. We further assume that for each geometric point s of S, the fiber $\mathcal{C}_s$ has at most nodal singularities. Let $\sigma_1, \sigma_2\colon S \longrightarrow \mathcal{C}$ be two disjoint sections such that for each geometric point s of S, $\sigma_1(s)$ and $\sigma_2(s)$ lie in the smooth locus of the fiber $\mathcal{C}_s$. Define $\mathcal{C}^{gl} := \mathcal{C}/\sigma_1 \sim \sigma_2$, denote the quotient map by $gl\colon \mathcal{C} \longrightarrow \mathcal{C}^{gl}$, and let $\sigma := gl \circ \sigma_1 = gl \circ \sigma_2$. Recall that for each line bundle L on $\mathcal{C}^{gl}$ we have a short exact sequence

$$0 \longrightarrow L \longrightarrow gl_* gl^* L \longrightarrow \sigma_* \mathcal{O}_S \longrightarrow 0. \tag{2.1}$$

Next, recall that the canonical line bundle $\omega_{\mathcal{C}/S}$ of a family of nodal curves, defined as $\det(\Omega^1_{\mathcal{C}/S})$, admits the following description (cf. [Knu83, p.163]). Every section α of $\omega_{\mathcal{C}/S}$, when restricted to the fiber $\mathcal{C}_s$ over a geometric point s of S, is a meromorphic 1-form α_s on the normalization of $\mathcal{C}_s$. Moreover, α_s has at most simple poles at the pre-images $\{p_{\pm,s}\}$ of the nodes $\{p_s\}$ and

$$res_{p_{+,s}}\alpha_s + res_{p_{-,s}}\alpha_s = 0$$

for each node p_s of the fiber $\mathcal{C}_s$. Along with Nakayama's Lemma, this implies that we have a canonical exact sequence of $\mathcal{O}_{\mathcal{C}^{gl}}$-modules

$$0 \longrightarrow \omega_{\mathcal{C}^{gl}/S} \longrightarrow gl_*\omega_{\mathcal{C}/S}(\sigma_1+\sigma_2) \longrightarrow \sigma_*\mathcal{O}_S \longrightarrow 0. \tag{2.2}$$

Choosing $L = \omega_{\mathcal{C}^{gl}/S}$ and taking the obvious vertical maps from (2.1) to (2.2), an application of the 5-lemma tells us that $gl_*gl^*\omega_{\mathcal{C}^{gl}/S} \cong gl_*\omega_{\mathcal{C}/S}(\sigma_1+\sigma_2)$.

Lemma 2.2. *In the situation above, let $D \subset \mathcal{C}^{gl}$ be a divisor such that $D \longrightarrow S$ is a flat map of degree d, and such that for each geometric point s of S, the fiber D_s is supported on the smooth locus of the fiber $\mathcal{C}^{gl}_s$. Then for each $k \geq 1$, there is a short exact sequence*

$$0 \longrightarrow \omega_{\mathcal{C}^{gl}/S}(D)^{\otimes k} \longrightarrow gl_*\omega_{\mathcal{C}/S}(D+\sigma_1+\sigma_2)^{\otimes k} \longrightarrow \sigma_*\mathcal{O}_S \longrightarrow 0.$$

Proof. If we take $L = \omega_{\mathcal{C}^{gl}/S}(D)^{\otimes k}$, then (2.1) becomes

$$0 \longrightarrow \omega_{\mathcal{C}^{gl}/S}(D)^{\otimes k} \longrightarrow gl_*gl^*\omega_{\mathcal{C}^{gl}/S}(D)^{\otimes k} \longrightarrow \sigma_*\mathcal{O}_S \longrightarrow 0.$$

It remains to show that $gl_*gl^*\omega_{\mathcal{C}^{gl}/S}(D)^{\otimes k} \cong gl_*\omega_{\mathcal{C}/S}(D+\sigma_1+\sigma_2)^{\otimes k}$.

Using Nakayama's Lemma, it suffices to check that this isomorphism holds at each geometric point s of S. Let $U_s \subset \mathcal{C}_s$ be an open set such that either both or neither of the points $\sigma_1(s)$ and $\sigma_2(s)$ are in U_s. So long as both $gl^*\omega_{\mathcal{C}^{gl}_s}(D_s)^{\otimes k}$ and $\omega_{\mathcal{C}_s}(D_s+\sigma_1(s)+\sigma_2(s))^{\otimes k}$ agree on every such U_s, the pushforwards will be isomorphic. By the above discussion, we see that

$$\Gamma(U_s, gl^*\omega_{\mathcal{C}^{gl}_s}(D_s)) \cong \Gamma(U_s, \omega_{\mathcal{C}_s}(D_s+\sigma_1(s)+\sigma_2(s)))$$

for each such U_s, and therefore

$$\Gamma(U_s, gl^*\omega_{\mathcal{C}^{gl}_s}(D_s)^{\otimes k}) \cong \Gamma(U_s, \omega_{\mathcal{C}_s}(D_s+\sigma_1(s)+\sigma_2(s))^{\otimes k})$$

as required. □

Now let $\mathcal{C}$ be a nodal curve over a field κ and let $\nu\colon \mathcal{C}^\nu \longrightarrow \mathcal{C}$ be its normalization. Recall that for any line bundle L on $\mathcal{C}$ we have a short exact sequence

$$0 \longrightarrow L \longrightarrow \nu_*\nu^*L \longrightarrow \mathcal{O}_N \longrightarrow 0$$

analogous to (2.1). By reasoning analogous to the proof of Lemma 2.2, we also have the following.

