TWISTS AND BRAIDS FOR GENERAL 3-FOLD FLOPS

WILL DONOVAN AND MICHAEL WEMYSS

Abstract. Given a quasi-projective 3-fold X with only Gorenstein terminal singularities, we prove that the flop functors beginning at X satisfy Coxeter-type braid relations, with the combinatorics controlled by a certain simplicial hyperplane arrangement. This incorporates known special cases with degree 3 braid relations into a general theory, with higher degree relations occurring even for two smooth rational curves meeting at a point. Considering compositions of flops then yields a new pure braid group action on the derived category of any such 3-fold that admits algebraically flopping curves. We also construct a group action in the more general case where individual curves may flop analytically, but not algebraically, and furthermore we lift the action to a form of affine pure braid group under the additional assumption that X is $\mathbb{Q}$-factorial.

Along the way, we produce two new twist autoequivalences of the derived category of X. One uses commutative deformations of the scheme-theoretic fibre of a flopping contraction, and the other uses noncommutative deformations of the fibre with reduced scheme structure, generalising constructions of [T07, DW1] which considered only the case when the flopping locus is irreducible. For type A flops of irreducible curves, we show that the two autoequivalences are related, but that in other cases they are very different, with the noncommutative twist being linked to birational geometry via the Bridgeland–Chen [B, C02] flop–flop functor.

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1. Introduction

The derived category, through the Bridgeland stability manifold and its autoequivalences, is one of the fundamental tools that enriches and furthers our understanding of birational geometry, especially in low dimension. Amongst other things it is widely expected, and proved in many cases, that the derived category allows us to run the minimal model program [BM, BO, B, C02, T13, W] and track minimal models [CI, T08, W], to illuminate new and known symmetries [D13, DS1, HLS, ST, T07], and to understand wall crossing phenomena [CI, C14, N], in both a conceptually and computationally easier manner.

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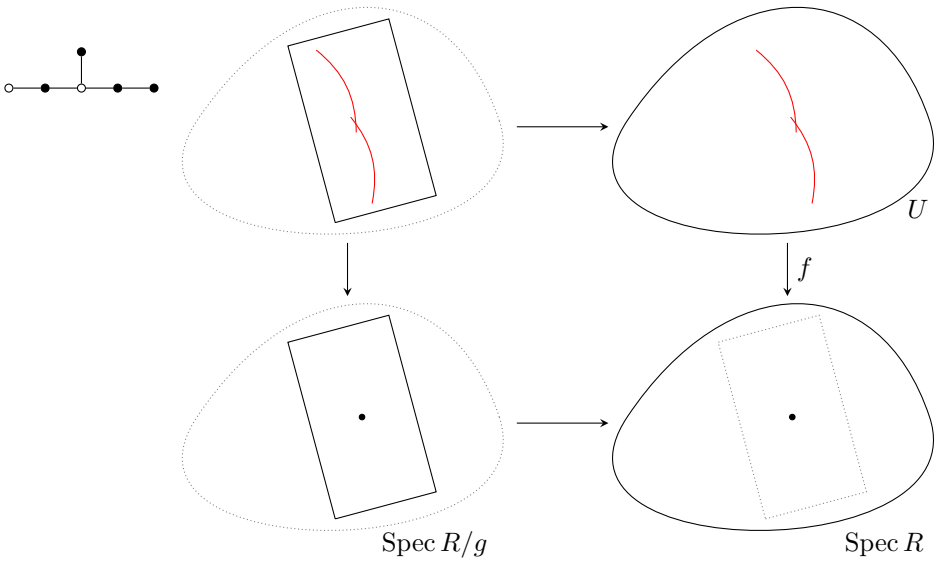
Consider a general algebraic 3-fold flopping contraction $f\colon X \to X_{\mathrm{con}}$, where X has only Gorenstein terminal singularities. It is known that there are derived equivalences between X and its flops [B, C02], and a key result of this paper is that these functors satisfy higher degree braid relations, with combinatorics controlled by the real hyperplane arrangements appearing in [W, §5–§7], which need not be Coxeter.

