Perverse Schobers

Mikhail Kapranov, Vadim Schechtman

November 12, 2014

The notion of a perverse sheaf, introduced in [BBD], has come to play a central role in algebraic geometry and representation theory. In particular, appropriate categories of perverse sheaves provide "categorifications" of various representation spaces, these spaces being recovered as the Grothedieck groups of the categories.

The goal of this paper is to suggest the possibility of categorifying the very concept of a perverse sheaf. In other words, we propose to develop a theory of perverse sheaves not of vector spaces but of *triangulated categories*.

Given a complex manifold X, an analytic Whitney stratification $\mathcal{S} = (X_{\alpha})_{\alpha \in A}$ of X and a ground field **k**, one has the category $\operatorname{Perv}(X, \mathcal{S})$ of perverse sheaves of **k**-vector spaces on X smooth with respect to S. Traditionally, there have been two ways of looking at $\operatorname{Perv}(X, \mathcal{S})$:

- (1) General definition: as an abelian subcategory in the triangulated category $D^b_{\text{constr}}(X, \mathcal{S})$ of constructible complexes of sheaves of **k**-vector spaces on X, smooth with respect to \mathcal{S} .
- (2) Quiver description (for some particular (X, S)): as a category of diagrams of some given type formed by vector spaces $(V_i)_{i \in I}$ and maps between them subject to certain relations. These diagrams have the following features:
 - (2a) Arrows come in pairs $V_i \rightleftharpoons V_j$ having the same ends but opposite directions. This reflects the (Verdier) self-duality of Perv (X, \mathcal{S}) .
 - (2b) In most cases, the relations contain 2 or 3 summands, with coefficients ± 1 .

So far, there is no obvious direct way to categorify the approach (1) since it is not clear what are complexes of triangulated categories.

On the other hand, we observe that the features (2a) and (2b) are of the kind that immediately suggest a categorical generalization. We can replace vector spaces V_i by triangulated categories \mathcal{V}_i and arrows by exact functors. The pairs of opposite arrows in (2a) can be interpreted as adjoint pairs of functors, 2-term relations as isomorphisms of functors and 3-term ones as exact triangles in appropriate functor categories (see Appendix for the precise framework in which these make sense). By forming Grothendieck groups $V_i = K_0(\mathcal{V}_i) \otimes \mathbf{k}$ of such a diagram of categories, one would then obtain a quiver in the original sense, i.e., a perverse sheaf. The idea that the cone can be seen as a categorical analog of the difference, lies, of course, at the very foundations of algebraic K-theory, especially in the Waldhausen approach.

This strongly suggests that there should be meaningful objects which can be understood as "perverse sheaves of triangulated categories" and which give usual perverse sheaves by passing to the Grothendieck groups. We propose to call such hypothetical objects *perverse Schobers* (or, sometimes, for brevity, simply *Schobers*), using the German analog¹ of the English word "stack" which would be the correct (but overused) term for speaking of "sheaves of categories".

In this paper we work out several basic examples of quiver descriptions of perverse sheaves and define, in an *ad hoc* way, what should be the perverse Schobers in these situations. In the simplest case, we propose, in §1, to identify perverse Schobers on a disk with one allowed singular point, with *spherical functors* of [A1][AL2].

For a disk with several allowed singular points we propose, in §2, a definition in terms of certain diagrams of spherical functors, and explain the invariance properties of such a definition. Among other things, we reformulate the classical Picard-Lefschetz formula as a general statement about perverse sheaves on a disk, and then lift it to a distinguished triangle associated to a perverse Schober and a certain configuration of paths. Such "Picard-Lefschetz triangles" should therefore be considered as fundamental features of perverse Schobers. It is natural to expect analogous features in

¹A literal Russian analog would be the word cror. We learned the term "Schober" from W. Soergel.

the case $\dim(X) > 1$ and even consider them as "codimension 1 data" of a perverse Schober.

A series of examples of spherical functors is provided by representation theory. More precisely, for a reductive group G we have spherical functors acting in the derived category of sheaves on G/B and satisfying the relations of the corresponding braid group $Br(\mathfrak{g})$. In §4 we review these examples and suggest a conjectural interpretation in terms of perverse Schobers on \mathfrak{h}/W the space of conjugacy classes of semisimple elements in the Lie algebra \mathfrak{g} .

Most of the known quiver descriptions of $\operatorname{Perv}(X, \mathcal{S})$ can be obtained using a choice of "cuts" which are certain totally real subvarieties $K \subset X$ (of real dimension equal to $\dim_{\mathbb{C}} X$). In §5 we summarize the features of such cuts and note that Lagrangian varieties, used in constructing Fukaya categories [FO³][Se], provide a reasonable class of candidates for cuts. One can therefore expect Fukaya-categorical constructions to have a bearing on the problem of classification of perverse sheaves.

Additionally, we discuss the idea of defining "Fukaya categories with coefficients". This idea was proposed by M. Kontsevich in order to study the usual Fukaya category of a manifold by fibering it over a manifold of smaller dimension. We suggest that perverse Schobers should be considered as the right "coefficient data" for such a definition, just like sheaves are natural coefficient data for defining cohomology.

We would like to thank A. Bondal, V. Ezhov, M. Finkelberg, P. Schapira, W. Soergel, Y. Soibelman and B. Toën for useful discussions and correspondence. V.S. is grateful to the Kavli IPMU for hospitality and support during the visit when this paper was finished. The work of M. K. was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

1 Perverse Schobers on a disk: spherical functors

A. Disk with one marked point. Let Δ be the unit disk in \mathbb{C} and $Perv(\Delta, 0)$ be the category of perverse sheaves on Δ with the only possible singularity at 0. The most iconic example of a quiver description of perverse sheaves is the following classical statement [Be] [GGM].

Theorem 1.1. Perv(Δ , 0) is equivalent to the category \mathcal{P}_1 of quadruples (Φ, Ψ, u, v) where $\Phi, \Psi \in \text{Vect}_{\mathbf{k}}$ and

$$(1.2) \qquad \Phi \xleftarrow{v}{\overleftarrow{}} \Psi$$

are linear maps such that

(1.3) $T_{\Psi} := \mathrm{Id}_{\Psi} - vu \text{ is an isomorphism.}$

Exercise 1.4. Show that T_{Ψ} is an isomorphism iff T_{Φ} : $\mathrm{Id}_{\Phi} - uv$ is an isomorphism.



Figure 1: Defining Φ and Ψ topologically.

One way of constructing an explicit equivalence is as follows [GGM]. Choose a base point b on the boundary of Δ and connect it with 0 by a simple arc K, see Fig. 1. Then to $\mathcal{F} \in \text{Perv}(\Delta, 0)$ we associate the spaces

$$\Phi(\mathcal{F}) := \underline{\mathbb{H}}_{K}^{1}(\mathcal{F})_{0} \simeq \mathbb{H}_{K}^{1}(\Delta, \mathcal{F}) \quad \text{(vanishing cycles)},$$

$$\Psi(\mathcal{F}) := \mathcal{F}_{b} \simeq \underline{\mathbb{H}}_{K}^{1}(\mathcal{F})_{b} \quad \text{(nearby cycles)}.$$

(We recall that $\mathcal{F}|_{\Delta-\{0\}}$ is a local system in degree 0). The map $v = v_{\mathcal{F}}$ is the generalization map [GM][Cu] for the constructible sheaf $\underline{\mathbb{H}}_{K}^{1}(\mathcal{F})$ on K, and u is the composition

(1.5)
$$\mathcal{F}_b \xrightarrow{\text{counterclockwise}} H^0(\Delta - K, \mathcal{F}) \xrightarrow{\delta} \mathbb{H}^1_K(\Delta, \mathcal{F}).$$

Remark 1.6. Further, we have the following elementary statements which we recall here as indicative of a certain pattern.

- (1) $\underline{\mathbb{H}}_{K}^{i}(\mathcal{F}) = 0 \text{ for } i \neq 1.$
- (2) The sheaf $\mathcal{R}(\mathcal{F}) = \underline{\mathbb{H}}_{K}^{1}(\mathcal{F})$ on K is constant on $K \{0\}$ so it has only two essentially different stalks $\mathcal{R}(\mathcal{F})_{0} = \Phi(\mathcal{F})$ and $\mathcal{R}(\mathcal{F})_{b} = \Psi(\mathcal{F})$.
- (3) Each of the two stalks, considered as a functor $\text{Perv}(\Delta, 0) \rightarrow \text{Vect}_{\mathbf{k}}$, is an exact functor which takes Verdier duality to vector space duality.
- (4) The map $u = u_{\mathcal{F}}$ is the dual $u_{\mathcal{F}} = (v_{\mathcal{F}^*})^*$.

B. Spherical functors. As a natural categorical analog of the data (1.2)-(1.3) we would like to suggest the following remarkable concept introduced by R. Anno [A1].

Definition 1.7. Let

$$S: \mathcal{D}_0 \longrightarrow \mathcal{D}_1$$

be an exact functor between triangulated categories (see Appendix for conventions). Assume that S admits a left adjoint L and a right adjoint R, so that we have the unit and counit natural transformations

$$SR \Rightarrow \mathrm{Id}_{\mathcal{D}_1}, \quad LS \Rightarrow \mathrm{Id}_{\mathcal{D}_0},$$
$$\mathrm{Id}_{\mathcal{D}_0} \Rightarrow RS, \quad \mathrm{Id}_{\mathcal{D}_1} \Rightarrow SL,$$

whose cones will be denoted by

$$T_1 = \operatorname{Cone}\{SR \Rightarrow \operatorname{Id}_{\mathcal{D}_1}\}, \quad T'_1 = \operatorname{Cone}\{\operatorname{Id}_{\mathcal{D}_1} \Rightarrow SL\}[-1] \quad \text{(the twist functors)}, \\ T_0 = \operatorname{Cone}\{\operatorname{Id}_{\mathcal{D}_0} \Rightarrow RS\}[-1], \quad T'_0 = \operatorname{Cone}\{LS \Rightarrow \operatorname{Id}_{\mathcal{D}_0}\} \quad \text{(the cotwist functors)}.$$

We call S a spherical functor, if:

(SF1) T_1 is an equivalence.

(SF2) The composition $R \to RSL \to T_0L[1]$ is an isomorphism. In other words, composition with T_0 identifies R and L.

In this case T'_1 is quasi-inverse to T_1 and T'_0 is quasi-inverse to T_0 .

More precisely, each of the pairs of adjoint functors

(1.8)
$$\mathcal{D}_0 \xrightarrow{S}_R \mathcal{D}_1, \quad \mathcal{D}_1 \xrightarrow{S}_L \mathcal{D}_0$$

can be regarded as an analog of (1.2). Further, (SF1) is an analog of (1.3), the adjunction unit allowing us to take the "categorical difference", i.e., the cone. Further still, the categorical analog of Exercise 1.4 can be found in the following result of Anno and Logvinenko [AL2].

