OU-HET 846
IPMU14-0xxx

Two-dimensional superconfomal field theories from Riemann surfaces with boundary

Koichi Nagasaki[1] and Satoshi Yamaguchi[2]

[1]Kavli IPMU (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan
kohichi.nagasaki@ipmu.jp
[2]Department of Physics, Osaka University, 1-1 Machikaneyama, Toyonaka, Osaka 560-0043, Japan
yamaguch@het.phys.sci.osaka-u.ac.jp

Abstract

We consider a 2-dimensional conformal field theory (CFT) obtained from twisted compactification of the 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory on a Riemann surface with boundary. We find the boundary conditions to preserve some of the supersymmetry. In particular an $\mathcal{N} = (2, 2)$ superconformal field theory is obtained from supersymmetry breaking due to the boundary from $\mathcal{N} = (4, 4)$. In this case we calculate the central charge of the CFT and show its dependence on the topology of the Riemann surface.

1 Introduction and summary

We often find an interesting relationship between a geometry and a supersymmetric quantum field theory by compactifying a higher dimensional conformal field theory. The class S theories [1] are famous examples which are obtained by compactification of 6-dimensional (2,0) superconformal field theories (SCFTs) by Riemann surfaces. AGT correspondence [2, 3] is a relation between a class S theory and a 2-dimensional CFT on the Riemann surface. SCFTs obtained from a d-dimensional theory compactified on various manifolds are studied, for example, [4, 5, 6, 7, 8, 9, 10].

It is also interesting to consider a Riemann surface with boundary. However for the class S theories it seems difficult to introduce a boundary of the Riemann surface since an M5-brane cannot have a supersymmetric boundary.

In this paper we construct 2-dimensional CFTs obtained from compactification of 4-dimensional gauge theories on Riemann surfaces with boundary. To realize a boundary theory we consider type IIB superstring in this paper. Our gauge theory is a 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory (SYM) realized on the world-volume of D3-branes. These D3-branes can end on D5-branes or NS5-branes, and thus can have a boundary.

The 2-dimensional CFTs obtained from compactification on closed Riemann surfaces [4] are studied by using c-extremization [11, 12]. This method is an analogue to a-maximization in 4-dimensions [13, 14] and F-maximization in 3-dimensions [15]. For a-maximization its gravity dual is studied in [16, 17, 18, 19, 20].

In this paper we study the 4-dimensional $\mathcal{N}=4$ SYM on $\mathbb{R}^{1,1}\times\Sigma_o$ where Σ_o is a Riemann surface with a boundary. In the low energy limit this theory is expected to become a 2-dimensional CFT. We find a class of boundary conditions at the boundary of Σ_o which preserve some of the supersymmetry, following the strategy of [21]. The boundary is a geodesic and preserves the $\mathcal{N}=(0,1),\ (1,1),\ (2,2)$ supersymmetry out of the $\mathcal{N}=(0,2),\ (2,2),\ (4,4)$ original bulk supersymmetry, respectively. It is an interesting future work to study more general boundary conditions as in [21, 22, 23, 24, 25] and S-duality. In this paper we also show some attempt to find a different class of boundary conditions.

Among these theories we calculate the central charge for the $\mathcal{N}=(2,2)$ case because in this case the central charge is related to the 't Hooft anomaly coefficients which are invariant under the RG flow [26]. We obtain a positive central charge only when the Euler number χ_o of Σ_o is negative. In this case the central charge is written as

$$c=3d_G|\chi_o|, \tag{1.1}$$

where d_G is the dimension of the gauge group. This theory has the $\mathcal{N}=(2,2)$ superconformal symmetry with $c=3\times$(integer). Therefore this theory seems to be a sigma model with a Calabi-Yau target space. Further study of this theory, in particular the relationship with the theory of [4], is also an interesting problem.

Another interesting future work is to investigate the realization in the string theory and AdS/CFT correspondence [27]. Our setup is realized by D3-branes wrapping on a holomorphic cycle in a local Calabi-Yau manifold and ending on a 5-brane system [28, 29, 21, 22, 23, 24, 25].

The construction of this paper is as follows: In section 2 we introduce a twisted compactification of 4-dimensional gauge theories following [30, 31, 32]. In section 3 we find a condition for preserving supersymmetry and calculate the central charge.

2 Twisted compactification of $\mathcal{N}=4$ SYM

We first review a 4-dimensional $\mathcal{N}=4$ SYM on a curved spacetime and twisting following [11, 12]. In subsection 2.1, first we obtain the action on the flat spacetime. In subsections 2.2 and 2.3 we introduce a closed Riemann surface with constant curvature and twist the theory. We also show how many supersymmetries are preserved by compactification on closed Riemann surfaces.

2.1 $\mathcal{N}=4$ SYM on the flat spacetime

The 4-dimensional $\mathcal{N}=4$ SYM action on the flat spacetime is obtained by the trivial dimensional reduction from the 10-dimensional SYM. It contains a 10-dimensional vector field $A_M,\ M=0,1,\ldots,9$ and a 10-dimensional Majorana-Weyl spinor Ψ, which satisfies $\Gamma_{0123456789}\Psi=\Psi$. Both of them are in the adjoint representation of the gauge group G. The vector field is decomposed into a 4-dimensional vector $A_\mu,\ \mu=0,1,2,3,$ and 6 scalars $\Phi_A=A_A,\ A=4,\ldots,9$ in 4-dimensions. The action is written as

$$S=\frac{1}{{g_{\rm YM}}^2}\int d^4x\mathrm{Tr}'\left\{-\frac{1}{4}F_{MN}F^{MN}+\frac{i}{2}\overline{\Psi}\Gamma^M D_M\Psi\right\}, \tag{2.1}$$

where $g_{\rm YM}$ is the 4-dimensional gauge coupling. F_{MN} $M,N=0,1,\ldots,9$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \qquad \mu,\nu = 0,1,2,3, \tag{2.2}$$

$$F_{\mu A} = -F_{A\mu} = \partial_\mu \Phi_A + i[A_\mu, \Phi_A] =: D_\mu \Phi_A, \tag{2.3}$$

$$F_{AB} = i[\Phi_A, \Phi_B]. \tag{2.4}$$

The covariant derivative for Ψ is defined as

$$D_\mu \Psi = \partial_\mu \Psi + i[A_\mu, \Psi], \qquad D_A \Psi = i[\Phi_A, \Psi]. \tag{2.5}$$