Roughly speaking, by taking the fundamental group associated to the complexified complement of the above hyperplane arrangement, we obtain new pure braid group actions on the derived category of X. However the situation is complicated by the fact that flops of individual curves in the fibres of f may exist analytically, but not algebraically. To deal with this, we develop suitable new twist autoequivalences. Given the flopping contraction f, for each subset J of curves in its fibres we produce in full generality a twist autoequivalence, and in the case when the curves J are algebraically floppable we show that our twist recovers the (inverse of the) flop–flop functor. We call these twists the J-twists, and the fact that they exist regardless of whether the curves J flop algebraically allows us to obtain a group action on $\mathrm{D^b}(\mathrm{coh}\, X)$, with appropriate elements naturally corresponding to all possible choices of J.

It turns out that the above J-twists are obtained by twisting over the noncommutative deformations of parts of the exceptional locus of the contraction, equipped with its reduced subscheme structure. We also produce another new twist autoequivalence, twisting instead over the deformations of the scheme-theoretic exceptional locus. We show that for a type A irreducible curve, these two twists are related, but in other cases they are very different. We explain why both must be understood in order to construct affine braid group actions on the derived category of X.

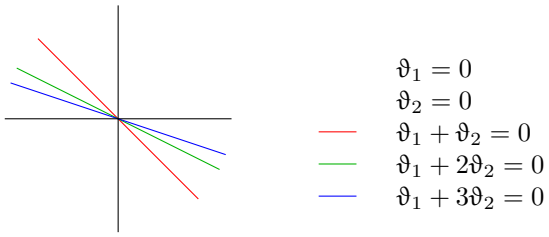
1.1. **Flops and Braids.** In known special cases, braiding is already known to hold. For example, in certain toric type A cases, degree 3 braid relations $\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \cong \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2$ exist between flops. This case is investigated in [S03, DS2], where the braiding of flops corresponds to analogous degree 3 relations between Seidel–Thomas twists on surfaces, as studied in [ST].

In this paper, we explain how the general picture is quite different to these special cases. As a motivating example consider a minimal model $f\colon U \to \operatorname{Spec} R$, where R is an isolated 3-fold cDV singularity with precisely two irreducible curves above the unique singular point. Suppose that a generic hyperplane section $g \in R$ cuts

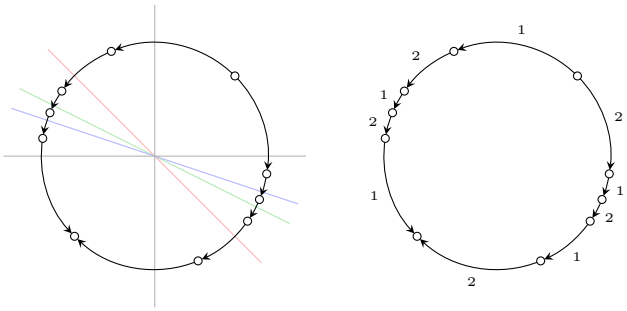


to an ADE surface singularity R/g of Type E_6. By Reid's general elephant principle, the left-hand morphism is a partial crepant resolution of singularities, so is dominated by the minimal resolution. In the above picture, following Katz [K91] we describe the top-left scheme by taking the minimal resolution and blowing down the curves corresponding to the black vertices in the E_6 Dynkin diagram.

We remark that even though the dual graph of f is the Type A configuration $\circ\!-\!\circ$, by [P83] or [W, 7.2] the chamber structure controlling the number of minimal models of $\operatorname{Spec} R$ comes from the hyperplanes given below, not the 3 hyperplanes of the familiar A_2 arrangement.



It follows that in this example there are precisely 10 minimal models, so in particular the degree 3 braiding of flops cannot possibly hold, since all minimal models are connected by flops and so such a braiding would imply that $\operatorname{Spec} R$ has only 6 minimal models. We can however hope for higher length braid relations, given in the above example by



$$\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \cong \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2.$$

This example demonstrates that the braiding of 3-fold flops is not controlled by the dual graph, and in particular, a Type A dual graph does not imply Type A braiding phenomena. Instead, 3-fold geometry is controlled by the generic hyperplane section, and the same applies to the braiding of flop functors. This shows that in dimension greater than two there cannot be an axiomatic approach to braiding of twists along the lines of [ST], since the axiomatics of the dual graph are not enough to encode the extra information coming from the generic hyperplane section.

It is not the dual graph but a GIT chamber structure that controls the combinatorics of the braiding, and there are two different ways to describe this chamber structure. The first, due to Pinkham [P83], is by taking an intersection in some large root system. This gives a hyperplane arrangement, but is cumbersome to calculate in practice, and is not well suited to functorial arguments. The second, in [W], is by tracking moduli spaces under mutation, and by taking a generic hyperplane section this reduces the combinatorics to knitting on surface singularities. The latter method is much more convenient for the purposes of this paper, since our strategy to establish functorial braiding is to track skyscrapers under flop functors, which is precisely what moduli tracking does.