Lemma 2.3. *In the situation above, let D be a divisor on $\mathcal{C}$. Denote by N the divisor of nodes on $\mathcal{C}$, and denote by P the divisor of pre-images of nodes in the normalization $\mathcal{C}^\nu$. Then for each $k \geq 1$, there is a short exact sequence of $\mathcal{O}_{\mathcal{C}}$-modules*

$$0 \longrightarrow \omega_{\mathcal{C}}(D)^{\otimes k} \longrightarrow \nu_*\omega_{\mathcal{C}^\nu}(D+P)^{\otimes k} \longrightarrow \mathcal{O}_N \longrightarrow 0.$$

Proposition 2.4 (Riemann–Roch)**.** *Let κ be a field, and let $\mathcal{C}$ be a curve of arithmetic genus g over κ, with at most nodal singularities. Let L be a line bundle on $\mathcal{C}$ of total degree d. Then we have*

$$h^0(\mathcal{C}, L) - h^1(\mathcal{C}, L) = d - g + 1.$$

Proof. Let $\nu : \mathcal{C}^\nu \longrightarrow \mathcal{C}$ be the normalization of $\mathcal{C}$ and let N be the divisor of nodes in $\mathcal{C}$. The sequence (2) gives a long exact sequence on cohomology

$$0 \longrightarrow H^0(\mathcal{C}, L) \longrightarrow H^0(\mathcal{C}^\nu, \nu^* L) \longrightarrow H^0(N, \mathcal{O}_N) \longrightarrow H^1(\mathcal{C}, L) \longrightarrow H^1(\mathcal{C}^\nu, \nu^* L) \longrightarrow 0.$$

Exactness then implies that

$$h^0(\mathcal{C}, L) - h^0(\mathcal{C}^\nu, \nu^* L) + j - h^1(\mathcal{C}, L) + h^1(\mathcal{C}^\nu, \nu^* L) = 0,$$

where j is the length of N. We can rearrange terms and apply the smooth Riemann–Roch theorem:

$$\begin{aligned} h^0(\mathcal{C}, L) - h^1(\mathcal{C}, L) &= h^0(\mathcal{C}^\nu, \nu^* L) - h^1(\mathcal{C}^\nu, \nu^* L) - j \\ &= \sum_a d_a - \sum_a g_a + \ell - j \end{aligned}$$

where ℓ is the number of irreducible components C_a^ν of $\mathcal{C}^\nu$, d_a is the degree of L restricted to the component $\mathcal{C}_a^\nu$, and g_a is the geometric genus of $\mathcal{C}_a^\nu$. Using that $d = \sum_a d_a$ and $g = \sum_a g_a - (\ell - 1) + j$, we conclude the result. ☐

Lemma 2.5. *Let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n)$ be a stable marked curve over a field κ. Then, for $k \geq 2$, we have*

$$h^0\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) = N_{g,n}^k + 1\,(:= (2k-1)(g-1) + kn)\,.$$

Proof. Stability and $k \geq 2$ imply that $\omega_{\mathcal{C}} \otimes \left(\omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes -k}\right)$ has negative degree on each component of $\mathcal{C}$, and thus

$$H^1\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) = 0.$$

If $\mathcal{C}$ is smooth (or even just irreducible), then by Riemann–Roch, we have

$$\begin{aligned} h^0\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) &= k(2g - 2 + n) - g + 1 \\ &= (2k-1)(g-1) + nk. \end{aligned}$$

For non-smooth $\mathcal{C}$, let N be the divisor of nodes of $\mathcal{C}$ and let j be the length of N. Let $\nu : \mathcal{C}^\nu \longrightarrow \mathcal{C}$ be a normalization of $\mathcal{C}$, and let P be the divisor of pre-images of the nodes. Because $H^1\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) = 0$ (as we showed above), Lemma 2.3 shows that we have a short exact sequence
(2.3)

$$0 \longrightarrow H^0\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) \to H^0\left(\mathcal{C}^\nu, \omega_{\mathcal{C}^\nu}\left(\textstyle\sum_{i=1}^n \sigma_i + P\right)^{\otimes k}\right) \longrightarrow \kappa^j \longrightarrow 0\,.$$

Write $\mathcal{C}^\nu$ as a union of its irreducible components $\mathcal{C}^\nu = \bigcup_{a=1}^\ell \mathcal{C}_a^\nu$. Denote by $\{\sigma_{(a,i)}\}$ the set of marked points on the component $\mathcal{C}_a^\nu$, and define $n_a := |\{\sigma_{(a,i)}\}|$. Denote by g_a the geometric genus of $\mathcal{C}_a^\nu$. Denote by P_a the restriction of P to $\mathcal{C}_a^\nu$, and define $p_a := \deg(P_a)$. Then:

$$\begin{aligned} h^0\left(\mathcal{C}, \omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right) &= h^0\left(\mathcal{C}^\nu, \omega_{\mathcal{C}^\nu}\left(\sum_{i=1}^n \sigma_i + P\right)^{\otimes k}\right) - j, \\ h^0\left(\mathcal{C}^\nu, \omega_{\mathcal{C}^\nu}\left(\sum_{i=1}^n \sigma_i + P\right)^{\otimes k}\right) &= \sum_{a=1}^{\ell} h^0\left(\mathcal{C}_a^\nu, \omega_{\mathcal{C}_a^\nu}\left(\sum_{(a,i)} \sigma_{(a,i)} + P_a\right)^{\otimes k}\right), \end{aligned}$$

and

$$\begin{aligned} h^0\left(\mathcal{C}_i^\nu, \omega_{\mathcal{C}_a^\nu}\left(\sum_{(a,i)} \sigma_{(a,i)} + P_a\right)^{\otimes k}\right) &= \deg\left(\omega_{\mathcal{C}_a^\nu}\left(\sum_{(a,i)} \sigma_{(a,i)} + P_a\right)^{\otimes k}\right) - g_a + 1 \\ &= k(2g_a - 2 + n_a + p_a) - g_a + 1 \\ &= (2k-1)(g_a - 1) + k(n_a + p_a). \end{aligned}$$

Substituting back, we get

$$\begin{aligned} h^0\left(\mathcal{C}^\nu, \omega_{\mathcal{C}^\nu}\left(\sum_{i=1}^n \sigma_i + P\right)^{\otimes k}\right) &= \sum_{a=1}^{\ell} h^0\left(\mathcal{C}_a^\nu, \omega_{\mathcal{C}_a^\nu}\left(\sum_{(a,i)} \sigma_{(a,i)} + P_a\right)^{\otimes k}\right) \\ &= \sum_{a=1}^{\ell}((2k-1)(g_a - 1) + k(n_a + p_a)) \\ &= (2k-1)\sum_{a=1}^{\ell}(g_a - 1) + k\sum_{a=1}^{\ell}(n_a + p_a) \\ &= (2k-1)\left(\sum_{a=1}^{\ell} g_a - \ell\right) + k(n + 2j). \end{aligned}$$

