AROUND THE UNIFORM RATIONALITY

ILYA KARZHEMANOV

Abstract. We prove that there exist rational but not uniformly rational smooth algebraic varieties. The proof is based on computing a certain numerical obstruction developed in the case of compactifications of affine spaces. We show that for some particular compactifications this obstruction behaves differently compared to the uniformly rational situation.

1. Introduction

1.1. Let X be a complex projective manifold of dimension $n \ge 2$. Recall that rationality of X (i. e. the existence of a birational map $X \dashrightarrow \mathbb{P}^n$) provides a Zariski open subset $U \subset X$ isomorphic (as an affine scheme) to a domain in $\mathbb{C}^n$. Any rational X obviously carries a family of (*very free* in terminology of e. g. [65]) rational curves and one may try to obtain a family of holomorphic maps $\mathbb{C} \longrightarrow X$ (called *sprays* in [39]) such that near each of its points X is *(algebraically) h-Runge* (see [39] for the precise definition and results). The main expectation is that Zariski locally near *every* point X should actually look like as an open subset $U \subseteq \mathbb{C}^n$. One refers to the latter property as *uniform rationality* (of X), the notion introduced recently in [3] (following [39]), where some examples and basic properties of uniformly rational (or u. r. for short) manifolds have been established. The ultimate goal was to approach the following:

Question G (cf. [3], [39])**.** Is it true that every rational manifold is uniformly rational?

Note that spherical (e. g. toric) varieties and blowups of u. r. varieties at smooth centers are easily seen to be u. r. (all this is contained in [3], together with examples of X being the intersection of two quadrics, small resolution of a singular cubic threefold, and some other instances).[1)] This immediately gives positive answer to Question G in the case when $n = 2$. It is also easy to see that all points on a rational manifold X which may not admit affine neighborhoods $U \subseteq \mathbb{C}^n$ form a locus of codimension ≥ 2 (compare with Proposition 2.12 below).

The goal of the present paper is to prove

Theorem 1.2. *In the previous notation, there exists rational, but not uniformly rational, X whenever n is at least* 4.

MS 2010 classification: 14E08, 14M20, 14M27.

Key words: rational variety, uniform rationality, compactification.

[1)]We do not treat here two very interesting questions (both discussed in [3]) on *rectifiability* of divisorial families and on the local regularization of an arbitrary birational map $X \dashrightarrow \mathbb{P}^n$. Instead we rather concentrate on the "negative side" of Question G (see below). In addition, recall that small resolution of a Lefschetz cubic is (Moishezon and) not u. r., which shows that projectivity assumption on X is crucial for Question G to be of any content.

Thus Theorem 1.2 answers Question G negatively. But still it would be interesting to find out whether a sufficiently large power $X^N := X \times X \times \dots$ of any rational manifold X is u. r. (same question for the "stabilization" $X \times \mathbb{P}^N$ of X). See Appendix below for other plausible questions.

1.3. We proceed with a description of the proof of Theorem 1.2. First of all, in view of the above discussion, it is reasonable to treat only rational manifolds which are "minimal" in certain sense, like those that are not blowups of other manifolds for instance. The most common ones are compactifications of *affine* $\mathbb{C}^n$ with $\mathrm{Pic} \simeq \mathbb{Z}$ (see **2.1** and **A.5** below for the setup and examples). Next, one may guess that being u. r. for a rational manifold X results in "homogeneity property" for the underlying set of points (compare with [39, 3.5.E‴]). The latter means (ideally) that an appropriate test function $f : X \longrightarrow \mathbb{R}$ ("Gibbs distribution") must be constant on uniformly rational X. More precisely, as soon as just the points on X are concerned, it is natural to look just for such f that are conformly invariant (these constitute a class of the so-called *asymptotic invariants* of X).

Now, if L is an ample (or more generally, nef) line bundle on X, then the very first candidate for f one can think of would be the *Seshadri constant* $s_L(\cdot)$ of L. Namely, for any point $o \in X$ let us consider the blowup $\sigma : Y \longrightarrow X$ of o, with exceptional divisor $E := \sigma^{-1}(o)$, and then put

$$f(o) := s_L(o) := \max\{\varepsilon \in \mathbb{R} \mid \text{the divisor } \sigma^* L - \varepsilon E \text{ is nef}\}.$$

We will also write simply $s(o)$ instead of $s_L(o)$ when $\mathrm{Pic}\, X = \mathbb{Z} \cdot L$ (resp. when L is clear from the context). Further, conformal invariance of $f = s_L(\cdot)$ may be seen via another definition of it as follows (see [15]):

$$s_L(o) := \sup \frac{\mathrm{mult}_o\, \mathcal{M}}{k}, \tag{1.4}$$

where the supremum is taken over all $k \in \mathbb{Z}$ and linear subsystems $\mathcal{M} \subseteq |kL|$ having *isolated* base locus near o. One of the key ingredients in establishing the expression (1.4) is the Poincaré – Lelong formula

$$\mathrm{mult}_o\, D = \lim_{r \to 0} \frac{1}{r^{2n-2}} \mathrm{Vol}\,(D \cap B(r)) = \lim_{r \to 0} \frac{1}{r^{2n-2}} \int_{D \cap B(r)} \omega^{2n-2}$$

for the multiplicity of a hypersurface $D \subset X$ at the point o, written in terms of volumes of intersections with small balls $B(r)$ centered at o.

Recall that the basic play ground for our approach are those X containing $U := \mathbb{C}^n$ as a Zariski open subset. We also require the boundary $\Gamma := X \setminus U$ to be (of pure codimension 1 and) irreducible. Assuming such X uniformly rational, we claim that $s(\cdot)$ attains the same value at some points on U and Γ, respectively (see Proposition 2.12). One thus gets a relatively simple numerical criterion to test uniform rationality of the manifolds in question. The main issue then is to find a particular X for which this obstruction actually gives something non-trivial.

For $n = 3$, as we show in **A.5** (see Remark A.13), one does not obtain anything interesting (although we demonstrate here that being u. r. confirms the results Section 2). Anyway, we construct the needed examples (for any $n \geq 4$) in Section 3. The idea behind our construction is to mimic the one for the fourfold V_5^4 from Example A.10. Namely, we start by blowing up $\mathbb{P}^n$ at a smooth cubic of dimension $n - 2$ and contracting the proper transform of hyperplane, which gives a *singular* n-fold Y. The only singularity of Y happens to be of the form $\mathbb{C}^n/(\mathbb{Z}/2)$ and so an appropriate double covering $X \longrightarrow Y$ makes (Γ Cartier and) Y smooth. Here X is an index $n - 3$ Fano manifold having $\mathrm{Pic}\, X = \mathbb{Z} \cdot \mathcal{O}_X(\Gamma)$ (we keep the same notation for the images of Γ on Y, X, etc). It remains then to estimate the function $s(\cdot)$ on X and to show that X indeed compactifies $\mathbb{C}^n$.

The first issue is resolved in Corollary 3.6 by an explicit computation, where we show that $s(\cdot) = 1$ on Γ, while $s(\cdot) \geq 2$ on the complement $X \setminus \Gamma$. In turn, the second issue (that is $X \setminus \Gamma \simeq \mathbb{C}^n$) reduces to finding a cubic polynomial P such that the double cover of $\mathbb{C}^n$ with ramification in P, i. e. $\operatorname{Spec} \mathbb{C}[\mathbb{C}^n][\sqrt{P}]$, is also isomorphic to $\mathbb{C}^n$. Theorem 1.2 for the given X now follows from Corollary 3.10 by combining the just mentioned properties of X with Corollary 2.15.

Remark 1.5. The assumption on Γ to be irreducible is crucial in our approach (cf. Remark 2.8). In fact, the surface $X := \mathbb{F}_1$ is uniformly rational (as a toric surface) and compactifies $\mathbb{C}^2$, with Γ being the union of a (-1)-curve Z and a ruling R of the natural projection $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$. Then one can easily see that $s(o) = 3$ (with respect to $-K_X = 2Z + 3R$) for any point $o \notin Z$. Otherwise we have $s(o) = 1$ – in contradiction with what happens for irreducible Γ. Anyhow, X is an *equivariant* compactification of $\mathbb{C}^2$, and it would be interesting to find out whether all such compactifications of $\mathbb{C}^n$ are u. r. (see [46] for their structural theory).

Finally, in the Appendix below we have made an attempt to *explain* the appearance of X, as well as the role that asymptotic invariants play here. For the latter, we formulate a heuristic principle behind, which we support further by several examples and comparisons with the previous work. We believe such discussion is of some importance, as it helps one to build a certain intuition for the class of similar geometric problems, leading to a better understanding of the phenomenon of rationality, say. (N. B. The results of this part of the paper are *not* used in the proof of Theorem 1.2 and carry just an expository significance.)

Conventions. All varieties, unless stated otherwise, are defined over the complex field $\mathbb{C}$ and assumed to be normal and projective. We will be using freely standard notation, notions and facts (although we recall some of them for convenience) from [57], [65], [66] and [67].

Acknowledgments. I am grateful to C. Birkar, F. Bogomolov, A. I. Bondal, S. Galkin, Yu. G. Prokhorov, M. Romo, and J. Ross for their interest and helpful comments. Some parts of the paper were written during my visits to CIRM, Università degli Studi di Trento (Trento, Italy), Cambridge University (Cambridge, UK) and Courant Institute (New York, US). The work was supported by World Premier International Research Initiative (WPI), MEXT, Japan, and Grant-in-Aid for Scientific Research (26887009) from Japan Mathematical Society (Kakenhi).

2. Beginning of the proof of Theorem 1.2: an obstruction

2.1. Let X be a Fano manifold with $\operatorname{Pic} X \simeq \mathbb{Z}$ compactifying $\mathbb{C}^n$. In other words, there exists an affine open subset $U \subset X$, $U \simeq \mathbb{C}^n$, such that $\operatorname{Pic} X = \mathbb{Z} \cdot \mathcal{O}_X(\Gamma)$ for the boundary $\Gamma := X \setminus U$. We will also assume that Γ is an *irreducible* hypersurface.

Fix one particular such $X \neq \mathbb{P}^n$ (see **A.5** and Section 3 below for some examples). Let H be a generator of $\operatorname{Pic} X$ and $x_1, \ldots, x_n$ be affine coordinates on U. Then, for $r \gg 1$, there exist sections $s_i \in H^0(X, H^r)$ such that $s_i = x_i$ on U. Indeed, with H^r very ample, $s_i\big|_U$ induce an identification $U = \mathbb{C}^n$. We may also assume without loss of generality that $X \subset \mathbb{P}^{\dim |H^r|}$ is projectively normal.

Now pick a point $p \in \Gamma$ and a rational function $t \in \mathcal{O}_{X,p} \subset \mathbb{C}(U)$ defining Γ in an affine neighborhood $U' \subset X$ of p.

Lemma 2.2. *We have $t^{-1} \in \mathbb{C}[U]$. More precisely, $t^{-1}\big|_U$ is an irreducible polynomial in $x_1, \ldots, x_n$.*

Proof. Rational function t does not have any zeroes on $U = X \setminus \Gamma$ by construction. Hence t^{-1} is a polynomial $\in \mathbb{C}[U]$. Its irreducibility follows from that of Γ. □

2.3. Suppose that X *is* uniformly rational. Let $U' \ni p$ be as above. Then U' embeds into $\mathbb{C}^n$.

Let $s \in H^0(X, H)$ be the section whose zero locus equals Γ. By definition of H we have $s\big|_{U'} = t$ and $s\big|_U = 1$ (cf. Lemma 2.2), so that the functions $s\big|_{U'}, s\big|_U \in \mathbb{C}(x_1, \ldots, x_n)$ are identified on $U' \cap U$ via $s\big|_{U'} = ts\big|_U$. This yields

$$y_i := s_i\big|_{U'} = t^r x_i \in \mathcal{O}_{X,p} \tag{2.4}$$

for all i. Indeed, both line bundles $H^r\big|_{U'}$ and $H^r\big|_U$ are trivial on U' and U, respectively, for $s_i\big|_{U'}$ and $s_i\big|_U$ regarded as rational functions on $\mathbb{C}^n$, satisfying by construction $s_i\big|_{U'} \in \mathbb{C}[U']$ and $s_i\big|_U = x_i$. Then $H^r\big|_{U'}$ and $H^r\big|_U$ are glued over $U' \cap U$ via the multiplication by t^r as (2.4) indicates.

