ON THE CUT-AND-PASTE PROPERTY OF ALGEBRAIC VARIETIES

ILYA KARZHEMANOV

Abstract. We show that the above-named property (after M. Larsen and V. Lunts) does not hold in general.

1. Introduction

1.1. Fix some ground field $\mathbf{k} \subseteq \mathbb{C}$. Consider the ring $K_0(\mathrm{Var}_{\mathbf{k}})$ of $\mathbf{k}$-varieties, generated (over $\mathbb{Z}$) by the isomorphism classes $[X]$ of various quasi-projective such X, with multiplication given by $[X] \cdot [Y] := [X \times_{\mathbf{k}} Y]$ (for all Y) and addition $[X] + [Y]$ being just the formal sum subject to the relation $[X] = [X \setminus Y] + [Y]$ whenever $Y \subseteq X$ is a closed subset. In particular, the class of a point can be identified with the unity $1 \in K_0(\mathrm{Var}_{\mathbf{k}})$, so that $[\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}] = \mathbb{L} - 1$ for $[\mathbb{A}^1_{\mathbf{k}}] := \mathbb{L}$. We will suppress the reference to $\mathbf{k}$ in what follows for notations' simplicity.

The ring $K_0(\mathrm{Var})$, or more precisely, *motivic measures* with values in $K_0(\mathrm{Var})$, was used in [8] for example to show (via the celebrated *motivic integration*) that birationally isomorphic Calabi-Yau manifolds have equal Hodge numbers (see [3], [4], [11], [12], [9], [10] and [5] for other results, applications and references). Despite this success, however, the structure of $K_0(\mathrm{Var})$ is still poorly understood, although it is known that $K_0(\mathrm{Var})$ is not a domain (see [13]) and one has an explicit description of "conical" subrings in $K_0(\mathrm{Var})$ (see [7]); see also [9] and [1] for a description of quotients of $K_0(\mathrm{Var})$ by some ideals related to $\mathbb{L}$ (note that it is still not known whether $\mathbb{L}$ is a 0-divisor in $K_0(\mathrm{Var})$). The present paper provides another (tiny) contribution to this interesting subject.

Namely, given two varieties X and Y let's say (following [9]) that they satisfy the *cut-and-paste property* if $[X] = [Y]$ in $K_0(\mathrm{Var})$ and additionally $X = \coprod_{i=1}^{k} X_i$ (resp. $Y = \coprod_{i=1}^{k} Y_i$) for some $k \geq 1$ and constructible subsets $X_i \subseteq X$ (resp. $Y_i \subseteq Y$), with $X_i \simeq Y_i$ for all i.

Theorem 1.2. *There exist two smooth projective varieties X and Y, with $[X] = [Y]$, which violate the cut-and-paste property.*[1)]

In order to prove Theorem 1.2 it suffices to exhibit such X and Y, not birational to each other, yet satisfying $[X] = [Y]$. Specifically, the idea is to take *birationally rigid* X and Y (provided, say, by the paper [14]), and then show that $[X] = [Y]$. For the latter, we want both X, Y to be "simple", e.g. fibred over $\mathbb{P}^1$ onto surfaces all having the same class in $K_0(\mathrm{Var})$ (this is also inline with [14]). The equality $[X] = [Y]$, for certain X and Y, can then be established via direct technical (though elementary) argument (see **2.1, 2.5** below).

[1)] This also answers Question (b) in [6, **3.G‴**] at negative due to the results from [2].

2. Proof of Theorem 1.2

2.1. Fix two cubic forms $G := G(x_0, \ldots, x_3)$ and $F := F(x_0, \ldots, x_3)$ over $\mathbf{k}$. We will assume G generic and the equation $F = 0$ define a cubic surface (in $\mathbb{P}^3$) with an ordinary double point as the only singularity. We also choose F generic among such forms.