Using that $g = \sum_{a=1}^{\ell} g_a - (\ell - 1) + j$, we have

$$h^0\left(\mathcal{C}^\nu, \omega_{\mathcal{C}^\nu}\left(\sum_{i=1}^n \sigma_i + P\right)^{\otimes k}\right) = (2k-1)(g-1) + j + kn.$$

In light of the exact sequence (2.3), this implies the result. □

3. Gluing Embedded Curves

Definition 3.1. Let S be a scheme, and let $k \geq 6$. A *marked, k-log canonically embedded curve over S* consists of the data $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$, where

(1) $\pi\colon \mathcal{C} \longrightarrow S$ is a flat, projective morphism of relative dimension 1, not necessarily connected,
(2) the pair $(\mathcal{C}, \{\sigma_i\})$ is a disjoint union of stable marked curves over S,
(3) η denotes a projective embedding $\eta\colon \mathcal{C} \longrightarrow \mathbb{P}_S^N$ over S by a *complete* linear system,
(4) and φ denotes an isomorphism $\varphi\colon \eta^*\mathcal{O}_{\mathbb{P}_S^N}(1) \xrightarrow{\cong} \omega_{\mathcal{C}/S}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}$.

Our goal in this section is to prove the following.

Theorem 3.2 (Gluing Embedded Curves)**.** *Let S be a scheme, and $k \geq 6$. Let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be a marked, k-log-canonically embedded curve over S. Denote by ℓ_{σ_1,σ_2} the line $\mathbb{P}^1_S \hookrightarrow \mathbb{P}^N_S$ spanned by σ_1 and σ_2. Then:*

(1) *There exists a natural section*

$$\gamma\colon S \longrightarrow \ell_{\sigma_1,\sigma_2}.$$

(2) *The projection from γ gives an embedding*

$$\mathcal{C}^{gl} := \mathcal{C}/\sigma_1 \sim \sigma_2 \xrightarrow{\eta^{gl}} \mathbb{P}^{N-1}_S.$$

(3) *The isomorphism φ determines an isomorphism*

$$(\eta^{gl})^*\mathcal{O}_{\mathbb{P}^{N-1}_S}(1) \xrightarrow[\cong]{\varphi^{gl}} \omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} .$$

Remark 3.3. We choose σ_1 and σ_2 for notational convenience. Our proof applies equally well to any choice of i and j.

Proof. To prove the theorem, we need to establish the claims 1–3.

1. Constructing the Section and the Isomorphism.

Lemma 3.4 (Claim 1)**.** *Let S be a scheme, and let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be a marked, k-log-canonically embedded curve over S. Denote by ℓ_{σ_1,σ_2} the line $\mathbb{P}^1_S \hookrightarrow \mathbb{P}^N_S$ spanned by σ_1 and σ_2. Then there exists a natural section*

$$\gamma\colon S \longrightarrow \ell_{\sigma_1,\sigma_2}.$$

Proof. Define $\mathcal{C}^{gl} := \mathcal{C}/\sigma_1 \sim \sigma_2$. Denote the quotient map by $gl\colon \mathcal{C} \longrightarrow \mathcal{C}^{gl}$, and let $\sigma := gl \circ \sigma_1 = gl \circ \sigma_2$. By Lemma 2.2, we have a short exact sequence

$$0 \longrightarrow \omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} \longrightarrow gl_*\omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k} \longrightarrow \sigma_*\mathcal{O}_S \longrightarrow 0 .$$

Pushing this sequence forward to S along the projection $\pi^{gl}\colon \mathcal{C}^{gl} \longrightarrow S$, we obtain a long exact sequence

$$0 \longrightarrow \pi^{gl}_*\omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} \longrightarrow \pi^{gl}_* gl_*\omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k} \longrightarrow \pi^{gl}_*\sigma_*\mathcal{O}_S$$

$$\longrightarrow R^1\pi^{gl}_*\omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} \longrightarrow R^1\pi^{gl}_* gl_*\omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k} \to 0 .$$

Because $k \geq 2$, degree considerations combine with Grothendieck–Riemann–Roch to show that the higher direct image sheaves vanish. Because $\pi^{gl}\sigma = 1_S$ and $\pi^{gl}gl = \pi$, we can rewrite the cohomology long exact sequence as the short exact sequence

$$0 \longrightarrow \pi^{gl}_*\omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} \longrightarrow \pi_*\omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k} \longrightarrow \mathcal{O}_S \longrightarrow 0 .$$

Dualizing and projectivizing, we obtain the sequence

$$S \xrightarrow{\gamma} \mathbb{P}_S(\pi_*\omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k})^\vee \dashrightarrow \mathbb{P}_S(\pi^{gl}_*\omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k})^\vee$$

where the dashed arrow indicates the projection from the point γ.

The first map gives the desired section

$$S \xrightarrow{\gamma} \mathbb{P}_S(\pi_* \omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k})^\vee \ .$$

We must still show that γ factors through ℓ_{σ_1,σ_2}. We have the map

$$\mathcal{C} \xrightarrow{\eta} \mathbb{P}_S(\pi_* \omega_{\mathcal{C}/S}\left(\textstyle\sum_{i=1}^n \sigma_i\right)^{\otimes k})^\vee \dashrightarrow \mathbb{P}_S(\pi_*^{gl} \omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k})^\vee$$

which comes from restricting the linear system $\pi_* \omega_{\mathcal{C}/S}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}$ to the sections in $\pi_*^{gl} \omega_{\mathcal{C}^{gl}/S}\left(\sum_{i=3}^n \sigma_i\right)^{\otimes k}$. These sections, by construction, agree on σ_1 and σ_2. Thus, this composition factors through $\mathcal{C}^{gl}$, and γ factors through ℓ_{σ_1,σ_2}. □

The proof of the previous lemma immediately implies the following.