Lemma 2.5. *In the previous setting, if $y_i \neq const$ for all i, then $y_1, \ldots, y_n$ are local parameters on $U' \subseteq \mathbb{C}^n$ generating the maximal ideal of the $\mathbb{C}$-algebra $\mathcal{O}_{X,p}$.*

Proof. Notice that

$$\mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(U) = \mathbb{C}(U') = \mathbb{C}(y_1, \ldots, y_n)$$

by construction, i. e. $x_i = y_i/t^r$ (resp. y_i) are (birational) coordinates on U', defined everywhere out of Γ (resp. everywhere on U'). This implies that the morphism $\xi : \mathbb{C}^n \cap (t^{-1} \neq 0) \longrightarrow (U' \subseteq \mathbb{C}^n)$, given by

$$(x_1, \ldots, x_n) \mapsto (y_1 = x_1 t^r, \ldots, y_n = x_n t^r),$$

is birational.[2)]

Functions y_i do not have common codimension 1 zero locus on U (cf. Lemma 2.2). Hence ξ does not contract any divisors. In particular, ξ^{-1} is well-defined near $\xi(\Gamma)$ by Hartogs, which shows that $y_1, \ldots, y_n$ are the claimed local parameters. □

Lemma 2.6. *$y_i = const$ for at most one i.*

Proof. Indeed, otherwise (2.4) gives $s_i = s_j$ on X for some $i \neq j$, a contradiction. □

Lemma 2.7. *Let $y_1 = const$. Then $t^r, y_2, \ldots, y_n \in \mathcal{O}_{X,p}$ are local parameters on $U' \subseteq \mathbb{C}^n$ generating the maximal ideal of the $\mathbb{C}$-algebra $\mathcal{O}_{X,p}$.*

Proof. One may assume that $y_1 = 1$. Then $y_i \neq$ const for all $i \geq 2$ by Lemma 2.6, and a similar argument as in the proof of Lemma 2.5 shows that birational morphism $\eta : \mathbb{C}^n \cap (t^{-1} \neq 0) \longrightarrow (U' \subseteq \mathbb{C}^n)$, given by

$$\eta : (x_1, \ldots, x_n) \mapsto (t^r = 1/x_1, y_2 = x_2 t^r, \ldots, y_n = x_n t^r),$$

does not contract any divisors. Hence again $t^r, y_2, \ldots, y_n$ are the asserted local parameters. □

[2)]More specifically, dividing all the x_i by x_1, say, one may assume $x_i = y_i$ on $U' \subseteq \mathbb{C}^n$ for all $i \geq 2$. Then ξ is simply the multiplication of x_1 by t^r.

Remark 2.8. An upshot of the previous considerations is that the whole "analysis" on X, encoded in the line bundle H, can be captured just by *two* charts, like U and U', with a transparent gluing (given by t) on the overlap $U \cap U'$. Let us stress one more time that this holds under the assumption that X is u. r. In addition, as will also be seen in **2.9** below, similar property does not extend directly to the case of X with reducible boundary Γ (compare Proposition 2.12 and Remark 1.5).

2.9. Let $h \in H^0(X, H^r)$ be any section. One may write

$$h\big|_U = \sum_{0 \le i_1 + \ldots + i_n \le m} a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}, \tag{2.10}$$

where $a_{i_1,\ldots,i_n} \in \mathbb{C}$, $m = m(h) \ge 0$ and i_j are non-negative integers. Now, it follows from (2.4) and Lemmas 2.5, 2.7 that

$$h\big|_{U'} = \sum_{0 \le i_1 + \ldots + i_n \le m} a_{i_1,\ldots,i_n} y_1^{i_1} \ldots y_n^{i_n} t^{r - r(i_1 + \ldots + i_n)}. \tag{2.11}$$

Conversely, starting with *any* function on U' as in (2.11), with $m \le 1$, we can find $h \in H^0(X, H^r)$ such that $h\big|_{U'} =$ RHS of (2.11) (cf. Remark 2.8). Indeed, in this way we get a global section of H^r, regular away the codimension ≥ 2 locus $X \setminus U \cup U'$ (recall that Γ is irreducible), hence regular on the entire X.

This discussion condensates to the next

Proposition 2.12. *There exist a point $o \in U$ and a point $p = p(o) \in \Gamma \cap U'$ such that for any hypersurface $\Sigma \sim r\Gamma$,*[3] *having prescribed multiplicity* $\operatorname{mult}_o \Sigma > 0$ *at o, there is a hypersurface $\hat{\Sigma} \sim r\Gamma$ such that* $\operatorname{mult}_p \hat{\Sigma} \ge \operatorname{mult}_o \Sigma$.

Proof. Set $o \in U = \mathbb{C}^n$ to be the origin with respect to x_i.

Lemma 2.13. *The loci $H_i := (s_i = 0)$, $1 \le i \le n$, have a common intersection point, denoted p, on Γ.*

Proof. Assume the contrary. Then all $y_i \ne \text{const}$ (cf. Lemma 2.6), for otherwise $\Gamma = (s_1 = 0)$, say, and so $\cap H_i = \deg \Gamma \ne 0$. Further, by construction $\cap H_i$ is a (reduced) point, which immediately gives $X = \mathbb{P}^n$ (recall that H^r is very ample according to the setting of **2.1**), a contradiction. □

Let the section $h \in H^0(X, H^r)$ correspond to Σ. We may assume without loss of generality all but one $a_{i_1,\ldots,i_n}$ in (2.10) and (2.11) to be zero. Let also p be as in Lemma 2.13. Then, since $t(p) = 0$ by definition, one may take $\hat{\Sigma} := \Sigma$ whenever all $y_i \ne \text{const}$. Finally, if $y_1 = 1$ (and $i_1 \ne 0$), say, then from (2.4) and Lemmas 2.6, 2.7 we obtain

$$\operatorname{mult}_o \Sigma = i_1 \operatorname{mult}_o t^{-r} + i_2 + \ldots + i_n \le \operatorname{mult}_o x_1 = 1.$$

It is thus suffices to take any $\hat{\Sigma} \ni p$. □

Remark 2.14. The proof of Proposition 2.12 shows that both Σ and $\hat{\Sigma}$ can actually be taken to vary in some linear systems, having isolated base loci near o and p, respectively. Furthermore, the value $s(o)$ is attained on a linear system $\mathcal{M}$ (cf. **1.3**), with isolated base point at o, iff the value $s(p)$ is attained on a similar linear system for p and $\hat{\Sigma}$.

Corollary 2.15. *For $o \in U$ and $p \in \Gamma$ as above we have $s(p) \ge s(o)$.*

[3] "$\sim$" denotes the linear equivalence of divisors on X.

Proof. Fix some $f := f_i^{(j)} \in \mathcal{M}$ and $k := k_i$ as in (1.4). We may assume w. l. o. g. that $r = 1$ because $s_{H^r}(\cdot) = rs(\cdot)$. We may also take $f = f(h_1, \ldots, h_n)$ to be a homogeneous polynomial in some $h_i \in H^0(X, H)$ for $X \subset \mathbb{P}^{\dim|H|}$ being projectively normal (cf. **2.1**). Let $m_i(k) := \mathrm{mult}_o\, h_i$ be such that $\mathrm{mult}_o\, f = \sum m_i(k)$. One may assume that sup lim of $\sum m_i(k)/k$ exists and equals $s(o)$.

Now, Proposition 2.12 provides some sections $\hat{h}_1, \ldots, \hat{h}_n \in H^0(X, H)$, having $\mathrm{mult}_p\, \hat{h}_i \geq m_i(k)$ for all i. Then we obtain $\hat{f} := f(\hat{h}_1, \ldots, \hat{h}_n) \in H^0(X, H^k)$ and $\mathrm{mult}_p\, \hat{f} \geq \sum m_i(k)$. Thus by (1.4) and Remark 2.14 we get $s(p) \geq s(o)$ as wanted. □

With Corollary 2.15 we conclude our construction of a necessary condition for the manifold $X \supset \mathbb{C}^n$ in **2.1** to be u. r. Let us now construct those X that do not pass through this simple obstruction.

3. End of the proof of Theorem 1.2

3.1. The following construction was motivated by that of the fourfold V_5^4 in Example A.10 (see Appendix).

Take the projective space $\mathbb{P} := \mathbb{P}^n$, $n \geq 4$, with a hyperplane $H \subset \mathbb{P}$ and a cubic hypersurface $S \subset H$. Let $\sigma : V \longrightarrow \mathbb{P}$ be the blowup of (the ideal defining) S. More specifically, for the reasons that will become clear in **3.8** below, we assume $V \subset \mathbb{P}^1 \times \mathbb{P}$ to be given by equation

$$wt_0 = (wx_1^2 + F)t_1,$$

where t_i are projective coordinates on the first factor and H, S are given by $w = 0, w = F(x_1, \ldots, x_n) = 0$, respectively, in projective coordinates w, x_i on $\mathbb{P}$. Furthermore, we take F in the form

$$x_1x_3^2 + x_2^2x_4 + F_3,$$

with a general homogeneous cubic $F_3 \in \mathbb{C}[x_3, \ldots, x_n]$. This easily shows (Bertini) that V is *smooth*.

Put $E := \sigma^{-1}S$ and $H^* := \sigma^*H$. Notice that σ resolves the indeterminacies of the linear system $|3H - S|$. Let $\varphi : V \longrightarrow Y$ be the corresponding morphism onto some variety Y with very ample divisor $\mathcal{O}_Y(1)$ pulling back to $3H^* - E$.

Lemma 3.2. *φ is birational and contracts the divisor $H_V := \sigma_*^{-1}H \simeq \mathbb{P}^{n-1}$ to a point.*

Proof. By construction of $|3H - S|$ map φ coincides with the Veronese embedding (with respect to $2H$) on an affine open subset in $\mathbb{P}$. Hence φ is birational.

Now let $Z \subset \mathbb{P}$ be the image of a curve contracted by φ. Suppose that $Z \not\subset H$. Then, since $\sigma_*^{-1}Z$ is contracted by φ, we have $3H \cdot Z = \deg S \cdot Z$. On the other hand, we obviously have $\deg S \cdot Z \leq H \cdot Z$, a contradiction. Thus every curve contracted by φ belongs to $H_V \simeq \mathbb{P}^{n-1}$. □

Note that Y has exactly one singular point (cf. Lemma 3.3 below). More precisely, $\varphi \circ \sigma^{-1}$ induces an isomorphism between $\mathbb{P} \setminus H \simeq \mathbb{C}^n$ and $Y \setminus \varphi(E)$, so that Y can be singular only at the point $o := \varphi(H_V)$ on the boundary $\varphi(E)$ (cf. Lemma 3.2).

Further, we want to modify Y into a *smooth* n-fold (our X below), yet preserving the properties $\mathbb{C}^n \subset Y$ and $\mathrm{Pic}\, Y = \mathbb{Z}$. Let us start with the following technical observation:

Lemma 3.3. *Singularity $o \in Y$ is locally analytically of the form $\mathbb{C}^n/\mu_2$ for the 2-cyclic group μ_2 acting diagonally on $\mathbb{C}^n$.*

Proof. Recall that $K_V = -(n+1)H^* + E$ and φ contracts $H_V = \sigma_*^{-1}H \sim H^* - E$ to the point o.

