Orthogonal pairs for Lie algebra sl(6)

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1 Introduction

An *orthogonal pair* in a semisimple Lie algebra is a pair of Cartan subalgebras which are orthogonal with respect to the Killing form. Description of orthogonal pairs in a given Lie algebra is an important step in the classification of *orthogonal decompositions*, i.e. decompositions of the Lie algebra into the sum of Cartan subalgebras pairwise orthogonal with respect to the Killing form.

Orthogonal decompositions come up firstly in the theory of integer lattices in the paper by Thompson [14]. Then the theory of such bases was substentially developed by mathematicians [9]. The classification problem of orthogonal pairs in $sl(n)$ is closely related to the classification of complex Hadamard matrices of order n [9], [2].

Independently, a unitary version of orthogonal pairs appeared in quantum theory under the name of mutually unbiased bases [2], objects of constant use in Quantum Information Theory, Quantum Tomography, etc. This makes a link of the subject to various vibrant probelms in Mathematical Physics.

One of the reasons why mutually unbiased bases are important in practice is that they provide a crucial mathematical tool that allows to transfer quantum information with minimal loss of it in the channel. Reliable protocols in quantum channels, such as protocol BB84, are based on a choice of maximal number of mutually unbiased bases in the relevant vector space of quantum states of transmited particles. Protocol BB84, which utilize 3 such bases in a 2 dimensional vector space, allows us to significantly extend the distance between the source and the receiver of quantum information. Clearly, big number of bases in a higher dimensional spaces is of tremendous importance in constructing reliable protocols in quantum channels.

Also, in quantum teleportation, it is important to check the result of purity of teleportation by means of Quantum Tomography. The Quantum Tomography with minimal error bar is again based on mutually unbiased bases.

Despite of simple definition, the classification of orthogonal pairs is a very hard problem of algebraic geometric origin. We will consider pairs in the Lie algebra $sl(n, \mathbb{C})$. According to the famous Winnie-the-Pooh conjecture [7], orthotogonal decompositions are possible in this algebra when n is a power of prime number only. This suggests the idea that the behavior of the objects under the study strongly depend on the arithmetic properties of the number n. For $n = 1, 2, 3$, there is a unique, up to natural symmetries, orthogonal pair. For $n = 5$, there are three of them [8], [11], while, for $n = 4$ (the first non-prime integer), there is a one dimensional family of pairs parameterized by a rational curve.

The first positive integer which is not a power of prime is $n = 6$. Winnie-the-Pooh conjecture is open even for this case. Researchers in the quantum information theory have independently come to the unitary version of the Winnie-the-Pooh conjecture, which claims non-existence of $n + 1$ mutually unbiased bases in the n-dimensional

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complex space [7] when n is not a power of prime. The case $n = 6$ is the subject of problem number 13 in the popular list of problems in Quantum Information Theory [12].

In this paper we construct a 4-dimensional family of orthogonal pairs in Lie algebra $sl(6, \mathbb{C})$. The existence of such a family was conjectured by the authors (unpublished) and independently by mathmatical physicists [13],[10]. Despite of many efforts the proof of the existence of the family was not available until now.

In [1], we interpreted orthogonal pairs and decompositions as representations of the algebra $B(\Gamma)$ for a suitable choice of graph Γ (see section 2). These algebras are so-called *homotopes* of the Poincare groupoids of graph Γ considered as a topological space. In the course of the prove of the main result of the paper, we present various relevant algebras as free products of two algebras over a third one and explore these facts for describing the moduli spaces of their representations.

The key point in the proof is, probably, section 7, where we consider the moduli spaces, X, of 6 dimensional representations of $B(\Gamma)$, where Γ is a full bipartite graph of length $(3, 3)$. We define 3 functions on X which determine a map $X \to \mathcal{U}$, where $\mathcal{U}$ is a three dimensional affine space. The advantage of this map is that the original problem of describing orthogonal pairs in $sl(6, \mathbb{C})$ can be interpreted in terms of gluing four copies of X in such a way that everything is basically done over $\mathcal{U}$. If A is a 6×6 matrix that conjugates one Cartan subalgebra in the orthogonal pair to the other one, then this is about presenting this matrix in 4 blocks of 3×3 matrices. This reduces the problem to the study of the fibres of the above map. After factorization by permutation group $S_3 \times S_3$, the fibre is actually isomorphic to (an open affine subset in) two disjoint copies of an elliptic curve.

This leads us to study of the geometry of the elliptic fibration. Namely, we study the interplay of relevant involutions acting on the elliptic fibres. This part is based on heavy use of algebraic geometry. Eventually, it allows us to show the existence of the 4-dimensional family. Note that the proof is based on formula 174 which probably needs more conceptual explanation than just verification.

In order to study the moduli space of representations of $B(\Gamma)$, we introduce more general algebra $\mathbf{Pr}(\Gamma)$. This algebra is a homotope of the path algebra of the double quiver $\mathbf{Q}_\Gamma$ constructed from graph Γ by replacing every edge of Γ by two arrows with opposite orientation. Algebra $\mathbf{Pr}(\Gamma)$ is generated by idempotents x_v labelled by the vertices of Γ. They satisfy relations which are weaker than those for algebra $B(\Gamma)$. The moduli spaces of representations for $\mathbf{Pr}(\Gamma)$ is naturally fibered over the moduli of representations for $B(\Gamma)$.

Orthogonal pairs in $sl(n)$ correspond to representations for algebra $B(\Gamma)$ where Γ is the complete bipartite graphs $\Gamma_{n,n}$. We study algebra $\mathbf{Pr}(\Gamma_{k,n})$ for complete bipartite graph $\Gamma_{k,n}$. We consider a quotient of $\mathbf{Pr}(\Gamma_{k,n})$ which we denote by $\mathbf{P}_{k,n}$. We prove that representation spaces and moduli varieties for $\mathbf{Pr}(\Gamma_{k,n})$ and $\mathbf{P}_{k,n}$ are smooth and irreducible. We calculate the dimensions of these varieties. Also, we prove that algebra $\mathbf{P}_{k,n}$ is a free product of algebras of $\mathbf{Pr}$ of smaller bipartite graphs over algebra $\mathbf{Pr}(\Gamma_{k,1})$. We show birational equivalence of representation spaces and moduli varieties for $\mathbf{Pr}(\Gamma_{k,n})$ and fibred product of representation spaces and moduli varieties for $\mathbf{Pr}$'s of smaller bipartite graphs over moduli variety of $\mathbf{Pr}(\Gamma_{k,1})$.

Then we consider algebras $B_{k,n}$ which are similar quotients of $B(\Gamma_{k,n})$. We use results on algebra $\mathbf{Pr}(\Gamma_{k,n})$ to get similar results for $B(\Gamma_{k,n})$. Analogously, We prove that $B_{n,n}$ and $B_{k,n}$ are free products of B's of smaller bipartite graphs over some algebras $\mathcal{A}_n$ and $\widetilde{\mathcal{A}}_k$, respectively. We get a birational equivalence of representation spaces and moduli varieties for $B(\Gamma_{k,n})$ and fibred product similar to the case of $\mathbf{Pr}$.

We construct Morita equivalence of the algebra $\mathcal{A}_n$ awith the deformed preprojective algebra for arbitrary quiver. The deformed preprojective algebras are intensively studied by many authors (cf. [5], [3]). Using a result of Crawley-Boewey [4], we check the required conditions for representation space and moduli space for $B_{n,n}$. Thus, we get an important fact, a birational equivalence of the moduli space for $B_{n,n}$ and the fibred product of B's of smaller bipartite graphs over moduli variety of $\mathcal{A}_n$. Moreover, using properties of flat morphisms, we get a similar birational equivalence for $B_{k,n}$.

At the end of the paper, we construct a birational immersion of the moduli space for $\mathcal{A}_n$ into the fibred product of $\widetilde{\mathcal{A}}_k$. Together with some other technical results this allows us to finish the proof.

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2 Basic definitions and preliminary results.

Firstly, in this section we formulate definitions of orthogonal pair in Lie algebra, generalized hadamard matrices, their connection. Also, we remind the famous Winnie-the-Pooh conjecture formulated by Kostrikin et all in []. In the second subsection we recall the notion of algebraic unbiasedness and mutually unbiased basis in Hermitian space. In the third subsection we formulate the definition of reduced Temperley-Lieb algebra of graph. Also, we light the connection between orthogonal pairs (decompositions) and representations of temperley-Lieb algebra of arbitrary graph. In the fourth subsection we introduce the algebra $\mathbf{Pr}(\Gamma)$ the generalization of $B(\Gamma)$. Also, we formulate some trivial properties of $\mathbf{Pr}(\Gamma)$. In the last subsection we recall the notion of representation space and moduli variety and note some properties of these objects. For fixed algebra C, we will consider moduli varieties of two types: representation space, i.e. space of all algebraic homomorphisms from C to $M_n(F)$, and moduli variety, i.e quotient of representation space by natural $\mathrm{PGL}_n(F)$ - action.

2.1 Orthogonal Cartan subalgebras and generalized Hadamard matrices.

Consider a simple Lie algebra L over an algebraically closed field F of characteristic zero. Let K be the Killing form on L. In 1960, J.G.Thompson, in course of constructing integer quadratic lattices with interesting properties, introduced the following definitions.

Definition. Two Cartan subalgebras H_1 and H_2 in L are said to be *orthogonal* if $K(h_1, h_2) = 0$ for all $h_1 \in H_1, h_2 \in H_2$.

There is the classification problem for pair of Cartan subalgebras in $sl(n)$. Reader could find some previous results about this problem and closely related problem of classification of generalized Hadamard matrices later.

Definition. Decomposition of L into the direct sum of Cartan subalgebras $L = \oplus_{i=1}^{h+1} H_i$ is said to be *orthogonal* if H_i is orthogonal to H_j, for all $i \neq j$.

We will study pair of orthogonal Cartan subalgebras of $sl(n)$ and orthogonal decompositions of $sl(n)$ up to action of $GL_n(F)$ by conjugation.

Intensive study of orthogonal decompositions has been undertaken since then (see the book **??** and references therein). For Lie algebra $sl(n)$, A.I. Kostrikin and co-authors **??** arrived to the following conjecture, called *Winnie-the-Pooh Conjecture* (cf. *ibid.* where, in particular, the name of the conjecture is explained by a wordplay in the Milne's book in Russian translation).

Conjecture 1. *Lie algebra $sl(n)$ has an orthogonal decomposition if and only if $n = p^m$, for a prime number p.*

The conjecture has proved to be notoriously difficult. Even the non-existence of an orthogonal decomposition for $sl(6)$, when $n = 6$ is the first number which is not a prime power is still open.

Further, let us recall the connection between orthogonal pairs in $sl(n)$ and generalized Hadamard matrices of order n. Firstly, remind the definition of generalized Hadamard matrices. Let $\mathcal{N}$ be the set of $n \times n$ matrices with non-zero entries. A matrix $A = \{a_{ij}\}$ from $\mathcal{N}$ is said to be a *generalized Hadamard* matrix if

$$\sum_{j=1}^{n} \frac{a_{ij}}{a_{kj}} = 0. \tag{1}$$

for all $i \neq k$.

This condition can be recast by means of *Hadamard involution* $h : \mathcal{N} \to \mathcal{N}$ defined by

$$h : a_{ij} \mapsto \frac{1}{n a_{ji}}. \tag{2}$$

Proposition 2. *A is a generalized Hadamard matrix if and only if A is invertible and $h(A) = A^{-1}$.*

Proof. Indeed, (1) is equivalent to $A \cdot h(A) = 1$. □

Remark. Sometimes, generalized Hadamard matrices are named as *Type-$\|$ matrices* (cf. [?]) or *orthogonal-inverse matrices* (cf. []).

For any two Cartan subalgebras in a simple Lie algebra, one is known to be always a conjugate of the other by an automorphism of the Lie algebra. For the case of $sl(n)$, Cartan subalgebras are conjugate by an element of $GL_n(F)$, i.e. if (H, H') is pair of Cartan subalgebras, then $H' = AHA^{-1}$, for $A \in GL_n(F)$. The transition matrix A is uniquely defined when we fix basis $\{e_i\}$ and $\{f_i\}$ such that H consists of diagonal matrices in the first basis and H' does in the second basis. The freedom of choice for one basis is given by the normalizer in $GL_n(F)$ of one Cartan subalgebra, i.e. the group of monomial matrices. Therefore, the transition matrix A is defined up to transformations

$$A' = M_1 A M_2, \tag{3}$$

where M_1 and M_2 are invertible monomial matrices.

Proposition 3. *[?] Two Cartan subalgebras H and AHA^{-1} form an orthogonal pair of Cartan subalgebras in $sl(n)$ if and only if A is a generalized Hadamard matrix.*

2.2 Algebraic unbiasedness and mutually unbiased bases and configurations of lines in a Hermitian space

In this subsection, we will remind the notion of algebraic unbiasedness, mutually unbiased bases and complex Hadamard matrices.

Two minimal (i.e. rank 1) projectors p and q in V are said to be *algebraically unbiased* if

$$tr(pq) = \frac{1}{n} \tag{4}$$

Equivalently, this reads as one of the two (equivalent) algebraic relations:

$$pqp = \frac{1}{n}p, \tag{5}$$

$$qpq = \frac{1}{n}q. \tag{6}$$

We will also consider *orthogonal* projectors. Orthogonality of p and q is algebraically expressed as

$$pq = qp = 0 \tag{7}$$

Two maximal (i.e. of cardinality n) sets of minimal orthogonal projectors $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ are said to be *algebraically unbiased* if p_i and q_j are algebraically unbiased for all pairs (i, j).

Let $sl(V)$ be the Lie algebra of traceless operators in V. Killing form is given by the trace of product of operators. A Cartan subalgebra H in V defines a unique maximal set of minimal orthogonal projectors in V. Indeed, H can be extended to the Cartan subalgebra H' in $gl(V)$ spanned by H and the identity operator E. Rank 1 projectors in H' are pairwise orthogonal and comprise the required set. We say that these projectors are *associated* to H.

If p is a minimal projector in H', then trace of p is 1, hence, $p - \frac{1}{n}E$ is in H. If projectors p and q are associated to orthogonal Cartan subalgebras, then

$$\mathrm{Tr}(p - \frac{1}{n}E)(q - \frac{1}{n}E) = 0,$$

which is equivalent to p and q to be algebraically unbiased.

Therefore, an orthogonal pair of Cartan subalgebras is in one-to-one correspondence with two algebraically unbiased maximal sets of minimal orthogonal projectors. Similarly, orthogonal decompositions of $sl(n)$ correspond to $n+1$ of pairwise algebraically unbiased sets of minimal orthogonal projectors. This will lead us to the representation theory of reduced Temperley-Lieb algebras which we study in the next section.

More explicitly, algebraic unbiasedness can be expressed as follows. Let projectors p and q be given as

$$p = e \otimes x, \quad q = f \otimes y,$$

where e and f are in V and x and y are in V^*. The equations $p^2 = p$ and $q^2 = q$ imply:

$$(e, x) = 1, \quad (f, y) = 1, \tag{8}$$

where $(-, -)$ stands for the pairing between vectors and covectors. Then the algebraic unbiasedness of p and q reads:

$$(x, f)(y, e) = \frac{1}{n}. \tag{9}$$

Orthogonality conditions (7) reads:

$$(x, f) = 0, \quad (y, e) = 0. \tag{10}$$

The terminology of unbiased bases first appeared in physics. It is a unitary version of the algebraic unbiasedness introduced above.

Let V be an n dimensional complex space with a fixed Hermitian metric $\langle \ , \ \rangle$. Two orthonormal Hermitian bases $\{e_i\}$ and $\{f_j\}$ in V are *mutually unbiased* if, for all (i, j),

$$|\langle e_i, f_j\rangle|^2 = \frac{1}{n} \tag{11}$$

Consider the orthogonal projectors p_i and q_j, corresponding to these bases, defined by:

$$p_i(-) = e_i \otimes \langle -, e_i\rangle, \quad q_j(-) = f_j \otimes \langle -, f_j\rangle.$$

Then, the condition (9) is satisfied for them, hence they are algebraically unbiased. Note that these operators are rank 1 Hermitian projectors, and, being such, are defined by non-zero vectors in their images. We say that two rank 1 projectors are *unbiased* if they are algebraically unbiased Hermitian projectors.

We can regard algebraic unbiasedness as the complexification of unbiasedness. ???? Fix a Hermitian form on V. The Hermitian involution gives a new duality on the set of algebraic configurations:

$$p_i \mapsto p_i^\dagger \tag{12}$$

As we know(??), the duality induces an anti-holomorphic involution on the variety of algebraically unbiased minimal projectors.

Since mutually unbiased bases are algebraically unbiased, they are related to orthogonal Cartan subalgebras in $sl(n)$. Given m pairwise mutually unbiased bases $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m$ in a Hermitian space V, we obtain m Cartan subalgebras $H_1, H_2, ..., H_m$ in $sl(n)$ which are pairwise orthogonal with respect to the Killing form. In particular, a collection of $n + 1$ mutually unbiased bases in a Hermitian vector space of dimension n gives rise to an orthogonal decomposition of $sl(n)$. This fact was noticed by P.Oscar Boykin, Pham Huu Tiep, Meera Sitharam and Pawel Wocjan in [2].

Let $\mathcal{B}$ an orthonormal basis in $\mathbb{C}^n$. Matrix $A = (a_{ij})$ is said to be *complex Hadamard* if bases $\mathcal{B}$ and $A(\mathcal{B})$ are mutually unbiased. Let A and C be a complex Hadamard matrices. We will say that A is equivalent to C if $A = M_1CM_2$ for some unitary monomial matrices M_1, M_2.

There exists the following relation between complex Hadamard matrices and generalized Hadamard ones: A is a complex Hadamard if and only if A is a generalized Hadamard and $|a_{ij}| = 1$. As we know, there is an anti-holomorphic involution on the variety of generalized Hadamard matrices. Fixed points of this involution is a variety of complex Hadamard matrices. Therefore, if we construct d-dimensional complex algebraic variety of generalized Hadamard matrices, then we get d-dimensional real variety of complex Hadamard matrices.

2.3 Reduced Temperley-Lieb algebra of graph $B_r(\Gamma)$, orthogonal pairs and decompositions in $sl(n)$.

The above discussion of the problem on orthogonal decompositions and algebraically unbiased projectors motivates the study of representation theory for algebras $B(\Gamma)$, which we introduce here. Under some specialization of parameters, these algebras become quotients of more familiar Temperley-Lieb algebras of graphs. The latter are, in their turn, quotients of Hecke algebras of graphs.

Let Γ be a connected simply laced graph with no loop (i.e. no edge with coinciding ends). Denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of the graph. Let $\mathbb{K}$ be a commutative ring $\mathbb{K} = F[r, r^{-1}]$, where F is a field of characteristic zero. We define reduced Temperley-Lieb algebra $B(\Gamma)$ as a unital algebra over Generators x_i of $B(\Gamma)$, except for 1, are numbered by all vertices i of Γ. They subject relations:

- $x_i^2 = x_i$, for every i in $V(\Gamma)$,
- $x_ix_jx_i = rx_i$, $x_jx_ix_j = rx_j$, if i and j are adjacent in Γ,
- $x_ix_j = x_jx_i = 0$, if there is no edge connecting i and j in Γ.

For fixed $r \in F^*$, we will use the notation $B_r(\Gamma)$. Clearly, any automorphism of graph Γ induces automorphism of algebra $B(\Gamma)$. It can be shown that algebra $B(\Gamma)$ is a quotient of Temperley-Lieb algebra $TL(\Gamma)$ of graph Γ (cf.??).

Fix $r_{ij} \in F^*$ for any non-oriented edge (ij). Denote by $\mathbf{r}$ the collection of all r_{ij}. Consider algebra $B_{\mathbf{r}}(\Gamma)$. Namely, for fixed $r_{ij} \in F^*$, let us define the algebra $B_{\mathbf{r}}(\Gamma)$ as unital algebra with generators x_v labeled by vertices of Γ with relations:

- $x_i^2 = x_i$ for every $i \in V(\Gamma)$,
- $x_ix_jx_i = r_{ij}x_i, x_jx_ix_j = r_{ij}x_j$ for adjacent vertices i, j,
- $x_ix_j = x_jx_i = 0$ for non-adjacent vertices i, j.

It is clear that if r_{ij} are the same for all edges ij and is r then $B_{\mathbf{r}}(\Gamma) = B_r(\Gamma)$.

Using relations and connectedness of graph Γ, we get that ranks of generators x_i under any representation are the same. We will say that representation ϱ of $B_r(\Gamma)$ has rank d iff rank of some (and hence all) x_i is d. We will study non-trivial representations of algebra $B(\Gamma)$ (i.e. representations of positive rank).

It is easy that group $Aut(\Gamma)$ acts on the variety of representations of $B(\Gamma)$. Denote by $\Gamma_m(n)$ the graph with m rows by n vertices in each row. Two vertices are adjacent iff they are in different rows. It is clear that automorphism group of $\Gamma_m(n)$ is the wreath product of symmetric groups $S_n \wr S_m$. Also, we will consider direct product of symmetric groups $S_n^{\times m}$ acting by permutations of vertices lying in the same rows. Thus, we can formulate the following theorem:

Theorem 4.

- *Non-ordered set of m orthogonal Cartan subalgebras $H_0,\dots,H_{m-1}$ of $sl(n)$ are in bijective correspondence with $S_n \wr S_m$-orbits of n-dimensional representations of the algebra $B_{\frac{1}{n}}(\Gamma_m(n))$.*
- *Ordered set of m orthogonal Cartan subalgebras $H_0,\dots,H_{m-1}$ of $sl(n)$ are in bijective correspondence with $S_n^{\times m}$-orbits of n-dimensional representations of the algebra $B_{\frac{1}{n}}(\Gamma_m(n))$. We have analogous statement for $GL_n(F)$-quotients:*
- *$GL_n(F)$-orbits of non-ordered set of m orthogonal Cartan subalgebras $H_0,\dots,H_{m-1}$ of $sl(n)$ are in bijective correspondence with $S_n \wr S_m$-orbits of n-dimensional modules of the algebra $B_{\frac{1}{n}}(\Gamma_m(n))$.*
- *$GL_n(F)$-orbits of ordered set of m orthogonal Cartan subalgebras $H_0,\dots,H_{m-1}$ of $sl(n)$ are in bijective correspondence with $S_n^{\times m}$-orbits of n-dimensional modules of the algebra $B_{\frac{1}{n}}(\Gamma_m(n))$.*

Proof. Let us show that n-dimensional representation of $B_{\frac{1}{n}}(\Gamma_m(n))$ has rank 1. Actually, we have m sets of n orthogonal projectors of the same rank. Thus, these projectors has rank 1. It is easy that n-dimensional representation of $B_{\frac{1}{n}}(\Gamma_m(n))$ defines m sets of pairwise algebraically unbiased sets of minimal projectors. Straightforward check proves the theorem. ☐

2.4 Standard orthogonal pair in $sl(n)$, Heisenberg relation and deformation.

In this subsection we give some examples of orthogonal pairs in $sl(n)$ related to Heisenberg group and its deformations.

It is well-known that Cartan subalgebra H of Lie algebra $sl(n)$ has basis $X, ..., X^{n-1}$, where X satisfy to relations: $X^n = 1$ and $\mathrm{Tr}X^i = 0, i = 1, ..., n-1$. Adding identity element, we get associative commutative algebra $\widehat{H}$ with basis $1, X, ..., X^{n-1}$. Consider pair of Cartan subalgebras (H_0, H_1). As we know, there are bases $1, X, ..., X^{n-1}; X^n = 1, \mathrm{Tr}X^i = 0, i = 1, ..., n-1$ and $1, Y, ..., Y^{n-1}; Y^n = 1, \mathrm{Tr}Y^j = 0, j = 1, ..., n-1$ of associative subalgebras $\widehat{H_0}$ and $\widehat{H_1}$ respectively. We will say that pair of Cartan subalgebras (H_0, H_1) is *standard* (cf.??) iff X, Y satisfy to Heisenberg relation:

$$XY = \epsilon YX, \tag{13}$$

where ϵ is a primitive root of 1 of degree n. It is well-known that standard pair (H_0, H_1) is orthogonal. Actually, if X, Y such that $XY = \epsilon YX$ then $\mathrm{Tr}(XY) = \epsilon\mathrm{Tr}(YX) = \epsilon\mathrm{Tr}(XY)$. Hence, $\mathrm{Tr}(XY) = 0$. Analogously, one can prove that $\mathrm{Tr}(X^iY^j) = 0$ for $i, j = 1, ..., n-1$.

As we know, any two Cartan subalgebras are conjugate. Consider standard pair (H_0, H_1). Let H_0 be a subalgebra of diagonal matrices. Choose generator X as diagonal matrix of type: $\mathrm{diag}(1, \epsilon, ..., \epsilon^{n-1})$. Thus, $Y = AXA^{-1}$ for some matrix A. It can be shown in usual way that matrix A (up to permutation of rows and columns) has the following view:

$$A = (a_{ij} = \epsilon^{(i-1)(j-1)})_{i,j=1,...,n} = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & \epsilon & ... & \epsilon^{n-1} \\ ... & ... & ... & ... \\ 1 & \epsilon^{n-1} & ... & \epsilon \end{pmatrix} \tag{14}$$

It is well-known that matrix X and Y define the irreducible representation of Heisenberg group, which is a central extension of $\mathbb{Z}_n \oplus \mathbb{Z}_n$ by $\mathbb{Z}_n$.

Assume that $n = km$. Consider deformation of Heisenberg relation of the following type:

$$X^kY = \epsilon^kYX^k, XY^m = \epsilon^mY^mX. \tag{15}$$

Proposition 5. *Consider Cartan subalgebras* $H_0 =< X, ..., X^{n-1} >_F$, $H_1 =< Y, ..., Y^{n-1} >_F$, *where* $X^n = Y^n = 1$, $\mathrm{Tr}X^i = \mathrm{Tr}Y^i = 0, i = 1, ..., n-1$ *and* X, Y *satisfy to relation (15). Then pair of Cartan subalgebras* (H_0, H_1) *is orthogonal.*

Proof. We have to prove that $\mathrm{Tr}X^iY^j = 0$ for $i, j = 1, ..., n-1$. Consider the case: $j = am, a = 1, ..., k-1$. Applying relation (15), we obtain the following identity: $X^iY^{am} = \epsilon^{am}X^{i-1}Y^{am}X$. $\mathrm{Tr}(X^iY^{am}) = \epsilon^{am}\mathrm{Tr}(X^{i-1}Y^{am}X) = \epsilon^{am}\mathrm{Tr}(X^iY^{am})$. Thus, $\mathrm{Tr}(X^iY^{am}) = 0$ for any $i = 1, ..., n-1$. Further, consider the case $j \neq am$. $X^iY^j = X^{i-k}X^kY^j = \epsilon^{kj}X^{i-k}Y^jX^k$. Because of $j \neq am$, we get that $kj \neq 0(\mathrm{modn})$. Hence, $\mathrm{Tr}(X^iY^j) = \epsilon^{kj}\mathrm{Tr}(X^{i-k}Y^jX^k) = \epsilon^{kj}\mathrm{Tr}(X^iY^j)$. Therefore, we get that $\mathrm{Tr}(X^iY^j) = 0$ for all $i, j = 1, ..., n-1$. □

Orthogonal pair of Cartan subalgebras (H_0, H_1) of $sl(n)$ is said to be *(k,m)- weak standard* if there are bases $X^i, i = 1, ..., n-1$ of H_0 and $Y^j, j = 1, ..., n-1$ satisfying to relation (15).

For studying of (k,m) - weak standard orthogonal pairs, we will introduce the group $\widehat{G}$ and its quotient G. Denote by $[k, m]$ and (k, m) the l.c.m. and g.c.d of k and m respectively. Consider group $\widehat{G}$ with generators x, y, t and defining relations: $x^n = y^n = t^{[k,m]} = 1, x^ky = t^{\frac{k}{(k,m)}}yx^k, xy^m = t^{\frac{m}{(k,m)}}y^mx, xt = tx, yt = ty$. It is evident that group $\widehat{G}$ is a central extension:

$$0 \longrightarrow \mathbb{Z}_{[k,m]} \longrightarrow \widehat{G} \longrightarrow G_1 \longrightarrow 1, \tag{16}$$

where G_1 is generated by x, y satisfying to relations: $x^n = y^n = 1, x^ky = yx^k, xy^m = y^mx$. It is easy that element x^k and y^m are in the center of G_1. Thus, G_1 is a central extension:

$$0 \longrightarrow \mathbb{Z}_m \oplus \mathbb{Z}_k \longrightarrow G_1 \longrightarrow \mathbb{Z}_m * \mathbb{Z}_k \longrightarrow 1, \tag{17}$$

where $\mathbb{Z}_m * \mathbb{Z}_k$ is a free product of cyclic groups. Denote by a, b the generators of $\mathbb{Z}_m$ and $\mathbb{Z}_k$ respectively. Also, we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k \longrightarrow \widehat{G} \longrightarrow \mathbb{Z}_m * \mathbb{Z}_k \longrightarrow 1. \tag{18}$$

Consider natural morphism: $\mathbb{Z}_m * \mathbb{Z}_k \to \mathbb{Z}_m \oplus \mathbb{Z}_k$. As we know (see [**?**]), kernel of this morphism is a free group F of rank $(m-1)(k-1)$ with generators $a^i b^j a^{-i} b^{-j}, i = 1, ..., m-1, j = 1, ..., k-1$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k & \longrightarrow & H_1 & \longrightarrow & F & \longrightarrow & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k & \longrightarrow & \widehat{G} & \longrightarrow & \mathbb{Z}_m * \mathbb{Z}_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_m \oplus \mathbb{Z}_k & \xrightarrow{=} & \mathbb{Z}_m \oplus \mathbb{Z}_k & \longrightarrow & 0
\end{array} \tag{19}$$

H_1 is a subgroup of G generated by $x^k, y^m, t, x^i y^j x^{-i} y^{-j}, i = 1, ..., m-1, j = 1, ..., k-1$. Since F is a free group, upper sequence is split. And hence, H_1 is a semidirect product. Since x^k, y^m are central, we obtain that $x^i y^j x^{-i} y^{-j} \cdot x^k = x^k \cdot x^i y^j x^{-i} y^{-j}$ and $x^i y^j x^{-i} y^{-j} \cdot y^m = y^m \cdot x^i y^j x^{-i} y^{-j}$, i.e. action of F on $\mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k$ is trivial. Thus, H_1 is a direct product $F \times \mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k$.

Further, consider n-dimensional representation ρ of $\widehat{G}$ corresponding to orthogonal pair. In this case, we have $\rho(t) = \epsilon^{(k,m)} \cdot 1, \mathrm{Tr}\rho(x^i) = 0, i = 1, ..., m-1$, $\mathrm{Tr}\rho(y^j) = 0, j = 0, ..., m-1$ and $\mathrm{Tr}\rho(x^i y^j) = 0$ for $i, j = 1, ..., m-1$. Let us restrict ρ to subgroup $\mathbb{Z}_m \oplus \mathbb{Z}_k$ generated by x^k and y^m. Using vanishing of the traces, we get that this restriction is a regular representation of $\mathbb{Z}_m \oplus \mathbb{Z}_k$ and $\rho(F(\mathbb{Z}_m \oplus \mathbb{Z}_k))$ is a n-dimensional commutative diagonalizable subalgebra, i.e. there is a basis in which matrices from $\rho(F(\mathbb{Z}_m \oplus \mathbb{Z}_k))$ are diagonal. As we know elements $x^i y^j x^{-i} y^{-j}, i = 1, ..., m-1, j = 1, ..., k-1$ commute with x^k and y^m, then one can show that $\rho(x^i y^j x^{-i} y^{-j})$ are commuting matrices. Let us consider the quotient of $\widehat{G}$ by commutativity relation of $x^i y^j x^{-i} y^{-j}, i = 1, ..., m-1, j = 1, ..., k-1$. Denote this quotient by G. We have the following exact sequence for G:

$$0 \longrightarrow H = \mathbb{Z}^{\oplus(k-1)(m-1)} \oplus \mathbb{Z}_{[k,m]} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_k \longrightarrow G \longrightarrow G_2 = \mathbb{Z}_m \oplus \mathbb{Z}_k \longrightarrow 0 \tag{20}$$

One can construct representations of group G as follows: fix one-dimensional representation (or character) χ of H with condition $\chi : t \mapsto \epsilon^{(k,m)}$. By proposition 5, we get that $F[G]$ - module $F[G] \otimes_{F[H]} \chi$ defines weak standard orthogonal pair. Variety of characters of H is $\mathrm{Hom}(H, F^*) = (F^*)^{(k-1)(m-1)} \oplus \mathrm{Hom}(\mathbb{Z}_k, F^*) \oplus \mathrm{Hom}(\mathbb{Z}_m, F^*)$. There is an action of $G_2 = \mathbb{Z}_m \oplus \mathbb{Z}_k$ on H (and hence on $\mathrm{Hom}(H, F^*)$). It is clear that orthogonal pairs are equivalent iff corresponding characters of H are in the same orbit of G_2 - orbit. It can be shown in usual way that

Proposition 6. *(k,m) - Weak standard orthogonal pairs in $sl(n)$ are parameterized by algebraic torus $T(k, m) = (F^*)^{(k-1)(m-1)}$.*

Further, consider (m,k) - weak standard orthogonal pairs. The same arguments show us that (m,k) - weak standard orthogonal pairs are parameterized by torus $T(m, k) = (F^*)^{(k-1)(m-1)}$.

Proposition 7. *If $(k, m) = 1$, then intersection of two tori $T(k, m) \cap T(m, k)$ in $X(n, n)$ is a standard pair.*

Proof. Consider the relations: $X^k Y = \epsilon^k Y X^k$, $X^m Y = \epsilon^m Y X^m$. Because of $(k, m) = 1$, there are $a, b \in \mathbb{Z}$ such that $ak + bm = 1$. Thus, $XY = X^{ak+bm} Y = \epsilon^{ak+bm} Y X^{ak+bm} = \epsilon Y X$. □

Let us consider the case $n = 6 = 2 \cdot 3$. In this case relation (15) has the following view:

$$X^2 Y = \epsilon^2 Y X^2, XY^3 = -Y^3 X, \tag{21}$$

where ϵ is a primitive root of unity of degree 6. As we know, X and Y are parameterized by two-dimensional algebraic torus $T(2,3)$. Find generalized hadamard matrices parameterized by this torus. One can show in usual way that these matrices $A(a,b), a,b \in F^*$ (up to permutation of columns and rows) have the following type:

$$A(a,b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a\epsilon & b\epsilon^2 & -1 & a\epsilon^4 & b\epsilon^5 \\ 1 & \epsilon^2 & \epsilon^4 & 1 & \epsilon^2 & \epsilon^4 \\ 1 & -a & b & -1 & a & -b \\ 1 & \epsilon^4 & \epsilon^2 & 1 & \epsilon^4 & \epsilon^2 \\ 1 & a\epsilon^5 & b\epsilon^4 & -1 & a\epsilon^2 & b\epsilon \end{pmatrix}, \tag{22}$$

where $a, b \in F^*$. It is easy that generalized Hadamard matrices corresponding to $T(3,2)$ are $A^t(a,b)$ (up to permutation of columns and rows).

Remark. This example will play important role in the proof of main result of this paper.

3 n-dimensional representations of $B_{\mathbf{r}}(\Gamma_{k,n})$ and fibred products.

Let us introduce the notions of variety of representations and moduli variety of algebra A. Variety of representations of A is affine variety $\mathrm{Hom}_{alg}(A, \mathrm{M}_n(F))$. We will denote this variety by $\mathbf{Rep}_n(A)$. It is easy that there is a well-defined action of group $\mathrm{GL}_n(F)$ on $\mathbf{Rep}_n(A)$. It is well-known that there is algebraic quotient $\mathbf{Rep}_n(A)/\mathrm{GL}_n(F)$. This quotient is called moduli variety of A. We will denote moduli variety by $\mathcal{M}_n(A)$.

In this section we consider n-dimensional representations of reduced Temperley-Lieb algebra $B_{\mathbf{r}}(\Gamma_{k,n})$ for complete bipartite graph $\Gamma_{k,n}$. Firstly, we will introduce an algebra $\mathbf{Pr}(\Gamma)$. This algebra is a natural generalization of $B_{\mathbf{r}}(\Gamma)$. Further, we will prove that these representations are representations of the natural quotient $B_{k,n}$. For this purpose, we will introduce algebra $\widetilde{\mathcal{A}}_k(r_i)$. We will prove that $B_{k,n}$ is a free product of $B_{\mathbf{r}}(\Gamma_{k,m})$ and $B_{\mathbf{r}}(\Gamma_{k,n-m})$ over $\widetilde{\mathcal{A}}_k(r_i)$. Using these arguments, we deduce that variety $\mathbf{Rep}_n B_{k,n}$ is a fibred product of $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,m})$ and $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,n-m})$ over $\mathbf{Rep}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$.

After that we will study some basic relation between $\mathcal{M}_n B_{k,n}$ and fibred product of $\mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})$ and $\mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})$ over $\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$.

3.1 Algebras $\mathbf{Pr}(\Gamma)$, $B_{\mathbf{r}}(\Gamma)$ and its relations to path algebras.

In this subsection we will introduce the algebra $\mathbf{Pr}(\Gamma)$ and will study its connection with $B_{\mathbf{r}}(\Gamma)$. Algebra $\mathbf{Pr}(\Gamma)$ is algebra over $\mathbb{K}[r_{ij}]$ with unit and generators x_v labeled by vertices of Γ with relations:

- $x_v^2 = x_v$ for every $v \in V(\Gamma)$
- $x_v x_w = x_w x_v = 0$ for non-adjacent vertices v, w.

It is clear that algebra $B_{\mathbf{r}}(\Gamma)$ is a quotient of $\mathbf{Pr}(\Gamma)$ for fixed $r_{ij} \in F^*$. Moreover, the algebras $\mathbf{Pr}(\Gamma)$ and $B_{\mathbf{r}}(\Gamma)$ are algebras with augmentation. Denote by $\mathbf{Pr}^+(\Gamma)$ and $B^+_{\mathbf{r}}(\Gamma)$ the respective ideals of augmentation.

Let us construct the double quiver $\mathbf{Q}_\Gamma$. The set of vertices of $\mathbf{Q}_\Gamma$ is the set $V(\Gamma)$. For any adjacent vertices i, j in the graph Γ, we will connect these vertices by opposite arrows a_{ij} and a_{ji}. For any path $\gamma \in \mathbf{Q}_\Gamma$, we will consider the element $x_\gamma \in \mathbf{Pr}(\Gamma)$ of form: $x_{i_1}...x_{i_k}$, where $i_1, ..., i_k$ are consecutive vertices of path γ.

Let us formulate (cf. [1]) proposition:

Proposition 8. *Algebra $\mathbf{Pr}(\Gamma)$ has F-basis of form $1, x_\gamma$ where γ runs over all pathes in $\mathbf{Q}_\Gamma$. Similarly, algebra $B_{\mathbf{r}}(\Gamma)$ has F-basis of form $1, x_\gamma$ where γ runs over homotopic classes of pathes in the graph Γ.*

Recall the construction of homotop $\widehat{A}_x$ of the algebra A by means of the element $x \in A$. Let x be the fixed element of algebra A. We will consider non-unital algebra A_x with multiplication $*_x$ defined by formula:

$$a_1 *_x a_2 = a_1 x a_2.$$

Formally adding the unit, we get the algebra $\widehat{A}_x$. We studied the properties of homotops in the article [1].

Consider the path algebras $F\mathbf{Q}_\Gamma$ and $F\Gamma$ of quiver $\mathbf{Q}_\Gamma$ and graph Γ respectively. It is clear that algebra $F\Gamma$ is a quotient of $F\mathbf{Q}_\Gamma$ by ideal generated by elements $a_{ij}a_{ji} - e_i$ for any arrows a_{ij}, a_{ji} and vertices i.

For $s_{ij} \in F^*$ such that $s_{ij}^2 = r_{ij}$, consider the elements

$$\Delta(\mathbf{Q}_\Gamma) = 1 + \sum s_{ij}a_{ij}, \quad \Delta(\Gamma) = 1 + \sum s_{ij}l_{ij}$$

sum is taken over all arrows a_{ij} of the quiver $\mathbf{Q}_\Gamma$ and all edges l_{ij} of the graph Γ. It is easy that algebras $B_{\mathbf{r}}(\Gamma)$ and $\mathbf{Pr}(\Gamma)$ are homotops of path algebras $F\Gamma$ and $F\mathbf{Q}_\Gamma$ by means of the elements $\Delta(\Gamma) \in F\Gamma$ and $\Delta(\mathbf{Q}_\Gamma) \in F\mathbf{Q}_\Gamma$ respectively. Evidently, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Pr}(\Gamma) & \xrightarrow{\psi_i} & F\mathbf{Q}_\Gamma \\ \downarrow & & \downarrow \\ B_{\mathbf{r}}(\Gamma) & \xrightarrow{\psi_i} & F\Gamma \end{array} \tag{23}$$

for $i = 1, 2$.

Also, note the following property of the algebras $B_{\mathbf{r}}(\Gamma)$ and $\mathbf{Pr}(\Gamma)$ (cf. [1]):

Proposition 9. *Homological dimension of categories* $\mathrm{B}_{\mathbf{r}}(\Gamma) - \mathrm{mod}$ *and* $\mathbf{Pr}(\Gamma) - \mathrm{mod}$ *is less or equal 2.*

3.2 Connection between representation of $\mathbf{Pr}(\Gamma)$ and $B_{\mathbf{r}}(\Gamma)$.

In this subsection we will consider representations of quiver $\mathbf{Q}_\Gamma$ and its relation to representation of $\mathbf{Pr}(\Gamma)$. As we know, representation of quiver Q with set of vertices Q_0 and set of arrows Q_1 has the following description. Denote by FQ_0 the subalgebra of FQ generated by all elements $e_v, v \in Q_0$. Denote by $\mathbf{Rep}_n Q$ and $\mathbf{Rep}_n Q_0$ varieties of n-dimensional representation of FQ and FQ_0 respectively. Algebra FQ_0 is isomorphic to direct sum: $\oplus_{v \in Q_0} F$. We have the surjective morphism of varieties:

$$f : \mathbf{Rep}_n Q \to \mathbf{Rep}_n Q_0 \tag{24}$$

Recall that variety $\mathbf{Rep}_n Q_0$ is the union of irreducible components. These components are parameterized by vectors $\vec{\alpha} = (\alpha_1, ..., \alpha_{|Q_0|}), \alpha_i \in \mathbb{N}_0, \sum_{i=1}^{|Q_0|} \alpha_i = n$. These vectors are called *dimension vectors*. We will denote by $\mathbf{Rep}_n Q_0[\vec{\alpha}]$ the component corresponding to dimension vector $\vec{\alpha}$. Denote by $\mathbf{Rep}_n Q[\vec{\alpha}]$ the subvariety $f^{-1}(\mathbf{Rep}_n Q_0[\vec{\alpha}])$ of $\mathbf{Rep}_n Q$. Thus, we have the following decomposition:

$$\mathbf{Rep}_n Q = \bigcup_{\vec{\alpha}} \mathbf{Rep}_n Q[\vec{\alpha}]. \tag{25}$$

Variety $\mathbf{Rep}_n Q[\vec{\alpha}]$ has the following description. Fix the representation $\varrho \in \mathbf{Rep}_n Q[\vec{\alpha}]$. Consider the space V of the representation ϱ. Space V is the direct sum $\oplus_{v \in Q_0} V_v$ of subspaces $V_v, v \in Q_0$. Elements $\varrho(e_v), v \in Q_0$ are orthogonal projectors: $\varrho(e_v) : V \to V_v$. Linear operators $\varrho(a_{ij})$ transform subspace V_j into V_i. Denote by $\mathbf{Gr}(\vec{\alpha}, V)$ the product $\prod_{v \in Q_0} \mathbf{Gr}(\alpha_v, V)$. Then variety $\mathbf{Rep}_n Q_0[\vec{\alpha}]$ is a dense open subvariety of $\mathbf{Gr}(\vec{\alpha}, V)$. The fiber of f is the product $\prod_{a_{ij} \in Q_1} \mathrm{Hom}(V_j, V_i)$.

Consider representation ρ of the algebra $\mathbf{Pr}(\Gamma)$. Denote by $\mathbf{Rep}_n \mathbf{Pr}(\Gamma)[\vec{\alpha}]$ the variety of n-dimensional $\mathbf{Pr}(\Gamma)$-representations satisfying to condition: $\mathrm{rank}\rho(x_v) = \alpha_v, v \in V(\Gamma)$. Let $|\vec{\alpha}| = \sum_{i=1}^{|V(\Gamma)|} \alpha_i$. Using morphism $\phi_i, i = 1, 2$, we have the morphisms of varieties:

$$\phi_i^* : \mathbf{Rep}_{|\vec{\alpha}|} \mathbf{Q}_\Gamma[\vec{\alpha}] \to \mathbf{Rep}_{|\vec{\alpha}|} \mathbf{Pr}(\Gamma)[\vec{\alpha}], i = 1, 2 \tag{26}$$

Similar statements for algebras $F\Gamma$ and $B_{\mathbf{r}}(\Gamma)$ are true.

We will say that representation ρ of $B_{\mathbf{r}}(\Gamma)$ is representation of rank α if $\mathrm{rank}(x_v) = \alpha$ for some vertex v. Note that, it follows from relations of $B_{\mathbf{r}}(\Gamma)$ that ranks of all x_v coincide. Denote by $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)[\alpha]$ the variety of n-dimensional $B_{\mathbf{r}}(\Gamma)$-representation of rank α.

Note some properties of morphisms: $\mathbf{Rep}_n\Gamma \to \mathbf{Rep}_n\mathbf{Q}_\Gamma[\vec{\alpha}]$ and $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)[\alpha] \to \mathbf{Rep}_n\mathbf{Pr}(\Gamma)[\vec{\alpha}]$. Using relations of algebras $F\Gamma$ and $B_{\mathbf{r}}(\Gamma)$, we get that images of $\mathbf{Rep}_n\Gamma$ and $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)$ are in the components of the $\mathbf{Rep}_n\mathbf{Q}_\Gamma[\vec{\alpha}]$ and $\mathbf{Rep}_n\mathbf{Pr}(\Gamma)[\vec{\alpha}]$ with condition: $\alpha_i = \alpha_j = \alpha$ for all $i, j \in V(\Gamma)$ respectively.

Thus, square (23) implies the commutative diagram of varieties:

$$\begin{array}{ccc} \mathbf{Rep}_n\mathbf{Pr}(\Gamma)[\vec{\alpha}] & \longleftarrow & \mathbf{Rep}_n\mathbf{Q}_\Gamma[\vec{\alpha}] \\ \uparrow & & \uparrow \\ \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)[\alpha] & \longleftarrow & \mathbf{Rep}_n\Gamma \end{array} \tag{27}$$

Further, define map from moduli varieties of algebras $\mathbf{Pr}(\Gamma)$ to affine space of dimension $|E(\Gamma)|$.

$$tr_\Gamma : \mathbf{Rep}_n\mathbf{Pr}(\Gamma)[1, ..., 1] \to F^{E(\Gamma)} \tag{28}$$

by formula:

$$tr_\Gamma : \rho \mapsto (\mathrm{Tr}\rho(x_i x_j)), (ij) \in E(\Gamma) \tag{29}$$

where ij runs over all non-oriented edges of graph Γ. Fix $r_{ij} \in F^*$ for any edge ij. Then $\mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)[1] = tr_\Gamma^{-1}(\{r_{ij}\}_{ij\in E(\Gamma)})$. It is clear tr is $\mathrm{GL_n(F)}$-equivariant map. Thus, we have the reduction:

$$\mathrm{Tr}_\Gamma : \mathcal{M}_n\mathbf{Pr}(\Gamma)[1, ..., 1] := \mathbf{Rep}_n\mathbf{Pr}(\Gamma)[1, ..., 1]/\mathrm{GL_n(F)} \to F^{E(\Gamma)} \tag{30}$$

and also, $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)[1] := \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma)[1]//\mathrm{GL_n(F)} = \mathrm{Tr}_\Gamma^{-1}(\{r_{ij}\}_{ij\in E(\Gamma)})$.

Consider complete bipartite graph $\Gamma_{k,m}$ with two rows of vertices. There are k vertices and m vertices in upper and lower rows respectively. Denote by $p_1, ..., p_k$ and $q_1, ..., q_m$ the generators of $\mathbf{Pr}(\Gamma_{k,m})$ corresponding to vertices of upper and lower rows respectively. Consider subalgebras $A_{<p_1,...,p_k>}$ and $A_{<q_1,...,q_m>}$ generated by projectors $p_1, ..., p_k$ and $q_1, ..., q_m$ respectively. It is clear that these subalgebras are $F^{\oplus k}$ and $F^{\oplus m}$ respectively. One can show that $\mathbf{Pr}(\Gamma_{k,n})$ is a free product of $F^{\oplus k}$ and $F^{\oplus m}$. Let us recall some facts about varieties of representations.

Lemma 10. *(cf. [?]) Let A_1, A_2, B be a finite-generated algebras. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{Rep}_n(A_1 *_B A_2) & \longrightarrow & \mathbf{Rep}_n(A_1) \\ \downarrow & & \downarrow \\ \mathbf{Rep}_n(A_2) & \longrightarrow & \mathbf{Rep}_n(B) \end{array} \tag{31}$$

Moreover, there is an isomorphism of representation spaces:

$$\mathbf{Rep}_n(A_1 *_B A_2) \cong \mathbf{Rep}_n(A_1) \times_{\mathbf{Rep}_n(B)} \mathbf{Rep}_n(A_2) \tag{32}$$

Corollary 11. *Consider dimension vector $\vec{\alpha} = (\alpha_1 = \mathrm{rank}p_1, ..., \alpha_k = \mathrm{rank}p_k, \alpha_{k+1} = \mathrm{rank}q_1, ..., \alpha_{k+m} = \mathrm{rank}q_m)$. Denote by $\vec{\alpha}_k = (\alpha_1, ..., \alpha_k)$ and $\vec{\alpha}_m = (\alpha_{k+1}, ..., \alpha_{k+m})$. Assume that $n \geq \sum_{i=1}^k \alpha_i$ and $n \geq \sum_{i=1}^m \alpha_{k+i}$. In this case, we have the following isomorphism of varieties:*

$$\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,m})[\vec{\alpha}] = \mathbf{Rep}_n F^{\oplus k}[\vec{\alpha}_k] \times \mathbf{Rep}_n F^{\oplus m}[\vec{\alpha}_m]. \tag{33}$$

Denote by $\mathrm{GL}_{\vec{\alpha}_k}(F)$ and $\mathrm{GL}_{\vec{\alpha}_m}(F)$ the groups $\mathrm{GL}_{\alpha_1}(F) \times ... \times \mathrm{GL}_{\alpha_k}(F) \times \mathrm{GL}_{n-\sum_{i=1}^k \alpha_i}(F)$ and $\mathrm{GL}_{\alpha_{k+1}}(F) \times ... \times \mathrm{GL}_{\alpha_{k+m}}(F) \times \mathrm{GL}_{n-\sum_{i=1}^m \alpha_{k+i}}(F)$ respectively. Then

$$\mathbf{Rep}_n F^{\oplus k}[\vec{\alpha}_k] = \mathrm{GL}_n(F)/\mathrm{GL}_{\vec{\alpha}_k}(F), \mathbf{Rep}_n F^{\oplus m}[\vec{\alpha}_m] = \mathrm{GL}_n(F)/\mathrm{GL}_{\vec{\alpha}_m}(F). \tag{34}$$

We get that $\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,m})$ *is irreducible and we have the following isomorphism of varieties:*

$$\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,m})[\vec{\alpha}] = \mathrm{GL}_n(F)/\mathrm{GL}_{\vec{\alpha}_k}(F) \times \mathrm{GL}_n(F)/\mathrm{GL}_{\vec{\alpha}_m}(F). \tag{35}$$

In particular, we have the formula for dimension of $\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,m})[\vec{\alpha}]$*:*

$$\dim_F \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,m})[\vec{\alpha}] = 2n^2 - \sum_{i=1}^{k+m} \alpha_i^2 - (n - \sum_{i=1}^{k} \alpha_i)^2 - (n - \sum_{i=1}^{m} \alpha_{k+i}^2). \tag{36}$$

3.3 Algebras $B_{k,n}$ as free products.

Denote by $p_1, ..., p_n$ and $q_1, ..., q_k, k \leq n$ the generators of the algebra $B_{\mathbf{r}}(\Gamma_{k,n})$ corresponding to vertices of lower and upper rows respectively. Let Q be an element $\sum_{i=1}^{k} q_i$.

We can formulate the following statement for algebra $B_{\mathbf{r}}(\Gamma_{k,n})$:

Proposition 12. *Any* $B_{\mathbf{r}}(\Gamma_{k,n})$*-representation of rank* s *has dimension more or equal* sn*. Assume that there exist* sn*-dimensional* $B_{\mathbf{r}}(\Gamma_{k,n})$ *- representations of rank* s*. Let* I *be an ideal of* $B_{\mathbf{r}}(\Gamma_{k,n})$ *generated by element* $\sum_{i=1}^{n} p_i - 1$*. Denote by* $B_{k,n}$ *the quotient* $B_{\mathbf{r}}(\Gamma_{k,n})/I$*. Then*

$$\sum_{j=1}^{n} r_{ij} = 1 \tag{37}$$

for any $i = 1, ..., k$*, and we have the isomorphism of varieties:*

$$\mathbf{Rep}_{sn} B_{k,n}[s] \cong \mathbf{Rep}_{sn} B_{\mathbf{r}}(\Gamma_{k,n})[s]. \tag{38}$$

Proof. Straightforward. □

It is well-known that any n-dimensional $B_{\mathbf{r}}(\Gamma_{k,n})$-representation ρ of rank 1 is an irreducible. Actually, it is easy to check that elements $\rho(p_i), i = 1, ..., n, \rho(p_i q_1 p_j), i \neq j$ form the basis of matrix algebra. It was proved by Ivanov D.N. ([**?**]).

Of course, we have the analogous statement for algebra $B_{\mathbf{r}}(\Gamma_{n,n})$.

Corollary 13. *Any* $B_{\mathbf{r}}(\Gamma_{n,n})$*-representation of rank* s *has dimension more or equal* sn*. Assume that there exist* sn*-dimensional* $B_{\mathbf{r}}(\Gamma_{n,n})$*-representations of rank* s*. Let* J *be an ideal of* $B_{\mathbf{r}}(\Gamma_{n,n})$ *generated by elements* $\sum_{i=1}^{n} p_i - 1$ *and* $\sum_{i=1}^{n} q_i - 1$*. Denote by* $B_{n,n}$ *the quotient* $B_{\mathbf{r}}(\Gamma_{k,n})/J$*. Then*

$$\sum_{j=1}^{n} r_{ij} = \sum_{i=1}^{n} r_{ij} = 1 \tag{39}$$

for any $i, j = 1, ..., n$*, and we have the isomorphism of varieties:*

$$\mathbf{Rep}_{sn} B_{n,n}[s] \cong \mathbf{Rep}_{sn} B_{\mathbf{r}}(\Gamma_{n,n})[s]. \tag{40}$$

Consider the case of non-trivial n-dimensional representations of $B_{k,n}$ and $B_{n,n}$. It can be shown in usual way that these representations has rank 1.

Fix a partition of n vertices of lower row into two complement subsets with m vertices and $n - m$ vertices respectively. With respect to this partition, we get the partition of generators $p_1, ..., p_n$ into two non-intersected subsets $p_{i_1}, ..., p_{i_m}$ and $p_{i_{m+1}}, ..., p_{i_n}$.