Lemma 3.5 (Claim 3)**.** *Let k, S and $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be as in the previous lemma. Then the isomorphism φ determines an isomorphism*

$$\mathbb{P}_S^{N-1} \xrightarrow{\cong} \mathbb{P}_S(\pi_*^{gl} \omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k})^\vee \ ,$$

and, thus, an isomorphism

$$(\eta^{gl})^* \mathcal{O}_{\mathbb{P}_S^{N-1}}(1) \xrightarrow[\cong]{\varphi^{gl}} \omega_{\mathcal{C}^{gl}/S}\left(\textstyle\sum_{i=3}^n \sigma_i\right)^{\otimes k} \ .$$

2. Verifying that Projecting Gives an Embedding. It remains to show that projecting from γ induces an embedding η^{gl} of the glued curve $\mathcal{C}^{gl}$. Because the projection from γ is a map over S, it suffices to check that it gives an embedding on fibers. Therefore, throughout this section, we assume that $S = Spec(\kappa)$ for a field κ.

The following proposition provides the basis for our approach.

Proposition 3.6. [Har77, Proposition IV.3.7] *Let κ be a field, let $\mathcal{C}$ be a curve in $\mathbb{P}^3_\kappa$, let γ be a κ-point not on $\mathcal{C}$, and let $\eta' : \mathcal{C} \longrightarrow \mathbb{P}^2_\kappa$ be the morphism determined by projection from γ. Then η' is birational onto its image and $\eta'(\mathcal{C})$ has at most nodes as singularities if and only if:*

1. *γ lies on only finitely many secants of $\mathcal{C}$,*
2. *γ is not on any tangent line of $\mathcal{C}$,*
3. *γ is not on any secant with coplanar tangent lines, and*
4. *γ is not on any multisecant of $\mathcal{C}$.*

Now let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be a marked k-log-canonically embedded curve over the field κ. We show that any κ-point $\gamma \in \ell_{\sigma_1,\sigma_2}$ on the line spanned by σ_1 and σ_2 satisfies an analogue of Proposition 3.6. A priori, it suffices to change the first requirement so that γ lies on a unique secant ℓ_{σ_1,σ_2}. However, we can prove a stronger statement. First, we rephrase the four criteria, using $\ell_{p,q}$ for the line between p and q. Then the conditions can be written (and strengthened) as:

1. for all $p, q \in \mathcal{C}$, $\ell_{p,q} \cap \ell_{\sigma_1,\sigma_2} = \emptyset$ unless $\{p, q\} \cap \{\sigma_1, \sigma_2\} \neq \emptyset$,
2. for all $p \in \mathcal{C}$, $T_p\mathcal{C} \cap \ell_{\sigma_1,\sigma_2} = \emptyset$ unless $p \in \{\sigma_1, \sigma_2\}$,
3. for all $p, q \in \mathcal{C}$, $T_p\mathcal{C} \cap T_q\mathcal{C} = \emptyset$ unless $p = q$, and
4. $\mathcal{C}$ has no multisecant.

We can further simplify as follows. First, we note that conditions 1–3 are special cases of the same thing. In particular, if we begin with condition 1, and take the limit as q approaches p, we arrive at condition 2, and as σ_1 approaches σ_2, we get

3. Second, $\mathcal{C}$ has no multisecants if and only if $\mathcal{C}$ has no trisecants. With these changes, the statement becomes the following.

Proposition 3.7. *Let $\mathcal{C}$ be a curve in $\mathbb{P}^N_\kappa$, and let $\sigma_1, \sigma_2 \in \mathcal{C}$. Then the projection from a point $\gamma \in \ell_{\sigma_1,\sigma_2} \setminus \{\sigma_1, \sigma_2\}$ is an isomorphism on $\mathcal{C} \setminus \{\sigma_1, \sigma_2\}$ if:*

(1) *$\mathcal{C} \subset \mathbb{P}^N_\kappa$ has no trisecant.*
(2) *No length 4 sub-scheme of $\mathcal{C}$ is contained in a plane.*

Proof. We begin by observing that the absence of trisecant lines and quadrisecant planes is significantly stronger than the minimal condition needed to ensure that the projection restricts to an embedding on $\mathcal{C} \setminus \{\sigma_1, \sigma_2\}$. Indeed, we are projecting from a point $\gamma \in \mathbb{P}^N_\kappa$, and we will prove that it suffices to show that γ lies on a unique secant line of $\mathcal{C}$, and that there are no trisecants through γ.

The fibers of the projection restricted to $\mathcal{C}$ are intersections with lines through γ. In other words, any line through γ intersecting $\mathcal{C}$ in more than one point is a fiber where the map is non-injective. The absence of a trisecant line guarantees that fibers consist of at most two points. The lack of a quadrisecant plane guarantees that there is only one line through γ which intersects the curve in at least two points. Thus, away from ℓ_{σ_1,σ_2}, the projection map is injective on the curve $\mathcal{C}$. To see that it is an isomorphism, we note that it will be an isomorphism at any point where the line intersects the curve transversely, so we just need to rule out the existence of tangent lines to $\mathcal{C}$ containing γ. Such a line would, along with ℓ_{σ_1,σ_2} give a length 4 sub-scheme of $\mathcal{C}$ contained in a plane (through γ); by hypothesis, none exists. $\square$

We now verify that all marked k-log-canonically embedded curves $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ satisfy the conditions of the proposition.