Our arguments work in general, and it turns out that the above E_6 example is misleadingly easy. To work in generality, with multiple curve configurations, the chamber structures are not even the root system of some Coxeter group and so to describe them effectively we must use the language of simplicial hyperplane arrangements.

1.2. **Braiding.** We first describe the case where the flopping contraction $f\colon X \to X_{\mathrm{con}}$ contracts precisely two irreducible curves, and furthermore that each curve is individually (algebraically) floppable. The general case is described next in §1.3. In the case where the curves intersect, passing to the formal fibre, using [W] we associate to f a chamber structure Θ_{A}, where A is a certain noncommutative algebra. We refer the reader to §2.4 and §3.1 for full details.

These two-curve configurations follow a predictable pattern based on the above motivating example. As in the example above, we write F_1 and F_2 for the Bridgeland–Chen flop functors associated to the two flopping curves in X. By abuse of notation, we also use F_1 and F_2 to denote flop functors at the varieties obtained after successive flops. The functor associated to the flop of the two curves together is denoted by F.

Theorem 1.1 (=3.6)**.** *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, contracting precisely two independently floppable irreducible curves. If X has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots}_{n} \cong \mathsf{F} \cong \underbrace{\mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \cdots}_{n}$$

for some $n \geq 2$. Furthermore:

(1) *If the curves intersect, n is half the number of chambers in Θ_{A}, and $n \geq 3$.*
(2) *If the curves are disjoint, then $n = 2$.*

The proof heavily uses moduli tracking, developed as part of the Homological MMP [W]. The strategy is to track skyscrapers $\mathcal{O}_x$, with the challenge being to show that the composition $(\mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1} \circ \mathsf{F}_1^{-1} \circ \cdots) \circ (\cdots \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2)$ applied to $\mathcal{O}_x$ is also a skyscraper. Constraints on the number of chambers in Θ_{A} follow by the methods of [W], and it is anticipated that $n \leq 8$ in 1.1(1).

When there are more than two curves in the connected chain, the braiding is still controlled by the hyperplane arrangement Θ_{A}, but to describe this requires more effort. We first prove, in 3.14, that in this setting Θ_{A} is always a simplicial hyperplane arrangement. This allows us to associate to Θ_{A} the Deligne groupoid, in the knowledge that the relations are determined by the codimension two walls [D72, 1.10, 1.12] (see also [CM]).

As a motivating example which is not a root system, consider the following example which arises from certain D_4 flops with three curves above the origin. It has 7 hyperplanes, and 32 chambers:

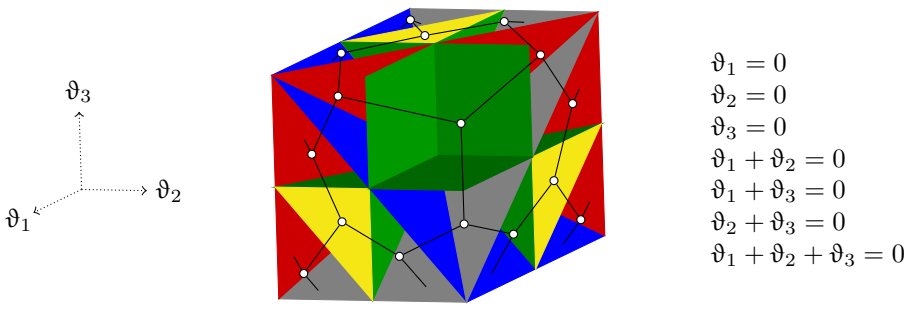


FIGURE 1. Chamber structure for the configuration 

In 3.17, we prove in the general setting that crashing through a codimension two wall corresponds to flopping two curves together, and so the relations in the Deligne groupoid $\mathbb{G}_{\mathrm{A}}$ can be verified using 1.1. In the above example, four of these relations are illustrated in Figure 2 below, corresponding to the four codimension two walls marked blue.