One can choose such divisors $D_1, \ldots, D_n$ on Y that $\sigma_*^{-1}D_i \sim H^*$ for all i and the pair

$$(V, \sum_{i=1}^{n} \sigma_*^{-1}D_i + H_V)$$

is log canonical. Note also that

$$K_V + \sum_{i=1}^{n} \sigma_*^{-1}D_i + H_V = 0.$$

Then we apply [67, Lemma 3.38] to deduce that the pair $(Y, \sum_{i=1}^{n} D_i)$ is log canonical.

It now follows from [66, 18.22] that $o \in Y$ is a toric singularity. In particular, it is of the form $\mathbb{C}^n/\mu_m$ for a cyclic group μ_m acting diagonally on $\mathbb{C}^n$, and it remains to show that $m = 2$.

For the latter, notice that $\sigma(D_i)$ are hyperplanes on $\mathbb{P}$, with the plane $\sigma(D_1)\cap\ldots\cap\sigma(D_{n-2})$, say, intersecting the cubic S at exactly 3 distinct points. This implies that $H_V \cap D_1 \cap \ldots \cap D_{n-2}$ is a (-2)-curve on the smooth surface $D_1 \cap \ldots \cap D_{n-2}$ and the equality $m = 2$ follows by varying D_i. □

Choose some generic hypersurface $R \in |3(3H^* - E)|$ and let $\pi : \tilde{V} \longrightarrow V$ be the double covering ramified in $R + H_V \sim 10H^* - 4E$. Variety $\tilde{V}$ is smooth, as so are R and H_V, with $R \cap H_V = \emptyset$. We also have

$$-K_{\tilde{V}} = -\pi^*(K_V + \frac{1}{2}(R + H_V)) = \pi^*((n-4)H^* + E) := (n-4)\tilde{H} + \tilde{E}$$

by the Hurwitz formula, where $\tilde{H}$ and $\tilde{E}$ are the pullbacks to $\tilde{V}$ of H^* and E, respectively.

It is immediate from the construction that the group $\operatorname{Pic}\tilde{V}$ is generated by $\mathcal{O}_{\tilde{V}}(\pi^{-1}H_V)$ and $\mathcal{O}_{\tilde{V}}(\tilde{E})$ (note that $\pi^*H_V = 2\pi^{-1}H_V$ because π ramifies in H_V). Indeed, since $\mathcal{O}_V(H^*)$ and $\mathcal{O}_V(E)$ generate $\operatorname{Pic}V$, with intersections $H^* \cap R$ and $E \cap R$ being irreducible, the line bundles $\mathcal{O}_{\tilde{V}}(\pi^{-1}H_V)$ and $\mathcal{O}_{\tilde{V}}(\tilde{E})$ are the claimed generators of $\operatorname{Pic}\tilde{V}$.

Lemma 3.4. *There exists a birational contraction $f : \tilde{V} \longrightarrow X$ of $\pi^{-1}H_V$, given by a multiple of the linear system $|\pi^*(3H^* - E)|$, onto some* smooth *variety X.*

Proof. Let $Z \subset \pi^{-1}H_V \simeq \mathbb{P}^{n-1}$ be a line. We have

$$K_{\tilde{V}} \cdot Z = -((n-4)\tilde{H} + \tilde{E}) \cdot Z = 3 - n < 0.$$

Then [67, Theorem 3.25] delivers the contraction f as stated. Finally, Lemma 3.3 yields

$$\pi^{-1}H_V \cdot Z = \frac{1}{2}\pi^*H_V \cdot Z = \frac{1}{2}H_V \cdot \pi(Z) = -1,$$

which implies that f is just the blowup of the smooth point $f(\pi^{-1}H_V) \in X$. □

It follows from Lemma 3.4 that X is a smooth Fano n-fold of index $n-3$. Namely, we have

$$-K_X = (n-3)f_*\tilde{H} = (n-3)f_*\tilde{E},$$

for $\operatorname{Pic}X = \mathbb{Z}\cdot\mathcal{O}_X(f_*\tilde{E})$.

Let us now find those curves on X having the smallest intersection number with $f_*\tilde{E}$:

Proposition 3.5. *For every curve $Z \subset X$ we have $f_*\tilde{E}\cdot Z \geq 1$ and equality is achieved when $\sigma(\pi(f_*^{-1}Z))$ is a point on $\mathbb{P}$. In other words, $f_*^{-1}Z \subset \tilde{E}$ is an elliptic curve, a. k. a. the preimage of a ruling on E.*

Proof. Notice first that

$$-K_{\tilde{V}} = \frac{n-3}{2}(3\tilde{H} - \tilde{E}) - \frac{n-1}{2}(\tilde{H} - \tilde{E}).$$

In particular, we get $f^*K_X = \frac{n-3}{2}(3\tilde{H} - \tilde{E})$, and hence $f_*\tilde{E} \cdot Z = a$ iff $-K_X \cdot Z = (n-3)a$ iff

$$\frac{n-3}{2}(3\tilde{H} - \tilde{E}) \cdot f_*^{-1}Z = (n-3)a$$

for any $a \in \mathbb{Z}$.

Further, if $\pi(f_*^{-1}Z)$ is a ruling on E, then $\tilde{H} \cdot f_*^{-1}Z = 0$ by definition and

$$\tilde{E} \cdot f_*^{-1}Z = \pi^*E \cdot f_*^{-1}Z = E \cdot \pi_*(f_*^{-1}Z) = 2E \cdot \pi(f_*^{-1}Z) = -2$$

by the projection formula, where $\pi(f_*^{-1}Z)$ has intersection index 4 with ramification divisor $R + H_V$, i. e. $f_*^{-1}Z$ is an elliptic curve. This implies that $a = 1$ for such Z and Proposition 3.5 follows. □

Corollary 3.6. *For every point $p \in X$ we have $s(p) = 1$ when $p \in f_*(\tilde{E})$ and $s(p) \geq 2$ otherwise.*

Proof. Consider the case when $p = f(\pi^{-1}H_V) \in f_*\tilde{E}$ first. Note that the Mori cone $\overline{NE}(\tilde{V}) \subset N_1(\tilde{V}) \otimes \mathbb{R} = \mathbb{R}^2$ is generated by the classes of a line in $\pi^{-1}H_V \simeq \mathbb{P}^{n-1}$ and an elliptic curve $Z \subset \tilde{E}$ as in Proposition 3.5. Now, by construction of $\tilde{V}$ via the blowup f of X at p we obtain that $s(p) = 1$, since divisor $\tilde{H}$ is nef and

$$f^*f_*\tilde{E} - \lambda\pi^{-1}H_V = (\frac{3}{2} - \frac{\lambda}{2})\tilde{H} + \frac{1}{2}(\lambda - 1)\tilde{E}$$

is nef only when $\lambda \leq 1$. Then the estimate $s(p) \geq 1$ holds for any other $p \in \tilde{E}$ due to the lower semi-continuity of the function $s(\cdot)$ on X (see [71, Example 5.1.11]). But $s(p) > 1$ can not occur for these p because otherwise the divisor $\sigma^*f_*\tilde{E} - \lambda E$ (we are using the notation from **1.3**), with $\lambda > 1$, intersects the curve $\sigma_*^{-1}Z$ as $1 - \lambda < 0$. Thus $s(\cdot) = 1$ identically on $f_*\tilde{E}$.

Recall further that π when considered on $\tilde{V} \setminus \pi^{-1}H_V \cup \tilde{E} = X \setminus f_*\tilde{E}$ is the double cover of $V \setminus H_V \cup E \simeq \mathbb{C}^n$ ramified in R. Also, the proper transform on V of any element $\Sigma \in |m\tilde{E}|, m \in \mathbb{Z}$, is an element from $|\frac{m}{2}(3\tilde{H} - \tilde{E})|$ which maps (via $\sigma \circ \pi$) onto some $\Sigma' \in |m(3H - S)|$ on $\mathbb{P}$. In particular, we get

$$\text{(3.7)} \qquad \operatorname{mult}_p \Sigma = \operatorname{mult}_{\sigma\circ\pi(p)} \Sigma' \qquad \text{or} \qquad \geq \operatorname{mult}_{\sigma\circ\pi(p)} \Sigma'$$

as long as $p \notin f_*\tilde{E}$ (for $p \in X$ identified with $f^{-1}(p) \in \tilde{V}$), depending on whether $p \notin R$ or $p \in R$, respectively.

Now take $m = 1$ and $\Sigma' \in |3H - S|$ satisfying $\operatorname{mult}_{\sigma\circ\pi(p)} \Sigma' = 2$. Such Σ' vary in a linear system on $\mathbb{P}$ with isolated base locus near p.[4)] This and (3.7) (cf. (1.4)) imply that $s(p) \geq 2$ for $f_*\tilde{E} \equiv \Sigma$ (numerically on X). □

3.8. It remains to show that $X \setminus f_*\tilde{E} \simeq \mathbb{C}^n$ for one particular R.

Identifying $\mathbb{P} \setminus H = V \setminus H_V \cup E = Y \setminus \varphi(E)$ with $\mathbb{C}^n = \mathbb{P}^n \cap (w = 1)$ via σ, φ we observe that there are elements $y_1, \ldots, y_n$ in $|3H^* - E|$, depending on the affine coordinates x_i, for which the assignment $x_i \mapsto y_i$, $1 \leq i \leq n$, induces an automorphism on $\mathbb{C}^n = \varphi \circ \sigma^{-1}(\mathbb{C}^n)$. Namely,

$$y_1 := x_1 + F,\ y_2 := x_2,\ y_3 := x_2x_3,\ \ldots,\ y_n := x_2x_n$$

satisfy this property, since one has induced isomorphism $\mathbb{C}(y_1, \ldots, y_n) \simeq \mathbb{C}(x_1, \ldots, x_n)$ (cf. the definition of F in **3.1**). This also shows (as $\sigma^*F\big|_E \neq 0$ identically) that one may assume $y_i\big|_E \neq 0$ identically for all i.

[4)]Indeed, if $x_1, \ldots, x_n, w$ are projective coordinates on $\mathbb{P}$, with $H = (w = 0)$ and $S = (w = F(x_1, \ldots, x_n) = 0)$ as in **3.1**, then we consider $\Sigma' := (F + wB = 0)$ for an arbitrary quadratic form $B = B(x_1, \ldots, x_n)$ and $p := [0 : \ldots : 0 : 1]$. The case of arbitrary $p \in \mathbb{C}^n$ is easily reduced to this one.

Further, the equation of R on $V \setminus H_V \cup E$ is a cubic polynomial in y_i, and we may take

$$R \cap (V \setminus H_V \cup E) := (P(y_1, \ldots, y_{n-1}) + y_n + 1 = 0)$$

for some generic P. Notice that this defines a *smooth* hypersurface in $\mathbb{C}^n$.

Expressing y_i in terms of x_i we identify $R \cap (V \setminus H_V \cup E)$ with a hypersurface in $\mathbb{P} \setminus H$. Then compactifying via w, we obtain that $R \subset V$ can only be singular at the locus $y_1 = \ldots = y_{n-1} = w = 0$, i. e. precisely at S.