Lemma 3.8. *Let $\mathcal{C} = \bigcup \mathcal{C}_a$ be a nodal curve (with irreducible components $\mathcal{C}_a$) of arithmetic genus g over a field κ. Let g_a be the geometric genus of the normalization $\mathcal{C}_a^\nu$ of $\mathcal{C}_a$. Let L be a line bundle of degree d on $\mathcal{C}$, let L_a be the pullback to $\mathcal{C}_a^\nu$ of L, and let $d_a := \deg L_a$. Assume that, for all a, $d_a \geq 2g_a + 2 + j_a$, where j_a is the number of preimages of nodes in $\mathcal{C}_a^\nu$. Then $\mathcal{C}$ has no trisecant lines when embedded by the complete linear system $|L|$.*

Proof. A trisecant is an effective divisor T of degree 3 that is contained in a line. For a curve embedded by the complete linear system of a line bundle L, this condition on T can be rewritten as $h^0(\mathcal{C}, L) - h^0(\mathcal{C}, L(-T)) = 2$. Riemann–Roch (Proposition 2.4) tells us that

$$\begin{aligned} h^0(\mathcal{C}, L) - h^1(\mathcal{C}, L) &= d - g + 1, \text{ and} \\ h^0(\mathcal{C}, L(-T)) - h^1(\mathcal{C}, L(-T)) &= d - g - 2. \end{aligned}$$

Applying Serre duality and then subtracting one from the other, we get

$$\big(h^0(\mathcal{C}, L) - h^0(\mathcal{C}, L(-T))\big) - \big(h^0(\mathcal{C}, \omega_\mathcal{C} \otimes L^{-1}) - h^0(\mathcal{C}, \omega_\mathcal{C}(T) \otimes L^{-1})\big) = 3.$$

This equation implies that, in order to show that T is not a trisecant, it suffices to show that

$$h^0(\mathcal{C}, \omega_\mathcal{C} \otimes L^{-1}) - h^0(\mathcal{C}, \omega_\mathcal{C}(T) \otimes L^{-1}) = 0.$$

In particular, it suffices to show that both terms vanish. A line bundle can be shown to have no global sections by checking that there is no component on which the degree is positive. Thus, we want $2g_a - 2 + j_a - d_a < 0$ and $2g_a - 2 + 3 + j_a - d_a < 0$. We see that $d_a \geq 2g_a + 2 + j_a$ suffices for both. $\square$

Lemma 3.9. *In the situation of Lemma 3.8, assume that, for all a, $d_a \geq 2g_a + 3 + j_a$. Then, when embedded by the complete linear system $|L|$, $\mathcal{C}$ has no quadrisecant planes.*

Proof. Let T be an effective divisor of degree 4 on $\mathcal{C}$ that is contained in a plane. Then, by a similar calculation as in the proof of Lemma 3.8, the divisor T must satisfy

$$h^0(\mathcal{C}, \omega_{\mathcal{C}}(T) \otimes L^{-1}) - h^0(\mathcal{C}, \omega_{\mathcal{C}} \otimes L^{-1}) = 1.$$

By the degree condition, $\omega_{\mathcal{C}} \otimes L^{-1}$ already has negative degree on each component, and so has no global sections. Therefore, T is contained in a plane if and only if

$$h^0(C, \omega_{\mathcal{C}}(T) \otimes L^{-1}) = 1.$$

However, we can compute that the degree on each component is

$$2g_a - 2 + j_a + 4 - d_a = 2g_a + 2 + j_a - d_a.$$

By hypothesis, this is negative. $\square$

A direct computation now shows that if $(\mathcal{C}, \{\sigma_i\}_{i=1}^n)$ is a disjoint union of stable curves, then $\deg \left(\omega_{\mathcal{C}} \left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right)$ satisfies the conditions of Lemma 3.9 (and thus Lemma 3.8) so long as $k \geq 6$.

Corollary 3.10 (Claim 2)**.** *Let $k \geq 6$. Let S be a scheme, and let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be a marked, k-log-canonically embedded curve over S. Then the projection from γ in $\mathbb{P}_S^N$ induces an embedding η^{gl} of $\mathcal{C}^{gl}$ in $\mathbb{P}_S^{N-1}$.*

This concludes the proof of Theorem 3.2. $\square$

4. Moduli of Pluri-Log-Canonically Embedded Curves

We now introduce a scheme $\overline{\mathcal{E}}_{g,n}^k$ parameterizing stable curves $\mathcal{C}$, of genus g, with n marked points $\{\sigma_i\}_{i=1}^n$ in the smooth locus of $\mathcal{C}$, equipped with an embedding into projective space by $\omega_{\mathcal{C}} \left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}$.

Definition 4.1 (Log-Canonical Hilbert scheme)**.** Fix $k \geq 6$. We define the *k-log-canonical Hilbert scheme of embedded marked curves* $\overline{\mathcal{E}}_{g,n}^k$ to be the scheme representing the functor

$$S \mapsto \{(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)\}$$

which maps a scheme S to the set of isomorphism classes of k-log-canonically embedded stable marked curves over S.

Remark 4.2. We see that $\overline{\mathcal{E}}_{g,n}^k$ is a scheme by a construction analogous to the construction of the moduli stack of stable curves $\overline{\mathcal{M}}_{g,n}$. First, we fix k, which determines the dimension N of the projective space that these curves map into as well as the degree of the map. This gives us a Hilbert polynomial $P(t)$, and we can look in $\mathrm{Hilb}_{P(t)}(\mathbb{P}^N)$ at the locus of curves $\mathcal{C}$ that are stable with respect to the markings on some divisor of degree n. Denote this scheme by $\mathcal{X}_g^k$ and the universal family over it by $\mathcal{X}_{g,1}^k$. Then, $\overline{\mathcal{E}}_{g,n}^k \subset \prod_{\mathcal{X}_g^k}^n \mathcal{X}_{g,1}^k$ is given by the locus where the marked points are distinct and are arranged in such a way that the curve is stable.

Remark 4.3. There is a map $\overline{\mathcal{E}}^k_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ which forgets the embedding. This map is a PGL $\left|\omega_{\mathcal{C}}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k}\right|$-bundle over the DM-stack $\overline{\mathcal{M}}_{g,n}$.

Every S-point of $\overline{\mathcal{E}}^k_{g,n+2}$ or $\overline{\mathcal{E}}^k_{g_1,n_1+1} \times \overline{\mathcal{E}}^k_{g_2,n_2+1}$ determines an embedded curve satisfying the conditions of Theorem 3.2.[2] Because the section γ in Theorem 3.2 is natural with respect to base change, Theorem 3.2 immediately implies the following.