Theorem 1.9. In addition to (SF1) and (SF2) consider the following two conditions:

- (SF3) T_0 is an equivalence.
- (SF4) The composition $LT_1[-1] \rightarrow LSR \rightarrow R$ is an isomorphism.

Then, any two of the conditions (SF1)-(SF4) imply the other two.

So we can consider a diagram (1.8), i.e., the data of a spherical functor, as the data defining a "perverse Schober" over $(\Delta, 0)$. By passing to K_0 (or to any K_i , or to any homological functor, Hochschild homology for example) we get a perverse sheaf over $(\Delta, 0)$.

B. Examples of spherical functors. We now give some examples, to be used later.

Example 1.10. Let \mathbb{S}^d be the *d*-dimensional sphere and $q : \mathbb{S}^d \to \text{pt}$ be the projection. We then have the functor

$$\mathcal{D}_0 := D^b(\operatorname{Vect}) = D^b(\operatorname{pt}) \xrightarrow{S=q^{-1}} D^b(\mathbb{S}^d) =: \mathcal{D}_1$$

with right adjoint $R = Rq_*$ and left adjoint $L \simeq Rq_*[-d]$. The second adjunction is the Poincaré duality. Formaly, it comes from the adjoint pair $(Rq_!, q^!)$ by noticing that $q^! \simeq q^{-1}[d]$ (since q is smooth orientable of relative dimension d) and that $Rq_! = Rq^*$ (since q is proper), see [KS2] for background. More intrinsically, L is canonically identified with the tensor product of $Rq_*[-d]$ and $H^d(\mathbb{S}^d, \mathbf{k})$, the 1-dimensional **k**-vector space spanned by global orientations of \mathbb{S}^d . **Proposition 1.11.** (a) S is a spherical functor.

(b) More generally, for any \mathbb{S}^d -fibration $q: Z \to Y$ of CW-complexes, the functor

$$\mathcal{D}_0 := D^b(Y) \xrightarrow{S=q^{-1}} D^b(Z) =: \mathcal{D}_1$$

is a spherical functor.

Proof: We prove (a), since (b), being a relative version, is proved in the same way.

The functor $T_1 = \text{Cone}\{SR \Rightarrow \text{Id}_{\mathcal{D}_1}\}\$ is the endomorphism of $D^b(S^d)$ defined as follows. Let

$$U = (\mathbb{S}^d \times \mathbb{S}^d) - \Delta \stackrel{j}{\hookrightarrow} \mathbb{S}^d \times \mathbb{S}^d$$

be the embedding of the complement of the diagonal, $\pi_1, \pi_2 : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ be the projections and $p_1, p_2 : U \to \mathbb{S}^d$ be their restrictions to U. Then

$$T_1(\mathcal{F}) = Rp_{2!}(p_1^*\mathcal{F}),$$

a formula remindful of the Fourier-Sato transform relating sheaves on dual spheres [SKK]. To see this, we write the functor in the RHS in terms of a "kernel", as

$$R\pi_{2*}((\pi_1^*\mathcal{F})\otimes_{\mathbf{k}}\mathcal{K}), \quad \mathcal{K}=j_!\underline{\mathbf{k}}_U[1] \in D^b(\mathbb{S}^d \times \mathbb{S}^d)$$

and note the exact sequence

$$0 \to \mathcal{K}[-1] \longrightarrow \underline{\mathbf{k}}_{\mathbb{S}^d \times \mathbb{S}^d} \longrightarrow \underline{\mathbf{k}}_{\Delta} \to 0,$$

in which $\underline{\mathbf{k}}_{\mathbb{S}^d \times \mathbb{S}^d}$ is the kernel for SR while $\underline{\mathbf{k}}_{\Delta}$ is the kernel for $\mathrm{Id}_{\mathcal{D}_1}$.

The twist $T'_1 = \operatorname{Cone} \{ \operatorname{Id}_{\mathcal{D}_1} \Rightarrow SL \} [-1]$ can be found explicitly as

$$T'_{1}(\mathcal{F}) = Rp_{2*}(p_{2}^{*}\mathcal{F}))[-1] = R\pi_{2*}((\pi_{1}^{*}\mathcal{F}) \otimes_{\mathbf{k}} \mathcal{K}'), \quad \mathcal{K}' = Rj_{*}\mathbf{k}_{U}[-1].$$

Now, the condition (SF1), i.e., the fact that T_1 and T'_1 are quasi-inverse to each other, can be established directly by finding the "composition" of the kernels \mathcal{K} and \mathcal{K}'

$$\mathcal{K} * \mathcal{K}' = R\pi_{13*}(\pi_{12}^{-1}\mathcal{K} \otimes_{\mathbf{k}} \pi_{23}^{-1}\mathcal{K}'), \quad \pi_{ij} : \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \times \mathbb{S}^d,$$

and showing that both $\mathcal{K} * \mathcal{K}'$ and $\mathcal{K}' * \mathcal{K}$ are isomorphic to $\underline{\mathbf{k}}_{\Delta}$ in degree 0. This amounts to the fact that for distinct points $x, y \in \mathbb{S}^d$ we have

$$H^{\bullet}\left(\mathbb{S}^{d} - \{x\}, j_{!}\left(\underline{\mathbf{k}}_{\mathbb{S}^{d} - \{x,y\}}\right)\right) = 0,$$

while for x = y we have $H^{\bullet}(\mathbb{S}^d - \{x\}) = \mathbf{k}$ and $H^{\bullet}_c(\mathbb{S}^d - \{x\}) = \mathbf{k}[-d].$

Further, the cotwist T_0 is the shift by (-d) tensored with $H^d(\mathbb{S}^d, \mathbf{k})$, the 1-dimensional vector space of orientations of \mathbb{S}^d . So it is an equivalence and (SF2) is also satisfied.

Example 1.12. Note the particular case d = 2, when $\mathbb{S}^2 = \mathbb{CP}^1$. Proposition 1.11 implies that for any \mathbb{P}^1 -fibration $q: Z \to Y$ of complex algebraic varieties, $q^{-1}: D^b(Y) \to D^b(Z)$ is a spherical functor. Another class of examples is provided by quaternionic geometry, since $\mathbb{HP}^1 = \mathbb{S}^4$.

Let us now mention some "coherent" examples.

Example 1.13. Recall that an *n*-dimensional smooth projective variety Z over **k** is called *Calabi-Yau* (in the strict sense), if

$$H^{i}(Z, \mathcal{O}_{Z}) = \begin{cases} \mathbf{k}, & \text{if } i = 0, \text{ or } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Let X be a smooth algebraic variety over \mathbf{k} , and $q: Z \to X$ be a smooth proper family of Calabi-Yau manifolds. Then the pullback functor $q^*: D^b_{\rm coh}(X) \to D^b_{\rm coh}(Z)$ is a spherical functor. The proof is similar to that of Proposition 1.11.

Example 1.14. ([A1]) Consider a diagram

$$Y \xleftarrow{\rho} D \xleftarrow{\imath} X$$

of smooth complex varieties X, D, Y, where ρ is a \mathbb{P}^1 -bundle and i an embedding of a divisor. We then have a diagram of adjoint functors

$$\mathcal{D}_{0} := D^{b}_{\mathrm{coh}}(Y) \xrightarrow{\substack{K = \rho_{*}i^{*} \\ R = \rho_{*}i^{!}}} D^{b}_{\mathrm{coh}}(X)$$

and $R = T_0 L$ where $T_0 = \text{Cone}\{\text{Id}_{\mathcal{D}_0} \Rightarrow RS\}.$

Lemma 1.15. (*R. Anno,* [A1]). *S* is spherical iff the intersection index of D with a generic fiber of ρ is (-2).

Example 1.16. A particular case of Example 1.14 and Lemma 1.15 is obtained for

$$Y = \mathrm{pt} \xleftarrow{\rho} D = \mathbb{P}^1 \xleftarrow{i} X = T^* \mathbb{P}^1$$

The corresponding spherical functor can be seen as a "quasi-classical approximation" to that in Example 1.12.

Remark 1.17. More precisely, for a complex algebraic manifold Z, the coherent derived category $D^b_{\text{coh}}(T^*Z)$ can be thought of as a quasi-classical approximation to the derived category $D^b(Z)$ of arbitrary sheaves on Z. Indeed, passing to solutions of \mathcal{D} -modules gives a functor between derived categories in the first line of the following table (a functor that restricts to the Riemann-Hilbert equivalence between constructible and holonomic regular derived categories). It can be compared with the second line which is an instance of Serre's theorem for the affine morphism $p: T^*Z \to Z$.

(Arbitrary) sheaves on ZCoherent
$$\mathcal{D}_Z$$
-modules on ZCoherent sheaves on T^*Z Coherent $p_*\mathcal{O}_{T^*Z} = \operatorname{gr}(\mathcal{D}_Z)$ -
modules on Z

Note that the functor $S = q^{-1}$ on sheaf-theoretic derived categories in Example 1.12 matches, after being interpreted in terms of \mathcal{D} -modules and passing to the associated graded, the functor $S = i_* \rho^*$ on coherent derived categories in Example 1.16.

2 Disk with several marked points

A. Quiver description of perverse sheaves. Let $B = \{b_1, \dots, b_n\}$ be a finite set of marked points in the unit disk Δ and $Perv(\Delta, B)$ be the category of perverse sheaves on Δ with possible singularities at B. We then have the following [GMV, Prop. 1.2].

Proposition 2.1. Perv (Δ, B) is equivalent to the category \mathcal{P}_n of diagrams formed by vector spaces $\Psi, \Phi_1, \cdots, \Phi_n$ and linear maps $\Phi_i \rightleftharpoons_{u_i}^{v_i} \Psi$ such that each $T_{\Psi,i} := \mathrm{Id}_{\Psi} - v_i u_i$ is an isomorphism.

The category \mathcal{P}_n can be seen as an amalgamation of n copies of the category \mathcal{P}_1 from Theorem 1.1. To construct an equivalence, we choose a base point $b \in \partial \Delta$ and a "system of cuts" K, i.e., a set of simple arcs $\{K_1, \dots, K_n\}$ with K_i connecting b with b_i and with different K_i meeting only at a small common interval near b, see Fig. 2. Notationally, we view K as the union $K = \bigcup K_i \subset \Delta$.



Figure 2: Equivalence depending on a system of cuts.

Given K, we have an equivalence $F_K : \operatorname{Perv}(\Delta, B) \to \mathcal{P}_n$ sending \mathcal{F} to

(2.2)
$$\Phi_i^K(\mathcal{F}) = \underline{\mathbb{H}}_K^1(\mathcal{F})_{b_i}, \quad \Psi^K(\mathcal{F}) = \mathcal{F}_b \simeq \underline{\mathbb{H}}_{K_i}^1(\mathcal{F})_b = \underline{\mathbb{H}}_K^1(\mathcal{F})_b.$$

Remark 2.3. We have the following elementary statements, continuing the pattern of Remark 1.6.