Tr' is a trace normalized as $\mathrm{Tr}' = \frac{1}{h^\vee}\mathrm{Tr}_{\rm adjoint}$ where $h^\vee$ is the dual Coxeter number. For example, $\mathrm{Tr}' = 2\mathrm{Tr}_{\rm fundamental}$ for $\mathrm{SU}(N)$. The action is rewritten as

$$S = \frac{1}{{g_{\rm YM}}^2} \int d^4x \mathrm{Tr}' \Big\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} D_\mu \Phi_A D^\mu \Phi^A + \frac{1}{4}[\Phi_A, \Phi_B][\Phi^A, \Phi^B] + \frac{i}{2}\overline{\Psi}\Gamma^\mu D_\mu \Psi - \frac{1}{2}\overline{\Psi}\Gamma^A[\Phi_A, \Psi] \Big\}. \tag{2.6}$$

This action is invariant under the supersymmetry transformation:

$$\delta A_M = i\bar{\epsilon}\Gamma_M \Psi, \;\; \delta\Psi = \frac{1}{2}\Gamma^{MN} F_{MN}\epsilon. \tag{2.7}$$

The parameters ϵ are Majorana-Weyl fermions satisfying

$$\Gamma_{0123456789}\epsilon = \epsilon. \tag{2.8}$$

Then the supersymmetry current is obtained as

$$J^\mu = \frac{i}{2}\mathrm{Tr}'\{2F^{\mu N}\Gamma_N - F_{KL}\Gamma^{KL\mu}\}\Psi = \frac{i}{2}\mathrm{Tr}'\{F_{KL}\Gamma^{KL}\Gamma^\mu \Psi\}. \tag{2.9}$$

2.2 Riemann surfaces

We will consider this 4-dimensional $\mathcal{N}=4$ SYM theory compactified on a compact Riemann surface Σ. In this paper we concentrate on a Riemann surface with constant curvature $R = 2\kappa$, where

$$\kappa = \begin{cases} +1 & (\mathbf{g}=0) \\ 0 & (\mathbf{g}=1) \\ -1 & (\mathbf{g}>1), \end{cases} \tag{2.10}$$

for a genus $\mathbf{g}$ closed Riemann surface. We denote the coordinates of this Riemann surface by (x^2, x^3), the vielbein by E^a, $a=2,3$, and the spin connection by Ω^{23}. The curvature 2-form is written as $R^{23} = d\Omega^{23}$, and thus the Gauss-Bonnet theorem reads

$$\int_\Sigma d\Omega^{23} = \frac{1}{2}\int_\Sigma \sqrt{g}R = 4\pi(1-\mathbf{g}). \tag{2.11}$$

For $\mathbf{g} \neq 1$ the volume of the Riemann surface is

$$\mathrm{vol}_\Sigma = 4\pi|1-\mathbf{g}|, \tag{2.12}$$

and the volume form is

$$d\mathrm{vol}_\Sigma = \kappa d\Omega^{23}. \tag{2.13}$$

2.3 Twisted gauge theory on the curved spacetime

Now we consider the 4-dimensional $\mathcal{N}=4$ SYM theory on a curved spacetime with the metric $g_{\mu\nu}$ and a background SO(6) gauge field $\mathcal{A}_\mu=\frac{1}{2}\mathcal{A}_\mu^{AB}M_{AB}$, where $M_{AB},\ A,B=4,\cdots,9$ are the SO(6) generators. The action becomes

$$S=\frac{1}{{g_{\rm YM}}^2}\int d^4x\sqrt{g}\mathrm{Tr}'\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}D'_\mu\Phi_A D'^\mu\Phi^A+\frac{1}{4}[\Phi_A,\Phi_B][\Phi^A,\Phi^B] \\ +\frac{i}{2}\overline{\Psi}\Gamma^\mu D'_\mu\Psi-\frac{1}{2}\overline{\Psi}\Gamma^A[\Phi_A,\Psi]\Big\}, \tag{2.14}$$

where the covariant derivative D'_μ includes the spin connection and the SO(6) gauge field

$$D'_\mu\Phi_A:=\partial_\mu\Phi_A+i[A_\mu,\Phi_A]+\sum_B\mathcal{A}_\mu^{AB}\Phi_B, \tag{2.15}$$

$$D'_\mu\Psi:=\partial_\mu\Psi+i[A_\mu,\Psi]+\frac{1}{4}\Omega_\mu^{ab}\Gamma_{ab}-i\mathcal{A}_\mu\Psi. \tag{2.16}$$

Here $\mathcal{A}_\mu\Psi:=\frac{i}{4}\mathcal{A}_\mu^{AB}\Gamma_{AB}\Psi$. In order to preserve the supersymmetry, a parameter of the SUSY transformation (2.7) should satisfy the Killing spinor equation. The twisted Killing spinor equation is

$$D'_\mu\epsilon:=\left(\partial_\mu+\frac{1}{4}\Omega_\mu^{ab}\Gamma_{ab}-i\mathcal{A}_\mu\right)\epsilon=0. \tag{2.17}$$

We choose the external gauge field $\mathcal{A}_\mu$ in $\mathrm{SO}(2)^3\subset\mathrm{SO}(6)$, such that the field strength,

$$F=d\mathcal{A},\ \ \mathcal{A}=\mathcal{A}_\mu dx^\mu, \tag{2.18}$$

satisfies

$$F=\begin{cases}-T\ d\mathrm{vol}_\Sigma & (\mathbf{g}\neq 1)\\ -T\frac{2\pi}{\mathrm{vol}_\Sigma}d\mathrm{vol}_\Sigma & (\mathbf{g}=1).\end{cases} \tag{2.19}$$

Here T is an $\mathrm{SO}(2)^3$ generator

$$T=a_1T_1+a_2T_2+a_3T_3, \tag{2.20}$$

where a_i are parameters of twisting and $T_i,\ i=1,2,3$ are generators expressed in the spinor representation

$$T_1=\frac{i}{2}\Gamma^{45},\ \ T_2=\frac{i}{2}\Gamma^{67},\ \ T_3=\frac{i}{2}\Gamma^{89}. \tag{2.21}$$