We have a natural morphisms: $\phi : B_{\mathbf{r}}(\Gamma_{k,m}) \to B_{\mathbf{r}}(\Gamma_{k,n}) \to B_{k,n} = B_{\mathbf{r}}(\Gamma_{k,n})/I_P$ and $\phi' : B_{\mathbf{r}}(\Gamma_{k,n-m}) \to B_{\mathbf{r}}(\Gamma_{k,n}) \to B_{k,n}$ defined by composition of embeddings of graphs and natural projection. Let $r_j, j = 1, ..., k$ be the sum $\sum_{k=1}^{m} r_{i_k j}$. Consider the subalgebra $\widetilde{\mathcal{A}}_k(r_i) := A_{\langle P; q_1, ..., q_k \rangle}$ of algebra $B_{\mathbf{r}}(\Gamma_{k,s})$ generated by elements $P = \sum_{j=1}^{m} p_{i_j}$ and $q_i, i = 1, ..., k$.

Lemma 14. *Algebra $\widetilde{\mathcal{A}}_k(r_i)$ has a defining relations: $P^2 = P, q_i^2 = q_i, q_i P q_i = r_i q_i, i = 1, ..., k$.*

Proof. As we know (cf. [1]) algebra $B_{\mathbf{r}}(\Gamma_{k,m})$ has a basis $p_{i_1} q_{j_1} \dots p_{i_s} q_{j_s}$ with $i_k \neq i_{k+1}$ and $j_k \neq j_{k+1}$ and elements that can be obtained from these products by removing the first and the last factor. Let us show that the elements of the form

$$1; q_{i_1} P q_{i_2} P \dots q_{i_s}; \quad P q_{i_2} P \dots q_{i_s}; \quad q_{i_1} P q_{i_2} P \dots P; \quad P q_{i_2} P \dots q_{i_s} P \tag{41}$$

with $i_k \neq i_{k+1}$ for all $k \leq s-1$ in all these expressions form a basis of the algebra $A = \widetilde{\mathcal{A}}_k(r_i)$.

Introduce the filtration on A by defining $F_i A$ to be the subspace spanned by all elements in the list (41) with the number of factors q_i's and P in the products to be less than or equal to i. Also, we have a filtration $F_i B(\Gamma_{k,s})$ on $B(\Gamma_{k,s})$. The basis in $F_i B(\Gamma_{k,s})$ is compatible with filtration, hence it gives a basis in $F_i B(\Gamma_{k,s}) / F_{i-1} B(\Gamma_{k,s})$ (consisting of products of projectors of length i). Clearly, $F_i A \subset F_i B(\Gamma_{k,m})$, hence we have a map:

$$\varphi : F_i A / F_{i-1} A \to F_i B(\Gamma_{k,m}) / F_{i-1} B(\Gamma_{k,m})$$

The quotient $F_i A / F_{i-1} A$ is generated by expressions in (41) of length precisely i. One can easily see that these elements are mapped into linearly independent elements in $F_i B(\Gamma_{k,m}) / F_{i-1} B(\Gamma_{k,m})$, because the image under φ of any two different such elements is a linear combinations of disjoint subsets of elements in the basis for $F_i B(\Gamma_{k,m}) / F_{i-1} B(\Gamma_{k,m})$. By induction on i (starting from $F_0 A$), we get that elements (41) are linearly independent, hence they form a basis in A. Note that the same argument also proves the strict compatibility with filtration:

$$F_i A = A \cap F_i B(\Gamma_{k,m}).$$

Thus, we get the required statement. □

Denote by i the monomorphism: $i_m : \widetilde{\mathcal{A}}_k(r_i) \to B_{\mathbf{r}}(\Gamma_{k,m})$. Let r'_i be $1 - r_i$. Consider subalgebra $A_{\langle P', q_1, \dots, q_n \rangle}(r'_i)$ of $B_{\mathbf{r}}(\Gamma_{k,n-m})$ generated by $P' = \sum_{i=m+1}^{n} p_i$ and q_i. Analogous to lemma 14, this subalgebra isomorphic to $\widetilde{\mathcal{A}}_k(r'_i)$. Further, there exists isomorphism: $\tau : \widetilde{\mathcal{A}}_k(r_i) \cong \widetilde{\mathcal{A}}_k(r'_i)$ defined by correspondence: $P \mapsto 1 - P', q_i \mapsto q_i$. Hence, we have the monomorphism: $i' : \widetilde{\mathcal{A}}_k(r_i) \to B_{\mathbf{r}}(\Gamma_{k,n-m})$ defined by formula:

$$P \mapsto 1 - P' \mapsto 1 - \sum_{i=m+1}^{n} p_i, \quad q_i \mapsto q_i.$$

One can check that the following diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_k(r_i) & \xrightarrow{\ i\ } & B_{\mathbf{r}}(\Gamma_{k,m}) \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle \phi} \\ B_{\mathbf{r}}(\Gamma_{k,n-m}) & \xrightarrow{\ \phi'\ } & B_{k,n} \end{array} \tag{42}$$

is commutative.

Proposition 15. *Consider the partition of set $p_1, ..., p_n$ into two complement subsets $p_{i_1}, ..., p_{i_m}$ and $p_{i_{m+1}}, ..., p_{i_n}$ and algebras $B_{\mathbf{r}}(\Gamma_{k,m})$ and $B_{\mathbf{r}}(\Gamma_{k,n-m})$. Algebra $B_{k,n}$ is a free product of $B_{\mathbf{r}}(\Gamma_{k,m})$ and $B_{\mathbf{r}}(\Gamma_{k,n-m})$ over $\widetilde{\mathcal{A}}_k(r_i)$.*

Proof. We have a morphism: $B_{\mathbf{r}}(\Gamma_{k,s}) *_{\widetilde{\mathcal{A}}_k(r_i)} B_{\mathbf{r}}(\Gamma_{k,n-s}) \to B_{k,n}$. This morphism is surjective, because images of $B_{\mathbf{r}}(\Gamma_{k,s})$ and $B_{\mathbf{r}}(\Gamma_{k,n-s})$ generate $B_{k,n}$. By definition of morphisms ϕ and ϕ', we get that $p_i = \phi(p_i), i = 1, ..., s; p_{i+s} = \phi'(p_i), i = s+1, ..., n; q_j = \phi(q_j) = \phi'(q_j)$.

Obviously, $p_i^2 = p_i, q_j^2 = q_j, p_i q_j p_i = r_{ij} p_i, q_j p_i q_j = r_{ij} q_j, q_i q_j = 0$ for $i \neq j$, $p_i p_j = 0$ for $i, j \in \{1, ..., s\}$ and for $i, j \in \{s+1, ..., n\}$ are relations in free product $B_{\mathbf{r}}(\Gamma_{k,s}) *_{\widetilde{\mathcal{A}}_k(r_i)} B_{\mathbf{r}}(\Gamma_{k,n-s})$.

Let us prove that $p_i p_j = 0$ for all $i \neq j$. Let $i \in \{1, ..., s\}$ and $j \in \{s+1, ..., n\}$. Evidently, $p_i = p_i P = p_i \sum_{m=1}^{s} p_m = p_i$ and $p_j = (1 - P) p_j = \sum_{m=s+1}^{n} p_m p_j = p_j$. Hence, $p_i p_j = p_i P (1 - P) p_j = 0$. Analogously, $p_j p_i = 0$. Hence, relations of algebra $B_{k,n}$ are satisfied. Using surjectivity of morphism, we obtain the required statement. □

Denote by $\mathcal{A}_n(r_i)$ the unital algebra with generators $P; q_1, ..., q_n$ and relations $P^2 = P, q_i^2 = q_i, q_iPq_i = r_iq_i, \sum_{i=1}^n q_i = 1$. Denote by I_m and I_{n-m} the ideals generated by element $\sum_{i=1}^n q_i - 1$ in algebras $B_{\mathbf{r}}(\Gamma_{n,m})$ and $B_{\mathbf{r}}(\Gamma_{n,n-m})$ respectively. Analogous to $\widetilde{\mathcal{A}}_k$, we have monomorphism: $i : \mathcal{A}_n(r_i) \to B_{\mathbf{r}}(\Gamma_{n,m})/I_m = B_{n,m}$, isomorphism $\tau : \mathcal{A}_n(r_i) \cong \mathcal{A}_n(r_i'), r_i' = 1 - r_i$ and, hence, monomorphism $i' : \mathcal{A}_n(r_i) \to B_{\mathbf{r}}(\Gamma_{n,n-m})/I_{n-m} = B_{n,n-m}$. Clearly, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_n(r_i) & \xrightarrow{i} & B_{n,m} \\ \downarrow{i'} & & \downarrow{\phi} \\ B_{n,n-m} & \xrightarrow{\phi'} & B_{n,n} \end{array} \tag{43}$$

We get the following statement for algebra $B_{n,n}$:

Corollary 16. *Consider the algebra $B_{n,n}$. Fix a partition of the set $p_1, ..., p_n$ into two complement subsets $p_{i_1}, ..., p_{i_m}$ and $p_{i_{m+1}}, ..., p_{i_n}$. Then algebra $B_{n,n}$ is a free product of algebras $B_{n,m} = B_{\mathbf{r}}(\Gamma_{n,m})/I_m$ and $B_{n,n-m} = B_{\mathbf{r}}(\Gamma_{n,n-m})/I_{n-m}$ over algebra $\mathcal{A}_n(r_i)$.*

Proof. Analogous to proof of proposition 15. □

3.4 Fibred products.

It is clear that morphisms i and i' define morphisms $i^* : \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{n,m}) \to \mathbf{Rep}_n \mathcal{A}_n(r_i)$, $i'^* : \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{n,n-m}) \to \mathbf{Rep}_n \mathcal{A}_n(r_i)$. It is easy that n-dimensional representation of $B_{\mathbf{r}}(\Gamma_{n,m})$ is a representation of rank 1. Also, note that representations of algebra $\mathcal{A}_n(r_i)$ and $\widetilde{A}_k(r_i)$ are parameterized by dimension vectors consisting of ranks of generators. It is easy that morphism i^* transforms $B_{\mathbf{r}}(\Gamma_{n,m})$-representation of rank 1 to n-dimensional representation of $\mathcal{A}_n(r_i)$ with dimension vector $(1, ..., 1, m)$, i.e. $\mathrm{rank} q_i = 1, \mathrm{rank P} = m$. Analogous, we have the similar arguments for algebra $\widetilde{\mathcal{A}}_k(r_i)$. Denote by $\mathbf{Rep}_n \mathcal{A}_n(r_i)[\vec{1}, m]$ the variety of representations of algebra $\mathcal{A}_n(r_i)$ with dimension vector $(1, ..., 1, m)$.

Using lemma 10 and proposition 12, we get the following:

Corollary 17. *We have the isomorphisms of varieties:*

- $$\mathbf{Rep}_n B_{k,n}[1] = \mathbf{Rep}_n B_{k,n} \cong \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathbf{Rep}_n \widetilde{A}_k(r_i)[\vec{1},m]} \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1], \tag{44}$$

- $$\mathbf{Rep}_n B_{n,n}[1] = \mathbf{Rep}_n B_{k,n} \cong \mathbf{Rep}_n B_{n,m} \times_{\mathbf{Rep}_n \mathcal{A}_n(r_i)[\vec{1},m]} \mathbf{Rep}_n B_{n,n-m}. \tag{45}$$

Remark. Of course, there is a trivial generalization of this fact for case of sn-dimensional representations of rank s.

Further, let us study the quotient of fibred product. Let Y_1, Y_2, Z be an affine G-varieties for some reductive algebraic group G. Assume that we have G-morphisms: $f_i : Y_i \to Z, i = 1, 2$. Thus, $Y_1 \times_Z Y_2$ is an affine G-variety too. Therefore, we can consider algebraic quotients $Y_i/G, i = 1, 2$ and Z/G, i.e. $\mathrm{Spec} F[Y_i]^G, i = 1, 2$, $\mathrm{Spec} F[Z]^G$. Also, we have an algebraic quotient $Y_1 \times_Z Y_2/G = \mathrm{Spec}(F[Y_1] \otimes_{F[Z]} F[Y_2])^G$. One can construct the following morphism: $p : (Y_1 \times_Z Y_2)/G \to Y_1/G \times_{Z/G} Y_2/G$. In this subsection we will study this natural morphism.

Denote by G_x the stabilizer of point x. Fix points $y_1 \in Y_1, y_2 \in Y_2$. It is easy that $G_{y_1} \subseteq G_{f_1(y_1)}$ and $G_{y_2} \subseteq G_{f_2(y_2)}$. Denote by Gx the orbit of point x. Note that morphism: $p : (Y_1 \times_Z Y_2)/G \to Y_1/G \times_{Z/G} Y_2/G$ defined by rule: $p : G(y_1, y_2) \mapsto (Gy_1, Gy_2)$. Denote by $\pi, \pi_i, i = 1, 2$ the natural morphisms: $\pi : Y_1 \times_Z Y_2 \to (Y_1 \times_Z Y_2)/G$, $\pi_i : Y_i \to Y_i/G, i = 1, 2$ and Also, consider subvarieties $Y_i^{(0)} = \{y_i | \pi_i^{-1}(\pi_i(y_i)) - \quad \mathit{closed} \quad \mathit{orbit}\}$. One can show that $Y_i^{(0)}/G$ is a geometric quotient. It is clear that $\pi_i(Y_i^{(0)})$ are open subvarieties of Y_i/G.

Lemma 18. *Consider point* $(Gy_1, Gy_2) \in Y_1^{(0)}/G \times_{Z/G} Y_2^{(0)}/G$ *such that* $y_i \in Y_i^{(0)}, z \in Z$*, where* $f_1(y_1) = f_2(y_2) = z$*. Assume that* $p^{-1}(Gy_1, Gy_2)$ *consists of closed orbits. Then fiber* $p^{-1}(Gy_1, Gy_2)$ *is isomorphic to variety of double classes:* $G_{y_1}\backslash G_z/G_{y_2}$*.*

Proof. One can show that $p^{-1}(Gy_1, Gy_2) = G_z\backslash(G_z/G_{y_1} \times G_z/G_{y_2})$. The rest is trivial. □

Corollary 19. *Consider a component* C *of* $Y_1 \times_Z Y_2$ *which contains a point* $(y_1, y_2), f_1(y_1) = f_2(y_2) = z$ *satisfying to condition:* $|G_{y_1}\backslash G_z/G_{y_2}| = 1$*. Then restriction of* p *to* C *is a birational morphism.*

Consider subvariety $\{y \in Y_1/G \times_{Z/G} Y_2/G | p^{-1}(y) = \emptyset\}$ of $Y_1/G \times_{Z/G} Y_2/G$. It can be shown that this subvariety can be non-empty. Actually, consider point $(y_1, y_2) \in Y_1 \times_Z Y_2$ such that $f_1(y_1) \neq f_2(y_2)$ and $\overline{f_1(y_1)^G} \cap \overline{f_2(y_2)^G} \neq \emptyset$. In this case, we get the point $(\pi_1(y_1), \pi_2(y_2)) \in Y_1/G \times_{Z/G} Y_2/G$ such that $p^{-1}(\pi_1(y_1), \pi_2(y_2)) = \emptyset$.

Let us apply these arguments to the case of $\mathbf{Rep}_n B_{n,n}$. As we know, in this case any n-dimensional non-trivial representations of algebras $B_{n,n}$, $B_{n,m}$ and $B_{n,n-m}$ of rank 1 are irreducible. Therefore, quotients $\mathbf{Rep}_n B_{n,n}$, $\mathbf{Rep}_n B_{n,m}$ and $\mathbf{Rep}_n B_{n,n-m}$ by $\mathrm{GL}_n(F)$ are geometric, i.e. all $\mathrm{GL}_n(F)$ - orbits are closed. Hence, $\mathbf{Rep}_n^{(0)} B_{n,n} = \mathbf{Rep}_n B_{n,n}$, $\mathbf{Rep}_n^{(0)} B_{n,m} = \mathbf{Rep}_n B_{n,m}$ and $\mathbf{Rep}_n^{(0)} B_{n,n-m} = \mathbf{Rep}_n B_{n,n-m}$.

Consider the variety $\mathbf{Rep}_n B_{n,n}$. We have the following morphism: $f_n : \mathcal{M}_n B_{n,n} \to \mathcal{M}_n B_{n,m} \times_{\mathcal{M}_n \mathcal{A}_n(r_i)[\vec{1},m]} \mathcal{M}_n B_{n,n-m}$. We get that

$$f_n^{-1}(\rho_1, \rho_2) = G_\psi/F^*,$$

for $\rho_1 \in \mathbf{Rep}_n B_{n,m}, \rho_2 \in \mathbf{Rep}_n B_{n,n-m}, \psi \in \mathbf{Rep}_n \mathcal{A}_n(r_i)$ such that $i^*(\rho_1) = i'^*(\rho_2) = \psi$.

We have to study representation theory of the algebras $\mathcal{A}_n(r_i)$ and $\widetilde{\mathcal{A}}_k(r_i)$ for further studying of morphisms $f_n : \mathcal{M}_n B_{n,n} \to \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{n,m}) \times_{\mathcal{M}_n \mathcal{A}_n(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{n,n-m})$ and $f_n' : \mathcal{M}_n B_{k,n} \to \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1]$.

4 Deformed preprojective algebra and algebra $\mathcal{A}_n(r_i)$.

In this section we will study algebra $\mathcal{A}_n$ and its connection with deformed preprojective algebra of some quiver. Namely, we will prove that these algebras are Morita equivalent. Using representation theory of deformed preprojective algebra, we obtain that representation and moduli variety $\mathbf{Rep}_n \mathcal{A}_n[(\vec{1}, m)]$ and $\mathcal{M}_n \mathcal{A}_n[(\vec{1}, m)]$ are irreducible for any $r_i \in F, i = 1, ..., n, \sum_{i=1}^n r_i = m$. Also, we will calculate dimensions of $\mathcal{M}_n \mathcal{A}_n(r_i)[\vec{1}, m]$. In first subsection we recall notions and facts about deformed preprojective algebra. In second subsection we will prove Morita equivalence of $\mathcal{A}_n(r_i)$ and deformed preprojective algebra $\Pi_{\vec{\lambda}}(\mathcal{Q})$ for some quiver $\mathcal{Q}$ and $\vec{\lambda} = (-r_1, ..., -r_n, 1)$.

4.1 Roots and deformed preprojective algebra.

In this subsection we will recall the main properties and notions of quiver and introduce deformed preprojective algebra. In this subsection, we will consider free-loop quivers. Although, one can generalize all notions and facts in the case of quiver with loops.

Let Q be a quiver with k vertices. Thus, $Q_0 = \{1, ..., k\}$. Assume Q has no loops. The description of quiver Q encoded by $k \times k$-matrix χ_Q:

$$(\chi_Q)_{ij} = \delta_{ij} - \#\{arrows \quad from \quad i \quad to \quad j\} \tag{46}$$

Let $\mathbb{Z}Q_0$ be a free abelian group generated by vertices. For each vector $\vec{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{Z}Q_0$ let $supp(\vec{\alpha}) = \{i \in Q_0 | \quad \alpha_i \neq 0\}$. We say that $supp(\vec{\alpha})$ is connected if the full subquiver of Q with vertex set $supp(\vec{\alpha})$ is connected.

Recall that we have Euler form $\chi_Q : \mathbb{Z}Q_0 \times \mathbb{Z}Q_0 \to \mathbb{Z}$ defined by formula:

$$\chi_Q(\vec{\alpha}, \vec{\beta}) = \vec{\alpha} \cdot \chi_Q \cdot \vec{\beta}^t, \vec{\alpha}, \vec{\beta} \in \mathbb{Z}Q_0. \tag{47}$$

Its symmetrization is called by *Tits form* T_Q, i.e. $T_Q(\vec{\alpha},\vec{\beta}) = \chi_Q(\vec{\alpha},\vec{\beta}) + \chi_Q(\vec{\beta},\vec{\alpha})$. Denote by q_Q the quadratic form: $q_Q(\alpha) = \frac{1}{2}T_Q(\vec{\alpha},\vec{\alpha}) = \chi_Q(\vec{\alpha},\vec{\alpha})$. Denote by $\vec{\epsilon_i}$ the coordinate vector corresponding to vertex $i \in Q_0$. Denote by Π the set of vectors $\epsilon_i, i \in Q_0$. The matrix $A_{ij} = T_Q(\vec{\epsilon_i},\vec{\epsilon_j})$ is a Generalized Cartan Matrix (at least when Q has no loops), and so there is an associated Kac-Moody Lie algebra. This algebra has a root system associated to it. For vertex $i \in Q_0$, there is a reflection:

$$refl_i : \mathbb{Z}Q_0 \to \mathbb{Z}Q_0, \quad refl_i(\vec{\alpha}) = \vec{\alpha} - T_Q(\vec{\alpha},\vec{\epsilon_i})\vec{\epsilon_i}. \tag{48}$$

It is clear $T_Q(\vec{\alpha},\vec{\beta}) = T_Q(refl_i(\vec{\alpha}), refl_i(\vec{\beta}))$ for any $i \in Q_0$. The Weyl group is the subgroup $W \subset Aut(\mathbb{Z}Q_0)$ generated by the $refl_i, i \in Q_0$. The set $\Phi_{re}(Q) = \bigcup_{w\in W(Q)} w(\Pi)$ is called *real roots.* It is easy $q_Q(\vec{\alpha}) = 1$.

The fundamental region is

$$F_Q = \{\vec{\alpha} \in \mathbb{N}Q_0 | \quad T_Q(\vec{\alpha},\vec{\epsilon_i}) \le 0 \quad for \quad all \quad \vec{\epsilon_i} \in \Pi \quad and \quad \vec{\alpha} \quad has \quad a \quad connected \quad support\} \tag{49}$$

The set $\Phi_{im}(Q) = \bigcup_{w\in W(Q)} w(F_Q) \cup w(-F_Q)$ is called *imaginary roots.* Clearly, $q_Q(\vec{\alpha}) \le 0$ for any imaginary root $\vec{\alpha}$. Finally, the root system of Q is defined as $\Phi(Q) = \Phi_{re}(Q) \cup \Phi_{im}(Q)$. An element $\vec{\alpha} \in \Phi(Q) \cap \mathbb{N}Q_0$ is called *positive root.* A non-zero element $\vec{\alpha} \in \mathbb{Z}Q_0$ is called *indivisible* if $\gcd(\alpha_i) = 1$. Clearly any real root is indivisible, and if $\vec{\alpha}$ is a real root, only $\pm\vec{\alpha}$ are roots. On the other hand every imaginary root is a multiple of an indivisible root, and all other nonzero multiples are also roots. Recall the connection between roots and indecomposable representations of quiver.

Theorem 20. *(Kac)*

- *If there is an indecomposable representation of Q with dimension vector $\vec{\alpha}$, then $\vec{\alpha}$ is a root.*
- *If $\vec{\alpha}$ is a positive real root there is a unique indecomposable representation with dimension vector $\vec{\alpha}$ (up to isomorphism).*
- *If $\vec{\alpha}$ is a positive imaginary root then there are infinitely many indecomposables with dimension vector $\vec{\alpha}$ (up to isomorphism).*

Further, we will define deformed preprojective algebra. For free-loop quiver Q, let us construct a *double* quiver Q^d, that is to an each arrow $a \in Q_1$ we add an opposite arrow $a^* \in Q_1^d$. Define commutator c as element $\sum_{a\in Q_1}[a, a^*] \in FQ^d$. For the weight $\vec{\lambda} = (\lambda_1, ..., \lambda_k) \in F^k$ we define deformed preprojective algebra:

$$\Pi_{\vec{\lambda}}(Q) = FQ^d/(c - \sum_{i=1}^{k} \lambda_i e_i) \tag{50}$$

Multiply all arrows by non-zero $t \in F^*$, we get the isomorphism of preprojective algebras:

$$\Pi_{\vec{\lambda}}(Q) \cong \Pi_{t\vec{\lambda}}(Q). \tag{51}$$

We know already that vector $\vec{\alpha} = (\alpha_1, ..., \alpha_k)$ is a dimension vector of $\Pi_{\vec{\lambda}}$-representation iff $\vec{\lambda}\cdot\vec{\alpha} = \sum_{i=1}^{k} \lambda_i\alpha_i = 0$.

Fix $\varrho \in \mathbf{Rep}Q^d[\vec{\alpha}]$. As we know, the space of ϱ decompose into direct sum of $V_i = \mathrm{Im}\varrho(e_i), i = 1, ..., k$ of dimension $\alpha_i, i = 1, ..., k$ respectively. Consider the algebra $\mathrm{End}[\vec{\alpha}] = \oplus\mathrm{End}(V_i)$. We can define momentum map:

$$\mu_{\vec{\alpha}} : \mathbf{Rep}Q^d[\vec{\alpha}] \to \mathrm{End}[\vec{\alpha}] \tag{52}$$

by following formula:

$$\mu_{\vec{\alpha}}(\varrho) = \sum_{a\in Q_1} \varrho(a)\varrho(a^*) - \varrho(a^*)\varrho(a) \in \mathrm{End}[\vec{\alpha}]. \tag{53}$$

Thus, $\mathbf{Rep}\Pi_{\vec{\lambda}}[\vec{\alpha}] = \mu_{\vec{\alpha}}^{-1}(\sum_{i=1}^{k} \lambda_i\varrho(e_i))$.

Recall the definition of λ-Schur roots. Let us define $p_Q(\vec{\alpha})$ as $1 - q_Q(\vec{\alpha})$. We have the following inequality:

$$p_Q(\vec{\alpha}) \ge 0 \tag{54}$$

Definition The set $S_{\vec{\lambda}}$ of λ-Schur roots is defined to be a set of $\vec{\alpha} \in \mathbb{N}^k$ such that $p_Q(\vec{\alpha}) \geq p_Q(\vec{\beta_1})+...+p_Q(\vec{\beta_r})$ for all decompositions $\vec{\alpha} = \vec{\beta_1} + ... + \vec{\beta_r}$ with $\vec{\beta_i}$ positive roots satisfying to $\vec{\lambda} \cdot \vec{\beta_i} = 0$.

We will use the following result from the representation theory of deformed preprojective algebras.

Theorem 21. *(Crawley-Boewey) Let $(\vec{\lambda}, \vec{\alpha})$ are such that $\vec{\alpha} \in S_{\vec{\lambda}}$. Then $\mathbf{Rep}_{|\vec{\alpha}|}\Pi_{\vec{\lambda}}[\vec{\alpha}]$ is a reduced and irreducible complete intersection of dimension $|\vec{\alpha}|^2 - 1 + 2p_Q(\vec{\alpha})$. And general element of $\mathbf{Rep}_{|\vec{\alpha}|}\Pi_{\vec{\lambda}}[\vec{\alpha}]$ is a simple representation. Thus, $\dim_F \mathcal{M}_{|\vec{\alpha}|}\Pi_{\vec{\lambda}}[\vec{\alpha}] = 2p_Q(\vec{\alpha})$.*

4.2 Morita-equivalence of algebra $\mathcal{A}_n(r_i)$ and deformed preprojective algebra.

In this subsection we will prove that algebras $\mathcal{A}_n$ and preprojective algebra of some quiver are Morita equivalent.

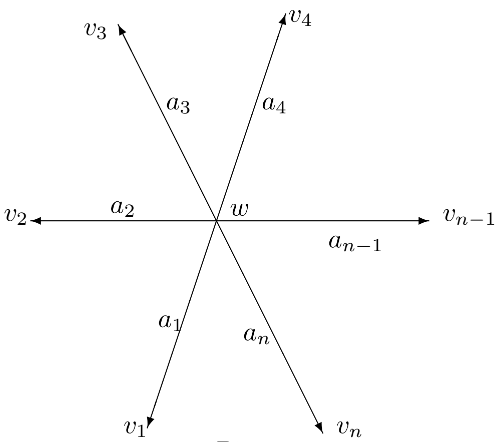
Recall the following useful Morita's theorem:

Theorem 22. *(Morita) Consider algebra A and left A-ideal I. Let I be a direct summand of free left module A. Assume, we have the identity:*

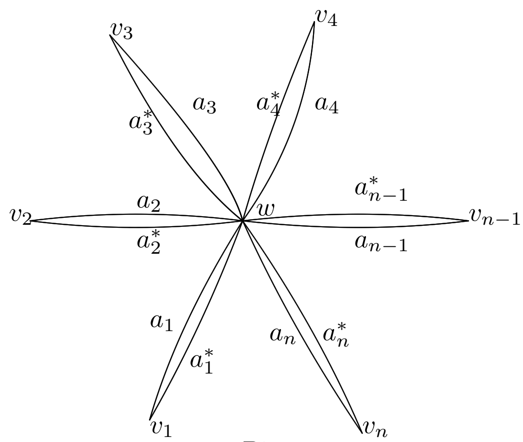
$$AIA = A. \tag{55}$$

Then algebras A and $\mathrm{End}_A(I)$ are Morita equivalent. In particular, consider the idempotent $e \in A$. If $AeA = A$, then algebras eAe and A are Morita equivalent.

Consider quiver $\mathcal{Q}$ with $n+1$ vertices, which we denote by $v_1, ..., v_n, w$. Arrows of the $\mathcal{Q}$ are a_i with source w and target v_i, i.e. $|\mathcal{Q}_1| = n$(see picture 1). Adding opposite arrows a_i^*, we get the double quiver $\mathcal{Q}^d$(see picture 2) and path algebra $F\mathcal{Q}^d$.



Picture 1



Picture 2

Let $\vec{\lambda} = (\lambda_{v_1}, ..., \lambda_{v_n}, \lambda_w)$. Thus, we can construct deformed preprojective algebra $\Pi_{\vec{\lambda}} = \Pi_{\vec{\lambda}}(\mathcal{Q})$. As we know this algebra is a quotient of $F\mathcal{Q}^d$ by ideal J generated by element

$$x = \sum_{i=1}^{n}[a_i^*, a_i] - \sum_{i=1}^{n} \lambda_{v_i} e_{v_i} - \lambda_w e_w. \tag{56}$$

Thus, algebra $\Pi_{\vec{\lambda}}$ has the following relations:

$$a_i a_i^* = -\lambda_{v_i} e_{v_i}, \sum_{i=1}^{n} a_i^* a_i = (\sum_{i=1}^{n} a_i^*)(\sum_{i=1}^{n} a_i) = \lambda_w e_w \tag{57}$$

and relations of the quiver path algebra.

Denote by E the sum $\sum_{i=1}^n e_{v_i}$. Note that $\Pi_{\vec{\lambda}} E \Pi_{\vec{\lambda}} = \Pi_{\vec{\lambda}}$. It follows immediately from relations of $\Pi_{\vec{\lambda}}$. Using this fact and Morita's theorem, we get the Morita equivalence of algebras $E\Pi_{\vec{\lambda}}E$ and $\Pi_{\vec{\lambda}}$.

Proposition 23. *Assume $\lambda_w \neq 0$. Then there is an isomorphism of algebras:*

$$E\Pi_{\vec{\lambda}}E \cong \mathcal{A}_n(r_i), \tag{58}$$

where $r_i = -\frac{\lambda_{v_i}}{\lambda_w}, i = 1, ..., n$. Thus, algebra $\mathcal{A}_n(r_i)$ is Morita-equivalent to deformed preprojective algebra $\Pi_{\vec{\lambda}}$.

Proof. Using multiplication by non-zero element, we can suppose that $\lambda_w = 1, \lambda_{v_i} = -r_i, i = 1, ..., n$. Algebra $\Pi_{\vec{\lambda}}$ is a quotient $F\mathcal{Q}^d/J$, where $J = F\mathcal{Q}^d x F\mathcal{Q}^d$. Using Morita equivalence, we get that algebra $E\Pi_{\vec{\lambda}}E$ is a quotient of $EF\mathcal{Q}^dE$ by ideal $J' = EF\mathcal{Q}^d xF\mathcal{Q}^dE$. Consider generators of $EF\mathcal{Q}^dE$ - ideal J'. It is trivial that $x = ExE + e_w x e_w$, and $e_w x e_w = (\sum_{i=1}^n a_i^*)(\sum_{i=1}^n a_i) - e_w$. Ideal J is generated by ExE and $e_w x e_w$, and hence, J' is generated by ExE and subspace $EF\mathcal{Q}^d e_w x e_w F\mathcal{Q}^d E$. Further, it is easy $EF\mathcal{Q}^d e_w = EF\mathcal{Q}^d E(\sum_{i=1}^n a_i)$ and $e_w EF\mathcal{Q}^d E = (\sum_{i=1}^n a_i^*)EF\mathcal{Q}^d E$. Thus, we get that

$$EF\mathcal{Q}^d e_w x e_w F\mathcal{Q}^d E = EF\mathcal{Q}^d E(\sum_{i=1}^n a_i)((\sum_{i=1}^n a_i^*)(\sum_{i=1}^n a_i) - e_w)(\sum_{i=1}^n a_i^*)F\mathcal{Q}^d E. \tag{59}$$

Therefore, algebra $E\Pi_{\vec{\lambda}}E$ is generated by elements $e_i, i = 1, ..., n$ and $a_i^* a_j$ for all $i, j = 1, ..., n$ with relations:

$$a_i^* a_i = r_i e_i, \quad (\sum_{i=1}^n a_i)(\sum_{i=1}^n a_i^*)(\sum_{i=1}^n a_i)(\sum_{i=1}^n a_i^*) = (\sum_{i=1}^n a_i)(\sum_{i=1}^n a_i^*)$$

Elements $e_i, i = 1, ..., n$ and $(\sum_{i=1}^n a_i)(\sum_{i=1}^n a_i^*)$ are generators of the algebra $E\Pi_{\vec{\lambda}}E$.

Let us consider the map $\psi : \mathcal{A}_n(r_i) \to E\Pi_{\vec{\lambda}}E$ given by correspondence:

$$q_i \mapsto e_i, \quad P \mapsto (\sum_{i=1}^n a_i)(\sum_{j=1}^n a_j^*). \tag{60}$$

Direct checking shows us that ψ is a homomorphism of algebras. Using previous arguments, we get that ψ is isomorphism.

□

Fix dimension vector $\vec{\alpha} = (\alpha_{v_1}, ..., \alpha_{v_n}, \alpha_w) \in \mathbb{N}_0^{n+1}$ such that $(\vec{\alpha}, \vec{\lambda}) = 0$. $\mathcal{M}_{|\vec{\alpha}|}\Pi_{\vec{\lambda}}[\vec{\alpha}]$ the variety of $\vec{\alpha}$-modules of deformed preprojective algebra $\Pi_{\vec{\lambda}}$. Let $\vec{\alpha}_v$ be a vector $(\alpha_{v_1}, ..., \alpha_{v_n})$ Consider variety $\mathcal{M}_{|\vec{\alpha}_v|}\mathcal{A}_n(r_i)[\vec{\alpha}]$ of $\alpha_{v_1} + ... + \alpha_{v_n}$-dimensional $\mathcal{A}_n$ - modules with properties:

$$\mathrm{rank} q_i = \alpha_{v_i}, i = 1, ..., n \quad \mathrm{rankP} = \alpha_w.$$

Using Morita equivalence, we get the isomorphism of varieties:

$$\mathcal{M}_{|\vec{\alpha}|}\Pi_{\vec{\lambda}}[\vec{\alpha}] \cong \mathcal{M}_{|\vec{\alpha}_v|}\mathcal{A}_n(r_i)[\vec{\alpha}]. \tag{61}$$

Remark. Also, let us consider the unital algebra $\mathcal{C}(r_i)$ with generators $s_i, i = 1, ..., n$ and relations

$$s_i^2 = r_i s_i, \quad \sum_{i=1}^n s_i = 1. \tag{62}$$

It can be shown in standard way that this algebra isomorphic to algebra $e_w \Pi_{\vec{\lambda}} e_w$ and, thus, we get the Morita equivalence of algebras: $\mathcal{C}(r_i)$ and $\Pi_{\vec{\lambda}}$.

For dimension vector $\vec{\alpha} = (\alpha_{v_1}, ..., \alpha_{v_n}, \alpha_w)$ denote by $\mathcal{M}_{\alpha_w}\mathcal{C}(r_i)[\vec{\alpha}_v]$ the variety of $\alpha_w \times \alpha_w$ matrices $S_i, i = 1, ..., n$, such that $\mathrm{rank} S_i = \alpha_{v_i}$ and satisfying to relations (62).

Using this equivalence, we get the isomorphism of moduli varieties:

$$\mathcal{M}_{\vec{\alpha}}\Pi_{\vec{\lambda}} \cong \mathcal{M}_{\alpha_w}\mathcal{C}(r_i)[\vec{\alpha}_v] \tag{63}$$

4.3 Crawley-Boewey condition for dimension vector $\vec{\alpha} = (1, ..., 1, m)$.

In this subsection we will study the properties of variety $\mathcal{M}_{n,(1,...,1,m)}(\mathcal{A}_n)$ via Morita equivalence with deformed preprojective algebra $\Pi_{\vec{\lambda}}$ of the quiver Q.

Proposition 24. *Consider dimension vector $\vec{\alpha} = (1, ..., 1, m)$ for $m \in \{2, ..., n-2\}$. Then vector $\vec{\alpha} = (1, ..., 1, m)$ is a $\vec{\lambda}$ - Schur root (i.e. $\vec{\alpha} \in \Sigma_{\vec{\lambda}}$) for any vector $\vec{\lambda} = (-r_1, ..., -r_n, 1)$ such that $r_1 + ... + r_n = m$.*

Proof. Recall that we have to prove that $p_Q(\vec{\alpha}) > p_Q(\vec{\beta_1}) + ... + p_Q(\vec{\beta_s})$ for any non-trivial decomposition $\vec{\alpha} = \vec{\beta_1} + ...\vec{\beta_s}$, where $\vec{\beta_i}, i = 1, ..., s$ are positive roots and $(\vec{\beta_i}, \vec{\lambda}) = 0$. It is clear that last component of $\vec{\beta_i}$ is $m_i \in \{0, ..., m\}$. Among other components there are n_i 1's and $n - n_i$ zeroes. We have the following relations: $\sum_{i=1}^s m_i = m$, $\sum_{i=1}^s n_i = n$. It is clear that matrix χ_Q has the following form:

$$\chi_Q = \begin{pmatrix} 1 & -1 & -1 & ... & -1 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \tag{64}$$

It can be shown in usual way that $p_Q(\vec{\alpha}) = 1 - \chi_Q(\vec{\alpha}, \vec{\alpha}) = (n - m - 1)(m - 1)$ and $p_Q(\vec{\beta_i}) = 1 - \chi_Q(\vec{\beta_i}, \vec{\beta_i}) = (n_i - m_i - 1)(m_i - 1)$. Thus, we have to prove the following inequality:

$$(n - m - 1)(m - 1) > \sum_{i=1}^{s} (n_i - m_i - 1)(m_i - 1).$$

Transform it as follows:

$$(n - m)m - m - (n - m) + 1 > \sum_{i=1}^{s} (n_i - m_i)m_i - \sum_{i=1}^{s} n_i + s.$$

Finally, we get

$$(n - m)m > \sum_{i=1}^{s} (n_i - m_i)m_i + s - 1. \tag{65}$$

for $n_i, m_i \in \mathbb{N}_0$ such that $\sum_{i=1}^s n_i = n$ and $\sum_{i=1}^s m_i = m$. Further, let us give some remarks about n_i and m_i. Fix root $\vec{\beta_i} = (\beta_{i,1}, ..., \beta_{i,n}, \beta_{i,n+1} = m_i)$. We have two cases: $m_i = 0$ or $m_i > 0$. In the first case, using non-triviality of β_i, we get $n_i > 0$. In the second case, $\vec{\lambda} \cdot \vec{\beta_i} = m_i + \sum_{l=1}^n r_l \beta_{i,l} = 0$. Hence, $n_i > 0$. Thus, $n_i > 0$ for any root $\vec{\beta_i}$. Using inequality (54), we obtain:

$$(n_i - m_i - 1)(m_i - 1) = (n_i - m_i)m_i - n_i + 1 \geq 0$$

Thus, $(n_i - m_i)m_i \geq 0$. It means that $n_i \geq m_i$ for all $i = 1, ..., s$. Also, we have $n - m \geq 2$ and $m \geq 2$.

Let us prove the following lemma.

Lemma 25. *Let X, Y be integers and $X, Y \geq 2$. For any $s \geq 2$ and any partitions $X = x_1 + ... + x_s$ and $Y = y_1 + ... + y_s$ satisfying to conditions:*

- $x_i, y_i \in \mathbb{N}_0$,
- $x_i^2 + y_i^2 > 0$ *for any* $i = 1, ..., s$

we have the following inequality:

$$XY > \sum_{i=1}^{s} x_i y_i + s - 1. \tag{66}$$

Proof. of lemma. Fix partition $\theta = (X = x_1 + ... + x_s, Y = y_1 + ... + y_s)$. For simplicity, let us denote by $f(\theta)$ the $\sum_{i=1}^{s} x_i y_i + s - 1$. Without loss of generality, let us assume that $x_i, y_i > 0$ for $i = 1, ..., k_1$, $x_i > 0, y_i = 0$ for $i = k_1 + 1, ..., k_2$ and $x_i = 0, y_i > 0$ for $i = k_2 + 1, ..., s$:

$$X = x_1 + ... + x_{k_1} + x_{k_1+1} + ... + x_{k_2} + 0 + ... + 0.$$

$$Y = y_1 + ... + y_{k_1} + 0 + ... + 0 + y_{k_2+1} + ... + y_s.$$

Denote by X_0 and Y_0 the sums $\sum_{i=1}^{k_1} x_i$ and $\sum_{i=1}^{k_1} y_i$ respectively. By X_1 and Y_1 we denote sums $\sum_{i=k_1+1}^{k_2} x_i$ and $\sum_{i=k_2+1}^{s} y_i$. It is clear $X = X_0 + X_1 \geq 2, Y = Y_0 + Y_1 \geq 2$ and $X_1 \geq k_2 - k_1, Y_1 \geq s - k_2$. Note that $k_1 + X_1 + Y_1 - 1 \geq s - 1$. Let us prove that

$$XY > \sum_{i=1}^{k_1} x_i y_i + k_1 + X_1 + Y_1 - 1. \tag{67}$$

Let us write XY in the following manner:

$$XY = (X_0 + X_1)(Y_0 + Y_1) = X_0 Y_0 + X_1 Y_0 + X_0 Y_1 + X_1 Y_1.$$

We will consider three cases: $k_1 > 1$, $k_1 = 1$, $k_1 = 0$. Let us consider the first case. We have the following inequality: $X_0 Y_0 \geq \sum_{i=1}^{k_1} x_i y_i + k_1(k_1 - 1)$ (because of $x_i y_j \geq 1$ for all $i, j = 1, ..., k_1$), $X_1 Y_0 \geq k_1 X_1, X_0 Y_1 \geq k_1 Y_1$. Thus, we obtain:

$$XY \geq \sum_{i=1}^{k_1} x_i y_i + k_1(k_1 - 1) + k_1 X_1 + k_1 Y_1 + X_1 Y_1$$

Therefore, inequality (66) transforms to

$$k_1(k_1 - 1) + k_1 X_1 + k_1 Y_1 + X_1 Y_1 > k_1 - 1 + X_1 + Y_1$$

We can transform this inequality as follows:

$$(k_1 + X_1 - 1)(k_1 + Y_1 - 1) > 0$$

Therefore, first case is proved.

Second case. If $k_1 = 1$, then we have the partitions: $X = x_1 + x_2 + ... + x_{k_2} + 0 + ... + 0$ and $Y = y_1 + 0 + ... + 0 + y_{k_2+1} + ... + y_s$. Also, $X_1 = \sum_{i=2}^{k_2} x_i$, $Y_1 = \sum_{i=k_2+1}^{s} y_i$ and $x_1 + X_1 \geq 2$, $y_1 + Y_1 \geq 2$.

It is easy $XY = x_1 y_1 + x_1 Y_1 + X_1 y_1 + X_1 Y_1$. We can rewrite inequality (67) as follows:

$$x_1 Y_1 + y_1 X_1 + X_1 Y_1 > X_1 + Y_1.$$

This inequality is true, because $x_1 + X_1 \geq 2$ and $y_1 + Y_1 \geq 2$.

Last case $k_1 = 0$. We have $X = x_1 + ... + x_{k_2} + 0 + ... + 0$, $Y = 0 + ... + 0 + y_{k_2+1} + ... + y_s$. Inequality (67) transforms to:

$$XY > X + Y - 1.$$

It is true, because $X, Y \geq 2$. Lemma is proved □

To the end of proof of the proposition, let us apply lemma in the case $y_i = m_i, x_i = n_i - m_i, i = 1, ..., s$. □

Corollary 26. *Fix $m \in \{2, ..., n-2\}$. For any $r_1, ..., r_n \in F$ such that $r_1 + ... + r_n = m$, general representation of algebra $\mathcal{A}_n(r_i)$ with dimension vector $(1, ..., 1, m)$ is simple. Also, variety $\mathcal{M}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$ and $\mathbf{Rep}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$ are irreducible and have dimensions $2(n-m-1)(m-1)$ and $2(n-m-1)(m-1) + n^2 - 1$ respectively.*

Also, using Morita equivalence, we have the following:

Corollary 27. *General element of* $\mathbf{Rep}_n\mathcal{A}_n(r_i)[(\vec{1},m)]$ *is a simple representation.*

Proof. Using theorem of Crawley-Boewey and proposition 24, we obtain that general $\Pi_{\vec{\lambda}}$-representation of dimension vector $(1,...,1,m)$ is an irreducible for any $\vec{\lambda}=(-r_1,...,-r_n,1)$ such that $r_1+...+r_n=m$. Applying Morita-equivalence, we get the required. □

Let us calculate automorphism group of n-dimensional $\mathcal{A}_n(r_i)$ - representation ρ. Let us introduce the notion graph $G_n(\rho)$ of the representation ρ. This graph has n vertices labeled by $i, i=1,...,n$ corresponding to generators $q_i, i=1,...,n$. There is edge between vertices i and j iff $\rho(q_iPq_j)\neq 0$ or $\rho(q_jPq_i)\neq 0$. It is easy that if we have two isomorphic representations ρ' and ρ, then $G_n(\rho)\cong G_n(\rho')$. It means that we have well-defined notion of graph of $\mathcal{A}_n(r_i)$-module.

Proposition 28. *Assume that* $r_i\neq 0$*. Consider* n*-dimensional representation* ρ *of* $\mathcal{A}_n(r_i)$ *of dimension vector* $(1,...,1,m)$*. Then group* $Aut_{\mathcal{A}_n(r_i)}(\rho)$ *is an algebraic torus and the following statements are equivalent:*

- $Aut_{\mathcal{A}_n(r_i)}(\rho)=(F^*)^s, s\leq m,$
- *graph* $G_n(\rho)$ *has* s *connected components.*

Proof. It is easy that vector space of ρ has a basis $v_1,...,v_n$ such that $q_iv_j=\delta_{ij}v_i$. Further, consider $f\in Aut_{\mathcal{A}_n(r_i)}(\rho)$. Then $f(v_i)=\alpha_iv_i$, where $\alpha_i\neq 0$. Thus, $Aut_{\mathcal{A}_n(r_i)}(\rho)$ is a subgroup of algebraic torus $(F^*)^n$. Assume that $\rho(q_iP)v_j=x_{ij}v_i$ for some $x_{ij}\in F$ and any i,j. We get the following identity:

$$f(\rho(q_iP)v_j)=\alpha_j\rho(q_iP)v_j=\alpha_jx_{ij}v_i. \tag{68}$$

From other hand, we obtain the following:

$$f(\rho(q_iP)v_j)=f(x_{ij}v_i)=\alpha_ix_{ij}v_i. \tag{69}$$

Hence, $x_{ij}(\alpha_i-\alpha_j)=0$ for any i,j. Further, consider graph G_n of representation ρ. This graph has n vertices labeled by $q_i, i=1,...,n$. There is edge between q_i and q_j iff $x_{ij}\neq 0$. Note that G_n is connected iff $\alpha_i=\alpha_j$ for any i,j, i.e. $Aut_{\mathcal{A}_n(r_i)}(\rho)=F^*$. Therefore, we get that $Aut_{\mathcal{A}_n(r_i)}(\rho)=(F^*)^s$ iff graph G_n has s connected components. Also, it can be shown in usual way that if $Aut_{\mathcal{A}_n(r_i)}(\rho)=(F^*)^s$ then $\mathrm{rank}P\geq s$. □

Corollary 29. *If* $r_i\neq 0,1, i=1,...,n$ *then for any representation* ρ *graph* $G_n(\rho)$ *has no components consisting of one vertex.*

Proof. Assume that one connected component has one vertex. Without loss of generality, number of this vertex is 1. Then we have the following relations: $q_1Pq_j=0, j=2,...,n$ and $q_jPq_1=0, j=2,...,n$. Using relation $\sum_{j=1}^n q_j=1$, we get that $q_1P(1-q_1)=0$. Multiply by P from right side, we obtain: $(1-q_1)Pq_1P$. Calculating trace, we get:

$$\mathrm{Tr}(1-q_1)Pq_1P=\mathrm{Tr}Pq_1P-\mathrm{Tr}q_1Pq_1P=\mathrm{Tr}Pq_1-r_1\mathrm{Tr}Pq_1=(1-r_1)r_1\neq 0 \tag{70}$$

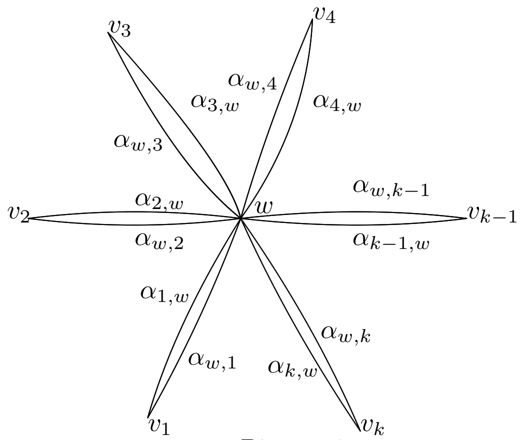
Contradiction. □

5 Algebra $\widetilde{\mathcal{A}}_k(r_i)$ and its moduli variety.

In this section we will study algebra $\widetilde{\mathcal{A}}_k(r_i)$. Assume that $r_i\neq 0, i=1,...,k$.

5.1 Homological properties of algebra $\widetilde{\mathcal{A}}_k(r_i)$.

Denote by $e_w, e_i, i = 1, ..., k$ the trivial path in vertices. Denote by FQ the path algebra of quiver Q. Consider two-sided ideal I generated by elements $\alpha_{i,w}\alpha_{w,i} - r_i e_i$. Denote by Δ_Q the element $e_w + \sum_{i=1}^{k}(e_i + \alpha_{i,w} + \alpha_{w,i})$. One can prove that $\widetilde{A}_k(r_i)$ is a homotop of S_Q with element Δ_Q. One can show that Δ_Q is well-tempered element (cf.[1]) Consider the quotient $S_Q = FQ/I$, where Q is a quiver with vertices labeled by w,v_1,...,v_k, arrows $\alpha_{i,w}$ and α_{w,v_i}. Arrows $\alpha_{i,w}, \alpha_{w,i}, i = 1, ..., k$ connect vertices v_i, w and w, v_i respectively (see picture). Note that deformed preprojective algebra is a quotient of S_Q by relation: $\sum_{i=1}^{n} \alpha_{w,i}\alpha_{i,w} - r_w e_w$.



Picture 2

Further, consider algebra S_Q. Let us apply Morita's theorem to algebra S_Q and idempotent e_w. It is easy that $S_Q e_w S_Q = S_Q$. Thus, algebras S_Q and $e_w S_Q e_w$ are Morita equivalent. One can show that algebra $e_w S_Q e_w$ is generated by $s_i = \alpha_{wi}\alpha_{iw}, i = 1, ..., k$. It is easy that

$$s_i^2 = r_i s_i. \tag{71}$$

It can be shown in usual way that algebra $e_w S_Q e_w$ is an unital algebra generated by s_i satisfying to relations (71). One can show that this algebra is isomorphic to $\mathbf{Pr}(\Gamma[k])$, where $\Gamma[k]$ is complete graph with k vertices. Using subsection 3.1, we obtain that $\mathbf{Pr}(\Gamma[k])$ is a homotop of path algebra of double quiver $\mathbb{Q}_{\Gamma[k]}$ with k vertices. Using properties of homotopes, we get the following proposition:

Proposition 30. *Hochschild dimension of $\widetilde{\mathcal{A}}_k(r_i)$ is 2.*

Proof. Applying theorem (cf [1]) and Morita invariance of Hochshild dimension, we get the required statement. □

Of course, we have exact sequence of $\widetilde{\mathcal{A}}_k(r_i)$ - bimodules:

$$0 \longrightarrow \widetilde{\mathcal{A}}_k^{+}(r_i) \longrightarrow \widetilde{\mathcal{A}}_k(r_i) \xrightarrow{\epsilon} F \longrightarrow 0 \tag{72}$$

where ϵ is augmentation, i.e. $\epsilon(1) = 1, \epsilon(P) = \epsilon(q_i) = 0$. Using basis of $\mathcal{A}_k(r_i)$, we get that $\widetilde{\mathcal{A}}_k(r_i)$ - bimodule $\widetilde{\mathcal{A}}_k^{+}(r_i)$ is a projective left $\widetilde{A}_k(r_i)$ - module, and we have the following isomorphism:

$$\widetilde{\mathcal{A}}_k^{+}(r_i) \cong \oplus_{i=1}^{k} \widetilde{\mathcal{A}}_k(r_i) q_i \oplus \widetilde{\mathcal{A}}_k(r_i) P \tag{73}$$

This augmentation has the following modification:

$$0 \longrightarrow \widetilde{\mathcal{A}}_k^{++}(r_i) \longrightarrow \widetilde{\mathcal{A}}_k(r_i) \xrightarrow{\epsilon_A} FP \oplus F(1-P) \longrightarrow 0 \tag{74}$$

Algebra $FP \oplus F(1-P)$ is an unital algebra generated by P. "Augmentation" ϵ_A is defined by formula:

$$\epsilon_A(1) = 1, \epsilon_A(P) = P, \epsilon_A(q_i) = 0 \tag{75}$$

It is easy that ϵ_A is a homomorphism of algebras.

Let us prove the following proposition:

Proposition 31. *$\widetilde{\mathcal{A}}_k^{++}(r_i)$ is a projective $\widetilde{\mathcal{A}}_k(r_i)$ - module.*

Proof. Let us restrict the ϵ_A to $\widetilde{\mathcal{A}}_k^{+}(r_i)$. Denote this restriction by ϵ'_A. Therefore, we have the following exact sequence:

$$0 \longrightarrow \widetilde{\mathcal{A}}_k^{++}(r_i) \longrightarrow \widetilde{\mathcal{A}}_k^{+}(r_i) \xrightarrow{\epsilon'_A} FP \longrightarrow 0. \tag{76}$$

It is easy that $\widetilde{\mathcal{A}}_k(r_i)q_i \subset \widetilde{\mathcal{A}}_k^{++}(r_i)$. Thus, we have the induced map: $\widetilde{\mathcal{A}}_k(r_i)P \to F$. It can be shown in usual way that kernel of this map is a $\widetilde{\mathcal{A}}_k(r_i)$ - module: $\oplus_{i=1}^k \widetilde{\mathcal{A}}_k(r_i)q_iP$. Therefore, we get the following isomorphism of left $\widetilde{\mathcal{A}}_k(r_i)$ - modules:

$$\widetilde{\mathcal{A}}_k^{++}(r_i) \cong \oplus_{i=1}^k \widetilde{\mathcal{A}}_k(r_i)q_iP \oplus \widetilde{\mathcal{A}}_k(r_i)q_i. \tag{77}$$

Also, we have the similar decomposition of $\widetilde{\mathcal{A}}_k^{++}(r_i)$ as right module. It is easy that $\frac{1}{r_i}q_iP$ is an idempotent, hence $\widetilde{\mathcal{A}}_k(r_i)q_iP$ is a projective $\widetilde{\mathcal{A}}_k(r_i)$ - module. The rest is trivial. □

There are two 1-dimensional $\widetilde{\mathcal{A}}_k(r_i)$-modules FP and $F(1-P)$. One can check that $q_i, i = 1, ..., k$ act trivially on FP, P acts as identity operator. Also, $P, q_i, i = 1, ..., k$ act trivially on $F(1-P)$. Proposition 31 show us that exact sequence (76) as projective resolutions of FP respectively. Of course, sequence (72) is a projective resolution of $F(1-P)$.