- (1) $\underline{\mathbb{H}}_{K}^{i}(\mathcal{F}) = 0 \text{ for } i \neq 1.$
- (2) Each stalk of $\mathcal{R}(\mathcal{F})$, considered as a functor $\operatorname{Perv}(\Delta, B) \to \operatorname{Vect}_{\mathbf{k}}$, is an exact functor which takes Verdier duality to vector space duality.

Remark 2.4. The space of cohomology with support $\mathbb{H}^1_K(\Delta, \mathcal{F})$ can be seen as "uniting" all the spaces of vanishing cyclies $\Phi^K_i(\mathcal{F})$. As common in singularity theory, we can imagine that \mathcal{F} obtained as a deformation of a perverse sheaf $\mathcal{G} \in \operatorname{Perv}(\Delta, 0)$ with only one (but more complicated) singular point at 0. Then $\mathbb{H}^1_K(\Delta, \mathcal{F})$ recovers $\Phi(\mathcal{F})$. Note that a categorification of the space of vanishing cycles of an isolated singular point of a function is provided by the Fukaya-Seidel category [Se], and the method of construction adopted in *loc. cit.* uses precisely a deformation into several Morse critical points. Therefore, the space $\mathbb{H}^1_K(\Delta, \mathcal{F})$ for a perverse sheaf $\mathcal{F} \in \operatorname{Perv}(\Delta, B)$, can be seen as a de-categorified analog of the Fukaya-Seidel category.

The equivalence F_K depends only on the isotopy class of K. Unlike the one point case, there are now many such classes, forming a set which we denote C. It is acted upon simply transitively by the Artin braid group

$$\operatorname{Br}_{n} = \langle s_{1}, \cdots, s_{n-1} | s_{i}s_{i+1}s_{i} = s_{i+1}s_{i}s_{i+1} \rangle.$$

Indeed,

$$\operatorname{Br}_n = \pi_0 \operatorname{Diff}^+(\Delta; B, b)$$

is the group of isotopy classes of diffeomorphisms of Δ , preserving orientation, preserving b as a point and B as a set. The equivalences F_K for different $K \in \mathcal{C}$ are connected by self-equivalences f_{σ} of \mathcal{P}_n :

(2.5)
$$\mathcal{P}_{n} \xrightarrow{f_{\sigma}} \mathcal{P}_{n}, \quad \sigma \in \operatorname{Br}_{n},$$
$$F_{K} \xrightarrow{F_{\sigma(K)}} \operatorname{Perv}(\Delta, B)$$

The self-equivalence f_{s_i} corresponding to a generator s_i of Br_n , is given by [GMV, Prop. 1.3]:

(2.6)

$$\begin{aligned}
f_{s_i}(\Psi, \Phi_j, u_i, v_j) &= (\Psi, \Phi'_j, u'_j, v'_j), \\
\Psi' &= \Psi, \quad \Phi'_j = \Phi_j, \quad u'_j = u_j, \quad v'_j = v_j, \quad j \neq i, i+1, \\
\Phi'_{i+1} &= \Phi_i, \quad \Phi'_i = \Phi_{i+1}, \\
u'_i &= u_{i+1}, \quad v'_i = v_{i+1}, \quad u'_{i+1} = u_i T_{\Psi,i+1}, \quad v'_{i+1} = T_{\Psi,i+1}^{-1} v_i.
\end{aligned}$$

Remarks 2.7. (a) Note that perverse sheaves being a topological concept, the group $\text{Diff}^+(\Delta; B, b)$ naturally acts on the category $\text{Perv}(\Delta, B)$ from the first principles, the action descending to that of Br_n .

(b) We can turn Proposition 2.1 and formulas (2.6) around to produce an intrinsic (i.e., not tied to any particular K and manifestly Br_n -equivariant) definition of $\operatorname{Perv}(\Delta, B)$ which does not appeal to any pre-existing concept of a perverse sheaf. More precisely, we can define an object $P \in \operatorname{Perv}(\Delta, B)$ to be a system of objects $P_K \in \mathcal{P}_n, K \in \mathcal{C}$ and compatible isomorphisms $f_{\sigma}(P_K) \to P_{\sigma(K)}, \sigma \in \operatorname{Br}_n$ so that each particular P_K is just a particular "shadow" of a more intrinsic object P.

B. Schobers on a disk with several marked points. To give an "invariant" definition of a perverse Schober on (Δ, B) , we adopt the approach of Remark 2.7(b). That is, for each system of cuts $K \in \mathcal{C}$ we define a *Kcoordinatized Schober* to be a system of *n* spherical functors with a common target

$$\mathfrak{S}_K = \{S_i : \mathcal{D}_i \longrightarrow \mathcal{D}, \quad i = 1, \cdots, n\}.$$

Each \mathfrak{S}_K gives rise to spherical reflection functors $T_i = \operatorname{Cone}\{S_i R_i \Rightarrow \operatorname{Id}_{\mathcal{D}}\}, i = 1, \cdots, n$. According to our convention on working with triangulated categories in terms of dg-enhancements, see Appendix, all *K*-coordinatized Schobers form an ∞ -category which we denote $\operatorname{Sch}_K(\Delta, B)$. We define equivalences $\mathbf{f}_{\sigma} : \operatorname{Sch}_K(\Delta, B) \to \operatorname{Sch}_{\sigma(K)}(\Delta, B), \sigma \in \operatorname{Br}_n$ on generators $s_i \in \operatorname{Br}_n$ by the direct analog of (2.6):

(2.8)
$$\begin{aligned} \mathbf{f}_{s_i} \{ S_i : \mathcal{D}_i \longrightarrow \mathcal{D} \} &= \{ S'_i : \mathcal{D}'_i \longrightarrow \mathcal{D}' \}, \\ \mathcal{D}' &= \mathcal{D}, \ \mathcal{D}'_j = \mathcal{D}_j, \ S'_j = S_j, \quad j \neq i, i+1, \\ \mathcal{D}'_{i+1} &= \mathcal{D}_i, \ \mathcal{D}'_i = \mathcal{D}_{i+1}, \\ S'_i &= S_{i+1}, \ S'_{i+1} = T_{i+1}^{-1} S_i. \end{aligned}$$

We then extend to arbitrary $\sigma \in Br_n$ by verifying the braid relations for the \mathbf{f}_i which is done in exactly the same way as for (2.6).

By definition, a perverse Schober on (Δ, B) is a system $\mathfrak{S} = (\mathfrak{S}_K)_{K \in \mathcal{C}}$ of coordinatized Schobers and compatible identifications $\mathbf{f}_{\sigma}(\mathfrak{S}_K) \to \mathfrak{S}_{\sigma(K)}$. The datum \mathfrak{S}_K will be referred to as the *K*-shadow of \mathfrak{S} . We denote by $\mathrm{Sch}(\Delta, B)$ the ∞ -category of perverse Schobers on (Δ, B) . **C. The Picard-Lefschetz formula.** Underlying classical Picard-Lefschetz theory, there is a general statement about perverse sheaves on a disk which we now formulate.

Let $\mathcal{F} \in \text{Perv}(\Delta, B)$. In the approach of (2.2), "the" space of vanishing cycles of \mathcal{F} at some $b_i \in B$ can be defined in terms of a small segment of an arc terminating in b_i (which does not, *a priori*, have to be a part of a system of cuts). Let now γ be a simple arc joining two marked points b_i and b_k and not passing through any other marked points, as in Figure 3.



Figure 3: The Picard-Lefschetz situation.

We can then define the spaces of vanishing and nearby cycles of \mathcal{F} relative to γ :

$$\Phi_{i,\gamma} = \underline{\mathbb{H}}_{\gamma}^{1}(\mathcal{F})_{b_{i}}, \quad \Phi_{k,\gamma} = \underline{\mathbb{H}}_{\gamma}^{1}(\mathcal{F})_{b_{k}},$$
$$\Psi_{\gamma} = \underline{\mathbb{H}}_{\gamma}^{1}(\mathcal{F})_{\text{gen}} \text{ (generic stalk)}.$$

which are connected by the maps

$$\Phi_{i,\gamma} \xrightarrow{v_{i,\gamma}} \Psi_{\gamma} \xrightarrow{v_{k,\gamma}} \Phi_{k,\gamma}$$

The definition of these maps is similar to §1A: the maps v are generalization maps, and the maps u are obtained by counterclockwise continuation, as in (1.5). We define the *transition map* along γ as

$$M_{ik}(\gamma) = u_{k,\gamma} \circ v_{i,\gamma} : \Phi_{i,\gamma} \longrightarrow \Phi_{k,\gamma}.$$

The data of the $\Phi_i(\mathcal{F})$, $i = 1, \dots, n$ as local systems on the circles around b_i and of all the $M_{ik}(\gamma)$ describe the image of \mathcal{F} in the localization of Perv(Δ, B) by the subcategory of constant sheaves [GMV, §2].

One can say that "abstract Picard-Lefschetz theory" is the study of how $M_{ik}(\gamma)$ changes when we replace γ by a different (non-isotopic) arc γ' .

More precisely, assume that γ' is obtained from γ by an "elementary move" past another marked point b_j so that the bigon formed by γ' and γ contains b_j and two arcs α and β , but no other marked points as in Fig. 3. In particular, we assume that the closed path obtained by following γ from b_i to b_k and then γ' from b_k to b_i , has orientation compatible with the standard (counterclockwise) orientation of Δ . Note that homotopy inside the bigon and the *clockwise* rotation around b_i give identifications

(2.9)
$$\Phi_{i,\gamma} \simeq \Phi_{i,\beta} \simeq \Phi_{i,\gamma'}, \quad \Phi_{k,\gamma} \simeq \Phi_{k,\alpha} \simeq \Phi_{k,\gamma'}, \quad \Phi_{j,\beta} \simeq \Phi_{j,\alpha},$$

so we can consider them as single spaces denoted by Φ_i, Φ_k and Φ_j respectively.

Proposition 2.10 (Picard-Lefschetz formula for perverse sheaves). We have the equality of linear operators $\Phi_i \to \Phi_k$:

$$M_{ik}(\gamma') = M_{ik}(\gamma) + M_{jk}(\alpha)M_{ij}(\beta).$$

This statement is a version of [GMV, Prop. 2.4], formulated in a more invariant way and without localizing by constant sheaves. It holds for perverse sheaves on any oriented surface. It is convenient to give two proofs of Proposition 2.10.

Invariant proof: To eliminate the need for the first two identifications in (2.9), let us deform the paths $\alpha, \beta, \gamma, \gamma'$ so that:

- γ, β, γ' have a common segment $[b_i, b'_i]$ near b_i .
- γ, α, γ' have a common segment $[b'_k, b_k]$ near b_k . See Fig. 4.