Corollary 4.4. *For each $k \geq 6$, there exist maps*

$$\overline{\mathcal{E}}^k_{g_1,n_1+1} \times \overline{\mathcal{E}}^k_{g_2,n_2+1} \longrightarrow \overline{\mathcal{E}}^k_{g_1+g_2,n_1+n_2} \qquad \textit{and} \qquad \overline{\mathcal{E}}^k_{g,n+2} \longrightarrow \overline{\mathcal{E}}^k_{g+1,n}\ ,$$

which fit into commuting squares

$$\begin{tikzcd} \overline{\mathcal{E}}^k_{g_1,n_1+1} \times \overline{\mathcal{E}}^k_{g_2,n_2+1} \arrow[r] \arrow[d] & \overline{\mathcal{E}}^k_{g_1+g_2,n_1+n_2} \arrow[d] \\ \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \arrow[r] & \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2} \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \overline{\mathcal{E}}^k_{g,n+2} \arrow[r] \arrow[d] & \overline{\mathcal{E}}^k_{g+1,n} \arrow[d] \\ \overline{\mathcal{M}}_{g,n+2} \arrow[r] & \overline{\mathcal{M}}_{g+1,n}\ . \end{tikzcd}$$

5. Modular Operads of Embedded Curves

In this section, we prove Theorem 1.2. For the reader familiar with modular operads, we remark that, given the above construction of the gluing maps, the only non-trivial point which remains is to prove that the gluing maps are associative. For the rest of our readers, we begin by recalling the definition of modular operads and stating what it is we need to show. Readers familiar with these notions should feel free to skip the following paragraph.

Review of Modular Operads. Our goal in this paragraph is to provide a minimal list of things one must produce to exhibit a modular operad. For a more elegant and thorough treatment, we refer the reader to the article [GK98], which we take as our primary reference.

Definition 5.1. Denote by S_n the permutation group on n elements $\{1, \ldots, n\}$.[3] For $\pi \in S_m$, $\rho \in S_n$, and $1 \leq i \leq m$, denote by

$$\pi \circ_i \rho \in S_{m+n-1}$$

the permutation which re-orders $\{i, \ldots, i+n-1\}$ according to ρ and then re-orders the set of sets $\{\{1\}, \ldots, \{i-1\}, \{i, \ldots, i+n-1\}, \{i+n\}, \ldots, \{m+n-1\}\}$ by π. Explicitly,

$$(\pi \circ_i \rho)(j) := \begin{cases} \pi(j) & j < i \text{ and } \pi(j) < \pi(i), \\ \pi(j) + n - 1 & j < i \text{ and } \pi(j) > \pi(i), \\ \pi(i) + \rho(j-i+1) - 1 & i - 1 < j < i + n, \\ \pi(j-n+1) & j \geq i+n \text{ and } \pi(j-n+1) < \pi(i), \\ \pi(j-n+1) + n - 1 & j \geq i+n \text{ and } \pi(j-n+1) > \pi(i). \end{cases}$$

[2] For $\overline{\mathcal{E}}^k_{g_1,n_1+1} \times \overline{\mathcal{E}}^k_{g_2,n_2+1}$, take the disjoint union of the factors.

[3] By convention, S_0 is the trivial group, i.e. the group of automorphisms of the empty set.

Let $(\mathcal{D}, \otimes, \sigma)$ be a symmetric monoidal category[4] such that $\otimes$ preserves coproducts, for example $\mathcal{D}$ could be the category of DM-stacks with the Cartesian product. We further assume that there exists an initial object $0 \in \mathcal{D}$; e.g. 0 could be the DM-stack $\emptyset$.

Definition 5.2. An operad in $(\mathcal{D}, \otimes, \sigma)$ consists of:

(1) for each non-negative integer $n \in \mathbb{N}$, an object $\mathcal{P}(n) \in \mathcal{D}$ with a homomorphism $S_n \longrightarrow Aut(\mathcal{P}(n))$, and
(2) for each $1 \leq i \leq m$, a map $\circ_i \colon \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$.

We require that these satisfy the following conditions. For $\pi \in S_m$, $\rho \in S_n$, and $1 \leq i \leq m$, we require

$$(\pi \circ_i \rho) \cdot \circ_i = \circ_{\pi(i)} \cdot (\pi \otimes \rho) \tag{5.1}$$

as maps $\mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$.

For $1 \leq i < j \leq \ell$, we require

$$\circ_{j+m-1} \cdot (\circ_i \otimes 1_{\mathcal{P}(n)}) = \circ_i \cdot (\circ_j \otimes 1_{\mathcal{P}(m)}) \cdot (1_{\mathcal{P}(\ell)} \otimes \sigma) \tag{5.2}$$

and for $1 \leq i \leq \ell$ and $1 \leq j \leq m$, we require

$$\circ_{i+j-1} \cdot (\circ_i \otimes 1_{\mathcal{P}(n)}) = \circ_i \cdot (1_{\mathcal{P}(\ell)} \otimes \circ_j) \tag{5.3}$$

as maps $\mathcal{P}(\ell) \otimes \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(\ell+m+n-2)$.

We define a map of operads $\mathcal{P}^1 \longrightarrow \mathcal{P}^2$ in the obvious manner, i.e. it consists of a collection of equivariant maps $\mathcal{P}^1(n) \longrightarrow \mathcal{P}^2(n)$ for all $n \in \mathbb{N}$ which intertwine the various maps $\circ_i$ for $\mathcal{P}^1$ and $\mathcal{P}^2$.

Remark 5.3. To make sense of these axioms, it is helpful to picture $\mathcal{P}(n)$ as a collection of labels for trees with one outgoing leaf and n incoming leaves marked $1, \ldots, n$; the group S_n acts by permuting the markings of the incoming leaves. In this picture, the map $\circ_i$ corresponds to gluing the outgoing leaf of a tree in $\mathcal{P}(n)$ to the i^{th}-incoming leaf of a tree in $\mathcal{P}(m)$ to obtain a tree in $\mathcal{P}(n+m-1)$. Axiom (5.1) requires that if we first relabel and then glue, this is equivalent to gluing first and then relabelling in the natural fashion. Axioms (5.2) and (5.3) require the gluing of three trees to be associative in the natural fashion.