Lemma 3.9. *R is smooth and $R \cap H_V = \emptyset$.*

Proof. After the blowup σ the only singularities on E that $R = (P + w^2 y_n + w^3 = 0)$ can have belong to the locus $E \cap \bigcap_{i=1}^{n-1} (y_i = 0) \cap H_V$. But the latter is empty by the choice of y_i. Hence R is smooth. The statement about $R \cap H_V$ follows from Lemma 3.2 and the fact that $y_i\big|_E \neq 0$ identically. □

Lemma 3.9 implies that $\tilde{V}$ is smooth. Then so is X (cf. Lemma 3.4) and on the open chart $\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} = X \setminus f_* \tilde{E}$ morphism $\pi : \tilde{V} \longrightarrow V$ coincides with the projection of

$$\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} = (T^2 = P(y_1, \ldots, y_{n-1}) + y_n + 1) \subset \mathbb{C}^{n+1}$$

onto $\mathbb{C}^n = V \setminus H_V \cup E$ (having affine coordinates y_i). This yields $\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E} \simeq \mathbb{C}^n$, if one takes $T, y_1, \ldots, y_{n-1}$ as generators of the affine algebra $\mathbb{C}[\tilde{V} \setminus \pi^{-1} H_V \cup \tilde{E}]$.

Finally, since the defining equation of $R\big|_E$ is $P(y_1, \ldots, y_{n-1}) = 0$, with generic P, both cycles $R\big|_E$ and $\tilde{E}$ are irreducible. Thus $f_* \tilde{E}$ is also irreducible and so X is the pertinent compactification of $\mathbb{C}^n$ (with the boundary divisor $\Gamma = f_* \tilde{E}$). This concludes the construction of X.

Theorem 1.2 now follows from the next

Corollary 3.10. *X is not uniformly rational.*

Proof. Notice first that X satisfies all the assumptions (except possibly for u. r.) made in **2.1** of Section 2. Then Corollary 2.15 applies to X once we assume the latter to be actually u. r. We obtain $s(p) \geq s(o)$ for some $p \in f_* \tilde{E}$ and $o \in X \setminus f_* \tilde{E}$. At the same time, Corollary 3.6 gives $s(p) = 1$ and $s(o) \geq 2$, a contradiction. Hence X can not be u. r. □

Appendix: discussion and examples

A.1. Comments on the proof. Let us briefly summarize what we have encountered in the course of the proof of Theorem 1.2.

First of all, we started with a particular class of Fano manifolds with Pic $\simeq \mathbb{Z}$ and, assuming those to be uniformly rational, we have deduced a geometric condition as explained in Remark 2.8. This provides some control on how our u. r. manifold X is glued out of affine domains $\subseteq \mathbb{C}^n$.

The next natural step is to "measure" what happens with a given point $o \in X$ when transported from one domain to another. We believe that *probabilistic* viewpoint is the most suitable here for obtaining such measurements. Namely, as has already been mentioned in **1.1** of Introduction, one may expect certain conformally invariant functions (a. k. a. distributions) $f : X \longrightarrow \mathbb{R}$ should enter the game. Informally, each value $f(o)$ is a "probability to observe the point $o \in X$", and so it is not prohibited to consider X *on a large scale* (hence the conformal invariance of f), as well as to apply Central Limit Theorem type of arguments (compare with e.g. [17, 2.3.2], [32], [60], [36] and [14]). This dictates the motto that points on u.r. X should be "equally distributed" w.r.t. f. We confirm this, in a way, by Corollary 2.15 when $f = s(\cdot)$.

Finally, in **3.8** of Section 3 we construct such X, violating our probabilistic principle (hence not uniformly rational). Roughly, the strategy was to find those compactifications of $\mathbb{C}^n$, for which all the lines are contained in the boundary Γ (cf. **2.1**).

Such compactifications seem to be not known (compare with examples in **A.5** below). This also makes one curious whether all these X (including also the u.r. ones) are constructed by the standard *extraction-contraction-cyclic covering* operation (or a sequence of these), applied to some weighted projective space, as it was in Section 3. Although this sort of operations had been widely used in birational geometry (description of canonical singularities, construction of log flips, etc, as in [93] for example) we are not aware of any systematic applications of them to the problems of classification of algebraic varieties. Perhaps our probabilistic view point might again be of some use here.

More specifically, various f as above should impose strong (numerical) restrictions on u.r. X, thus bounding the class of such manifolds. This might require, however, finer versions of f to be developed, as compared to $s(\cdot)$. Unfortunately, it is not clear which particular refinement one should choose, as there are plenty of them (cf. **A.3**). Yet let us conclude this item by mentioning two counterparts of $s(\cdot)$ which may bring further perspective to the subject.

Namely, one may relax the conditions on $\mathcal{M}$ in (1.4) and define a similar quantity $m_L(o)$ (called the *mobility threshold* of L at o), or again just $m(o)$ when L is clear, in the following way:

$$m_L(o) := \sup \frac{\operatorname{mult}_o \mathcal{M}}{k},$$

where sup is taken over all k and *mobile* linear subsystems $\mathcal{M} \subseteq |kL|$. One may equally consider only those $\mathcal{M}$ that give various rational maps ("observables") $X \dashrightarrow \mathbb{P}^1$.

Recall that function $s(\cdot)$ was used (by J.-P. Demailly, C.S. Seshadri et al) in order to establish the celebrated ampleness criterion for (nef) line bundles, obtain various "Fujita-type" theorems for global forms with logarithmic poles, and so on (the ultimate reading on the subject are [71] and [72]). The function $m(\cdot)$ (introduced by A. Corti in [91]) is less known and can be used for example to formulate in a compact way the main result of [56] as the bound $m(\cdot) \le 2$ (for any smooth quartic threefold $X := X_4 \subset \mathbb{P}^4$). There are other birational properties of (Mori fiber spaces) X guident by the behavior of $m(\cdot)$ and related invariants (see e.g. [6], [82] and also Remark A.35 below for some complementary results).

The preceding discussion indicates a link between differential and birational geometries, and the last asymptotic invariant we mention in this regard (cf. **A.3** and **A.14**) is the so-called *global log canonical threshold*, defined as follows:

$$\operatorname{lct}(o) := \sup \{\lambda \mid \text{the pair } (X, \lambda D) \text{ is log canonical at } o \text{ for any Weil divisor } D \sim_{\mathbb{Q}} -K_X\}$$

for any $o \in X$. Apparently, the quantity $\inf\limits_{o \in X} \operatorname{lct}(o)$ has also a differential-geometric interpretation (see [95], [9]), being the *alpha-invariant* $\alpha(X) := \sup \left\{ \lambda \mid \sup\limits_{\phi} \int_X \exp(-\lambda\phi)\omega^n < \infty \right\}$, where the inner sup is taken over all ω-plurisubharmonic (w.r.t. to a Kähler form $\omega \in H^2(X, \mathbb{Z})$) functions ϕ on X having $\sup\limits_X \phi = 0$.

Remark A.2. It would be interesting to apply the previous reasoning to other classes of Fano manifolds in order to decide whether they are u. r. and classify all such. Some natural candidates can be found in e. g. [69], [48], [30], [13], [12], [21], [63], [89], [68], [23] (cf. [17, 4] and Remark A.37 below). Recall also those manifolds mentioned in Remark 1.5. It would be equally interesting to develop some sort of a *numerical characterization* for u. r. manifolds. Reducing possibly again to the case of $\mathbb{C}^n$-compactifications, one may approach this problem from either birational point of view (as in [4], [59], [61] for example), or by studying the intrinsic geometry – minimal curves, say, or the anticanonical degree – of these manifolds (compare with [47], [51], [49], [52], [76], [98] and [11] for instance; see also [20], [50], [53], [99] for an illustration in the case of homogeneous spaces and some connections with $s(\cdot), m(\cdot), \mathrm{lct}(\cdot)$, etc). This may be related to similar differential-geometric problems (characterization via various types of curvatures or Morse profiles as in Conjecture A.4 below), arithmetic ones (behavior of height zeta-functions as in [1], [97], [5] and others), and to the derived geometry as well (see e. g. [31] for a characterization of the fourfold from [100]).

A.3. Heuristics. Functions similar to $s(\cdot), m(\cdot), \mathrm{lct}(\cdot)$ had already appeared in different areas of mathematics and proved to be extremely useful. It is still interesting (and important perhaps) to give a systematic account for all such functions and how they influence (birational for instance) geometry of a manifold (compare with the discussion in **A.1**). In the forthcoming examples we have tried to extract some common features of the way these (conformly invariant) functions enter the geometry, how they are computed, and what geometric properties they obstruct. This is by no means an extensive account and we refer to the papers [33], [45], [42], [43] as a sample, where the reader will find an overwhelming discussion (from much more general grounds) of matters similar to the present ones.

Firstly, let us mention the *d-conformal volume* $V_c(d, M)$ of a compact Riemannian manifold M, defined as the infimum over all branched conformal immersions $\phi : M \longrightarrow S^d$ (to the unit sphere in $\mathbb{R}^{d+1}$) of suprema of volumes of $(g\circ\phi)(M)$ for all conformal diffeomorphisms g of S^d (see [73]). The quantity $V_c(M) := \lim\limits_{d\to\infty} V_c(d, M)$ is called *conformal volume* of M. One can show that $2V_c(M) \ge \lambda_1 \mathrm{Vol}_M$ for the first Laplacian eigenvalue λ_1 of a compact surface M (see [73, Theorem 1] and corollaries thereof). Moreover, equality $2V_c(M) = \lambda_1 \mathrm{Vol}_M$ holds iff M is a minimal surface in the unit sphere, with coordinates all being λ_1-eigen functions. (See [73] for a similar story on L^2-estimates for the mean curvature of M.) As for higher dimensions, we refer to [35] containing related "Ahlfors-type" considerations, applied to the *quasi-conformal* maps from $\mathbb{R}^n$ into convex manifolds. Finally, the papers [44], [41], [38] provide more results and ideas on the subject. Conformal volume and its relation with λ_1 were used, for instance, as obstructions to the exitance of maps between Riemann surfaces (see e. g. the proof of the *Surface Coverings Theorem* in [38, §4] or that of [44, Theorem 2.A_1]).[5)]

Next we recall the notion of *topological entropy*. Namely, given a (cubical for instance) partition Π of a compact topological manifold M one defines $\mathrm{ent}\,\Pi := \log \#\Pi$, where $\#$ is the number of elements in a partition. Let $\Pi(m), m \ge 1$, be the partition obtained by subdividing each "cube" from Π into $m^{\dim M}$ smaller "cubes". Then with any continuous self-map $f : M \longrightarrow M$ one associates an inf (denoted $\mathrm{ent}\, f$) of all $e \in \mathbb{R}$ such that there exist $k \gg 1$ for which $\lim\limits_{i\to\infty} \sup i^{-1} \mathrm{ent}\,(\Pi(m) \cap f^k(\Pi(m)) \cap \ldots \cap f^{ki}(\Pi(m))) \le ek$ (with arbitrary $m \ge 1$). Note that $\mathrm{ent}\, f$ does not depend on Π (hence on the metric, if any, on M) and is thus an asymptotic invariant of M. More similarity between $\mathrm{ent}\, f$ (especially when $f = \mathrm{id}$) and invariants $s(\cdot), m(\cdot)$, etc is provided by the

5)Observe an analogy with discussion in **A.1**: heuristically, taking inf in the definition of $V_c(d, M)$, say, corresponds to applying a "CLT reasoning" (i. e. one takes inf over an infinite number of "repetitions"), while sup corresponds to certain "mass concentration" on the resulting limit object (space, structure, etc).

Yomdin's theorem (and its proof), as discussed in [34]. (Roughly, the entropy measures the rate of growth of the quantity $\mathrm{Vol}(\square)^{1/m}$, $\square \in \Pi(m)$, when $m \to \infty$; compare with the notion of the C^r*-size* in [34, 3] of a subset $Y \subset \mathbb{R}^m$ and also with the Poincaré – Lelong formula in **1.3** above.) We refer to [34, 2.7], [40] and [2] (cf. [41, §4]) for further discussion on significance of the entropy for (algebraic) geometry/topology of M, as well as some computations of $\mathrm{ent}\, f$ for different f and M.