Also, note that we can find connection between algebra $\widetilde{\mathcal{A}}_k(r_i)$ and $\mathbf{Pr}(\Gamma[k])$ more directly. Namely, if we consider the following subspace $\widetilde{\mathcal{A}}_k(r_i)P\widetilde{\mathcal{A}}_k(r_i)$, i.e two-sided ideal of $\widetilde{\mathcal{A}}_k(r_i)$ generated by P. It can be shown in usual way that $\widetilde{\mathcal{A}}_k(r_i)P\widetilde{\mathcal{A}}_k(r_i) = \widetilde{\mathcal{A}}_k^{+}(r_i)$. Actually, using relation $q_iPq_i = r_iq_i$, we can get all $q_i, i = 1, ..., k$ and hence, we can get any element of $\widetilde{\mathcal{A}}_k^{+}(r_i)$. Consider algebra $P\widetilde{\mathcal{A}}_k(r_i)P$. One can show that this algebra is isomorphic to $\mathbf{Pr}(\Gamma[k])$. This construction is similar to construction of Morita-equivalence of fundamental group and Poincare grouppoid. Also, if we consider two-sided ideal of $\widetilde{\mathcal{A}}_k(r_i)$ generated by $\sum_{i=1}^k q_i$, then we get the following identity: $(\sum_{i=1}^k q_i)\widetilde{\mathcal{A}}_k(r_i)(\sum_{i=1}^k q_i) = \widetilde{\mathcal{A}}_k^{++}(r_i)$. Also, we obtain that algebra $(\sum_{i=1}^k q_i)\widetilde{\mathcal{A}}_k(r_i)(\sum_{i=1}^k q_i)$ is isomorphic to path algebra $F\mathbb{Q}_{\Gamma[k]}$ of double quiver $\mathbb{Q}_{\Gamma[k]}$ with k vertices.

5.2 Endomorphisms and automorphisms of $\widetilde{\mathcal{A}}_k(r_i)$-modules.

Consider n-dimensional $\widetilde{\mathcal{A}}_k(r_i)$-module V. Applying functor $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(-, V)$ to sequence (74), we get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(FP \oplus F(1-P), V) \longrightarrow V \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\widetilde{\mathcal{A}}_k^{++}(r_i), V) \tag{78}$$

$$\longrightarrow \mathrm{Ext}^1_{\widetilde{\mathcal{A}}_k(r_i)}(FP \oplus F(1-P), V) \longrightarrow 0. \tag{79}$$

Also, applying functor $- \otimes_{\widetilde{\mathcal{A}}_k(r_i)} V$ to (74), we get the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^{\widetilde{\mathcal{A}}_k(r_i)}(FP \oplus F(1-P), V) \longrightarrow \widetilde{\mathcal{A}}_k^{++}(r_i) \otimes_{\widetilde{\mathcal{A}}_k(r_i)} V \longrightarrow V \tag{80}$$

$$\longrightarrow (FP \oplus F(1-P)) \otimes_{\widetilde{\mathcal{A}}_k(r_i)} V \longrightarrow 0. \tag{81}$$

Denote by Ker_P, Ker_{1-P}, Coker_P and Coker_{1-P} the $\widetilde{\mathcal{A}}_k(r_i)$ - modules: $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(FP, V)$, $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(F(1-P), V)$, $FP \otimes_{\widetilde{\mathcal{A}}_k(r_i)} V$ and $F(1-P) \otimes_{\widetilde{\mathcal{A}}_k(r_i)} V$ respectively. It is easy that Coker_P and Ker_P

are direct sum of several copies of FP's, Coker_{1-P} and Ker_{1-P} are direct sum of several copies of $F(1-P)$'s. Denote by Im the image of $\widetilde{\mathcal{A}}_k^{++}(r_i)\otimes_{\widetilde{\mathcal{A}}_k(r_i)} V$ in V. $\widetilde{\mathcal{A}}_k(r_i)$-module Im has the following description: consider subspace of V generated by $\mathrm{Im}q_j, j=1,...,k$ and $\mathrm{Im}Pq_j, j=1,...,k$. It is easy that this subspace is $\widetilde{\mathcal{A}}_k(r_i)$ - submodule and one can show that this submodule is Im. It is clear that we have the exact sequence:

$$0 \longrightarrow \mathrm{Im} \longrightarrow V \longrightarrow \mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P} \longrightarrow 0 \tag{82}$$

Lemma 32. *Any $\widetilde{\mathcal{A}}_k(r_i)$-endomorphism g of V induces $\widetilde{\mathcal{A}}_k(r_i)$-endomorphisms g' and g'' of* Im *and* $\mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P}$ *respectively. Also, we have the following diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im} & \longrightarrow & V & \longrightarrow & \mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} & & \\ 0 & \longrightarrow & \mathrm{Im} & \longrightarrow & V & \longrightarrow & \mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P} & \longrightarrow & 0 \end{array} \tag{83}$$

Also, g'' transforms Coker_P *into* Coker_P *and* Coker_{1-P} *into* Coker_{1-P}.

Proof. It is sufficient to prove that restriction of g to Im is endomorphism of Im. As we know, Im is generated by $\mathrm{Im}Pq_j$ and $\mathrm{Im}q_j$. It is clear that $g(Pq_jv)=Pq_jg(v)$ and $g(q_jv)=q_jg(v)$. Therefore, g preserves Im. Denote this endomorphism of Im by g'. Thus, we have induced endomorphism g'' of $\mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P}$. Also, it is easy that g'' preserves Coker_P and Coker_{1-P}. □

Proposition 33. *We have the following exact sequence:*

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V) \oplus \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_{1-P}, V) \longrightarrow \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}V \longrightarrow \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}\mathrm{Im}. \tag{84}$$

Moreover, $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V) = \mathrm{Hom}_F(\mathrm{Coker}_P, \mathrm{Ker}_P)$ *and* $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_{1-P}, V) = \mathrm{Hom}_F(\mathrm{Coker}_{1-P}, \mathrm{Ker}_{1-P})$.

Proof. Applying functor $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(-, V)$ to sequence (82), we get the following sequence:

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V) \oplus \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_{1-P}, V) \longrightarrow \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}V \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, V). \tag{85}$$

Let us prove that $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, V) = \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}\mathrm{Im}$. Applying functor $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, -)$ to sequence (82), we get the sequence:

$$0 \longrightarrow \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}\mathrm{Im} \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, V) \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, \mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P}). \tag{86}$$

Direct calculations show us that $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}, \mathrm{Coker}_P \oplus \mathrm{Coker}_{1-P}) = 0$. Further, calculate $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V)$. Denote by $coker_P$ the dimension of Coker_P. As we know, $\mathrm{Coker}_P \cong FP^{coker_P}$. Using projective resolution, we get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V) \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\widetilde{\mathcal{A}}_k^{+}(r_i), V)^{coker_P} \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\widetilde{\mathcal{A}}_k^{++}(r_i), V)^{coker_P} \tag{87}$$

It can be shown in usual way that $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, V) = \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P, \mathrm{Ker}_P) = \mathrm{Hom}_F(\mathrm{Coker}_P, \mathrm{Ker}_P)$. One can prove analogous statement for $\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_{1-P}, V)$. □

Consider n-dimensional $B_{k,n}$-module V. Using diagram (42), we get $B_{\mathbf{r}}(\Gamma_{k,m})$ - module ϕ^*V, $B_{\mathbf{r}}(\Gamma_{k,n-m})$ - module ϕ'^*V and $\widetilde{\mathcal{A}}_k(r_i)$ - module $i^*\circ\phi^*V = i'^*\circ\phi'^*V$. Denote these modules by V. Consider $B_{\mathbf{r}}(\Gamma_{k,m})$ - module V. As we know from [1], we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(\mathrm{Coker}', \mathrm{Ker}') \longrightarrow \mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}V \longrightarrow F \longrightarrow 0, \tag{88}$$

where Coker$'$ and Ker$'$ are $B_{\mathbf{r}}(\Gamma_{k,m})$ - modules $\mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(F,V)$ and $F\otimes_{B_{\mathbf{r}}(\Gamma_{k,m})} V$ respectively. Note that this exact sequence is split, i.e. $\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V) = F\oplus \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(\mathrm{Coker}',\mathrm{Ker}')$. It can be shown in usual way that $i^*\mathrm{Coker}' = \mathrm{Coker}_{1-P}$ and $i^*\mathrm{Ker}' = \mathrm{Ker}_{1-P}$. One can consider the case of algebra $B_{\mathbf{r}}(\Gamma_{k,n-m})$. Namely, we have the exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(\mathrm{Coker}'',\mathrm{Ker}'') \longrightarrow \mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}V \longrightarrow F \longrightarrow 0, \tag{89}$$

where Coker$''$ and Ker$''$ are $B_{\mathbf{r}}(\Gamma_{k,n-m})$ - modules $\mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(F,V)$ and $F\otimes_{B_{\mathbf{r}}(\Gamma_{k,n-m})} V$ respectively. It can be shown in usual way that $i'^*\mathrm{Coker}'' = \mathrm{Coker}_P$ and $i'^*\mathrm{Ker}'' = \mathrm{Ker}_P$. Also, note that

$$\mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(FP, F(1-P)) = \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(F(1-P), FP) = 0. \tag{90}$$

It is easy that if $s\in \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(\mathrm{Coker}',\mathrm{Ker}')$, then we can define element of $\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)$ as follows. We have natural morphisms: $V\to \mathrm{Coker}'$ and $\mathrm{Ker}'\to V$. Thus, we have the element of $\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)$ defined as composition:

$$V \longrightarrow \mathrm{Coker}' \xrightarrow{s} \mathrm{Ker}' \longrightarrow V. \tag{91}$$

We will denote this endomorphism by $\widehat{s}$.

Thus, we can define composition of $s_1, s_2 \in \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(\mathrm{Coker}',\mathrm{Ker}')$. Analogously, one can define composition in the case of algebras $B_{\mathbf{r}}(\Gamma_{k,n-m})$ and $\widetilde{\mathcal{A}}_k(r_i)$. It is easy that natural morphisms:

$$\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V) \to \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}(V), \mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V) \to \mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}(V) \tag{92}$$

are ring monomorphisms.

Proposition 34. *Subrings* $\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)$ *and* $\mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)$ *of* $\mathrm{End}_{\widetilde{\mathcal{A}}_k(r_i)}(V)$ *commute.*

Proof. Actually, consider $\alpha_1 1 + s_1 \in \mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,m})}(V), \alpha_2 1 + s_2 \in \mathrm{End}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)$, where $s_1 \in \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,m})}(\mathrm{Coker}',\mathrm{Ker}') = \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_{1-P},\mathrm{Ker}_{1-P})$ and $s_2 \in \mathrm{Hom}_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(\mathrm{Coker}'',\mathrm{Ker}'') = \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Coker}_P,\mathrm{Ker}_P)$. As we know, endomorphism $\widehat{s_1}\circ\widehat{s_2}$ is defined as composition:

$$V \longrightarrow \mathrm{Coker}_P \xrightarrow{s_2} \mathrm{Ker}_P \longrightarrow V \longrightarrow \mathrm{Coker}_{1-P} \xrightarrow{s_1} \mathrm{Ker}_P \longrightarrow V \tag{93}$$

As we know, $\mathrm{Ker}_P = \oplus FP$ and $\mathrm{Coker}_{1-P} = \oplus F(1-P)$. Thus, composition $\mathrm{Ker}_P \to V \to \mathrm{Coker}_{1-P}$ is zero by (90). Hence, $\widehat{s_1}\circ\widehat{s_2} = 0$. Similarly, $\widehat{s_2}\circ\widehat{s_1} = 0$. Therefore,

$$(\alpha_1 1 + s_1)\circ(\alpha_2 1 + s_2) = \alpha_1\alpha_2 1 + \alpha_1 s_2 + \alpha_2 s_1 = (\alpha_2 1 + s_2)\circ(\alpha_1 1 + s_1), \alpha_i \in F, i = 1,2. \tag{94}$$

Note that we have the following identity:

$$(\alpha_1 1 + s_1)\circ(\alpha_2 1 + s_2) = \alpha_1(\alpha_2 1 + s_2) + \alpha_2(\alpha_1 1 + s_1) - \alpha_1\alpha_2 1. \tag{95}$$

□

Recall the following trivial facts. It is well known that for any algebra A and A-module V: $Aut_A(V)$ is a group of units of $\mathrm{End}_A(V)$. Also, note that for any algebra A and A - module V group $Aut_A(V)$ has central subgroup F^*.

Consider the following groups: $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V)$, $Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)$, $Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)$ and $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im})$. It is easy that we can consider $Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)$, $Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)$ as subgroups of $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V)$. Also, we have a natural group homomorphism: $f : Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V) \to Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im})$. Fix element $g\in Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V)$ of the following type:

$$g = \alpha 1 + s_1 + s_2, s_1 \in \mathrm{Hom}_F(\mathrm{Coker}_P,\mathrm{Ker}_P), s_2 \in \mathrm{Hom}_F(\mathrm{Coker}_{1-P},\mathrm{Ker}_{1-P}) \tag{96}$$

Using formula (95), we get the following factorization of g:

$$g = (\alpha_1 1 + \frac{1}{\alpha_2} s_1) \circ (\alpha_2 1 + \frac{1}{\alpha_1} s_2) \in Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V) \cdot Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V), \tag{97}$$

where $\alpha_1\alpha_2 = \alpha$. Consider quotients $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V)/F^*$, $Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)/F^*$, $Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)/F^*$ and $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im})/F^*$. We have natural morphism: $f : Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V)/F^* \to Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)/F^*$. Formula (97) means that $\mathrm{Ker} f = Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V)/F^* \times Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V)/F^*$. Using these arguments, we get the following proposition:

Proposition 35. *Fix $B_{k,n}$ - module V. Consider V as module over $B_{\mathbf{r}}(\Gamma_{k,m})$, $B_{\mathbf{r}}(\Gamma_{k,n-m})$ and $\widetilde{\mathcal{A}}_k(r_i)$ - module by commutative diagram (42). We have the following immersion of the varieties:*

$$Aut_{B_{\mathbf{r}}(\Gamma_{k,m})}(V) \backslash Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V) / Aut_{B_{\mathbf{r}}(\Gamma_{k,n-m})}(V) \subseteq Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im})/F^* \tag{98}$$

We can introduce the notion of graph $G_k(\rho)$ of $\widetilde{\mathcal{A}}_k(r_i)$ - representation ρ. This notion is quite similar to notion of graph of $\mathcal{A}_n(r_i)$-representation ρ. Graph $G_k(\rho)$ has k vertices labeled by $i, i = 1, ..., k$. Two vertices i and j are connected by edge iff $\rho(q_iPq_j) \neq 0$ or $\rho(q_jPq_i) \neq 0$. Also, we have well-defined notion of graph of $\widetilde{\mathcal{A}}_k(r_i)$-module.

Let us formulate the following proposition:

Proposition 36. *Assume that $r_i \neq 0$. Consider $\widetilde{\mathcal{A}}_k(r_i)$-representation ρ of dimension vector $(1, ..., 1, m)$. Suppose that ρ satisfies to condition: space of representation is generated by $\mathrm{Im}\rho(q_i), \mathrm{Im}\rho(Pq_i)$. Then group $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\rho)$ is a subgroup of algebraic torus $(F^*)^k$. Assume that $(F^*)^s \subseteq Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\rho)$ then graph $G_k(\rho)$ has at least s connected components.*

Proof. Denote by V_ρ the $\widetilde{\mathcal{A}}_k(r_i)$ - module corresponding to ρ. Consider $f \in Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\rho)$. Denote by $v_i, i = 1, ..., k$ the eigenvectors of $\rho(q_i), i = 1, ..., k$. It is easy that $f(v_i) = \alpha_i v_i$ for some $\alpha_i \in F^*, i = 1, ..., k$. Consider $Pv_i, i = 1, ..., k$. Clearly, $f(\rho(P)v_i) = \alpha_i\rho(P)v_i$. Since space V_ρ is generated by $\mathrm{Im}\rho(q_i), \mathrm{Im}\rho(Pq_i)$, we get the definition of f on V_ρ. Therefore, we have immersion of groups: $Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\rho) \subset (F^*)^k$.

We have the following relation: $\rho(q_i)\rho(P)v_j = x_{ij}v_i$ for some $x_{ij} \in F$. Applying f, we get the following:

$$f(\rho(q_i)\rho(P)v_j) = \rho(q_i)\rho(P)f(v_j) = \alpha_j\rho(q_i)\rho(P)v_j = \alpha_j x_{ij} v_i. \tag{99}$$

From other hand, $f(\rho(q_i)\rho(P)v_j) = f(x_{ij}v_i) = x_{ij}\alpha_i v_j$. Therefore, $x_{ij}(\alpha_i - \alpha_j) = 0$. Note that if $x_{ij} = 0$, then $\rho(q_iPv_j) = \rho(q_iPq_jv_j) = 0$ and $\rho(q_iPq_j) = 0$. The rest is easy. □

Also, let us note the following useful property:

Proposition 37. *Consider $\widetilde{\mathcal{A}}_k(r_i)$ - modules Im and V from exact sequence (82). Then graphs $G_k(\mathrm{Im})$ and $G_k(V)$ are isomorphic.*

Proof. It is easy that $G_k(\mathrm{Im}) \subseteq G_k(V)$. Thus, We have to show that $G_k(V) \subseteq G_k(\mathrm{Im})$. Recall that submodule Im is generated by vectors $q_jv, v \in V$ and $Pq_jv, v \in V$. It is easy that if $q_iPq_jv \neq 0$ for some vector $v \in V$, then $q_iPq_jv = q_iPq_j(q_jv) \neq 0$ for some vector $q_jv \in \mathrm{Im}$. Thus, if vertices i and j are connected in $G_k(V)$, then they are connected in $G_k(\mathrm{Im})$ and, hence we have proved the required statement. □

5.3 Properties of $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$.

Consider algebras $\mathbf{Pr}(\Gamma_{n,1})$, $\mathbf{Pr}(\Gamma_{k,1})$. It is easy that $\mathcal{A}_n(r_i)$, $\widetilde{\mathcal{A}}_k(r_i)$ are quotients of $\mathbf{Pr}(\Gamma_{n,1})$ and $\mathbf{Pr}(\Gamma_{k,1})$ by relations: $q_iPq_i - r_iq_i, i = 1, ..., n, \sum_{i=1}^n q_i - 1$ and $q_iPq_i - r_iq_i, i = 1, ..., k$ respectively. It can be shown in usual way that $\mathbf{Rep}_n\mathcal{A}_n(r_i)[\vec{1}, m]$ and $\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$ are fibers of morphisms:

$$tr_{n,1} : \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{n,1})[\vec{1}, m] \to F^{n-1}, tr_{k,1} : \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] \to F^k \tag{100}$$

respectively. Morphisms $tr_{n,1}$ and $tr_{k,1}$ are defined by formulas: $tr_{n,1} : \rho \mapsto (Tr\rho(Pq_1), ..., Tr\rho(Pq_n))$ and $tr_{k,1} : \rho \mapsto (Tr\rho(Pq_1), ..., Tr\rho(Pq_k))$. Also, F^{n-1} is a affine space with coordinates $r_1, ..., r_n$ and relation $r_1 + ...r_n = m$. F^k is a affine space with coordinates $r_1, ..., r_k$.

Fix $m \in \{2, ..., n-2\}$. Consider following commutative diagram of varieties:

$$\begin{array}{ccc} \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{n,1})[(\vec{1}, m)] & \xrightarrow{tr_{n,1}} & F^{n-1} \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle pr_2} \\ \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[(\vec{1}, m)] & \xrightarrow{tr_{k,1}} & F^k \end{array} \quad (101)$$

Morphisms pr_1, pr_2 are natural projections defined by formulas: $pr_1 : (q_1, ..., q_n, P) \mapsto (q_1, ..., q_k, P)$ and $pr_2 : (r_1, ..., r_n) \mapsto (r_1, ..., r_k)$. As we know from corollary 11, varieties $\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{n,1})[(\vec{1}, m)]$ and $\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[(\vec{1}, m)]$ are irreducible. Clearly, morphisms pr_1 and pr_2 are surjective.

Lemma 38. *Morphism $tr_{k,1}$ are surjective.*

Proof. As we know, fibers of morphism $tr_{n,1}$ are varieties $\mathbf{Rep}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$. Recall that algebra $\mathcal{A}_n(r_i)$ and $\Pi_{\vec{\lambda}}$ with vector $\vec{\lambda} = (-r_1, ..., -r_n, 1)$ are Morita-equivalent. It is well-known that there exists $\Pi_{\vec{\lambda}}$-representation of dimension vector $(1, ..., 1, m)$ iff $r_1 + ... + r_n = m$. Thus, $\mathbf{Rep}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$ is non-empty for any $r_1, ..., r_n$ such that $r_1 + ... + r_n = m$. Composition of morphisms $tr_{\Gamma_{n,1}} \circ pr_2$ is surjective, and, hence, morphism $tr_{k,1}$ is surjective. □

Proposition 39. *Consider general n-dimensional representation ρ with dimension vector $(\vec{1}, m)$ of algebra $\mathbf{Pr}(\Gamma_{k,1})$. Then we have the following cases:*

- *if $n > k + m, m > k$, then $\dim_F Aut_{\mathbf{Pr}(\Gamma_{k,1})}(\rho) = (n - m - k)^2 + (m - k)^2 + 1$,*
- *if $n \leq m + k, m > k$, then $\dim_F Aut_{\mathbf{Pr}(\Gamma_{k,1})}(\rho) = (m - k)^2 + 1$,*
- *if $n > m + k, m \leq k$, then $\dim_F Aut_{\mathbf{Pr}(\Gamma_{k,1})}(\rho) = (n - m - k)^2 + 1$,*
- *if $n \leq m + k, m \leq k$, then $\dim_F Aut_{\mathbf{Pr}(\Gamma_{k,1})}(\rho) = 1$.*

Proof. Fix representation $\rho \in \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m]$. Denote by V_ρ the $\widetilde{\mathcal{A}}_k(r_i)$ - module corresponding to ρ. Assume that $r_i = \mathrm{Tr}\rho(Pq_i) \neq 0$, then we can consider ρ as representation of $\widetilde{\mathcal{A}}_k(r_i)$. In this case, using sequence (84), we get the following:

$$\dim_F Aut_{\widetilde{\mathcal{A}}_k(r_i)}(V_\rho) \leq \dim_F \mathrm{Hom}_F(\mathrm{Coker}_P, \mathrm{Ker}_P) + \dim_F(\mathrm{Coker}_{1-P}, \mathrm{Ker}_{1-P}) + \dim_F Aut_{\widetilde{\mathcal{A}}_k(r_i)}(\mathrm{Im}), \quad (102)$$

where Im is a $\widetilde{\mathcal{A}}_k(r_i)$-submodule of V_ρ generated by $\mathrm{Im}\rho(q_j)$ and $\mathrm{Im}\rho(Pq_j)$. Consider subvariety $U \subset \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$ of all representations $\rho \in \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$ satisfying to conditions:

- $\dim_F \mathrm{Coker}_P = \dim_F \mathrm{Ker}_P = m - k$ if $m \geq k$ and zero if $m < k$,
- $\dim_F \mathrm{Coker}_{1-P} = \dim_F \mathrm{Ker}_{1-P} = n - m - k$ if $n \geq m + k$ and zero if $n < m + k$
- $Aut_{\widetilde{\mathcal{A}}_k(r_i)}\mathrm{Im} = F^*$.

If $Aut_{\widetilde{\mathcal{A}}_k(r_i)}\mathrm{Im} = F^*$, then inequality (102) transforms to identity. It can be shown in usual way that subvariety U is dense in $\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$. Further, since $tr_{k,1}$ is surjective, we get that $tr_{k,1}^{-1}(U)$ is dense in $\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m]$. □

Corollary 40. *Consider variety $\mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m]$. Then we have the following cases:*

- *if* $n > k+m, m > k$, *then* $\dim_F \mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] = 2k(n-1)+2m(n-m)-k(k-1)-n^2+(n-m-k)^2+(m-k)^2+1 = k^2-k+1$,
- *if* $n \le m+k, m > k$, *then* $\dim_F \mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] = 2k(n-1)+2m(n-m)-k(k-1)-n^2+(m-k)^2+1 = 2kn-k+2mn-m^2-n^2-2mk+1$,
- *if* $n > m+k, m \le k$, *then* $\dim_F \mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] = 2k(n-1)+2m(n-m)-k(k-1)-n^2+(n-m-k)^2+1 = -k-m^2+2mk+1$,
- *if* $n \le m+k, m \le k$, *then* $\dim_F \mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] = 2k(n-1)+2m(n-m)-k(k-1)-n^2+1 = 2kn-k+2mn-2m^2-k^2-n^2+1$.

Proposition 41. *Variety* $\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$ *is irreducible for any* $(r_1, ..., r_k) \in F^k$. *And, hence, variety* $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$ *is irreducible too.*

Proof. Fix point $pt = (r_1, ..., r_k) \in F^k$. Recall that $tr^{-1}_{\Gamma_{k,1}}(pt) = \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$. Denote by U' the affine space $pr_2^{-1}(pt)$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccc} tr^{-1}_{\Gamma_{n,1}}(U) & \longrightarrow & U' \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_2} \\ \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)] & \longrightarrow & pt \end{array} \qquad (103)$$

Consider surjective morphism $tr_{\Gamma_{n,1}} : tr^{-1}_{\Gamma_{n,1}}(U') \to U'$. Using corollary 26, we obtain that for any point $u = (r_1, ..., r_n) \in U'$, variety $tr^{-1}_{\Gamma_{n,1}}(u) = \mathbf{Rep}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$ is irreducible and has dimension $2(n-m-1)(m-1)$. Recall the following property of morphisms: if Y is irreducible, morphism $f : X \to Y$ is a dominant, all fibers are irreducible and has the same dimension, then X is an irreducible variety. Using this property, we get $tr^{-1}_{\Gamma_{n,1}}(U')$ is irreducible for any irreducible $U' \subseteq F^{n-1}$. Because of morphism pr_1 is surjective, we obtain that variety $\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$ is irreducible. □

As we know from proposition 26, morphism $tr_{n,1}$ is equidimensional. One can prove that morphisms pr_1 and pr_2 are equidimensional. Using irreducibility of the varieties and surjectivity of the morphisms, we get that morphism $tr_{k,1}$ is equidimensional.

Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{n,1})[(\vec{1}, m)] & \xrightarrow{\pi_{n,1}} & \mathcal{M}_n\mathbf{Pr}(\Gamma_{n,1})[(\vec{1}, m)] & \xrightarrow{\mathrm{Tr}_{n,1}} & F^{n-1} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle p_{\mathcal{M}}} & & \downarrow{\scriptstyle pr_2} \\ \mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[(\vec{1}, m)] & \xrightarrow{\pi_{k,1}} & \mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[(\vec{1}, m)] & \xrightarrow{\mathrm{Tr}_{k,1}} & F^k, \end{array} \qquad (104)$$

where $\pi_{n,1}, \pi_{k,1}$ are natural surjective morphisms, morphisms $\mathrm{Tr}_{n,1}, \mathrm{Tr}_{k,1}$ are defined obviously. Clearly, $\mathrm{Tr}_{n,1} \circ \pi_{n,1} = tr_{n,1}$, $\mathrm{Tr}_{k,1} \circ \pi_{k,1} = tr_{k,1}$.

Proposition 42. *Morphism* $\mathrm{Tr}_{k,1}$ *is equidimensional. For* $k < n$ *and any* $r_i \in F, i = 1, ..., k$

$$\dim_F \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)] = k(2n-k-2) + 2m(n-m). \qquad (105)$$

There are several possibilities for $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$*:*

- *if* $n \le m+k$, $m \le k$, *then* $\dim_{\mathrm{F}} \mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)] = k(2n-k-2) + 2m(n-m) - n^2 + 1$.
- *if* $n > m+k$, $m \le k$, *then* $\dim_{\mathrm{F}} \mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)] = k(2n-k-2) + 2m(n-m) - n^2 + (n-m-k)^2 + 1 = (2k-m-1)(m-1)$,

- *if* $n \leq m+k,\ m>k,$ *then* $\dim_{\mathrm{F}}\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1},m)] = k(2n-k-2)+2m(n-m)-n^2+(m-k)^2+1 = (2k+m-n-1)(n-m-1),$

- *if* $n>m+k,\ m>k,$ *then* $\dim_{\mathrm{F}}\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1},m)] = k(2n-k-2)+2m(n-m)-n^2+(n-m-k)^2+(m-k)^2+1 = (k-1)^2.$

Proof. Since $tr_{n,1}$ and pr_2 are surjective and equidimensional, we get that $pr_2\circ tr_{n,1} = tr_{k,1}\circ pr_1$ is equidimensional.

Let us formulate the following useful obvious statement: Assume that X, Y, Z are irreducible varieties. Morphisms $f: X\to Z$, $g: X\to Y$ and $h: Y\to Z$ are surjective morphisms with relation: $f = h\circ g$. Suppose that f is equidimensional. Then h and g are equidimensional.

Using this statement and surjectivity of pr_1 and $tr_{k,1}$, we obtain that pr_1 and $tr_{k,1}$ are equidimensional. Also, we get that $Tr_{k,1}$ and $\pi_{k,1}$ are equidimensional. Also, we get that morphism $p_{\mathcal{M}}$ is surjective and equidimensional.

As we know from corollary 11, $\dim_F\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{n,1})[(\vec{1},m)] = n(n-1)+2m(n-m)$. Using dimension of fiber, we get $\dim_F\mathbf{Rep}_n\mathbf{Pr}(\Gamma_{k,1})[(\vec{1},m)] = n(n-1)+2m(n-m)-(n-k)(n-k-1)$. Using equidimensionality of $tr_{k,1}$, we obtain that

$$\dim_F\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1},m)] = k(2n-k-2)+2m(n-m).$$

Analogous arguments prove the rest. □

Let us come back to algebra $F\mathbb{Q}_{\Gamma[k]}$ from subsection 5.1. Denote by $i, i=1,...,k$ and β_{ij} the vertices and arrows of the quiver $\mathbb{Q}_{\Gamma[k]}$. For any $i=1,...,k$, let us denote by e_i the projectors corresponding to vertex v_i. As we know, we have the following isomorphism of the algebras:

$$F\mathbb{Q}_{\Gamma[k]} \cong (\sum_{i=1}^{k} q_i)\widetilde{\mathcal{A}}_k(r_i)(\sum_{i=1}^{k} q_i). \tag{106}$$

defined by rule:

$$e_i \mapsto q_i, \beta_{ij}\mapsto q_iPq_j \tag{107}$$

Thus, we have the following isomorphism:

$$\mathrm{Tr}F\mathbb{Q}_{\Gamma[k]} = F\mathbb{Q}_{\Gamma[k]}/[F\mathbb{Q}_{\Gamma[k]}, F\mathbb{Q}_{\Gamma[k]}] \cong \mathrm{Tr}\widetilde{\mathcal{A}}_k^{++}(r_i) = \widetilde{\mathcal{A}}_k^{++}(r_i)/[\widetilde{\mathcal{A}}_k^{++}(r_i), \widetilde{\mathcal{A}}_k^{++}(r_i)] \tag{108}$$

Actually, consider element q_iP. We have the following identity: $q_iP - r_iq_i = q_iP - q_iPq_i = [q_iP, q_i]$. Therefore, any element of type $Pq_{i_1}P...Pq_{i_s}P$ can be expressed as follows: $q_{i_1}P...Pq_{i_1} + \quad commutators$. It is easy that $\mathrm{Tr}\widetilde{\mathcal{A}}_k(r_i) = F\mathrm{Tr}1\oplus F\mathrm{Tr}P\oplus\mathrm{Tr}\widetilde{\mathcal{A}}_k^{++}(r_i)$. Denote by $\mathcal{M}\mathbb{Q}_{\Gamma[k]}[\vec{1}]$ the variety of $F\mathbb{Q}_{\Gamma[k]}$ - modules of dimension vector $\vec{1}$. We have the following result:

Proposition 43.

- $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]\subset\mathcal{M}\mathbb{Q}_k[\vec{1}]$.
- *if* $m\geq k, n\geq m+k,$ *then* $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]\cong\mathcal{M}\mathbb{Q}_k[\vec{1}]$.

Proof. Prove the first statement. There is a functor: $\theta_1: \widetilde{\mathcal{A}}_{\mathrm{k}}(\mathrm{r_i})-\mathrm{mod}\to \mathrm{F}\mathbb{Q}_{\Gamma[\mathrm{k}]}-\mathrm{mod}$ defined by correspondence:

$$\theta_1: V\mapsto \mathrm{Hom}_{\widetilde{\mathcal{A}}_k(r_i)}(\widetilde{\mathcal{A}}_k(r_i)\sum_{i=1}^{k} q_i, V). \tag{109}$$

Consider morphism $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]\to\mathcal{M}F\mathbb{Q}_{\Gamma[k]}[\vec{1}]$ defined by this correspondence. It is well-known from geometric invariant theory that point of moduli variety corresponds to closed orbit. Using this statement, we get that two $\widetilde{\mathcal{A}}_k(r_i)$ - modules V_1 and V_2 such that $[V_1]=[V_2]\in\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)$ iff their characters are the same, i.e. $\mathrm{Tr}_{V_1}(x) = \mathrm{Tr}_{V_2}(x)$ for any $x\in\widetilde{\mathcal{A}}_k(r_i)$. Let us prove that if V_1 and V_2 corresponds to fixed $W\in\mathcal{M}F\mathbb{Q}_{\Gamma[k]}[\vec{1}]$, then characters of V_1 and V_2 are the same. Using isomorphism (108), we get that $\mathrm{Tr}_{V_1}(x) = \mathrm{Tr}_{V_2}(x)$ for any $x \in \widetilde{\mathcal{A}}_k^{++}(r_i)$ and $\mathrm{Tr}1 = n, \mathrm{Tr}P = m$. Thus, we have proved the first statement.

Using isomorphism $F\mathbb{Q}_{\Gamma[k]} \cong \sum_{i=1}^k q_i \widetilde{\mathcal{A}}_k(r_i) \sum_{i=1}^k q_i$, we get the functor: $\theta_2 : \mathrm{F}\mathbb{Q}_{\Gamma[k]} - \mathrm{mod} \to \widetilde{\mathcal{A}}_{\mathrm{k}}(\mathrm{r_i}) - \mathrm{mod}$ defined by rule:

$$\theta_2 : W \mapsto \widetilde{\mathcal{A}}_k(r_i) \sum_{i=1}^k q_i \otimes_{F\mathbb{Q}_{\Gamma[k]}} W. \tag{110}$$

Fix $F\mathbb{Q}_{\Gamma[k]}$-module W. In this case $\theta_2(W) \in \mathcal{M}_{2k}\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, k]$. We can consider the following $\widetilde{\mathcal{A}}_k(r_i)$-module: $FP^{\oplus(m-k)} \oplus F(1-P)^{\oplus(n-m-k)} \oplus \theta_2(W) \in \mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$. Since θ_1 and θ_2 are adjoint, we get the required statement. □

As we know, we have analogous statement for algebra $B_{\mathbf{r}}(\Gamma)$(cf.BZ). Let $\Delta(\Gamma)$ be a Laplacian of graph Γ. Let us formulate the following useful proposition for $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)[1]$:

Proposition 44. *We have the following immersion:* $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)[1] \subset (F^*)^{rkH_1(\Gamma)}$. *Variety* $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)[1]$ *is given by condition* $\mathrm{rank}\Delta(\Gamma) \leq n$. *In particular, if* $|V(\Gamma)| \leq n$, *then* $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)[1] \cong (F^*)^{rkH_1(\Gamma)}$.

Also, note the following useful property of varieties $\mathcal{M}_n B_{\mathbf{r}}(\Gamma)$: if $n_1 \leq n_2$, then there is an immersion: $\mathcal{M}_{n_1} B_{\mathbf{r}}(\Gamma)[\alpha] \to \mathcal{M}_{n_2} B_{\mathbf{r}}(\Gamma)[\alpha]$ defined as follows. Fix $B_{\mathbf{r}}(\Gamma)$-module W of dimension n_1. Consider direct sum: $W \oplus F^{n_2-n_1}$, where F is a trivial $B_{\mathbf{r}}(\Gamma)$-module. One can show that this correspondence is an immersion.

Recall that we have morphisms $i : \widetilde{\mathcal{A}}_k(r_i) \to B_{\mathbf{r}}(\Gamma_{k,m})$ and $\phi : B_{\mathbf{r}}(\Gamma_{k,m}) \to F\Gamma_{k,m}$. Thus, we have the following useful morphism:

$$\mathrm{Tr}F\mathbb{Q}_k \cong \mathrm{Tr}\widetilde{A}_k^{++}(r_i) \to \mathrm{Tr}B_{\mathbf{r}}^{+}(\Gamma_{k,m}) \cong \mathrm{Tr}\Gamma_{k,m}, \tag{111}$$

where $\mathrm{Tr}\Gamma_{k,m}$ is a vector space of free loops in the graph $\Gamma_{k,m}$. Also, we have homomorphisms of symmetric algebras:

$$S^{\bullet}\mathrm{Tr}F\mathbb{Q}_k = S^{\bullet}\mathrm{Tr}\widetilde{A}_k^{++}(r_i) \to S^{\bullet}\mathrm{Tr}B_{\mathbf{r}}(\Gamma_{k,m}) = S^{\bullet}\mathrm{Tr}\Gamma_{k,m} \tag{112}$$

Let us describe this morphism in coordinates. Consider element $i(\mathrm{Tr}q_{i_1}P...q_{i_l}P) \in S^{\bullet}\mathrm{Tr}B_{\mathbf{r}}(\Gamma_{k,m})$. It is easy that

$$i(\mathrm{Tr}q_{i_1}P...q_{i_l}P) = \mathrm{Tr}q_{i_1}i(P)...q_{i_l}i(P) = \mathrm{Tr}q_{i_1}(p_1 + ... + p_m)...q_{i_l}(p_1 + ... + p_m) \tag{113}$$

Thus, we get that $i(\mathrm{Tr}q_{i_1}...Pq_{i_s}P)$ is a product of s elements:

$$\mathrm{Tr}q_{i_1}(p_1+...+p_m)...q_{i_l}(p_1+...+p_m)q_{i_1} = c(\mathbf{s})\mathrm{Tr}q_{i_1}(p_1+...+p_m)q_{i_2}p_1q_{i_1} \cdot q_{i_1}p_1q_{i_2}(p_1+...+p_m)q_{i_3}p_1q_{i_1}... \tag{114}$$

$q_{i_1}p_1q_{i_s}(p_1 + ... + p_m)q_{i_1} = c(\mathbf{r})\mathrm{Tr}q_{i_1}(p_1 + ... + p_m)q_{i_2}p_1 \cdot \mathrm{Tr}q_{i_1}p_1q_{i_2}(p_1 + ... + p_m)... \cdot \mathrm{Tr}q_{i_1}p_1q_{i_l}(p_1 + ... + p_m)q_{i_1}$, where $c(\mathbf{r}) = r_{1,1}^{l-1}r_{1,2}...r_{1,l}$. Analogous statement for $i'(\mathrm{Tr}q_{i_1}...Pq_{i_l}P)$ is true. One can describe formula (114) in terms of path algebras.

5.4 Relation between $\mathcal{M}_n\mathcal{A}_n(r_i)$ and $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)$.

Also, note the following relation between varieties $\mathcal{M}_n\mathcal{A}_n(r_i)$ and $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)$. Consider partition of set $(1, ..., n)$ into two complement subsets $(1, ..., k) \cup (k+1, ..., n)$. Consider morphisms of algebras:

$$i_1 : \widetilde{\mathcal{A}}_k(r_i) \to \mathcal{A}_n(r_i), i_2 : \widetilde{\mathcal{A}}_{n-k}(r_i) \to \mathcal{A}_n(r_i), \tag{115}$$

defined by natural way. For unification, denote by $\widetilde{\mathcal{A}}$ the unital algebra generated by elements P, Q with relations: $P^2 = P, Q^2 = Q$. Clearly, $\widetilde{\mathcal{A}} \cong \mathbf{Pr}(\Gamma_{1,1})$. We have the following morphisms: $j_1 : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_k(r_i), j_2 : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_{n-k}(r_i)$ defined by correspondences: $j_1 : (P, Q) \mapsto (P, q_1 + ... + q_k), j_2 : (P, Q) \mapsto (P, 1 - \sum_{i=k+1}^n q_i)$. It is easy that we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_k(r_i) & \xrightarrow{i_1} & \mathcal{A}_n(r_i) \\ \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle i_2} \\ \widetilde{\mathcal{A}} & \xrightarrow{j_2} & \widetilde{\mathcal{A}}_{n-k}(r_i) \end{array} \tag{116}$$

It can be shown in usual way that

$$\mathcal{A}_n(r_i) \cong \widetilde{\mathcal{A}}_k(r_i) *_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}}_{n-k}(r_i) \tag{117}$$

Therefore, we obtain the following isomorphism of the varieties:

$$\mathbf{Rep}_n\mathcal{A}_n(r_i)[\vec{1}, m] \cong \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] \times_{\mathbf{Rep}_n\widetilde{\mathcal{A}}[k,m]} \mathbf{Rep}_n\widetilde{\mathcal{A}}_{n-k}(r_i)[\vec{1}, m], \tag{118}$$

where $\mathbf{Rep}_n\widetilde{\mathcal{A}}[k, m]$ is a variety of projectors P, Q of rank k, m respectively. Consider $\mathrm{GL}_n(F)$ - invariant divisor D_r of $\mathbf{Rep}_n\widetilde{\mathcal{A}}[k, m]$ defined by relation $\mathrm{Tr}PQ = \sum_{i=1}^{k} r_i = r$. It is clear that

$$\mathbf{Rep}_n\mathcal{A}_n(r_i)[\vec{1}, m] \cong \mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] \times_{D_r} \mathbf{Rep}_n\widetilde{\mathcal{A}}_{n-k}(r_i)[\vec{1}, m] \tag{119}$$

Of course, these results are true for another partition with obvious substitutions.

Further, let us consider quotients of these varieties by $\mathrm{GL}_n(F)$. We have morphism:

$$\pi : \mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1}, m] \to \mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] \times_{\mathcal{D}_r} \mathcal{M}_n\widetilde{\mathcal{A}}_{n-k}(r_i)[\vec{1}, m], \tag{120}$$

where $\mathcal{D}_r$ is a quotient $D_r/\mathrm{GL}_n(F)$.

Recall general fact about ring of $\mathrm{GL}_n(F)$ - invariant functions. Consider algebra A. Then $\mathcal{O}(\mathcal{M}_n(A))^{\mathrm{GL}_n(F)}$ is generated by functions $\mathrm{Tr}a, a \in A$. Using isomorphism (108), we get that generators of $\mathcal{O}(\mathcal{M}_n(\widetilde{\mathcal{A}}_k(r_i)))^{\mathrm{GL}_n(F)}$ are necklaces in quiver $\mathbb{Q}_k$, i.e. equivalence classes of cycles in quiver $\mathbb{Q}_k$. We will say that necklace is a *generating* if this necklace has no self-intersections. It can be shown in usual way that we can choose generating necklaces as generators of $\mathcal{O}(\mathcal{M}_k\mathbb{Q}_{\Gamma[k]}[\vec{1}])$. We can describe this fact in the following terms:

Corollary 45. *Ring* $\mathcal{O}(\mathcal{M}_n\mathcal{A}_n(r_i)[(\vec{1}, m)]$ *is generated by elements* $\mathrm{Tr}Pq_{i_1}Pq_{i_2}...Pq_{i_s}, s \leq n$. *Also,* $\mathcal{O}(\mathcal{M}_n\mathcal{A}_n(r_i))[(\vec{1}, m)]$ *is generated by elements* $\mathrm{Tr}Pq_{i_1}Pq_{i_2}...Pq_{i_s}, s \leq k$.

Denote by $t_{(i_1,...,i_m)}$ the function $\mathrm{Tr}(Pq_{i_1}Pq_{i_2}...Pq_{i_m})$. Using isomorphism (108), we can consider $t_{i_1,...,i_m}$ as necklace in quiver $\mathbb{Q}_k$, i.e. cycle up to cyclic permutation of vertices. Using irreducibility of variety $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$, we get the following

Proposition 46. *Assume* $k \geq 3$. *Then field of rational functions* $F(\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)])$ *has the following generators:* $t_{(i_1,i_2)}, t_{(i_1,i_2,i_3)}$, $i_1, i_2, i_3 = 1, ..., k$, $i_1 \neq i_2, i_1 \neq i_3, i_2 \neq i_3$.

Proof. As we know, variety $\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[(\vec{1}, m)]$ is a subvariety of $\mathcal{M}_n\mathbb{Q}_k[\vec{1}]$. It can be shown in usual way that generators of $F(\mathcal{M}_n\mathbb{Q}_k[\vec{1}])$ are necklaces of length less or equal 3. Actually, show that necklace $\mathrm{Tr}\beta_{12}\beta_{23}\beta_{34}\beta_{41}$ can be expressed in terms of necklace of length at more 3. Consider following element: $\mathrm{Tr}\beta_{12}\beta_{23}\beta_{34}\beta_{41} \cdot \mathrm{Tr}\beta_{13}\beta_{31} \in S^\bullet\mathrm{Tr}\mathbb{Q}_{\Gamma[k]}$. Using equivalence relation, we get the following formula: $\mathrm{Tr}\beta_{31}\beta_{12}\beta_{23} \cdot \mathrm{Tr}\beta_{34}\beta_{41}\beta_{13} = \mathrm{Tr}\beta_{12}\beta_{23}\beta_{34}\beta_{41} \cdot \mathrm{Tr}\beta_{13}\beta_{31}$, and hence,

$$\mathrm{Tr}\beta_{12}\beta_{23}\beta_{34}\beta_{41} = \frac{\mathrm{Tr}\beta_{31}\beta_{12}\beta_{23} \cdot \mathrm{Tr}\beta_{34}\beta_{41}\beta_{13}}{\mathrm{Tr}\beta_{13}\beta_{31}} \tag{121}$$

The rest is easy. □

Let $Par(k, n)$ be the set of partition of set $(1, ..., n)$ into two complement subset consisting of k and $n - k$ elements. For any partition $\theta = (i_1, ..., i_k) \cup (i_{k+1}, ..., i_n)$, denote by $\widetilde{\mathcal{A}}^\theta_k(r_i)$ and $\widetilde{\mathcal{A}}^\theta_{n-k}(r_i)$ the algebras generated by $P; q_{i_1}, ..., q_{i_k}$ and $P; q_{i_{k+1}}, ..., q_{i_n}$ respectively. As we know, $\mathcal{A}_n(r_i) \cong \widetilde{\mathcal{A}}^\theta_k(r_i) *_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}}^\theta_{n-k}(r_i)$, where morphisms are defined obviously. Therefore, we have the following morphism:

$$\pi^\theta : \mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1}, m] \to \mathcal{M}_n\widetilde{\mathcal{A}}^\theta_k(r_i)[\vec{1}, m] \times_{\mathcal{D}_r} \mathcal{M}_n\widetilde{\mathcal{A}}^\theta_{n-k}(r_i)[\vec{1}, m] \tag{122}$$

It is easy that π defined by formula (120) is π^θ for $\theta = (1, ..., k) \cup (k + 1, ..., n)$.

Define morphism Π as follows:

$$\Pi = \prod_{\theta \in Par(k,n)} \pi^\theta : \mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1}, m] \to \prod_{\theta \in Par(k,n)} \mathcal{M}_n\widetilde{\mathcal{A}}^\theta_k(r_i)[\vec{1}, m] \times_{\mathcal{D}_r} \mathcal{M}_n\widetilde{\mathcal{A}}^\theta_{n-k}(r_i)[\vec{1}, m]. \tag{123}$$

Proposition 47. *Morphism* Π *is a birational isomorphism of* $\mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1},m]$ *on its image, i.e. morphism* Π *from (123) is a birational immersion.*

Proof. Consider open subvariety U of $\mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1},m]$ defined by relations $t_{(i_1,i_2)} \neq 0$ for all possible i_1, i_2. Since variety $\mathcal{M}_n\mathcal{A}_n(r_i)[\vec{1},m]$ is irreducible, then U is dense in it. Using proposition 46, the ring $\mathcal{O}(U)$ is generated by $t^{\pm 1}_{(i_1,i_2)}, t_{(i_1,i_2,i_3)}$. Denote by U_θ the image of U under π^θ. Morphism $\pi^{\theta *} : \mathcal{O}(U_\theta) \to \mathcal{O}(U)$ is an injective. Moreover, $\bigotimes_\theta \mathcal{O}(U_\theta) = \mathcal{O}(\prod_\theta U_\theta)$ contains all $t_{(i_1)}, t^{\pm 1}_{(i_1,i_2)}, t_{(i_1,i_2,i_3)}$, and hence, natural morphism: $\mathcal{O}(\prod_\theta U_\theta) \to \mathcal{O}(U)$ is surjective. It means that morphism

$$\Pi : U \to \prod_\theta U_\theta \tag{124}$$

is an immersion. Therefore, we get the required statement. □

Remark. Note that we don't require that variety $\mathcal{M}_n\widetilde{\mathcal{A}}^\theta_k(r_i)[\vec{1},m] \times_{\mathcal{D}_r} \mathcal{M}_n\widetilde{\mathcal{A}}^\theta_{n-k}(r_i)[\vec{1},m]$ is irreducible. Of course, these results have obvious generalizations on the case of variety $\mathcal{M}_n\mathbf{Pr}(\Gamma_{k,1})[\vec{1},m]$.

Also, note the following useful result. Without loss of generality, consider a partition $\{1,...,n\} = \{1,...,m\} \cup \{m+1,...,n\}$. In this case, we have the isomorphisms: $B_{k,n} \cong B_{\mathbf{r}}(\Gamma_{k,m}) *_{\widetilde{\mathcal{A}}_k(r_i)} B_{\mathbf{r}}(\Gamma_{k,n-m})$ and $\mathcal{A}_n(r_i) \cong \widetilde{\mathcal{A}}_m(r_i) *_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}}_{n-m}(r_i)$. We can define morphisms: $\widetilde{\mathcal{A}}_m(r_i) \to B_{\mathbf{r}}(\Gamma_{k,m})$, $\widetilde{\mathcal{A}}_{n-m}(r_i) \to B_{\mathbf{r}}(\Gamma_{k,n-m})$ and $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}_k(r_i)$ by formulas: $(P; q_1,...,q_m) \mapsto (p_1+...+p_k; q_1,...,q_m)$, $(P; q_{m+1},...,q_n) \mapsto (p_1+...+p_k; q_{m+1},...,q_n)$ and $(P;Q) \mapsto (P; q_1+...+q_k)$. One can check that we have the following commutative diagram:

$$\begin{array}{ccccc} & \widetilde{\mathcal{A}}_m(r_i) & \longrightarrow & & B_{\mathbf{r}}(\Gamma_{k,m}) \\ \nearrow & & & \nearrow & \\ \widetilde{\mathcal{A}} & \longrightarrow & \widetilde{\mathcal{A}}_k(r_i) & & \\ \searrow & & & \searrow & \\ & \widetilde{\mathcal{A}}_{n-m}(r_i) & \longrightarrow & & B_{\mathbf{r}}(\Gamma_{k,n-m}) \end{array} \tag{125}$$

Therefore, we have a well-defined morphism:

$$\widetilde{\mathcal{A}}_m(r_i) *_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}}_{n-m}(r_i) \to B_{\mathbf{r}}(\Gamma_{k,m}) *_{\widetilde{\mathcal{A}}_k(r_i)} B_{\mathbf{r}}(\Gamma_{k,n-m}). \tag{126}$$

Also, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_n(r_i) & \longrightarrow & B_{k,n} \\ \uparrow \cong & & \uparrow \cong \\ \widetilde{\mathcal{A}}_m(r_i) *_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}}_{n-m}(r_i) & \longrightarrow & B_{\mathbf{r}}(\Gamma_{k,m}) *_{\widetilde{\mathcal{A}}_k(r_i)} B_{\mathbf{r}}(\Gamma_{k,n-m}) \end{array} \tag{127}$$

Further, let us apply functor **Rep** to this commutative diagram. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Rep}_n B_{k,n} & \longrightarrow & \mathbf{Rep}_n\widetilde{\mathcal{A}}_n(r_i)[\vec{1},k] \\ \downarrow \cong & & \downarrow \cong \\ \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathbf{Rep}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathbf{Rep}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1] & \longrightarrow & \mathbf{Rep}_n\widetilde{\mathcal{A}}_m(r_i)[\vec{1},k] \times_{\mathbf{Rep}_n\widetilde{\mathcal{A}}[m,k]} \mathbf{Rep}_n\widetilde{\mathcal{A}}_{n-m}(r_i)[\vec{1},k] \end{array} \tag{128}$$

Also, we can take quotient by $\mathrm{GL}_n(F)$ - action. Therefore, we get the following proposition:

Proposition 48. *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}_nB_{k,n}[1] & \longrightarrow & \mathcal{M}_n\widetilde{\mathcal{A}}_n(r_i)[\vec{1},k] \\ \downarrow & & \downarrow \\ \mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,n-m}) & \longrightarrow & \mathcal{M}_n\widetilde{\mathcal{A}}_m(r_i)[\vec{1},k] \times_{\mathcal{M}_n\widetilde{\mathcal{A}}[m,k]} \mathcal{M}_n\widetilde{\mathcal{A}}_{n-m}(r_i)[\vec{1},k] \end{array} \tag{129}$$

Remark. This proposition will play important role in the proof of main result of this paper.

6 Moduli varieties $\mathcal{M}_nB_{k,n}$ and $\mathcal{M}_nB_{n,n}$.

In this section we will study properties of morphisms $\mathcal{M}_nB_{k,n} \to \mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,n-m})$ and $\mathcal{M}_nB_{n,n} \to \mathcal{M}_nB_{n,m} \times_{\mathcal{M}_n\widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_nB_{n,n-m}$.

6.1 Description of moduli variety $\mathcal{M}_nB_{k,n}$.

In this subsection we will consider representations of $B_{\mathbf{r}}(\Gamma)$ for some graph Γ. Fix $\mathbf{r} = (r_{ij} \in F^*), (ij) \in E(\Gamma)$.