The condition for existing covariantly constant spinors is, from eq. (2.17),

$$D'_\mu\epsilon=0\ \Rightarrow\ [D'_2,D'_3]\epsilon=0\ \Rightarrow\ \left(\frac{1}{2}d\Omega_{23}\Gamma^{23}-id\mathcal{A}\right)\epsilon=0. \tag{2.22}$$

Using the relations (2.13) and (2.19),

$$\left(\frac{1}{2}\kappa\ d\mathrm{vol}_\Sigma\cdot\Gamma^{23}+id\mathrm{vol}_\Sigma\cdot T\right)\epsilon=0. \tag{2.23}$$

Finally, substituting eq.(2.20), the supersymmetry condition is

$$\left(-\kappa i\Gamma^{23}+a_1i\Gamma^{45}+a_2i\Gamma^{67}+a_3i\Gamma^{89}\right)\epsilon=0. \tag{2.24}$$

The amount of the supersymmetry depends on the number of the non-zero parameters among a_i, $i=1,2,3$. Let us classify them here:

1. All a_i are non-zero: $(-\kappa\Gamma^{23} + a_1\Gamma^{45} + a_2\Gamma^{67} + a_3\Gamma^{89})\,\epsilon = 0$.
   In this case the number of the supersymmetries is $\mathcal{N} = (0,2)$. The constraint for the parameters a_i is
$$a_1 + a_2 + a_3 = \kappa. \tag{2.25}$$

2. Two of a_i are non-zero: $(-\kappa\Gamma^{23} + a_1\Gamma^{45} + a_2\Gamma^{67})\,\epsilon = 0$.
   In this case the number of the supersymmetries is $\mathcal{N} = (2,2)$. The constraint for the parameters a_i is
$$a_1 + a_2 = \kappa. \tag{2.26}$$

3. One of a_i is non-zero: $(-\kappa\Gamma^{23} + a_1\Gamma^{45})\,\epsilon = 0$.
   In this case the number of the supersymmetries is $\mathcal{N} = (4,4)$. The constraint for the non-zero parameter a_1 is
$$a_1 = \kappa. \tag{2.27}$$

4. No background field: $(-\kappa\Gamma^{23})\,\epsilon = 0$.
   In this case the number of the supersymmetries is $\mathcal{N} = (8,8)$. This situation is realized only for the zero curvature case $\kappa = 0$, i.e. $\mathbf{g} = 1$.

These results are summarized in Table 1.

# of $a_i \neq 0$	$\mathcal{N}$	$\mathbf{g}$
3	$(0,2)$	all
2	$(2,2)$	all
1	$(4,4)$	all
0	$(8,8)$	1

Table 1: Remaining supersymmetries for closed Riemann surfaces.

3 SUSY boundary condition and central charge

In this section we introduce a boundary on the Riemann surface. We assume that the boundary is a geodesic. First we explain this assumption is appropriate and simplifies our argument. After that we study the boundary condition for preserving some supersymmetries. We obtain the central charge when the $\mathcal{N} = (2,2)$ supersymmetry is preserved. We also show an attempt to find other class of boundary conditions.

3.1 Shape of the Boundary

In this paper we focus on Riemann surfaces with one boundary. We also assume that these surfaces have constant curvature. In this paper we only consider a geodesic boundary for simplicity. There could be a non-geodesic boundary which preserves some supersymmetry, although we do not find an example. The analysis is rather simple for the geodesic boundary for the following reasons. Let (x^2, x^3) the coordinates of the Riemann surface and the geodesic boundary $x^3 = 0$. Then we can choose a gauge such that locally $\mathcal{A}_2 = \mathcal{A}_3 = 0$ on the boundary since $\mathcal{A}$ is proportional to Ω^{23} and we can choose the gauge $\Omega^{23} = 0$ on a geodesic. Then terms including the external gauge field $\mathcal{A}_\mu$ in the covariant derivative (2.15) can be omitted and $D' = D$ is satisfied at least locally. However we cannot ignore the holonomy along the boundary. The boundary condition must be consistent with this holonomy. Another reason for choosing the geodesic boundary is that we want to use the doubling trick later. If the boundary is a geodesic, one can join together the Riemann surface and a copy of it with the opposite orientation to construct a closed surface with constant curvature.

Let us see the holonomy of this external gauge field along the boundary. First for simplicity we consider an S^2 with a boundary at the equator — a northern (or southern) hemisphere S^2_+ (S^2_-). This holonomy is given by

$$\oint_{\partial S^2_+} \mathcal{A} = \int_{S^2_+} d\mathcal{A} = \int_{S^2_+} F = \text{Magnetic flux.} \tag{3.1}$$

Here we use Stokes' theorem to express it as an integral of the gauge field strength. This integral gives a magnetic flux through the surface S^2_+. Due to the Dirac quantization condition, the integral of magnetic flux on the S^2 is an integral multiplication of 2π. Now this gauge field is distributed isotropically. Then the integral only over the northern hemisphere (3.1) gives an integer or a half integer times 2π. We can use the same strategy for a general Riemann surface Σ_o with one geodesic boundary. Let Σ be the closed Riemann surface made by gluing Σ_o and a copy of it with the opposite orientation $\overline{\Sigma}_o$ along their boundaries. Notice that the genus $\mathbf{g}$ of Σ is an even number and thus it is not 1. The holonomy along this boundary can be written by using eqs. (2.19), (2.20), (2.12) as

$$H := \exp\left(i\oint_{\partial\Sigma_o}\mathcal{A}\right) = \exp\left(i\int_{\Sigma_o}F\right) = \exp\left(\frac{i}{2}\int_{\Sigma}F\right) = \prod_{i=1,2,3}\exp(-i\pi n_i T_i), \tag{3.2}$$

where $n_i := 2|1-\mathbf{g}|a_i$ are integers [12]. Later we use the fact

$$H^2 = \exp\left(i\int_\Sigma F\right) = 1 \tag{3.3}$$

following from the Dirac quantization condition. The boundary condition considered in this paper later (3.11) is consistent with this holonomy (3.2).