Another instance is the Borsuk – Ulam theorem and its vast generalizations in [37], concerning continuous maps $f : S^n \longrightarrow \mathbb{R}^k$ and volumes (with respect to a given concave measure μ on the sphere S^n) of ε-neighborhoods of the f-fibers, $\varepsilon \geq 0$, – the so-called *waists* $\mathrm{wst}\,(S^n \longrightarrow \mathbb{R}^k, \varepsilon)$. Basically, one estimates $\mathrm{wst}\,(S^n \longrightarrow \mathbb{R}^k, \varepsilon)$ from below in terms of the usual Euclidean $\mathrm{Vol}\,(S^{n-k} + \varepsilon)$, which is again a reminiscence of CLT and concentration of $\mu\big|_{S^{n-k}}$ at the center of mass of S^{n-k}.

Our next illustration concerns one instance of $\mathrm{lct}(\cdot)$ (cf. the end of **A.1**). Namely, the beautiful idea from [10] claims that all birational maps between two manifolds X and Y of general type 1 : 1 correspond to linear isometries between the pseudo-normed spaces $(H^0(X, mK_X), \langle\langle\ \rangle\rangle_m)$ and $(H^0(Y, mK_Y), \langle\langle\ \rangle\rangle_m)$, some $m \in \mathbb{N}$. Here $\langle\langle\eta\rangle\rangle_m$, for any form $\eta \in H^0(X, mK_X)$, equals the volume of X w. r. t. the (normalized) density induced by η. In turn, the asymptotics of $\langle\langle\eta_0 + t\eta\rangle\rangle_m$, for any fixed η_0 and variable $t \in \mathbb{C}$, is governed by the log canonical threshold of the divisor $D_0 := (\eta_0 = 0)$, and the points on both X and Y can be interpreted (roughly) as the loci where the value $\mathrm{lct}(D_0)$ is attained (see [10, Section 4]). This already suffices to recover birational maps from the stated linear isometries.

Last, but not the least, subject is on the *weight function* $\mathrm{wt}_\Delta : \widehat{\mathcal{X}_\eta} \longrightarrow \mathbb{R} \cup \{+\infty\}$, which one associates to a smooth algebraic variety X and a regular function f on X with the divisor $\Delta := (f = 0)$ (see [78, §6.1]). Here $\widehat{\mathcal{X}_\eta}$ denotes the completion (w. r. t. the t-adic topology) of the scheme $X \times_{\mathbb{C}[t]} \mathrm{Spec}\, \mathbb{C}[[t]]$ along the special fiber. Note that the dual complex of a log resolution (see [67, 2.3] for definitions) of the pair (X, Δ) naturally embeds into the $\mathbb{C}((t))$-analytic space $\widehat{\mathcal{X}_\eta}$ and is a deformation retract of the latter. This defines a *Berkovich skeleton* $\mathrm{Sk}\,\mathcal{X}$ in the $\mathbb{C}((t))$-analytic space X (polyhedron $\mathrm{Sk}\,\mathcal{X}$ is unique up to the embedding into X). The crucial property of $\mathrm{Sk}\,\mathcal{X}$ is that it, to some extent, captures the geometry of (X, Δ). Namely, the function wt_Δ turns out to be piecewise affine on the faces of $\mathrm{Sk}\,\mathcal{X}$, equal to the log discrepancy function $a(\star, X, \Delta) + 1$, weighted by $\mathrm{mult}_\star\Delta$, on the vertices of $\mathrm{Sk}\,\mathcal{X}$, and finally the minimal value of wt_Δ is attained on a certain face of $\mathrm{Sk}\,\mathcal{X}$ (see [79, Theorem 4.6] for precise formulations). Moreover, the latter face is a deformation retract of $\mathrm{Sk}\,\mathcal{X}$, which makes it plausible to think of (the "Morse function") wt_Δ and its properties on X as the right general framework for studying $s(\cdot), m(\cdot), \mathrm{lct}(\cdot)$ and related asymptotic invariants.

We conclude the present discussion by illustrations from Kähler geometry. This aims to (partially) justify the relation between birational and differential geometries pointed out in **A.1** (compare also with [80]). Again, it is impossible to give a more or less complete account here, so we will briefly mention just two instances (see [85] and references therein for an extensive collection of relevant notions and facts).

The first instance is a numerical characterization of the Kähler cone of an arbitrary compact Kähler manifold X (see [16, Theorem 0.1]). Basic idea is to replace (via the "mass concentration") any given nef $(1,1)$-class $\alpha \in H^2(X, \mathbb{R})$, satisfying $\displaystyle\int_X \alpha^n > 0$, by an analytic cycle Z whose δ-function determines a Kähler current on X. This allows one restrict to analytic subsets on X and argue by induction on the dimension.

The second instance is the problem of existence of extremal metrics on X. Once again, this a priori analytic problem (of convergence of the Kähler – Ricci flow for instance) can be replaced essentially by estimating, only in terms of $\dim X$, the injectivity radii of certain geodesic balls in X, which relates this subject to our earlier "microlocal" discussion. One may even reduce to the purely algebro-geometric problem on whether X is *stable* (for a given projective embedding). We will not specify this deep and beautiful notion here, referring to the survey *op.cit*, but mention only that existence of a constant scalar curvature metric on $X \subset \mathbb{P}^N$, with the group of automorphisms $\operatorname{Aut} X$ being discrete, implies the Chow – Mumford (or CM for short) stability of such X (see [18], [19]). Let us also indicate that various notions of stability are governed by certain (asymptotic) numerical invariants of a manifold X, such as the Futaki invariant, Chow character, alpha-invariant and the *Bergman function* in the case of CM-(semi)stability (see [74]).

The following is in line with what has been said above:

Conjecture A.4. *If the function $s(\cdot)$ (resp. $m(\cdot)$) is* measurable *with respect to $\omega_{FS}\big|_X$ for a projective embedding $X \subset \mathbb{P}^N$ of a complex manifold X, having* $\operatorname{Pic} X \simeq \mathbb{Z}$, *then X is CM-semistable.*

In the forthcoming examples we will provide some evidence for Conjecture A.4. Notice however that the assumption $\operatorname{Pic} \simeq \mathbb{Z}$ is really crucial here because of the examples of unstable surfaces (with $\operatorname{Pic} > \mathbb{Z}$), constructed in [92], and the results of [81].

A.5. Compactifications of $\mathbb{C}^n$. Let $G(3,7)$ be the Grassmannian of 3-dimensional linear subspaces in $\mathbb{C}^7$ and $\mathcal{U} \longrightarrow G(3,7)$ be the tautological bundle. Then given three global sections $\sigma_1, \sigma_2, \sigma_3$ of $\wedge^2\mathcal{U}^*$ in general position, the locus

$$X := G(3,7) \cap (\sigma_1 = \sigma_2 = \sigma_3 = 0)$$

is a smooth threefold. This is an example of a Fano threefold *of principle series* (i. e. $-K_X$ is ample and $\operatorname{Pic} X = \mathbb{Z} \cdot K_X$). Note that $(-K_X)^3 = 22$.

Example A.6. Let V_d be the space of binary forms of degree d. We may regard $\mathbb{C}^7 = V_6$ as a representation of $SL(2,\mathbb{C})$. Then there is a unique $SL(2,\mathbb{C})$-invariant linear subspace $V_2 \subset \bigwedge^2 \mathbb{C}^7$, spanned by some σ_i as above, so that the corresponding $X := X_{22}$ admits a regular $PSL(2,\mathbb{C})$-action. One computes $H^0(X, -K_X) = \mathbb{C} \oplus V_{12}$ as $SL(2,\mathbb{C})$-modules. This yields a point $p \in X$, invariant under the icosahedron subgroup $A_5 \subset PSL(2,\mathbb{C})$, and hence (Zariski) locally near p threefold X looks like a $PSL(2,\mathbb{C})$-orbit of an A_5-invariant form $\in \mathbb{P}(V_{12})$, which gives an open $PSL(2,\mathbb{C})$-orbit ($\simeq SL(2,\mathbb{C})/A_5$) on X. Furthermore, there is a unique 2-dimensional $PSL(2,\mathbb{C})$-orbit $F \subset X$, the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the morphism given by a linear subsystem in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(11,1)|$.[6] One finds that F is singular along a rational normal curve [7] (the image of the diagonal $\subset \mathbb{P}^1 \times \mathbb{P}^1$) and every line ℓ on $X \subset \mathbb{P}^{13}$ is contained in F (and is tangent to $\operatorname{Sing} F$). One also has $N_{\ell/X} = \mathcal{O}(1) \oplus \mathcal{O}(-2)$ for the normal bundle of (arbitrary) $\ell \subset X$. Finally, there is a surface $\Gamma \subset X$, singular along a line (hence $\Gamma \neq F$), such that $X \setminus \Gamma \simeq \mathbb{C}^3$.

Example A.7. Let $X = X_{22}$ be as in Example A.6. Fix a line $\ell \subset \Gamma$ and consider the *double projection* $\pi : X \dashrightarrow X_5$ (i. e. π is given by the linear system $|-K_X - 2\ell|$):

$$\begin{array}{ccc} Y & \overset{\chi}{\dashrightarrow} & Y^+ \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma^+} \\ X & \overset{\pi}{\dashrightarrow} & X_5 \end{array}$$

[6] F coincides with the union of orbits $PSL(2,\mathbb{C})x^{11}y \cup PSL(2,\mathbb{C})x^{12}$ for $x^{11}y, x^{12} \in V_{12}$.

[7] Notice that $\operatorname{Sing} F = PSL(2,\mathbb{C})x^{12}$ in the notation from the previous footnote.

Here σ is the blowup of ℓ, χ is a K_Y-flop, $X_5 \subset \mathbb{P}^6$ is a *del Pezzo threefold* (i. e. $-K_{X_5} = 2H$ with $\mathrm{Pic}\, X_5 = \mathbb{Z}\cdot H$) such that $H^3 = 5$ and $X_5 \setminus (\sigma^+ \circ \chi)_* E \simeq \mathbb{C}^3$ for the surface $E := \sigma^{-1}(\ell)$. Furthermore, in the notation of Example A.6 one may regard X_5 as the closure of a $PSL(2, \mathbb{C})$-orbit in $\mathbb{P}(V_6)$, so that $PSL(2, \mathbb{C}) \subseteq \mathrm{Aut}\, X_5$. Notice however that $\Gamma := (\sigma^+ \circ \chi)_* E \sim H$ is *not* a $PSL(2, \mathbb{C})$-orbit because otherwise its complement ($= \mathbb{C}^3$) would satisfy $\mathrm{Pic}\,(PSL(2, \mathbb{C})/S_4) \simeq \mathbb{Z}/2\mathbb{Z}$ for the octahedron subgroup $S_4 \subset PSL(2, \mathbb{C})$. Yet there are lines on X_5 that sweep out the 2-dimensional orbit F on X and have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.[8)] The surface $F \sim 2H$ admits a similar description to that in Example A.6. In particular, its normalization coincides with $\mathbb{P}^1 \times \mathbb{P}^1$ (for the normalization morphism given by a linear subsystem in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(5, 1)|$) and F is singular along the image of the diagonal, with $\mathrm{Sing}\, F$ being the 1-dimensional $PSL(2, \mathbb{C})$-orbit. Moreover, the map π^{-1} is given by the linear system $|3H - 2C|$, where $C \subset F$ is a rational normal curve of degree 5, the image of one of the rulings on $\mathbb{P}^1 \times \mathbb{P}^1$. Finally, X_5 is unique and can be obtained as a linear section of the Grassmannian $G(2, 5) \subset \mathbb{P}^9$, with $H = \mathcal{O}_X(1)$.