Consider variety $\mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,m})[1]$ for $k+m>n$. As we know from BZ, $\mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,m})$ is a subvariety of $(F^*)^{(k-1)(m-1)}$. Let us describe this subvariety in terms of the Laplacian of the graph $\Gamma_{k,m}$. Consider matrix of Laplacian Δ of graph $\Gamma_{k,m}$:

$$\Delta = \begin{pmatrix} 1 & 0 & ... & 0 & s_{11} & s_{12} & ... & s_{1m} \\ 0 & 1 & ... & 0 & s_{21} & s_{22}x_{22} & ... & s_{2m}x_{2m} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & s_{k1} & s_{k2}x_{k2} & ... & s_{km}x_{km} \\ s_{11} & s_{21} & ... & s_{k1} & 1 & 0 & ... & 0 \\ s_{12} & \frac{s_{22}}{x_{22}} & ... & \frac{s_{k2}}{x_{k2}} & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ s_{1m} & \frac{s_{2m}}{x_{2m}} & ... & \frac{s_{km}}{x_{km}} & 0 & 0 & ... & 1 \end{pmatrix} \tag{130}$$

Denote by E_k and E_m the identity matrices of size k and m respectively.

Lemma 49. *Consider matrix of the following type:*

$$\Delta = \begin{pmatrix} E_m & A \\ B & E_k \end{pmatrix} \tag{131}$$

Then $\mathrm{rank}\Delta \le n$ *iff* $\mathrm{rank}(BA - E_k) \le n-m$.

Proof.

$$\begin{pmatrix} E_m & -A \\ 0 & E_k \end{pmatrix} \cdot \begin{pmatrix} E_m & A \\ B & E_k \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ B & -BA+E_k \end{pmatrix} \tag{132}$$

□

Corollary 50. *Variety* $\mathcal{M}_nB_{\mathbf{r}}(\Gamma_{k,m})[1] \subset (F^*)^{(k-1)(m-1)}$ *is defined by condition:*

$$\mathrm{rank}(\begin{pmatrix} s_{11} & s_{12} & ... & s_{1m} \\ s_{21} & s_{22}x_{22} & ... & s_{2m}x_{2m} \\ ... & ... & ... & ... \\ s_{k1} & s_{k2}x_{k2} & ... & s_{km}x_{km} \end{pmatrix} \cdot \begin{pmatrix} s_{11} & s_{21} & ... & s_{k1} \\ s_{12} & \frac{s_{22}}{x_{22}} & ... & \frac{s_{k2}}{x_{k2}} \\ ... & ... & ... & ... \\ s_{1n} & \frac{s_{2n}}{x_{2n}} & ... & \frac{s_{kn}}{x_{kn}} \end{pmatrix} - \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix}) \le n-m \tag{133}$$

Further, apply this lemma to the case of $\mathcal{M}_n B_{k,n}$. It can be shown in usual way that rank$\Delta \geq n$. Also, Δ has rank n iff

$$\begin{pmatrix} s_{11} & s_{12} & ... & s_{1n} \\ s_{21} & s_{22}x_{22} & ... & s_{2n}x_{2n} \\ ... & ... & ... & ... \\ s_{k1} & s_{k2}x_{k2} & ... & s_{kn}x_{kn} \end{pmatrix} \begin{pmatrix} s_{11} & s_{21} & ... & s_{k1} \\ s_{12} & \frac{s_{22}}{x_{22}} & ... & \frac{s_{k2}}{x_{k2}} \\ ... & ... & ... & ... \\ s_{1n} & \frac{s_{2n}}{x_{2n}} & ... & \frac{s_{kn}}{x_{kn}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix} \tag{134}$$

As we know from subsection 3.3, we have the following relations:

$$\sum_{j=1}^{n} s_{ij}^2 = \sum_{j=1}^{n} r_{ij} = 1, i = 1, ..., k \tag{135}$$

Thus, we get $k(k-1)$ equations defining variety $\mathcal{M}_n B_{k,n} \subset (F^*)^{(k-1)(n-1)}$.

Remark. Consider case $r_{ij} = \frac{1}{n}$ and $k = n$. It is easy that these equations coincide with equations defining generalized Hadamard matrix.

Let us formulate the following useful proposition:

Proposition 51. *For any irreducible component C of $\mathcal{M}_n B_{k,n}$, we have the following inequality:*

$$\dim_F C \geq (k-1)(n-1) - k(k-1) = (n-k-1)(k-1). \tag{136}$$

Proof. Straightforward. □

6.2 The fibred product.

In this subject we will study properties of morphisms: $f'_n : \mathcal{M}_n B_{k,n} \to \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1]$ and $f_n : \mathcal{M}_n B_{n,n} \to \mathcal{M}_n B_{n,m} \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{n,n-m}$.

Remark. Note that if $k \leq \frac{n}{2}$, then we can choose m such that $k + m \leq n, k + n - m \leq n$. Using proposition 44, morphism $f_{k,n}(m)$ has the following view:

$$f'_n : \mathcal{M}_n B_{k,n} \to (F^*)^{(k-1)(m-1)} \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} (F^*)^{(k-1)(n-m-1)} \tag{137}$$

Recall that $\mathcal{M}_n B_{k,n}$ is a subvariety defined by equations (134). Consider composition of morphisms: $\widetilde{\mathcal{A}}_k(r_i) \to B_{\mathbf{r}}(\Gamma_{k,m}) \to F\Gamma_{k,m}$. Using proposition 44, we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{k+m} B_{\mathbf{r}}(\Gamma_{k,m})[1] & \xrightarrow{\cong} & (F^*)^{(k-1)(m-1)} \\ \downarrow & \swarrow & \\ \mathcal{M}_{k+m} \widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] & & \end{array} \tag{138}$$

If $n < k + m$, then we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] & \xrightarrow{s_1} & (F^*)^{(k-1)(m-1)} \\ \downarrow i^* & \swarrow & \\ \mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] & & \end{array} \tag{139}$$

where s_1 is an immersion. Also, one can consider similar commutative diagram for algebra $B_{\mathbf{r}}(\Gamma_{k,n-m})$. Denote by s_2 the natural morphism: $\mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1] \to (F^*)^{(k-1)(n-m-1)}$. One can show that we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] & \xrightarrow{s_1} & (F^*)^{(k-1)(m-1)} \\ \downarrow{\scriptstyle i^*} & \swarrow & \\ \mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m] & & \\ \uparrow{\scriptstyle i'^*} & \nwarrow & \\ \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1] & \xrightarrow{s_2} & (F^*)^{(k-1)(n-m-1)} \end{array} \tag{140}$$

Therefore, we have well-defined fibred product: $(F^*)^{(k-1)(m-1)} \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} (F^*)^{(k-1)(n-m-1)}$ and immersion $S = s_1 \times s_2 : M_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1] \to (F^*)^{(k-1)(m-1)} \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} (F^*)^{(k-1)(n-m-1)}$.

Denote the coordinates of $(F^*)^{(k-1)(n-1)}$, $(F^*)^{(k-1)(m-1)}$ and $(F^*)^{(k-1)(n-m-1)}$ by $x_{2,2}, ..., x_{k,n}$, $z_{2,2}, ..., z_{k,m}$ and $y_{2,2}, ..., y_{k,n-m}$ respectively. Morphism pr_m is defined by formula:

$$pr_m : \begin{pmatrix} x_{2,2} & ... & x_{k,2} \\ ... & ... & ... \\ x_{2,n} & ... & x_{k,n} \end{pmatrix} \mapsto \left(\begin{pmatrix} x_{2,2} & ... & x_{k,2} \\ ... & ... & ... \\ x_{2,m} & ... & x_{k,m} \end{pmatrix}, \begin{pmatrix} \frac{x_{2,m+2}}{x_{2,m+1}} & ... & \frac{x_{k,m+2}}{x_{k,m+1}} \\ ... & ... & ... \\ \frac{x_{2,n}}{x_{2,m+1}} & ... & \frac{x_{k,n}}{x_{k,m+1}} \end{pmatrix} \right) \tag{141}$$

We have the following commutative diagram:

$$\begin{array}{ccccc} & & (F^*)^{(k-1)(n-1)} & & \\ & \overset{\subseteq}{\nearrow} & & \overset{pr_m}{\searrow} & \\ \mathcal{M}_n B_{k,n} & & \longrightarrow & & (F^*)^{(k-1)(m-1)} \times (F^*)^{(k-1)(n-m-1)} \\ & \underset{f'_n}{\searrow} & & \underset{S'}{\nearrow} & \\ & & M_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1] & & \end{array} \tag{142}$$

Morphism S' is a composition of S and natural immersion. Thus, morphism f'_n is a map of elimination of $k-1$ variables.

Describe fibred product $\mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})[1] \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})[1]$ as subvariety of the algebraic torus $(F^*)^{(k-1)(m-1)} \times (F^*)^{(k-1)(n-m-1)}$. For simplicity, if $i = 1$ or $j = 1$, then $x_{i,j} = 1, z_{i,j} = 1, y_{i,j} = 1$. Express these elements in terms of $z_{i,j}$. We will use the following notation:

$$h_{i_1,i_2} = \sum_{j=1}^{m} s_{i_1,j} s_{i_2,j} z_{i_1,j}/z_{i_2,j}, h'_{i_1,i_2} = \sum_{j=m+1}^{n} s_{i_1,j} s_{i_2,j} y_{i_1,j-m}/y_{i_2,j-m}. \tag{143}$$

Using formula (114) and definitions of homomorphisms i and i', one can obtain that

$$i(\mathrm{Tr} q_{i_1} ... P q_{i_l} P) = c(\mathbf{r}) \cdot h_{i_1,i_2} h_{i_2,i_3} ... h_{i_l,i_1}, i'(\mathrm{Tr} q_{i_1} ... P q_{i_l} P) = c'(\mathbf{r})(-1)^s \cdot h'_{i_1,i_2} h'_{i_2,i_3} ... h'_{i_l,i_1} \tag{144}$$

As we know from proposition 43, $\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]$ is a subvariety of $\mathcal{M}_k \mathbb{Q}_{\Gamma[k]}[\vec{1}]$. We can associate equation with any necklace $(i_1, ..., i_s)$ of $\mathbb{Q}_{\Gamma[k]}$ as follows:

$$c(\mathbf{r}) h_{i_1,i_2} ... h_{i_s,i_1} = c'(\mathbf{r})(-1)^s h'_{i_1,i_2} ... h'_{i_s,i_1}. \tag{145}$$

Using corollary 45, we can choose only generating necklaces.

Similar results for morphism f_n are true.

6.3 Subvarieties E_1 and E_2.

Define two subvarieties $E_1(f'_n)$ and $E_2(f'_n)$ of the fibred product as follows:

$$E_1(f'_n) = \{x \in \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m}) | \dim_F {f'_n}^{-1}(x) \geq 1\} \tag{146}$$

$$E_2(f'_n) = \{x \in \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m}) | {f'_n}^{-1}(x) = \emptyset\} \tag{147}$$

Consider point $x = (x_1, x_2) \in \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})$ such that $i^*(x_1) = i'^*(x_2) = x' \in \mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m]$. Using proposition 36, we get that fiber ${f'_n}^{-1}(x)$ is an algebraic torus of dimension less or equal $k-1$. Also, condition ${f'_n}^{-1}(x) \subseteq Aut_{\widetilde{\mathcal{A}}_k(r_i)}(x'')/F^*$, where x'' is a $\widetilde{\mathcal{A}}_k(r_i)$ - submodule of x' generated by $\mathrm{Im} q_i$ and $\mathrm{Im} P q_i$ (proposition 36). Using proposition 37, we get that if $\dim_F {f'_n}^{-1}(x) \geq l-1$ then graph $G_k(x'') = G_k(x')$ has at least l connected components. Thus, we have the following filtration of E_1:

$$E_1^{(k-1)}(f'_n) \subseteq ... \subseteq E_1^{(1)}(f'_n) = E_1(f'_n) \tag{148}$$

where $E_1^{(i)}(f'_n) = \{x \in E_1 | \dim_F {f'_n}^{-1}(x) \geq i\}$.

For fixed partition θ: $\{1, ..., k\}$ into $s+1$ non-intersecting subsets $I_1, ..., I_{s+1}$ consider $C'(\theta)$ the subvariety of $(F^*)^{(k-1)(m-1)} \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} (F^*)^{(k-1)(n-m-1)}$ defined by equations:

$$\sum_{i=1}^{m} s_{l_1,i} s_{l_2,i} z_{l_1,i}/z_{l_2,i} = 0, \sum_{i=1}^{m} s_{l_1,i} s_{l_2,i} z_{l_2,i}/z_{l_1,i} = 0, \tag{149}$$

$$\sum_{i=m+1}^{n} s_{l_1,i} s_{l_2,i} y_{l_1,i-m}/y_{l_2,i-m} = 0, \sum_{i=m+1}^{n} s_{l_1,i} s_{l_2,i} y_{l_2,i-m}/y_{l_1,i-m} = 0. \tag{150}$$

for any $l_1 \in I_{k_1}$ and $l_2 \in I_{k_2}$, $k_1 \neq k_2$.

Denote by $C(\theta) = S^{-1}(C'(\theta)) \subset \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})$.

Proposition 52. $E_1^{(s)}(f'_n) \subset \bigcup_\theta C(\theta)$, *where θ runs over all partitions of $\{1, ..., k\}$ into $s+1$ non-intersecting subsets.*

Proof. Using proposition 36, we get that $q_i P q_j = 0$ for any i, j from different subsets. Thus, we have to express condition $q_i P q_j$ in terms of algebra $B_{\mathbf{r}}(\Gamma_{k,m})$. We get that $q_{l_1}(p_1 + ... + p_m) q_{l_2} = 0$. Therefore, $q_{l_1}(p_1 + ... p_m) q_{l_2} p_1 = 0$ and $\mathrm{Tr}(q_{l_1}(p_1 + ... + p_m) q_{l_2} p_1) = \sum_{i=1}^{m} s_{i,l_1} s_{i,l_2} x_{i,l_1}/x_{i,l_2} = 0$. Analogously, one can obtain another equations. ☐

Further, consider subvariety $E_2(f'_n)$. It is easy that $x = (x_1, x_2) \in E_2(f'_n)$ iff there is non-empty intersection of closures of $\mathrm{GL}_n(F)$ - orbits of $i^*(x_1)$ and $i'^*(x_2)$ and $i^*(x_1) \neq i'^*(x_2)$. In this case, there is a semisimple representation $x'' \in \overline{i^*(x_1)} \cap \overline{i'^*(x_2)}$. Therefore, if $x \in E_2(f'_n)$ then x'' has non-trivial stabilizer. It is easy that characters of x and x'' are the same.

Proposition 53. *Let ρ be a representation of $\widetilde{\mathcal{A}}_k(r_i)$. Consider* $\mathrm{GL}_n(F)$ *- orbit of ρ - $O(\rho)$. Assume that there is a semisimple $\widetilde{\mathcal{A}}_k(r_i)$ - representation $\rho'' \in \overline{O(\rho)}$ with non-trivial stabilizer. Then there are two complement subsets I, J of $\{1, ..., k\}$ satisfying to condition:* $\mathrm{Tr}\rho(q_{i_1} P ... q_{i_s} P) = 0$ *if $\{i_1, ..., i_s\} \cap I \neq \emptyset$ and $\{i_1, ..., i_s\} \cap J \neq \emptyset$.*

Proof. Using proposition 36, we get that there are at least two subsets I, J such that $I \cup J = \{1, ..., k\}, I \cap J = \emptyset$ and $\rho''(q_i P q_j) = \rho''(q_j P q_i) = 0$ for any $i \in I, j \in J$. Thus, $\mathrm{Tr}\rho''(q_{i_1} P ... q_{i_s} P) = 0$ if $\{i_1, ..., i_s\} \cap I \neq \emptyset$ and $\{i_1, ..., i_s\} \cap J \neq \emptyset$. Since characters of ρ and ρ'' are the same, we get the required statement. ☐

For fixed partition θ: $\{1, ..., k\} = I \cup J, I \cap J = \emptyset$, we will consider subvariety $D(\theta) \subset \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)} \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,n-m})$ defined by equations:

$$i^*(\mathrm{Tr} q_{i_1} P ... q_{i_s} P) = 0 = i'^*(\mathrm{Tr} q_{i_1} P ... q_{i_s} P), \tag{151}$$

if $\{i_1, ..., i_s\} \cap I \neq \emptyset$ and $\{i_1, ..., i_s\} \cap J \neq \emptyset$. Using proposition 53, we get the following:

Corollary 54. *$E_2(f'_n) \subset \bigcup_\theta D(\theta)$, where union taken over all possible partitions of $\{1,...,k\}$ into two non-empty complement subsets I, J.*

Fix partition $\theta = I \cup J$. Consider subvarieties $D'_1(\theta) \subset (F^*)^{(k-1)(m-1)}, D'_2(\theta) \subset (F^*)^{(k-1)(n-m-1)}$ defined by equations:

$$h_{i_1,i_2}...h_{i_s,i_1} = 0 \tag{152}$$

and

$$h'_{i_1,i_2}...h'_{i_s,i_1} = 0 \tag{153}$$

respectively. Denote by $D_i(\theta), i = 1,2$ the $s_i^{-1}(D'_i(\theta)), i = 1,2$ respectively. It is easy that

$$D(\theta) = D_1(\theta) \times_{\mathcal{M}_n \widetilde{\mathcal{A}}_k(r_i)[\vec{1},m]} D_2(\theta). \tag{154}$$

6.4 Combinatorial description of E_2.

In this subsection we will introduce the notion of maximal θ-subquivers for fixed partition θ. Using this notion, we get the description of components of $D_1(\theta)$ and $D_2(\theta)$.

Firstly, consider the following description of the varieties. Consider polynomial ring $F[h_{i,j}], i \neq j, i,j = 1,...,k$. We have a homomorphism of rings: $H_1 : F[h_{i,j}] \to F[z_{i,j}]$ defined by formulas (143). Thus, we have the morphism of affine varieties: $H_1^* : (F^*)^{(k-1)(m-1)} \to F^{k(k-1)}$. Also, we can define morphism: $H_1^* \circ s_1 : \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m}) \to (F^*)^{(k-1)(m-1)} \to F^{k(k-1)}$. We can consider variety $D''_1(\theta) \subset F^{k(k-1)}$ defined by equations (152). It is easy that $D'_1(\theta) = (H_1^*)^{-1}(D''_1(\theta))$ and $D_1(\theta) = (H_1^* \circ s_1)^{-1}(D''_1(\theta))$. Analogously, define $H_2 : F[h_{i,j}] \to F[y_{i,j}]$. In this case, $D'_2(\theta) = (H_2^*)^{-1}(D''_2(\theta))$ and $D_2(\theta) = (H_2^* \circ s_2)^{-1}(D''_2(\theta))$.

We have the following combinatorial description of the set of the equations defining of $D''_1(\theta)$. Consider complete double quiver $\mathbb{Q}_{\Gamma[k]}$. For any subquiver Q of $\mathbb{Q}_{\Gamma[k]}$, denote by $V(Q)$ and $Arr(Q)$ the sets of vertices and arrows of the quiver Q respectively. We will consider subquivers of $\mathbb{Q}_{\Gamma[k]}$ such that $V(Q) = V(\mathbb{Q}_{\Gamma[k]}) = \{1,...,k\}$. We can associate with any subquiver $Q \subset \mathbb{Q}_{\Gamma[k]}$ variety $M(Q) \subset (F)^{k(k-1)}$ as follows: $M(Q)$ is a subvariety of $(F)^{k(k-1)}$ defined by equations $h_{i,j} = 0$ for $a_{i,j} \in Arr(\mathbb{Q}_{\Gamma[k]}) \setminus Arr(Q)$. It is clear that if $Q' \subseteq Q''$, then $M(Q') \subseteq M(Q'')$.

Fix the partition of vertices of the $\mathbb{Q}_{\Gamma[k]}$: $\theta = I \cup J$. Let us define the notion of θ - subquiver of $\mathbb{Q}_{\Gamma[k]}$. The subquiver Q of $\mathbb{Q}_{\Gamma[k]}$ is said to be a *θ-subquiver* if $V(Q) = \{1,...,k\}$ and Q satisfy to condition: there are no cycles $c = (i_1,...,i_s) \in Q$ such that $(i_1,...,i_s) \cap I \neq \emptyset, (i_1,...,i_s) \cap J \neq \emptyset$.

We can define partial order on the set of subquivers of $\mathbb{Q}_{\Gamma[k]}$ by natural way. Restrict this partial order to set of θ - subquivers. It leads us to notion of *maximal θ - subquiver* of $\mathbb{Q}_{\Gamma[k]}$. Denote by $\mathrm{Max}(\theta)$ the set of θ-maximal subquivers of $\mathbb{Q}_{\Gamma[k]}$.

Proposition 55.

- *For any θ-subquiver $Q \subset \mathbb{Q}_k$, we have the following immersion:*

$$M(Q) \subset D''_1(\theta). \tag{155}$$

- *We have the following identity for $D''_1(\theta)$:*

$$D''_1(\theta) = \bigcup_{Q \in \mathrm{Max}(\theta)} M(Q) \tag{156}$$

Proof. First statement is easy. Prove the second statement. With any irreducible component C of $D''_1(\theta)$ we can associate the subquiver $Q(C) \subset \mathbb{Q}_{\Gamma[k]}$ as follows: $a_{ij} \in Q(C)$ iff ideal of component C contains h_{ij}. It can be shown in usual way that $Q(C)$ is a θ - subquiver and $C \subset M(Q(C))$. Thus, $D''_1(\theta) \subset \bigcup M(Q)$, where union is taken over all θ-subquivers. It is easy that we can take only θ-maximal subquivers. □

This proposition motivates us to study maximal θ-subquivers. For this purpose, introduce the notion of *linear connected component* of a quiver $\mathbb{Q}$. We will say that set of vertices I generates linear connected component if

- for any pair vertices i_1, i_2 there are path from i_1 to i_2 and path from i_2 to i_1

- for any vertices $j \notin I, i \in I$ there is no path from j to i or there is not path from i to j.

We can define the equivalence relation on the set of vertices as follows: $i \sim j$ iff there are path from i to j and path from j to i. It is easy that linear connected component is an equivalence class. We will denote by $\{I\}$ the linear connected component (briefly l.c.c.) generating by vertex set I. Consider two l.c.c. $\{I_1\}$ and $\{I_2\}$. We will say that $\{I_1\} > \{I_2\}$ if there is an arrow from some vertex $i_1 \in I_1$ to some vertex $i_2 \in I_2$. It follows from definition of linear connectedness that if this order is well-defined. Let us formulate the following trivial property of l.c.c.:

Proposition 56.

- *Set of l.c.c. of quiver $\mathbb{Q}$ is partially ordered.*
- *Quiver $\mathbb{Q}$ is connected iff set of l.c.c. is linear ordered.*

Proof. It is easy. □

Consider maximal θ-subquiver Q. It is clear that there are no l.c.c. of Q which contains vertices from I and J. Thus, one can consider decomposition of I and J into union of vertex sets of l.c.c. of Q:

$$I = \cup_{i=1}^{l_1} I_i, J = \cup_{j=1}^{l_2} J_j \tag{157}$$

We have the following useful property of maximal θ - subquivers:

Proposition 57. *Fix partition $\theta = I \cup J$. Consider θ-subquiver $Q \subset \mathbb{Q}_{\Gamma[k]}$. We have the decomposition of vertex set $\{1, ..., k\}$ of Q into union of vertex sets of l.c.c. (157). Q is maximal θ - subquiver of $\mathbb{Q}_{\Gamma[k]}$ iff*

- *Q is connected.*
- *Consider a pair of l.c.c. $\{K_1\}, \{K_2\} : \{K_1\} > \{K_2\}$. Then for any pair of vertices $v_1 \in K_1, v_2 \in K_2$ there is an arrow from v_1 to v_2.*

Proof. It is easy that if Q satisfy to conditions, then Q is a maximal θ-subquiver. Converse statement is easy too. □

Remark. Assume that we have the following ordering on the set of l.c.c. of maximal θ - subquiver Q: $\{I_1\} > ... > \{I_{k_1}\} > \{J_1\} > ... > \{I_{k_1+1}\} > ... > \{J_{l_2}\}$. Consider another partition: $\theta' : I_1 \bigcup K, K = \bigcup_{i=2}^{l_1} I_i \cup \bigcup_{j=1}^{l_2} J_j$. It is easy that Q is θ' - subquiver of $\mathbb{Q}_k$, but not maximal.

Thus, problem of finding $\bigcup D(\theta)$ has the following parts:

- one have to classify all maximal θ-subquivers for any partition θ of $\{1, ..., k\}$,
- one have to calculate $(H_1^* \circ s_1)^{-1}(M(Q)) \subset \mathcal{M}_n B_{\mathbf{r}}(\Gamma_{k,m})$ for maximal θ-subquiver Q.

Consider the case of $\mathcal{M}_n B_{n,n}$. Assume that $r_i \neq 0, 1$. Subvariety E_1 has description quite similar to case of $\mathcal{M}_n B_{k,n}$. Consider subvariety $E_2 \subset \mathcal{M}_n B_{n,m} \times_{\mathcal{M}_n \mathcal{A}_n(r_i)[\vec{1},m]} \mathcal{M}_n B_{n,n-m}$. Also, we have to study all partitions and corresponding them maximal θ-subquivers. Let us note the following property of morphism $H_1^* \circ s_1$ in case of $\mathcal{M}_n B_{n,n}$:

Proposition 58. *Fix partition θ. Consider maximal θ - subquivers Q with condition: there is a l.c.c. of Q consisting of one vertex. Then $(H_1^* \circ s_1)^{-1}(M(Q)) = \emptyset$.*

Proof. Consider maximal θ - subquiver Q with l.c.c. consisting one vertex i. Then for any vertex j we have the following identity $h_{i,j} h_{j,i} = \mathrm{Tr} P q_i P q_j = 0$. In the case of $\mathcal{M}_n B_{n,n}$, we have the identity $\sum_{i=1}^n q_i = 1$. Thus, $r_i = \mathrm{Tr} P q_i P(q_i + \sum_{j \neq i} q_j) = \mathrm{Tr} P q_i P q_i = r_i^2$. Thus, if $r_i \neq 0, 1$, then $(H_1^* \circ s_1)^{-1}(M(Q)) = \emptyset$. □

This proposition means that we can consider only maximal θ-subquivers with condition: any l.c.c. has more than 1 vertex.

6.5 Varieties $\mathcal{M}_6B_{3,6}$, $\mathcal{M}_6B_{6,6}$ and fibred products.

In this subsection we will apply results of subsections 6.2 and 6.3 in the case of $\mathcal{M}_6B_{3,6}$ and $\mathcal{M}_6B_{6,6}$.

Fix $r_{ij} = \frac{1}{6}$. Thus, $r_i = \frac{1}{2}$. Consider case of $\mathcal{M}_6B_{3,6}$. Denote by $X(3,6)$, $X(3,3)$ and $Y(3)$ the varieties $\mathcal{M}_6B_{3,6}$, $\mathcal{M}_6B(\Gamma_{3,3})$ and $\mathcal{M}_6\widetilde{\mathcal{A}}_3(1/2)$ respectively. As we know, $X(3,3) = (F^*)^4$ and $Y(3) = \mathcal{M}_3\mathbb{Q}_{\Gamma[3]}[1]$. There are only 5 generating necklaces in the quiver $\mathbb{Q}_{\Gamma[3]}$. They correspond to the following elements: $A = \mathrm{Tr}Pq_1Pq_2$, $B = \mathrm{Tr}Pq_1Pq_3$, $C = \mathrm{Tr}Pq_2Pq_3$, $\alpha = \mathrm{Tr}Pq_1Pq_2Pq_3$ and $\beta = \mathrm{Tr}Pq_1Pq_3Pq_2$. One can check that $Y(3)$ is defined by equation:

$$ABC = \alpha\beta. \tag{158}$$

In this case, we get the following commutative diagram:

$$\begin{array}{ccc} X(3,3)\times_{Y(3)} X(3,3) & \longrightarrow & X(3,3) \\ \downarrow & & \downarrow{\scriptstyle \sigma\circ i^*} \\ X(3,3) & \xrightarrow{\;i^*\;} & Y(3) \end{array} \tag{159}$$

where σ acts on $Y(3)$ by the rule: $\sigma : P \mapsto 1 - P$.

Consider fibred product: $X(3,3)\times_{Y(3)} X(3,3)$. Let us formulate the following:

Lemma 59. *Any irreducible component of $X(3,3)\times_{Y(3)} X(3,3)$ has dimension more or equal 3.*

Proof. Straightforward. □

We have natural morphism: $f_6' : X(3,6) \to X(3,3)\times_{Y(3)} X(3,3)$. We will study properties of this morphism. Namely, we will calculate varieties $E_1(f_6')$ and $E_2(f_6')$.

We obtain the following result:

Proposition 60.
- *Subvariety $E_1(f_6')$ consists of finite set of points,*
- *dimension of any component of $E_2(f_6')$ is less or equal 3.*

Proof. See Appendix A. □

Denote by $C_i, i = 1, ..., s$ and $C_i', i = 1, ..., s'$ the components of $X(3,3)\times_{Y(3)} X(3,3)$ and $X(3,6)$ which dimension more or equal to 4.

Corollary 61.
- *$s = s'$*
- *there is a bijection $i \leftrightarrow j$ between set $C_i, i = 1, ..., s$ and $C_j', j = 1, ..., s'$ such that $\overline{f_6'(C_i)} = C_j'$ and restriction of f_6' to C_i is a birational morphism.*

Remark We will prove that there is only one 4-dimensional irreducible component of $X(3,3)\times_{Y(3)} X(3,3)$ in the Section ??. Therefore, we get that $X(3,6)$ is a 4-dimensional and irreducible.

Consider the second case. For simplicity, denote by $X(6,6)$ and $Y(6)$ the varieties $\mathcal{M}_6B_{6,6}$ and $\mathcal{M}_6\mathcal{A}_6(1/2)[\vec{1},3]$ respectively. Similar to subsection 10, we can define involution σ on $Y(6)$. We have the following commutative diagram:

$$\begin{array}{ccc} X(3,6)\times_{Y(3)} X(3,6) & \longrightarrow & X(3,6) \\ \downarrow & & \downarrow{\scriptstyle \sigma\circ i_6^*} \\ X(3,6) & \xrightarrow{\;i_6^*\;} & Y(6) \end{array} \tag{160}$$

Therefore, we have the morphism: $f_6 : X(6,6) \to X(3,6)\times_{Y(6)} X(3,6)$.

Proposition 62.
- *Variety $E_1(f_6)$ consists of finite set of points.*
- $\dim_F E_2(f_6) \leq 3$.

Proof. See Appendix B. □

Denote by $C = \bigcup_{i=1}^{s} C_i$, $C' = \bigcup_{i=1}^{s'} C'_i$ the union of four-dimensional irreducible components of $X(6,6)$ and the union of four-dimensional irreducible components of $X(3,6) \times_{Y(6)} X(3,6)$ respectively. Using proposition 62, we obtain the following result:

Proposition 63. • *$s = s'$,*

- *there is a bijection: $i \leftrightarrow j$ such that $\overline{f_6(C_i)} = C'_j$ and $f_6|_{C_i}, i = 1, ..., s$ is a birational isomorphism.*

Conjecture 64. *$s = s' = 1$, i.e. there is only one four-dimensional irreducible component of $X(6,6)$.*

Remark. We will prove that $s > 0$ in Section ??.

7 The case of graph $\Gamma_{3,3}$.

In this section we will consider the case of $B_r(\Gamma_{3,3})$, i.e. $r_{ij} = r$ for $i, j = 1, 2, 3$. Let $p_1, p_2, p_3, q_1, q_2, q_3$ be the generators of $B_r(\Gamma_{3,3})$. Variety $\mathcal{M}_6 B_r(\Gamma_{3,3})[1]$ parameterizes 6-dimensional $B_r(\Gamma_{3,3})$-modules of rank 1. Let P and Q be the elements $\sum_{i=1}^{3} p_i$ and $\sum_{i=1}^{3} q_i$ respectively. Consider unital algebra $\widetilde{\mathcal{A}}_3(3r)$ with generators w_1, w_2, w_3, W with relations: $w_i^2 = w_i, W^2 = W, w_i W w_i = 3r w_i, i = 1, 2, 3$. In this subsection, we will consider the following morphisms of algebras: $\psi_{1,2} : \mathcal{A}_r^3 \to B_r(\Gamma_{3,3})$ given by formulas:

$$\psi_1 : w_i \mapsto q_i, W \mapsto P \tag{161}$$

$$\psi_2 : w_i \mapsto p_i, W \mapsto Q \tag{162}$$

It is evident that there is the involution τ on the algebra $B_r(\Gamma_{3,3})$ defined by rule: $p_i \leftrightarrow q_i, i = 1, 2, 3$. And hence, $\psi_2 = \psi_1 \circ \tau$.

For simplicity, we will use the following notation:

$$X = \mathcal{M}_6 B_r(\Gamma_{3,3})[1], Y(3) = \mathcal{M}_6 \mathcal{A}_3(3r)[(1,1,1,3)] \tag{163}$$

Also, we can consider variety $\mathcal{D}$ parameterizing $\mathrm{GL}_6(\mathrm{F})$-orbits of pair of the projectors (P, Q) of rank 3 with relation $\mathrm{Tr} PQ = \mathrm{Tr}(p_1 + p_2 + p_3)(q_1 + q_2 + q_3) = 9r$. In this subsection, we will study properties of morphisms: $X \to Y(3)$ and $X \to Y(3) \times_{\mathcal{D}} Y(3)$.

7.1 Preliminary remarks.

Consider algebra $\mathbf{Pr}(\Gamma_{k,m})$ with generators $p_1, ..., p_k, q_1, ..., q_m$. Algebras $\mathbf{Pr}(\Gamma_{k,1})$, $\mathbf{Pr}(\Gamma_{1,m})$ and $\mathbf{Pr}(\Gamma_{1,1})$ are generated by elements $P = \sum_{i=1}^{k} p_i, q_1, ..., q_m$, $p_1, ..., p_k, Q = \sum_{i=1}^{m} q_i$ and P, Q respectively.

One can show that

$$\mathbf{Pr}(\Gamma_{k,m}) \cong \mathbf{Pr}(\Gamma_{k,1}) *_{\mathbf{Pr}(\Gamma_{1,1})} \mathbf{Pr}(\Gamma_{1,m}). \tag{164}$$

Thus, we have the isomorphism of varieties:

$$\mathbf{Rep}_{m+k}\mathbf{Pr}(\Gamma_{k,m})[\vec{1}] \cong \mathbf{Rep}_{m+k}\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] \times_{\mathbf{Rep}_{m+k}\mathbf{Pr}(\Gamma_{1,1})[k,m]} \mathbf{Rep}_{m+k}\mathbf{Pr}(\Gamma_{1,m})[k, \vec{1}]. \tag{165}$$

For moduli varieties there is the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{k,m})[\vec{1}] & \longrightarrow & \mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] \\ \downarrow & & \downarrow \\ \mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{1,m})[k, \vec{1}] & \longrightarrow & \mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{1,1})[k, m] \end{array} \tag{166}$$

The dimension counting shows us that this diagram is not a fibred product. Also, using considering of projections: $\mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{k,m})[\vec{1}] \to F^{km}$, $\mathcal{M}_{k+m}\mathbf{Pr}(\Gamma_{k,1})[\vec{1}, m] \to F^k$, $\mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{1,m})[k, \vec{1}] \to F^m$ and $\mathcal{M}_{m+k}\mathbf{Pr}(\Gamma_{1,1})[k, m] \to F$, we get the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{k+m,1}B_{\mathbf{r}}(\Gamma_{k,m}) & \longrightarrow & \mathcal{M}_{m+k}\widetilde{\mathcal{A}}_k(r_i)[\vec{1}, m] \\ \downarrow & & \downarrow \\ \mathcal{M}_{m+k}\widetilde{\mathcal{A}}_m(r_i)[k, \vec{1}] & \longrightarrow & \mathcal{D} \end{array} \tag{167}$$

Let us come back to the case $k = m = 3, r_{ij} = r \neq 0$. In this situation, we can identify algebras $\widetilde{\mathcal{A}}_m(3r)$ and $\widetilde{\mathcal{A}}_k(3r)$ via involution $\tau : p_i \leftrightarrow q_i, i = 1, 2, 3$. Using notation, we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{pr} & Y \\ {\scriptstyle pr\circ\tau}\downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{D} \end{array} \tag{168}$$

As we know from proposition 44, $\mathcal{M}_{k+m}B_{\mathbf{r}}(\Gamma_{k,m})[1] = (F^*)^{(k-1)(m-1)}$. Thus, $X = (F^*)^4$. Coordinates of X may be chosen as follows:

$$x_1 = \frac{1}{r^2}\mathrm{Tr}p_1q_1p_2q_2, x_2 = \frac{1}{r^2}\mathrm{Tr}p_1q_1p_3q_2, y_1 = \frac{1}{r^2}\mathrm{Tr}p_1q_1p_2q_3, y_2 = \frac{1}{r^2}\mathrm{Tr}p_1q_1p_3q_3. \tag{169}$$

Clearly, $X = \mathrm{Spec}F[x_1^{\pm1}, x_2^{\pm1}, y_1^{\pm1}, y_2^{\pm1}]$. It can be shown in usual way that $\tau : x_1 \mapsto \frac{1}{x_1}, y_2 \mapsto \frac{1}{y_2}, x_2 \mapsto \frac{1}{y_1}, y_1 \mapsto \frac{1}{x_2}$. As we know, commutative ring $\mathcal{O}(Y)$ is generated by $a_{(i_1,i_2)} = \frac{1}{r^2}\mathrm{Tr}(Pq_{i_1}Pq_{i_2})$ and $a_{(i_1,i_2,i_3)} = \frac{1}{r^3}\mathrm{Tr}(Pq_{i_1}Pq_{i_2}Pq_{i_3})$ for $i_1, i_2, i_3 = 1, 2, 3$. We take the coefficients $\frac{1}{r^2}$ and $\frac{1}{r^3}$ for simplicity of calculations. Also, recall from subsection 6.5:

$$Y = \mathrm{Spec}F[a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, a_{(1,3,2)}]/\langle a_{(1,2)}a_{(1,3)}a_{(2,3)} = a_{(1,2,3)}a_{(1,3,2)}\rangle$$

Describe the variety $\mathcal{D}$ in terms of traces. Recall that $\mathcal{D}$ is the variety of projectors P and Q of rank 3 and satisfying to condition: $\mathrm{Tr}PQ = 9r$. One can show that $\mathcal{D} = F^2 = \mathrm{Spec}F[\frac{1}{r^2}\mathrm{Tr}PQPQ, \frac{1}{r^3}\mathrm{Tr}PQPQPQ]$.

7.2 Identities for projectors $p_1, p_2, p_3; q_1, q_2, q_3$.

In this subsection we prove some identity for projectors $p_1, p_2, p_3, q_1, q_2, q_3$ of rank 1 with conditions: $\mathrm{Tr}p_iq_j = r$. We have the following formulas:

$$\frac{1}{r^2}\mathrm{Tr}p_iq_jp_1q_1 = \frac{1}{r^2}\mathrm{Tr}p_1q_1p_iq_jp_1, \tag{170}$$

and

$$\frac{1}{r^2}\mathrm{Tr}p_iq_1p_1q_jp_1\frac{1}{r^2}\mathrm{Tr}p_1q_1p_iq_jp_1 = 1. \tag{171}$$

Note the following useful property of projectors $p_1, p_2, p_3; q_i, q_j$ of rank 1 with condition $\mathrm{Tr}p_iq_j = r$ for all i, j.

Lemma 65. *Consider projectors: $p_1, p_2, p_3; q_1, q_2$ of rank 1 with condition $\mathrm{Tr}p_iq_j = r$. Then we have the following identity:*

$$\frac{1}{r^2}\mathrm{Tr}(p_1 + p_2 + p_3)q_1(p_1 + p_2 + p_3)q_2 = \frac{1}{r^3}\mathrm{Tr}(p_1(q_1 + q_2)p_2(q_1 + q_2)p_3(q_1 + q_2)) + 1 = \tag{172}$$

$$\frac{1}{r^3}\mathrm{Tr}(p_1(q_1 + q_2)p_3(q_1 + q_2)p_2(q_1 + q_2)) + 1$$

Proof. Using relations: $p_i = \frac{1}{r^2} p_i q_1 p_1 q_1 p_i$, we get:

$$\frac{1}{r^3}\mathrm{Tr} p_1(q_1+q_2)p_2(q_1+q_2)p_3(q_1+q_2) = \frac{1}{r^7}\mathrm{Tr}(p_1(q_1+q_2)p_2q_1p_1 \cdot p_1q_1p_2(q_1+q_2)p_3q_1p_1 \cdot p_1q_1p_3(q_1+q_2)p_1) =$$

$$\frac{1}{r^2}\mathrm{Tr} p_1(q_1+q_2)p_2q_1p_1 \cdot \frac{1}{r^3}\mathrm{Tr} p_1q_1p_2(q_1+q_2)p_3q_1p_1 \cdot \frac{1}{r^2}\mathrm{Tr} p_1q_1p_3(q_1+q_2)p_1 = (1+\frac{1}{x_1})(1+\frac{x_1}{x_2})(1+x_2).$$

Moreover,

$$(1+\frac{1}{x_1})(1+\frac{x_1}{x_2})(1+x_2) = (1+x_1+x_2)(1+\frac{1}{x_1}+\frac{1}{x_2}) - 1.$$

We can transform the right expression into:

$$\frac{1}{r^2}\mathrm{Tr}(p_1+p_2+p_3)q_1(p_1+p_2+p_3)q_2 - 1 \tag{173}$$

This proves our statement.

□

Analogously, we get the similar formula for p_1, p_2, p_3, q_i, q_j for any i, j and for $q_1, q_2, q_3; p_i, p_j, i \neq j, i, j = 1, 2, 3$.

Further, let us formulate the following proposition:

Proposition 66. *Consider projectors $p_1, p_2, p_3; q_1, q_2, q_3$ of rank 1 with condition $Trp_iq_j = r$. Denote by P and Q the sums $p_1+p_2+p_3$ and $q_1+q_2+q_3$ respectively. Then we hold the following identity:*

$$\prod_{(i,j)\in\{1,2,3\}} (\frac{1}{r^2}\mathrm{Tr}(Pq_iPq_j) - 1) = \prod_{(i,j)\in\{1,2,3\}} (\frac{1}{r^2}\mathrm{Tr}(Qp_iQp_j) - 1), \tag{174}$$

where product is taken over all non-ordered pairs $(i,j) \in \{1,2,3\}$.

Proof. Using relation (172), we obtain the following formula:

$$(\frac{1}{r^2}\mathrm{Tr} Pq_1Pq_2 - 1)(\frac{1}{r^2}\mathrm{Tr} Pq_2Pq_3 - 1)(\frac{1}{r^2}\mathrm{Tr} Pq_3Pq_1 - 1) =$$

$$\frac{1}{r^3}\mathrm{Tr} p_1(q_1+q_2)p_2(q_1+q_2)p_3(q_1+q_2) \cdot \frac{1}{r^3}\mathrm{Tr} p_1(q_2+q_3)p_2(q_2+q_3)p_3(q_2+q_3) \cdot \frac{1}{r^3}\mathrm{Tr} p_1(q_1+q_3)p_2(q_1+q_3)p_3(q_1+q_3) =$$

$$(1+\frac{1}{x_1})(1+\frac{x_1}{x_2})(1+x_2) \cdot (1+\frac{x_1}{y_1})(1+\frac{y_1}{x_1}\frac{x_2}{y_2})(1+\frac{y_2}{x_2}) \cdot (1+y_1)(1+\frac{y_2}{y_1})(1+\frac{1}{y_2}) =$$

$$(1+\frac{1}{x_1})(1+\frac{x_1}{y_1})(1+y_1) \cdot (1+\frac{x_1}{x_2})(1+\frac{y_1}{x_1}\frac{x_2}{y_2})(1+\frac{y_2}{y_1}) \cdot (1+x_2)(1+\frac{y_2}{x_2})(1+\frac{1}{y_2}) =$$

$$\frac{1}{r^3}\mathrm{Tr} q_1(p_1+p_2)q_2(p_1+p_2)q_3(p_1+p_2) \cdot \frac{1}{r^3}\mathrm{Tr} q_1(p_2+p_3)q_2(p_2+p_3)q_3(p_2+p_3) \cdot \frac{1}{r^3}\mathrm{Tr} q_1(p_1+p_3)q_2(p_1+p_3)q_3(p_1+p_3) =$$

Using proposition 65, we get the required identity:

$$(\frac{1}{r^2}\mathrm{Tr} Pq_1Pq_2 - 1)(\frac{1}{r^2}\mathrm{Tr} Pq_2Pq_3 - 1)(\frac{1}{r^2}\mathrm{Tr} Pq_3Pq_1 - 1) =$$

$$(\frac{1}{r^2}\mathrm{Tr} Qp_1Qp_2 - 1)(\frac{1}{r^2}\mathrm{Tr} Qp_2Qp_3 - 1)(\frac{1}{r^2}\mathrm{Tr} Qp_3Qp_1 - 1)$$

□

7.3 Properties of the map $X \to Y \times_{\mathcal{D}} Y$.

Denote by u_1, u_2, u_3 the following elements:

$$u_1 = a_{(1,2)} + a_{(1,3)} + a_{(2,3)} = \frac{1}{r^2}(\mathrm{Tr}Pq_1Pq_2 + \mathrm{Tr}Pq_1Pq_3 + \mathrm{Tr}Pq_2Pq_3), \tag{175}$$

$$u_2 = a_{(1,2,3)} + a_{(1,3,2)} = \frac{1}{r^3}(\mathrm{Tr}Pq_1Pq_2Pq_3 + \mathrm{Tr}Pq_1Pq_3Pq_2), \tag{176}$$

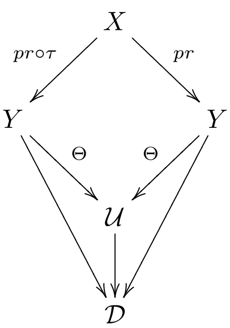
$$u_3 = (a_{(1,2)} - 1)(a_{(1,3)} - 1)(a_{(2,3)} - 1) = (\frac{1}{r^2}\mathrm{Tr}Pq_1Pq_2 - 1)(\frac{1}{r^2}\mathrm{Tr}Pq_2Pq_3 - 1)(\frac{1}{r^2}\mathrm{Tr}Pq_3Pq_1 - 1). \tag{177}$$

One can consider elements u_i as elements of $F[x_1^{\pm 1}, x_2^{\pm 1}, y_1^{\pm 1}, y_2^{\pm 1}]$. It can be shown in usual way that $\tau(u_i) = u_i, i = 1, 2, 3$.

Element u_1 is a $\mathrm{Tr}PQPQ$ up to constant. Expression u_2 is a linear combination of $\mathrm{Tr}PQPQPQ$, $\mathrm{Tr}PQPQ$ and constant. Also, element u_3 is described in proposition 66.

Consider 3-dimensional affine space $\mathcal{U} = \mathrm{Spec}F[u_1, u_2, u_3]$. There exists a natural surjective map: $\mathcal{U} \to \mathcal{D}$. There are natural surjective maps: $\Theta : Y \to \mathcal{U}$.

We obtain that variety $Y \times_{\mathcal{U}} Y$ is a divisor of the $Y \times_{\mathcal{D}} Y$, $\dim_F Y \times_{\mathcal{U}} Y = 5, \dim_F Y \times_{\mathcal{D}} Y = 6$. Thus, we get the following commutative diagram:



(178)

Consider natural map: $pr_{12} = (pr, pr \circ \tau) : X \to Y \times_{\mathcal{D}} Y$. Using commutative diagram (178), we obtain the following proposition:

Proposition 67. $pr_{12}(X) \subset Y \times_{\mathcal{U}} Y \subset Y \times_{\mathcal{D}} Y$.

Consider action of symmetric group S_3 acting by permutations of the projectors p_i on the variety X. We have injection: $j : F[X]^{S_3} \to F[X]$ and injection twisted by involution τ: $\tau \circ j : F[X]^{S_3} \to F[X]$. Thus, we can consider the intersection $F[X]^{S_3}$ and $\tau(F[X]^{S_3})$ in the ring $F[X]$. It is easy $\tau(F[X]^{S_3}) = F[X]^{\tau S_3 \tau^{-1}} = F[X]^{\tau S_3 \tau}$. Moreover, group $\tau S_3 \tau$ acts on the X by permutations of the projectors q_i. Therefore, one can check that intersection $F[X]^{S_3} \cap \tau(F[X]^{S_3}) = F[X]^{S_3 \times S_3}$, where $S_3 \times S_3$ acts on the X by permutations of p_i and q_j. Rings $F[X]^{S_3}$ and $\tau(F[X]^{S_3})$ are isomorphic. Identify these rings via isomorphism τ. Also, note that $\tau(S_3 \times S_3)\tau = S_3 \times S_3$ in the group $Aut(\Gamma_{3,3})$. Thus, we have the well-defined involution τ on the $F[X]^{S_3 \times S_3}$ such that we have the following commutative diagram:

$$\begin{array}{ccc} F[X]^{S_3 \times S_3} & \xrightarrow{i} & F[X] \\ \tau \uparrow & & \uparrow \tau \\ F[X]^{S_3 \times S_3} & \xrightarrow{i} & F[X] \end{array} \tag{179}$$

where i is standard injection. For i we have the decomposition $j \circ i_1$, where $i_1 : F[X]^{S_3 \times S_3} \to F[X]^{S_3}$ and $j : F[X]^{S_3} \to F[X]$, here S_3 is the group acting by permutations of p_i. Using relation $\tau \circ i \circ \tau = i$, we obtain the decomposition $i = (\tau \circ j) \circ (i_1 \circ \tau)$. Thus, we get the following commutative diagram:

$$\begin{array}{ccc} F[X]^{S_3} & \xrightarrow{\ j\ } & F[X] \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle \tau\circ j} \\ F[X]^{S_3\times S_3} & \xrightarrow{i_1\circ\tau} & F[X]^{S_3} \end{array} \tag{180}$$

Further, consider immersion: $i_2 : F[Y] \to F[X]^{S_3}$. It is easy that this immersion is compatible with action another symmetric group S_3. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} F[Y] & \xrightarrow{\ i_2\ } & F[X]^{S_3} \\ {\scriptstyle \psi}\uparrow & & \uparrow{\scriptstyle i_1} \\ F[Y]^{S_3} & \xrightarrow{\ \theta\ } & F[X]^{S_3\times S_3} \end{array} \tag{181}$$

Here θ is the immersion induced by i_2, ψ is standard immersion.

Moreover, we can consider the situation of the immersion $i_1 \circ \tau : F[X]^{S_3\times S_3} \to F[X]^{S_3}$. In this situation, we have the following commutative diagram:

$$\begin{array}{ccc} F[X]^{S_3\times S_3} & \xrightarrow{i_1\circ\tau} & F[X]^{S_3} \\ {\scriptstyle \tau\circ\theta}\uparrow & & \uparrow{\scriptstyle i_2} \\ F[Y]^{S_3} & \xrightarrow{\ \psi\ } & F[Y] \end{array} \tag{182}$$

Actually, using relation $i_1 \circ \theta = i_2 \circ \psi$, we get the relation: $(i_1 \circ \tau) \circ (\tau \circ \theta) = i_2 \circ \psi$. Also, direct checking show us that the following diagram:

$$\begin{array}{ccc} F[X]^{S_3\times S_3} & \xleftarrow{\ \theta\ } & F[Y]^{S_3} \\ {\scriptstyle \tau\circ\theta}\uparrow & & \uparrow \\ F[Y]^{S_3} & \longleftarrow & F[\mathcal{U}] \end{array} \tag{183}$$

Here injection: $F[\mathcal{U}] \to F[Y]^{S_3}$ is given by elements u_1, u_2, u_3.

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc} F[X] & \xleftarrow{\ j\ } & F[X]^{S_3} & \xleftarrow{\ i_2\ } & F[Y] \\ {\scriptstyle \tau\circ j}\uparrow & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle \psi} \\ F[X]^{S_3} & \xleftarrow{i_1\circ\tau} & F[X]^{S_3\times S_3} & \xleftarrow{\ \theta\ } & F[Y]^{S_3} \\ {\scriptstyle i_2}\uparrow & & \uparrow{\scriptstyle \tau\circ\theta} & & \uparrow \\ F[Y] & \xleftarrow{\ \psi\ } & F[Y]^{S_3} & \longleftarrow & F[\mathcal{U}] \end{array} \tag{184}$$

Further, let us apply to this diagram the functor Spec. Also, let us denote by $\mathcal{X}$, $\mathcal{Y}$ the varieties $X/S_3 = \mathrm{Spec} F[X]^{S_3}$ and $Y/S_3 = \mathrm{Spec} F[Y]^{S_3}$ respectively. Thus, we get the following proposition:

Proposition 68. *There is the following commutative diagram:*

$$
\begin{array}{ccccc}
X & \xrightarrow{\pi} & X/S_3 & \xrightarrow{\phi} & Y \\
\downarrow{\scriptstyle \pi\circ\tau} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
X/S_3 & \xrightarrow{\tau\circ\pi_1} & \mathcal{X} & \xrightarrow{\psi} & \mathcal{Y} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi\circ\tau} & & \downarrow{\scriptstyle \eta} \\
Y & \xrightarrow{\pi_1} & \mathcal{Y} & \xrightarrow{\eta} & \mathcal{U}
\end{array} \tag{185}
$$

where π, π_1 are standard factorization maps, ϕ is a well-defined map: $X/S_3 \to Y$. Also, we have the following identity: $pr = \phi \circ \pi$.

7.4 General fibers of the morphism pr.

In this subsection we will prove that the morphism pr has degree 12.

Let us express the morphism pr in coordinates:

$$a_{(1,2)} = (1 + \frac{1}{x_1} + \frac{1}{x_2})(1 + x_1 + x_2), \tag{186}$$

$$a_{(1,3)} = (1 + \frac{1}{y_1} + \frac{1}{y_2})(1 + y_1 + y_2) \tag{187}$$

and

$$a_{(2,3)} = (1 + \frac{y_1}{x_1} + \frac{y_2}{x_2})(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2}). \tag{188}$$

Analogous to this formula, we obtain the following expressions:

$$a_{(1,2,3)} = (1 + x_1 + x_2)(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2})(1 + \frac{1}{y_1} + \frac{1}{y_2}) \tag{189}$$

and

$$a_{(1,3,2)} = (1 + \frac{1}{x_1} + \frac{1}{x_2})(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2})(1 + y_1 + y_2). \tag{190}$$

Fix a general point $P = (A = a_{(1,2)}, B = a_{(1,3)}, C = a_{(2,3)}, \alpha = a_{(1,2,3)}, \beta = a_{(1,3,2)})$. Then for calculation of fiber $pr_1^{-1}(P)$ we have to compute a number of solutions of the system of equations (186), (187), (188), (189) and (190) for the point P.

Assume that $A, B, C \neq 0$. Hence, $\alpha, \beta \neq 0$. Thus, we can simplify formulas (189) and (190) as follows:

$$(1 + x_1 + x_2)(1 + \frac{1}{y_1} + \frac{1}{y_2}) = \frac{\alpha}{C}(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2}), \tag{191}$$

$$(1 + \frac{1}{x_1} + \frac{1}{x_2})(1 + y_1 + y_2) = \frac{\beta}{C}(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}). \tag{192}$$

For the calculation of fiber $pr^{-1}(P)$, let us compactify of $(F^*)^4$. We will choose the following compactification: $(F^*)^4$ is an open subvariety of $\mathbb{P}^2_t \times \mathbb{P}^2_z$ as follows. We will denote by $((t_0 : t_1 : t_2), (z_0 : z_1 : z_2))$ the coordinates of $\mathbb{P}^2_t \times \mathbb{P}^2_z$. We have the following formulas:

$$x_1 = \frac{t_1}{t_0}, \quad x_2 = \frac{t_2}{t_0}, \quad y_1 = \frac{z_0}{z_1}, \quad y_2 = \frac{z_0}{z_2}. \tag{193}$$

for coordinates of $(F^*)^4$ and $\mathbb{P}^2_t \times \mathbb{P}^2_z$.