3.2 Boundary condition

Let us here consider the boundary conditions which preserve some part of the supersymmetry. For preserving the supersymmetry the current component normal to the boundary must be zero at the boundary ($x^3 = 0$). From eq. (2.9) this condition is expressed as

$$\bar{\epsilon}J^3 = 0 \quad \Leftrightarrow \quad \mathrm{Tr}'\left(\bar{\epsilon}F_{KL}\Gamma^{KL}\Gamma^3\Psi\right) = 0. \tag{3.4}$$

In this condition we can replace D' by D since we can choose the gauge where $\mathcal{A} = 0$ at the boundary. Thus we can employ the same strategy as [21] (see also [24]). Define the following matrices:

$$B_0 = \Gamma^{468579}, \tag{3.5}$$

$$B_1 = \Gamma^{3468}, \tag{3.6}$$

$$B_2 = \Gamma^{3579}, \tag{3.7}$$

and redefine the scalar fields

$$(X_4, X_6, X_8) := (\Phi_4, \Phi_6, \Phi_8), \tag{3.8}$$

$$(Y_5, Y_7, Y_9) := (\Phi_5, \Phi_7, \Phi_9). \tag{3.9}$$

The boundary condition (3.4) is decomposed into the following equations as done in [21]:

$$\begin{aligned}
\mathrm{Tr}'\bar{\epsilon}(\Gamma^{\mu\nu}F_{\mu\nu} + 2\Gamma^{3\mu}F_{3\mu})\Gamma^3\Psi &= 0,\\
\mathrm{Tr}'\bar{\epsilon}(2\Gamma^{3a}D_3X_a + \Gamma^{ab}[X_a, X_b])\Gamma^3\Psi &= 0,\\
\mathrm{Tr}'\bar{\epsilon}(2\Gamma^{3m}D_3Y_m + \Gamma^{mn}[Y_m, Y_n])\Gamma^3\Psi &= 0,\\
\mathrm{Tr}'\bar{\epsilon}\Gamma^{\mu a}D_\mu X_a\Gamma^3\Psi &= 0,\\
\mathrm{Tr}'\bar{\epsilon}\Gamma^{\mu m}D_\mu Y_m\Gamma^3\Psi &= 0,\\
\mathrm{Tr}'\bar{\epsilon}\Gamma^{am}[X_a, Y_m]\Gamma^3\Psi &= 0,
\end{aligned} \tag{3.10}$$

where $\mu, \nu = 0, 1, 2,\ a, b = 4, 6, 8$, and $m, n = 5, 7, 9$. An example of the boundary condition is the NS5-brane like boundary condition

$$D_3X_a = 0,\ Y_m = 0,\ F_{\mu 3} = 0, \tag{3.11}$$

for the bosonic fields. For the fermionic fields we impose

$$B_2\Gamma^3\Psi = -\Gamma^3\Psi \tag{3.12}$$

at the boundary. Actually the NS5-brane like boundary conditions (3.11) and (3.12) preserve the supersymmetry if the parameter ϵ satisfies

$$B_2\epsilon = \epsilon. \tag{3.13}$$

The conditions (3.10) is verified. The condition (3.13) for ϵ kills half of the supersymmetry as follows. If an $i\Gamma^{23}$ eigenvector ϵ_1 satisfies (2.24), $B_2\epsilon_1$ also satisfies (2.24) and they are independent. Therefore among the linear combinations of these two independent parameters, one combination $\epsilon = (1 + B_2)\epsilon_1$ satisfies the condition (3.13). Since ϵ_1 and $B_2\epsilon_1$ have the same chirality (Γ^{01} eigenvalue), the preserved supersymmetry is as follows.

1. $\mathcal{N} = (0, 2)$ bulk $\Rightarrow \mathcal{N} = (0, 1)$.
2. $\mathcal{N} = (2, 2)$ bulk $\Rightarrow \mathcal{N} = (1, 1)$.
3. $\mathcal{N} = (4, 4)$ bulk $\Rightarrow \mathcal{N} = (2, 2)$.

Let us verify the boundary conditions (3.11) and (3.12) are consistent with the holonomy (3.2). For the vector representation $(H\Phi)_A = \pm\Phi_A$, so the conditions for the bosons (3.11) are consistent. The consistency of the condition for the fermions (3.13) is verified by (3.3) $H^2 = 1$ and $B_2\Gamma^3 H\Psi = H^{-1}B_2\Gamma^3\Psi$.

3.3 $\mathcal{N} = (4,4)$ case and the central charge

The case where the bulk $\mathcal{N} = (4,4)$ supersymmetry is broken to $\mathcal{N} = (2,2)$ by the boundary is interesting because of the R-symmetry of the $\mathcal{N} = 2$ superconformal symmetry. In this case $a_2 = a_3 = 0$ and $a_1 = \kappa$, and T in eq. (2.20) becomes

$$T = \kappa \frac{i}{2} \Gamma^{45}. \tag{3.14}$$

The preserved supersymmetry parameters satisfy eq. (2.24), which is rewritten as

$$\Gamma^{2345} \epsilon = -\epsilon. \tag{3.15}$$

Then the exact central charge is obtained from the 't Hooft anomaly coefficient as in [12]. However in our case the situation is much simpler since there is only one candidate U(1) symmetry Q^R for the R-symmetry

$$Q^R = \frac{i}{2} \Gamma^{68} + \frac{i}{2} \Gamma^{79}. \tag{3.16}$$

This is determined such that for the right moving SUSY parameters ϵ ($\Gamma^{01}\epsilon = +\epsilon$) satisfy $Q^R \epsilon = \pm\epsilon$ and the left moving ones satisfy $Q^R \epsilon = 0$. The right moving central charge is expressed as

$$c = 3\mathrm{Tr}_{\text{Weyl fermion}} (\Gamma^{01} (Q^R)^2). \tag{3.17}$$

In the above expression $\mathrm{Tr}_{\text{Weyl fermion}}$ means counting the number of the 2-dimensional Weyl fermions.