We denote by $\overline{\gamma}, \overline{\gamma}'$ the parts of γ and γ' lying between b'_i and b'_k , by $\overline{\beta}$ the part of β between b'_i and b_j , and by $\overline{\alpha}$ the part of α between b_k and b'_j .



Figure 4: The Picard-Lefschetz situation, deformed.

The points b'_i, b'_k being smooth for \mathcal{F} , both sides of our putative equality factor through the maps

$$v_{[b_i,b'_i]}: \Phi_i \longrightarrow \mathcal{F}_{b'_i}, \quad u_{[b'_k,b_k]}: \mathcal{F}_{b'_i} \longrightarrow \Phi_k.$$

Our statement would therefore follow from the next lemma which is an invariant version of the statement that the map (1.3) is indeed the monodromy around 0.

Lemma 2.11. We have an equality of operators $\mathcal{F}_{b'_i} \to \mathcal{F}_{b'_k}$:

$$T_{\overline{\gamma}'} = T_{\overline{\gamma}} + v_{\overline{\alpha}} R u_{\overline{\beta}},$$

where:

- (1) $u_{\overline{\beta}} : \mathcal{F}_{b'_i} \to \underline{\mathbb{H}}^1_{\overline{\beta}}(\mathcal{F})_{b_j}$ is the coboundary of the counterclockwise continuation map, cf. (1.5).
- (2) $R: \underline{\mathbb{H}}^{1}_{\overline{\beta}}(\mathcal{F})_{b_{j}} \to \underline{\mathbb{H}}^{1}_{\overline{\alpha}}(\mathcal{F})_{b_{j}}$ is the identification obtained by deforming, by clockwise rotation aroung b_{j} , the path $\overline{\beta}$ into the path $\overline{\alpha}$, i.e., R is the third identification in (2.9).
- (3) $v_{\overline{\alpha}} : \underline{\mathbb{H}}^{1}_{\overline{\alpha}}(\mathcal{F})_{b_{j}} \to \underline{\mathbb{H}}^{1}_{\overline{\alpha}}(\mathcal{F})_{\text{gen}} \simeq \mathcal{F}_{b'_{k}}$ is the generalization map of the sheaf $\underline{\mathbb{H}}^{1}_{\overline{\alpha}}(\mathcal{F}).$

Proof of the lemma: Let U be a small disk around b_j . We fix $a \in \mathcal{F}_{b'}$ and compare the two sides of the putative equality when applied to a. For this, let $s_a \in \Gamma(U - \overline{\beta}, \mathcal{F})$ be the section obtained by continuing a on the left side of $\overline{\beta}$ towards and around b_j . Then $u_{\overline{\beta}}(a)$ is equal to the class of s_a in $\underline{\mathbb{H}}^1_{\overline{\beta}}(\mathcal{F})_{b_j}$. Next, $Ru_{\overline{\beta}}(a)$ is similarly represented by the section t_a obtained from s_a by continuously moving the branch cut clockwise from $\overline{\alpha}$ to $\overline{\beta}$. This means that $t_a = s_a$ on the left side of $\overline{\alpha} \cup \overline{\beta}$. Finally, $v_{\overline{\alpha}} Ru_{\overline{\beta}}(a)$ is obtained as the difference of the two boundary values if t_a when continued along both sides of $\overline{\alpha}$ all the w ay to b'_k . It remains to notice that these boundary values are equal, in virtue of the above, to $T_{\overline{\gamma}}(a)$ and $T_{\overline{\gamma}'}(a)$.

Proof using shadows: Choose a system of cuts K (depicted by dotted lines in Fig. 3) adopted to our situation. We assume that the arcs K_i, K_j and K_k are positioned as in the figure, i.e., that γ together with K_i and K_k form a triangle containing K_j , α and β and not containing any other marked points. We orient each K_{ν} to run from b to b_{ν} . Consider the quiver $F_K(\mathcal{F}) =$ (Ψ, Φ_i, u_i, v_i) . Note that we have isotopies of oriented paths rel. B:

$$\alpha \sim K_k * K_j^{-1}, \ \beta \sim K_j * K_i^{-1}, \ \gamma' \sim K_k * K_i^{-1},$$
$$\gamma \sim K_k * \partial \Delta * K_i^{-1}.$$

Here * means composition of the paths and $\partial \Delta$ is the boundary circle of Δ , oriented anticlockwise and run from b to b. We can use these isotopies to calculate the transition maps, obtaining

$$M_{ij}(\beta) = u_{j,K}v_{i,K}, \quad M_{jk}(\alpha) = u_{k,K}v_{j,K}, \quad M_{ik}(\gamma') = u_{k,K}v_{i,K},$$
$$M_{ik}(\gamma) = u_{k,K}T_{j,\Psi}v_{i,k},$$

and the claim follows from the identity $T_{i,\Psi} = \mathrm{Id} - v_i u_i$.

D. The Picard-Lefschetz triangle. Let \mathfrak{S} be a perverse Schober on (Δ, B) . The transition maps constructed in n°C categorify to functors between triangulated categories. More precisely, if γ is an oriented arc joining b_i and b_k as above, then we have a diagram of triangulated categories and spherical functors which depends "canonically" (i.e., up to a contractible set of choices) only on the isotopy class of γ rel. B:

(2.12)
$$\Phi_{i,\gamma}(\mathfrak{S}) \xrightarrow{S_{i,\gamma}} \Psi_{\gamma}(\mathfrak{S}) \xleftarrow{S_{k,\gamma}} \Phi_{k,\gamma}(\mathfrak{S}).$$

To define it, we choose $K' \in \mathcal{C}$ so that K'_i (oriented from b to b_i), together with γ and $(K'_k)^{-1}$ form a positively oriented triangle not containing any other marked points. (Note: the choice of K as in Fig.3 is not good.) We consider the K'-shadow $\mathfrak{S}_{K'} = \{S_{\nu}^{K'} : \mathcal{D}_{\nu}^{K'} \to \mathcal{D}^{K'}\}_{\nu=1}^{n}$ of \mathfrak{S} and define

$$\Psi_{\gamma}(\mathfrak{S}) = \mathcal{D}^{K'}, \ \Phi_{i,\gamma}(\mathfrak{S}) = \mathcal{D}_{i}^{K'}, \ \Phi_{k,\gamma}(\mathfrak{S}) = \mathcal{D}_{k}^{K'},$$
$$S_{i,\gamma} = S_{i}^{K'}, \ S_{k,\gamma} = S_{k}^{K'}.$$

It is straightforward to see (by looking at the subgroup in Br_n permuting all K' with our property) that this definition is indeed "canonical" in the sense described. So we consider the data (2.12) as intrinsically associated to \mathfrak{S} and γ .

Denote $R_{k,\gamma}$ the right adjoint to the spherical functor $S_{k,\gamma}$. Define now the *transition functor*

$$\mathcal{M}_{ik}(\gamma) = R_{k,\gamma} \circ S_{i,\gamma} : \Phi_{i,\gamma}(\mathfrak{S}) \longrightarrow \Phi_{k,\gamma}(\mathfrak{S}).$$

Consider now a situation when we have arcs $\gamma, \gamma', \alpha, \beta$ depicted in Fig. 3. Using identifications of categories similar to (2.9), we can speak about triangulated categories $\Phi_i(\mathfrak{S})$ and $\Phi_k(\mathfrak{S})$.

Proposition 2.13 (Picard-Lefschetz triangle). We have a canonical triangle of exact functors $\Phi_i(\mathfrak{S}) \to \Phi_k(\mathfrak{S})$:

$$\mathcal{M}_{ik}(\gamma) \longrightarrow \mathcal{M}_{ik}(\gamma') \longrightarrow \mathcal{M}_{jk}(\alpha) \circ \mathcal{M}_{ij}(\beta) \longrightarrow \mathcal{M}_{ik}(\gamma)[1].$$

Proof: This is obtained identically to the "shadow" proof of Proposition 2.10 with the identity $T_{j,\Psi} = \text{Id} - v_j u_j$ replaced by the triangle coming from the fact that $T_j = \text{Cone}\{\text{Id} \to R_j S_j\}$.

E. Schobers on a Riemann surface. Let Σ be an oriented topological surface, possibly with boundary $\partial \Sigma$, and $B \subset \Sigma$ is a finite set not meeting $\partial \Sigma$. One can then define a perverse Schober on (Σ, B) by decomposing Σ as $\Delta \cup_C U$, where $\Delta \subset \Sigma$ is a closed disk with boundary circle $C = \partial \Delta$ which contains all points of B, and U is the closure of $\Sigma - \Delta$. Then a perverse Schober \mathfrak{S} is, by definition, a datum of:

- (1) A perverse Schober \mathfrak{S}_{Δ} on (Δ, B) , defined as in n°B.
- (2) A local system of triangulated categories \mathfrak{S}_U on U identified with \mathfrak{S}_Δ over C.

While one can work with objects thus defined (for instance, one can construct transition functors and Picard-Lefschet triangles for arcs not necessarily contained in Δ), a more intrinsic definition is desirable.

3 Spherical functors and spherical pairs

A. Symmetric description of $Perv(\Delta, 0)$. In [KS1, §9] we have given a different quiver description of $Perv(\Delta, 0)$ obtained as a particular case of a general result for real hyperplane arrangements:

Proposition 3.1. The category $Perv(\Delta, 0)$ is equivalent to the category formed by diagrams of vector spaces

(3.2)
$$E_{-} \underbrace{\stackrel{\gamma_{-}}{\underset{\delta_{-}}{\leftarrow}}}_{\delta_{-}} E_{0} \underbrace{\stackrel{\gamma_{+}}{\underset{\delta_{+}}{\leftarrow}}}_{\delta_{+}} E_{+}$$

satisfying the two following conditions:

- (1) $\gamma_{-}\delta_{-} = \operatorname{Id}_{E_{-}}, \ \gamma_{+}\delta_{+} = \operatorname{Id}_{E_{+}}.$
- (2) The maps $\gamma_{-}\delta_{+}: E_{+} \to E_{-}, \ \gamma_{+}\delta_{-}: E_{-} \to E_{+}$ are invertible.

This is obtained by choosing not one but two base points $b_+, b_- \in \partial \Delta$ and considering a cut K which joins b_+ with b_- and passes through 0, as in depicted in Fig. 5.



Figure 5: A symmetric cut.

The spaces E_{\pm}, E_0 are obtained as the stalks of the sheaf $\underline{\mathbb{H}}_K^1(\mathcal{F})$ at b_{\pm} and 0 respectively, the maps γ_{\pm} are the generalization maps for this sheaf, and the δ_{\pm} can be obtained by duality.