We get the following system of equations:

$$(t_0+t_1+t_2)(t_0t_1+t_1t_2+t_2t_0)=At_0t_1t_2, \tag{194}$$

$$(z_0+z_1+z_2)(z_0z_1+z_1z_2+z_2z_0)=Bz_0z_1z_2, \tag{195}$$

$$(t_0t_1z_2+t_1t_2z_0+t_2t_0z_1)(z_0z_1t_2+z_1z_2t_0+z_2z_0t_1)=Ct_0t_1t_2z_0z_1z_2, \tag{196}$$

$$(t_0+t_1+t_2)(z_0+z_1+z_2)=\frac{\alpha}{C}(t_0z_0+t_1z_1+t_2z_2), \tag{197}$$

$$(z_0z_1+z_1z_2+z_2z_0)(t_0t_1+t_1t_2+t_2t_0)=\frac{\beta}{C}(z_0z_1t_0t_1+z_1z_2t_1t_2+z_2z_0t_2t_0). \tag{198}$$

Under our assumption we can omit the equation (196). We have to compute a intersection number of divisors (194),(195),(197) and (198). After computing we have to except the points at the divisors $t_0t_1t_2=0$ and $z_0z_1z_2=0$. For simplicity, denote by α' and β' the expressions $\frac{\alpha}{C}$ and $\frac{\beta}{C}$ respectively.

Firstly, let us study equation (194). Consider the rational map:

$$p:\mathbb{P}^2_t \dashrightarrow \mathbb{P}^1,$$

defined by correspondence: $p:(t_0:t_1:t_2)\mapsto((t_0+t_1+t_2)(t_0t_1+t_1t_2+t_2t_0):t_0t_1t_2)$. Consider the surface $\mathcal{E}\subset\mathbb{P}^2_t\times\mathbb{P}^1$ which is a closure of graph of the morphism p. This surface is called Beauville elliptic family (cf. [?]). We have the natural projection: $\mathcal{E}\to\mathbb{P}^1$. This projection has 6 sections defined by points $F_1(1:0:0),F_2(0:1:0),F_3(0:0:1),G_1(0:-1:1),G_2(1:0:-1),G_3(1:-1:0)$. It is well-known that fiber $\mathcal{E}_A, A\in\mathbb{P}^1$ is an elliptic curve, iff $A\neq(0:1),(1:1),(9:1),(1:0)$. Analogously, second equation defines elliptic curve $\mathcal{E}_B$. Intersection of $\mathcal{E}_A\cap\{t_0t_1t_2=0\}$ is a divisor $2\sum_{i=1}^3F_i+\sum_{i=1}^3G_i$. Denote by $\sim_L$ the linear equivalence of divisors. Let us formulate some property of points $F_i,G_i,i=1,2,3$.

Lemma 69. *Assume $A\neq(0:1),(1:1),(9:1),(1:0)$. Consider elliptic curve $\mathcal{E}_A$. Then we have the following relations for points $F_i;G_i,i=1,2,3$:*

$$G_1+G_2+G_3\sim_L 3G_i, i=1,2,3.$$

$$F_i+F_j\sim_L G_i+G_j, i,j=1,2,3,$$

$$2F_i\sim_L 2G_i, i=1,2,3,$$

Proof. Consider the intersection of line given by equation $(A-1)t_0-t_1-t_2=0$ and $\mathcal{E}_A$. It can be shown in usual way that intersection is triple point G_1. Analogously, $G_i,i=1,2,3$ are flex points. Therefore, $3G_i\sim_L 3G_j$. Points G_1, G_2 and G_3 lie on the line $t_0+t_1+t_2=0$. Hence, $G_1+G_2+G_3\sim_L 3G_i$. Further, points F_1,F_2 and G_3 lie on the line $t_2=0$. Thus, $F_1+F_2+G_3\sim_L 3G_i$. □

It is well known that for fixed point $P\in\mathcal{E}_A$ elements $X-P,X\in\mathcal{E}_A$ form the group $Pic^0(\mathcal{E}_A)$. Consider elliptic curve $\mathcal{E}_A\subset\mathbb{P}^2$. As we know, there is an usual group law on the $\mathcal{E}_A$. Recall that usual group law on the plane elliptic curve is defined by flex point. It is easy that flex points of $\mathcal{E}_A$ are $G_i,i=1,2,3$. Without loss of generality, let us choose the point G_1. Map $\mathcal{E}_A\to Pic^0(\mathcal{E}_A)$ defined by correspondence: $X\mapsto X-G_1$ is an isomorphism of the groups. From relations, we get that elements $0,G_2-G_1,G_3-G_1,F_1-G_1,F_2-G_1,F_3-G_1$ form the group $\mathbb{Z}_6$. It is easy that F_1-G_1 is element of second order, G_2-G_1,G_3-G_1 are elements of third order, F_2-G_1,F_3-G_1 are elements of sixth order. It is well known that Beauville family

$$(t_0+t_1+t_2)(t_0t_1+t_0t_2+t_1t_2)=At_0t_1t_2$$

is the family of the elliptic curves with fixed structure of sixth order (cf. [?]).

Further, let us express the natural action of group S_3 on the elliptic curve of $\mathcal{E}_A$ in terms of automorphisms of the curve. It can be shown in usual way that permutation of x_0 and x_1 is the automorphism: $P\mapsto 2F_1-P,P\in\mathcal{E}_A$, cyclic permutation $(0,1,2):x_0\mapsto x_1\mapsto x_2\mapsto x_0$ is the automorphism: $P\mapsto P+G_2-G_1,P\in\mathcal{E}_A$. Also, map $\varsigma:x_i\mapsto\frac{1}{x_i},i=0,1,2$ is the automorphism: $\varsigma:P\mapsto P+F_1-G_1$.

If $A, B \neq 0, 1, 9, \infty$ first and second equation define the product of elliptic curves $\mathcal{E}_A \times \mathcal{E}_B$. Equations (197), (198) define divisors $D_{1,\alpha'}, D_{2,\beta'} \subset \mathcal{E}_A \times \mathcal{E}_B$ respectively. Recall that product of elliptic curves $\mathcal{E}_A \times \mathcal{E}_B$ has divisors: $\mathcal{E}_A \times pt$ and $pt \times \mathcal{E}_B$ which are not numerically equivalent. These divisors are called *vertical and horizontal* respectively. We will say that divisor D of $\mathcal{E}_A \times \mathcal{E}_B$ has *type* (a, b) if $D \cdot (pt \times \mathcal{E}_B) = a$ and $D \cdot (\mathcal{E}_A \times pt) = b$. We obtain that $D_{1,\alpha'}$ is divisor of type (3,3), $D_{2,\beta'}$ is divisor of type (6,6).

Lemma 70. *For any $\alpha', \beta' \in \mathbb{P}^1$ divisor $D_{1,\alpha'}$ is linear equivalent to $3G_1 \times \mathcal{E}_B + \mathcal{E}_A \times 3G_1$, Divisor $D_{2,\beta'}$ of $\mathcal{E}_A \times \mathcal{E}_B$ is reducible and we have the following identity:*

$$D_{2,\beta'} = D'_{2,\beta'} + \sum_{i=1}^{3} F_i \times \mathcal{E}_B + \sum_{i=1}^{3} \mathcal{E}_A \times F_i. \tag{199}$$

Thus, for any $\alpha', \beta' \in \mathbb{P}^1$ divisors $D_{1,\alpha'}$ and $D'_{2,\beta'}$ are not linear equivalent, in particular, are not equal.

Proof. First statement is trivial. Consider the divisor $D_{2,\beta'}$ of $\mathcal{E}_A \times \mathcal{E}_B$ defined by (198). It is easy that $\mathcal{E}_A \times F_i, i = 1, 2, 3$ and $F_i \times \mathcal{E}_B$ are components of $D_{2,\beta'}$. Thus, $D_{2,\beta'} = D'_{2,\beta'} + \sum_{i=1}^{3} F_i \times \mathcal{E}_B + \sum_{i=1}^{3} \mathcal{E}_A \times F_i$. Clearly, divisors $D_{1,\alpha'}$ and $D_{2,\beta'}$ are linear equivalent to $3G_1 \times \mathcal{E}_B + \mathcal{E}_A \times 3G_1$ and $6G_1 \times \mathcal{E}_B + \mathcal{E}_A \times 6G_1$ respectively. Using lemma 69, we get the following linear equivalences of divisors:

$$D'_{2,\beta'} \sim_L (F_1 + F_2 + F_3) \times \mathcal{E}_B + \mathcal{E}_A \times (F_1 + F_2 + F_3) \nsim 3G_1 \times \mathcal{E}_B + \mathcal{E}_A \times 3G_1$$

□

Consider morphism $\varsigma^{\times 2} = \varsigma \times \varsigma : \mathcal{E}_A \times \mathcal{E}_B \to \mathcal{E}_A \times \mathcal{E}_B$. It is easy to see that $\varsigma \times \varsigma$ transforms linear system of divisors $\{D_{1,\alpha'}\}_{\alpha' \in \mathbb{P}^1}$ into linear system $\{D_{2,\beta'}\}_{\beta' \in \mathbb{P}^1}$.

Lemma 71. *For general A, B, α, β divisors $D_{1,\alpha'}$ and $D_{2,\beta'}$ are irreducible.*

Proof. Using transformation $\varsigma^{\times 2}$, it is enough to prove that $D_{1,\alpha'}$ is an irreducible for general α'. Consider linear system of divisors $\{D_{1,\alpha'}\}_{\alpha' \in \mathbb{P}^1}$. By theorem of Bertini, general divisor of this linear system is smooth outside of base locus. Let us compute the base locus of the system. It is given by system of the equations: $(t_0 + t_1 + t_2)(z_0 + z_1 + z_2) = 0$ and $t_0 z_0 + t_1 z_1 + t_2 z_2 = 0$ in the variety $\mathcal{E}_A \times \mathcal{E}_B$. There are 6 points: $(1 : -1 : 0) \times (0 : 0 : 1), (1 : 0 : -1) \times (0 : 1 : 0), (0 : 1 : -1) \times (1 : 0 : 0), (0 : 0 : 1) \times (1 : -1 : 0), (0 : 1 : 0) \times (1 : 0 : -1), (1 : 0 : 0) \times (0 : 1 : -1)$, 6 points: $(1 : -1 : 0) \times (1 : 1 : w), (1 : 0 : -1) \times (1 : w; 1), (0 : 1 : -1) \times (w : 1 : 1)$, where w satisfy to relation $(2w + 1)(w + 2) = Bw$ and 6 points $(1 : 1 : v) \times (1 : -1 : 0), (1 : v : 1) \times (1 : 0 : -1), (v : 1 : 1) \times (0 : 1 : -1)$, where v satisfy to relation: $(2u + 1)(u + 2) = Au$. Consider the point $(1 : 0 : 0) \times (0 : 1 : -1)$. Let us prove that general divisor $D_{1,\alpha'}$ is smooth in this point. For this purpose, let us consider the affine coordinate chart $V = F^4_{x_1,x_2,y_0,y_2}$, where $x_1 = \frac{t_1}{t_0}, x_2 = \frac{t_2}{t_0}, y_0 = \frac{z_0}{z_1}, y_2 = \frac{z_2}{z_1}$. Consider the intersection $\mathcal{E}_A \times \mathcal{E}_B \cap V$. Then divisor $D_{1,\alpha'}$ in this affine chart is given by the system of the equations:

$$(1 + x_1 + x_2)(x_1 + x_2 + x_1 x_2) = A x_1 x_2,$$
$$(y_0 + 1 + y_2)(y_0 + y_2 + y_0 y_2) = B y_0 y_2,$$
$$(1 + x_1 + x_2)(y_0 + 1 + y_2) = \alpha'(y_0 + x_1 + x_2 y_2).$$

One can calculate the matrix of the jacobian of this system in the point $(0, 0) \times (0, -1)$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & B-1 & -1 \\ -\alpha' & \alpha' & 1-\alpha' & 1 \end{pmatrix} \tag{200}$$

It is easy that for general B, α' rank of this matrix is 3. Analogously, one can consider another points. Hence, the general divisor $D_{1,\alpha'}$ is smooth, and hence, irreducible. Also, divisor $D_{2,\beta'}$ is irreducible as transform of $D_{1,\alpha}$. □

Corollary 72. *Morphism pr is dominant. For general point P fiber $pr^{-1}(P)$ consists of finite set of points, moreover $|pr^{-1}(P)| \leq 12$.*

Proof. Actually, we have that $D_{1,\alpha'} \cdot D_{2,\beta'} = 18$ and we have 6 common points in the "infinite" part. □

7.5 Varieties Y and its quotient $\mathcal{Y}$.

In this subsection we will study morphism $\Theta : Y \to \mathcal{U}$ and morphism $\eta : \mathcal{Y} \to \mathcal{U}$.

Proposition 73. *Morphism $\Theta : Y \to \mathcal{U}$ is a fibration of curves of genus 4.*

Proof. Fix a general point $\mathbf{u} = (u_1, u_2, u_3)$. Fiber $Y_{\mathbf{u}}$ is defined by system of equations:

$$a_{(1,2)} + a_{(1,3)} + a_{(2,3)} = u_1,$$

$$a_{(1,2,3)} + a_{(1,3,2)} = u_2,$$

$$(a_{(1,2)} - 1)(a_{(1,3)} - 1)(a_{(2,3)} - 1) = u_3,$$

$$a_{(1,2,3)}a_{(1,3,2)} = a_{(1,2)}a_{(1,3)}a_{(2,3)}.$$

Let us compactify these equations in the following way: affine space will be considered as open dense subvariety of $\mathbb{P}^5$. Thus, first two equations define 3-dimensional linear subspace $\mathcal{L}$ of $\mathbb{P}^5$. Third and second equations define a 2-dimensional subspace $\mathcal{V}$ of $H^0(\mathbb{P}^5, \mathcal{O}(3))$, i.e. pencil of cubics. These pencil has a singular element which is a union of linear space and quadric $\mathcal{Q}$. Actually, we can express this quadric in terms of local coordinates:

$$\mathcal{Q} : a_{(1,2,3)}a_{(1,3,2)} - a_{(1,2)}a_{(1,3)} - a_{(1,2)}a_{(2,3)} - a_{(1,3)}a_{(2,3)} + u_1 - 1 = u_3. \quad (201)$$

Consider projective space $\mathbb{P}^5$ with homogenous coordinates $(z_0 : z_1 : z_2 : z_3 : z_4 : z_5)$. For simplicity, let us change the variables in the following manner: $a_{(1,2,3)} = \frac{z_1}{z_0}, a_{(1,3,2)} = \frac{z_2}{z_0}, a_{(1,2)} = \frac{z_3}{z_0}, a_{(1,3)} = \frac{z_4}{z_0}, a_{(2,3)} = \frac{z_5}{z_0}$. Thus, the we get the following quadric $\mathcal{Q}$:

$$z_1z_2 - z_3z_4 - z_3z_5 - z_4z_5 - (u_1 - 1 - u_3)z_0^2 = 0 \quad (202)$$

and cubic $\mathcal{C}$:

$$z_0z_1z_2 = z_3z_4z_5 \quad (203)$$

Let us prove some properties of the intersection of general quadric and cubic:

Lemma 74. *For general $\mathbf{u} \in \mathcal{U}$ intersection $\mathcal{Q}$, $\mathcal{C}$ and 3-dimensional linear space $\mathcal{L}$ is non-singular.*

Proof. Space $\mathcal{L}$ is given by equations:

$$z_3 + z_4 + z_5 - u_1z_0 = 0, z_1 + z_2 - u_2z_0 = 0 \quad (204)$$

Consider matrix of the jacobian of the equations defining intersection of $\mathcal{Q}$, $\mathcal{C}$ and $\mathcal{L}$:

$$J = \begin{pmatrix} -u_1 & 0 & 0 & 1 & 1 & 1 \\ -u_2 & 1 & 1 & 0 & 0 & 0 \\ -z_1z_2 & -z_0z_2 & -z_0z_1 & z_4z_5 & z_3z_5 & z_3z_4 \\ -2(u_1 - 1 - u_3)z_0 & -z_2 & -z_1 & z_4 + z_5 & z_3 + z_5 & z_3 + z_4 \end{pmatrix} \quad (205)$$

Rows of J correspond to variables $z_0, ..., z_5$ respectively. It can be shown in usual way that statement of the lemma is equivalent to condition $\text{rank}J = 4$ for general $\mathbf{u} \in \mathcal{U}$. We will denote by J_{i_1,i_2,i_3,i_4} the submatrix of J with rows with numbers $i_1, i_2, i_3, i_4 \in \{1, ..., 6\}, i_1 < i_2 < i_3 < i_4$. Consider the varieties V_{i_1,i_2,i_3,i_4} defined by equations $\det J_{i_1,i_2,i_3,i_4} = 0, i_1, i_2, i_3, i_4 \in \{1, ..., 6\}$ respectively. It is easy that singularities of the intersection $\mathcal{C} \cap \mathcal{Q} \cap \mathcal{L}$ are in the $\bigcap_{(i_1,i_2,i_3,i_4),1 \le i_1 < i_2 < i_3 < i_4 \le 6} V_{i_1,i_2,i_3,i_4}$.

Further, one can show that $\det J_{3,4,5,6} = \det J_{2,4,5,6} = (z_3 - z_4)(z_4 - z_5)(z_5 - z_3), \det J_{2,3,4,5} = (z_4 - z_3)(z_5 - z_0)(z_1 - z_2), \det J_{2,3,4,6} = (z_5 - z_3)(z_4 - z_0)(z_1 - z_2), \det J_{2,3,5,6} = (z_5 - z_4)(z_3 - z_0)(z_1 - z_2)$. It is easy that each of the varieties V_{i_1,i_2,i_3,i_4} is a union of the three projective hyperspaces. There are several cases. Consider the case $z_5 = z_4, z_1 = z_2$. In this situation we have the line l: $z_1 = \frac{u_2z_0}{2}, z_2 = \frac{u_2}{z_0}, z_3 = -2z_5 + u_1z_0, z_4 = z_5$. And we have to consider the intersection of this line with quadric $\mathcal{Q}$ and cubic $\mathcal{C}$. One can show that the intersection of the line l, quadric $\mathcal{Q}$ and cubic $\mathcal{C}$ is empty for general u_1, u_2, u_3. One can solve the rest analogously. Lemma is proved

□

Analogous to lemma, standard arguments show us that intersection of 3-dimensional space and cubic is non-singular 2-dimensional cubic. Also, intersection of 3-dimensional space and quadric is non-singular quadric. Thus, we get the intersection of cubic and quadric in 3-dimensional space. Using lemma and generality of $\mathbf{u}$, we get that this intersection is complete and non-singular. Thus, cubic defines non-singular divisor of type $(3,3)$ on quadric $\mathcal{Q}$, i.e. curve of genus 4. □

Further, we will study morphism $\eta : \mathcal{Y} \to \mathcal{U}$. Action of S_3 on the ring $\mathcal{O}(Y)$ is defined as follows. Direct calculations show us that permutations $(1,2)$ and $(1,2,3)$ act on the ring $\mathcal{O}(Y)$ by formulas:

$$(1,2) : a_{(1,2)} \mapsto a_{(1,2)}, a_{(1,3)} \leftrightarrow a_{(2,3)}, a_{(1,2,3)} \leftrightarrow a_{(1,3,2)} \tag{206}$$

and

$$(1,2,3) : a_{(1,2)} \mapsto a_{(2,3)}, a_{(2,3)} \mapsto a_{(1,3)}, a_{(1,3)} \mapsto a_{(1,2)}, a_{(1,2,3)} \mapsto a_{(1,2,3)}, a_{(1,3,2)} \mapsto a_{(1,3,2)} \tag{207}$$

We will study $\mathcal{O}(Y)^{S_3}$ the ring of S_3-invariants. Using Noether's theorem, this ring is finitely generated. Direct calculations show us that we can choose generators of this ring in following way: $u_1, u_2, u_3, v = (a_{(1,2,3)} - a_{(1,3,2)})^2, w = (a_{(1,2)} - a_{(2,3)})(a_{(2,3)} - a_{(1,3)})(a_{(1,3)} - a_{(1,2)})(a_{(1,2,3)} - a_{(1,3,2)})$. There is a relation:

$$-16w^2 + v^4 + c_1v^3 + c_2v^2 + c_3v = 0. \tag{208}$$

Here, $c_i \in F[u_1, u_2, u_3], i = 1, 2, 3$.

Using computing system Maple, we get the following formulas:

$$c_1 = 6u_1 + u_1^2 + 12u_3 - 3u_2^2 - 15, \tag{209}$$

$$c_2 = 96u_3 + 48u_3^2 - 24u_1 - 12u_2^2u_1 + 3u_2^4 - 24u_2^2u_3 - 24u_3u_1 - 16u_1^2 + 8u_1^3 + 30u_2^2 + 8u_3u_1^2 - 2u_2^2u_1^2 + 48, \tag{210}$$

$$c_3 = 64 + 192u_3^2 - 48u_2^2 - 384u_3u_1 - 32u_3u_1^3 - u_2^6 + 24u_2^2u_1 + 64u_3^3 - 96u_2^2u_3 + 6u_2^4u_1 + 208u_1^2 - 8u_1^3u_2^2 - \tag{211}$$
$$48u_3^2u_2^2 + u_2^4u_1^2 + 12u_3u_2^4 + 224u_3u_1^2 - 8u_3u_1^2u_2^2 - 96u_1^3 + 16u_3^2u_1^2 + 24u_3u_1u_2^2 + 16u_2^2u_1^2 -$$
$$192u_3^2u_1 - 192u_1 + 192u_3 + 16u_1^4 - 15u_2^4.$$

Clearly, the affine curve (208) is a non-singular for general $\mathbf{u} \in \mathcal{U}$. Fix general point $\mathbf{u} \in \mathcal{U}$. Also, let us note that after standard compactification of fibre $\mathcal{Y}_{\mathbf{u}}$, we get the quartic curve in $\mathbb{P}^2$. This curve is singular at infinity. For simplicity, let us construct birational morphism $\mathcal{Y} \to \widetilde{\mathcal{Y}}$ given by the following formula:

$$u_i \mapsto u_i, i = 1, 2, 3 \quad v \mapsto v, \quad w \mapsto t = \frac{w}{v} = \frac{(a_{(1,2)} - a_{(2,3)})(a_{(2,3)} - a_{(1,3)})(a_{(1,3)} - a_{(1,2)})}{(a_{(1,2,3)} - a_{(1,3,2)})}. \tag{212}$$

It easy that variety $\widetilde{\mathcal{Y}}$ is given by equation:

$$-16t^2v + v^3 + c_1v^2 + c_2v + c_3 = 0, \tag{213}$$

where c_i are given by formulas (209),(210),(211). It is clear that this birational isomorphism is compatible with S_3 - action.

Also, we can transform $\widetilde{\mathcal{Y}}$ by the following substitution:

$$v_1 = ABC = \frac{u_2^2 - v}{4}.$$

We obtain the following form of the $\widetilde{\mathcal{Y}}$ given by formula:

$$(-u_2^2 + 4v_1)w^2 - 4v_1^3 + c_1'v_1^2 + c_2'v_1 + c_3' = 0, \tag{214}$$

where

$$c_1' = -15 + u_1^2 + 12u_3 + 6u_1, \tag{215}$$

$$c_2' = 4u_1^2 - 24u_3 + 6u_1 - 12u_3^2 - 12 - 2u_1^3 + 6u_3u_1 - 2u_3u_1^2, \tag{216}$$

$$c_3' = 4(u_3 - u_1 + \frac{u_1^2}{4} + 1)(1 + u_3 - u_1)^2. \tag{217}$$

Let us note the following symmetry γ of the variety Y defined by correspondences: $\gamma : a_{(i,j)} \mapsto a_{(i,j)}, i, j = 1, 2, 3$ and $\gamma : a_{(1,2,3)} \mapsto -a_{(1,3,2)}, a_{(1,3,2)} \mapsto -a_{(1,2,3)}$. It is easy that action of γ is compatible with S_3-action. Thus, there is an action of γ on the $\mathcal{Y}$ defined by rule: $\gamma : u_1 \mapsto u_1, u_2 \mapsto -u_2, u_3 \mapsto u_3, w \mapsto w, v \mapsto v$, and analogously, we can define action of γ on the $\widetilde{\mathcal{Y}}$. Consider natural projection: $\eta : \widetilde{\mathcal{Y}} \to \mathcal{U}$. It is easy that we can define action of the involution γ on $\mathcal{U}$ compatible with action on $\widetilde{\mathcal{Y}}$.

Proposition 75. *Morphism: $\eta : \widetilde{\mathcal{Y}} \to \mathcal{U}$ is elliptic fibration. Involution γ provides the isomorphism of fibers $\widetilde{\mathcal{Y}}_{\mathbf{u}}$ and $\widetilde{\mathcal{Y}}_{\gamma(\mathbf{u})}$ for $\mathbf{u} \in \mathcal{U}$.*

Proof. Fix general point $\mathbf{u} \in \mathcal{U}$. Consider fiber of the variety $\widetilde{\mathcal{Y}}$ over $\mathbf{u}$ given by formula (214). Consider natural compactification of the curve $\widetilde{\mathcal{Y}}_{\mathbf{u}}$ in the projective plane $\mathbb{P}^2$ with homogenous coordinates $(t_0 : t_1 : t_2)$. Put $w = \frac{t_1}{t_0}, v_1 = \frac{t_2}{t_0}$. Thus, we get the following cubic curve:

$$(4t_2 - u_2^2 t_0)t_1^2 = -4t_2^3 + c_1' t_2^2 t_0 + c_2' t_2 t_0^2 + c_3' t_0^3. \tag{218}$$

Studying natural projection: $\mathbb{P}^2 \to \mathbb{P}^1_{(t_0:t_2)}$, we obtain that this projection defines the covering of degree 2 of compactification $\widetilde{\mathcal{Y}}_{\mathbf{u}}$ onto $\mathbb{P}^1$ with ramification. One can check that for general point $\mathbf{u}$ ramification divisor has degree 4. Using non-singularity and degree of ramification divisor, we get that cubic curve is elliptic. It is easy that automorphism γ preserves the curve. □

7.6 Degree of morphism pr, function fields $F(X)^{S_3}$ and $F(\mathcal{X})$.

In this section we will study birational properties of varieties X/S_3 and $\mathcal{X}$. In particular, we will study ramification divisor of morphism ϕ.

Expressing the variable $a_{(1,2,3)}$ from the equation $a_{(1,2)}a_{(2,3)}a_{(1,3)} = a_{(1,2,3)}a_{(1,3,2)}$, we get the following isomorphism of function fields:

$$F(Y) \cong F(a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}) \tag{219}$$

and

$$F(\mathcal{Y}) \cong F(u_1, u_2, u_3, v, t; 16t^2v - v^3 - c_1v^2 - c_2v - c_3 = 0) \tag{220}$$

where $c_i, \in F[u_1, u_2, u_3], i = 1, 2, 3$ and v, t are defined by (212). It is easy that $F(X) = F(x_1, x_2, y_1, y_2)$. Recall that general fiber of pr is less or equal 12. Thus, we have for function fields the following inequality: $|F(X) : F(Y)| \leq 12$.

Fix a general point $(A = a_{(1,2)}, B = a_{(1,3)}, C = a_{(2,3)}, \alpha = a_{(1,2,3)}, \beta = a_{(1,3,2)}$. Let us study a fibers of the morphisms ϕ and pr over general point for obtaining of ramification divisors. For this purpose, consider the following system of equations (194), (195), (197), (198). Recall that first and second equation define elliptic curves $\mathcal{E}_A$ and $\mathcal{E}_B$ respectively.

For general $B, \frac{\alpha}{C}, \frac{\beta}{C}$ system of equations (195), (197), (198) defines a curve $\mathcal{S}$. We can consider this curve as intersection of two divisors D_1 and D_2 into 3-dimensional variety $\mathcal{E}_B \times \mathbb{P}^2$. We will study the intersection of curve $\mathcal{S}$ and curve $\mathcal{E}_A$. For this purpose, let us project this curve onto $\mathbb{P}^2$. Denote this projection map by $\pi_{\mathbb{P}^2}$.

It can be shown in usual way that projection of $\pi_{\mathbb{P}^2}$ is a S_3 - equivariant morphism. Actually, $(t_0 : t_1 : t_2) \in \pi_{\mathbb{P}^2}(\mathcal{C})$ iff $\sigma(t_0 : t_1 : t_2) \in \pi_{\mathbb{P}^2}(\mathcal{S})$ for any permutation $\sigma \in S_3$. Consider elementary symmetric polynomials: $\sigma_1 = t_0 + t_1 + t_2, \sigma_2 = t_0t_1 + t_0t_2 + t_1t_2, \sigma_3 = t_0t_1t_2$ We can choose the coordinate of quotient $\mathbb{P}^2_t/S_3$ as follows $(s_1 = \sigma_1^3 : s_2 = \sigma_1\sigma_2 : s_3 = \sigma_3)$. It is clear that curve $\mathcal{E}_A$ transforms into curve given by formula $s_2 = As_3$. Using Maple, we can check that curve $\pi_{\mathbb{P}^2}(\mathcal{S})$ transforms into curve S' given by equation:

$$a_1s_1s_3^3 + a_2s_2^3s_3 + a_3s_1s_2^3 + a_4s_1^2s_2s_3 + a_5s_1s_2^2s_3 + a_6s_1^2s_3^2 + a_7s_1s_2s_3^2 + a_8s_2^4 = 0, \tag{221}$$

with $a_i \in \mathbb{C}[A, B, C, \alpha, \beta, ABC = \alpha\beta]$:

$$a_1 = -B^2\alpha^3\beta^3 + 9\beta^3\alpha^3B,$$

$$a_2 = -B^2C^2\beta^3\alpha + 6BC^2\beta^3\alpha + B^2C^3\beta^3 + 3\alpha^3\beta^3 + 9C\beta^3\alpha^2 - 2BC\beta^3\alpha^2,$$
$$a_3 = BC^3\beta^2\alpha - BC^4\beta\alpha - B^2C^6 - C^2\alpha^2\beta^2 + BC^3\alpha^2\beta + B^2C^5\beta - B^2C^4\alpha\beta + B^2C^5\alpha,$$
$$a_4 = -2BC^2\beta^2\alpha^2 + C\beta^3\alpha^2 - B^2C^4\alpha^2 - 2BC^3\alpha^2\beta + 3C^2\alpha^2\beta^2 + B^2C^3\alpha^2\beta,$$
$$a_5 = -9C^2\alpha^2\beta^2 - 3BC^3\beta^2\alpha + 14BC^2\beta^2\alpha^2 + BC\beta^3\alpha^2 - \alpha^3\beta^3 - 6C\alpha^3\beta^2 - 2BC^2\alpha^3\beta$$
$$-3BC^3\alpha^2\beta - 6C\beta^3\alpha^2 + BC\alpha^3\beta^2 + 9BC^4\beta\alpha - B^2C^2\alpha^2\beta^2 - 2BC^2\beta^3\alpha,$$
$$a_6 = 3\alpha^3\beta^3 + 6BC^2\alpha^3\beta - B^2C^2\alpha^3\beta + 9C\alpha^3\beta^2 - 2BC\alpha^3\beta^2 + B^2C^3\alpha^3,$$
$$a_7 = -9\alpha^3\beta^3 + B^2C\beta^3\alpha^2 - 3BC\alpha^3\beta^2 - 3BC\beta^3\alpha^2 + B^2C\alpha^3\beta^2 -$$
$$18BC^2\beta^2\alpha^2 - B\beta^3\alpha^3,$$
$$a_8 = -B^2C^4\beta^2 - 2BC^2\beta^2\alpha^2 + B^2C^3\alpha\beta^2 + C\alpha^3\beta^2 + 3C^2\alpha^2\beta^2 - 2BC^3\beta^2\alpha.$$

Thus, consider the fiber of ϕ over general point (A, B, C, α, β) of variety Y. This fiber is the intersection of curves: $s_2 = As_3$ and $\pi_{\mathbb{P}^2}(\mathcal{S})$. Omitting point $(1:0:0)$, we get the following equation:

$$s_1^2(Aa_4 + a_6) + s_1s_3(a_1 + Aa_7 + A^2a_5 + A^3a_3) + s_3^2(A^3a_2 + A^4a_8) = 0. \tag{222}$$

It can be shown that intersection of $s_2 = As_3$ and S' is 2. Thus, intersection of $\mathcal{E}_A$ and $\pi_{\mathbb{P}^2}(\mathcal{S})$ is 12. Therefore, we have proved the following proposition:

Proposition 76. *Degrees of the morphisms pr and ϕ are 12 and 2 respectively.*

Recall that we have the intersection index of $D_{1,\alpha'}$ and $D'_{2,\beta'}$ is 18. Thus, for general point P there are 6 points of $D_{1,\alpha'} \cap D'_{1,\beta'}$ in the "infinite" part: $t_0t_1t_2 = 0, z_0z_1z_2 = 0$. One can calculate the intersection of $D_{1,\alpha}$ with $t_0t_1t_2 = 0$ and with $z_0z_1z_2 = 0$. We will study the points up to common permutation of t_i and z_i There are several points:

- $(1:0:0) \times (0:-1:1)$
- $(1:0:0) \times (1:z':z'')$, where z', z'' are different roots of the equation: $\alpha'(\alpha'-1) - (\alpha'-A)(\alpha'-1)z + (\alpha'-A)z^2 = 0$
- $(1:-1:0) \times (0:0:1)$
- $(1:-1:0) \times (1:1:z)$, where z is a root of the equation: $(2+z)(2z+1) = Az$
- $(1:z':z'') \times (1:0:0)$, where z', z'' are different roots of the equation: $\alpha'(\alpha'-1) - (\alpha'-B)(\alpha'-1)z + (\alpha'-B)z^2 = 0$
- $(1:1:z) \times (1:-1:0)$, where z is a root of the equation: $(2+z)(2z+1) = Bz$

Let us apply the involution $\varsigma^{\times 2}$ to divisor $D_{1,\beta'}$. It is easy that $\varsigma^{\times 2}$ preserves the divisors $t_0t_1t_2 = 0$ and $z_0z_1z_2 = 0$. One can see that for general A, B, α', β' there are 6 common points $(1:-1:0) \times (0:0:1), (1:0:-1) \times (0:1:0), (0:1:-1) \times (1:0:0), (1:0:0) \times (0:1:-1), (0:1:0) \times (1:0:-1), (0:0:1) \times (1:-1:0)$. Using proposition 76, we get that intersection indexes of these 6 common points are 1 for general $(A, B, \alpha', \beta') \in Y$.

Remark. As we know, morphism *pr* is dominant. Note that *pr* is not surjective. For instance, one can check that if $-A\beta' + AB - 3\alpha'\beta' - \alpha' B = 0$, then intersection indexes of the 6 common points are 3, and divisors $D_{1,\alpha'}$ and $D'_{1,\beta'}$ are irreducible. Thus, for point $P_0 \in \{-A\beta' + AB - 3\alpha'\beta' - \alpha' B = 0\}$ the fiber $pr^{-1}(P_0)$ is empty.

Further, let us study function field of the varieties X, Y, $\mathcal{X}$, $\mathcal{Y}$. We obtain that $F(X) = F(Y, x_1, x_2)$, where x_1, x_2 satisfy to $(1 + x_1 + x_2)(1 + \frac{1}{x_1} + \frac{1}{x_2}) = A$ and equation (222) expressed in the variables x_1, x_2. Thus, variables y_1, y_2 are rational functions over $a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}$ and x_1, x_2. One can show that $F(X) = F(Y, x_1)$, where x_1 satisfy to polynomial relation of degree 12 over $a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}$.

As we know, $|F(X)^{S_3} : F(Y)| = 2$, i.e. $F(X)^{S_3}$ is a quadratic extension of $F(Y)$. Put $h = s_1/s_3$. h can be described in terms of x_i's as follows:

$$h = \frac{(1 + x_1 + x_2)^3}{x_1 x_2}. \tag{223}$$

We have the relation:

$$h^2(Aa_4 + a_6) + h(a_1 + Aa_7 + A^2 a_5 + A^3 a_3) + A^3 a_2 + A^4 a_8 = 0. \tag{224}$$

Using the isomorphism (219), we obtain the isomorphism of function field

$$F(X)^{S_3} = F(x_1, x_2, y_1, y_2)^{S_3} \cong F(a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}, h),$$

where $a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}, h$ are described in terms of x_1, x_2, y_1, y_2 by formulas: (186),(189),(190),(223) and h satisfy to relation (224). Denote by d the discriminant of (224). Evidently, $F(x_1, x_2, y_1, y_2)^{S_3} \cong F(a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}, \sqrt{d})$. Of course, the choice of d is non-unique. Direct calculations show us that d can be chosen $S_3 \times S_3$ - invariant. Moreover, using Maple, we can choose d as element of $F[u_1, u_2, u_3]$:

$$d = 325 - 27u_2^2 + 2u_3u_1^2 + u_2^2u_1^2 + 138u_1^2 - 380u_1 + 30u_2 + 326u_3 - 56u_3u_1 + u_3^2 - 2u_3u_1u_2 - 2u_1u_2^2 - 4u_2^3 + \tag{225}$$
$$26u_2u_1^2 + 30u_3u_2 - 2u_1^3u_2 - 86u_1u_2 - 20u_1^3 + u_1^4.$$

Thus, we have proved the following proposition:

Proposition 77. • *We have the following isomorphism for function fields*

$$F(X)^{S_3} \cong F(a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}, \sqrt{d}) \tag{226}$$

where d is given by (225),

• *There exists the isomorphism of function fields:*

$$F(X)^{S_3 \times S_3} = F(\mathcal{X}) \cong F(u_1, u_2, u_3, t, v, \sqrt{d}; 16t^2v - v^3 - c_1v^2 - c_2v - c_3 = 0) \tag{227}$$

Using this proposition, we obtain that fiber $\mathcal{X}_{\mathbf{u}}$ over general point $\mathbf{u} \in \mathcal{U}$ is a union of two isomorphic elliptic curves $C_1 \cup C_2$. This curves correspond to different values $\sqrt{d}$. Also, it can be shown that the fiber $X_{\mathbf{u}}$ over general point $\mathbf{u} \in \mathcal{U}$ is a union of two isomorphic curves of genus 37. Let $\mathcal{V}$ be the hypersurface in the affine space $F^4_{(u_1,u_2,u_3,s)}$ defined by equation: $s^2 = d$, where d is defined by formula (225). There exists a morphism $X \to \mathcal{V}$, which fibers are connected, and natural projection $\mathcal{V} \to \mathcal{U}$, which is a covering of degree 2.

Corollary 78. *Consider morphism:*

$$\Phi = \Theta \circ pr : X \to \mathcal{U}.$$

This morphism has the following Stein factorization: $X \to \mathcal{V} \to \mathcal{U}$, i.e. fibers of the maps $X \to \mathcal{V}$ and $\mathcal{V} \to \mathcal{U}$ are connected and discrete respectively.

Analogously, we have quite similar Stein decomposition for morphism $\Theta \circ pr \circ \tau$.

7.7 Involutions on the $\mathcal{X}$.

In this subsection we will study involutions on the variety $\mathcal{X}$ and their properties.

Recall that we have the involution $\tau : X \to X$ given by rule: $\tau : p_i \leftrightarrow q_i, i = 1, 2, 3$ and there is a well-defined involution $\tau : \mathcal{X} \to \mathcal{X}$. We have the birational involutions $j : X/S_3 \to X/S_3$ defined as automorphisms of the coverings ϕ of degree 2. Also, recall that there is well-defined maps $j : \mathcal{X} \to \mathcal{X}$.

Consider the involution $\kappa : X \to X$ defined by formula: $x_i \mapsto \frac{1}{x_i}, y_i \mapsto \frac{1}{y_i}, i = 1, 2$. It can be shown in usual way that we can define involution $\kappa : \mathcal{X} \to \mathcal{X}$ such that the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & X \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\kappa} & \mathcal{X} \end{array}$$

is commutative.

Let us study the relations between involutions j, κ, τ on the $\mathcal{X}$. As we know,

$$F(\mathcal{X}) \cong F(u_1, u_2, u_3, v, t, \sqrt{d}; 16t^2v - v^3 - c_1v^2 - c_2v - c_3 = 0)$$

Proposition 79. *Consider the involutions τ, j, κ acting on the variety $\mathcal{X}$. $\tau \circ \kappa = \kappa \circ \tau$. Let us express action of the involutions in coordinates:*

$$\tau(u_i) = u_i, j(u_i) = u_i, \kappa(u_i) = u_i, i = 1, 2, 3, \tag{228}$$

$$\tau(\sqrt{d}) = -\sqrt{d}, \tag{229}$$

$$j(v) = v, j(t) = t, j(\sqrt{d}) = -\sqrt{d}, \tag{230}$$

$$\kappa(t) = -t, \kappa(v) = v, \kappa(\sqrt{d}) = -\sqrt{d}. \tag{231}$$

Proof. Consider the involutions τ and κ defined on the X. It is easy that they commutes. Thus, τ and κ commutes as involutions acting on $\mathcal{X}$. Further, let us consider expression of element $\sqrt{d}$ in coordinates x_1, x_2, y_1, y_2. One can show that $\tau(\sqrt{d}) = -\sqrt{d}, \kappa(\sqrt{d}) = -\sqrt{d}$. Direct calculations prove the rest of the statement. □

Recall that automorphism of $\phi \circ \tau$ is the involution $\tau \circ j \circ \tau$. Denote by t', v' the elements $\tau(t), \tau(v)$ of the function field $F(\mathcal{X})$. It is easy that $\tau \circ j \circ \tau(t') = t', \tau \circ j \circ \tau(v') = v', \tau \circ j \circ \tau(\sqrt{d}) = -\sqrt{d}$.

Proposition 80. *Morphism $pr_{12} = (pr, pr \circ \tau) : X \to \widetilde{\mathcal{Y}} \times_{\mathcal{U}} \widetilde{\mathcal{Y}}$ is a birational immersion, i.e. varieties X and $pr_{12}(X)$ are birationally isomorphic.*

Proof. It is sufficient to prove that the involutions $j \neq \tau \circ j \circ \tau$. Actually, it means that map $(\psi, \psi \circ \tau) : \mathcal{X} \to \widetilde{\mathcal{Y}} \times_{\mathcal{U}} \widetilde{\mathcal{Y}}$ is a birational immersion and hence, pr_{12} is. Consider divisor $D \subset \mathcal{X}$ consisting of points $x \in \mathcal{X}$ such that $j \circ \kappa(x) = x$. Also, consider the divisor $\tau(D)$ consisting of points such that $\tau \circ j \circ \kappa(x) \circ \tau = x$. Using commutativity of κ and τ, we get that $\tau(D)$ is divisor of the points x satisfying to $\tau \circ j \circ \tau \circ \kappa(x) = x$. Divisors D and $\tau(D)$ are given by equations: $t = 0$ and $t' = 0$ respectively. We can consider divisors D' and $\tau(D')$ in X the preimages of D and $\tau(D)$ under natural projection $X \to \mathcal{X}$ respectively. It is easy D and $\tau(D)$ are given by equations:

$$(1 + x_1 + x_2)(1 + \frac{1}{y_1} + \frac{1}{y_2})(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}) - (1 + y_1 + y_2)(1 + \frac{1}{x_1} + \frac{1}{x_2})(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2}) = 0$$

and

$$(1 + x_2 + y_2)(1 + \frac{1}{x_1} + \frac{1}{y_1})(1 + \frac{x_1}{x_2} + \frac{y_1}{y_2}) - (1 + x_1 + y_1)(1 + \frac{1}{x_2} + \frac{1}{y_2})(1 + \frac{x_2}{x_1} + \frac{y_2}{y_1}) = 0$$

respectively. Further, denote by f and g the elements $\frac{x_1}{y_2} - \frac{y_2}{x_1} + \frac{x_2y_1}{x_1} - \frac{x_1}{x_2y_1} + \frac{y_2}{x_2y_1} - \frac{x_2y_1}{y_2}$ and $\frac{x_2}{y_1} - \frac{y_1}{x_2} + \frac{x_1y_2}{x_2} - \frac{x_2}{y_2x_1} + \frac{y_1}{x_1y_2} - \frac{x_1y_2}{y_1}$ respectively. It can be shown in usual way that $\tau(f) = -f$ and $\tau(g) = g$. Also, it is easy that divisors D and $\tau(D)$ are given by equation $f + g = 0$ and $-f + g = 0$ respectively. Consider intersection of D and $\tau(D)$. It is easy that intersection $D \cap \tau(D)$ is given by $f = 0$ and $g = 0$. Using following expressions for f and g:

$$f = (\frac{1}{y_2} - \frac{1}{x_1})(\frac{x_1}{x_2} - y_1)(x_2 - \frac{y_2}{y_1}) = 0, g = (\frac{1}{y_1} - \frac{1}{x_2})(y_2 - \frac{y_1}{x_1})(x_1 - \frac{x_2}{y_2}) = 0$$

we get 9 two-dimensional components of $D \cap \tau(D)$. Thus, $j \neq \tau \circ j \circ \tau$.

□

As we know, τ provides automorphism of function field $F(\mathcal{X})$. Consider the elements $t' = \tau(t)$, $v' = \tau(v)$. It is easy that t', v' are rational function of variables $t, v, \sqrt{d}, u_1, u_2, u_3$. Proposition 80 shows us that these functions are non-trivially depends on $\sqrt{d}$.

Fix a general point $\mathbf{u} = (u_1, u_2, u_3)$. As we know, fiber of variety $\mathcal{X}$ over $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{U}$ is a union of isomorphic elliptic curves, i.e. $\mathcal{X}_{\mathbf{u}} = C_1 \cup C_2$ which are corresponds to $\sqrt{d}$ and $-\sqrt{d}$ respectively. Denote by E the fiber $\widetilde{\mathcal{Y}}_{\mathbf{u}}$. Let us identify curves C_1 and C_2 by means of the involution j. Also, fix a point $P \in E$, let us denote by $(P, \sqrt{d})$ and $(P, -\sqrt{d})$ the fiber of morphism $\phi : \mathcal{X}_{\mathbf{u}} \to \widetilde{\mathcal{Y}}_{\mathbf{u}}$ over point $P \in \widetilde{\mathcal{Y}}_{\mathbf{u}} = E$.

Consider morphism $j \circ \tau$ acting on the $\mathcal{X}$. As we know, $j \circ \tau(u_i) = u_i, j \circ \tau(\sqrt{d}) = \sqrt{d}$. Thus, $j \circ \tau$ defines automorphisms of the curves C_1 and C_2. Thus, we obtain two automorphisms of the elliptic curve E. As we know, there are two types of the automorphisms of E:

- shift: $P \mapsto P + S, P \in E$ for fixed element $S \in Pic^0(E)$,
- reflection: $P \mapsto 2R - P, P \in E$ for fixed point $R \in E$.

Let two automorphisms be of second type. Thus,

$$j \circ \tau : (P, \sqrt{d}) \mapsto (2R_+ - P, \sqrt{d}); (P, -\sqrt{d}) \mapsto (2R_- - P, -\sqrt{d})$$

for some fixed points $R_+, R_- \in E$. Therefore, $j \circ \tau \circ j \circ \tau$ is identity morphism. Thus, $j = \tau \circ j \circ \tau$, and hence, we get contradiction with proposition 80.

Let one of the automorphisms is of first type, other is of second type. Thus,

$$j \circ \tau : (P, \sqrt{d}) \mapsto (2R - P, \sqrt{d}); (P, -\sqrt{d}) \mapsto (P + S, -\sqrt{d})$$

for some fixed point $R \in E$ and fixed element $S \in Pic^0(E)$. Thus, involution τ is given by formula:

$$\tau : (P, \sqrt{d}) \mapsto (2R - P, -\sqrt{d}); (P, -\sqrt{d}) \mapsto (P + S, \sqrt{d}).$$

We get contradiction with fact: $\tau^2 = 1$. Actually, automorphism

$$\tau^2 : (P, \sqrt{d}) \mapsto (2R - P + S, \sqrt{d}); (P, -\sqrt{d}) \mapsto (2R - P - S, -\sqrt{d})$$

is not identity element for any $R \in E$ and $S \in Pic^0(E)$.

Thus, two automorphisms are of first type:

$$j \circ \tau : (P, \sqrt{d}) \mapsto (P + S_1, \sqrt{d}); (P, -\sqrt{d}) \mapsto (P + S_2, -\sqrt{d})$$

for fixed elements $S_1, S_2 \in Pic^0(E)$. Further, we get the following formula for automorphism τ:

$$\tau : (P, \sqrt{d}) \mapsto (P + S_1, -\sqrt{d}); (P, -\sqrt{d}) \mapsto (P + S_2, \sqrt{d}).$$

Therefore,

$$\tau^2 : (P, \sqrt{d}) \mapsto (P + S_1 + S_2, \sqrt{d}); (P, -\sqrt{d}) \mapsto (P + S_2 + S_1, -\sqrt{d}).$$

And hence, $S_1 + S_2 \sim_L 0$. Denote by S the element S_1. Thus, we obtain the following formula for τ:

$$\tau : (P, \sqrt{d}) \mapsto (P + S, -\sqrt{d}); (P, -\sqrt{d}) \mapsto (P - S, \sqrt{d}) \tag{232}$$

Thus, we have proved the following proposition:

Proposition 81. *Consider birational immersion: $\mathcal{X} \to \widetilde{Y} \times_{\mathcal{U}} \widetilde{Y}$. Fix a general point $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{U}$. Consider fibers $\mathcal{X}_{\mathbf{u}} = C_1 \cup C_2$, $\widetilde{\mathcal{Y}}_{\mathbf{u}} = E$. Curves C_1 and C_2 are divisors in the $E \times E$. Then C_1 and C_2 are divisors of type $(P, P + S), P \in E$ and $(P, P - S), P \in E$ for some fixed element $S \in Pic^0(E)$.*

Fix point $\mathbf{u} \in \mathcal{U}$. Thus, there is a point $R \in E$ such that $S \sim_L R - P_0$, where $P_0 = (0 : 1 : 0)$ is inflection point. From irreducibility $\mathcal{X}$ it follows that the point R don't define the section of fibration $\mathcal{Y} \to \mathcal{U}$. In particular, point R depends on point $\mathbf{u} \in \mathcal{U}$.

8 Case of graph $\Gamma_{3,6}$.

In this section we will study the variety $X(3,6)$ of 6-dimensional representations of algebra $B_{\frac{1}{6}}(\Gamma_{3,6})$.

8.1 Previous remarks.

In this subsection we recall the varieties, which we will study and some their properties and results. Also, we will formulate main results and some ideas of proof.

Let us recall the varieties and their notation:

- $X(6,6)$ is the variety of the projectors $p_1,...,p_6;q_1,...,q_6$ of rank 1 with relations:

$$p_ip_j = q_iq_j = 0, \sum_{i=1}^{6} q_i = \sum_{i=1}^{6} p_i = 1, \mathrm{Tr} p_iq_j = \frac{1}{6}.$$

  up to $GL_6(F)$ - conjugacy, i.e. $X(6,6) := \mathcal{M}_6 B_{\frac{1}{6}}(\Gamma_{6,6})[1]$.

- $X(3,6)$ is the variety of the projectors $p_1,p_2,p_3;q_1,...,q_6$ of rank 1 with relations:

$$p_ip_j = q_iq_j = 0, \sum_{i=1}^{6} q_i = 1, \mathrm{Tr} p_iq_j = \frac{1}{6}$$

  up to $GL_6(F)$ - conjugacy, i.e. $X(3,6) := \mathcal{M}_6 B_{\frac{1}{6}}(\Gamma_{3,6})[1]$.

- $X(3,3)$ is the variety of the projectors $p_1,p_2,p_3;q_1,q_2,q_3$ of rank 1 with relations:

$$p_ip_j = q_iq_j = 0, \mathrm{Tr} p_iq_j = \frac{1}{6}$$

  up to $GL_6(F)$ - conjugacy, i.e. $X(3,3) := \mathcal{M}_6 B_{\frac{1}{6}}(\Gamma_{3,3})[1]$. As we know, $X(3,3) \cong (F^*)^4$.

- $Y(6)$ is the variety of projectors $P;q_1,...,q_6$, where P is the projector of rank 3 and $q_1,...,q_6$ are orthogonal projectors of rank 1 with relations:

$$\sum_{i=1}^{6} q_i = 1, q_iq_j = 0, \mathrm{Tr} Pq_i = \frac{1}{2}.$$

  up to $GL_6(F)$ - conjugacy, i.e. $Y(6) := \mathcal{M}_6 \widetilde{A}_6(\frac{1}{2})[\vec{1},3]$.

- $Y(3)$ is the variety of projectors $P;q_1,q_2,q_3$, where P is the projector of rank 3 and q_1,q_2,q_3 are orthogonal projectors of rank 1 with relations:

$$q_iq_j = 0, \mathrm{Tr} Pq_i = \frac{1}{2}.$$

  up to $GL_6(F)$ - conjugacy, i.e. $Y(3) := \mathcal{M}_6 A_3(\frac{1}{2})[\vec{1},3]$

- Y is the variety of projectors $P;Q$ of rank 3 with relation $\mathrm{Tr} PQ = \frac{3}{2}$

Also we have well-defined action of symmetric groups by permutations of p_i and q_j. We have the following actions: group $S_3^{(p)}$ acts on $X(3,6)$ by permutation of p_i. Also, we have the actions of $S_3^{(p)}$ and $S_3^{(q)}$ on $X(3,3)$ by permutation of p_i and q_i respectively. Thus, consider the following quotients:

- variety Z is a quotient of $X(3,6)$ by action of group $S_3^{(p)}$
- variety $\mathcal{X}$ is a quotient of $X(3,3)$ by action of group $S_3^{(p)} \times S_3^{(q)}$

- variety $\mathcal{Y}(3)$ is a quotient of $Y(3)$ by action of S_3.

Moreover, we have the following natural maps:

- $pr_1 : X(6,6) \to X(3,6)$ defined by rule: $(p_1, ..., p_6; q_1, ..., q_6) \mapsto (p_1, p_2, p_3; q_1, ..., q_6)$,
- $pr_2 : X(3,6) \to X(3,3)$ defined by rule: $(p_1, p_2, p_3; q_1, ..., q_6) \mapsto (p_1, p_2, p_3; q_1, q_2, q_3)$,
- $\phi_1 : X(3,6) \to Y(6)$ defined by rule: $(p_1, p_2, p_3; q_1, ..., q_6) \mapsto (p_1 + p_2 + p_3; q_1, ..., q_6)$,
- $\phi_2 : X(3,3) \to Y(3)$ defined by rule: $(p_1, p_2, p_3; q_1, q_2, q_3) \mapsto (p_1 + p_2 + p_3; q_1, q_2, q_3)$,
- $\psi_1 : Y(6) \to Y(3)$ defined by rule: $(P; q_1, ..., q_6) \mapsto (P; q_1, q_2, q_3)$,
- $\psi_2 : Y(3) \to Y$ defined by rule: $(P; q_1, q_2, q_3) \mapsto (P; q_1 + q_2 + q_3)$.