Consider the locus $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ given by an equation

$$\alpha(t_0, t_1)G + \beta(t_0, t_1)F = 0 \tag{2.2}$$

of bidegree $(3, m)$ for G, F as above and some $m \geq 3$, with x_i (resp. t_i) being projective coordinates on $\mathbb{P}^3$ (resp. on $\mathbb{P}^1$). We also assume both forms α, β to be generic.

Lemma 2.3. *X is a smooth 3-fold.*

Proof. Indeed, by Bertini theorem applied to the linear system of divisors (2.2), all possible singularities of X can lie only on the surface $\mathfrak{B} := (G = F = 0) \subset \mathbb{P}^3 \times \mathbb{P}^1$. This surface is smooth as being isomorphic to $\mathbb{P}^1\times$ the smooth curve $(G = F = 0) \subset \mathbb{P}^3$. In particular, locally analytically near every point on $\mathfrak{B}$ we may assume both G, F to be some local coordinates. The claim now easily follows by taking partial derivatives of (2.2) w. r. t. t_i and G, F. $\square$

Further, X carries a natural fibration $X \longrightarrow \mathbb{P}^1$ on cubic surfaces induced by the projection of $\mathbb{P}^3 \times \mathbb{P}^1$ onto the second factor. We will refer to X as a *pencil* (of cubic surfaces).

Let $y_0 := t_0^m, y_1 := t_0^{m-1}t_1, \ldots, y_m := t_1^m$ be the monomials entering α and β with some $\mathbf{k}$-coefficients $\alpha_0, \ldots, \alpha_m$ and $\beta_0, \ldots, \beta_m$, respectively. We may regard y_i as new transcendental variables. Consider the locus $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^m \times \mathbb{A}^1$ given by equation

$$L_\alpha G + (L_\beta + \lambda)F = 0,$$

where $\mathbb{A}^1 = \operatorname{Spec} \mathbf{k}[\lambda]$ and the linear forms $L_\alpha := L_\alpha(y_0, \ldots, y_m), L_\beta := L_\beta(y_0, \ldots, y_m)$ are obtained from α, β, respectively, via replacing each monomial with the corresponding y_i, i.e., $L_\alpha = \sum \alpha_i y_i$ and $L_\beta = \sum \beta_i y_i$.

Put $t_0 := 1$ (resp. $y_0 := 1$) and consider the open loci

$$X_0 := X \cap (t_0 = 1) \setminus (GF = 0) \subset X$$

and

$$\mathcal{X}_0 := \mathcal{X} \cap (y_0 = 1) \setminus (GF = 0) \subset \mathcal{X}.$$

Notice that $X_0 \subset \mathcal{X}_0 \times \mathbb{A}^1$ as a closed subset, with (tautological) equations $y_1 = t, \ldots, y_m = t^m$, where $t := t_1$.

In what follows, we will not distinguish between X_0 (resp. $\mathcal{X}_0$) and any affine scheme, whose reduced structure equals X_0 (resp. $\mathcal{X}_0$). Then we have

Lemma 2.4. *$X_0 \times \mathbb{A}^{m+1} \simeq \mathcal{X}_0 \times \mathbb{A}^1$ and $(X_0 \times \mathbb{A}^{m+1}) \cap (y_i = 0) \simeq (\mathcal{X}_0 \times \mathbb{A}^1) \cap (y_i = t)$ for all i, where $\mathbb{A}^{m+1}$ (resp. $\mathbb{A}^1$) is provided with affine coordinates $y_1, \ldots, y_m, \lambda$ (resp. t).*