Remarks 3.3. (a) Note that in this description the maps $P_{\pm} = \delta_{-}\gamma_{-}$ and $P_{\pm} = \delta_{+}\gamma_{+}$ are projectors in E_{0} , that is $P_{\pm}^{2} = P_{\pm}$. We can consider E_{\pm} as subspaces in E_{0} which are the images of P_{\pm} .

(b) The pattern of Remarks 1.6 and 2.3 continues here: $\underline{\mathbb{H}}_{K}^{i}(\mathcal{F}) = 0$ for $i \neq 1$ and each stalk of $\underline{\mathbb{H}}_{K}^{1}$ considered as a functor into Vect_k, takes Verdier duality into vector space duality.

B. Reminder on semi-orthogonal decompositions. A categorification of diagrams (3.2) is naturally formulated in the language of semi-orthogonal decompositions of triangulated categories [BK1] [KuL] which we now recall.

Let \mathcal{B} be a full triangulated subcategory of a triangulated category \mathcal{A} , with $i: \mathcal{B} \to \mathcal{A}$ being the embedding functor. We denote by

$${}^{\perp}\mathcal{B} = \{A \in \mathcal{A} | \operatorname{Hom}(A, B) = 0 \ \forall B \in \mathcal{B} \},\$$
$$\mathcal{B}^{\perp} = \{A \in \mathcal{A} | \operatorname{Hom}(B, A) = 0 \ \forall B \in \mathcal{B} \}$$

the left and right orthogonals to \mathcal{B} We say that \mathcal{B} is *left admissible*, resp. *right admissible*, if *i* has a left adjoint **i*, resp. a right adjoint *i**. If \mathcal{B} is left (resp. right) admissible, we have a semi-orthogonal decomposition

$$\mathcal{A} = \langle {}^{\perp}\mathcal{B}, \mathcal{B} \rangle, \quad \text{ resp. } \mathcal{A} = \langle \mathcal{B}, \mathcal{B}^{\perp} \rangle$$

which means that each object $A \in \mathcal{A}$ is included in functorial exact triangles

$$C \longrightarrow A \longrightarrow B \rightarrow C[1], \quad C \in^{\perp} \mathcal{B}, \quad B = {}^{*}i(A) \in \mathcal{B},$$

resp. $B' \longrightarrow A \longrightarrow D' \rightarrow B'[1], \quad B' = i^{*}(A) \in \mathcal{B}, \quad D' \in \mathcal{B}^{\perp}.$

In particular, ${}^{\perp}\mathcal{B} = \operatorname{Ker}({}^{*}i)$, resp. $\mathcal{B}^{\perp} = \operatorname{Ker}(i^{*})$. We will call ${}^{*}i$ the projection on \mathcal{B} along ${}^{\perp}\mathcal{B}$ and i^{*} the projection on \mathcal{B} along \mathcal{B}^{\perp} .

If \mathcal{B} is left admissible, then ${}^{\perp}\mathcal{B}$ is right admissible, and $({}^{\perp}\mathcal{B}){}^{\perp} = \mathcal{B}$. Similarly, for a right admissible \mathcal{B} we have that \mathcal{B}^{\perp} is left admissible and ${}^{\perp}(\mathcal{B}^{\perp}) = \mathcal{B}$.

C. Spherical pairs. Let \mathcal{E}_0 be a triangulated category and $\mathcal{E}_+, \mathcal{E}_- \subset \mathcal{E}_0$ be a pair of left admissible subcategories, so that we have the diagrams of embeddings

$$\mathcal{E}_{-} \xrightarrow{\delta_{-}} \mathcal{E}_{0} \xleftarrow{\delta_{+}} \mathcal{E}_{+}, \quad {}^{\perp}\mathcal{E}_{-} \xrightarrow{j_{-}} \mathcal{E}_{0} \xleftarrow{j_{+}} {}^{\perp}\mathcal{E}_{+}$$

with δ_{\pm} having a left adjoint γ_{\pm} and j_{\pm} having a right adjoint k_{\pm} .

Definition 3.4. The pair of left admissible subcategories \mathcal{E}_{\pm} is called a *spherical pair*, if:

(SP1) The compositions

$$\gamma_+\delta_-: \mathcal{E}_- \longrightarrow \mathcal{E}_+, \quad \gamma_-\delta_+: \mathcal{E}_+ \longrightarrow \mathcal{E}_-$$

are equivalences.

(SP2) The compositions

$$k_+j_-: {}^{\perp}\mathcal{E}_- \longrightarrow {}^{\perp}\mathcal{E}_+, \quad k_-j_+: {}^{\perp}\mathcal{E}_+ \longrightarrow {}^{\perp}\mathcal{E}_-$$

are equivalences.

Remark 3.5. Note that unlike a spherical functor, the definition of a spherical pair does not appeal to any enhancement of the triangulated category \mathcal{E}_0 , as no functorial cones are taken. In the following (as elsewhere in the paper) we will, however, assume that we are in an enhanced situation as described in Appendix.

D. From a spherical pair to a spherical functor. Let \mathcal{E}_{\pm} be a spherical pair. Consider the diagram

$$\mathcal{D}_0 = {}^{\perp}\mathcal{E}_- = \operatorname{Ker}(\gamma_+) \xrightarrow{S} \mathcal{E}_+ = \mathcal{D}_1,$$

where $S = \gamma_+|_{\perp \mathcal{E}_+}$.

Proposition 3.6. *S* is a spherical functor.

This follows from Theorem 1.9 (which gives that (SF1) and (SF3) imply sphericity) and from the next more precise statement.

Proposition 3.7. (a) The functor $R : \mathcal{E}_+ \to {}^{\perp}\mathcal{E}_-$ defined as the composition $\mathcal{E}_+ \xrightarrow{\delta_+} \mathcal{E}_0 \xrightarrow{k_-} {}^{\perp}\mathcal{E}_-$ is right adjoint to S.

(b) The functor

$$T_1 = \operatorname{Cone} \{ SR \Rightarrow \operatorname{Id}_{\mathcal{D}_1} \} : \mathcal{D}_1 = \mathcal{E}_+ \longrightarrow \mathcal{E}_+ = \mathcal{D}_1$$

is identified with the composition $\gamma_+ \circ \delta_- \circ \gamma_- \circ \delta_+ : \mathcal{E}_+ \to \mathcal{E}_+$. In particular, it is invertible.

(c) The functor

$$T_0 = \operatorname{Cone} \{ \operatorname{Id}_{\mathcal{D}_1} \Rightarrow RS \} [-1] : \mathcal{D}_0 = {}^{\perp} \mathcal{E}_- \longrightarrow {}^{\perp} \mathcal{E}_- = \mathcal{D}_0$$

is identified with the composition $k_+ \circ j_- \circ k_- \circ j_+ : {}^{\perp}\mathcal{E}_- \to {}^{\perp}\mathcal{E}_-$. In particular, it is invertible.

Proof: (a) S is the composition $\gamma_+ \circ j_-$. By definition, $\gamma_+ = *\delta_+$ and therefore $\delta_+ = \gamma_+^*$. Therefore $S^* = j_-^* \circ \gamma_+^* = k_- \circ \delta_+$.

(b) Let us make $k_{-} = j_{-}^{*} : \mathcal{E}_{0} \to {}^{\perp}\mathcal{E}_{-}$ into an endofunctor of \mathcal{E}_{0} by composing it with the embedding $j_{-} :{}^{\perp}\mathcal{E}_{-} \to \mathcal{E}_{0}$. Then the functorial triangle for the orthogonal decomposition $\mathcal{E}_{0} = \langle {}^{\perp}\mathcal{E}_{-}, \mathcal{E}_{-} \rangle$ gives that

(3.8)
$$j_{-} \circ k_{-} = \operatorname{Cone}\{\operatorname{Id}_{\mathcal{E}_{0}} \Rightarrow \delta_{-}\gamma_{-}\}[-1].$$

We now write $SR = S \circ k_{-} \circ \delta_{+}$ and note that $S = \gamma_{+} \circ j_{-}$, so SR is the composition

$$\mathcal{E}_+ \xrightarrow{\delta_+} \mathcal{E}_0 \xrightarrow{k_-} {}^{\perp} \mathcal{E}_- \xrightarrow{j_-} \mathcal{E}_0 \xrightarrow{\gamma_+} \mathcal{E}_+.$$

Identifying δ_+ on the left and γ_+ on the right with the corresponding arrows in the proposed formula for T_1 , we reduce part (b) of our proposition to the statement that

$$\delta_{-} \circ \gamma_{-} = \operatorname{Cone}\{j_{-} \circ k_{-} \Rightarrow \operatorname{Id}_{\mathcal{E}_{0}}\},\$$

which is equivalent to (3.8).

(c) is proved similarly to (b).

We do not know whether each spherical functor can be obtained from a spherical pair. but the following construction provides some geometric hint.

Example 3.9 (Polar coordinates). Let Y be a CW-complex and $p: V \to Y$ be a real vector bundle. Denote by $i: Y \hookrightarrow V$ the embedding of the zero section, and by $j: V^{\circ} \hookrightarrow V$ the embedding of the complement of the zero section. Let $q: \mathbb{S} = V^{\circ}/\mathbb{R}^*_{>0} \to Y$ be the spherical bundle associated to V, and $\tau: V^{\circ} \to \mathbb{S}$ the natural projection. Let also $\pi = q\tau: V^{\circ} \to Y$ be the composite projection.

Let $\mathcal{E}_0 = \mathcal{E}_0(V) := D^b_{\text{conic}}(V) \subset D^b(V)$ be the full subcategory of $\mathbb{R}^*_{>0^-}$ conic complexes, i.e., of complexes \mathcal{F} such that each $\underline{H}^i(\mathcal{F})$ is constant on each orbit of $\mathbb{R}^*_{>0}$ in V, see [KS2]. Consider the subcategories $\mathcal{E}_{\pm} = \mathcal{E}_{\pm}(V) \subset \mathcal{E}_0$ defined as follows. First, let

$$\mathcal{E}_+ = \mathcal{E}_+(V) := j_! \pi^{-1} D^b(\mathbb{S}) \simeq D^b(\mathbb{S}), \text{ so that } {}^{\perp} \mathcal{E}_+ = i_* D^b(Y) \simeq D^b(Y).$$

Second, let $\mathcal{E}_{-} = \mathcal{E}_{-}(V)$ consist of complexes \mathcal{F} with $Rp_*\mathcal{F} = 0$, so that ${}^{\perp}\mathcal{E}_{-} = p^{-1}D^b(Y) \simeq D^b(Y)$. Then $(\mathcal{E}_{+}, \mathcal{E}_{-})$ is a spherical pair, and the functor

$$D^b(Y) \simeq {}^{\perp}\mathcal{E}_- \stackrel{S=\gamma_+j_-}{\longrightarrow} \mathcal{E}_+ \simeq D^b(\mathbb{S})$$

is identified with the spherical functor q^{-1} from Example 1.10.