Further, denote some involutions on the varieties:

- involutions $\sigma^{(p)} : p_i \leftrightarrow p_{i+3}, i = 1,2,3; q_j \leftrightarrow q_j, j = 1, ..., 6$, $\sigma^{(q)} : q_j \leftrightarrow q_{j+3}, j = 1,2,3; p_i \leftrightarrow p_i, i = 1, ..., 6$, $\tau : p_i \leftrightarrow q_i, i = 1, ..., 6$ act on $X(6,6)$. It is easy that $\sigma^{(q)} = \tau \circ \sigma^{(p)} \circ \tau$.
- Also, we can define action of $\sigma^{(q)}$ on $X(3,6)$.
- We can define action of τ on $X(3,3)$,
- involutions $\sigma_P : P \mapsto 1 - P, q_j \leftrightarrow q_j, j = 1, ..., 6$ acts on $Y(6)$. Denote this involution by $\sigma_P^{(6)}$. Also we can define action of $\sigma^{(q)}$ on $Y(6)$.
- We can define action of $\sigma_P^{(3)}$ on $Y(3)$ by formula: $P \mapsto 1 - P$. It is easy that $\sigma_P^{(3)} \circ \psi = \psi \circ \sigma_P^{(6)}$.
- Involutions $\tau : P \leftrightarrow Q$, $\sigma_P : P \mapsto 1-P, Q \mapsto Q$ and $\sigma_Q : P \mapsto P, Q \mapsto 1-Q$. One can check that $\sigma_P = \sigma_Q$ as involution on Y. We will denote this involution by σ.

It is trivial that action of $\sigma^{(q)}$ on $X(6,6)$ and $X(3,6)$ commute with map pr_1. Analogously, we have the same properties in another cases.

Also, we have the following commutative diagrams:

- $$\begin{array}{ccc} X(6,6) & \xrightarrow{pr_1 \circ \sigma^{(p)}} & X(3,6) \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \sigma_P^{(6)} \circ \phi_1} \\ X(3,6) & \xrightarrow{\phi_1} & Y(6) \end{array} \quad . \tag{233}$$

  By theorem ??, variety $X(6,6)$ is birational isomorphic to fibred product $X(3,6) \times_{Y(6)} X(3,6)$.

- $$\begin{array}{ccc} X(3,6) & \xrightarrow{pr_2 \circ \sigma^{(q)}} & X(3,3) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \sigma_P^{(3)} \circ \phi_2 \circ \tau} \\ X(3,3) & \xrightarrow{\phi_2 \circ \tau} & Y(3) \end{array} \tag{234}$$

  By theorem ??, variety $X(3,6)$ is birational isomorphic to fibred product $\widetilde{X} = X(3,3) \times_{Y(3)} X(3,3)$. Denote by ζ the birational isomorphism: $X(3,6) \to \widetilde{X}$. Also, we have the following commutative diagram:

  $$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p''} & X(3,3) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \sigma_P^{(3)} \circ \phi_2 \circ \tau} \\ X(3,3) & \xrightarrow{\phi_2 \circ \tau} & Y(3) \end{array} \tag{235}$$

where p', p'' are natural projection. This commutative diagram is a fibred product. There is an action of involution $\sigma^{(q)}$ given by following formula:

$$\sigma^{(q)}(x_1, x_2) = (x_2, x_1),$$

where $(x_1, x_2) \in \widetilde{X}$. It is trivial that $p'' = p' \circ \sigma^{(q)}$.

•

$$\begin{array}{ccc} Y(6) & \xrightarrow{\sigma^{(q)}\circ\psi_1} & Y(3) \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \sigma_Q\circ\psi_2} \\ Y(3) & \xrightarrow{\psi_2} & Y \end{array} \tag{236}$$

We have the well-defined map: $\psi : Y(6) \to Y(3) \times_Y Y(3)$.

Further, let us introduce the following varieties and morphisms:

- there is an action of $S_3^{(p)}$ on $\widetilde{X}$. Denote by $\widetilde{Z}$ the quotient $\widetilde{X}/S_3^{(p)}$. It is trivial that variety Z and $\widetilde{Z}$ are birationally isomorphic. Denote by $\widetilde{\zeta}$ the birational isomorphism: $Z \to \widetilde{Z}$. Also, note the following isomorphism:
$$\widetilde{Z} \cong X(3,3)/S_3^{(p)} \times_{\mathcal{Y}(3)} X(3,3)/S_3^{(p)} \tag{237}$$
- there is a natural action of group $G = S_3 \times S_3 \times S_3$ by permutations of $p_i, i = 1,2,3$, $q_i, i = 1,2,3$ and $q_i, i = 4,5,6$ on $\widetilde{X}$. Denote the quotient $\widetilde{X}/G$ by $\widetilde{\mathcal{X}}$. It is easy that $\widetilde{\mathcal{X}}$ is a fibred product $\mathcal{X} \times_{\mathcal{Y}(3)} \mathcal{X}$. Also, we have the morphism: $\Phi' : \widetilde{\mathcal{X}} \to \mathcal{Y}(3) \times_Y \mathcal{Y}(3)$.

Moreover, we have the following natural morphisms:

- morphism $\phi_1 : X(3,6) \to Y(6)$ has the following decomposition:
$$X(3,6) \xrightarrow{\ \pi\ } Z \xrightarrow{\ \mu\ } Y(6), \tag{238}$$
where π is a natural projection, μ is a natural morphism.
- also morphism $\Phi : \widetilde{X} \to Y(3) \times_Y Y(3)$ has the following decomposition:
$$\widetilde{X} \xrightarrow{\ \widetilde{\pi}\ } \widetilde{Z} \xrightarrow{\ \widetilde{\mu}\ } Y(3) \times_Y Y(3), \tag{239}$$
where $\widetilde{\pi}$ is a natural projection, $\widetilde{\mu}$ is a natural morphism.

It is easy that we have the following commutative diagram:

$$\begin{array}{ccccc} X(3,6) & \xrightarrow{\pi} & Z & \xrightarrow{\mu} & Y(6) \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \widetilde{\zeta}} & & \downarrow{\scriptstyle \psi} \\ \widetilde{X} & \xrightarrow{\widetilde{\pi}} & \widetilde{Z} & \xrightarrow{\widetilde{\mu}} & Y(3) \times_Y Y(3) \end{array} \tag{240}$$

Further, note the following relation between $\widetilde{Z}$ and $\widetilde{\mathcal{X}}$. Using isomorphism (237), we can define morphism: $\widetilde{Z} \to \widetilde{\mathcal{X}}$ by natural factorization of action of group $S_3 \times S_3$. These two symmetric group permute $q_i, i = 1,2,3$ and $q_i, i = 4,5,6$ respectively. Analogously, we get the morphism: $Y(3) \times_Y Y(3) \to \mathcal{Y}(3) \times_Y \mathcal{Y}(3)$. One can show that the following diagram:

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{\mu}} & Y(3) \times_Y Y(3) \\ {\scriptstyle \pi\times\pi}\downarrow & & {\scriptstyle \pi\times\pi}\downarrow \\ \widetilde{\mathcal{X}} & \xrightarrow{\Phi'} & \mathcal{Y}(3) \times_Y \mathcal{Y}(3) \end{array} \tag{241}$$

is commutative.

Consider the morphism Φ. It is trivial that morphism $\Phi : \widetilde{X} \to Y(3) \times_Y Y(3)$ is a composition of natural projections and ϕ_2. Namely, we have the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{p'} & X(3,3) & \xrightarrow{\phi_2} & Y(3) \\ \downarrow{\scriptstyle p'\circ\sigma^{(q)}} & & & & \\ X(3,3) & & & & \downarrow{\scriptstyle \psi_2} \\ \downarrow{\scriptstyle \phi_2} & & & & \\ Y(3) & & \xrightarrow{\sigma\circ\psi_2} & & Y \end{array} \tag{242}$$

8.2 Previous properties of the variety $X(3,6)$.

In this subsection we will formulate previous properties of $X(3,6)$, which we will use for proof of its irreducibility.

First of all, let us make note about dimension of any irreducible component of $X(3,6)$. As we know from (generalized Hadamard matrices)?? variety $X(3,6)$ is subvariety of $(F^*)^{10}$ given by equations:

$$1 + z_1 + ... + z_5 = 0, 1 + t_1 + ... + t_5 = 0, \tag{243}$$

$$1 + \frac{1}{z_1} + ... + \frac{1}{z_5} = 0, 1 + \frac{1}{t_1} + ... + \frac{1}{t_5} = 0, \tag{244}$$

$$1 + \frac{z_1}{t_1} + ... + \frac{z_5}{t_5} = 0, 1 + \frac{t_1}{z_1} + ... + \frac{t_5}{z_5} = 0, \tag{245}$$

where $t_i, z_i, i = 1, ..., 5$ are coordinates in $(F^*)^{10}$. There is a description of these coordinates as traces of elements $p_1 q_1 p_i q_j, i = 2, 3, j = 2, ..., 6$. Therefore, dimension of any irreducible component of $X(3,6)$ is more or equal 4.

Note that $\dim_F X(3,6) = 4$ follows from birationality of $X(3,6)$ and $\widetilde{X}$. Actually, map $\phi_2 : X(3,3) \to Y(3)$ is dominant and finite in general point. Thus, we get that $\dim_F \widetilde{X} = 4$.

Firstly, let us prove the following:

Proposition 82. *Image of any irreducible component of $\widetilde{X}$ under p' (and, hence under p'') is dense in $X(3,3)$.*

Proof. Evidently, for any irreducible component $p'(\widetilde{X})$ and $p''(\widetilde{X})$ are both dense in $X(3,3)$ or both subvarieties in $X(3,3)$. Consider irreducible component $\widetilde{X}_1$ of $\widetilde{X}$. Assume that $p'(\widetilde{X}_1)$ and $p''(\widetilde{X}_1)$ are both subvarieties in $X(3,3)$. Then dimension of fibers of the restrictions p' and p'' on $\widetilde{X}_1$ are more than 0. Consider commutative diagram:

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{p'\circ\sigma^{(q)}} & p'(\widetilde{X}_1) \\ \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle \sigma_P\circ\phi_2\circ\tau} \\ p'(\widetilde{X}_1) & \xrightarrow{\phi_2\circ\tau} & Y_1, \end{array} \tag{246}$$

where variety $Y_1 = \phi_2 \circ \tau \circ p'(\widetilde{X}_1)$. Then for general point $y \in Y_1$ fiber $(\phi_2 \circ \tau)^{-1}(y)$ has dimension more than 0. We studied properties of morphism ϕ_2 in the section ??. As we know, dimension of fiber $(\phi_2 \circ \tau)^{-1}(y)$ is not more than 1. Thus, fibers of the maps pr' and pr'' have dimension 1. Because of $\dim_F \widetilde{X}_1 = 4$, we obtain that $\dim_F Y_1 = 2$. Consider the subvariety $S = \{y \in Y(3) | \dim_F (\phi_2)^{-1}(y) = 1\}$ of the $Y(3)$. Therefore, $Y_1 \subseteq S$. Let us prove the following lemma, contradicting with $\dim_F Y_1 = 2$.

Lemma 83. $\dim_F S = 1$.

Proof of the lemma. As we know, map ϕ_2 is given by formulas (186), (187), (188), (189), (190). Consider point $y = (A = a_{(1,2)}, B = a_{(1,3)}, C = a_{(2,3)}, \alpha = a_{(1,2,3)}, \beta = a_{(1,3,2)}) \in S$. Using results and notation of section ??, we have several cases:

- all curves E_A, E_B, E_C are elliptic,
- only two curves among E_A, E_B, E_C are elliptic,
- only one curve is elliptic.
- all curves are rational.

First case mean that $(A, B, C) \neq (0, 1, 9)$. As we know, the variety $(\phi_2)^{-1}(y)$ is the intersection $D_{1,\alpha'} \cap D'_{2,\beta'}$ in the product $E_A \times E_B$. Here $D_{1,\alpha'}$ and $D'_{2,\beta'}$ are defined by formulas (??) and $\alpha' = \frac{\alpha}{C}, \beta' = \frac{\beta}{C}$. Thus, divisors $D_{1,\alpha'}$ and $D'_{2,\beta'}$ are reducible. As we know, these divisors are of type $(3, 3)$. Hence, one of component of $D_{1,\alpha'}$ is of type $(1, 1)$ or $(1, 2)$. It means that curves E_A and E_B are isomorphic or 2-isogenous. By symmetry, we get the same property for curves E_A and E_C. By theorem Bertini, general divisor $D_{1,\alpha'}$ is irreducible. Thus, there are: one relation between A and B, because of E_A and E_B are isomorphic or 2-isogenous, one relation between A and C because of E_A and E_B are isomorphic or 2-isogenous, one relation between α and A, B, C because divisor $D_{1,\alpha'}$ is reducible. Thus, we obtain 1-dimensional variety of points $y \in Y(3, 3)$ such that $\dim_F \phi_2^{-1}(y) = 1$ and E_A, E_B, E_C are elliptic curves.

Consider the second case. Without loss of generality, suppose that E_A and E_B are elliptic. Assume that $C \neq 0$, i.e. $C = 1$ or 9. We can consider this case analogous to first one. Assume that $C = 0$. Then we have the following relations:

$$(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2})(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}) = 0,$$

Assume $1 + \frac{x_1}{y_1} + \frac{x_2}{y_2} = 0, 1 + \frac{y_1}{x_1} + \frac{y_2}{x_2} \neq 0$. It means that $\alpha \neq 0, \beta = 0$. Consider the equation:

$$(1 + x_1 + x_2)(1 + \frac{1}{y_1} + \frac{1}{y_2})(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}) = \alpha.$$

We can rewrite this equation by means of change the variables: $x_i \mapsto \frac{1}{x'_i}$. We obtain the following equation:

$$(1 + x'_1 + x'_2)(1 + y_1 + y_2) = \frac{AB}{\alpha}(1 + x'_1 y_1 + x'_2 y_2).$$

Arguments similar to first case show that there is the relation between A and B describing isomorphism or 2-isogenous of E_A and E_B. Thus, in this case $\dim_F S \leq 1$. Further, suppose that $1 + \frac{x_1}{y_1} + \frac{x_2}{y_2} = 0, 1 + \frac{y_1}{x_1} + \frac{y_2}{x_2} = 0$. Arguments quite similar to first case show us that E_A and E_B are isomorphic or 2-isogenous. Thus, we have proved the second case.

Consider third case. Assume $A = 0, B = 0$ and $1 + x_1 + x_2 = 0, 1 + \frac{1}{y_1} + \frac{1}{y_2} = 0$. Then $\alpha = 0$. We obtain the following equations:

$$(1 + \frac{x_1}{y_1} + \frac{x_2}{y_2})(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}) = C, (1 + \frac{1}{x_1} + \frac{1}{x_2})(1 + y_1 + y_2) = \frac{\beta}{C}(1 + \frac{y_1}{x_1} + \frac{y_2}{x_2}).$$

Solving $1 + x_1 + x_2 = 0, 1 + \frac{1}{y_1} + \frac{1}{y_2} = 0$, we get that first equation defines non-singular curve of genus 4 for general C. Thus, we obtain $\dim_F S \leq 1$ in this case. Analogous arguments prove the rest of third case.

Fourth case mean that $A, B, C = 0, 1, 9$. Thus, we get $\dim_F S \leq 1$. The lemma is proved.

Thus, we get $\dim_F Y_1 \leq 1$. Hence, image of any component of $\widetilde{X}$ under p' and p'' is dense in $X(3, 3)$. $\square$

8.3 Function fields $F(X(3,3))$, $F(X(3,3))^{\mathbb{Z}_3}$, $F(X(3,3))^{S_3}$ as extensions of $F(Y(3))$.

Firstly, let us study function fields of $X(3, 3)$ and its quotients.

Consider action of group S_3 on $X(3, 3)$ by permutations of $p_i, i = 1, 2, 3$. There is a normal subgroup $\mathbb{Z}_3 \lhd S_3$ and extensions of fields:.

$$F(Y(3)) \subset F(X(3, 3))^{S_3} \subset F(X(3, 3))^{\mathbb{Z}_3} \subset F(X(3, 3)) \quad (247)$$

Using subsection ??, we have the isomorphisms:

$$F(Y(3)) = F(a_{(1,2,3)}, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}), F(X(3,3))^{S_3} = F(a_{(1,2,3)}, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, h), \quad (248)$$

$$F(X(3,3)) = F(x_1, x_2, y_1, y_2) \quad (249)$$

Also, we have quadratic extension: $F(X(3,3)^{S_3} \subset F(X(3,3))^{\mathbb{Z}_3}$. Thus, $F(X(3,3))^{\mathbb{Z}_3} = F(a_{(1,2,3)}, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, h, w)$, where w satisfy to quadratic relation. Let us find this relation. As we know, one can choose the generators $a_{(1,2,3)}, a_{(1,2)}, a_{(2,3)}, a_{(1,3)}, h$ of the function field $F(X(3,3))^{S_3}$, where $a_{(1,2)}, a_{(1,3)}, a_{(1,2,3)}, a_{(1,3,2)}, h$ are described in terms of x_1, x_2, y_1, y_2 by formulas: (186),(189),(190),(223) and satisfy to relation (224).

Consider the function field $F(X(3,3))^{\mathbb{Z}_3}$. The generators of this field are $a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w$, where w is given in terms of x_1, x_2, y_1, y_2 by formula:

$$w = (x_1 - x_2)(1 - \frac{1}{x_1})(1 - \frac{1}{x_2}).$$

Therefore, we get the following relation:

$$w^2 = a_{(1,2)}^2 + 18a_{(1,2)} - 27 - 4(h + \frac{a_{(1,2)}^3}{h}). \quad (250)$$

Thus, $F(\Sigma)_3^{\mathbb{Z}} = F(\alpha, h, w)$, where α, h, w satisfy to relations (259) and (250). Further, consider the field $F(X(3,3))$. This field is cubic extension of $F(X(3,3))^{\mathbb{Z}_3}$. Let us show that generators of this field are $a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w, f$, where f is given by formula:

$$f = x_1 + \epsilon\frac{x_2}{x_1} + \epsilon^2\frac{1}{x_2},$$

where ϵ is primitive 3-root of unity. Consider the element $g = x_1 + \epsilon^2\frac{x_2}{x_1} + \epsilon\frac{1}{x_2}$. Evidently, $fg \in F(X(3,3))^{\mathbb{Z}_3}$. Moreover, $x_1 + \frac{x_2}{x_1} + \frac{1}{x_2} \in F(X(3,3))^{\mathbb{Z}_3}$. It can be shown in usual way that x_1, x_2 can be described in terms of $a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w, f$. As we know from subsection ??, y_1, y_2 are rational function over $a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, x_1, x_2$. One can show that we have the following relation:

$$f^3 = 1/8(a_{(1,2)} - 3 + w)^3 - 3(1-\epsilon)(h - 3a_{(1,2)} + 3) - 3(1-\epsilon^2)(\frac{a_{(1,2)}^3}{h} - 3a_{(1,2)} + 3). \quad (251)$$

Thus, $F(X(3,3)) \cong F(a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w, f)$ satisfying to (224), (250), (251).

We have proved the following proposition:

Proposition 84. *We have the following isomorphisms of function fields:*

- $$F(X(3,3))^{S_3} = F(a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h), \quad (252)$$

  where $h, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}$ satisfy to (224),

- $$F(X(3,3))^{\mathbb{Z}_3} = F(a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w), \quad (253)$$

  where $w, h, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}$ satisfy to (224), (250),

- $$F(X(3,3)) = F(a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}, h, w, f), \quad (254)$$

  where $f, w, h, a_{(1,2)}, a_{(1,3)}, a_{(2,3)}, a_{(1,2,3)}$ satisfy to (224), (250), (251).

8.4 General fibers of $X(3,3)$, $X(3,3)/\mathbb{Z}_3$, $X(3,3)/S_3$ and $Y(3)$ over $F^3 = F_{(a_{(1,2)},a_{(1,3)},a_{(2,3)})}$.

Let us consider the variety $Y(3)$. As we know, this variety is given by equation: $a_{(1,2,3)}a_{(1,3,2)} = a_{(1,2)}a_{(2,3)}a_{(1,3)}$, where $a_{(i,j)} = 4\mathrm{Tr}Pq_iPq_j$, $a_{(i,j,k)} = 8\mathrm{Tr}Pq_iPq_jPq_k$. Consider the involution σ_P defined early. In the coordinates $a_{(i,j)}, a_{(i,j,k)}$ σ_P-action could be described as follows:

$$\sigma_P : a_{(i,j)} \mapsto a_{(i,j)}, a_{(i,j,k)} \mapsto -a_{(i,j,k)}, i,j,k = 1,2,3. \tag{255}$$

We have the following decomposition of p' into sequence of the following natural maps:

$$X(3,3) \xrightarrow{\theta_1} X(3,3)/\mathbb{Z}_3 \xrightarrow{\theta_2} X(3,3)/S_3 \xrightarrow{\theta} Y(3). \tag{256}$$

Consider affine space $F^3 = SpecF[a_{(1,2)}, a_{(1,3)}, a_{(2,3)}]$. Thus, we have dominant map: $Y(3) \to F^3$. Fix general point $pt = (A,B,C) \in F^3$. Consider fibres of the varieties $X(3,3)$, $X(3,3)/\mathbb{Z}_3$, $X(3,3)/S_3$, $Y(3)$ over pt. In this situation fiber of $Y(3)$ over pt is affine line F^1_α with coordinate $\alpha = a_{(1,2,3)}$. Compactify this fiber as projective line $\mathbb{P}^1_\alpha$. Let us compactify fibres of $X(3,3)$, $X(3,3)/\mathbb{Z}_3$, $X(3,3)/S_3$ and denote they by Σ, $\Sigma/\mathbb{Z}_3$, Σ/S_3 respectively. Therefore, we have the following natural maps:

$$\Sigma \xrightarrow{\theta_1} \Sigma/\mathbb{Z}_3 \xrightarrow{\theta_2} \Sigma/S_3 \xrightarrow{\theta} \mathbb{P}^1_\alpha. \tag{257}$$

Using subsection 8.3, we have the following description of the function field of the algebraic curve Σ/S_3:

$$F(\Sigma)^{S_3} \cong F(h,\alpha), \tag{258}$$

where h and α satisfy to relation:

$$h^2\alpha^2 p_1(\alpha) + h\alpha p_2(\alpha) + p_3(\alpha) = 0. \tag{259}$$

Polynomials $p_1(\alpha), p_2(\alpha), p_3(\alpha)$ are given by formulas:

$$p_1(\alpha) = (A^2C + \alpha^2 - CA\alpha + 3A\alpha)(BA^2 - AB\alpha + 3A\alpha + \alpha^2),$$

$$p_2(\alpha) = -A^3(9B\alpha^2A + 2\alpha^4 + 6\alpha^2A - B\alpha^3A + B\alpha^2A^2 + 9CA\alpha^2 + 3BC^2A\alpha - 14BCA\alpha^2 - B^2CA^2\alpha + 3C\alpha^3 + 6CA^2B\alpha$$
$$+B^2CA\alpha^2 - AC\alpha^3 + A^2C\alpha^2 + 2B^2C^2A^2 + B^3\alpha^2 - 9B^2\alpha^2 - B^3\alpha CA + 3B^2\alpha CA - C^2\alpha BA^2 - C^3\alpha BA + C^2\alpha^2 BA$$
$$+3B\alpha^3 - B^2\alpha^3 + C^3\alpha^2 - C^2\alpha^3 + 18B\alpha^2C + B^2\alpha^2A + C^2\alpha^2A - 9C^2\alpha^2),$$

$$p_3(\alpha) = A^6(-CB\alpha + C^2B + 3C\alpha + \alpha^2)(B^2C - CB\alpha + 3B\alpha + \alpha^2).$$

It is easy that equation (259) is the relation (224). Note that degrees of polynomials p_1, p_2, p_3 are 4. Consider the following homogenous coordinates $(h_0 : h_1), (\alpha_0 : \alpha_1)$ of the product $\mathbb{P}^1_h \times \mathbb{P}^1_\alpha$. We have the following identity for affine coordinates $h = \frac{h_1}{h_0}, \alpha = \frac{\alpha_1}{\alpha_0}$. Thus, Σ/S_3 is the divisor of $\mathbb{P}^1_h \times \mathbb{P}^1_\alpha$ of type $(2,6)$. Evidently, map $\psi : \Sigma/S_3 \to \mathbb{P}^1_\alpha$ is the natural projection. For general (A,B,C) one can show that Σ/S_3 has two singularities $((1:0) \times (0:1)) \in \mathbb{P}^1_h \times \mathbb{P}^1_\alpha$ and $((0:1) \times (1:0)) \in \mathbb{P}^1_h \times \mathbb{P}^1_\alpha$ which are double points. Hence, genus of Σ/S_3 is 3. Further, let us calculate the ramification divisor $\mathcal{D}_\theta \subset \mathbb{P}^1_\alpha$ of the morphism ψ. We have studied this divisor in the subsection 7.6. Substitute affine coordinate α by $\frac{\alpha_1}{\alpha_0}$, we get that $\mathcal{D}_\theta$ is given by equation:

$$\alpha_1^2\alpha_0^2(p_2^2(\alpha_0,\alpha_1) - 4p_1(\alpha_0,\alpha_1)p_2(\alpha_0,\alpha_1)). \tag{260}$$

One can show that divisor $\mathcal{D}_\theta$ has several components. Proposition 77 shows that equation (225) is the only one component with multiplicity one. This equation is in the terms of the coordinates $u_i, i = 1,2,3$. Recall that $u_1 = A + B + C, u_2 = \alpha + ABC/\alpha, u_3 = (A-1)(B-1)(C-1)$. Consider the equations (250), (224). For fixed (A,B,C) we can rewrite relation (250) in the following manner:

$$w^2 = A^2 + 18A - 27 - 4(h + \frac{A^3}{h}). \tag{261}$$

Further, let us consider the following compactification of $\Sigma/\mathbb{Z}_3$. Curve $\Sigma/\mathbb{Z}_3 \subset \mathbb{P}^1_w \times \mathbb{P}^1_h \times \mathbb{P}^1_\alpha$ is given by equations (259), (261) in affine coordinates. It is trivial that natural map θ_2 is induced by projection $\mathbb{P}^1_w \times \mathbb{P}^1_h \times \mathbb{P}^1_\alpha \to \mathbb{P}^1_h \times \mathbb{P}^1_\alpha$. Consider ramification divisor of θ_2. Denote it by $\mathcal{D}_{\theta_2}$. By definition, $\mathcal{D}_{\theta_2} \subset \Sigma/S_3$. It is easy that $\mathcal{D}_{\theta_2}$ is given by equation:

$$h_0h_1((A^2+18A-27)h_0h_1 - 4h_1^2 - 4A^3h_0^2). \tag{262}$$

Finally, consider the following compactification of the curve Σ. Curve $\Sigma \subset \mathbb{P}^1_f \times \mathbb{P}^1_w \times \mathbb{P}^1_h \times \mathbb{P}^1_\alpha$ is given by the equations (224), (250), (251). Consider the ramification divisor $\mathcal{D}_{\theta_1} \subset \Sigma/\mathbb{Z}_3$. It is easy that this divisor is given by equation:

$$w_0^3h_0h_1(\frac{1}{8}((A-3)w_0+w_1)^3h_0h_1 - w_0^3(3(1-\epsilon)(h_1+(-3A+3)h_0)h_1+3(1-\epsilon^2)(A^3h_0+(-3A+3)h_1)h_0)). \tag{263}$$

8.5 Irreducibility of $X(3,6)$.

In this subsection we will prove that variety $X(3,6)$ is irreducible. Using birationality, it is sufficient to prove the irreducibility of $\widetilde{X}$.

Recall that there is the decomposition of p' into sequence of the natural morphism (256). We can define involution $\sigma^{(q)}$ on the $\widetilde{X}/\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\widetilde{X}/S_3 \times S_3$. Thus, we get the following diagram:

$$\begin{array}{ccccccc}
\widetilde{X} & \xrightarrow{\theta_1} & \widetilde{X}/\mathbb{Z}_3 & \xrightarrow{\theta_2} & \widetilde{X}/S_3 & \xrightarrow{\theta} & X(3,3) \\
\downarrow{\scriptstyle \theta_1\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\
\widetilde{X}/\mathbb{Z}_3 & \xrightarrow{\sigma^{(q)}\circ\theta_1} & \widetilde{X}/\mathbb{Z}_3\times\mathbb{Z}_3 & \xrightarrow{\theta_2} & \widetilde{X}/\mathbb{Z}_3\times S_3 & \xrightarrow{\theta} & X(3,3)/\mathbb{Z}_3 \\
\downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} \\
\widetilde{X}/S_3 & \xrightarrow{\theta_1} & \widetilde{X}/\mathbb{Z}_3\times S_3 & \xrightarrow{\sigma^{(q)}\circ\theta_2} & \widetilde{X}/S_3\times S_3 & \xrightarrow{\theta} & X(3,3)/S_3 \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta} \\
X(3,3) & \xrightarrow{\theta_1} & X(3,3)/\mathbb{Z}_3 & \xrightarrow{\theta_2} & X(3,3)/S_3 & \xrightarrow{\sigma_P\circ\theta} & Y(3)
\end{array} \tag{264}$$

We denote by $\theta_i, i = 1,2$ and θ all maps of factorizations by the same groups. One can show that this commutative diagram and any square is fibred product.

Fix the general point $pt = (A,B,C) \in F^3$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\Sigma\times_{\mathbb{P}^1_\alpha}\Sigma & \xrightarrow{\theta_1} & \Sigma/\mathbb{Z}_3\times_{\mathbb{P}^1_\alpha}\Sigma & \xrightarrow{\theta_2} & \Sigma/S_3\times_{\mathbb{P}^1_\alpha}\Sigma & \xrightarrow{\theta} & \Sigma \\
\downarrow{\scriptstyle \theta_1\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \theta_1} \\
\Sigma/\mathbb{Z}_3\times_{\mathbb{P}^1_\alpha}\Sigma & \xrightarrow{\sigma^{(q)}\circ\theta_1} & \Sigma/\mathbb{Z}_3\times_{\mathbb{P}^1_\alpha}\Sigma/\mathbb{Z}_3 & \xrightarrow{\theta_2} & \Sigma/\mathbb{Z}_3\times_{\mathbb{P}^1_\alpha}\Sigma/S_3 & \xrightarrow{\theta} & \Sigma/\mathbb{Z}_3 \\
\downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} \\
\Sigma/S_3\times_{\mathbb{P}^1_\alpha}\Sigma & \xrightarrow{\theta_1} & \Sigma/\mathbb{Z}_3\times_{\mathbb{P}^1_\alpha}\Sigma/S_3 & \xrightarrow{\sigma^{(q)}\circ\theta_2} & \Sigma/S_3\times_{\mathbb{P}^1_\alpha}\Sigma/S_3 & \xrightarrow{\theta} & \Sigma/S_3 \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\circ\sigma^{(q)}} & & \downarrow{\scriptstyle \theta} \\
\Sigma & \xrightarrow{\theta_1} & \Sigma/\mathbb{Z}_3 & \xrightarrow{\theta_2} & \Sigma/S_3 & \xrightarrow{\sigma_P\circ\theta} & \mathbb{P}^1_\alpha
\end{array} \tag{265}$$

Assume that variety $\widetilde{X}$ is reducible, i.e. $\widetilde{X} = \cup_{j=1}^s \widetilde{X}_j$. Using proposition 82, we get that for any component $\widetilde{X}_j$ variety $pr'(\widetilde{X}_j)$ is dense in $Y(3)$. Thus, for general point $pt = (A,B,C) \in F^3$ curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ is reducible. Let us prove that $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ is irreducible. It will be sufficient for irreducibility of $\widetilde{X}$.

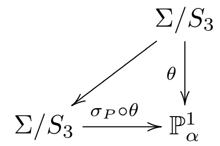
One can show that curves Σ and Σ/S_3 are irreducible.

Proposition 85. *For general point $pt = (A, B, C) \in F^3$, curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ is irreducible.*

Proof. Our proof has the following steps:

- $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$ is irreducible curve,
- $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$ is irreducible curve,
- $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3$ is irreducible curve,
- $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ is irreducible one.

First step. As we know, maps ψ and $\psi \circ \sigma^{(q)}$ are coverings of degree 2. Assume that $\Sigma/S_3 \times_{F^1_\alpha} \Sigma/S_3$ is reducible. Using proposition 82, we obtain that there are 2 components. Each component gives us the map $\Sigma/S_3 \to \Sigma/S_3$ such that the following diagram:



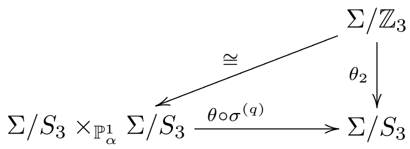
(266)

Thus, we have the following relation for ramification divisor $\mathcal{D}_\theta$:

$$\sigma_P(\mathcal{D}_\theta) = \mathcal{D}_\theta \tag{267}$$

Recall that this divisor is defined by equation (225). Recall that this equation is expressed in terms of variables u_1, u_2, u_3, where $u_1 = A + B + C$, $u_2 = \alpha + \beta$, $u_3 = (A-1)(B-1)(C-1)$. Using relation $\beta = \frac{ABC}{\alpha}$, we get that $\sigma_P(u_i) = u_i, i = 1, 3, \sigma_P(u_2) = -u_2$. Consider equation (225) as polynomial over u_2. One can see that u_2-degree of (225) is 3. Also, for general u_1, u_3 all coefficients of (225) are nonzero. Hence, $\sigma_P(\mathcal{D}_\theta) \neq \mathcal{D}_\theta$. Contradiction. Therefore, $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$ is irreducible.

Second step. Consider the curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$. Assume that this curve is reducible. As we know, $\psi : \Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3 \to \Sigma$ is covering of degree 2. Using proposition 82, we get that there are only 2 components of $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$. Each component is isomorphic to Σ. Consider the covering $\phi_1 : \Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3 \to \Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$. This covering has degree 3. Under assumption of reducibility of $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$, we obtain that curve $\Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$ has two components. Each component is isomorphic to $\Sigma/\mathbb{Z}_3$. Analogous to first step, we obtain that each component defines the map such that the following diagram:



(268)

is commutative. Hence, ramification divisors of the maps $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3 \to \Sigma/S_3$ and $\Sigma/\mathbb{Z}_3 \to \Sigma/S_3$ coincide. It can be shown in usual way that ramification divisor of $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3 \to \Sigma/S_3$ is $\theta^{-1}(\sigma_P(\mathcal{D}_\theta))$. Thus, coincidence of the divisors means that $\theta^{-1}(\sigma_P(\mathcal{D}_\theta)) = \mathcal{D}_{\theta_2}$. And hence, $\sigma_P(\mathcal{D}_\theta) = \theta(\mathcal{D}_{\theta_2})$. Divisor $\mathcal{D}_{\theta_1}$ is reducible. Component of $\mathcal{D}_{\theta_2}$ with multiplicity one is defined by equation: $(A^2 + 18A - 27)h_0h_1 - 4h_1^2 - 4A^3h_0^2$. It is easy that for general A, we get that there are two components $h_1 = th_0$ and $h_1 = \frac{A^3}{t}h_0$, where c is a root of equation: $(A^2 + 18A - 27) = 4(t + \frac{A^3}{t})$. Denote these two components by $\theta(\mathcal{D}_{\theta_2})'$ and $\theta(\mathcal{D}_{\theta_2})''$ respectively. Further, these components are defined by equations:

$$\alpha^2 t^2 p_1(\alpha) + \alpha t p_2(\alpha) + p_3(\alpha) = 0, \tag{269}$$

$$\alpha^2 A^6 p_1(\alpha) + \alpha A^3 t p_2(\alpha) + p_3(\alpha) t^2 = 0, \tag{270}$$

respectively. Hence, $\deg\theta(\mathcal{D}_{\theta_2}) = 12$. Recall that $\deg\mathcal{D}_\theta = 6$. It contradicts with coincidence. Therefore, curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$ is irreducible. Second step is proved.

Third step. Assume that curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3$ is reducible. Using second step, we get that there are two components of $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3$. Thus, curve $\Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3$ has two components. Each component defines the isomorphism: $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3 \to \Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3$ such that the following diagram:

$$\begin{array}{ccc} & & \Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3 \\ & \swarrow \cong & \downarrow \theta_2 \\ \Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3 & \xrightarrow{\sigma^{(q)}\circ\theta_2} & \Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3 \end{array} \qquad (271)$$

Denote by $\widetilde{\mathcal{D}}_{\theta_2}$ the ramification divisor of map: $\Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3 \to \Sigma/S_3 \times_{\mathbb{P}^1_\alpha} \Sigma/S_3$. Therefore, we have the following relation: $\sigma^{(q)}(\widetilde{\mathcal{D}}_{\theta_2}) = \widetilde{\mathcal{D}}_{\theta_2}$. It is easy that $\widetilde{\mathcal{D}}_{\theta_2} = \theta^{-1}(\mathcal{D}_{\theta_2})$. Using universality of fibred product, we obtain that

$$\theta(\mathcal{D}_{\theta_2}) = \sigma_P(\theta(\mathcal{D}_{\theta_2})) \qquad (272)$$

As we know from second step, $\theta(\mathcal{D}_{\theta_2})$ has two components $\theta(\mathcal{D}'_{\theta_2}$ and $\theta(\mathcal{D}''_{\theta_2}$, which are given by equations (269) and (270) respectively. We have two cases: $\sigma_P(\theta(\mathcal{D}'_{\theta_2}) = \theta(\mathcal{D}'_{\theta_2})$ and $\sigma_P(\theta(\mathcal{D}'_{\theta_2}) = \theta(\mathcal{D}''_{\theta_2})$. Consider coefficients p'_6, p'_5 and p''_6, p''_5 at α^6 and α^5 of (269) and (270). One can see that these coefficients of the polynomials (269) and (270) are $p'_6 = t^2; p'_5 = A(t(6-B-C)-2A^3)$ and $p''_6 = A^6; p''_5 = A^6(A(6-B-C)-2t)$ respectively. It is evident, σ_P transforms $p'_5 \mapsto -p'_5, p'_6 \mapsto p'_6; p''_5 \mapsto -p''_5, p''_6 \mapsto p''_6$. First case means that $p'_5 p'_6 = 0$, second case means that $p''_6 p'_5 + p'_6 p''_5 = 0$. One can see that for general (A, B, C) first and second cases are impossible. Third step is proved.

Fourth step. Assume curve $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ is reducible. Using third step, we obtain that $\Sigma \times_{\mathbb{P}^1_\alpha} \Sigma$ has three components. Each component defines map: $\Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma \to \Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma$ such that the following diagram:

$$\begin{array}{ccc} & & \Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma \\ & \swarrow & \downarrow \theta_1 \\ \Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma & \xrightarrow{\sigma^{(q)}\circ\theta_1} & \Sigma/\mathbb{Z}_3 \times_{\mathbb{P}^1_\alpha} \Sigma/\mathbb{Z}_3 \end{array} \qquad (273)$$

is commutative. It can be shown in usual way that

$$\theta_2 \circ \theta_1(\mathcal{D}_\theta) = \sigma_P \circ \theta_2 \circ \theta_1(\mathcal{D}_\theta). \qquad (274)$$

Direct checking in style of third step shows that for general A, B, C it is not true. Fourth step is proved. □

Corollary 86. *Variety $X(3,6)$ is irreducible.*

8.6 Properties of the morphism: $\phi_1 : X(3,6) \to Y(6)$.

In this subsection we will prove that morphism $\mu : Z \to Y(6)$ is a birational immersion.

Consider morphism $\psi_2 : Y(3) \to Y$. As we know from subsection ??, we have the following decomposition:

$$Y(3) \xrightarrow{\psi'_2} \mathcal{U} \xrightarrow{\psi''_2} Y \qquad (275)$$

where $\mathcal{U}$ is affine space F^3 with coordinates u_1, u_2, u_3 defined in subsection??. Also, note the following property of this decomposition:

$$\begin{array}{ccccc} Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} & \xrightarrow{\psi''_2} & Y \\ \sigma_P \downarrow & & \sigma_P \downarrow & & \sigma \downarrow \\ Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} & \xrightarrow{\psi''_2} & Y \end{array} \qquad (276)$$

One can show that σ_P acts on $\mathcal{U}$ by rule: $\sigma_P : u_1 \mapsto u_1, u_2 \mapsto -u_2, u_3 \mapsto u_3$. This action coincides with action of γ from subsection ??. Also, recall that we have the following commutative diagram for $X(3,3)$:

$$\begin{array}{ccc} X(3,3) & \xrightarrow{\phi_2} & Y(3) \\ {\scriptstyle \phi_2\circ\tau}\downarrow & & \downarrow{\scriptstyle \psi'_2} \\ Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} \end{array} \tag{277}$$

Using these diagrams, we obtain that the diagram (242) can be rewritten in the following manner:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{p'\circ\sigma^{(q)}} & X(3,3) & \xrightarrow{\phi_2} & Y(3) \\ \Big| & & {\scriptstyle \phi_2\circ\tau}\downarrow & & \downarrow{\scriptstyle \psi'_2} \\ {\scriptstyle p'}\Big| & & Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} \\ \Big| & & {\scriptstyle \sigma_P}\downarrow & & \downarrow{\scriptstyle \sigma_P} \\ \downarrow & & & & \\ X(3,3) & \xrightarrow{\phi_2\circ\tau} & Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} \\ {\scriptstyle \phi_2}\downarrow & & {\scriptstyle \psi'_2}\downarrow & /\!\!/ & \downarrow{\scriptstyle \psi''_2} \\ Y(3) & \xrightarrow{\psi'_2} & \mathcal{U} & \xrightarrow{\psi''_2} & Y \end{array} \tag{278}$$

Thus, we have proved the following proposition:

Proposition 87. *Image of variety $\widetilde{X}$ under map Φ is in the subvariety $Y(3)\times_{\mathcal{U}} Y(3)$, i.e. we have the following commutative diagram:*

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow & \searrow^{\Phi} & \\ Y(3)\times_{\mathcal{U}} Y(3) & \hookrightarrow & Y(3)\times_Y Y(3) \end{array} \tag{279}$$

Corollary 88. *Similar statement for varieties $\widetilde{Z}$, $\widetilde{\mathcal{X}}$ and morphisms $\widetilde{\mu}$, Φ' are true. Also, we get that the following diagram:*

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{\mu}} & Y(3)\times_{\mathcal{U}} Y(3) \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi\times\pi} \\ \widetilde{\mathcal{X}} & \xrightarrow{\Phi'} & \mathcal{Y}(3)\times_{\mathcal{U}} \mathcal{Y}(3) \end{array} \tag{280}$$

is commutative.

Proposition 89. *Morphism $\widetilde{\mu} : \widetilde{Z} \to Y(3)\times_Y Y(3)$ is a birational immersion.*

Proof. It is sufficient to prove that morphism Φ' is a birational immersion. Consider the following commutative diagram:

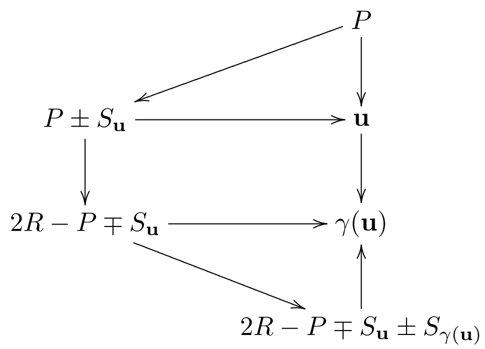
$$\begin{array}{ccccc} \widetilde{\mathcal{X}} & \xrightarrow{\Phi'} & \Phi'(\widetilde{\mathcal{X}}) & \hookrightarrow & \mathcal{Y}(3)\times_{\mathcal{U}} \mathcal{Y}(3) \\ \downarrow{\scriptstyle p'} & & & & \downarrow{\scriptstyle pr} \\ \mathcal{X} & & \xrightarrow{\phi_2} & & \mathcal{Y}(3) \end{array} \tag{281}$$

where pr is natural projection. As we know, $\deg p' = 2, \deg\phi_2 = 2$. Hence, we have the following cases: $\deg\Phi' = 1, 2, 4$. Let us prove that $\deg\Phi' = 1$. It is sufficient to prove that map $\deg\Phi' = 1$, i.e. Φ' is a birational immersion.

Fix general point $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{U}$. Denote by $\widetilde{\mathcal{X}}_{\mathbf{u}}$, $\mathcal{X}_{\mathbf{u}}$ and $\mathcal{E}_{\mathbf{u}}$ the fibers of $\widetilde{\mathcal{X}}$, $\mathcal{X}$ and $\mathcal{Y}(3)$ over $\mathbf{u}$ respectively. Thus, fiber of $\mathcal{Y}(3) \times_{\mathcal{U}} \mathcal{Y}(3)$ over $\mathbf{u}$ is a product $\mathcal{E}_{\mathbf{u}} \times \mathcal{E}_{\mathbf{u}}$. Statement of the proposition means that for general point $\mathbf{u} \in \mathcal{U}$ morphism Φ'' is a immersion of the fiber $\widetilde{\mathcal{X}}_{\mathbf{u}} = \mathcal{X}_{\mathbf{u}} \times_{\mathcal{E}_{\mathbf{u}}} \mathcal{X}_{\mathbf{u}}$ into $\mathcal{E}_{\mathbf{u}} \times \mathcal{E}_{\mathbf{u}}$. As we know from subsection ??, for general point $\mathbf{u} \in \mathcal{U}$ curve $\mathcal{E}_{\mathbf{u}}$ is elliptic curve given by (??). Fiber $\mathcal{X}_{\mathbf{u}}$ is the union of divisors $(P, P + S_{\mathbf{u}})$ and $(P, P - S_{\mathbf{u}}), P \in \mathcal{E}_{\mathbf{u}}$ for $S_{\mathbf{u}} \in Pic^0(\mathcal{E}_{\mathbf{u}})$. Recall that there is the symmetry γ of $Y(3)$ defined by rule: $a_{(i,j)} \mapsto a_{(i,j)}, a_{(1,2,3)} \mapsto -a_{(1,3,2)}, a_{(1,3,2)} \mapsto -a_{(1,2,3)}$. Identify the fibers $\mathcal{E}_{\mathbf{u}}$ and $\mathcal{E}_{\gamma(\mathbf{u})}$ via γ. It can be shown in usual way that under this identification, involution $\sigma_P : \mathcal{Y}(3) \to \mathcal{Y}(3)$ has the following description:

$$\sigma_P : (P, \mathbf{u}) \mapsto (2R - P, \gamma(\mathbf{u})), \tag{282}$$

where $(P, \mathbf{u}) \in \mathcal{E}_{\mathbf{u}}$, $R = (0 : 1 : 0)$ is inflection point in the compactification of $\mathcal{E}_{\mathbf{u}}$ as cubic curve in $\mathbb{P}^2$. Fix the point $P \in \mathcal{E}_{\mathbf{u}}$. Thus, we obtain the following diagram:



(283)

Condition $\deg\Phi' = 4$ means that all points $2R - P \mp S_{\mathbf{u}} \pm S_{\gamma(\mathbf{u})}$ coincide. It is clear that it means $2S_{\mathbf{u}} = 0$. But it contradicts with proposition ??. Thus, this case is impossible.

Consider case $\deg\Phi' = 2$. In this case, we get $2R - P - S_{\mathbf{u}} - S_{\gamma(\mathbf{u})} = 2R - P + S_{\mathbf{u}} + S_{\gamma(\mathbf{u})}$ and $2R - P + S_{\mathbf{u}} - S_{\gamma(\mathbf{u})} = 2R - P - S_{\mathbf{u}} + S_{\gamma(\mathbf{u})}$. Assume that $P = R$. Thus, points $R + S_{\mathbf{u}} + S_{\gamma(\mathbf{u})}$ and $R - S_{\mathbf{u}} + S_{\gamma(\mathbf{u})}$ are points of second order of the cubic curve $\mathcal{E}_{\mathbf{u}}$. This pair of points is defined over $\mathcal{O}(\mathcal{U})$. As we know, second order's points of cubic curve are in the line. Therefore, third point of second order defines the section of fibration $\mathcal{Y}(3) \to \mathcal{U}$. Consider the fibration $\mathcal{Y}(3) \to \mathcal{U}$. As we know, this fibration is given by equation (??). Points of second order of cubic curve $\mathcal{E}_{\mathbf{u}}$ are intersection of $\mathcal{E}_{\mathbf{u}}$ and line $w = 0$ in $\mathbb{P}^2$. Denote this intersection by V. Variety V is given by equation:

$$-4v_1^3 + c_1' v_1^2 + c_2' v_1 + c_3' = 0, \tag{284}$$

where $c_1', c_2', c_3' \in F[u_1, u_3]$ are given by formulas (??). As we know, one can transform variety V into (??). It is clear that V is irreducible. Thus, polynomial from formula (284) is irreducible over $\mathbb{F}$ and hence over $\mathbb{F}(u_1, u_3)$. Therefore, there are no points of second order of cubic curve $\mathcal{E}_{\mathbf{u}}$ which are sections over $\mathcal{U}$. Thus, case $\deg\Phi' = 2$ is impossible. There is only one possible case $\deg\Phi' = 1$.

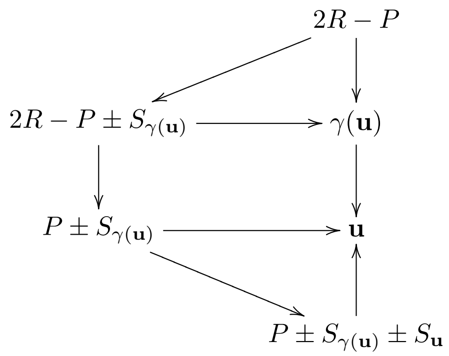
□

Corollary 90. *Morphism μ from diagram (238) is a birational immersion.*

Let us prove the $\sigma_P^{(3)}$-invariance of the image $\Phi(\widetilde{X}) \subset Y(3) \times_Y Y(3)$. Recall that there is a well-defined involution $\sigma_P^{(3)}$ acting on $Y(3) \times_Y Y(3)$. Also, recall that there is a well-defined involution $\sigma_P^{(3)}$ on the variety $\mathcal{Y}(3) \times_Y \mathcal{Y}(3)$. It is sufficient to prove that $\Phi'(\widetilde{\mathcal{X}})$ is $\sigma_P^{(3)}$ - invariant for $\sigma_P^{(3)}$ -invariance of the $\Phi(\widetilde{X})$. Fix general point $\mathbf{u} \in \mathcal{U}$. Consider the fiber of the variety $\mathcal{Y}(3) \times_{\mathcal{U}} \mathcal{Y}(3)$ over $\mathbf{u}$. As we know, this fiber is a product of isomorphic elliptic curves $\mathcal{E}_{\mathbf{u}} \times \mathcal{E}_{\mathbf{u}}$. Also, consider the fibers $\widetilde{\mathcal{X}}_{\mathbf{u}}$ and $\widetilde{\mathcal{X}}_{\sigma_P^{(3)}(\mathbf{u})}$. As we know from proof of the proposition 89, the fiber $\widetilde{\mathcal{X}}_{\mathbf{u}}$ is a union of four elliptic curves of the following type: $(P, 2R - P \pm S_{\mathbf{u}} \pm S_{\gamma(\mathbf{u})})$, where $P \in \mathcal{Y}(3)_{\mathbf{u}} = \mathcal{E}_{\mathbf{u}}$, $2R - P \pm S_{\mathbf{u}} \pm S_{\gamma(\mathbf{u})} \in \mathcal{Y}(3)_{\gamma(\mathbf{u})} = \mathcal{E}_{\mathbf{u}}$. After applying $\sigma_P^{(3)}$, we obtain that $\sigma_P^{(3)}(\widetilde{\mathcal{X}}_{\mathbf{u}}) =$

$(2R-P, P \pm S_{\mathbf{u}} \pm S_{\gamma(\mathbf{u})})$, where $2R-P \in \mathcal{Y}(3)_{\gamma(\mathbf{u})}, P \pm S_{\mathbf{u}} \pm S_{\gamma(\mathbf{u})} \in \mathcal{Y}(3)_{\gamma(\mathbf{u})}$. Let us check that $\sigma_P^{(3)}(\widetilde{\mathcal{X}}_{\mathbf{u}}) = \widetilde{\mathcal{X}}_{\sigma_P^{(3)}(\mathbf{u})} = \widetilde{\mathcal{X}}_{\gamma(\mathbf{u})}$.

Let us start from point $2R-P \in \mathcal{Y}(3)_{\gamma(\mathbf{u})}$. Using properties of $\widetilde{\mathcal{X}}$, we obtain the following diagram:



(285)

Thus, we obtain that $\sigma_P^{(3)}(\widetilde{\mathcal{X}}_{\mathbf{u}}) = \widetilde{\mathcal{X}}_{\gamma(\mathbf{u})}$, and hence, $\Phi'(\widetilde{\mathcal{X}})$ is $\sigma_P^{(3)}$ - invariant subvariety of $\mathcal{Y}(3) \times_Y \mathcal{Y}(3)$. Therefore, we have proved the following theorem:

Theorem 91. *Image $\Phi(\widetilde{X}) = \widetilde{\mu}(\widetilde{Z}) \subset Y(3) \times_Y Y(3)$ is $\sigma_P^{(3)}$ - invariant.*

9 Variety of orthogonal pairs in $sl(6)$.

9.1 Previous remarks.

Fix two partitions θ_1 and θ_2 of $\{1,2,3,4,5,6\}$ into two complement subsets. Without loss of generality, assume that $\theta_1 = (1,2,3) \cup (4,5,6)$ and $\theta_2 = (1,2,4) \cup (3,5,6)$. Denote by ρ the permutation $(3,4)$. Denote by $\Gamma_{3,2}$ the complete bipartite graph with 3 and 2 vertices in upper and down rows respectively.

Recall that we denote by $A_{<t_1,...,t_s>}$ the subalgebra of $B(\Gamma)$ generated by elements $t_1, ..., t_s \in B(\Gamma)$. Denote by $q_1, ..., q_6, p_1, p_2, p_3$ the generators of $B_r(\Gamma_{3,6})$. Thus, we have the following natural maps:

$$j : A_{<q_1,q_2,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,2}) \to A_{<q_1,q_2,q_3,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,3}), \tag{286}$$

$$j' : A_{<q_1,q_2,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,2}) \to A_{<q_1,q_2,q_4,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,3}), \tag{287}$$

$$i : A_{<q_1,q_2,q_3,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,3}) \to B_r(\Gamma_{3,6}), i' : A_{<q_1,q_2,q_4,p_1,p_2,p_3>} \cong B_r(\Gamma_{3,3}) \to B_r(\Gamma_{3,6}) \tag{288}$$

defined obviously. It is trivial that we have the following commutative diagram:

$$\begin{array}{ccc} A_{<q_1,q_2,p_1,p_2,p_3>} & \xrightarrow{j} & A_{<q_1,q_2,q_3,p_1,p_2,p_3>} \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle i} \\ A_{<q_1,q_2,q_4,p_1,p_2,p_3>} & \xrightarrow{i'} & B_r(\Gamma_{3,6}) \end{array} \tag{289}$$

Analogously, we have the following commutative diagram:

$$\begin{array}{ccc} A_{<q_5,q_6,p_1,p_2,p_3>} & \xrightarrow{j} & A_{<q_4,q_5,q_6,p_1,p_2,p_3>} \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle i} \\ A_{<q_3,q_5,q_6,p_1,p_2,p_3>} & \xrightarrow{i'} & B_r(\Gamma_{3,6}) \end{array} \tag{290}$$

Consider algebra $B_{3,6}$ which is a quotient of the $B_{\frac{1}{6}}(\Gamma_{3,6})$ by ideal I generated by the element $\sum_{i=1}^{6} q_i - 1$. As we know

$$B_{3,6} \cong A_{<q_1,q_2,q_3,p_1,p_2,p_3>} *_{A_{<Q;p_1,p_2,p_3>}} A_{<q_4,q_5,q_6,p_1,p_2,p_3>}, \tag{291}$$

where $Q = q_1 + q_2 + q_3$. Also, we have the isomorphism of algebras:

$$B_{3,6} \cong A_{<q_1,q_2,q_4,p_1,p_2,p_3>} *_{A_{<Q';p_1,p_2,p_3>}} A_{<q_3,q_5,q_6,p_1,p_2,p_3>}, \tag{292}$$

where $Q' = q_1 + q_2 + q_4$. It is trivial that one can get second isomorphism from first one by composition with automorphism ρ. Identify variety $\mathcal{M}_6(A_{<q_1,q_2,q_3,p_1,p_2,p_3>} *_{A_{<Q;p_1,p_2,p_3>}} A_{<q_4,q_5,q_6,p_1,p_2,p_3>})$ and $\mathcal{M}_6(A_{<q_1,q_2,q_4,p_1,p_2,p_3>} *_{A_{<Q';p_1,p_2,p_3>}} A_{<q_3,q_5,q_6,p_1,p_2,p_3>})$ with $X(3,3) \times_{Y(3)} X(3,3)$. It is easy to see that ρ is a birational involution of $X(3,3) \times_{Y(3)} X(3,3)$. Note that second identification is obtained from first one by composition with ρ. Standard arguments shows that isomorphisms (291) and (292) correspond to birational morphisms: $\zeta, \zeta \circ \rho : X(3,6) \to X(3,3) \times_{Y(3)} X(3,3)$.