Proof. Consider $\mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[\lambda, z]$ and identify $\mathcal{X}_0 \times \mathbb{A}^1$ with the closed subset $(y_1 = z - t - \lambda) \subset \mathcal{X}_0 \times \mathbb{A}^2$. Let's construct an isomorphism $\varphi : X_0 \times \mathbb{A}^{m+1} \xrightarrow{\sim} \mathcal{X}_0 \times \mathbb{A}^1$ as follows:

$$[x_0 : \ldots : x_3] \times t \times (y_1, \ldots, y_m, \lambda) \mapsto [x_0 : \ldots : x_3] \times (y_1', y_2', \ldots, y_m', \lambda') \times t \times y_1,$$
$$y_1' := y_1 + t + \lambda,\ y_2' := y_2 + t^2,\ \ldots,\ y_m' := y_m + t^m,\ \lambda' := (-\beta_1 - \alpha_1 \tfrac{G(x)}{F(x)})\lambda - L_\beta(y) + \beta_0 - \tfrac{G(x)}{F(x)}(L_\alpha(y) - \alpha_0),$$

where $x := (x_0, \ldots, x_3)$ and similarly for y.[2)]

The obtained morphism $\varphi : X_0 \times \mathbb{A}^{m+1} \longrightarrow \mathcal{X}_0 \times \mathbb{A}^1$ is easily seen to be dominant (by dimension count) and one-to-one (for it is this on the fibers = cubic surfaces of the form (2.2)). Indeed, we may fix x_i (all i), y_1 and t. Then φ reduces to an *affine* map on $x \times t \times \mathbb{A}^m$ sending $(y_2, \ldots, y_m, \lambda)$ to $(y_2', \ldots, y_m', \lambda')$ as is indicated above. (The corresponding equation $L_\alpha(y')G(x) + (L_\beta(y') + \lambda')F(x) = 0$ turns into $y_1' = y_1 + t + \lambda$.) It's now immediate from the construction that $\varphi\big|_{x \times t \times \mathbb{A}^m}$ is an isomorphism (of possibly non-reduced schemes).

Thus φ is an isomorphism. The last assertion of lemma is evident. $\square$

2.5. Consider another locus $\tilde{X} \subset \mathbb{P}^3 \times \mathbb{P}^1$, defined similarly as X with the same G, F, but having other (still generic) degree m forms $\tilde{\alpha}(t_0, t_1), \tilde{\beta}(t_0, t_1)$. All the previous gadgets such as $\mathcal{X}, X_0$, etc. are defined verbatim for $\tilde{X}$, and we will distinguish them (from those for X) by simply putting extra $\tilde{}$.

One may assume w. l. o. g. the fiber over the point $[0 : 1]$ for $X \longrightarrow \mathbb{P}^1$ (resp. for $\tilde{X} \longrightarrow \mathbb{P}^1$) to be *smooth*. Further, observe that both X and $\tilde{X}$ have the same number of singular fibers, all being of that type as the surface $(F = 0)$ (cf. the beginning of **2.1**). Indeed, the pencils $X, \tilde{X}$ correspond to smooth rational curves ($C, \tilde{C}$, say) in the space of all cubics in $\mathbb{P}^3$, with F corresponding to generic point in the (closed) locus Σ of all singular such cubics. But C and $\tilde{C}$ can be chosen to intersect Σ only at generic points (for instance, use the parameter count and generality of $X, \tilde{X}$), and the claim follows as $\chi_{\text{top}}(X) = \chi_{\text{top}}(\tilde{X}) = 18 - s$ for $s =$ the number of singular fibers in each of $X, \tilde{X}$.[3)]

Remark 2.6. More precisely, letting $m := 1$ in the definition of X (see (2.2)), one obtains the blowup $\operatorname{Bl}_Z(\mathbb{P}^3)$ of $\mathbb{P}^3$ at the curve $Z := (G = F = 0)$. This immediately gives $\chi_{\text{top}} = -14$ for such X. In general, there is a finite morphism $X \longrightarrow \operatorname{Bl}_Z(\mathbb{P}^3)$ of degree m and with only *simple* branch locus, concentrated on smooth fibers. Then we get $\chi_{\text{top}}(X) = m(-14 - 9(2m-2)) + 9(2m-2)(m-1) = -14m - 9(2m-2) = -32m + 18$ and thus $s = 32m$.