Example A.8. Let us mention two more examples of compactifications of $\mathbb{C}^3$ with $\mathrm{Pic} \simeq \mathbb{Z}$ and an infinite group of automorphisms. We proceed with notation of Example A.7 and consider the section $\langle C \rangle \sim H$ of X_5 by the linear span $\langle C \rangle$ of the curve $C \subset X_5$. We have $(\sigma^+ \circ \chi)_*^{-1} \langle C \rangle = E$ and the surface $(\sigma^{-1} \circ \chi)_*^{-1} E^+ = \Gamma$, where $E^+ := \sigma^{+-1}(C)$, has multiplicity 3 along the line ℓ. Furthermore, one can run this construction the other way around, i. e. start from X_5 and a $\mathbb{C}_+$-invariant curve C of degree 5, say, which yields a threefold X_{22}^a with $\mathbb{C}_+ \subseteq \mathrm{Aut}\, X_{22}^a$ as a finite index (algebraic) subgroup. This X_{22}^a is *not* isomorphic to X by construction. However, since all rational normal curves on X_5 of degree 5, not contained in F, are projectively equivalent on X_5 and $\langle C \rangle$ is the unique hyperplane section passing through the given C, one can easily see that $X_{22}^a \setminus \Gamma \simeq X_5 \setminus \langle C \rangle \simeq \mathbb{C}^3$ and $\mathrm{mult}_\ell\, \Gamma = 3$ as earlier. Similarly, starting from $\mathbb{C}^*$-invariant C we obtain a threefold X_{22}^m, with $\mathbb{C}^* \subseteq \mathrm{Aut}\, X_{22}^m$ as a finite index subgroup.

Remark A.9. At this stage (for dimensions $n \leq 3$), all compactifications of $\mathbb{C}^n$ with $\mathrm{Pic} \simeq \mathbb{Z}$ we have met so far are *not* $(\mathbb{C}_+)^n$-equivariant, except for $\mathbb{P}^n$ and the quadric. In fact, this is a typical situation, as is pointed out in Remark A.11 below. On the other hand, in higher dimensions there are such equivariant compactifications of $\mathbb{C}^n$ different from $\mathbb{P}^n$ and the quadric, as well as non-equivariant ones (see Example A.10 and Remark A.13 for further discussion; compare also with [22] and Remark 1.5 above). This suggests that higher-dimensional compactifications of $\mathbb{C}^n$ may perform rather unexpectedly.

Example A.10. Our last example is the fourfold $X := V_5^4 \subset \mathbb{P}^7$ of index 3 compactifying $\mathbb{C}^4$ and having $\mathrm{Pic}\, X = \mathbb{Z} \cdot H$ for some divisor H. Recall that X embeds into $\mathbb{P}^7$ via $|H|$ as a codimension 2 linear section of $G(2, 5) \subset \mathbb{P}^9$. Furthermore, X contains a unique plane $\Pi \sim \sigma_{2,2}$ (the corresponding Schubert cell) and all other planes ($\sim \sigma_{3,1}$) on X sweep out a divisor $R \sim H$, $\mathrm{Sing}\, R = \Pi$, such that the linear projection $X \dashrightarrow \mathbb{P}^4$ from Π is birational and contracts R to the twisted cubic. One can see that $X \setminus R \simeq \mathbb{C}^4$ and the group $\mathrm{Aut}\, X$ is an extension of $PSL(2, \mathbb{C})$ by $\mathbb{C}_+^5$ (with $\mathbb{C}^4 \subset X$ being an open $\mathrm{Aut}\, X$-orbit). Note also that every plane $\neq \Pi$ on X intersects Π at some line tangent to a fixed conic $C \subset \Pi$. Then the $\mathrm{Aut}\, X$-orbits are $X \setminus R$, $R \setminus \Pi$, $\Pi \setminus C$

8) All other lines on X_5, not contained in F, have normal bundle $\mathcal{O} \oplus \mathcal{O}$.

and C. Moreover, if $\Pi' \subset X$ is a $\sigma_{3,1}$-plane, then there is a commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{\sigma}\swarrow & & \searrow^{\varphi} \\ X & \underset{\pi}{\dashrightarrow} & Q \subset \mathbb{P}^4 \end{array}$$

where π is the projection from Π', σ is the blowup of Π', Q is a smooth quadric and φ induces a $\mathbb{P}^1$-bundle structure everywhere on W except for the fiber $= \sigma_*^{-1}\Pi$. There is hyperplane section $\Lambda \sim H$ of X passing through Π', singular along a line ℓ (so that Λ is swept by all the lines on X intersecting ℓ) and such that $X \setminus \Lambda \simeq Q \setminus \pi(\Lambda) \simeq \mathbb{C}^4$, where $\pi(\Lambda)$ is a cone. Finally, projection from generic $\ell \subset X, \ell \not\subset R$, yields a similar diagram of maps as above, but with π now being birational onto a 4-dimensional quadric Q such that the proper transform of a tangent cone to Q is a hyperplane section $\Lambda \subset X$ with only one singular point and $X \setminus \Lambda \simeq \mathbb{C}^4$.

Remark A.11. The preceding constructions are well-known and go back to papers [77], [70], [54], [55], [24], [25], [26], [28], [27], [29], [17], [86], [87] and [89]. Let us also mention that the threefolds $\mathbb{P}^3$, $Q \subset \mathbb{P}^4$, X_{22} and X_5 are the only (up to deformations) small compactifications of $\mathbb{C}^3$ (see [27], [84], [83]). In particular, this sets up the problem of finding all such compactifications of $\mathbb{C}^n$ for $n \leq 3$, since the case when $n \leq 2$ is obvious. In dimension $n = 4$ the picture is less complete, although the fourfolds $\mathbb{P}^4$, Q and V_5^4 exhaust all examples of compactifications of $\mathbb{C}^4$, having $\mathrm{Pic} \simeq \mathbb{Z}$ and Fano index ≥ 3 (see [84]).

Amongst the Fano manifolds we have considered there are "less homogeneous" ones – those not covered by lines. This makes one guess that such manifolds are *not* u. r. (cf. the discussion at the beginning of **A.1**). However, as Proposition A.12 shows, one does not obtain the needed examples this way (nevertheless the present constructions were the main motivation for those in Section 3).

Proposition A.12. *Manifolds from Examples A.6, A.8 (and more generally, every Fano threefold of principle series, having degree* 22*) and A.7, A.10 are uniformly rational.*

Proof. We begin with $X := X_5$. In the notation of Example A.7, it suffices to show $\mathrm{Sing}\, F \not\subset \Gamma$, so that the group $\mathrm{Aut}\, X$ "moves" the open chart $X \setminus \Gamma \simeq \mathbb{C}^3$ to cover any point on X. But the property $\mathrm{Sing}\, F \not\subset \Gamma$ is evident, for varying Γ, because there is no $\mathrm{Aut}\, X$-invariant rational normal curve $C \subset X$ of degree 5 (recall that $\langle C \rangle = \Gamma$).

Let us now turn to $X := X_{22}$ (this in fact can be *any* Fano threefold from the corresponding family). Pick a point $o \in X$. In the notation of Examples A.6, A.7, the map $\pi : X \dashrightarrow X_5$ is not defined only at ℓ and possibly at some other lines intersecting it. In other words, π is not defined only at a finite number of lines (cf. [57, Proposition 4.2.2, (iv)]), so that one may assume π to be *biregular* near o. Now recall that X_5 is unique and u. r. Then, since σ^+ is the blowup of $C \subset X_5$, the threefold Y^+ is u. r. as well (see [3, Proposition 2.6], [39, 3.5.E]). Thus, for X identified with Y^+ near o, there is a Zariski open subset $\subseteq \mathbb{C}^3$ on X containing o. Hence X is also u. r. because o was chosen arbitrarily.

Finally, to prove that $X := V_5^4$ is u. r. we recall that $X \setminus \Lambda \simeq \mathbb{C}^4$ for any $\sigma_{3,1}$-plane, with notation as in Example A.10. By description of the $\mathrm{Aut}\, X$-orbits on X, it suffices to show that $C \not\subset \Lambda$, since in this case the locus $X \setminus \Lambda$ will cover (via the $\mathrm{Aut}\, X$-action) any point on X. But the property $C \not\subset \Lambda$ is evident because $\Lambda \cap \Pi = \Pi' \cap \Pi$ is a line tangent to C.

Proposition A.12 is completely proved. ∎

Remark A.13. Proposition A.12 together with Propositions A.15 and A.19 below confirm the results of Section 2 (cf. the discussion in **1.3**). Furthermore, varieties X_{22}, $X_5, \ldots$ admit the "distribution-like" functions $s(\cdot)$ and $m(\cdot)$, which agrees with Conjecture A.4. More precisely, semistability for X_{22} and X_5 follows from the (highly non-trivial) papers [17], [18], [19], [9], while for X_{22}^a, X_{22}^m and V_5^4 this is not known.[9)] It would be interesting to explore these matters further (cf. Remark A.2).

A.14. Some computations. We are now going to carry out several computations for the functions $s(\cdot)$ and $m(\cdot)$ on some (not necessarily Fano, rational, etc) manifolds X with $\operatorname{Pic} X \simeq \mathbb{Z}$. Let us start by proving the next

Proposition A.15. *For the threefolds from Examples A.6, A.8 (and more generally, for every Fano threefold of principle series, having degree* 22 *and variety of lines F) the following holds:*

- $s(o) = 2$ *for any* $o \in X \setminus F$ *(resp.* $s(p) = 1$ *for any* $p \in F$*);*
- $m(o) = 3$ *for any* $o \in X \setminus F$ *(resp.* $m(p) = 8/3$ *for any* $p \in F$*).*

Proof. Fix $X := X_{22}$ for clarity (this in fact can be *any* Fano threefold from the corresponding family) and consider the function $s(\cdot)$ first. Take any $o \in X$ away from F. Then the blowup $\sigma : Y \longrightarrow X$ of o resolves the indeterminacies of the linear projection $X \dashrightarrow \mathbb{P}^9$ from the tangent space $T_{o,X}$. This shows that divisor $H - \lambda E$ is nef for all $\lambda \leq 2$ (we are using the notation from **1.3**) and so $s(o) \geq 2$.

Lemma A.16. $s(o) = 2$ *and* $s(p) = 1$ *for any* $p \in F$.

Proof. The first equality follows from $(H - \lambda E) \cdot \sigma_*^{-1} C < 0$ for all $\lambda > 2$ and a conic $C \subset X$ passing through o. Similarly, for $\lambda > 1$ and a line $\ell \subset X$ containing p, we get $s(p) \leq 1$. It remains to take a smooth hyperplane section of $X \subset \mathbb{P}^{13}$ passing through p and apply (1.4). ☐

We proceed with computing $m(p)$ for $p \in F$ (cf. **A.1**). Consider the double projection $\pi : X \dashrightarrow X_5$ from a line $\ell \ni p$ (see Example A.7). Notice that this yields $m(p) \geq 2$. In particular, if $\mathcal{M} \subseteq |kH|, k \geq 1$, is a mobile linear system with generic element $M \in \mathcal{M}$ passing through p, then one may assume that

$$\lambda := \frac{\operatorname{mult}_\ell M}{k} \geq 1.$$

Lemma A.17. $\lambda \leq 2$ *and the equality is attained on* $\mathcal{M} := |H - 2\ell|$.