The number of the chiral fermions can be counted by the index theorem as in [11]. In this paper we use the doubling trick to map the problem to the index theorem in the closed Riemann surface. We take the Riemann surface Σ_o and its orientation flipped one $\overline{\Sigma}_o$, and join them together, $\Sigma_o \bigcup \overline{\Sigma}_o =: \Sigma$, so that the boundary is corresponding (See Figure 1). Originally Ψ includes four 4-dimensional Weyl spinors. Half of them satisfying $\Gamma^{6789}\Psi = -\Psi$ have charge ± 1 of Q^R and the others are neutral. Let us denote these two charged 4-dimensional Weyl spinors $\Psi_\pm$ which satisfy $i\Gamma^{45}\Psi_\pm = \pm\Psi_\pm$ and $B_2 \Psi_\pm = \Psi_\mp$. These two fermions on $\mathbb{R}^{1,1} \times \Sigma_o$ are treated as a fermion Ψ_c on $\mathbb{R}^{1,1} \times \Sigma$. Ψ_c is defined as

$$\Psi_c = \begin{cases} \Psi_-(z), & (\mathrm{Im}(z) \geq 0) \\ \Psi_+(z^*), & (\mathrm{Im}(z) \leq 0). \end{cases} \tag{3.18}$$

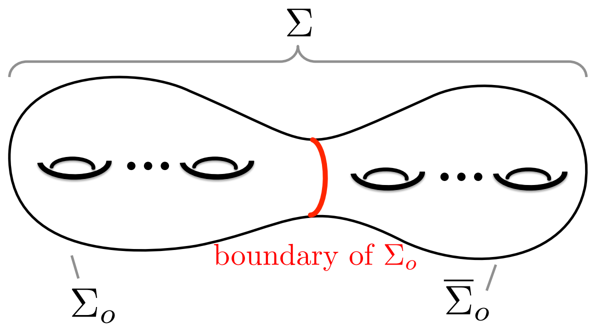

Figure 1: (Doubling trick) We construct a closed surface Σ by taking Σ_o and its orientation flipped copy $\overline{\Sigma}_o$.

Here we use the complex coordinate $z = x^2 + ix^3$ of Σ such that Σ_o is parametrized by $\mathrm{Im}z \geq 0$, $\overline{\Sigma}_o$ is parametrized by $\mathrm{Im}z \leq 0$ and $z \to z^*$ is the symmetry which exchanges Σ_o and $\overline{\Sigma}_o$. Actually this Ψ_c is continuous at the boundary due to the boundary condition (3.12). Furthermore, we define extended spin connections and gauge fields

$$\Omega^{23}_{\bar{z}}(z) := \begin{cases} \Omega^{23}_{\bar{z}}(z) & (\mathrm{Im}(z) \geq 0) \\ -\Omega^{23}_{z}(z^*) & (\mathrm{Im}(z) \leq 0), \end{cases} \qquad \mathcal{A}^{45}_{\bar{z}}(z) := \begin{cases} \mathcal{A}^{45}_{\bar{z}}(z) & (\mathrm{Im}(z) \geq 0) \\ -\mathcal{A}^{45}_{z}(z^*) & (\mathrm{Im}(z) \leq 0). \end{cases} \tag{3.19}$$

Then according to the above definitions, the Dirac equations for $\Psi_\pm$ on $\mathbb{R}^{1,1} \times \Sigma_o$ are equivalent to the one for Ψ_c on $\mathbb{R}^{1,1} \times \Sigma$

$$\Gamma^\mu D'_\mu \Psi_c(z) = 0. \tag{3.20}$$

We denote the number of 2-dimensional right(left)-moving massless fermions by $n_{R(L)}$ for the 4-dimensional Weyl fermion Ψ_c. The index theorem gives the difference of these numbers and it is rewritten using eqs. (2.19), (2.12) :

$$n_R - n_L = -\frac{1}{2\pi} \int_\Sigma \mathrm{Tr}_{\Psi_c} F = t2|\mathbf{g} - 1|, \tag{3.21}$$

where Tr_{Ψ_c} is taken in the representation to Ψ_c, and t is the eigenvalue of T for the fermion Ψ_c which is given by $t = -\kappa/2$ using eqs. (3.18), (3.14). Taking the multiplicity of the Lie algebra into account we obtain the result

$$\begin{aligned} c &= 3d_G(n_R - n_L) \\ &= -3d_G\kappa|\mathbf{g} - 1|, \end{aligned} \tag{3.22}$$

where d_G is the dimension of the gauge group. This expression gives the positive c only when $\kappa = -1$, $(\mathbf{g} > 1)$. In this case

$$c = 3d_G|\chi_o|. \tag{3.23}$$

In the final expression we use the Euler number of the original Riemann surface with the boundary. We are now considering a case where the Riemann surface has only one boundary, $\mathbf{b} = 1$. Then the Euler number of the original surface, Σ_o, is $\chi_o = 2 - 2\mathbf{g}/2 - \mathbf{b} = 1 - \mathbf{g}$.

3.4 Candidate for other types of boundary condition

In this subsection, we examine boundary conditions different from the NS5-like shown in the previous subsections. We show some cases where the original bulk supersymmetries are $\mathcal{N} = (0, 2)$, $(2, 2)$ and $(4, 4)$. We study how these supersymmetries are broken when introducing the boundary.

In this subsection we use the following notation for the SUSY parameters ϵ_I, $I = 1, \cdots, 8$. We diagonalize $\Gamma^{01}, i\Gamma^{M,M+1}$, $M = 2, 4, 6, 8$ and denote the eigenvalues as follows:

$$\begin{cases} \Gamma^{01}\epsilon_I = \lambda^0_I \epsilon_I \\ i\Gamma^{M,M+1}\epsilon_I = \lambda^M_I \epsilon_I \end{cases} \quad \Rightarrow \quad \begin{cases} \overline{\epsilon_I}\Gamma^{01} = -\lambda^0_I \overline{\epsilon_I} \\ \overline{\epsilon_I}(i\Gamma^{M,M+1}) = -\lambda^M_I \overline{\epsilon_I} \end{cases}, \tag{3.24}$$

where eigenvalues λ^0_I, λ^M_I take values $+1$ or -1 and are summarized in Table 2.