Observe further that the scheme $S := X \cap (GF = 0)$ equals the union of Z and a disjoint sum of m copies of the surfaces $(G = 0), (F = 0) \subset \mathbb{P}^3$. Thus S is independent of $X, \tilde{X}$. With this setup we argue as follows:

Proposition 2.7. $[X] \cdot \mathbb{L}^k = [\tilde{X}] \cdot \mathbb{L}^k$ *for all* $1 \le k \le m+1$. *Furthermore, there is an isomorphism* $X_0 \times \mathbb{A}^{m+1} \simeq \tilde{X}_0 \times \mathbb{A}^{m+1}$, *and similarly for* m *instead of* $m+1$.

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2) Heuristically, we "spread out" the locus $X_0 \subset \mathcal{X}_0 \times \mathbb{A}^1$ over the whole $\mathcal{X}_0 \times \mathbb{A}^1$, shifting it by y_i and λ.

3) In the latter argument, we have used that χ_{top} (a. k. a. $\chi_{\text{ét}}$) of the cubic surface $(G = 0)$ equals 9, while $\chi_{\text{top}} = 8$ for the surface $(F = 0)$ (with the obvious equality $\chi_{\text{top}}(X) = \chi_{\text{top}}(\tilde{X})$). In particular, since $[(G = 0)] = [\mathbb{P}^2] + 6\mathbb{L}$ and $[(F = 0)] = \mathbb{L}^2 + 4\mathbb{L} + [\mathbb{P}^1]$, both $X' := X \setminus \{\text{all singular fibers}\}$ and $\tilde{X}' := \tilde{X} \setminus \{\text{all singular fibers}\}$ satisfy $[X'] - [\tilde{X}'] = [X] - [\tilde{X}]$, with all the fibers on X' (resp. on $\tilde{X}'$) having the same class ($= [\mathbb{P}^2] + 6\mathbb{L}$) in $K_0(\mathrm{Var})$. It is then tempting to propose that $[X'] = [\tilde{X}']$, but we don't have a rigorous argument for the proof, unfortunately.

Proof. Firstly, since

$$[X] = [X_0] + [\text{the fiber over } [0:1]] + [S]$$

(same for $\tilde{X}$) and [any smooth cubic surface] $= [\mathbb{P}^2] + 6\mathbb{L}$, it suffices to consider $X_0, \tilde{X}_0$ in place of $X, \tilde{X}$, respectively.

Further, we have $\mathcal{X}_0 \simeq \tilde{\mathcal{X}}_0$, since one may perform a linear transformation of y_i (resp. of λ) which brings L_α to $L_{\tilde{\alpha}}$ (resp. $L_\beta + \lambda$ to $L_{\tilde{\beta}} + \lambda$). Then Lemma 2.4 yields $X_0 \times \mathbb{A}^{m+1} \simeq \tilde{X}_0 \times \mathbb{A}^{m+1}$ and so $[X_0] \cdot \mathbb{L}^{m+1} = [\tilde{X}_0] \cdot \mathbb{L}^{m+1}$. In the same way we obtain $[X_0] \cdot \mathbb{L}^m = [\tilde{X}_0] \cdot \mathbb{L}^m$ (using the second assertion of Lemma 2.4). The claim now follows by induction on the number of variables y_i in the linear forms $L_\alpha, L_{\tilde{\alpha}}$, etc. □

Corollary 2.8. $[X] \cdot \mathbb{L}^k \cdot (\mathbb{L} - 1)^{m-k} = [\tilde{X}] \cdot \mathbb{L}^k \cdot (\mathbb{L} - 1)^{m-k}$ *for all* $0 \le k < m$.

Proof. Again, it suffices to consider $X_0, \tilde{X}_0$ in place of $X, \tilde{X}$.