Proof. Put $H^+ := (\chi \circ \sigma^{-1})_* H$, $M^+ := (\chi \circ \sigma^{-1})_* M$ and $E^+ := \chi_* E$ (in the notation of Example A.7). One can write

$$\frac{1}{k} M^+ \equiv H^+ + (1 - \lambda) E^+$$

on Y^+. Notice also that contraction $\sigma^+ : Y^+ \longrightarrow X_5$ is given by the linear system $|H^+ - E^+| = (\chi \circ \sigma^{-1})_* |H - 2\ell|$ and $\sigma^{+-1}(C) \equiv H^+ - 2E^+$. This shows that once $\lambda > 2$, we get $M^+ \cdot Z < 0$ for every curve Z contracted by σ^+, which implies that the proper transform of $\sigma^{+-1}(C)$ on X is a fixed component of $\mathcal{M}$, a contradiction. The last assertion of lemma is evident. ☐

[9)] Actually, since the group $\operatorname{Aut} V_5^4$ is not reductive, fourfold V_5^4 does not admit the Kähler – Einstein metric (see [75]). The same holds for X_{22}^a and X_{22}^m due to [96]. Thus the results of [18] and [19] are not applicable here.

It follows from Lemma A.17 that $\mathcal{M}_Y := \sigma_*^{-1}\mathcal{M} \subseteq |k(H_Y - E)|$ for $H_Y := \sigma_*^{-1}H = \sigma^*H - E$. Put $R := \sigma^{-1}(p)$ (i. e. R is a fiber on the ruled surface $E \simeq \mathbb{F}_3$)[10) and also

$$\mu := \frac{\operatorname{mult}_R M_Y}{k}$$

for $M_Y := \sigma_*^{-1}M$.

Lemma A.18. $m(p) = 8/3$. *More precisely, we have* $\mu \le 2/3$, *with equality attained on* $\mathcal{M}_Y = |3(H_Y - E) - 2R|$.

Proof. Note that

$$\frac{\operatorname{mult}_p M}{k} = \lambda + \mu \le 2 + \mu.$$

Hence it suffices to maximize μ. Now, in the notation from the proof of Lemma A.17, since σ^+ is the blowup of C, one computes

$$\sigma^{+-1}(C)^3 = K_{X_5} \cdot C + 2 - 2g(C) = -8.$$

We also have

$$H^{+3} = 18, \qquad H^{+2} \cdot E^+ = 3, \qquad H^+ \cdot E^{+2} = -2,$$

which together with $\sigma^{+-1}(C) \equiv H^+ - 2E^+$ gives $E^{+3} = -2$.

Further, since $H \cdot \ell = 1$, we have $R \equiv E \cdot (H + E)$. Also, as E^+ is a ruled surface obtained from E by elementary modifications, the curve $R^+ := \chi_* R$ is smooth and $\equiv E^+ \cdot (H^+ + E^+)$. Then we get

$$(H^+ - E^+) \cdot R^+ = (H^{+2} - E^{+2}) \cdot E^+ = 5.$$

This shows that $\sigma^+(R^+)$ is a smooth rational curve of degree 5.

On the other hand, from the construction of inverse $\pi^{-1} : X_5 \dashrightarrow X$ in Example A.7 we deduce that $\sigma \circ \chi^{-1} : Y^+ \dashrightarrow X$ is given by the linear system $|3(H^+ - E^+) - 2(H^+ - 2E^+)|$. This yields $\mu \le 2/3$ (by the same argument as in the proof of Lemma A.17), if we assume for a moment that $\sigma^+(R^+) = C$.

In general, both curves $\sigma^+(R^+)$ and $C \subset \sigma^+(E^+)$ are projectively equivalent, which provides a mobile linear system of cubic hypersurfaces in $\mathbb{P}^6$ passing through $\sigma^+(R^+)$ with multiplicity 2, thus maximizing μ to 2/3. □

We conclude by computing $m(o)$ for $o \in X \setminus F$. Recall that there is a *triple projection* $\pi : X \dashrightarrow \mathbb{P}^3$ from o (i. e. π is given by the linear system $|-K_X - 3o|$). In this case one has a similar diagram as in Example A.7, but with X_5 now being replaced by $\mathbb{P}^3$ and σ being the blowup of o (see e. g. [57, §4.5] or [94]). Then, arguing exactly as in the proof of Lemma A.17 we obtain that $m(o) = 3$, which concludes the proof of Proposition A.15. □

Let us now consider the case of X_5 from Example A.7:

Proposition A.19. *For* $X := X_5$ *the following holds:*

- $s(o) = 1$ *for any* $o \in X \setminus F$ *(resp.* $s(p) = 1$ *for any* $p \in F$*);*
- $m(o) = 2$ *for any* $o \in X \setminus F$ *(resp.* $m(p) = 2$ *for any* $p \in F$*).*

Proof. As in the proof of Proposition A.15, we start with $s(o), o \in X \setminus F$, keeping the same notation as before/in the proof of Lemma A.16. Notice that the threefold Y is (at least) a weak Fano because the divisor $-K_Y = 2(\sigma^*H - E)$ is nef and big.

[10)For generic X in the family containing X_{22}, one has $E \simeq \mathbb{F}_1$, which does not affect however the forthcoming arguments and shows again that $m(p) = 8/3$.

Lemma A.20. *The divisor $-K_Y$ is not ample.*

Proof. The blowup $\sigma : Y \longrightarrow X$ resolves indeterminacies of the linear projection $\pi : X \dashrightarrow \mathbb{P}^5$ from o. Furthermore, since $(\sigma^*H - E)^3 = 4$, the image $X' := \pi(X)$ is an intersection of two quadrics. Threefold X' is not isomorphic to Y because the latter is smooth and $\operatorname{rk}\operatorname{Pic} Y = 2$. In particular, we get $K_Y \cdot Z = 0$ for some curve $Z \subset Y$ contracted to X', hence the assertion. $\square$

Lemma A.20 shows that $s(o) = 1$ and the same argument as in the proof of Lemma A.16 gives $s(p) = 1$ for any $p \in F$.

Let us now compute $m(p)$ for $p \in F$. Pick a line $\ell \ni p$ (cf. Example A.7). Consider the linear projection $\pi : X \dashrightarrow \mathbb{P}^4$ from ℓ and the blowup $\sigma : Y \longrightarrow X$ of ℓ. It is easy to see that $(\sigma^*H - \sigma^{-1}(\ell))^3 = 2$, i. e. $X' := \pi(X)$ is a quadric. Moreover, from the description of the family of lines on X we obtain that X' is the cone over a smooth quadric $Q \subset \mathbb{P}^3$, with the (-2)-curve on $\sigma^{-1}(\ell) \simeq \mathbb{F}_2$ contracted to the vertex.

Lemma A.21. *$m(p) = 2$.*

Proof. Let $\mathcal{M} \subseteq |kH|, k \geq 1$, be a mobile linear system. Then, in the same notation as in/after the proof of Lemma A.17, we may assume that $\lambda \geq 1$.

Further, one computes

$$\sigma^*H \cdot E^2 = -1, \qquad E^3 = 0, \qquad (\sigma^*H - E)^2 \cdot E = 2,$$

and the latter shows that the image of the surface $E := \sigma^{-1}(\ell)$ on X' is a hyperplane section (passing through the vertex). In particular, the linear system $\pi_*\mathcal{M}$ on X' is cut out by hypersurfaces of degree $2k - k\lambda$, which gives $\lambda \leq 2$. Hence, as in the proof of Proposition A.15, we may assume that $\mathcal{M} \subseteq |kH - k\lambda\ell|$ and $\sigma_*^{-1}\mathcal{M}$ contains the ruling $R := \sigma^{-1}(p) \subset E$.

Notice that the image $\pi \circ \sigma(R)$ is a generating line on the cone X'. Then $\pi_*\mathcal{M} \subseteq |\mathcal{O}_{X'}(2k - k\lambda)|$ can be considered as lifted from a linear system on the base surface Q. The latter yields the multiplicity of $\pi_*\mathcal{M}$ along the line $\pi \circ \sigma(R)$ does not exceed $2k - k\lambda$. Thus, sticking again to the notation around Lemma A.17, we get $\mu \leq 2 - \lambda$. All together this gives $m(p) \leq \lambda + \mu \leq 2$ and finally $m(p) = 2$. $\square$

We proceed with computing $m(o)$ for $o \in X \setminus F$. Recall that there is a double projection $\pi : X \dashrightarrow \mathbb{P}^2$ from o (cf. the arguments after the proof of Lemma A.18). In this case one has a similar diagram as in Example A.7, but with X_5 now being replaced by $\mathbb{P}^2$ and σ being the blowup of o. Then, with the same notation as in the proof of Lemma A.17, the divisor E^+ is a multisection of the conic bundle $\sigma^+ : X^+ \longrightarrow \mathbb{P}^2$.

Lemma A.22. *$m(o) = 2$ and the equality is attained on $|H - 2o|$.*

Proof. Note that σ^+ is given by the linear system $|H^+ - E^+|$. Then, assuming that $\lambda > 2$, we obtain

$$(H^+ + (1 - \lambda)E^+) \cdot Z = (2 - \lambda)E^+ \cdot Z < 0$$

for every curve $Z \subset Y^+$ contracted by σ^+. This shows that the linear system $|k(H - \lambda o)|, k \gg 1$, can not be mobile. Thus we get $\lambda \leq 2$ and $m(o) = 2$ (the last assertion of lemma is obvious). $\square$

Lemma A.22 finishes the proof of Proposition A.19. $\square$

Remark A.23. Observe that the fourfold $X := V_5^4$ from Example A.10 admits a hyperplane section $= X_5$ passing through every point $o \in X$. This immediately gives $s(o) \leq 1$ and $m(o) \leq 2$, while the opposite estimates $s(o) \geq 1, m(o) \geq 2$ are evident.

A.24. Let us turn to a couple of non-rational examples. We first treat the case of generic quartic threefold $X := X_4 \subset \mathbb{P}^4$ and show that $m(\cdot) = 3/2$ identically on such X.

Remark A.25. To save the space we will skip the computation of the function $s(\cdot)$ on $X = X_4$ – as the very definition suggests, $s(\cdot)$ is finer than $m(\cdot)$, hence more difficult to compute. However, at a quick glance, it is tempting to propose that $s(o) = 4/3$ for every point $o \in X$ not on a line $\subset X$, and $s(o) = 1$ otherwise (for the former, one has to analyze the base locus of the linear system $|3H - 4E|$ as below, while the part with $s(o) = 1$ is clear).

Fix an arbitrary point $o \in X$. Consider the blowup $\sigma : Y \longrightarrow X$ of o and put $E := \pi^{-1}(o)$, $H := \pi^*(-K_X)$.

Lemma A.26 (I. Cheltsov)**.** *If o does not lie on a line on X, then $m(o) = 3/2$.*[11)]

Proof. Write the equation of X in the form

$$w^3x + w^2q_2 + wq_3 + q_4 = 0,$$

for some projective coordinates w, x, y, z, t, so that $q_i = q_i(x, y, z, t)$ are homogeneous forms of degree i and o has all coordinates $= 0$, except for $w = 1$.

Suppose that $m(o) > 3/2$. Then, with notation as above, the linear system $|nH - mE|$ does not have fixed components for some $m, n \in \mathbb{N}$ such that $m/n > 3/2$ (cf. **A.1**).

Let $\mathcal{M}$ be the linear system on X cut out by various cubics

$$xh^2 + (wx + q_2)h_1 + (w^2x + wq_2 + q_3)\lambda = 0,$$

where $h_i = h_i(x, y, z, t)$ are homogeneous of degree i and $\lambda \in \mathbb{C}$ is arbitrary. Then, since $q_i(0, y, z, t)$ do not have common zeroes (by the assumption on o), we get $\operatorname{Bs} \mathcal{M} = \emptyset$ for the base locus of $\mathcal{M}$.

Let T be the tangent hyperplane section at o and $D \in |nH - mE|, M \in \mathcal{M}$ some generic elements. Then we have

$$12n = M \cdot T \cdot \sigma_* D \geq \operatorname{mult}_o M \cdot \operatorname{mult}_o T \cdot \operatorname{mult}_o \sigma_* D = 8m,$$

which implies that $m/n \leq 3/2$, a contradiction.