		λ_I^0	λ_I^2	λ_I^4	λ_I^6	λ_I^8
I	1	$-$	$+$	$+$	$+$	$+$
	2	$-$	$-$	$-$	$-$	$-$
	3	$+$	$+$	$+$	$+$	$-$
	4	$+$	$-$	$-$	$-$	$+$
	5	$-$	$+$	$+$	$-$	$-$
	6	$-$	$-$	$-$	$+$	$+$
	7	$+$	$+$	$+$	$-$	$+$
	8	$+$	$-$	$-$	$+$	$-$

Table 2: Eigenvalues of ϵ_I.

3.4.1 $\mathcal{N}=(0,2)$ case

The SUSY parameters preserved in the bulk are ϵ_1 and ϵ_2.

The current condition (3.4) for these generators is

$$\mathrm{Tr}'\,\bar{\epsilon}\left(F_{01}\Gamma^{01}+F_{23}\Gamma^{23}+F_{45}\Gamma^{45}+F_{67}\Gamma^{67}+F_{89}\Gamma^{89}\right)\Gamma^3\Psi=0, \tag{3.25}$$

$$\mathrm{Tr}'\,\bar{\epsilon}\left(F_{M,N}\Gamma^{M,N}+F_{M,N+1}\Gamma^{M,N+1}+F_{M+1,N}\Gamma^{M+1,N}+F_{M+1,N+1}\Gamma^{M+1,N+1}\right)\Gamma^3\Psi=0, \tag{3.26}$$
$$(M,N)=(0,2),(0,4),(0,6),(0,8),(2,4),(2,6),(2,8),(4,6),(4,8),(6,8).$$

We impose the boundary condition for the fermion field:

$$-i\Gamma^{23}\Psi=\Psi. \tag{3.27}$$

From the first equation (3.25),

$$\begin{aligned}&\mathrm{Tr}'\,\overline{\epsilon_I}\left(F_{01}\Gamma^{01}+F_{23}\Gamma^{23}+F_{45}\Gamma^{45}+F_{67}\Gamma^{67}+F_{89}\Gamma^{89}\right)\Gamma^3(-i\Gamma^{23}\Psi)=0\\ \leftrightarrow\ &\mathrm{Tr}'\,\overline{\epsilon_I}(i\Gamma^{23})\left(F_{01}\Gamma^{01}+F_{23}\Gamma^{23}+F_{45}\Gamma^{45}+F_{67}\Gamma^{67}+F_{89}\Gamma^{89}\right)\Gamma^3\Psi=0,\quad (I=1,2).\end{aligned} \tag{3.28}$$

The lefthand side is trivially zero for ϵ_1 which satisfies $\overline{\epsilon_1}(i\Gamma^{M,M+1})=-\overline{\epsilon_1}$. For ϵ_2 this equation gives the condition

$$\mathrm{Tr}'\overline{\epsilon_2}\Big(F_{01}-i\left(F_{23}+F_{45}+F_{67}+F_{89}\right)\Big)\Gamma^3\Psi=0. \tag{3.29}$$

Then,

$$F_{01}=0,\qquad F_{23}+F_{45}+F_{67}+F_{89}=0. \tag{3.30}$$

For the second equation (3.26) of (M,N)=$(0,2)$, $(2,4)$, $(2,6)$ and $(2,8)$ are trivially zero for the case of ϵ_2 in the same way and in the cases (M,N)=$(0,4)$,(0,6), $(0,8)$,(4,6), $(4,8)$ and $(6,8)$ this equation becomes trivial for ϵ_1.

The condition for the supersymmetry generated by ϵ_I to be preserved is summarized as follows:

(i) Supersymmetry generated by ϵ_1

$$\begin{cases}F_{0,M}+F_{1,M}=0 & (M=2,3),\\ F_{2,M}-F_{3,M+1}=F_{2,M+1}+F_{3,M}=0 & (M=4,6,8).\end{cases} \tag{3.31}$$

(ii) Supersymmetry generated by ϵ_2

$$\begin{cases} F_{0,1}=0, \;\; F_{23}+F_{45}+F_{67}+F_{89}=0 \\ F_{0,M}+F_{1,M}=0 & (M=4,5,6,7,8,9), \\ F_{M,N}-F_{M+1,N+1}=F_{M,N+1}+F_{M+1,N}=0 & ((M,N)=(4,6),(4,8),(6,8)). \end{cases} \tag{3.32}$$

Let us define complex fields

$$Z_1 := \Phi_1 + i\Phi_2, \;\; Z_2 := \Phi_3 + i\Phi_4, \;\; Z_3 := \Phi_5 + i\Phi_6. \tag{3.33}$$

We define coordinates on the 2d CFT and the Riemann surface and redefine gauge field on them.

$$\text{On } (x^0,x^1) : \; x^0 \pm x^1 =: x^{\pm}, \;\; A_{x^{\pm}} := \frac{1}{2}\,(A_0 \pm A_1), \tag{3.34}$$

$$\text{On } (x^2,x^3) : \; x^2 \pm ix^3 =: w^{\pm}, \;\; A_{w^{\pm}} := \frac{1}{2}\,(A_2 \mp iA_3). \tag{3.35}$$

Then, the following new derivatives can be defined:

$$\frac{1}{2}\Big(D_0 \pm D_1\Big) = \left(\frac{\partial}{\partial x^{\pm}} + [A_{x^{\pm}}, *]\right) =: D_{x^{\pm}}, \tag{3.36}$$

$$\frac{1}{2}\Big(D_2 \mp iD_3\Big) = \left(\frac{\partial}{\partial w^{\pm}} + [A_{w^{\pm}}, *]\right) =: D_{w^{\pm}}. \tag{3.37}$$

Using these notations the supersymmetry conditions (3.31), (3.32) are respectively rewritten as follows.

1. Supersymmetry generated by ϵ_1:

$$(3.31) \Rightarrow \begin{cases} F_{0M}+F_{1M}=0 \;\; (M=2,3), \\ D_{w^-}Z_A = 0. \end{cases} \tag{3.38}$$

2. Supersymmetry generated by ϵ_2:

$$(3.32) \Rightarrow \begin{cases} F_{01}=0, \;\; F_{23} = -\frac{i}{2}\sum_i [Z_i, \overline{Z}_i], \\ D_{x^+}Z_i = 0, \\ [Z_i, Z_j] = 0. \end{cases} \tag{3.39}$$

In the second case we find that this equation looks like a Hitchin system [33]. For more details of these types of equations, see [34].