Set $L_i := \mathbb{A}^m \cap (y_i = 0), 1 \le i \le m$, and observe that $X_0 \times \mathbb{A}^k \times (\mathbb{A}^1 \setminus \{0\})^{m-k} \simeq X_0 \times \mathbb{A}^m \setminus \bigcup_{i=1}^{m-k} X_0 \times L_i$ (resp. $\tilde{X}_0 \times \mathbb{A}^k \times (\mathbb{A}^1 \setminus \{0\})^{m-k} \simeq \tilde{X}_0 \times \mathbb{A}^m \setminus \bigcup_{i=1}^{m-k} \tilde{X}_0 \times L_i$) for all $k < m$. Moreover, since $X_0 \times L_i \simeq \tilde{X}_0 \times L_i$ (see Proposition 2.7) and the schemes $X_0 \times L_i$ (resp. $\tilde{X}_0 \times L_i$) are the same for all i, the isomorphisms $X_0 \times L_i \simeq \tilde{X}_0 \times L_i$ glue together and give

$$X_0 \times \mathbb{A}^m \setminus \bigcup_{i=1}^{m-k} X_0 \times L_i \simeq \tilde{X}_0 \times \mathbb{A}^m \setminus \bigcup_{i=1}^{m-k} \tilde{X}_0 \times L_i. \tag{2.9}$$

Namely, each isomorphism $X_0 \times L_i \simeq \tilde{X}_0 \times L_i$ extends to that between $X_0 \times \mathbb{A}^m$ and $\tilde{X}_0 \times \mathbb{A}^m$, mapping y_i to y_i. Then, given any two $0 \le i, j < m$, we glue two copies of $X_0 \times \mathbb{A}^m$ (resp. of $\tilde{X}_0 \times \mathbb{A}^m$) over $X_0 \times \mathbb{A}^m \setminus L_i$ and $X_0 \times \mathbb{A}^m \setminus L_j$, respectively (same for $\tilde{X}_0$), where the gluing isomorphism just interchanges y_i and y_j. The so obtained variety contains $X_0 \times L_i \cup X_0 \times L_j$ whose complement coincides with $X_0 \times \mathbb{A}^m \setminus (X_0 \times L_i \cup X_0 \times L_j)$ (and similarly for $\tilde{X}_0$). We also have $X_0 \times \mathbb{A}^m \setminus (X_0 \times L_i \cup X_0 \times L_j) \simeq \tilde{X}_0 \times \mathbb{A}^m \setminus (\tilde{X}_0 \times L_i \cup \tilde{X}_0 \times L_j)$ by construction. Iterating this procedure yields (2.9).

The pertinent equalities now follow from the fact that

$$[X_0 \times \mathbb{A}^m \setminus \bigcup_{i=1}^{m-k} X_0 \times L_i] = [X_0] \cdot (\mathbb{L}^m - [\bigcup_{i=1}^{m-k} L_i]) = [X_0] \cdot \mathbb{L}^k \cdot (\mathbb{L} - 1)^{m-k}$$

(same for $\tilde{X}_0$). □

Letting $k = 0$ in Corollary 2.8 we get the identity $[X] \cdot (\mathbb{L} - 1)^m = [\tilde{X}] \cdot (\mathbb{L} - 1)^m$. This together with Proposition 2.7 gives $[X] = [\tilde{X}]$ (we have expanded $(\mathbb{L} - 1)^m = \mathbb{L}^m - m\mathbb{L}^{m-1} + \ldots + (-1)^m$ in (commutative) $K_0(\mathrm{Var})$ in the usual way).

2.10. Both X and $\tilde{X}$ satisfy the genericity assumption from [14, §1] concerning the pencils of degree 3 del Pezzo surfaces (cf. Lemma 2.3 and **2.5** above). Hence $X, \tilde{X}$ are not birational to each other (see [14, Corollary 2.1,(i)]). On the other hand, we have shown that $[X] = [\tilde{X}]$, and Theorem 1.2 follows.

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Kavli IPMU (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan
E-mail address: ilya.karzhemanov@ipmu.jp