Thus we get $m(o) \leq 3/2$. Equality $m(o) = 3/2$ now follows because the section of X by general quadric

$$\lambda x^2 + \mu(wx + q_2) = 0, \tag{A.27}$$

$\lambda, \mu \in \mathbb{C}$, is irreducible and has multiplicity 3 at o. □

According to Lemma A.26 we may (and will) assume in what follows that there are some (possibly multiple) lines $L_i \subset X$ passing through o. Let the surface $M \subset X$ be given by equation (A.27) for generic $\lambda, \mu \in \mathbb{C}$. Then we have $M_Y := \sigma_*^{-1} M \in |2H - 3E|$ and $T_Y := \sigma_*^{-1} T = H - 2E$. Furthermore, since the forms $q_2(0, y, z, t)$ and $q_3(0, y, z, t)$ are coprime (use [90, Proposition 1] and generality of X), we get

$$\operatorname{Bs} |M_Y| = M_Y \cdot T_Y = \sum_{i=1}^{k} L_{Y,i} + \Xi,$$

[11)]One does not need here the genericity assumption about X.

some $k \geq 1$, where $L_{Y,i} := \sigma_*^{-1} L_i$ and Ξ is an effective residual 1-cycle. Moreover, since the map τ (see the diagram (A.30) below) is an isomorphism near Ξ by construction and the linear system $|mM_{Y^+}|, m \gg 1$, is basepoint-free (see the argument right after the proof Lemma A.32), we obtain

$$\text{(A.28)} \qquad \operatorname{Bs}|mM_Y| = \sum_{i=1}^{k} L_{Y,i}.$$

Further, notice that the linear system $|m(H-E)|, m \gg 1$, determines a small birational morphism $f := \Phi_{|m(H-E)|} : Y \longrightarrow W$, which contracts only $L_{Y,i}$ and such that divisor M_Y is f-negative (for $M \cdot L_{Y,i} = (2H-3E) \cdot L_{Y,i} = -1$).

Lemma A.29. *T_Y has (at worst) canonical singularities.*

Proof. By generality of X the surface T does not have *bad points* in the sense of [8, Proposition 2.1.1], which implies that singularities of T and T_Y are canonical (cf. [67, Theorem 4.57]). □

It follows from the inversion of adjunction (see [67, 5.4]) and Lemma A.29 that the pair $(Y, T_Y + \varepsilon M_Y), 0 < \varepsilon \ll 1$, is purely log terminal near all the curves $L_{Y,i}$. Furthermore, we have $K_Y + T_Y + \varepsilon M_Y \equiv \varepsilon M_Y$, and hence one obtains a *pl flip* (see e. g. [66]):

$$\text{(A.30)} \qquad \begin{array}{ccc} Y & \overset{\tau}{\dashrightarrow} & Y^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & W & \end{array}$$

Here τ is an isomorphism in codimension 1 and for every curve $Z \subset Y^+$ contracted by f^+ we have $(K_{Y^+} + T_{Y^+} + \varepsilon M_{Y^+}) \cdot Z > 0$, where $T_{Y^+} := \tau_* T_Y, M_{Y^+} = \tau_* M_Y$. Furthermore, threefold Y^+ is $\mathbb{Q}$-factorial, the pair $(Y^+, T_{Y^+} + \varepsilon M_{Y^+})$ is purely log terminal (see [67, Proposition 3.36, Lemma 3.38]), $K_{Y^+} + T_{Y^+} \sim 0$ and the pair (Y^+, T_{Y^+}) is canonical (see [88, Lemma 3.1]).

Lemma A.31. *T_{Y^+} is a normal surface with $K_{T_{Y^+}} \sim 0$ and canonical singularities.*

Proof. Since the pair (Y^+, T_{Y^+}) is canonical, the surface T_{Y^+} is normal with canonical singularities, having $K_{T_{Y,1}} \sim 0$ (see e. g. [67, Proposition 5.51]). □

Further, consider some resolution of indeterminacies

$$\begin{array}{ccc} & V & \\ & {\scriptstyle g}\swarrow \quad \searrow{\scriptstyle g^+} & \\ Y & \underset{\tau}{\dashrightarrow} & Y^+ \end{array}$$

of τ over W, where we may take g to be a composition of blowups at smooth centers. Put $T_V := g_*^{-1} T_Y = g_*^{+-1} T_{Y^+}$ and denote the restriction of g (resp. g^+) to T_V by the same symbol. Then we have

$$\begin{array}{ccc} & T_V & \\ & {\scriptstyle g}\swarrow \quad \searrow{\scriptstyle g^+} & \\ T_Y & \underset{\tau}{\dashrightarrow} & T_{Y^+} \end{array}$$

a resolution of indeterminacies of the birational map $T_Y \dashrightarrow T_{Y^+}$ induced by τ, which we again denote by the same letter.

Lemma A.32. *$\tau : T_Y \dashrightarrow T_{Y^+}$ is a* regular *birational morphism, which coincides with the contraction of all $L_{Y,i}$, i. e. $\tau = f\big|_{T_Y}$ on T_Y.*

Proof. Notice that one may take $g : T_V \longrightarrow T_Y$ (resp. $g^+ : T_V \longrightarrow T_{Y^+}$) to factor through the minimal resolution of the surface T_Y (resp. T_{Y^+}) near the g-exceptional (resp. g^+-exceptional) curves.

Suppose τ is not that as stated. Then it follows from Lemmas A.29, A.31 and the identities $K_{T_Y} = 0$, $K_{T_{Y^+}} = 0$ that $\mathrm{Exc}\,(g^+) \subseteq \mathrm{Exc}\,(g)$ for the exceptional loci, and hence either τ is an isomorphism or τ^{-1} is a sequence of contractions of (-2)-curves. But then from (A.28), with *disjoint* $L_{Y,i}$, we deduce that

$$M_{Y^+} \cdot \tau_* L_{Y,i} \leq M_Y \cdot L_{Y,i} < 0$$

for all those i for which $L_{Y,i}$ is not contracted, a contradiction. Hence $\tau_* L_{Y,i}$ are points for all i. □

Lemma A.32, relation (A.28) and the construction of τ imply that divisor M_{Y^+} is nef and $M_{Y^+}\big|_{T_{Y^+}} \equiv 0$. Moreover, as the pair $(Y^+, T_{Y^+} + \varepsilon M_{Y^+})$ is purely log terminal and $K_{Y^+} + T_{Y^+} + \varepsilon M_{Y^+} \equiv \varepsilon M_{Y^+}$, the linear system $|mM_{Y^+}|$ is basepoint-free for $m \gg 1$ (see [62, Theorem 1.1]).

Lemma A.33. *M_{Y^+} is big.*

Proof. Assume the contrary. Put $H^+ := \tau_* H$. Then H^+ and E^+ generate the class group $\mathrm{Cl}\,Y^+$. Now, if $|mM_{Y^+}|$ is a pencil, then the induced morphism $Y^+ \longrightarrow \mathbb{P}^1$ will be an extremal contraction, which implies (due to $M_{Y^+}\big|_{T_{Y^+}} \equiv 0$) that the classes of $M_{Y^+} \equiv 2H^+ - 3E^+$ and $T_{Y^+} \equiv H^+ - 2E^+$ are linearly dependent in $N^1(Y^+) \otimes \mathbb{Q}$. But this is clearly impossible.

Thus M_{Y^+} must have Iitaka dimension 2. Note that the cone $\overline{NE}(Y^+)$ is generated by two contractible extremal rays, R_1 and R_2, say. Then the first one corresponds to f^+, while $R_2 \equiv M_{Y^+}^2$ (i.e. R_2 determines the morphism given by $|mM_{Y^+}|$). Then by construction of τ we get $T_{Y^+} \cdot R_1 > 0$ and $T_{Y^+} \cdot R_2 < 0$. On the other hand, we have $T_{Y^+} \cdot R_2 = M_{Y^+}^2 \cdot T_{Y^+} \geq 0$, a contradiction. □

The identity $M_{Y^+}\big|_{T_{Y^+}} \equiv 0$ and Lemma A.33 imply that the morphism $\Phi_{|mM_{Y^+}|}$ is birational and contracts T_{Y^+} to a point.

Lemma A.34. *$m(o) \leqslant 3/2$.*

Proof. Suppose that $m(o) > 3/2$. Then there exists $\delta > 0$ such that generic element $D \in |m(H - (3/2 + \delta)E)|, m \gg 1$, is an irreducible surface on Y. But then, since $\tau : Y \dashrightarrow Y^+$ is an isomorphism in codimension 1, we find that $D^+ := \tau_* D \sim m(H^+ - (3/2 + \delta)E^+)$ is also an irreducible surface on Y^+. On the other hand, from the preceding properties of $\Phi_{|mM_{Y^+}|}$ we deduce that

$$D^+ \cdot Z = m\big(H^+ - \big(3/2 + \delta\big)E^+\big) \cdot Z = -\delta E^+ \cdot Z < 0$$

for every curve $Z \subset T_{Y^+}$, which implies that $T_{Y^+} \subset D^+$, i.e. $T_{Y^+} = D^+$, a contradiction. □

From Lemma A.34 and the fact that $|M_Y| \neq \emptyset$ we obtain $m(o) = 3/2$.

Remark A.35. The above discussion shows that $s(\cdot) \neq m(\cdot)$ on a smooth quartic threefold $X = X_4$. Indeed, if X caries a *non-linear Halphen pencil*, then $m(o) = 2$ w. r. t. a special point $o \in X$ (see [7]), which implies that the function $m(\cdot)$ is not lower semi-continuous (cf. Lemma A.26), hence not equal to $s(\cdot)$. One can also easily see that for $o \in L_i$ in the previous notation, threefold Y is a Mori dream space, which complements the results of e. g. [58], where the case of the blowup of some manifolds with $\mathrm{Pic} \simeq \mathbb{Z}$ at *generic* points was considered. It would be interesting to find out whether this Mds property holds for the blowup at *any* point on the manifolds in question (or obtain some sort of a characterization, in terms of $s(\cdot), m(\cdot)$, etc, of when does Mds occur).

The last case we consider is that of K3 surfaces. Let us just quote the following result:

Theorem A.36 (see [64])**.** *If S is a* K3 *surface with* $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$ *for some ample divisor L such that* (L^2) *is a square, then* $s(\cdot) = \sqrt{(L^2)}$ *identically on S.*

Once again, we indicate that Propositions A.15, A.19 (cf. Remark A.23), previous computations for general quartic threefolds and Theorem A.36 confirm (on the speculative level so far) Conjecture A.4,[12)] since the functions $s(\cdot)$ and $m(\cdot)$ are obviously measurable with respect to the usual volume form coming from projective embeddings of the manifolds we have considered.

Remark A.37. It would be intersecting to compute the functions $s(\cdot)$ and $m(\cdot)$ for the manifolds X constructed in Section 3 (by using Proposition 3.5, say, together with the technique of **A.14** and **A.24**). It might also be worth to test for these X all matters raised in Remark A.2 (like Conjecture A.4 for instance). Note at this point that $(-K_X)^n = 3 \cdot 2^{n-3}$, but $-K_X$ is *not* in general very ample, for otherwise X with $n = 5$ would contain an elliptic curve of degree 2. Hence one lacks the explicit projective embedding for X (in addition, X has non-trivial moduli and no automorphisms — these claims are easily seen from the previous constructions, — i. e. one should not expect any "symmetric" defining equations here).

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[12)] Note that CM-stability of general quartic threefolds follows from [8, Theorem 1.1.6] and [95, 18].

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Kavli IPMU (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan
E-mail address: ilya.karzhemanov@ipmu.jp