It is easy that we have the commutative diagram:

$$\begin{array}{ccc} A_{<p_1,p_2,p_3>} & \longrightarrow & A_{<q_1,q_2,p_1,p_2,p_3>} \\ \downarrow & & \downarrow \\ A_{<q_5,q_6,p_1,p_2,p_3>} & \longrightarrow & B_{3,6} \end{array} \tag{293}$$

Therefore, we get the natural morphism: $A_{<q_1,q_2,p_1,p_2,p_3>} *_{A_{<p_1,p_2,p_3>}} A_{<q_5,q_6,p_1,p_2,p_3>} \to B_{3,6}$. Using diagram (289), **??**, (**??**), we get the following diagram:

$$\begin{array}{ccc} & \nearrow & A_{<q_1,q_2,q_3,p_1,p_2,p_3>} *_{A_{<Q;p_1,p_2,p_3>}} A_{<q_4,q_5,q_6,p_1,p_2,p_3>} \cong B_{3,6} \\ A_{<q_1,q_2,p_1,p_2,p_3>} *_{A_{<p_1,p_2,p_3>}} A_{<q_5,q_6,p_1,p_2,p_3>} & & \updownarrow \rho \\ & \searrow & A_{<q_1,q_2,q_4,p_1,p_2,p_3>} *_{A_{<Q';p_1,p_2,p_3>}} A_{<q_3,q_5,q_6,p_1,p_2,p_3>} \cong B_{3,6} \end{array} \tag{294}$$

It is easy that there is unique 6-dimensional module of algebra $A_{<p_1,p_2,p_3>}$ such that rank of p_i is 1. Identifying varieties $\mathcal{M}_6(A_{<q_1,q_2,p_1,p_2,p_3>})$ and $\mathcal{M}_6(A_{<q_5,q_6,p_1,p_2,p_3>})$ with algebraic torus $X(3,2) = (F^*)^2$, we get the natural map: $X(3,6) \to X(3,2) \times X(3,2)$. Thus, we get the following commutative diagram:

$$\begin{array}{ccc} X(3,6) & \xrightarrow{\zeta} & \widetilde{X} \\ \downarrow{\scriptstyle \zeta\circ\rho} & & \downarrow \\ \widetilde{X} & \longrightarrow & X(3,2) \times X(3,2) \end{array} \tag{295}$$

Also, one can take the quotient by symmetric group $S_3^{(p)}$. Using commutativity ρ and $S_3^{(p)}$, we get the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\widetilde{\zeta}} & \widetilde{Z} \\ \downarrow{\scriptstyle \widetilde{\zeta}\circ\rho} & & \downarrow \\ \widetilde{Z} & \longrightarrow & X(3,2) \times X(3,2)/S_3^{(p)} \end{array} \tag{296}$$

Further, consider the map: $\mathcal{A}_6 = A_{<P;q_1,\ldots q_6>} \to B_{3,6}$, where $P = p_1 + p_2 + p_3$. Consider algebra $A_{<P;q_1,q_2>}$ and morphism: $A_{<P;q_1,q_2>} \to B_{\frac{1}{6}}(\Gamma_{3,2})$. Also, consider subalgebra $A_{<P>} = F \oplus F$. It is easy that we have natural morphisms: $A_{<P>} \to A_{<P;q_1,q_2>}$ and $A_{<P>} \to A_{<p_1,p_2,p_3,Q>}$ defined obviously. One can define the natural morphism: $A_{<P,q_1,q_2>} *_{A_{<P>}} A_{<P,q_5,q_6>} \to A_{<q_1,q_2,p_1,p_2,p_3>} *_{A_{<p_1,p_2,p_3>}} A_{<q_5,q_6,p_1,p_2,p_3>}$. It could be shown in usual way that there is a commutative diagram:

$$\begin{array}{ccc} A_{<q_1,q_2,p_1,p_2,p_3>} *_{A_{<p_1,p_2,p_3>}} A_{<q_5,q_6,p_1,p_2,p_3>} & \longrightarrow & B_{3,6} \\ \uparrow & & \uparrow \\ A_{<P,q_1,q_2>} *_{A_{<P>}} A_{<P,q_5,q_6>} & \longrightarrow & \mathcal{A}_6 \end{array} \tag{297}$$

Identify variety of modules $\mathcal{M}_6(A_{<P,q_1,q_2>})$ and $\mathcal{M}_6(A_{<P,q_5,q_6>})$,satisfying to condition: $rkP = 3, rkq_i = 1$, with affine space $Y(2) = F^1$, we obtain natural map: $Y(6) \to Y(2) \times Y(2)$.

Denote by π', π'_Y, π'' the natural morphisms $X(3,6) \to X(3,2) \times X(3,2)$, $Y(6) \to Y(2) \times Y(2)$ and $X(3,2) \times X(3,2) \to Y(2) \times Y(2)$ respectively. It is easy that

$$\pi' : (p_1, p_2, p_3; q_1, ..., q_6) \mapsto (p_1, p_2, p_3; q_1, q_2) \times (p_1, p_2, p_3; q_5, q_6) \tag{298}$$

$$\pi'_Y : (P; q_1, ..., q_6) \mapsto (P; q_1, q_2) \times (P; q_5, q_6) \tag{299}$$

and

$$\pi'' : (p_1, p_2, p_3; q_1, q_2) \times (p_1, p_2, p_3; q_5, q_6) \mapsto (p_1 + p_2 + p_3; q_1, q_2) \times (p_1 + p_2 + p_3; q_5, q_6) \tag{300}$$

Using technics of subsection??, we get the following commutative diagram of varieties:

$$\begin{array}{ccc} X(3,6) & \xrightarrow{\pi'} & X(3,2) \times X(3,2) \\ \phi_1 \downarrow & & \downarrow \pi'' \\ Y(6) & \xrightarrow{\pi'_Y} & Y(2) \times Y(2) \end{array} \tag{301}$$

Also, using standard arguments, we get the following commutative diagram:

$$\begin{array}{ccc} A_{<P,q_1,q_2>} *_{A_{<P>}} A_{<P,q_5,q_6>} & \longrightarrow & A_{<P,q_1,q_2,q_3>} *_{A_{<P,Q>}} A_{<P,q_4,q_5,q_6>} \\ \downarrow & & \downarrow \\ A_{<P,q_1,q_2,q_4>} *_{A_{<P,Q'>}} A_{<P,q_3,q_5,q_6>} & \longrightarrow & \mathcal{A}_6 \end{array} \tag{302}$$

where $Q = q_1 + q_2 + q_3$ and $Q' = q_1 + q_2 + q_4$. Identify varieties $\mathcal{M}_6(A_{<P,q_1,q_2,q_3>} *_{A_{<P,Q>}} A_{<P,q_4,q_5,q_6>})$ and $\mathcal{M}_6(A_{<P,q_1,q_2,q_4>} *_{A_{<P,Q'>}} A_{<P,q_3,q_5,q_6>})$ with $Y(3) \times_Y Y(3)$. Also, we can define birational involution ρ acting on $Y(3) \times_Y Y(3)$ transforming one identification to other one. Thus, we have the commutative diagram of varieties:

$$\begin{array}{ccc} Y(6) & \longrightarrow & Y(3) \times_Y Y(3) \\ \downarrow & & \downarrow \\ Y(3) \times_Y Y(3) & \longrightarrow & Y(2) \times Y(2) \end{array} \tag{303}$$

Consider map: $\widehat{\pi} : \widetilde{X} \to X(3,2) \times X(3,2)$ defined by rule:

$$\widehat{\pi} : (p_1, p_2, p_3; q_1, q_2, q_3) \times (p_1, p_2, p_3; q_4, q_5, q_6) \mapsto (p_1, p_2, p_3; q_1, q_2) \times (p_1, p_2, p_3; q_5, q_6) \tag{304}$$

It is easy that π' is a composition of birational morphism ζ and $\widehat{\pi}$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} X(3,6) & \xrightarrow{\zeta} & \widetilde{X} & \xrightarrow{\widehat{\pi}} & X(3,2) \times X(3,2) \\ \phi_1 \downarrow & & \downarrow \Phi & & \downarrow \pi'' \\ Y(6) & \xrightarrow{\psi} & Y(3) \times_Y Y(3) & \xrightarrow{\psi'} & Y(2) \times Y(2) \end{array} \tag{305}$$

Let us consider the quotient of the varieties in the higher row of diagram (305) by action of symmetric group $S_3^{(p)}$. There is a commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{\widehat{\zeta}} & \widetilde{Z} & \xrightarrow{\widehat{\pi}} & X(3,2)\times X(3,2)/S_3^{(p)} \\ \downarrow & & \downarrow{\scriptstyle \Phi'} & & \downarrow{\scriptstyle \pi''} \\ Y(6) & \xrightarrow{\psi} & Y(3)\times_Y Y(3) & \xrightarrow{\psi'} & Y(2)\times Y(2) \end{array} \tag{306}$$

Denote by Π the composition: $\pi''\circ\widehat{\pi}:\widetilde{Z}\to Y(2)\times Y(2)$. Also, using diagram (296), we get the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\widetilde{\zeta}} & \widetilde{Z} \\ \downarrow{\scriptstyle \widetilde{\zeta}\circ\rho} & & \downarrow{\scriptstyle \Pi} \\ \widetilde{Z} & \xrightarrow{\Pi} & Y(2)\times Y(2) \end{array} \tag{307}$$

9.2 Properties of morphism $\widehat{\pi}:\widetilde{X}\to X(3,2)\times X(3,2)$.

In this subsection we will prove that morphism $\widehat{\pi}$ is dominant and has degree 12.

As we know, variety $\widetilde{X}$ is a fibred product $X(3,3)\times_{Y(3)}X(3,3)$ and $X(3,3)=(F^*)^4$. $Y(3)$ is a hypersurface in $F^5_{A,B,C,\alpha,\beta}$ with coordinates defined by equation $ABC=\alpha\beta$. It is clear that $X(3,2)=(F^*)^2$. Define the coordinates in $X(3,3)\times_{Y(3)}X(3,3)$ as follows: $a_1=36\mathrm{Tr}p_1q_1p_2q_2, b_1=36\mathrm{Tr}p_1q_1p_3q_2, x_1=36\mathrm{Tr}p_1q_1p_2q_3, y_1=36\mathrm{Tr}p_1q_1p_3q_3$, $a_2=36\mathrm{Tr}p_1q_5p_2q_6, b_2=36\mathrm{Tr}p_1q_5p_3q_6, x_2=36\mathrm{Tr}p_1q_4p_2q_6, y_2=36\mathrm{Tr}p_1q_4p_3q_6$. Then morphism $\widehat{\pi}$ is given by formula:

$$\widehat{\pi}:(a_1,b_1,x_1,y_1)\times(a_2,b_2,x_2,y_2)\mapsto(a_1,b_1,a_2,b_2) \tag{308}$$

Fix a general point $(a_1=a,b_1=b,a_2=c,b_2=d)\in X(3,2)\times X(3,2)=(F^*)^4$. Let us prove that $\widehat{\pi}^{-1}(a,b,c,d)$ is non-empty. It can be shown in usual way that pre-image $\widehat{\pi}^{-1}(a,b,c,d)$ is a solution of the following system of equations:

$$(1+a+x_1)(1+\frac{1}{a}+\frac{1}{x_1})=(1+c+x_2)(1+\frac{1}{c}+\frac{1}{x_2}) \tag{309}$$

$$(1+b+y_1)(1+\frac{1}{b}+\frac{1}{y_1})=(1+d+y_2)(1+\frac{1}{d}+\frac{1}{y_2}) \tag{310}$$

$$(1+\frac{a}{b}+\frac{x_1}{y_1})(1+\frac{b}{a}+\frac{y_1}{x_1})=(1+\frac{c}{d}+\frac{x_2}{y_2})(1+\frac{d}{c}+\frac{y_2}{x_2}) \tag{311}$$

$$(1+a+x_1)(1+\frac{1}{b}+\frac{1}{y_1})(1+\frac{b}{a}+\frac{y_1}{x_1})=-(1+c+x_2)(1+\frac{1}{d}+\frac{1}{y_2})(1+\frac{d}{c}+\frac{y_2}{x_2}) \tag{312}$$

$$(1+\frac{1}{a}+\frac{1}{x_1})(1+b+y_1)(1+\frac{a}{b}+\frac{x_1}{y_1})=-(1+\frac{1}{c}+\frac{1}{x_2})(1+d+y_2)(1+\frac{c}{d}+\frac{x_2}{y_2}). \tag{313}$$

Let us simplify this system. For this purpose, introduce the following variables:

$$\alpha_1=\frac{1+a}{\sqrt{a}},\alpha_2=\frac{1+c}{\sqrt{c}},\beta_1=\frac{1+b}{\sqrt{b}},\beta_2=\frac{1+d}{\sqrt{d}},\gamma_1=\frac{a+b}{\sqrt{ab}},\gamma_2=\frac{c+d}{\sqrt{cd}}$$

$$x=\frac{x_1}{\sqrt{a}},y=\frac{y_1}{\sqrt{b}},z=\frac{x_2}{\sqrt{c}},w=\frac{y_2}{\sqrt{d}}.$$

One can check that there are relations between $\alpha_i,\beta_i,\gamma_i, i=1,2$:

$$\alpha_i^2+\beta_i^2+\gamma_i^2-\alpha_i\beta_i\gamma_i-4=0, i=1,2. \tag{314}$$

Thus, we obtain the following system:

$$(\alpha_1 + x)(\alpha_1 + \frac{1}{x}) = (\alpha_2 + z)(\alpha_2 + \frac{1}{z}) \tag{315}$$

$$(\beta_1 + y)(\beta_1 + \frac{1}{y}) = (\beta_2 + w)(\beta_2 + \frac{1}{w}) \tag{316}$$

$$(\gamma_1 + \frac{x}{y})(\gamma_1 + \frac{y}{x}) = (\gamma_2 + \frac{z}{w})(\gamma_2 + \frac{w}{z}) \tag{317}$$

$$(\alpha_1 + x)(\beta_1 + \frac{1}{y})(\gamma_1 + \frac{y}{x}) = -(\alpha_2 + z)(\beta_2 + \frac{1}{w})(\gamma_2 + \frac{w}{z}) \tag{318}$$

$$(\alpha_1 + \frac{1}{x})(\beta_1 + y)(\gamma_1 + \frac{x}{y}) = -(\alpha_2 + \frac{1}{z})(\beta_2 + w)(\gamma_2 + \frac{z}{w}) \tag{319}$$

Also, let us rewrite two last equations in the following manner:

$$(\alpha_1 + x)(\beta_1 + \frac{1}{y})(\gamma_2 + \frac{z}{w}) = -(\alpha_2 + z)(\beta_2 + \frac{1}{w})(\gamma_1 + \frac{x}{y}) \tag{320}$$

$$(\alpha_1 + \frac{1}{x})(\beta_1 + y)(\gamma_2 + \frac{w}{z}) = -(\alpha_2 + \frac{1}{z})(\beta_2 + w)(\gamma_1 + \frac{y}{x}). \tag{321}$$

Consider the following compactification of these equations: we will consider F^* with coordinates x, y, z, w as open dense subvariety of product $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(z_0:z_1)} \times \mathbb{P}^1_{(w_0:w_1)}$. One can describe non-homogenous coordinates in terms of homogenous ones as follows: $x = \frac{x_1}{x_0}, y = \frac{y_1}{y_0}, z = \frac{z_1}{z_0}, w = \frac{w_1}{w_0}$.

Denote by E_1, E_2 the curves in the product $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(z_0:z_1)}$ and $\mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(w_0:w_1)}$ given by formulas (315) and (316) respectively. It is easy that these curves are elliptic for general $\alpha_1, \alpha_2, \beta_1, \beta_2$. One can prove the following proposition:

Proposition 92. *For general $\alpha_1, \alpha_2, \beta_1, \beta_2$ elliptic curves E_1 and E_2 are not isogenous.*

Denote by $R_{00}, R_{01}, R_{10}, R_{11}$ the points $(0:1) \times (0:1), (0:1) \times (1:0), (1:0) \times (0:1), (1:0) \times (1:0)$ of curve E_1 respectively. We will denote by $R'_{00}, R'_{01}, R'_{10}, R'_{11}$ the same points of E_2. Denote by $Q_{00}, Q_{01}, Q_{10}, Q_{11}$ the points $(1:-\alpha_1) \times (1:-\alpha_2), (1:-\alpha_1) \times (-\alpha_2:1), (-\alpha_1:1) \times (1:-\alpha_2), (-\alpha_1:1) \times (-\alpha_2:1)$ of curve E_1. Also, denote by $Q'_{00}, Q'_{01}, Q'_{10}, Q'_{11}$ the points $(1:-\beta_1) \times (1:-\beta_2), (1:-\beta_1) \times (-\beta_2:1), (-\beta_1:1) \times (1:-\beta_2), (-\beta_1:1) \times (-\beta_2:1)$ of curve E_2. It is easy that

$$\{R_{00}, R_{01}, R_{10}, R_{11}\} = E_1 \cap \{x_0 x_1 z_0 z_1 = 0\}, \{R'_{00}, R'_{01}, R'_{10}, R'_{11}\} = E_2 \cap \{y_0 y_1 w_0 w_1 = 0\}$$

Using proposition 92, we obtain that $NS(E_1 \times E_2) = \mathbb{Z} \oplus \mathbb{Z}$, where $NS(E_1 \times E_2)$ is Neron-Severi group of $E_1 \times E_2$. One can check that there is an involution $\tau : E_1 \times E_2 \to E_1 \times E_2$ defined by

$$\tau : (x_0 : x_1) \times (y_0 : y_1) \times (z_0 : z_1) \times (w_0 : w_1) \mapsto (x_1 : x_0) \times (y_1 : y_0) \times (z_1 : z_0) \times (w_1 : w_0).$$

Recall that there are divisors of $E_1 \times E_2$ of two types: "horizontal" - $h_1 = point \times E_2$ and "vertical" - $h_2 = E_1 \times point$. Denote by D_1 and D_2 the divisors in $E_1 \times E_2$ given by (320), (321). Rewrite these equations in homogenous coordinates. We get D_1:

$$(\alpha_1 x_0 + x_1)(\beta_1 y_1 + y_0)(\gamma_2 w_1 z_0 + w_0 z_1) + (\alpha_2 z_0 + z_1)(\beta_2 w_1 + w_0)(\gamma_1 x_0 y_1 + x_1 y_0) = 0 \tag{322}$$

and D_2:

$$(\alpha_1 x_1 + x_0)(\beta_1 y_0 + y_1)(\gamma_2 w_0 z_1 + w_1 z_0) + (\alpha_2 z_1 + z_0)(\beta_2 w_0 + w_1)(\gamma_1 x_1 y_0 + x_0 y_1) = 0 \tag{323}$$

It is easy that $\tau(D_1) = D_2$. Also, denote by D the divisor given by equation (311). We will say that divisor D of $E_1 \times E_2$ is of type (a, b) iff $D \cdot h_2 = a, D \cdot h_1 = b$. Let us prove the following proposition:

Proposition 93. *D_1 and D_2 of $E_1 \times E_2$ are divisors of type $(4,4)$. Divisors D_1 and D_2 are reducible:*

$$D_1 = Q_{00} \times E_2 + E_1 \times Q'_{11} + D'_1, D_2 = Q_{11} \times E_2 + E_1 \times Q'_{00} + D'_2, \tag{324}$$

where $D'_i, i = 1,2$ are divisors of type $(3,3)$. In particular, $D'_1 \not\sim_L D'_2$. For general $\alpha_i, \beta_i, i = 1,2$ divisors D'_1, D'_2 are irreducible. $D'_1 \cdot D'_2 = 18$.

Proof. It is easy that $Q_{00} \times E_2 + E_1 \times Q'_{11}$ is a component of D_1. One can check that for general $\alpha_i, \beta_i, i = 1,2$ there are not a vertical and horizontal components in D'_1. Hence, if $D'_i, i = 1,2$ are reducible, then there are components of type $(1,1)$ or $(1,2)$. But it means that curve E_1 and E_2 are isomorphic or 2-isogenous. Using proposition 92, we get the required. □

Corollary 94. *For general point $(a,b,c,d) \in X(3,2) \times X(3,2)$ pre-image $\widehat{\pi}^{-1}(a,b,c,d)$ is a finite set. Thus, morphism $\widehat{\pi}$ is dominant.*

Let us calculate degree of $\widehat{pr}$. For this purpose, consider points of $D'_1 \cdot D'_2$ which lying in the $x_0x_1y_0y_1z_0z_1w_0w_1 = 0$. One can show that there are 4 points: $R_{00} \times R'_{00}, R_{11} \times R'_{11}, R_{01} \times R'_{10}, R_{10} \times R'_{01}$. Also, we have to find points of intersection $D'_1 \cap D'_2$ which lying in D. It can be shown in usual way that there are points $S_1 = Q_{10} \times Q'_{01}, S_2 = Q_{01} \times Q'_{10} \in D'_1 \cap D'_2$ not lying in D. For general point $(a,b,c,d) \in X(3,2) \times X(3,2)$ intersection multiplicities of these point is 1. Therefore, we have proved the following:

Proposition 95. *Degree of morhism $\widehat{\pi} : \widetilde{X} \to X(3,2) \times X(3,2)$ is 12.*

9.3 Properties of fibration $\Pi = \pi'' \circ \widehat{\pi} : \widetilde{Z} \to Y(2) \times Y(2)$.

In this subsection we will prove that general fibre of Π is a surface of general type.

Consider map $\widehat{\pi} : \widetilde{X} \to X(3,2) \times X(3,2)$. Let us introduce natural compactification of $X(3,2) \times X(3,2)$ as follows. $X(3,2) = (F^*)^2$ is an open subvariety of projective space $\mathbb{P}^2$, i.e. compactification of $X(3,2) \times X(3,2)$ is $\mathbb{P}^2 \times \mathbb{P}^2$. Denote by π'' the rational mapping: $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$ defined on $X(3,2) \times X(3,2)$..

Also, consider $S_3^{(p)}$ - invariant compactification $\widetilde{X}^c$ of $\widetilde{X}$ such that there is morphism: $\widehat{\pi}^c : \widetilde{X}^c \to \mathbb{P}^2 \times \mathbb{P}^2$. By construction, morphism $\widehat{\pi}^c$ coincide with $\widehat{\pi}$ on the open subvarieties $\widetilde{X}$ and $X(3,2) \times X(3,2)$. Degree of $\widehat{\pi}^c$ is 12.

Proposition 96. *Ramification divisor $\widehat{D} \subset \mathbb{P}^2 \times \mathbb{P}^2$ of $\widehat{\pi}^c$ is of type $(a,a), a \geq 22$.*

Proof. Let us make the following notes: Denote by $[pt], [line]$ the classes of point and line in $\mathbb{P}^2$. It is well-known that

$$H_0(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z}, H_2(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}, H_4(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, H_6(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}, H_8(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z}.$$

Generators of homology groups H_0, H_2, H_4, H_6, H_8 are $[pt] \times [pt]$; $[pt] \times [line], [line] \times [pt]$; $\mathbb{P}^2 \times [pt], [line] \times [line], [pt] \times \mathbb{P}^2$; $\mathbb{P}^2 \times [line], [line] \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^2$ respectively. It is well-known that $Pic(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}$.

Using obvious symmetry, we get that $\widehat{D}$ is homologically equivalent to $a([line] \times \mathbb{P}^2 + \mathbb{P}^2 \times [line])$. Let us prove that $a \geq 22$. Fix general line l in $\mathbb{P}^2$ and general point $P \in \mathbb{P}^2$. Consider the line $l \times P \subset \mathbb{P}^2 \times \mathbb{P}^2$. Consider curve $C = \widehat{pr}^{-1}(l \times P)$. As we know, $\widehat{\pi}|_C : C \to l \times P$ is a map of degree 12. Ramification divisor of $\widehat{\pi}|_C$ is the intersection $\widehat{D} \cap l \times P$. By Hurwitz's formula, we obtain the following formula:

$$2g_C - 2 = 12(-2) + deg(\widehat{D} \cap l \times P) = -24 + a, \tag{325}$$

where g_C is genus of curve C. Hence, $a = 22 + 2g_C \geq 22$. □

Let us prove the following proposition:

Proposition 97. *Fix a general $(A,B) \in \mathbb{P}^1 \times \mathbb{P}^1$. As we know, if $A, B \neq 0, 1, 9, \infty$ then $\pi''^{-1}(A,B)$ is a product of elliptic curves. Denote by $\widetilde{X}^c_{A,B}$ the fiber of $\widetilde{X}^c$ over A, B, then fiber $\widetilde{X}^c_{A,B}$ is a surface of general type.*

Proof. Consider map: $\pi'' : \mathbb{P}^2 \to \mathbb{P}^1$, where $\mathbb{P}^2$ and $\mathbb{P}^1$ are compactifications of $X(3,2)$ and $Y(2)$ respectively. It is easy that map: π'' in non-homogenous coordinates is given by formula: $(x,y) \mapsto (1+x+y)(1+\frac{1}{x}+\frac{1}{y})$. As we know from subsection??, this map defines the elliptic family. It is well-known that $A, B \neq 0,1,9,\infty$ fiber of this family is elliptic curve. Denote by $E_A \times E_B$ the fiber of π'' over A, B.

Further, we have the map: $\widehat{\pi}_{A,B} : \widetilde{X}^c_{A,B} \to E_A \times E_B$. Consider Stein factorization of $\widehat{\pi}_{A,B}$:

$$\widetilde{X}^c_{A,B} \xrightarrow{\widehat{\pi}_1} X^c_{A,B} \xrightarrow{\widehat{\pi}_2} E_A \times E_B \tag{326}$$

Morphisms $\widehat{\pi}_1$ and $\widehat{\pi}_2$ have connected fibers and discrete fibers respectively. Moreover, surface $X^c_{A,B}$ have no rational curves, hence it is minimal surface. Further, ramification divisor of $\widehat{\pi}$ coincides with ramification divisor of $\widehat{\pi}_2$. Thus, canonical class $K_{A,B}$ of $X^c_{A,B}$ is $\widehat{\pi}_2^{-1}(\widehat{D})$ and, using proposition 96, we get that $K^2_{A,B} > 0$. Since $X^c_{A,B}$ is covering of product of elliptic curves, it is irrational. Thus, $X^c_{A,B}$ is a surface of general type (cf.??). Therefore, $\widetilde{X}^c_{A,B}$ is surface of general type. □

Further, consider $\Pi : \widetilde{Z} \to Y(2) \times Y(2)$. As we know, morphism Π is a composition of $\widetilde{Z} \to X(3,2) \times X(3,2)/S_3^{(p)}$ and $X(3,2) \times X(3,2)/S_3^{(p)} \to Y(2) \times Y(2)$. Let us consider compactification $\widetilde{Z}^c$ such that $\Pi^c : \widetilde{Z}^c \to \mathbb{P}^1 \times \mathbb{P}^1$ is a composition of $\widetilde{Z}^c \to \mathbb{P}^2 \times \mathbb{P}^2/S_3^{(p)}$ and $\mathbb{P}^2 \times \mathbb{P}^2/S_3^{(p)} \to \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 98. *Fiber of $\mathbb{P}^2 \times \mathbb{P}^2/S_3^{(p)}$ over point $(A,B), A, B \neq 0,1,9,\infty$ is a K3 - surface.*

Proof. We have to prove that quotient of $E_A \times E_B$ by action of $S_3^{(p)}$ is a K3-surface. One can check that $\mathbb{Z}_3 \lhd S_3^{(p)}$ acts on $E_A \times E_B$ without fixed points. Thus, quotient of $E_A \times E_B$ by $\mathbb{Z}_3$ is a complex torus. Also, one can prove that quotient $S_3^{(p)}/\mathbb{Z}_3$ acts on $E_A \times E_B/\mathbb{Z}_3$ by formula: $x \mapsto -x, x \in E_A \times E_B/\mathbb{Z}_3$. Thus, quotient $E_A \times E_B/S_3^{(p)}$ is a Kummer surface. □

Moreover, we can prove the following proposition:

Proposition 99. *For general point $(A,B) \in \mathbb{P}^1 \times \mathbb{P}^1$ the fiber $\Pi^{-1}(A,B) = \widetilde{Z}_{A,B}$ is a surface of general type.*

Proof. The proof is quite similar to proof of proposition 97. □

9.4 Birational involutions of Z.

In this subsection we will study properties of some birational involutions of Z. We will denote by $Bir(X)$ the group of birational automorphisms of the variety X. Recall that permutation $\rho = (3,4)$ is a well-defined involution of $X(3,6)$. Using ζ, we can define birational involution $\zeta \circ \rho \circ \zeta^{-1} \in Bir(\widetilde{X})$. As we know, actions of S_3 and S_6 commute. Therefore, we have the well-defined involution $\rho \in Aut(Z)$ and birational involution $\widetilde{\zeta} \circ \rho \circ \widetilde{\zeta}^{-1} \in Bir(\widetilde{Z})$. As we know, $Y(6)$ is S_6 - variety. One can check that μ is S_6 - invariant morphism. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} & & Y(3) \times_Y Y(3) & & \\ & \nearrow^{\widetilde{\mu}\circ\widetilde{\zeta}} & & \nwarrow^{\psi} & \\ Z & & \xrightarrow{\quad\mu\quad} & & Y(6) \\ & \searrow_{\widetilde{\mu}\circ\widetilde{\zeta}\circ\rho} & & \swarrow_{\psi\circ\rho} & \\ & & Y(3) \times_Y Y(3) & & \end{array} \tag{327}$$

Actually, $\psi \circ \mu = \widetilde{\mu} \circ \widetilde{\zeta}$. Using S_6 - invariance of ρ, we obtain that $\psi \circ \rho \circ \mu = \psi \circ \mu \circ \rho = \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho$. Therefore, we get the required commutativity of diagram (327).

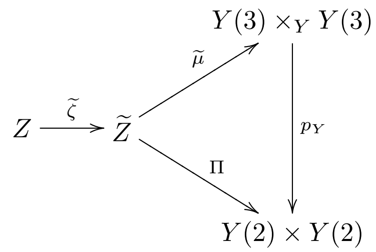
Proposition 100. *Assume $z, z' \in Z$ such that $\sigma_P^{(6)} \circ \mu(z) = \mu(z')$. Then $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta}(z) = \widetilde{\mu} \circ \widetilde{\zeta}(z')$ and $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho(z) = \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho(z')$.*

Proof. Applying ψ, we obtain that $\psi \circ \sigma_P^{(6)} \circ \mu(z) = \psi \circ \mu(z')$. Further, using relation $\sigma_P^{(3)} \circ \psi = \psi \circ \sigma_P^{(6)}$, we get that $\sigma_P^{(3)} \circ \psi \circ \mu(z) = \psi \circ \mu(z')$. Using commutativity of diagram (327), we get the required statement. Analogously, we obtain $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho(z) = \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho(z')$. □

As we know, $\widetilde{\mu} : \widetilde{Z} \to Y(3) \times_Y Y(3)$ is a birational morphism and image $\widetilde{\mu}(\widetilde{Z})$ is $\sigma_P^{(3)}$-invariant. Thus, we can define birational involution $\widetilde{\mu}^{-1} \circ \sigma_P^{(3)} \circ \widetilde{\mu} \in Bir(\widetilde{Z})$. Using birational morphism: $\widetilde{\zeta} : Z \to \widetilde{Z}$, we get birational involution $\sigma' = \widetilde{\zeta}^{-1} \circ \widetilde{\mu}^{-1} \circ \sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} \in Bir(Z)$. Also, we can define involution $\rho^{-1} \circ \sigma' \circ \rho \in Bir(Z)$.

Proposition 101. *Morphism $\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho \in Bir(Z)$ has finite order.*

Proof. Note the following properties of morphism: $\Pi \circ \widetilde{\zeta} : Z \to Y(2) \times Y(2)$: involution ρ acts on fibres of the morphism: $\Pi \circ \widetilde{\zeta}$. Let us prove that $\Pi \circ \sigma' = \Pi$. Actually, we can define involution σ_P on $Y(2) \times Y(2)$ by the rule: $P \mapsto 1 - P, q_i \mapsto q_i, i = 1, 2, 5, 6$. It is easy that natural morphism $p_Y : Y(3) \times_Y Y(3) \to Y(2) \times Y(2)$ satisfies to relation: $p_Y \circ \sigma_P^{(3)} = \sigma_P \circ p_Y$. Also, we have the following commutative diagram:



(328)

Further, $\Pi \circ \widetilde{\zeta} \circ \sigma' = p_Y \circ \widetilde{\mu} \circ \widetilde{\zeta} \circ \sigma' = p_Y \circ \sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} = \sigma_P \circ \Pi \circ \widetilde{\zeta}$. Study action of involution σ_P on $Y(2) \times Y(2)$. As we know, $Y(2) \times Y(2) = \mathcal{M}_6(< P; q_1, q_2 >) \times \mathcal{M}_6(< P; q_5, q_6 >) \cong F[\mathrm{Tr}Pq_1Pq_2, \mathrm{Tr}Pq_5Pq_6]$. One can calculate: $\sigma_P(\mathrm{Tr}Pq_1Pq_2) = \mathrm{Tr}(1-P)q_1(1-P)q_2 = \mathrm{Tr}Pq_1Pq_2$. Analogously, $\sigma_P(\mathrm{Tr}Pq_5Pq_6) = \mathrm{Tr}Pq_5Pq_6$ i.e. σ_P acts on $Y(2) \times Y(2)$ trivially. Thus, $\Pi \circ \widetilde{\zeta} \circ \sigma' = \Pi \circ \widetilde{\zeta}$. Therefore, involutions σ' and ρ act on the fibres of the morphism: $\Pi \circ \widetilde{\zeta} : Z \to Y(2) \times Y(2)$. Since general fibres of Π are surfaces of general type and birationality of $\widetilde{\zeta}$, we obtain that general fibres of $\Pi \circ \widetilde{\zeta}$ are surfaces of general type too. Recall the following property of surface of general type:

Proposition 102. *(cf.??) Let S be a surface of general type. Assume $\nu : S \to S'$ be a birational morphism, where S' is a minimal model. Then we have isomorphism: $\nu \circ Bir(S) \circ \nu^{-1} \cong Aut(S')$. Also, there is a constant c such that $|Bir(S)| = |Aut(S')| \leq c \cdot K_{S'}^2$.*

Therefore, group generated by ρ, σ' is finite, and hence, birational automorphism $\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho$ of Z has finite order. □

Let us formulate the following useful proposition:

Proposition 103. *We have the following relation for birational involutions σ' and ρ: $\sigma' \circ \rho = \rho \circ \sigma'$.*

Proof. See appendix. □

Corollary 104. *Involution σ' commutes with S_6 acting by permutations of $q_i, i = 1, ..., 6$.*

Proof. By proposition 103, σ' commutes with $\rho = (34)$. By construction, σ' commutes with $S_3 \times S_3$, where S_3's act by permutations of $q_i, i = 1, 2, 3$ and $q_i, i = 4, 5, 6$ respectively. Thus, σ' commutes with $(12), (23), (34), (45), (56)$, and hence with S_6. □

Let us prove the following important proposition:

Proposition 105. *Image* $\phi_1(X(3,6)) = \mu(Z) \subset Y(6)$ *is a* $\sigma_P^{(6)}$*-invariant and* $\mu^{-1} \circ \sigma_P^{(6)} \circ \mu = \sigma'$.

Proof. Recall that we proved early that there are birational immersion: $\widetilde{\mu} \circ \widetilde{\zeta} : Z \to Y(3) \times_Y Y(3)$. It means that there is open subvariety $U \subset Z$ such that restriction $\widetilde{\mu} \circ \widetilde{\zeta}|_U : U \to Y(3) \times_Y Y(3)$ is an immersion. Using commutativity of the upper triangle of the diagram (327), we get that restriction $\mu|_U : U \to Y(6)$ is an immersion too. Consider intersection $V = \cap_{\rho_1 \in S_6} \rho_1(U)$ - open subvariety of Z. It is clear that $\widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1|_V : V \to Y(3) \times_Y Y(3)$, $\mu|_V : V \to Y(6)$ are immersions. Recall following properties:

- for any $v \in V$ there is $v' \in V$ such that $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta}(v) = \widetilde{\mu} \circ \widetilde{\zeta}(v')$, i.e. $\sigma'(v) = v'$.
- $\mu \circ \rho_1 = \rho_1 \circ \mu$ for any $\rho_1 \in S_6$.
- $\psi \circ \sigma_P^{(6)} = \sigma_P^{(3)} \circ \psi$.
- $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1(v) = \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1(v') = \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1 \circ \sigma'(v)$.

Consider morphisms: $\prod \widetilde{\mu} = \prod_{\rho_1 \in S_6} \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1 : V \to \prod_{\rho_1 \in S_6} Y(3) \times_Y Y(3)$ and $\prod \psi = \prod_{\rho \in S_6} \psi \circ \rho_1 : Y(6) \to \prod_{\rho_1 \in S_6} Y(3) \times_Y Y(3)$. It is easy that the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\mu} & Y(6) \\ & \searrow^{\prod \widetilde{\mu}} & \downarrow{\scriptstyle \prod \psi} \\ & & \prod_{\rho_1 \in S_6} Y(3) \times_Y Y(3) \end{array} \tag{329}$$

As we know from ??, morphism $\prod \psi$ is a birational immersion. It is trivial that restriction of $\prod \widetilde{\mu}$ to V is an immersion. Hence, restriction of $\prod \psi \circ \mu$ to V is an immersion. Consider point $\prod \widetilde{\psi} \circ \mu(v), v \in V$. We will write point $\prod \widetilde{\psi} \circ \mu(v)$ in the following manner: $\prod \widetilde{\psi} \circ \mu(v) = (\psi \circ \rho_1 \circ \mu(v))_{\rho_1 \in S_6}$. Let us prove that $\sigma_P^{(6)} \circ \mu(v) = \mu(v')$.

Commutativity of $\sigma_P^{(3)}$ and S_6 means that $\sigma_P^{(3)} \circ \prod \widetilde{\mu}(v) = (\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1(v))_{\rho_1 \in S_6} = (\widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1 \circ \sigma'(v))_{\rho_1 \in S_6} = (\widetilde{\mu} \circ \widetilde{\zeta} \circ \rho_1(v'))_{\rho_1 \in S_6} = \prod \widetilde{\mu}(v')$. Thus,

$$\sigma_P^{(3)} \circ \prod \widetilde{\mu}(v) = \prod \widetilde{\mu}(v') = \prod \widetilde{\psi}(\mu(v')) \tag{330}$$

Further, using commutativity of diagram (329), we obtain that $\sigma_P^{(3)} \circ \prod \widetilde{\mu}(v) = \sigma_P^{(3)} \circ \prod \widetilde{\psi} \circ \mu(v) = (\sigma_P^{(3)} \circ \psi \circ \rho_1 \circ \mu(v))_{\rho_1 \in S_6} = (\psi \circ \sigma_P^{(6)} \circ \rho_1 \circ \mu(v))_{\rho_1 \in S_6}$. Using commutativity of $\sigma_P^{(6)}$ and S_6, we get that $(\psi \circ \sigma_P^{(6)} \circ \rho_1 \circ \mu(v))_{\rho_1 \in S_6} = (\psi \circ \rho_1 \circ \sigma_P^{(6)} \circ \mu(v))_{\rho_1 \in S_6} = \prod \widetilde{\psi}(\sigma_P^{(6)} \circ \mu(v))$. Using (330), we get that $\prod \widetilde{\psi}(\sigma_P^{(6)} \circ \mu(v)) = \prod \widetilde{\psi}(\mu(v'))$. Since $\prod \widetilde{\psi}$ is an immersion, we get that $\sigma_P^{(6)} \circ \mu(v) = \mu(v')$. Therefore, we get that for any v there is v' such that $\sigma_P^{(6)} \circ \mu(v) = \mu(v')$, i.e. image $\mu(V)$ is $\sigma_P^{(6)}$ - invariant. Also, we get the following identity: $\mu^{-1} \circ \sigma_P^{(6)} \circ \mu = \sigma'$. □

10 Appendix A: varieties $E_1(f_6')$ and $E_2(f_6')$.

In this section we will calculate dimensions of $E_1(f_6')$ and $E_2(f_6')$.

Let us calculate dimension of $E_1(f_6')$. Applying results of subsection 6.3, we get that there is a filtration: $E_1^{(2)}(f_6') \subset E_1^{(1)}(f_6') = E_1(f_6')$. As we know from proposition 52, we have the following immersion:

$$E_1^{(1)}(f_6') \subset \bigcup_\theta C(\theta), \tag{331}$$

where θ runs over all partitions of $\{1,2,3\}$ into union of two non-intersecting subsets. Without loss of generality, assume that $\theta = \{1\} \cup \{2,3\}$. Thus, $C(\theta)$ is defined by equations:

$$1 + z_{22} + z_{23} = 0, 1 + \frac{1}{z_{22}} + \frac{1}{z_{23}} = 0, 1 + z_{32} + z_{33} = 0, 1 + \frac{1}{z_{32}} + \frac{1}{z_{33}} = 0 \tag{332}$$

and the same system of equations for $y_{22}, y_{23}, y_{32}, y_{33}$. Let us formulate the following useful evident lemma:

Lemma 106. *System of equation:*

$$1+a+b=0, 1+\frac{1}{a}+\frac{1}{b}=0 \tag{333}$$

has two solutions: (ϵ,ϵ^2) and (ϵ^2,ϵ), where ϵ is a 3-th primitive root of unity.

Using this lemma, we get that there are only finite points satisfying to system (332).

In the case $E_1^{(2)}(f_6')$, we have the following system:

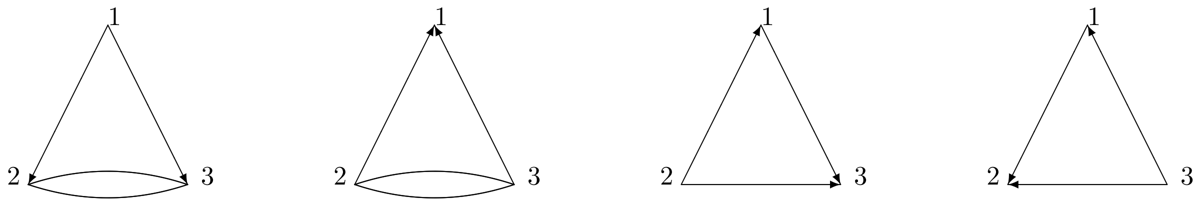
$$1+z_{22}+z_{23}=0, 1+\frac{1}{z_{22}}+\frac{1}{z_{23}}=0, 1+z_{32}+z_{33}=0, 1+\frac{1}{z_{32}}+\frac{1}{z_{33}}=0, 1+\frac{z_{22}}{z_{32}}+\frac{z_{23}}{z_{33}}=0, 1+\frac{z_{32}}{z_{22}}+\frac{z_{33}}{z_{23}}=0 \tag{334}$$

and the same system for y's.

We get that following proposition:

Proposition 107. *Subvariety E_1 consists of finite set of points.*

Further, consider subvariety $E_2(f_6')$. Without loss of generality, consider the case $\theta:\{1,2,3\}=1\cup 2,3$. It can be shown in usual way that there are 4 θ-maximal subquivers of $\mathbb{Q}_{\Gamma[3]}$: subquivers Q_1 and Q_2 have two l.c.c. with ordering: $\{1\}>\{2,3\}$ and $\{1\}<\{2,3\}$ respectively. Subquivers Q_3 and Q_4 have three l.c.c. with ordering: $\{2\}>\{1\}>\{3\}$ and $\{3\}>\{1\}>\{2\}$ respectively.



subquiver Q_1 subquiver Q_2 subquiver Q_3 subquiver Q_4

It is easy that morphisms $s_i, i=1,2$ are isomorphisms and hence, $D_i(\theta)=D_i'(\theta), i=1,2$. Components $(H_1^*)^{-1}M(Q_1)=(H_1^*\circ s_1)^{-1}M(Q_1)$ and $(H_1^*)^{-1}M(Q_2)$ are defined by equations:

$$1+z_{2,2}+z_{2,3}=0, 1+z_{3,2}+z_{3,3}=0 \tag{335}$$

and

$$1+\frac{1}{z_{2,2}}+\frac{1}{z_{2,3}}=0, 1+\frac{1}{z_{3,2}}+\frac{1}{z_{3,3}}=0 \tag{336}$$

respectively. Also, components $(H_1^*)^{-1}M(Q_3)$ and $(H_1^*)^{-1}M(Q_4)$ are given by equations:

$$1+\frac{1}{z_{2,2}}+\frac{1}{z_{2,3}}=0, 1+z_{3,2}+z_{3,3}=0, 1+\frac{z_{3,2}}{z_{2,2}}+\frac{z_{3,3}}{z_{2,3}}=0 \tag{337}$$

and

$$1+\frac{1}{z_{3,2}}+\frac{1}{z_{3,3}}=0, 1+\frac{z_{2,2}}{z_{3,2}}+\frac{z_{2,3}}{z_{3,3}}=0, 1+z_{2,2}+z_{2,3}=0 \tag{338}$$

respectively. It is easy that $\dim_F(H_1^*)^{-1}M(Q_1)=\dim_F(H_1^*)^{-1}M(Q_2)=2, \dim_F(H_1^*)^{-1}M(Q_3)=\dim_F(H_1^*)^{-1}M(Q_4)=1$.

We have similar description of components for $D_2(\theta)$. Denote corresponding components of $D_2(\theta)$ by $\hat{M}(Q_1)$, $\hat{M}(Q_2)$, $\hat{M}(Q_3)$ and $\hat{M}(Q_4)$. Consider components $M(Q_1)$ and $M(Q_2)$ of $D_1(\theta)$. Firstly, let us consider varieties $M(Q_i)\times_{Y(3)}\hat{M}(Q_j), i,j=1,2$. Study subvariety $M(Q_1)\times_{Y(3)}\hat{M}(Q_2)\subset X(3,3)\times_{Y(3)}X(3,3)$. We have the following equations for this subvariety: equations (335), equations of type (336) over y's:

$$1+\frac{1}{y_{2,2}}+\frac{1}{y_{2,3}}=0, 1+\frac{1}{y_{3,2}}+\frac{1}{y_{3,3}}=0 \tag{339}$$

and

$$(1+\frac{z_{2,2}}{z_{3,2}}+\frac{z_{2,3}}{z_{3,3}})(1+\frac{z_{3,2}}{z_{2,2}}+\frac{z_{3,3}}{z_{2,3}})=(1+\frac{y_{2,2}}{y_{3,2}}+\frac{y_{2,3}}{y_{3,3}})(1+\frac{y_{3,2}}{y_{2,2}}+\frac{y_{3,3}}{y_{2,3}}) \tag{340}$$

Expressing $z_{2,3}$ and $z_{3,3}$ in terms of $z_{2,2}$ and $z_{3,2}$ respectively, we get that $M(Q_1)$ is an open subvariety of $(F^*)^2$. Analogous statement for $\hat{M}(Q_2)$ is true. One can show that equation (340) is not trivial. Thus, we obtain that $\dim_F M(Q_1) \times_{Y(3)} \hat{M}(Q_2) = 3$. One can consider cases $M(Q_i) \times_{Y(3)} \hat{M}(Q_j), (i,j) = (1,1); (2,1); (2,2)$ analogously. It is easy that $\dim_F M(Q_i) \times_{Y(3)} D_2(\theta) \leq 3, i = 3,4$ and $\dim_F D_1(\theta) \times_{Y(3)} \hat{M}(Q_j) \leq 3, j = 3,4$.

Therefore, we obtain the following proposition:

Proposition 108. *Dimension of any component of $E_2(f_6')$ is less or equal to 3.*

11 Appendix B: Varieties $E_1(f_6)$ and $E_2(f_6)$.

11.1 Variety $E_1(f_6)$.

In this section we will study E_1 for morphism f_6.

Firstly, let us calculate dimension E_1. As we know from results of subsection 53, we have the following filtration of E_1:

$$E_1^{(2)}(f_6) \subset E_1^{(1)}(f_6) = E_1(f_6). \tag{341}$$

Recall that we have to consider partitions of $\{1,...,6\}$ onto non-intersecting subsets $I_1,...,I_{s+1}$ with condition $|I_j| \geq 2$ for all $j = 1,...,s+1$. Thus, $s+1 \leq 3$ and we have the following cases:

- partition: $\{1,...,6\} = I_1 \cup I_2$ and $|I_1| = 2, |I_2| = 4$.
- partition: $\{1,...,6\} = I_1 \cup I_2$ and $|I_1| = |I_2| = 3$.
- partition: $\{1,...,6\} = I_1 \cup I_2 \cup I_3$ and $|I_1| = |I_2| = |I_3| = 2$.

Third case corresponds to $E_1^{(2)}(f_6)$. It is easy that third case is a partial case of first one. Consider the first case. Without loss of generality, we can consider partition $\theta_1 = \{1,2\} \cup \{3,4,5,6\}$. Let us calculate dimension of $C(\theta_1)$. Let us write defining equations of $C'(\theta_1) = (F^*)^{10} \times_{Y(6)} (F^*)^{10}$:

$$1+z_{32}+z_{33}=0, 1+\frac{1}{z_{32}}+\frac{1}{z_{33}}=0, 1+z_{42}+z_{43}=0, 1+\frac{1}{z_{42}}+\frac{1}{z_{43}}=0, \tag{342}$$

$$1+z_{52}+z_{53}=0, 1+\frac{1}{z_{52}}+\frac{1}{z_{53}}=0, 1+z_{62}+z_{63}=0, 1+\frac{1}{z_{62}}+\frac{1}{z_{63}}=0 \tag{343}$$

$$1+\frac{z_{32}}{z_{22}}+\frac{z_{33}}{z_{23}}=0, 1+\frac{z_{22}}{z_{32}}+\frac{z_{23}}{z_{33}}=0, 1+\frac{z_{42}}{z_{22}}+\frac{z_{43}}{z_{23}}=0, 1+\frac{z_{22}}{z_{42}}+\frac{z_{23}}{z_{43}}=0, \tag{344}$$

$$1+\frac{z_{52}}{z_{22}}+\frac{z_{53}}{z_{23}}=0, 1+\frac{z_{22}}{z_{52}}+\frac{z_{23}}{z_{53}}=0, 1+\frac{z_{62}}{z_{62}}+\frac{z_{63}}{z_{23}}=0, 1+\frac{z_{22}}{z_{62}}+\frac{z_{23}}{z_{63}}=0, \tag{345}$$

Also, we have analogous system for y_{ij}. Further, let us calculate $C(\theta) = S^{-1}(C'(\theta))$. For this purpose, recall the equations defining $X(3,6)$:

$$1+z_{22}+z_{32}+z_{42}+z_{52}+z_{62}=0, 1+\frac{1}{z_{22}}+\frac{1}{z_{32}}+\frac{1}{z_{42}}+\frac{1}{z_{52}}+\frac{1}{z_{62}}=0 \tag{346}$$

$$1+z_{23}+z_{33}+z_{43}+z_{53}+z_{63}=0, 1+\frac{1}{z_{23}}+\frac{1}{z_{33}}+\frac{1}{z_{43}}+\frac{1}{z_{53}}+\frac{1}{z_{63}}=0 \tag{347}$$

$$1+\frac{z_{22}}{z_{23}}+\frac{z_{32}}{z_{33}}+\frac{z_{42}}{z_{43}}+\frac{z_{52}}{z_{53}}+\frac{z_{62}}{z_{63}}=0, 1+\frac{z_{23}}{z_{22}}+\frac{z_{33}}{z_{32}}+\frac{z_{43}}{z_{42}}+\frac{z_{53}}{z_{52}}+\frac{z_{63}}{z_{62}}=0 \tag{348}$$

We have similar system over y_{ij}. All these equations define $C(\theta_1)$ as subvariety of $(F^*)^{10} \times (F^*)^{10}$. Using lemma 106, we get that

Proposition 109. *$C(\theta_1)$ consists of finite set of points. And hence, component of $E_1(f_6)$ which corresponds to the first case, consists of finite set of points.*

Consider the second case. Without loss of generality, consider the following partition: $\theta_2 = \{1,...,6\} = \{1,2,3\} \cup \{4,5,6\}$. We have the following system of equations:

$$1 + z_{42} + z_{43} = 0, 1 + \frac{1}{z_{42}} + \frac{1}{z_{43}} = 0, 1 + z_{52} + z_{53} = 0, 1 + \frac{1}{z_{52}} + \frac{1}{z_{53}} = 0, \tag{349}$$

$$1 + z_{62} + z_{63} = 0, 1 + \frac{1}{z_{62}} + \frac{1}{z_{63}} = 0, 1 + \frac{z_{42}}{z_{22}} + \frac{z_{43}}{z_{23}} = 0, 1 + \frac{z_{22}}{z_{42}} + \frac{z_{23}}{z_{43}} = 0, \tag{350}$$

$$1 + \frac{z_{52}}{z_{22}} + \frac{z_{53}}{z_{23}} = 0, 1 + \frac{z_{22}}{z_{52}} + \frac{z_{23}}{z_{53}} = 0, 1 + \frac{z_{62}}{z_{22}} + \frac{z_{63}}{z_{23}} = 0, 1 + \frac{z_{22}}{z_{62}} + \frac{z_{23}}{z_{63}} = 0, \tag{351}$$

$$1 + \frac{z_{42}}{z_{32}} + \frac{z_{43}}{z_{33}} = 0, 1 + \frac{z_{32}}{z_{42}} + \frac{z_{33}}{z_{43}} = 0, 1 + \frac{z_{52}}{z_{32}} + \frac{z_{53}}{z_{33}} = 0, 1 + \frac{z_{32}}{z_{52}} + \frac{z_{33}}{z_{53}} = 0, \tag{352}$$

$$1 + \frac{z_{62}}{z_{32}} + \frac{z_{63}}{z_{33}} = 0, 1 + \frac{z_{32}}{z_{62}} + \frac{z_{33}}{z_{63}} = 0. \tag{353}$$

We have the same system for y_{ij}. Also, we have the system of type (346), (347), (348) for z_{ij} and y_{ij}. Using lemma 106, one can prove that

Proposition 110. *$C(\theta_2)$ consists of finite set of points. And hence, component of $E_1(f_6)$ which corresponds to the second case, consists of finite set of points.*

Therefore, we have the following:

Proposition 111. *Variety E_1 consists of finite set of points.*

11.2 Variety E_2 for $\mathcal{M}_6 B_{6,6}$ and fibred product.

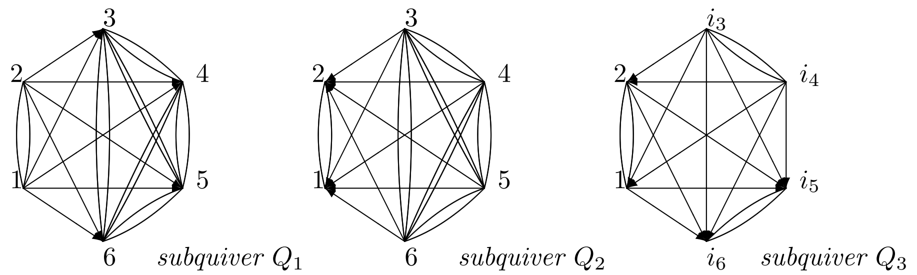
In this subsection we will study E_2 for morphism f_6.