3.4.2 $\mathcal{N}=(2,2)$ case

The supersymmetry parameters preserved in the bulk are $\epsilon_1,\ldots,\epsilon_4$ in Table 2. In this case we can use the same method to the previous $\mathcal{N}=(0,2)$ case. The normal component of the current satisfies:

$$\mathrm{Tr}'\,\bar{\epsilon}\left(F_{01}\Gamma^{01}+F_{23}\Gamma^{23}+F_{45}\Gamma^{45}+F_{67}\Gamma^{67}+F_{89}\Gamma^{89}\right)\Gamma^3\Psi = 0, \tag{3.40}$$

$$\mathrm{Tr}'\,\bar{\epsilon}\left(F_{M,N}\Gamma^{M,N}+F_{M,N+1}\Gamma^{M,N+1}+F_{M+1,N}\Gamma^{M+1,N}+F_{M+1,N+1}\Gamma^{M+1,N+1}\right)\Gamma^3\Psi = 0. \tag{3.41}$$

The first equation (3.40) becomes trivial for ϵ_I having eigenvalue $\lambda_I^2 = +1$ in the same way to $\mathcal{N} = (0,2)$ case and for ϵ_I having eigenvalue $\lambda_I^2 = -1$ this equation becomes

$$F_{01} = 0, \ \sum_{i=1}^{4} \lambda_I^{2i} F^{2i,2i+1} = 0. \tag{3.42}$$

The second equation (3.41) splits into two groups

$$\begin{aligned}
&(M,N) = (0,2),(2,4),(2,6),(2,8),\\
&(M,N) = (0,4),(0,6),(0,8),(4,6),(4,8),(6,8).
\end{aligned}$$

The former becomes trivial for $\lambda_I^2 = -1$ and the latter becomes trivial for $\lambda_I^2 = +1$. The nontrivial conditions are for $\lambda_I^2 = +1$

$$F_{02} - \lambda_I^0 F_{12} = F_{03} - \lambda_I^0 F_{13} = 0, \tag{3.43}$$

$$\begin{aligned}
&F_{M,N} - \lambda_I^M \lambda_I^N F_{M+1,N+1} = +\lambda_I^M F_{M+1,N} + \lambda_I^N F_{M,N+1} = 0\\
&\qquad\qquad (M,N) = (2,4)(2,6)(2,8),
\end{aligned} \tag{3.44}$$

and for $\lambda_I^2 = -1$

$$F_{0,M} - \lambda_I^0 F_{1,M} = 0, \qquad M = 4,5,6,7,8,9, \tag{3.45}$$

$$\begin{aligned}
&F_{M,N} - \lambda_I^M \lambda_I^N F_{M+1,N+1} = \lambda_I^M F_{M+1,N} + \lambda_I^N F_{M,N+1} = 0\\
&\qquad\qquad (M,N) = (4,6)(4,8)(6,8).
\end{aligned} \tag{3.46}$$

Summarizing the above, the supersymmetries generated by ϵ_I are respectively as follows:

1. ϵ_I $(\lambda_I^2 = +1)$

$$\begin{cases}
F_{0,M} - \lambda_I^0 F_{1,M} = 0 & (M = 2,3)\\
F_{M,N} - \lambda_I^M \lambda_I^N F_{M+1,N+1} = \lambda_I^M F_{M+1,N} + \lambda_I^N F_{M,N+1} = 0 & (M,N) = (2,4)(2,6)(2,8),
\end{cases} \tag{3.47}$$

2. ϵ_I $(\lambda_I^2 = -1)$

$$\begin{cases}
F_{01} = 0, \qquad \sum_{i=1}^{4} \lambda_I^{2i} F^{2i,2i+1} = 0 & \\
F_{0,M} - \lambda_I^0 F_{1,M} = 0 & (M = 4,5,6,7,8,9)\\
F_{M,N} - \lambda_I^M \lambda_I^N F_{M+1,N+1} = \lambda_I^M F_{M+1,N} + \lambda_I^N F_{M,N+1} = 0 & (M,N) = (4,6)(4,8)(6,8).
\end{cases} \tag{3.48}$$

The case we studied before in the subsection 3.4.1 corresponds to the case of $\lambda_I^0 = -1$ (eqs.(3.47)) in the current case.

3.4.3 $\mathcal{N} = (4,4)$ case

The supersymmetry parameters preserved in the bulk are $\epsilon_1, \ldots, \epsilon_8$ in Table 2. The conditions for the bosonic fields are

$$F_{0,1} = 0, \quad \sum_{i=1}^{4} \lambda_I^{2i} F_{2i,2i+1} = 0, \tag{3.49}$$

where $I = 2,4,6,8$ and

$$F_{0M} - \lambda_I^0 F_{1M} = 0, \tag{3.50}$$

where $M = 2,3$ for $I = 2,4,6,8$ while $M = 4,5,6,7,8,9$ for $I = 1,3,5,7$, and

$$F_{M,N} - \lambda_I^M \lambda_I^N F_{M+1,N+1} = \lambda_I^N F_{M,N} - \lambda_I^M F_{M+1,N+1} = 0, \tag{3.51}$$

where $(M,N) = (2,4), (2,6)$ and $(2,8)$ for $I =$ even, while $(M,N) = (4,6), (4,8)$ and $(6,8)$ for $I =$ odd. In the case of $\mathcal{N} = (2,2)$ after the breaking is an interesting case and the central charge is obtained only from the calculation of the 't Hooft anomaly, as shown in subsection 3.3.

Acknowledgement

We are pleased to thank Dongmin Gang, Kentaro Hori, Masahito Yamazaki, and Yutaka Yoshida for useful discussion. K.N. would like to thank the organizers of "the 7th Taiwan String Workshop" at National Taiwan University for giving him an opportunity to talk about this work. K.N. was supported in part by JSPS Research Fellowship for Young Scientists and JSPS KAKENHI Grant Number 13J02068. This work was supported in part by World Premier International Research Center Initiative (WPI), MEXT, Japan.

References

[1] D. Gaiotto, "N=2 dualities," *JHEP* **1208** (2012) 034, `arXiv:0904.2715 [hep-th]`.