Remark 3.10. In the situation of Example 3.9, let V^* be the vector bundle dual to V, so that the Fourier-Sato transform [KS2] gives an identification

$$\mathcal{E}_0(V) = D^b_{\text{conic}}(V) \xrightarrow{F} D^b_{\text{conic}}(V^*) = \mathcal{E}_0(V^*).$$

This identification takes the category $\mathcal{E}_{\pm}(V)$ to $\mathcal{E}_{\mp}(V^*)$.

4 Derived categories on the G/P and Schobers on symmetric products \mathfrak{h}/W .

A. The braid group action on $D^b(G/B)$ and $D_{\text{coh}}(T^*(G/B))$. Let \mathfrak{g} be a split reductive Lie algebra over \mathbb{C} , with Cartan subalgebra \mathfrak{h} and Weyl group W. We denote by $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ the real parts of \mathfrak{h} and \mathfrak{g} . Let $\Delta^{\text{sim}} \subset \Delta_+ \subset \Delta$ be the systems of simple and positive roots of \mathfrak{g} inside the set of all roots. The complex vector space \mathfrak{h} has an arrangement of hyperplanes $\{\alpha^{\perp}\}_{\alpha \in \Delta_+}$ and so has a natural stratification by *flats* of this arrangements, see [KS1].

We consider the complex manifold \mathfrak{h}/W . It has the induced stratification, denote it \mathcal{S} . The open stratum $(\mathfrak{h}/W)_0$, is the classifying space of the *braid* group of \mathfrak{g} , which we denote by $\operatorname{Br} = \operatorname{Br}(\mathfrak{g})$. By definition, Br is generated by elements $s_{\alpha}, \alpha \in \Delta^{\operatorname{sim}}$ subject to the relations defining the Weyl group Wwith the exception of the relations $s_{\alpha}^2 = 1$.

Example 4.1. In the case $\mathfrak{g} = \mathfrak{gl}_n$, the manifold \mathfrak{h}/W is space of monic polynomials f(z) of degree n in one variable. The stratification S is the stratification by types of coincidence of roots of f(z) and is labelled by (unordered) partitions of n. The group $Br(\mathfrak{gl}_n)$ is the usual Artin braid group Br_n .

Let $B \subset G$ be the standard Borel subgroup corresponding to the choice of Δ_+ . For any $I \subset \Delta^{\text{sim}}$ let $P_I \supset B$ be the standard parabolic subgroup generated by B and the exponents of the Chevalley generators associated to $(-\alpha), \alpha \notin I$. Thus $P_{\emptyset} = G$ and $P_{\Delta^{\text{sim}}} = B$. We write $P_{\alpha} = P_{\Delta^{\text{sim}} - \{\alpha\}}, \alpha \in \Delta^{\text{sim}}$, for the "next to minimal" parabolic subgroup associated to $\{\alpha\}$. We have the \mathbb{P}^1 -fibration

$$G/B \xrightarrow{q_{\alpha}} G/P_{\alpha}$$

and therefore, by Proposition 1.11, a spherical functor

$$D^b(G/P_\alpha) \stackrel{S_\alpha = q_\alpha^{-1}}{\longrightarrow} D^b(G/B)$$

and the corresponding twist functor $T_{\alpha}: D^b(G/B) \to D^b(G/B)$ written directly as

(4.2)
$$T_{\alpha}(\mathcal{F}) = R(p_{\alpha,2})_! p_{1,\alpha}^* \mathcal{F}[-1],$$
$$p_{\alpha,i} : \left(\left((G/B) \times_{G/P_{\alpha}} (G/B) \right) - \text{ rel. diagonal } \right) \longrightarrow G/B, \ i = 1, 2.$$

The following fact was known before the concept of a spherical functor was discovered, cf. [BBM].

Proposition 4.3. The functors T_{α} defined by (4.2), are equivalences which satisfy the relations of $Br(\mathfrak{g})$, i.e., define an action of $Br(\mathfrak{g})$ on the derived category $D^{b}(G/B)$.

Remarks 4.4. (a) Proposition 4.3 for $\mathfrak{g} = \mathfrak{gl}(n)$ has a quaternionic analog. We consider the space $\mathbb{H}F_n$ of complete flags of (left) quaternionic subspaces $V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{H}^n$, $\dim_{\mathbb{H}}(V_i) = i$. It fits into the \mathbb{HP}^1 -fibrations

$$q_i: \mathbb{H}F_n \longrightarrow \mathbb{H}F_n^{(i)} = \left\{ (V_1 \subset \dots \subset V_{i-1} \subset V_{i+1} \subset \dots \subset \mathbb{H}^n \right\}$$

over the spaces of next-to-complete flags. We note that $\mathbb{HP}^1 = \mathbb{S}^4$ is the 4sphere. By Proposition 1.11 this gives, for each $i = 1, \dots, n-1$, a spherical functor $S_i^{\mathbb{H}} = q_i^{-1} : D^b(\mathbb{H}F_n^{(i)}) \to D^b(\mathbb{H}F_n)$ and the induced twist automorphism $T_i^{\mathbb{H}}$ of $D^b(\mathbb{H}F_n)$. One can see directly by analyzing the Schubert correspondences in $\mathbb{H}F_n$, that the $T_i^{\mathbb{H}}$ satisfy the relations of the Artin braid group Br_n .

(b) One has also a real analog for any \mathfrak{g} , using the real loci of the flag varieties which fit into $\mathbb{RP}^1 = \mathbb{S}^1$ -fibrations.

A quasi-classical analog of the action in Proposition 4.3 has been constructed in [KhT, BR]. It uses the diagram

$$Y_{\alpha} := T^*(G/P_{\alpha}) \xleftarrow{\rho_{\alpha}} D_{\alpha} := T^*(G/P_{\alpha}) \times_{G/P_{\alpha}} G/B \xrightarrow{\imath_{\alpha}} T^*(G/B) := X$$

which produces a spherical functor

$$S^{\rm coh}_{\alpha} = i_{\alpha*}\rho^*_{\alpha} : D^b_{\rm coh}T^*(G/P_{\alpha}) \to D^b_{\rm coh}T^*(G/B)$$

It was proved in [KhT] for $\mathfrak{g} = \mathfrak{gl}_n$ and in [BR] for arbitrary \mathfrak{g} , that the corresponding twist functors T_{α}^{coh} define an action of $\operatorname{Br}(\mathfrak{g})$ on $D_{\operatorname{coh}}^b T^*(G/B)$ (which, in fact, extends to an action of the affine braid group).

These constructions can be seen as giving *local systems of triangulated* categories on $(\mathfrak{h}/W)_0 = K(\operatorname{Br}(\mathfrak{g}), 1)$ with general stalk being $D^b(G/B)$, resp. $D^b_{\operatorname{coh}}(T^*(G/B))$. We would like to suggest that these local systems extend to natural perverse Schobers on the entire \mathfrak{h}/W . For this, we review some features of usual perverse sheaves in this situation.

B. Perverse sheaves on \mathfrak{h}/W and double cubical diagrams. Denote $\operatorname{Perv}(\mathfrak{h}/W)$ the category of perverse sheaves on \mathfrak{h}/W smooth with respect to the stratification \mathcal{S} from n°A. A complete quiver description on $\operatorname{Perv}(\mathfrak{h}/W)$ is

not yet available. However, the results of [KS1] provide the following partial picture which aligns with the examples we considered earlier.

Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ be the real form of \mathfrak{h} . Inside \mathfrak{h}/W we consider the "real skeleton" (or "cut")

$$K = \mathfrak{h}_{\mathbb{R}}/W \subset \mathfrak{h}/W.$$

It can be thought of as a "curvilinear cone", the image of the dominant Weyl chamber $\mathfrak{h}_{\mathbb{R}}^+ \subset \mathfrak{h}_{\mathbb{R}}$. Let $r = |\Delta^{\text{sim}}|$ be the rank of \mathfrak{g} . Let $\mathcal{S}_{\mathbb{R}}$ be the stratification of K by the 2^r strata which are the images of the faces of $\mathfrak{h}_{\mathbb{R}}^+$. We denote these strata by $S_I, I \subset \Delta^{\text{sim}}$, so that $\dim_{\mathbb{R}}(S_I) = |I|$.

Note that each S_I is contractible. Therefore (cf. [GM]) a sheaf \mathcal{G} on K constructible with respect to $\mathcal{S}_{\mathbb{R}}$ can be recovered from the cubical diagram of vector spaces $\mathcal{G}_I = \Gamma(S_I, \mathcal{G})$ and generalization maps $\gamma_{II'} : \mathcal{G}_I \to \mathcal{G}_{I'}, I \subset I'$ which are required to form a representation of the poset $\mathbf{2}^{\Delta^{\text{sim}}}$ (so that the diagram is commutative).

The following is deduced by pulling \mathcal{F} back to a perverse sheaf on \mathfrak{h} smooth with respect to an arrangement of hyperplanes $\{\alpha^{\perp}\}_{\alpha\in\Delta_{+}}$ and applying the results of [KS1].

Proposition 4.5. (a) For $\mathcal{F} \in \text{Perv}(\mathfrak{h}/W)$ we have $\underline{\mathbb{H}}_{K}^{i}(\mathcal{F}) = 0$ for $i \neq \dim_{\mathbb{R}}(K)$, and the sheaf $\mathcal{R}_{\mathcal{F}} = \underline{\mathbb{H}}_{K}^{\dim(K)}(\mathcal{F})$ on K is constructible with respect to the stratification $\mathcal{S}_{\mathbb{R}}$.

(b) Let $I \subset \Delta^{\text{sim}}$. Denoting $E_I(\mathcal{F}) = \Gamma(S_I, \mathcal{R}_{\mathcal{F}})$, we have that E_I : Perv $(\mathfrak{h}/W) \to \text{Vect}_{\mathbf{k}}$ is an exact functor which takes the Verdier duality to the vector space duality.

Therefore we can associate to any $\mathcal{F} \in \text{Perv}(\mathfrak{h}/W)$ a double cubical diagram $E(\mathcal{F})$ formed by the vector spaces $E_I(\mathcal{F})$ and the maps

(4.6)
$$E_I(\mathcal{F}) \xrightarrow[\delta_{JI}]{\gamma_{IJ}} E_J(\mathcal{F}), \quad I \subset J.$$

Here γ_{IJ} is the generalization map for $\mathcal{E}_{\mathcal{F}}$, and δ_{JI} is the dual to the generalization map for $\mathcal{E}_{\mathcal{F}^*}$, where \mathcal{F}^* is the perverse sheaf Verdier dual to \mathcal{F} . Each of the collections (γ_{IJ}) , (δ_{JI}) forms a commutative cube. It seems very plausible that the functor $\mathcal{F} \mapsto E(\mathcal{F})$ from $\operatorname{Perv}(\mathfrak{h}/W)$ to the category of double cubical diagrams is fully faithful, i.e., \mathcal{F} can be recovered from $E(\mathcal{F})$.