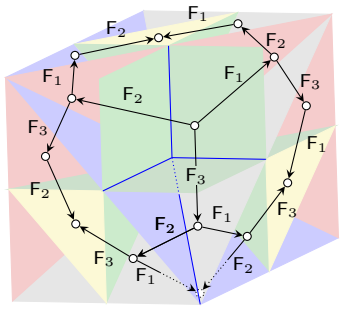


FIGURE 2. Four relations for the configuration 

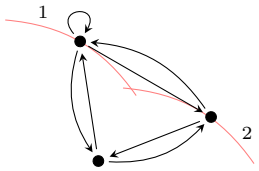
Theorem 1.2 (=3.19)**.** *Suppose that $X \to X_{\mathrm{con}}$ is an algebraic flopping contraction with precisely one connected chain of curves contracting to a point, where X has terminal Gorenstein singularities and each of the curves is individually (algebraically) floppable. Then:*

(1) *The flop functors form a representation of the Deligne groupoid $\mathbb{G}_{\mathrm{A}}$.*
(2) *The group $\pi_1(\mathbb{G}_{\mathrm{A}})$ acts on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.*

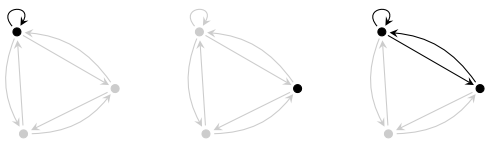
It is well-known (see 3.16) that any vertex group $\pi_1(\mathbb{G}_{\mathrm{A}})$ of the groupoid $\mathbb{G}_{\mathrm{A}}$ is isomorphic to the fundamental group of the complexified complement of the real hyperplane arrangement Θ_{A}, and that this can be presented as generators and relations [A, R82, S87]. However, our approach differs from that of Seidel–Thomas [ST] in that we do not use generators and relations of $\pi_1(\mathbb{G}_{\mathrm{A}})$ in our proof.

1.3. **New Intrinsic Autoequivalences.** In order to describe the case when curves are forced to flop together, and also to understand geometrically some of the generators of $\pi_1(\mathbb{G}_{\mathrm{A}})$, we need to associate intrinsic symmetries to a given flopping contraction $X \to X_{\mathrm{con}}$. For this, it is important to view the exceptional fibres of the contraction in two different ways: on the one hand taking the reduced scheme structure, with indexed components, and on the other taking the full scheme-theoretic fibre. It will turn out that suitably interpreted both choices yield new autoequivalences, with the former being naturally related to the individual flop, when that exists.

Considering the noncommutative ring A obtained by passing to the formal fibre, as in the previous subsection, taking factors we define certain finite dimensional algebras in 2.11. As explained in [W, 2.15], the presentation of A depends on the intersection theory of curves; an example is given below:



For any subset of the curves $J \subseteq \{1, 2\}$, we define A_J by factoring out all other idempotents:



A_1 A_2 $\mathrm{A}_{\{1,2\}}$

In a similar way A_0 is defined by factoring out the idempotents corresponding to both the curves; see 2.11 for full details. The algebras A_J defined for subsets J of the curves are a generalisation of the contraction algebras introduced in [DW1] to the case of multiple curves, since it is shown in [DW2] that A_J represents the functor of noncommutative deformations of the curves in J. By contrast, [DW2] also shows that A_0 represents both the commutative and noncommutative deformations of the whole scheme-theoretic fibre. Using these objects, in this paper we construct functors

$$J\mathsf{Twist}, F\mathsf{Twist}\colon \mathrm{D}(\mathrm{Qcoh}\, X) \to \mathrm{D}(\mathrm{Qcoh}\, X)$$

in 5.13, called the J-twist and the fibre twist, which twist around A_J and A_0 respectively.

Theorem 1.3 (=5.11)**.** *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction, where X is projective and has terminal Gorenstein singularities.*

(1) *For any subset J of the flopping curves, $J\mathsf{Twist}$ is an autoequivalence.*
(2) *If further X is $\mathbb{Q}$-factorial, then $F\mathsf{Twist}$ is an autoequivalence.*

In this level of generality both autoequivalences are new, but we remark that (1) recovers [DW1] in the special case when there is only one curve, and (2) recovers [T07] in the case where there is one curve of Type A. It is unclear whether the restriction to $\mathbb{Q}$-factorial in 1.3(2) is strictly necessary, but currently the proof requires this assumption.

After choosing a subset of curves J, in the case when the union of curves $\bigcup_{j\in J} C_j$ flops algebraically, the J-twist and the associated flop functors are related as follows.