[2] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett.Math.Phys.* **91** (2010) 167–197, `arXiv:0906.3219 [hep-th]`.

[3] N. Wyllard, "A_{N-1} conformal Toda field theory correlation functions from conformal N = 2 SU(N) quiver gauge theories," *JHEP* **0911** (2009) 002, `arXiv:0907.2189 [hep-th]`.

[4] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, "Topological reduction of 4-d SYM to 2-d sigma models," *Nucl.Phys.* **B448** (1995) 166–186, `arXiv:hep-th/9501096 [hep-th]`.

[5] T. Dimofte, D. Gaiotto, and S. Gukov, "Gauge Theories Labelled by Three-Manifolds," *Commun.Math.Phys.* **325** (2014) 367–419, `arXiv:1108.4389 [hep-th]`.

[6] I. Bah, C. Beem, N. Bobev, and B. Wecht, "AdS/CFT Dual Pairs from M5-Branes on Riemann Surfaces," *Phys.Rev.* **D85** (2012) 121901, arXiv:1112.5487 [hep-th].

[7] I. Bah, C. Beem, N. Bobev, and B. Wecht, "Four-Dimensional SCFTs from M5-Branes," *JHEP* **1206** (2012) 005, arXiv:1203.0303 [hep-th].

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. P. Warner, "Selfdual strings and N=2 supersymmetric field theory," *Nucl.Phys.* **B477** (1996) 746–766, arXiv:hep-th/9604034 [hep-th].

[9] S. Cecotti, C. Cordova, and C. Vafa, "Braids, Walls, and Mirrors," arXiv:1110.2115 [hep-th].

[10] T. Okazaki, "Membrane Quantum Mechanics," arXiv:1410.8180 [hep-th].

[11] F. Benini and N. Bobev, "Exact two-dimensional superconformal R-symmetry and c-extremization," *Phys.Rev.Lett.* **110** no. 6, (2013) 061601, arXiv:1211.4030 [hep-th].

[12] F. Benini and N. Bobev, "Two-dimensional SCFTs from wrapped branes and c-extremization," *JHEP* **1306** (2013) 005, arXiv:1302.4451 [hep-th].

[13] K. A. Intriligator and B. Wecht, "The Exact superconformal R symmetry maximizes a," *Nucl.Phys.* **B667** (2003) 183–200, arXiv:hep-th/0304128 [hep-th].

[14] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," *JHEP* **1205** (2012) 159, arXiv:1012.3210 [hep-th].

[15] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, "Contact Terms, Unitarity, and F-Maximization in Three-Dimensional Superconformal Theories," *JHEP* **1210** (2012) 053, arXiv:1205.4142 [hep-th].

[16] D. Martelli, J. Sparks, and S.-T. Yau, "The Geometric dual of a-maximisation for Toric Sasaki-Einstein manifolds," *Commun.Math.Phys.* **268** (2006) 39–65, arXiv:hep-th/0503183 [hep-th].

[17] D. Martelli, J. Sparks, and S.-T. Yau, "Sasaki-Einstein manifolds and volume minimisation," *Commun.Math.Phys.* **280** (2008) 611–673, arXiv:hep-th/0603021 [hep-th].

[18] A. Butti and A. Zaffaroni, "R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization," *JHEP* **0511** (2005) 019, arXiv:hep-th/0506232 [hep-th].

[19] R. Eager, "Equivalence of A-Maximization and Volume Minimization," *JHEP* **1401** (2014) 089, arXiv:1011.1809 [hep-th].

[20] Y. Tachikawa, "Five-dimensional supergravity dual of a-maximization," *Nucl.Phys.* **B733** (2006) 188–203, arXiv:hep-th/0507057 [hep-th].

[21] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," *J.Statist.Phys.* **135** (2009) 789–855, `arXiv:0804.2902 [hep-th]`.

[22] D. Gaiotto and E. Witten, "Janus Configurations, Chern-Simons Couplings, And The theta-Angle in N=4 Super Yang-Mills Theory," *JHEP* **1006** (2010) 097, `arXiv:0804.2907 [hep-th]`.

[23] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory," *Adv.Theor.Math.Phys.* **13** (2009) 721, `arXiv:0807.3720 [hep-th]`.

[24] A. Hashimoto, P. Ouyang, and M. Yamazaki, "Boundaries and defects of $\mathcal{N} = 4$ SYM with 4 supercharges. Part I: Boundary/junction conditions," *JHEP* **1410** (2014) 107, `arXiv:1404.5527 [hep-th]`.

[25] A. Hashimoto, P. Ouyang, and M. Yamazaki, "Boundaries and defects of $\mathcal{N} = 4$ SYM with 4 supercharges. Part II: Brane constructions and 3d $\mathcal{N} = 2$ field theories," *JHEP* **1410** (2014) 108, `arXiv:1406.5501 [hep-th]`.

[26] G. Hooft, *Recent developments in gauge theories.* NATO ASI series: Physics. Plenum Press, 1980.

[27] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200 [hep-th]`.

[28] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl.Phys.* **B492** (1997) 152–190, `arXiv:hep-th/9611230 [hep-th]`.

[29] T. Kitao, K. Ohta, and N. Ohta, "Three-dimensional gauge dynamics from brane configurations with (p,q) - five-brane," *Nucl.Phys.* **B539** (1999) 79–106, `arXiv:hep-th/9808111 [hep-th]`.

[30] K. J. Costello, "Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4," `arXiv:1111.4234 [math.QA]`.

[31] M. Bershadsky, C. Vafa, and V. Sadov, "D-branes and topological field theories," *Nucl.Phys.* **B463** (1996) 420–434, `arXiv:hep-th/9511222 [hep-th]`.

[32] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," *Int.J.Mod.Phys.* **A16** (2001) 822–855, `arXiv:hep-th/0007018 [hep-th]`.

[33] N. J. Hitchin, "The self-duality equations on a riemann surface," *Proc. London Math. Soc.* **(3) 55** no. 1, (1987) 59–126.

[34] D. Gaiotto, G. W. Moore, and A. Neitzke, "Wall-crossing, Hitchin Systems, and the WKB Approximation," `arXiv:0907.3987 [hep-th]`.