Example 4.7. For $\mathfrak{g} = \mathfrak{sl}_2$ we have $\mathfrak{h}/W = \mathbb{C}$ and $K = \mathbb{R}_{\geq 0}$ so the above reduces, very precisely, to the construction of §1A, except with the disk replaced by \mathbb{C} .

C. A double cubic diagram related to flag varieties. We now note that geometry of flag varieties provides a natural double cubic diagram of categories of the same shape as (4.6). More precisely, we have an ordinary cubical diagram of algebraic varieties G/P_I and projections $q_{IJ}: G/P_J \to G/P_I$, $I \subset J$. We have then the double cubical diagram formed by triangulated categories $D^b(G/P_I)$ and the adjoint pairs of functors

(4.8)
$$D^b(G/P_I) \xrightarrow[Rq_{IJ*}]{q_{IJ}} D^b(G/P_J) , \quad I \subset J.$$

As in Example 1.10, we can also consider the left adjoint to q_{IJ}^{-1} which differs from the right adjoint by a shift.

At the "quasi-classical level" one has a similar diagram formed by the categories $D^b_{\text{coh}}(T^*(G/P_I))$ and the functors between them obtained by translating q_{IJ}^{-1} and Rq_{IJ*} into the language of \mathcal{D} -modules and passing to the associated graded modules.

We would like to suggest that the diagram (4.8) comes from a more fundamental object: a perverse Schober on \mathfrak{h}/W extending the local system discussed in n°A.

Similarly for the quasi-classical analog with the $D^b_{\rm coh}(T^*(G/P_I))$. In this situation we have in fact more: an actions the affine braid group. These can possibly come from Schobers not on \mathfrak{h}/W but on T/W where T is the maximal torus in the algebraic group corresponding to \mathfrak{g} .

5 "Fukaya-style" approach to perverse sheaves: cuts, real skeletons and Langangian varieties

A. Maximally real cuts. Since our preliminary definitions of a perverse Schobers were based on quiver descriptions of perverse sheaves, let us look at some general features of such descriptions.

Let (X, \mathcal{S}) be a stratified complex manifold of dimension n. Obtaining a quiver description of $\operatorname{Perv}(X, \mathcal{S})$ requires, in particular, construction of many exact functors $\operatorname{Perv}(X, \mathcal{S}) \to \operatorname{Vect}_{\mathbf{k}}$. Indeeed, any component of the putative quiver must be such a functor. Arrows of the quiver are then natural transformations between these exact functors.

The common tool for that, used in examples in this paper, is a choice of a closed subset $K \subset X$ with the following property:

(Cut) For any $\mathcal{F} \in \text{Perv}(X, \mathcal{S})$ we have $\underline{\mathbb{H}}_{K}^{i}(\mathcal{F}) = 0, i \neq n$.

This property implies that the functor of abelian categories

$$\mathcal{R}: \mathcal{F} \longmapsto \mathcal{R}(\mathcal{F}) := \underline{\mathbb{H}}_{K}^{n}(\mathcal{F}), \quad \operatorname{Perv}(X, \mathcal{S}) \longrightarrow \operatorname{Sh}_{K}$$

is exact. So stalk of $\mathcal{R}(\mathcal{F})$ at any point $x \in K$ can be used as a component of a quiver, while generalization maps between the stalks provide some of the arrows.

We will call each K satisfying (Cut) an (admissible) cut for (X, S) and denote by $\mathcal{C}(X, S)$ the set of all such cuts. It is natural to look for a description of Perv(X, S) in terms of some data associated to all the cuts.

Recall that by the Riemann-Hilbert correspondence we can realize each $\mathcal{F} \in \operatorname{Perv}(X, \mathcal{S})$ as $\underline{R} \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ for a left \mathcal{D}_X -module \mathcal{M} whose characteristic variety satisfies the inclusion

$$\operatorname{Ch}(\mathcal{M}) \subset \Lambda_{\mathcal{S}} := \bigcup_{\alpha} \overline{T^*_{X_{\alpha}} X}.$$

We denote by $_{\mathcal{D}} \operatorname{Mod}(X, \mathcal{S})$ the category of such \mathcal{D}_X -modules. In these terms, a more detailed scenario (sufficient condition) for (Cut) to hold would be for K to satisfy the following two properties:

(Cut1) We have $\underline{\mathbb{H}}_{K}^{i}(\mathcal{O}_{X}) = 0$ for $i \neq n$.

(Cut2) Assuming (Cut1), the sheaf $\mathcal{B}_K = \underline{\mathbb{H}}_K^n(\mathcal{O}_X)$ (which, considered as a sheaf on X, is automatically a sheaf of left \mathcal{D}_X -modules), satisfies

$$\underline{\operatorname{Ext}}_{\mathcal{D}_X}^{j}(\mathcal{M},\mathcal{B}_K) = 0, \quad \forall \mathcal{M} \in \mathcal{D}\operatorname{Mod}(X,\mathcal{S}), \ j > 0.$$

The property (Cut1) is not related to a choice of S and holds for any *totally real subset* of X, see [Har]. More precisely, we recall, see [BER] for background:

Definition 5.1. (a) Let V be a \mathbb{C} -vector space of finite dimension n. A real subspace $L \subset V$ is called *totally real*, if $L \cap iL = 0$. We say that L is *maximally real*, if it is totally real of dimension n.

(b) Let X be a complex manifold of dimension n. A C^{∞} -submanifold $K \subset X$ is called *totally real*, resp. maximally real if for each $x \in K$ the subspace $T_x K \subset T_x X$ is totally real resp. maximally real.

The results of Harvey [Har] imply:

Proposition 5.2. Any closed subset of a totally real submanifold of X satisfies (Cut1).

More precisely, the result of *loc. cit.* is for arbitrary totally real *subsets* of X, a class of sets which includes totally real submanifolds [Har, §3.6 Ex. 2] and is closed under passing to closed subsets [Har, Cor. 3.2].

For example, \mathbb{R}^n , as well as $\mathbb{R}^n_{\geq 0}$ satisfies (Cut1). The sheaf $\mathcal{B}_{\mathbb{R}^n}$ is the sheaf of hyperfunctions of Sato [SKK].

We now consider the condition (Cut2). Sufficient criteria for it to hold were given by Lebeau [Le] and Honda-Schapira [HS]. The criterion of [HS] is based on the concept of *positive position* of two real Lagrangian submanifolds in a complex symplectic manifold such as T^*X . We do not recall this concept here, referring to [Sch] and references therein for more background. Informally, the essense of the criterion can be formulated like this.

(5.3) For (Cut2) to hold, K must be "maximally real with respect to
the stratification
$$S$$
", in particular, the intersection of K
with each stratum X_{α} should be maximally real in X_{α} .

Here is a precise but more restrictive statement which is a reformulation of Example 1 of [HS].

Proposition 5.4. Suppose S is such that Λ_S is contained in the union of $T^*_{Z_{\beta}}X$ for a collection of smooth closed complex submanifolds $Z_{\beta} \subset X$. Suppose $K \subset X$ is a maximally real analytic submanifold such that each $K \cap Z_{\beta}$ is maximally real in Z_{β} . Then K satisfies (Cut2).

B. Maximally real vs. Lagrangian cuts. Let V be a complex vector space of dimension n and $G_{\mathbb{R}}(n, V)$ the Grassmannian of real n-dimensional subspaces in V. We denote by $G^{\max}(V)$ the open subset in $G_{\mathbb{R}}(n, V)$ formed by maximally real subspaces.

Suppose V is equipped with a positive definite hermitian form h. Separating the real and imaginary parts $h = g + i\omega$, we have that ω is a symplectic form on V. Let $LG_{\omega}(V)$ be the closed subset in $G_{\mathbb{R}}(n, V)$ formed by subspaces Lagrangian with respect to ω . The following is well known.

Proposition 5.5. $LG_{\omega}(V)$ is contained in $G^{\max}(V)$ and the embedding is a homotopy equivalence. In other words, $LG_{\omega}(V)$ can be seen as a compact form of $G^{\max}(V)$.

Proof: For $V = \mathbb{C}^n$ with the standard hermitian form we have

$$LG_{\omega}(V) = O(2n)/U(n) \subset GL(2n, \mathbb{R})/GL(n, \mathbb{C}) = G^{\max}(V).$$

Let now X be a complex manifold equipped with a Kähler metric $h = g + i\omega$. Then ω makes X into a symplectic manifold, and we have

Corollary 5.6. Any Lagrangian submanifold of X is maximally real. \Box

Note that all the cuts used in this paper as well as in [GGM] [KS1], are Lagrangian. This suggests a possibility of describing more general Perv(X, S) in terms of data coming from Lagrangian cuts.

On the other hand, Lagrangian submanifolds of a Kähler manifold X are organized into a far-reaching structure: the Fukaya category $\mathbb{F}(X)$ (as well as its modifications such as the Fukaya-Seidel and wrapped Fukaya categories), see [FO³] [Se] for background. We briefly recall that $\mathbb{F}(X)$ is an (*a priori* partially defined) \mathbb{C} -linear A_{∞} category whose objects are, in the simplest setting, compact Lagrangian submanifolds $K \subset X$. The space $\text{Hom}(K_1, K_2)$ is defined when K_1 and K_2 meet transversely and in this case is formally spanned by the set $K_1 \cap K_2$ (with appropriate grading, see *loc. cit.*). The A_{∞} -composition

$$\mu_{K_1,\cdots,K_m}: \bigotimes_{i=1}^{m-1} \operatorname{Hom}(K_i, K_{i+1}) \longrightarrow \operatorname{Hom}(K_1, K_m), \quad m \ge 1,$$

is given by counting holomorphic² disks $D \subset X$ with boundary on $K_1 \cup \cdots \cup K_m$. Each D gives a contribution $\exp\left(-\int_D \omega\right)$ to the appropriate matrix element of μ_{K_1,\cdots,K_m} .

One can expect that the Fukaya structure on the collection of admissible Lagrangian cuts for (X, \mathcal{S}) has some significance for explicit description of $\operatorname{Perv}(X, \mathcal{S})$. In particular, a holomorphic disk D with boundary on the union of admissible cuts K_1, \dots, K_m may provide a link between the sheaves $\underline{\mathbb{H}}^n_{K_{\nu}}(\mathcal{F}), \mathcal{F} \in \operatorname{Perv}(X, \mathcal{S})$ for different ν , via the structure of perverse sheaves on a disk (§2).