Denote by S_{n+} the permutation group on $n+1$ letters $\{0, \ldots, n\}$. Denote by τ_n the cycle $(01 \cdots n)$.

Definition 5.4. A *cyclic operad* is an operad $\mathcal{P}$ in $(\mathcal{D}, \otimes, \sigma)$ such that, for each $n \in \mathbb{N}$, the S_n-action on $\mathcal{P}(n)$ extends to an S_{n+}-action[5] and such that

$$\tau_{n+m-1} \cdot \circ_m = \circ_1 \cdot (\tau_n \otimes \tau_m) \cdot \sigma \tag{5.4}$$

as maps $\mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$.

We define maps of cyclic operads in the obvious manner.

Remark 5.5. In the picture of Remark 5.3, the objects $\mathcal{P}(n)$ of a cyclic operad can be pictured as a collection of labels for trees with one outgoing and n incoming leaves, where we are also allowed the permute the outgoing leaf with the incoming leaves. Equivalently, we can view $\mathcal{P}(n)$ as a collection of labeled trees with $n+1$ leaves marked $0, \ldots, n$, where S_{n+} acts by permuting the markings on the leaves.

[4] σ denotes the symmetry isomorphism.

[5] Under the embedding $S_n \hookrightarrow S_{n+}$ corresponding to the inclusion $\{1, \ldots, n\} \subset \{0, \ldots, n\}$.

Axiom (5.4) requires that if we first relabel using the extra symmetry in S_{n+} and then glue, this is equivalent to gluing first and relabelling in the natural fashion.

Notation 5.6. If $\mathcal{P}$ is a cyclic operad, we write $\mathcal{P}((n+1))$ for the object $\mathcal{P}(n)$. In this notation, we have

$$\circ_i \colon \mathcal{P}((m)) \otimes \mathcal{P}((n)) \longrightarrow \mathcal{P}((m+n-2))$$

for $m, n > 1$. We will also consider cyclic operads $\mathcal{P}$ for which we define $\mathcal{P}((0))$. However, we do not assume the existence of maps $\circ_i$ with source $\mathcal{P}((m)) \otimes \mathcal{P}((n))$ for either m or n equal to 0.

Definition 5.7. A *stable cyclic operad* is a cyclic operad $\mathcal{P}$ such that for each non-negative integer $n \in \mathbb{N}$, there exists an S_n-equivariant decomposition

$$\mathcal{P}((n)) := \coprod_{g \in \mathbb{N}} \mathcal{P}((g, n))$$

such that $\mathcal{P}((g,n)) = 0$ if $n < 3 - 2g$, and such that, for all $1 \leq i \leq m$ and $n > 0$, the map $\circ_i$ restricts to a map

$$\mathcal{P}((g,m)) \otimes \mathcal{P}((h,n)) \longrightarrow \mathcal{P}((g+h, m+n-2)).$$

Remark 5.8. In a stable cyclic operad, we can picture the object $\mathcal{P}((g,n))$ as a collection of labels for dual graphs of stable curves of genus g with n marked points. In this picture, the maps $\circ_i$ correspond to gluing the first leg of a graph in $\mathcal{P}((h,n))$ to the i^{th} leg of a graph in $\mathcal{P}((g,m))$, and relabelling the remaining legs accordingly.

Definition 5.9. Let $n \geq 2$, let $\rho \in S_n$, and let $i \neq j \in \{1, \ldots, n\}$. Denote by $\rho_{\setminus\{i,j\}} \in S_{n-2}$ the induced bijection

$$\{1,\ldots,n-2\} \xrightarrow{\cong} \{1,\ldots,n\} \setminus \{i,j\} \xrightarrow{\rho} \{1,\ldots,n\} \setminus \{\rho(i),\rho(j)\} \xrightarrow{\cong} \{1,\ldots,n-2\},$$

where the first and last bijections are the canonical order-preserving bijections.

Definition 5.10. A *modular operad* is a stable cyclic operad $\mathcal{P}$ such that for each g, n and $i \neq j \in \{1, \ldots, n\}$, there exists a map

$$\xi_{ij} \colon \mathcal{P}((g,n)) \longrightarrow \mathcal{P}((g+1, n-2))$$

such that the following properties are satisfied. For each $\rho \in S_n$, we require

$$\rho_{\setminus\{i,j\}} \cdot \xi_{ij} = \xi_{\rho(i)\rho(j)} \cdot \rho \tag{5.5}$$

as maps $\mathcal{P}((g,n)) \longrightarrow \mathcal{P}((g+1, n-2))$.

For $1 \leq i \neq j \neq k \neq \ell \leq n$, we require

$$\xi_{ij} \cdot \xi_{k\ell} = \xi_{k\ell} \cdot \xi_{ij} \tag{5.6}$$

as maps $\mathcal{P}((g,n)) \longrightarrow \mathcal{P}((g+2, n-4))$.[6]

We further require

$$\xi_{12} \cdot \circ_m = \circ_m \cdot (\xi_{12} \otimes 1_{\mathcal{P}((h,n))}) \tag{5.7}$$

$$\xi_{m,m+1} \cdot \circ_m = \circ_m \cdot (1_{\mathcal{P}((g,m))} \otimes \xi_{12}) \tag{5.8}$$

[6] Note that we are abusing notation slightly on the left hand side of (5.6) by writing ξ_{ij} to denote the map which corresponds to the image of the pair i, j under the identification $\{1, \ldots, n\} \setminus \{k, \ell\} \cong \{1, \ldots, n-2\}$. An analogous abuse of notation also occurs on the right hand side.

and

$$\xi_{m-1,m} \cdot \circ_m = \xi_{m+n-2,m+n-1} \cdot \circ_{m-1} \cdot (1_{\mathcal{P}((g,m))} \otimes \tau_n) \tag{5.9}$$

as maps $\mathcal{P}((g,m)) \otimes \mathcal{P}((h,n)) \longrightarrow \mathcal{P}((g+h+1, m+n-4))$.

We define maps of modular operads in the obvious manner.