We have to consider the following two cases:

- $\{1,...,6\} = I_1 \cup I_2, |I_1| = 2, |I_2| = 4$.
- $\{1,...,6\} = I_1 \cup I_2, |I_1| = |I_2| = 3$.

Consider the first case. Without loss of generality, fix the following partition: $\theta_3 : \{1,...,6\} = \{1,2\} \cup \{3,4,5,6\}$.

Proposition 112. *Maximal θ-subquivers have the following view:*



Proof. As we know, any l.c.c. of the maximal θ-subquiver Q has at least two vertices. Thus, we have two cases:

- Q has 2 l.c.c. with 2 and 4 vertices,
- Q has 3 l.c.c., any l.c.c. has 2 vertices.

Considering of different ordering on the set of l.c.c. gives us the proof. □

Subquiver $Q_i, i = 1,2$ has two l.c.c. Also, there are orderings of l.c.c.: $\{1,2\} > \{3,4,5,6\}$ of Q_1 and $\{1,2\} < \{3,4,5,6\}$ of Q_2. Subquiver Q_3 have three l.c.c. with ordering: $\{i_3, i_4\} > \{1,2\} > \{i_5, i_6\}$.

Subvarieties $M(Q_1)$ and $M(Q_2)$ of $D_1''(\theta)$ are defined by equations:

$$1 + z_{3,2} + z_{3,3} = 0, 1 + z_{4,2} + z_{4,3} = 0, 1 + z_{5,2} + z_{5,3} = 0, 1 + z_{6,2} + z_{6,3} = 0, \quad (354)$$

$$1 + \frac{z_{3,2}}{z_{2,2}} + \frac{z_{3,3}}{z_{2,3}} = 0, 1 + \frac{z_{4,2}}{z_{2,2}} + \frac{z_{4,3}}{z_{2,3}} = 0, 1 + \frac{z_{5,2}}{z_{2,2}} + \frac{z_{5,3}}{z_{2,3}} = 0, 1 + \frac{z_{6,2}}{z_{2,2}} + \frac{z_{6,3}}{z_{2,3}} = 0, \quad (355)$$

and

$$1 + \frac{1}{z_{3,2}} + \frac{1}{z_{3,3}} = 0, 1 + \frac{1}{z_{4,2}} + \frac{1}{z_{4,3}} = 0, 1 + \frac{1}{z_{5,2}} + \frac{1}{z_{5,3}} = 0, 1 + \frac{1}{z_{6,2}} + \frac{1}{z_{6,3}} = 0, \quad (356)$$

$$1 + \frac{z_{2,2}}{z_{3,2}} + \frac{z_{2,3}}{z_{3,3}} = 0, 1 + \frac{z_{2,2}}{z_{4,2}} + \frac{z_{2,3}}{z_{4,3}} = 0, 1 + \frac{z_{2,2}}{z_{5,2}} + \frac{z_{2,3}}{z_{5,3}} = 0, 1 + \frac{z_{2,2}}{z_{6,2}} + \frac{z_{2,3}}{z_{6,3}} = 0 \quad (357)$$

respectively.

Without loss of generality, assume that $i_3 = 3, i_4 = 4, i_5 = 5, i_6 = 6$. In this case, subvariety $M(Q_3)$ is given by system of equations:

$$1 + \frac{1}{z_{3,2}} + \frac{1}{z_{3,3}} = 0, 1 + \frac{1}{z_{4,2}} + \frac{1}{z_{4,3}} = 0, 1 + \frac{z_{2,2}}{z_{3,2}} + \frac{z_{2,3}}{z_{3,3}} = 0, 1 + \frac{z_{2,2}}{z_{4,2}} + \frac{z_{2,3}}{z_{4,3}} = 0, \quad (358)$$

$$1 + z_{5,2} + z_{5,3} = 0, 1 + z_{6,2} + z_{6,3} = 0, 1 + \frac{z_{5,2}}{z_{2,2}} + \frac{z_{5,3}}{z_{2,3}} = 0, 1 + \frac{z_{6,2}}{z_{2,2}} + \frac{z_{6,3}}{z_{2,3}} = 0, \quad (359)$$

$$1 + \frac{z_{5,2}}{z_{3,2}} + \frac{z_{5,3}}{z_{3,3}} = 0, 1 + \frac{z_{6,2}}{z_{3,2}} + \frac{z_{6,3}}{z_{3,3}} = 0, 1 + \frac{z_{5,2}}{z_{4,2}} + \frac{z_{5,3}}{z_{4,3}} = 0, 1 + \frac{z_{6,2}}{z_{4,2}} + \frac{z_{6,3}}{z_{4,3}} = 0. \quad (360)$$

Consider subvariety $M(Q_1)$. Let us formulate the following useful lemma:

Lemma 113. *Consider system of equations over a_1, a_2:*

$$1 + a_1 + a_2 = 0, 1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} = 0, a_i, x_i \in F^* \quad (361)$$

- *If $(x_1 \neq 1$ and $x_2 \neq 1$ and $x_1 \neq x_2$, then this system has unique solution,*
- *if $(x_1, x_2) = (1,1)$, then solution of system has the following view: $(a_1, -1 - a_1), a_1 \neq 0, -1$,*
- *if $x_1 = x_2 \neq 1$ or $x_1 = 1$ or $x_2 = 1$, then system has no solution.*

Proof. Straightforward. □

Corollary 114. *We have the similar statement for system:*

$$1 + \frac{1}{a_1} + \frac{1}{a_2} = 0, 1 + \frac{x_1}{a_1} + \frac{x_2}{a_2} = 0 \quad (362)$$

Consider natural morphism: $p : S(Q_1) \to (F^*)^2_{z_{2,2}, z_{2,3}}$. Using this lemma, we obtain that if $z_{2,2} \neq z_{2,3}$, then preimage of p over $(z_{2,2}, z_{2,3})$ is unique and preimage of p over $(1,1)$ is 4-dimensional. Thus, $M(Q_1)$ has two-dimensional component $M^{(2)}(Q_1)$ and four-dimensional component $M^{(4)}(Q_1)$. Consider two-dimensional component $M^{(2)}(Q_1)$. Using lemma, we get the following equations for $(H_1^*)^{-1} M^{(2)}(Q_1)$:

$$z_{3,2} = z_{4,2} = z_{5,2} = z_{6,2}, z_{3,3} = z_{4,3} = z_{5,3} = z_{6,3}. \quad (363)$$

Consider $(H_1^* \circ s_1)^{-1}(M^{(2)}(Q_1))$. Using equations (346),(347),(348), we get the following system of equations:

$$1 + z_{2,2} + 4z_{3,2} = 0, 1 + \frac{1}{z_{2,2}} + \frac{4}{z_{3,2}} = 0, 1 + z_{2,3} + 4z_{3,3} = 0, 1 + \frac{1}{z_{2,3}} + \frac{4}{z_{3,3}} = 0, \quad (364)$$

$$1+\frac{z_{2,3}}{z_{2,2}}+4\frac{z_{3,3}}{z_{3,2}}=0, 1+\frac{z_{2,2}}{z_{2,3}}+4\frac{z_{3,2}}{z_{3,3}}=0. \tag{365}$$

One can check that this system has no solutions. Thus, $(H_1^*\circ s_1)^{-1}M^{(2)}(Q_1)=\emptyset$.

Consider four-dimensional component $M^{(4)}(Q_1)$. In this case, we have relation for $(H_1^*)^{-1}M^{(4)}(Q_1)$: $z_{2,2}=z_{2,3}=1$. Using equations (346),(347),(348), we get the following system of equations for $(H_1^*\circ s_1)^{-1}M^{(4)}(Q_1)$:

$$2+z_{3,2}+z_{4,2}+z_{5,2}+z_{6,2}=0, 2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}+\frac{1}{z_{5,2}}+\frac{1}{z_{6,2}}=0, \tag{366}$$

$$2+z_{3,3}+z_{4,3}+z_{5,3}+z_{6,3}=0, 2+\frac{1}{z_{3,3}}+\frac{1}{z_{4,3}}+\frac{1}{z_{5,3}}+\frac{1}{z_{6,3}}=0, \tag{367}$$

$$2+\frac{z_{3,2}}{z_{3,3}}+\frac{z_{4,2}}{z_{4,3}}+\frac{z_{5,2}}{z_{5,3}}+\frac{z_{6,2}}{z_{6,3}}=0, 2+\frac{z_{3,3}}{z_{3,2}}+\frac{z_{4,3}}{z_{4,2}}+\frac{z_{5,3}}{z_{5,2}}+\frac{z_{6,3}}{z_{6,2}}=0 \tag{368}$$

and

$$1+z_{3,2}+z_{3,3}=0, 1+z_{4,2}+z_{4,3}=0, 1+z_{5,2}+z_{5,3}=0, 1+z_{6,2}+z_{6,3}=0. \tag{369}$$

Show that equations (367) and (368) follow from (369) and (366). Denote by $S\subset(F^*)^4$ the surface defined by equations (366). For this purpose, let us prove that transformation $z_{i,2}\mapsto -1-z_{i,2}, i=3,4,5,6$ is a birational involution of S. It is easy that $2+(-1-z_{3,2})+...+(-1-z_{6,2})=-2-z_{3,2}-...-z_{6,2}=0$. We get the following second equation from (366):

$$2-\frac{1}{1+z_{3,2}}-...-\frac{1}{1+z_{6,2}}$$

Transforming this expression, we obtain:

$$2(1+z_{3,2})...(1+z_{6,2})-(1+z_{3,2})(1+z_{4,2})(1+z_{5,2})-...-(1+z_{4,2})(1+z_{5,2})(1+z_{6,2})=$$

$$z_{3,2}z_{4,2}z_{5,2}z_{6,2}(2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}+\frac{1}{z_{5,2}}+\frac{1}{z_{6,2}})-(2+z_{3,2}+z_{4,2}+z_{5,2}+z_{6,2})=0.$$

Also, consider equation: $1+z_{i,2}+z_{i,3}=0, i=3,4,5,6$. Transform it as follows: $1+\frac{z_{i,2}}{z_{i,3}}+\frac{1}{z_{i,3}}=0, i=3,4,5,6$. We get that $\frac{z_{i,2}}{z_{i,3}}=-1-\frac{1}{z_{i,3}}, i=3,4,5,6$. Thus,

$$2+\frac{z_{3,2}}{z_{3,3}}+...+\frac{z_{6,2}}{z_{6,3}}=2+(-1-\frac{1}{z_{3,3}})+...(-1-\frac{1}{z_{6,3}})=-2-\frac{1}{z_{3,3}}-...-\frac{1}{z_{6,3}}=0.$$

One can prove that $2+\frac{z_{3,3}}{z_{3,2}}+...+\frac{z_{6,3}}{z_{6,2}}=0$ analogously. Therefore, we have proved the following proposition:

Proposition 115. *$(H_1^*\circ s_1)^{-1}(M(Q_1))$ is birationally isomorphic to surface $S\subset(F^*)^4$ defined by equations (366).*

This surface is a hessian of nonsingular cubic surface ([?]). Consider natural projection: $p_1: S\to(F^*)^2_{z_{3,2},z_{4,2}}$. It is easy that degree of p_1 is 2.

Lemma 116.
- *If $(z_{3,2},z_{4,2})$ satisfy to $2+z_{3,2}+z_{4,2}\neq 0, 2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}\neq 0$, then $|p_1^{-1}(z_{3,2},z_{4,2})|$ is 1 or 2,*
- *if $(z_{3,2},z_{4,2})$ satisfy to $2+z_{3,2}+z_{4,2}=0, 2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}\neq 0$ or $2+z_{3,2}+z_{4,2}\neq 0, 2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}=0$, then $p_1^{-1}(z_{3,2},z_{4,2})=\emptyset$,*
- *if $(z_{3,2},z_{4,2})$ satisfy to $2+z_{3,2}+z_{4,2}=0, 2+\frac{1}{z_{3,2}}+\frac{1}{z_{4,2}}=0$, then $\dim_F p_1^{-1}(z_{3,2},z_{4,2})=1$.*

Proof. Straightforward. □

Proposition 117. *Surface S is an irreducible $K3$ - surface.*

Proof. Assume that S is reducible. Since S is defined by two equations in $(F^*)^4$, then dimension of every component is at least 2. Using lemma 116, we get that dimension of every component is 2. Consider natural compactification $\widetilde{S} \subset \mathbb{P}^4$ of the surface S. It is easy that if $\widetilde{S}$ is reducible, then singular locus of $\widetilde{S}$ has dimension at least 1. It can be checked in usual way that singular locus of $\widetilde{S}$ is finite set. Contradiction. Thus, S is an irreducible surface. Further, one can show that singular locus of $\widetilde{S}$ consists of ordinary double points. It is well-known that quartic surface with isolated double points is a $K3$ - surface. □

Also, one can prove analogous results in the case of Q_2.

Further, consider $(H_1^*)^{-1}M(Q_3)$. Using corollary 114, we get that if $(z_{2,2}, z_{2,3}) \neq (1,1)$ and $z_{2,2} \neq z_{2,3}$, then $z_{i,2}, z_{i,3}, i = 3,4,5,6$ can be expressed as rational functions of $z_{2,2}, z_{2,3}$, and $z_{3,2} = z_{4,2}, z_{3,3} = z_{4,3}, z_{5,2} = z_{6,2}, z_{5,3} = z_{6,3}$. Namely,

$$z_{3,2} = z_{4,2} = \frac{z_{2,2} - z_{2,3}}{z_{2,3} - 1}, z_{5,2} = z_{6,2} = \frac{z_{2,2}(z_{2,3} - 1)}{z_{2,2} - z_{2,3}}, \tag{370}$$

$$z_{3,3} = z_{4,3} = -\frac{z_{2,2} - z_{2,3}}{z_{2,2} - 1}, z_{5,3} = z_{6,3} = -\frac{z_{2,3}(z_{2,2} - 1)}{z_{2,2} - z_{2,3}}. \tag{371}$$

Also, we have the following transformation:

$$1 + \frac{z_{5,2}}{z_{3,2}} + \frac{z_{5,3}}{z_{3,3}} = \frac{z_{2,2}^2 - 6z_{2,2}z_{2,3} + z_{2,3}^2 + z_{2,2}z_{2,3}^2 + z_{2,2} + z_{2,3} + z_{2,3}z_{2,2}^2}{(z_{2,2} - z_{2,3})^2} = 0 \tag{372}$$

Thus, $(H_1^*)^{-1}M(Q_3) \subset D_1'(\theta)$ is a curve given by equation:

$$z_{2,2}^2 - 6z_{2,2}z_{2,3} + z_{2,3}^2 + z_{2,2}z_{2,3}^2 + z_{2,2} + z_{2,3} + z_{2,3}z_{2,2}^2 = 0. \tag{373}$$

Therefore,

$$\dim_F (H_1^* \circ s_1)^{-1} M(Q_3) \leq 1. \tag{374}$$

Remark.
It can be shown in usual way that $(H_1^*)^{-1}M(Q_3)$ is an irreducible singular rational curve with singularity $(1,1)$.

It is clear that we have similar results for $D_2(\theta)$. Denote by $\hat{M}(Q_1)$, $\hat{M}(Q_2)$ and $\hat{M}(Q_3)$ components of $D_2(\theta)$ corresponding to maximal θ-subquivers Q_1, Q_2 and Q_3. Using arguments similar to studying of $M(Q_i), i = 1,2$, we get that components $(H_2^* \circ s_2)^{-1}(\hat{M}(Q_i)), i = 1,2$ are $K3$ surfaces. This $K3$ surface is given by equations (366) in variables $y_{i,2}, i = 3,4,5,6$. Denote this surface by S'. Also, denote by p_1' the projection $S' \to (F^*)^2_{y_{3,2},y_{4,2}}$ We would like to prove that $\dim_F E_2(f_6) \leq 3$. Since $\dim_F (H_1^* \circ s_1)^{-1} M(Q_3) \leq 1$, we can consider only subvarieties $(H_1^* \circ s_1)^{-1} M(Q_i) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_j), i,j = 1,2$. Without loss of generality, consider subvariety $(H_1^* \circ s_1)^{-1} M(Q_i) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_j) \subset X(3,6) \times_{Y(6)} X(3,6)$. Consider the following composition of morphisms:

$$\begin{array}{ccc} (H_1^* \circ s_1)^{-1} M(Q_1) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_2) & \xrightarrow{\subseteq} & (H_1^* \circ s_1)^{-1} M(Q_1) \times (H_2^* \circ s_2)^{-1} \hat{M}(Q_2) \\ & & \downarrow{\scriptstyle p_1 \times p_1'} \\ & & (F^*)^2_{z_{3,2},z_{4,2}} \times (F^*)^2_{y_{3,2},y_{4,2}} \end{array} \tag{375}$$

Show that $(H_1^* \circ s_1)^{-1} M(Q_1) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_2)$ does not coincide with $(H_1^* \circ s_1)^{-1} M(Q_1) \times (H_2^* \circ s_2)^{-1} \hat{M}(Q_2)$. Consider divisor of $(F^*)^2_{z_{3,2},z_{4,2}} \times (F^*)^2_{y_{3,2},y_{4,2}}$ given by equation:

$$(1 + \frac{z_{3,2}}{z_{4,2}} + \frac{z_{3,3}}{z_{4,3}})(1 + \frac{z_{4,2}}{z_{3,2}} + \frac{z_{4,3}}{z_{3,3}}) = (1 + \frac{y_{3,2}}{y_{4,2}} + \frac{y_{3,3}}{y_{4,3}})(1 + \frac{y_{4,2}}{y_{3,2}} + \frac{y_{4,3}}{y_{3,3}}), \tag{376}$$

where $z_{3,3} = -1 - z_{3,2}, z_{4,3} = -1 - z_{4,2}$ (it follows from (354)), $y_{3,3} = -\frac{y_{3,2}}{y_{3,2}+1}, y_{4,3} = -\frac{y_{4,2}}{y_{4,2}+1}$ (356). Thus, we get:

$$(1 + \frac{z_{3,2}}{z_{4,2}} + \frac{1 + z_{3,2}}{1 + z_{4,2}})(1 + \frac{z_{4,2}}{z_{3,2}} + \frac{1 + z_{4,2}}{1 + z_{3,2}}) = (1 + \frac{y_{3,2}}{y_{4,2}} + \frac{1 + y_{3,2}}{1 + y_{4,2}})(1 + \frac{y_{4,2}}{y_{3,2}} + \frac{1 + y_{4,2}}{1 + y_{3,2}}). \tag{377}$$

Denote by T the divisor of $(F^*)^2_{z_{3,2},z_{4,2}} \times (F^*)^2_{y_{3,2},y_{4,2}}$ given by (377). It is easy that

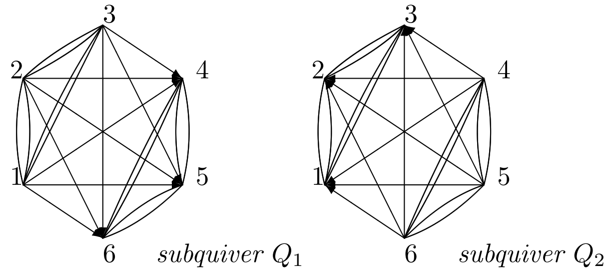
$$p_1 \times p_1'((H_1^* \circ s_1)^{-1} M(Q_1) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_2)) \subseteq T. \tag{378}$$

As we know $(H_1^* \circ s_1)^{-1} M(Q_1) \times (H_2^* \circ s_2)^{-1} \hat{M}(Q_2) = S \times S'$ is an irreducible variety and $p_1 \times p_1'$ is dominant. Thus, we get that $\dim_F (H_1^* \circ s_1)^{-1} M(Q_1) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_2) \leq 3$. Using similar arguments, one can show that $\dim_F (H_1^* \circ s_1)^{-1} M(Q_i) \times_{Y(6)} (H_2^* \circ s_2)^{-1} \hat{M}(Q_j) \leq 3, i, j = 1, 2$. Therefore, we have proved the following proposition:

Proposition 118. *Dimension of any component of $E_2(f_6)$ which corresponds to the first case, is less or equal to 3.*

Consider the second case. Without loss of generality, we can fix the partition $\theta : \{1, 2, 3\} \cup \{4, 5, 6\}$.

Proposition 119. *There are only two maximal θ-subquivers:*



Proof. Consider maximal θ-subquiver Q As we know, any l.c.c. of Q has at least two vertices. Thus, we have only two l.c.c. and every l.c.c. has three vertices. Considering different ordering on the set of l.c.c. gives us the proof. □

One can show that $(H_1^*)^{-1} M(Q_1)$ is given by equations:

$$1 + z_{4,2} + z_{4,3} = 0, 1 + \frac{z_{4,2}}{z_{2,2}} + \frac{z_{4,3}}{z_{2,3}} = 0, 1 + \frac{z_{4,2}}{z_{3,2}} + \frac{z_{4,3}}{z_{3,3}} = 0, \tag{379}$$

$$1 + z_{5,2} + z_{5,3} = 0, 1 + \frac{z_{5,2}}{z_{2,2}} + \frac{z_{5,3}}{z_{2,3}} = 0, 1 + \frac{z_{5,2}}{z_{3,2}} + \frac{z_{5,3}}{z_{3,3}} = 0, \tag{380}$$

$$1 + z_{6,2} + z_{6,3} = 0, 1 + \frac{z_{6,2}}{z_{2,2}} + \frac{z_{6,3}}{z_{2,3}} = 0, 1 + \frac{z_{6,2}}{z_{3,2}} + \frac{z_{6,3}}{z_{3,3}} = 0. \tag{381}$$

$(H_1^*)^{-1} M(Q_2)$ is defined by equations:

$$1 + \frac{1}{z_{4,2}} + \frac{1}{z_{4,3}} = 0, 1 + \frac{z_{2,2}}{z_{4,2}} + \frac{z_{2,3}}{z_{4,3}} = 0, 1 + \frac{z_{3,2}}{z_{4,2}} + \frac{z_{3,3}}{z_{4,3}} = 0, \tag{382}$$

$$1 + \frac{1}{z_{5,2}} + \frac{1}{z_{5,3}} = 0, 1 + \frac{z_{2,2}}{z_{5,2}} + \frac{z_{2,3}}{z_{5,3}} = 0, 1 + \frac{z_{3,2}}{z_{5,2}} + \frac{z_{3,3}}{z_{5,3}} = 0, \tag{383}$$

$$1 + \frac{1}{z_{6,2}} + \frac{1}{z_{6,3}} = 0, 1 + \frac{z_{2,2}}{z_{6,2}} + \frac{z_{2,3}}{z_{6,3}} = 0, 1 + \frac{z_{3,2}}{z_{6,2}} + \frac{z_{3,3}}{z_{6,3}} = 0. \tag{384}$$

Prove the following useful lemma:

Lemma 120. *Consider the following system of equations over variables a_1, a_2:*

$$1 + a_1 + a_2 = 0, 1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} = 0, 1 + \frac{a_1}{y_1} + \frac{a_2}{y_2} = 0, a_i, x_i, y_i \in F^*. \tag{385}$$

Then we have the following cases:

- *if*

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ 1 & \frac{1}{x_1} & \frac{1}{x_2}\\ 1 & \frac{1}{y_1} & \frac{1}{y_2}\end{pmatrix} = 2, \tag{386}$$

  then system (385) has at more one solution

- *if* $x_1 = x_2 = y_1 = y_2 = 1$*(i.e. rank of matrix is 1), then system (385) has the following solutions:* $(a_1, a_2 = -1 - a_1), a_1 \neq 0, -1$.

- *if*

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & \frac{1}{x_1} & \frac{1}{x_2}\\ 1 & \frac{1}{y_1} & \frac{1}{y_2}\end{pmatrix} \neq 0, \tag{387}$$

  then system (385) has no solution.

Proof. Straightforward. □

Corollary 121. *Consider system over variables* a_1, a_2*:*

$$1 + \frac{1}{a_1} + \frac{1}{a_2} = 0, 1 + \frac{x_1}{a_1} + \frac{x_2}{a_2} = 0, 1 + \frac{y_1}{a_1} + \frac{y_2}{a_2} = 0. \tag{388}$$

Then we have the following statements:

- *if*

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ 1 & x_1 & x_2\\ 1 & y_1 & y_2\end{pmatrix} = 2, \tag{389}$$

  then system (388) has at more one solution

- *if* $x_1 = x_2 = y_1 = y_2 = 1$*(i.e. rank of matrix is 1), then system (388) has the following solutions:* $(a_1, a_2 = -\frac{a_1}{1+a_1}), a_1 \neq 0, -1$.

- *if*

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & x_1 & x_2\\ 1 & y_1 & y_2\end{pmatrix} \neq 0, \tag{390}$$

  then system (388) has no solution.

For studying $(H_1^* \circ s_1)^{-1}M(Q_1)$, we will study two cases:

- $$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ 1 & x_1 & x_2\\ 1 & y_1 & y_2\end{pmatrix} = 2, \tag{391}$$

- $x_1 = x_2 = y_1 = y_2 = 1$

Consider the first case. In this case, we have the following relations: $z_{4,2} = z_{5,2} = z_{6,2}, z_{4,3} = z_{5,3} = z_{6,3}$. We have the following system of equations for $(H_1^* \circ s_1)^{-1}M(Q_1)$:

$$1 + z_{2,2} + z_{3,2} + 3z_{4,2} = 0, 1 + \frac{1}{z_{2,2}} + \frac{1}{z_{3,2}} + \frac{3}{z_{4,2}} = 0, \tag{392}$$

$$1 + z_{2,3} + z_{3,3} + 3z_{4,3} = 0, 1 + \frac{1}{z_{2,3}} + \frac{1}{z_{3,3}} + \frac{3}{z_{4,3}} = 0, \tag{393}$$

$$1+\frac{z_{2,3}}{z_{2,2}}+\frac{z_{3,3}}{z_{3,2}}+3\frac{z_{4,3}}{z_{4,2}}=0, 1+\frac{z_{2,2}}{z_{2,3}}+\frac{z_{3,2}}{z_{3,3}}+3\frac{z_{4,2}}{z_{4,3}}=0, \tag{394}$$

and equations (382). Prove that this system has no solutions. One can check the following identity:

$$(z_{2,2}+z_{3,2})(\frac{1}{z_{2,3}}+\frac{1}{z_{3,3}})(\frac{z_{2,2}}{z_{2,3}}+\frac{z_{3,2}}{z_{3,3}})=(z_{2,3}+z_{3,3})(\frac{1}{z_{2,2}}+\frac{1}{z_{3,2}})(\frac{z_{2,3}}{z_{2,2}}+\frac{z_{3,3}}{z_{3,2}}). \tag{395}$$

Using this identity, we get the following equation:

$$(1+3z_{4,2})(1+\frac{3}{z_{4,3}})(1+3\frac{z_{4,3}}{z_{4,2}})-(1+3z_{4,3})(1+\frac{3}{z_{4,2}})(1+3\frac{z_{4,2}}{z_{4,3}})=0 \tag{396}$$

Simplifying this equation, we obtain the following three cases:

- $z_{4,2}=1$,
- $z_{4,3}=1$,
- $z_{4,3}=z_{4,2}$.

Assume that $z_{4,3}=1$. In this case, we get that $z_{4,2}=-2$, $\frac{1}{z_{2,3}}+\frac{1}{z_{3,3}}=-4$, $\frac{1}{z_{2,2}}+\frac{1}{z_{3,2}}=\frac{1}{2}$, $1-\frac{2}{z_{2,2}}+\frac{1}{z_{2,3}}=0$ and $1-\frac{2}{z_{3,2}}+\frac{1}{z_{3,3}}=0$. Summarizing two last equations, we get

$$0=2-2(\frac{1}{z_{2,2}}+\frac{1}{z_{3,2}})+(\frac{1}{z_{2,3}}+\frac{1}{z_{3,3}})=2-2\cdot\frac{1}{2}+(-4)=-3 \tag{397}$$

Contradiction. Analogous arguments show that $(H_1^*)^{-1}M(Q_1)=\emptyset$ and $(H_1^*)^{-1}M(Q_2)=\emptyset$. Thus, we have proved the following proposition:

Proposition 122. *Component of $E_2(f_6)$ corresponds to the second case, is empty.*

Therefore, we have proved the following

Proposition 123. $\dim_F E_2(f_6)\le 3$.

12 Appendix C.

12.1 Local properties of standard orthogonal pair.

In this subsection we will construct point $z_0\in Z$ such that $\sigma'\circ\rho^{-1}\circ\sigma'\circ\rho$ acts on the tangent space $T_{z_0}Z$ trivially. This point corresponds to standard orthogonal pair up to permutation of rows.

Let us formulate conditions for point $z_0\in Z$ allowing to deduce that $\sigma'\circ\rho^{-1}\circ\sigma'\circ\rho$ acts on the tangent space $T_{z_0}Z$ trivially. Firstly, let us formulate conditions for determination of action $\mathrm{d}\sigma'$ on the tangent space $T_{z_0}Z$. Fix some point $pt\in X(6,6)$. Denote by z',z'' the images $pr_1(pt), pr_1\circ\sigma^{(p)}(pt)\in X(3,6)$ and z_0,z_1 the image of z',z'' under natural projection $\pi: X(3,6)\to Z$. Then $\sigma_P^{(6)}(\mu(z_0))=\mu(z_1)\in\mu(Z)$ and using proposition 100, we get that $\sigma_P^{(3)}\circ\widetilde{\mu}\circ\widetilde{\zeta}(z_0)=\widetilde{\mu}\circ\widetilde{\zeta}(z_1)$. Thus, $\sigma'(z_0)=z_1$. Consider differential of σ' at point z_0. We have the following formula:

$$\mathrm{d}\sigma'|_{z_0}=(\mathrm{d}\widetilde{\zeta})^{-1}\circ(\mathrm{d}\widetilde{\mu})^{-1}\circ\mathrm{d}\sigma_P^{(3)}\circ\mathrm{d}\widetilde{\mu}\circ\mathrm{d}\widetilde{\zeta}: T_{z_0}Z\to T_{z_1}Z. \tag{398}$$

Therefore, for definition of $\mathrm{d}\sigma'$ we need bijectivity of $\mathrm{d}\widetilde{\zeta}$ and injectivity of $\mathrm{d}\widetilde{\mu}$. Recall that $\mu\circ\zeta(Z)=\widetilde{\mu}(\widetilde{Z})$ is $\sigma_P^{(3)}$-invariant subvariety of $Y(3)\times_Y Y(3)$. It can be shown in usual way that $\mathrm{d}\sigma_P^{(3)}$ is an isomorphism. Thus, $\mathrm{d}\sigma_P^{(3)}(T_{\widetilde{\mu}\circ\widetilde{\zeta}(z_0)})=T_{\widetilde{\mu}\circ\widetilde{\zeta}(z_1)}$.

Note the following remarks.

- Assume that $\dim_F T_{z'}X(3,6) = \dim_F T_{\zeta(z')}\widetilde{X} = \dim_F T_{z_1}Z = \dim T_{z_0}Z = 4$, i.e. $z', \zeta(z'), z_1, \pi(z') = z_0$ are smooth points of $X(3,6), \widetilde{X}, Z$ and $\widetilde{Z}$ respectively. Using diagram (240), we obtain that $\mathrm{d}\widetilde{\pi}\circ \mathrm{d}\zeta = \mathrm{d}\widetilde{\zeta}\circ \mathrm{d}\pi$. Thus, if $\mathrm{d}\widetilde{\pi}$, $\mathrm{d}\zeta$ and $\mathrm{d}\pi$ are isomorphism, then $\mathrm{d}\widetilde{\zeta}$ is isomorphism too.

- If point $z' \in X(3,6)$ is smooth and stabilizer of z' under action of $S_3^{(p)}$ is trivial, then point $\pi(z') = z_0 \in Z$ is smooth and map $\mathrm{d}\pi$ is an isomorphism. Analogously, if point $\zeta(z') \in \widetilde{X}$ is smooth and stabilizer of $\zeta(z')$ under action of $S_3^{(p)}$ is trivial, then point $\widetilde{\zeta}(z_0)$ is smooth and $\mathrm{d}\widetilde{\pi}$ is an isomorphism.

- Using (240), we get the decomposition: $\mathrm{d}\Phi = \mathrm{d}\widetilde{\mu}\circ \mathrm{d}\widetilde{\pi}$. Thus, if $\mathrm{d}\Phi$ is injective and $\mathrm{d}\widetilde{\pi}$ is an isomorphism, then $\mathrm{d}\widetilde{\mu}$ is injective.

Thus, if we take non-singular point $z' \in X(3,6)$ such that stabilizer $\mathrm{St}_{S_3^{(p)}}(z') = 1$, $\mathrm{d}\zeta$ is an isomorphism, $\mathrm{d}\Phi$ is an immersion ,then $\mathrm{d}\sigma'$ is well-defined morphism: $T_{z_0}Z \to T_{z_1}Z$.

Firstly, let us check that morphism $\mathrm{d}\zeta$ is an isomorphism. For this purpose, let us describe the maps in suitable coordinates. Morphism: $(pr_1, pr_1\circ\sigma^{(p)}) : X(6,6) \to X(3,6)\times_{Y(6)} X(3,6)$ is birational. In terms of matrices, this birational morphism means decomposition of matrix of size 6×6 into two matrices of size 3×6, i.e. morphism: $pr = (pr_1, pr_1\circ\sigma^{(p)}) : X(6,6) \to X(3,6)\times_{Y(6)} X(3,6)$ is defined in terms of matrices by the following formula:

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & x_{11} & ... & x_{15} \\ ... & ... & ... & ... \\ 1 & x_{51} & ... & x_{55}\end{pmatrix} \mapsto (\begin{pmatrix} 1 & 1 & 1 \\ 1 & x_{11} & x_{12} \\ ... & ... & ... \\ 1 & x_{51} & x_{52}\end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{x_{14}}{x_{13}} & \frac{x_{15}}{x_{13}} \\ ... & ... & ... \\ 1 & \frac{x_{54}}{x_{53}} & \frac{x_{55}}{x_{53}}\end{pmatrix}) \tag{399}$$

In terms of matrices, birational morphism: $\zeta : X(3,6) \to \widetilde{X} = X(3,3)\times_{Y(3)} X(3,3)$ means the decomposition of matrix of type 3×6 into two matrices of type 3×3 in the following manner:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & x_{11} & x_{12} \\ ... & ... & ... \\ 1 & x_{51} & x_{52}\end{pmatrix} \mapsto (\begin{pmatrix} 1 & 1 & 1 \\ 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22}\end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{x_{41}}{x_{31}} & \frac{x_{42}}{x_{32}} \\ 1 & \frac{x_{51}}{x_{31}} & \frac{x_{52}}{x_{32}}\end{pmatrix}) \tag{400}$$

Consider diagram (235). Recall that $Y(3)$ is the subvariety of F^5 defined by equation $ABC = \alpha\beta$. Morphism $\phi_2\circ\tau$ is defined by formulas:

$$A = (1 + x_{11} + x_{21})(1 + \frac{1}{x_{11}} + \frac{1}{x_{21}}) \tag{401}$$

$$B = (1 + x_{12} + x_{22})(1 + \frac{1}{x_{12}} + \frac{1}{x_{22}}) \tag{402}$$

$$C = (1 + \frac{x_{11}}{x_{12}} + \frac{x_{21}}{x_{22}})(1 + \frac{x_{12}}{x_{11}} + \frac{x_{22}}{x_{21}}) \tag{403}$$

$$\alpha = (1 + x_{11} + x_{21})(1 + \frac{1}{x_{12}} + \frac{1}{x_{22}})(1 + \frac{x_{12}}{x_{11}} + \frac{x_{22}}{x_{21}}) \tag{404}$$

$$\beta = (1 + \frac{1}{x_{11}} + \frac{1}{x_{21}})(1 + x_{12} + x_{22})(1 + \frac{x_{11}}{x_{12}} + \frac{x_{21}}{x_{22}}) \tag{405}$$

We get the analogous formulas for $\frac{x_{41}}{x_{31}}, \frac{x_{51}}{x_{31}}, \frac{x_{42}}{x_{32}}, \frac{x_{52}}{x_{32}}$. Involution $\sigma_P^{(3)}$ is defined by rule: $\sigma_P^{(3)} : A \mapsto A, B \mapsto B, C \mapsto C, \alpha \mapsto -\alpha, \beta \mapsto -\beta$. Thus, for fixed point $t = (t_1, t_2) \in \widetilde{X} = X(3,3)\times_{Y(3)} X(3,3)$ such that $\phi_2\circ\tau(t_1) = \sigma_P^{(3)}\circ\phi_2\circ\tau(t_2) = y \in Y(3)$ we have the following isomorphism:

$$T_{t=(t_1,t_2)}\widetilde{X} = \mathrm{Ker}(\mathrm{d}\phi_2\circ\mathrm{d}\tau, -\mathrm{d}\sigma_{\mathrm{P}}^{(3)}\circ\mathrm{d}\phi_2\circ\mathrm{d}\tau) : T_{t_1}X(3,3)\oplus T_{t_2}X(3,3) \to T_yY(3) \tag{406}$$

Consider the point $pt \in X(6,6)$ given by matrix:

$$pt = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & -1 & \epsilon^4 & \epsilon^5 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \epsilon^5 & \epsilon^4 & -1 & \epsilon^2 & \epsilon \\ 1 & \epsilon^4 & \epsilon^2 & 1 & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon^4 & 1 & \epsilon^2 & \epsilon^4 \end{pmatrix}, \epsilon^6 = 1 \tag{407}$$

In this case,

$$z' = pr_1(pt) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & -1 & 1 \\ 1 & \epsilon^5 & \epsilon^4 \\ 1 & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon^4 \end{pmatrix}, z'' = pr_1 \circ \sigma^{(p)}(pt) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & -1 & 1 \\ 1 & \epsilon^5 & \epsilon^4 \\ 1 & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon^4 \end{pmatrix} = z' \tag{408}$$

Lemma 124. *z' is a smooth point of $X(3,6)$, i.e.* $\dim_F T_{z'}X(3,6) = 4$.

Proof. As we know, $X(3,6)$ is a subvariety of $(F^*)^{10}$ defined by equations (243), (244), (245). Thus, tangent space $T_{z'}X(3,6)$ is a kernel of matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & -1 & \epsilon^5 & \epsilon & \epsilon^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^5 & -1 & \epsilon & \epsilon^5 & \epsilon \\ \epsilon^4 & 1 & \epsilon^2 & \epsilon^4 & \epsilon^2 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & \epsilon^5 & -1 & \epsilon & \epsilon^2 & \epsilon^4 \end{pmatrix} \tag{409}$$

One can check that rank of this matrix is 6. Hence, z' is a smooth point. □

Recall that there are two-dimensional deformations of pt:

$$T(a,b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a\epsilon & b\epsilon^2 & -1 & a\epsilon^4 & b\epsilon^5 \\ 1 & -a & b & -1 & a & -b \\ 1 & a\epsilon^5 & b\epsilon^4 & -1 & a\epsilon^2 & b\epsilon \\ 1 & \epsilon^4 & \epsilon^2 & 1 & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon^4 & 1 & \epsilon^2 & \epsilon^4 \end{pmatrix}, T'(c,d) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & c\epsilon & \epsilon^2 & -c & \epsilon^4 & c\epsilon^5 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & d\epsilon^5 & \epsilon^4 & -d & \epsilon^2 & d\epsilon \\ 1 & c\epsilon^4 & \epsilon^2 & c & \epsilon^4 & c\epsilon^2 \\ 1 & d\epsilon^2 & \epsilon^4 & d & \epsilon^2 & d\epsilon^4 \end{pmatrix}, \tag{410}$$

where $a, b, c, d \in F^*$. As we know, $T(a,b) \cap T'(c,d) = pt$ in $X(6,6)$. It is well-known that $\dim_F T_{pt}X(6,6) = 4$ (cf.??). One can check that we have the following isomorphism of tangent spaces:

$$T_{pt}T(a,b) \oplus T_{pt}T'(c,d) = T_{pt}X(6,6). \tag{411}$$

Calculate $pr_1(T(a,b)) = t(a,b) \in X(3,6), pr_1 \circ \sigma^{(p)}(T(a,b)) = t(a,b) \in X(3,6); pr_1(T'(c,d)) = t'(c,d) \in X(3,6), pr_1 \circ \sigma^{(p)}(T(c,d)) = t'(\frac{1}{c}, \frac{1}{d}) \in X(3,6)$, where $t(a,b)$ and $t'(c,d)$ have the following type:

$$t(a,b) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a\epsilon & b\epsilon^2 \\ 1 & -a & b \\ 1 & a\epsilon^5 & b\epsilon^4 \\ 1 & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon^4 \end{pmatrix}, t'(c,d) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & c\epsilon & \epsilon^2 \\ 1 & -1 & 1 \\ 1 & d\epsilon^5 & \epsilon^4 \\ 1 & c\epsilon^4 & \epsilon^2 \\ 1 & d\epsilon^2 & \epsilon^4 \end{pmatrix} \tag{412}$$

It is easy that $t(a,b)\cap t'(c,d)=z'$ in $X(3,6)$. Also, we have the following isomorphism for tangent spaces:

$$T_{z'}t(a,b)\oplus T_{z'}t'(c,d)=T_{z'}X(3,6). \tag{413}$$

Calculate image of $z', t(a,b), t'(c,d)$ under morphism ζ:

$$\zeta(t(a,b))=(t_1(a,b)=\begin{pmatrix}1&1&1\\1&a\epsilon&b\epsilon^2\\1&-a&b\end{pmatrix}, t_2(a,b)=\begin{pmatrix}1&1&1\\1&\frac{1}{a\epsilon}&\frac{1}{b\epsilon^2}\\1&-\frac{1}{a}&\frac{1}{b}\end{pmatrix}) \tag{414}$$

$$\zeta(t'(c,d))=(t_1'(c,d)=\begin{pmatrix}1&1&1\\1&c\epsilon&\epsilon^2\\1&-1&1\end{pmatrix}, t_2'(c,d)=\begin{pmatrix}1&1&1\\1&\frac{c}{d\epsilon}&\epsilon^4\\1&-1&1\end{pmatrix}) \tag{415}$$

$$\zeta(z')=(\zeta(z')_1=t_1(1,1)=\begin{pmatrix}1&1&1\\1&\epsilon&\epsilon^2\\1&-1&1\end{pmatrix}, \zeta(z')_2=t_2(1,1)=\begin{pmatrix}1&1&1\\1&\epsilon^5&\epsilon^4\\1&-1&1\end{pmatrix}) \tag{416}$$

Let us note the following property of point $\zeta(z')$:

Lemma 125. *$\zeta(z')\in\widetilde{X}$ is a smooth point.*

Proof. Let us calculate tangent space $T_{\zeta(z')}\widetilde{X}$. Let $y\in Y(3)$ be the point $\phi_2\circ\tau(\zeta(z')_1)=\sigma_P^{(3)}\circ\phi_2\circ\tau(\zeta(z')_2)$. As we know from (406), we have to calculate $\mathrm{d}\phi_2$ at point $\zeta(z')_1$ and $\mathrm{d}\sigma_P^{(3)}\circ\mathrm{d}\phi_2\circ\mathrm{d}\tau$ at point $\zeta(z')_2$. As we know, $X(3,3)\cong(F^*)^4$ and $Y(3)\subset F^5$. It is easy that map: $(\mathrm{d}\phi_2\circ\tau,-\mathrm{d}\sigma_P^{(3)}\circ\mathrm{d}\phi_2\circ\mathrm{d}\tau):T_{\zeta(z')_1}X(3,3)\oplus T_{\zeta(z')_2}X(3,3)=F^8\to T_yY(3)\subset F^5$ is defined by matrix 8×5. Let us order variables as follows: rows correspond to coordinates A,B,C,α,β, columns correspond to coordinates $x_{11},x_{21},x_{12},x_{22},\frac{x_{41}}{x_{31}},\frac{x_{51}}{x_{31}},\frac{x_{42}}{x_{32}},\frac{x_{52}}{x_{32}}$ of $X(3,3)\times X(3,3)$. It is easy that $\mathrm{d}\sigma_P^{(3)}:T_yY(3)\to T_{\sigma_P^{(3)}(y)}Y(3)$ is a matrix:

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\end{pmatrix} \tag{417}$$

It can be shown in usual way that matrices of $\mathrm{d}\phi_2\circ\mathrm{d}\tau$ at points $\zeta(z')_1$ and $\zeta(z')_2$

$$\begin{pmatrix}0&\frac{\epsilon^2-1}{\epsilon}&0&0\\0&0&2(1-\epsilon^2)&\frac{\epsilon^2-1}{\epsilon}\\0&-\frac{\epsilon^2-1}{\epsilon}&0&-\frac{\epsilon^2-1}{\epsilon}\\0&0&2&2\epsilon^4\\0&0&2\epsilon^2&2\epsilon^5\end{pmatrix}, \begin{pmatrix}\frac{1-\epsilon^4}{\epsilon^4}&0&0&0\\0&0&-\frac{\epsilon^2-1}{\epsilon}&0\\-\frac{\epsilon^2-1}{\epsilon}&0&-\frac{\epsilon^2-1}{\epsilon}&0\\0&0&-2\epsilon^2&-2\\0&0&-2\epsilon&-2\epsilon\end{pmatrix} \tag{418}$$

Therefore, matrix of $(\mathrm{d}\phi_2\circ\mathrm{d}\tau|_{\zeta(z')_1},-\mathrm{d}\sigma_P^{(3)}\circ\mathrm{d}\phi_2\circ\mathrm{d}\tau|_{\zeta(z')_2})$ has the following view:

$$\begin{pmatrix}0&\frac{\epsilon^2-1}{\epsilon}&0&0&-\frac{1-\epsilon^4}{\epsilon^4}&0&0&0\\0&0&2(1-\epsilon^2)&\frac{\epsilon^2-1}{\epsilon}&0&0&\frac{\epsilon^2-1}{\epsilon}&0\\0&-\frac{\epsilon^2-1}{\epsilon}&0&-\frac{\epsilon^2-1}{\epsilon}&\frac{\epsilon^2-1}{\epsilon}&0&\frac{\epsilon^2-1}{\epsilon}&0\\0&0&2&2\epsilon^4&0&0&-2\epsilon^2&-2\\0&0&2\epsilon^2&2\epsilon^5&0&0&-2\epsilon&-2\epsilon\end{pmatrix} \tag{419}$$

One can see that rank of submatrix generated by second, third, forth and eighth columns is 4. Thus, point $\zeta(z')$ is a smooth. □

Using this lemma and some trivial computations, we get the following isomorphism of vector spaces:

$$T_{\zeta(z')}\widetilde{X} = T_{\zeta(z')}\zeta(t(a,b)) \oplus T_{\zeta(z')}\zeta(t'(c,d)). \tag{420}$$

Thus, $\mathrm{d}\zeta|_{z'}$ is an isomorphism

Secondly, consider morphism: $\mathrm{d}\Phi : T_{\zeta(z')}\widetilde{X} \to T_{\Phi\circ\zeta(z')}Y(3) \times_Y Y(3)$. As we know, morphism Φ is defined by the rule: $\Phi(\zeta(z')) = (\phi_2(\zeta(z')_1, \phi_2(\zeta(z')_2))$, where ϕ_2 is given by formulas:

$$A = (1 + x_{11} + x_{12})(1 + \frac{1}{x_{11}} + \frac{1}{x_{12}}) \tag{421}$$

$$B = (1 + x_{21} + x_{22})(1 + \frac{1}{x_{21}} + \frac{1}{x_{22}}) \tag{422}$$

$$C = (1 + \frac{x_{11}}{x_{21}} + \frac{x_{12}}{x_{22}})(1 + \frac{x_{21}}{x_{11}} + \frac{x_{22}}{x_{12}}) \tag{423}$$

$$\alpha = (1 + \frac{1}{x_{11}} + \frac{1}{x_{12}})(1 + x_{21} + x_{22})(1 + \frac{x_{11}}{x_{21}} + \frac{x_{12}}{x_{22}}) \tag{424}$$

$$\beta = (1 + x_{11} + x_{12})(1 + \frac{1}{x_{21}} + \frac{1}{x_{22}})(1 + \frac{x_{21}}{x_{11}} + \frac{x_{22}}{x_{12}}). \tag{425}$$

It is easy that $T_{\zeta(z')}\widetilde{X} \subset F^8$ and $T_{\Phi\circ\zeta(z')}Y(3) \times_Y Y(3) \subset F^{10}$. Therefore, $\mathrm{d}\Phi$ is defined by matrix of type 8×10. It is easy that $\mathrm{d}\Phi|_{\zeta(z')} = \mathrm{d}\phi_2|_{\zeta(z')_1} \oplus \mathrm{d}\phi_2|_{\zeta(z')_2}$ One can calculate that $\mathrm{d}\phi_2$ at points $\zeta(z')_1$ and $\zeta(z')_2$ are given by matrices:

$$\begin{pmatrix} 0 & -2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\epsilon^2 & 2\epsilon^5 & -2 & 2\epsilon^4 \\ 2\epsilon^5 & 2 & 2 & 2\epsilon^5 \end{pmatrix}, \begin{pmatrix} 0 & 2\frac{\epsilon^2-1}{\epsilon^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\epsilon^4 & -2\epsilon^4 & -2 & 2\epsilon^2 \\ -2\epsilon^4 & 2 & 2 & -2\epsilon^4 \end{pmatrix} \tag{426}$$

One can check that intersection of the kernel of matrix of $\mathrm{d}\Phi|_{\zeta(z')} = \mathrm{d}\phi_2|_{\zeta(z')_1} \oplus \mathrm{d}\phi_2|_{\zeta(z')_2}$ and $T_{\zeta(z')}\widetilde{X}$ is 0. Thus, restriction of $\mathrm{d}\Phi$ to $T_{\zeta(z')}\widetilde{X}$ is injective.

Thirdly, one can check that $\mathrm{St}_{S_3^{(p)}}(z')$ is trivial. Thus, $T_{z_0}Z$ is 4-dimensional. As we know, $pr_1 \circ \sigma^{(p)}(pt) = z'$. And hence, $\mu(z_0) = \sigma_P^{(6)}(\mu(z_0))$. Using proposition 100, we get that $\sigma_P^{(3)} \circ \widetilde{\mu} \circ \widetilde{\zeta}(z_0) = \widetilde{\mu} \circ \widetilde{\zeta}(z_0)$. Therefore, we obtain that $\mathrm{d}\sigma'(T_{z_0}Z) = T_{z_0}Z$. It is easy that there is a decomposition of $T_{z_0}Z$ into direct sum of $V_+ = \mathrm{d}\pi(T_{\zeta(z')}\zeta(t(a,b)))$ and $V_- = \mathrm{d}\pi(T_{\zeta(z')}\zeta(t'(c,d)))$. One can check that V_+ and V_- are subspaces corresponding to eigenvalue 1 and -1 of $\mathrm{d}\sigma'$ respectively.

Finally, consider involutions σ' and $\rho^{-1} \circ \sigma' \circ \rho$ of Z. Using proposition 100, we get the following identities:

$$\sigma'(\pi(t(a,b))) = \pi(t(a,b)), \sigma'(\pi(t'(c,d))) = \pi(t'(\frac{1}{c}, \frac{1}{d})) \tag{427}$$

for σ'. And

$$\rho^{-1} \circ \sigma' \circ \rho(\pi(t(a,b))) = \pi(t(a,b)), \rho^{-1} \circ \sigma' \circ \rho(\pi(t'(c,d))) = \pi(t'(\frac{1}{c}, \frac{1}{d})) \tag{428}$$

Therefore, $\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho$ acts on $\pi(t(a,b))$ and $\pi(t'(c,d))$ trivially. And hence, $\mathrm{d}(\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho)$ acts on $T_{z_0}Z$ trivially. Thus, we have proved the following proposition:

Proposition 126. *$\mathrm{d}(\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho)$ acts on $T_{z_0}Z$ trivially.*

As we know from ??, morphism $\sigma' \circ \rho^{-1} \circ \sigma' \circ \rho$ has finite order. Let us formulate following well-known property of morphism of finite order:

Proposition 127. *(cf.??) Let γ be the automorphism of finite order of variety V. Assume that $v \in V$ such that $\gamma(v) = v$ and $\mathrm{d}\gamma : T_v(V) \to T_v(V)$ is identity linear map. Then γ is identity.*

References

[1] Bondal A., Zhdanovskiy I., Representation theory for system of projectors and discrete Laplace operator, Preprint of IPMU, IPMU13-0001

[2] Boykin P.O., Sitharam M., Tiep P.H., Wocjan P. Mutually unbiased bases and orthogonal decompositions of Lie algebras, Quantum Inf. Comput. 7(2007), 371-382.

[3] Crawley-Boevey W., Noncommutative deformations of Kleinian singularities, Duke Math.J. 92(3)(1998), pp.605-635

[4] Crawley-Boevey W., Geometry of the moment map for representations of quvers, Compositio Math. 126(2001), 257-293

[5] Gan W.L., Ginzburg V., Deformed preprojective algebras and symplectic reflection for wreath products, J.Algebra, 283(2005), 350-363.

[6] U. Haagerup, Orthogonal maximal abelian *-subalgebras of the nn matrices and cyclic n-roots, Operator Algebras and Quantum Field Theory (Rome), 1996 (Cambridge, MA: International Press) pp 296-322.

[7] Kostrikin, A. I.; Kostrikin, I. A.; Ufnarovskii(, V. A. Orthogonal decompositions of simple Lie algebras (type A_n). (Russian) Analytic number theory, mathematical analysis and their applications. Trudy Mat. Inst. Steklov. 158 (1981), 105120, 229.

[8] Kostrikin A.I., Kostrikin I.A., Ufnarovsky V.A.; On the uniqueness of orthogonal decompositions of Lie algebras of type A_n and C_n "Matematiceskie issledovanija 74 (1983): 80-105.

[9] Kostrikin A.I., Pham Huu Tiep, Orthogonal decompositions and Integral lattices, Walter de Gruyter, 1994.

[10] Matolcsi M., Szollosi F., Towards a classification of 6×6 complex hadamard matrices, Open.Syst. Inf.Dyn. 15, 93(2008)

[11] K.Nomura, Type II Matrices of size five, Graphs and Combinatorics March 1999, Volume 15, Issue 1, pp 79-92.

[12] qig.itp.uni-hannover.de/qiproblems/Main Page

[13] Szollosi F., Complex Hadamard matrices of order 6: a four-parameter family, J. Lond. Math. Soc 85: 616-632 Part 3 (2012) arXiv:http:1008.0632

[14] Thompson J. A conjugacy theorem for E_8. J. Algebra, 1976, 38, N2, p.525-530.

http://chaos.if.uj.edu.pl/ karol/hadamard/