Remarks 5.7. (a) Any collection K_1, \dots, K_m of maximally real (not necessarily Lagrangian) submanifolds in X provides a natural boundary condition for holomorphic disks. It has been noticed [Se] that many ingredients of the Fukaya category construction have an "intrinsic" meaning and can be defined without explciit reference to the symplectic structure. In particular:

- The role of the grading on $\text{Hom}(K_1, K_2)$ is to make the count of disks possible by ensuring that in evaluating matrix elements of μ between basis vectors of the same degree, the index of the linearlized elliptic problem is 0 (so the corresponding linear $\overline{\partial}$ -operator is, generically, invertible).
- The quantity $\exp(-\int_D \omega)$ can be intepreted as the determinant of the invertible $\overline{\partial}$ -operator above.

They could therefore make sense for more general maximally real submanifolds. The usual technical reason for restricting to Lagrangian submanifolds in defining $\mathbb{F}(X)$ is that the Gromov compactness theorem (which, via the properties of 1-dimensional moduli spaces, is used to prove the A_{∞} -axioms)

²Here we leave aside the additional complication that it may be necessary to deform the complex structure on X to ensure generic position.

has originally been established only in that setting. However, see [FZ] for a recent generalization to the maximally real case.

(b) the above mentioned role of cuts (in particular, Lagrangian cuts) for description of perverse sheaves on X, is different from the classical "microlocal" point of view [KS2] which empasizes complex, conic Lagrangian subvarieties in T^*X rather than real ones in X itself. For the relation of the microlocal approach to the Fukaya category (of T^*X), see [NZ][Na]. A more general idea that the Fukaya category of any symplectic manifold should have an interpretation in terms of its geometric quantization, was proposed earlier in [BS].

C. Perverse Schobers as coefficients data for Fukaya categories. The Fukaya category $\mathbb{F}(X)$ of a symplectic manifold X can be seen as a categorification of its middle (co)homology (or, rather, of the part represented by Lagrangian cycles). Here "cohomology" is understood as $H^{\bullet}(X, \mathbb{C})$, the cohomology with constant coefficients, a particular case of a more general concept of $H^{\bullet}(X, \mathcal{F})$, the cohomology with coefficients in a sheaf \mathcal{F} or even more generally, in a complex of sheaves.

This point of view leads to the idea of introducing coefficients into the definition of the Fukaya category as well. It was proposed by M. Kontsevich with the goal of understanding the (usual) Fukaya category of a manifold by projecting it onto a manifold of smaller dimension.

In this direction we would like to suggest that perverse Schobers are the right coefficient data for defining Fukaya categories. That is, to a perverse Schober \mathfrak{S} on a Kähler manifold X there should be naturally associated a triangulated category $\mathbb{F}(X,\mathfrak{S})$, which for the constant Schober $D^b(\operatorname{Vect}_{\mathbb{C}})$ reduces to $\mathbb{F}(X)$. If we think about perverse sheaves in terms of some vector space data associated to Lagrangian cuts and then categorify these data to define Schobers, then it is natural to try to define $\mathbb{F}(X,\mathfrak{S})$ in terms of these categorified data.

For example, when X is a Riemann surface with marked points, the Fukaya category of X with coefficients in a constant ($\mathbb{Z}/2$ -graded) triangulated category \mathcal{A} was defined in [DK]. One can easily modify this definition when \mathcal{A} is replaced by a local system of triangulated categories on X, i.e., by the next simplest instance of a perverse Schober. We leave the case of an arbitrary perverse Schober on a Riemann surface (§2E) for future work.

A Conventions.

For the considerations of this paper to make sense, we should work in some framework of "refined", or "enhanced" triangulated categores having functorial cones. Let us describe one such framework, to be followed in the main body. (Another one would be that of stable ∞ -categories [Lu].)

All our categories will be k-linear, where k is a fixed base field. We recall that Tabuada [Ta1, Ta2] has introduced a *Morita model structure* on the category of dg-categories. Dg-categories fibrant with respect to this model structure are called *perfect*, see also [TV]. Perfect dg-categories form a subclass of pre-triangulated categories in the sense of [BK2]. That is, a perfect dg-category \mathcal{A} gives rise to a triangulated category $H^0(\mathcal{A})$ which is, in addition, idempotent complete.

In the main body of this paper the word "triangulated category" will always mean "a triangulated category \mathcal{D} together with an identification $\mathcal{D} \simeq H^0(\mathcal{A})$ where \mathcal{A} is a perfect dg-category", and "exact functor" will mean "an exact functor of triangulated categories obtained, by passing to H^0 , from a dg-functor between perfect dg-categories" and similarly for "natural transformation". With this understaning we will speak about "the" exact functor $\operatorname{Cone}\{T: F \Rightarrow G\}$ where T is a natural transformation of exact functors.

For a finite CW-complex Z we denote $D^b(Z)$ the bounded derived category of all sheaves of **k**-vector spaces on Z. For a smooth complex algebraic variety X we denote by $D^b_{\rm coh}(X)$ the bounded derived category of coherent sheaves on X.

The condition of perversity for constructible complexes on a complex manifold X is normalized in such a way that a constant sheaf in degree 0 is perverse.

References

- [A1] R. Anno Spherical functors, arXiv:0711.4409.
- [A2] R. Anno. Affine tangles and irreducible exotic sheaves, arXiv:0802.1070.
- [AL1] R. Anno, T. Logvinenko. Orthogonally spherical objects and spherical fibrations, arXiv:1011.0707.
- [AL2] R. Anno, T. Logvinenko. Spherical DG-functors, arXiv:1309.5035.
- [BER] M. Salah Bauendi, P. Ebenfeld, L. P. Rotschild. Real Submanifolds in Complex Space and Their Embeddings. Princeton Univ. Press, 1999.
- [Be] A. Beilinson. How to glue perverse sheaves. In: K-theory, arithmetic and geometry (Moscow, 1984), Lecture Notes in Math. 1289, Springer-Verlag, 1987, 42 - 51.
- [BBD] A. Beilinson, I. Bernstein, P. Deligne. Faisceaux Pervers, Astérisque 100, 1982.
- [BBM] A. Beilinson, R. Bezrukavnikov, I. Mirkovic. Tilting exercises. Mosc. Math. J. 4 (2004), 547–557.
- [BR] R. Bezrukavnikov, S. Riche. Affine braid group actions on derived categories of Springer resolutions. Ann. Sci. Éc. Norm. Supér. 45 (2012), 535-599.
- [BK1] A. I. Bondal, M. M. Kapranov. Representable functors, Serre functors and mutations. *Math. USSR Izv.* 35 (1990), 519-541.
- [BK2] A. I. Bondal, M. M. Kapranov. Enhanced triangulated categories. Math. USSR Sbornik, 70 (1991) 93-107.
- [BS] P. Bressler, Y. Soibelman. Mirror symmetry and deformation quantization. arXiv hep-th/0202128.
- [BT] C.Brav, H.Thomas, Braid groups and Kleinian singularities, Math. Ann. 351 (2011), 1005 - 1017.
- [CG] Neil Chriss, Victor Ginzburg, Representation Theory and Complex Geometry. Birkhäuser, 1997.
- [Cu] J. Curry. Sheaves, Cosheaves and Applications. arXiv:1303.3255.

- [DK] T. Dyckerhoff, M. Kapranov. Triangulated surfaces in triangulated categories. ArXiv: 1306.2545.
- [FZ] U. Frauenfelder, K. Zehmisch. Gromov compactness for holomorphic discs with totally real boundary conditions. arXiv:1403.6139.
- [FO³] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono. Lagrangian Intersection Floer Theory. Amer. Math. Soc. Publ. 2009.
- [GGM] A. Galligo, M. Granger, P. Maisonobe. D-modules et faisceaux pervers dont le support singulier est un croisement normal. Ann. Inst. Fourier Grenoble, 35 (1985), 1-48.
- [GM] S. Gelfand, R. D. MacPherson. Verma modules and Schubert cells: a dictionary. in: "Paul Dubreil and Marie-Paule Malliavin Algebra Seminar", 34th Year (Paris, 1981), pp. 1-50. Lecture Notes in Math. 924, Springer, Berlin-New York, 1982.
- [GMV] S. Gelfand, R. D. MacPherson, K. Vilonen. Perverse sheaves and quivers. Duke Math. J. 3 (1996), 621-643.
- [Har] R. Harvey. The theory of hyperfunctions on totally real subsets of a complex manifold with application to extension problems. Amer. J. Math. 91 (1969), 853-873.
- [HS] N. Honda, P. Schapira. A vanishing theorem for holonomic modules with positive characteristic varieties. *Publ. RIMS Kyoto Univ.* 26 (1990), 529-534.
- [KS1] M.Kapranov, V.Schechtman, Perverse sheaves over real hyperplane arrangements, arXiv:1403.5800.
- [KS2] M. Kashiwara, P. Schapira. Sheaves on manifolds. Springer, 1990.
- [KhT] M. Khovanov, R. Thomas. Braid cobordisms, triangulated categories, and flag varieties, arXiv:math/0609335.
- [KuL] A. Kuznetsov, V. Lunts. Categorical resolution of irrational singularities, arXiv:1212.6170.
- [Le] G. Lebeau. Annulation de la cohomologie hyperfonction de certains modules holonomes. C.R. Acad. Sci. Paris, Sér. A, t.290 (1980), 313-316.
- [Lu] J. Lurie. Stable ∞ -categories, arXiv math/0608228.

- [Na] D. Nadler. Fukaya categories as categorical Morse homology. SIGMA Symmetry Integrability Geom. Methods Appl. 10 (2014), Paper 018.
- [NZ] D. Nadler, E. Zaslow. Constructible sheaves and the Fukaya category. J. Amer. Math. Soc. 22 (2009) 233-286.
- [SKK] M. Sato, T. Kawai, M. Kashiwara. Hyperfunctions and pseudo-differential equations. Lecture Notes in Math. 287 (1973), 265 - 529.
- [Sch] P. Schapira. Conditions de positivité dans une variété symplectique complexe. Application a l'étude des microfonctions. Ann. Sci. ENS 14 (1991), 121-139.
- [Se] P. Seidel, Fukaya categories and Picard-Lefschetz theory, European Mathematical Society, Zurich, 2008.
- [ST] P. Seidel, R. Thomas, Braid group actions on derived categories of coherent sheaves, arXiv:math/0001043.
- [Ta1] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. C. R. Math. Acad. Sci. Paris 340 (2005) 15-19.
- [Ta2] G. Tabuada. Théorie homotopique des DG-catégories, arXiv:0710.4303.
- [TV] B. Toën, M. Vaquié. Moduli of objects in dg-categories. Ann. Sci. École Norm. Sup. (4) 40 (2007) 387-444.

M.K.: Kavli Institute for Physics and Mathematics of the Universe (WPI), 5-1-5 Kashiwanoha, Kashiwa-shi, Chiba, 277-8583, Japan. Email: mikhail.kapranov@ipmu.jp

V.S.: Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France, schechtman@math.ups-toulouse.fr