Remark 5.11. A modular operad is a stable cyclic operad with extra structure encoded by the maps ξ_{ij}. If we picture $\mathcal{P}((g,n))$ as a collection of labels for dual graphs of stable marked curves, then the maps ξ_{ij} correspond to gluing together the i and j legs of a graph Γ to obtain a new graph Γ'. As usual, Axiom (5.5) requires that relabelling and then gluing is equivalent to gluing first and relabelling in the natural fashion. Similarly, Axioms (5.6)–(5.9) require the various operations involving gluing two pairs of legs together to be associative in the natural fashion.

Proof of Theorem 1.2. Because the forgetful maps $\overline{\mathcal{E}}^k_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ are S_n-equivariant, by Corollary 4.4, it suffices to verify that the gluing maps on $\{\overline{\mathcal{E}}^k_{g,n}\}$ form a modular operad in order to conclude that the forgetful maps

$$\overline{\mathcal{E}}^k_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

determine a map of operads.

Further, the construction of the gluing maps in the proof of Theorem 3.2 immediately implies that the equivariance axioms (5.1), (5.4) and (5.5) are all satisfied. Therefore, it only remains prove that the gluing maps satisfy the associativity properties (5.2), (5.3) and (5.6)–(5.9). These will all be immediate consequences of the following lemma.

Lemma 5.12. *Let $k \geq 6$, let S be a scheme, and let $(\mathcal{C}, \{\sigma_i\}_{i=1}^n, \eta, \varphi)$ be a marked k-log-canonically embedded curve over S. Denote by*

$$(\mathcal{C}^{gl_{12,34}}, \{\sigma_i\}_{i=5}^n, \eta^{gl_{12,34}}, \varphi^{gl_{12,34}})$$

the embedded curve obtained by first gluing σ_1 to σ_2, and then gluing σ_3 to σ_4. Likewise, denote by

$$(\mathcal{C}^{gl_{34,12}}, \{\sigma_i\}_{i=5}^n, \eta^{gl_{34,12}}, \varphi^{gl_{34,12}})$$

the embedded curve obtained by first gluing σ_3 to σ_4, and then gluing σ_1 to σ_2. Then there exists a canonical isomorphism

$$(\mathcal{C}^{gl_{12,34}}, \{\sigma_i\}_{i=5}^n, \eta^{gl_{12,34}}, \varphi^{gl_{12,34}}) \cong (\mathcal{C}^{gl_{34,12}}, \{\sigma_i\}_{i=5}^n, \eta^{gl_{34,12}}, \varphi^{gl_{34,12}}).$$

Proof. The associativity of the classical gluing maps for curves guarantees the existence of a canonical isomorphism

$$(\mathcal{C}^{gl_{12,34}}, \{\sigma_i\}_{i=5}^n) \cong (\mathcal{C}^{gl_{34,12}}, \{\sigma_i\}_{i=5}^n).$$

Using this isomorphism, our observations about the vanishing of higher direct image sheaves imply that there exists a commuting diagram of $\mathcal{O}_S$-modules with exact rows and columns:

$$\begin{tikzcd}
\pi_*\omega_{\mathcal{C}^{gl_{12,34}}/S}\left(\sum_{i=5}^n \sigma_i\right)^{\otimes k} \arrow[r] \arrow[d] & \pi_*\omega_{\mathcal{C}^{gl_{34}}/S}\left(\sum_{i\neq 3,4} \sigma_i\right)^{\otimes k} \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[d, equal] \\
\pi_*\omega_{\mathcal{C}^{gl_{12}}/S}\left(\sum_{i=3}^n \sigma_i\right)^{\otimes k} \arrow[r] \arrow[d] & \pi_*\omega_{\mathcal{C}/S}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k} \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[d] \\
\mathcal{O}_S \arrow[r, equal] & \mathcal{O}_S \arrow[r] & 0
\end{tikzcd}.$$

Dualizing and projectivizing, we obtain a commuting diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & S \arrow[r, equal] \arrow[d] & S \arrow[d] \\
S \arrow[r] \arrow[d, equal] & \mathbb{P}_S(\pi_*\omega_{\mathcal{C}/S}\left(\sum_{i=1}^n \sigma_i\right)^{\otimes k})^\vee \arrow[r, dashed] \arrow[d, dashed] & \mathbb{P}_S(\pi_*\omega_{\mathcal{C}^{gl_{12}}/S}\left(\sum_{i=3}^n \sigma_i\right)^{\otimes k})^\vee \arrow[d, dashed] \\
S \arrow[r] & \mathbb{P}_S(\pi_*\omega_{\mathcal{C}^{gl_{34}}/S}\left(\sum_{i\neq 3,4} \sigma_i\right)^{\otimes k})^\vee \arrow[r, dashed] & \mathbb{P}_S(\pi_*\omega_{\mathcal{C}^{gl_{12,34}}/S}\left(\sum_{i=5}^n \sigma_i\right)^{\otimes k})^\vee
\end{tikzcd}$$

where the dashed arrows indicate the projections. The commutativity of the lower right square implies that $\eta^{gl_{12,34}} = \eta^{gl_{34,12}}$ and $\varphi^{gl_{12,34}} = \varphi^{gl_{34,12}}$. □

Remark 5.13. There is an equivalent, although more manifestly geometric, formulation of the above argument. Each of the pairs of points $\{\sigma_1, \sigma_2\}$ and $\{\sigma_3, \sigma_4\}$ lying over eventual nodes gives a point on the line between them. Then, projection from one point followed by projection from the image of the other, in either order, is the same map as projection from the line spanned by the two points. Here, the equivalence of the embeddings follows from the fact that these are just two factorizations of the same projection map, with one-dimensional center.

References

[GK98] E. Getzler and M. Kapranov. Modular operads. *Compos. Math.*, 110:65–126, 1998. 5

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *GTM*. Springer-Verlag, 1977. 3.6

[Knu83] F.F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2):161–199, 1983. 2.1, 2

[MW07] I. Madsen and M. Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007. 1

Kavli IPMU (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
E-mail address: satoshi.kondo@ipmu.jp
E-mail address: charles.siegel@ipmu.jp

Department of Mathematics, The University of Chicago
E-mail address: wolfson@math.uchicago.edu