Theorem 1.4 (=5.15)**.** *For a choice of subset J of the flopping curves, suppose that $\bigcup_{j\in J} C_j$ flops algebraically. Then $J\mathsf{Twist} \circ (\mathsf{F}_J \circ \mathsf{F}_J) \cong \mathrm{Id}$.*

Thus the J-twist gives an intrinsic characterisation of the inverse of the flop–flop functor, but has the advantage of always existing. This leads to our main result in the purely algebraic setting, which is the following.

Theorem 1.5 (=6.2)**.** *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction, where X is projective and has only Gorenstein terminal singularities. The subgroup K of $\pi_1(\mathbb{G})$ generated by the J-twists, as J ranges over all subsets of curves, acts on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.*

The groupoid $\mathbb{G}$ here may be defined as an appropriate cartesian product of groupoids $\mathbb{G}_{\mathrm{A}}$. It can happen that $K = \pi_1(\mathbb{G})$ (see 6.3), but verifying this in any level of generality seems group-theoretically difficult. Also, it is unclear geometrically whether the whole of $\pi_1(\mathbb{G})$ should act on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ in the algebraic setting, since it is not obvious whether all the other complete local functors in $\pi_1(\mathbb{G})$ have an intrinsic algebraic characterisation, or indeed whether this is needed. We discuss this briefly in §6. There also seems to be an affine version of this algebraic action, with the fibre twist playing the role of the affine element, however the combinatorics of this seems to be intriguingly new, and so we will return to it in the future.

For now, we simply remark that in the simplest situation, for a smooth rational curve of Type A, the fibre twist may be conjugate to the J-twist by a certain line bundle. The following theorem shows that in general the two twists are not conjugate in this way, and so both will be needed to understand the derived autoequivalence group.

Theorem 1.6 (=5.16)**.** *In the global projective flops setup of 2.2, for a contraction of a single irreducible curve to a point p, where in addition X is $\mathbb{Q}$-factorial, there exists a functorial isomorphism*

$$F\mathsf{Twist}(x \otimes \mathcal{F}) \cong J\mathsf{Twist}(x) \otimes \mathcal{F}$$

for some line bundle $\mathcal{F}$ on X, if and only if the following conditions hold.

(1) *The point p is cDV of Type A.*
(2) *There exists a line bundle $\mathcal{F}$ on X such that $\deg(\mathcal{F}|_{f^{-1}(p)}) = -1$.*

The existence of both the J-twists and the fibre twist relies on the mutation theory developed in [IW10, §6], then further in [DW1, W], together with the local-global methods of [DW1, §7]. The rough idea is that we first work locally on some suitable open subset U of X, on which there is a tilting bundle by [V]. Factors of this tilted algebra have idempotents which then gives rise to ideals; these ideals are naturally (algebraically) defined, and we prove they give tilting autoequivalences by reducing to the formal fibre. We then pull these ideals across to give a FM kernel on $U \times U$, then glue these functors into X, using the procedure developed in [DW1, §7]. We refer the reader to §4 and §5 for full details.

Along the way, especially when we restrict to $\mathbb{Q}$-factorial varieties to obtain and control the fibre twist, we use the theory of maximal modification algebras (=MMAs) as recalled in 2.16. These are the noncommutative version of minimal models. It is an open problem as to whether the property of being an MMA passes to localizations at maximal ideals (it is known that it does not pass to the completion), however in this paper we do establish and use the following, which may be of independent interest.

Theorem 1.7 (=2.17)**.** *Suppose that R is a three-dimensional Gorenstein normal domain over a field, and that Λ is derived equivalent to a $\mathbb{Q}$-factorial terminalization of* $\operatorname{Spec} R$*. Then Λ is an MMA, and further for all $\mathfrak{m} \in \operatorname{Max} R$, the algebra $\Lambda_{\mathfrak{m}}$ is an MMA of $R_{\mathfrak{m}}$.*

1.4. **Conventions.** We work over $\mathbb{C}$. Unqualified uses of the word *module* refer to right modules, and $\operatorname{mod} A$ denotes the category of finitely generated right A-modules. The full subcategory of finite length modules will be denoted $\mathsf{fl}\, A$. When left modules appear, we will either emphasise the fact that they are left modules, or consider them as objects of $\operatorname{mod} A^{\mathrm{op}}$. If $M \in \operatorname{mod} A$, we let $\operatorname{add} M$ denote all possible summands of finite sums of M. We say that M is a generator if $R \in \operatorname{add} M$. If $S, T \in \operatorname{mod} R$ where S is a summand of T, then we define the ideal $[S]$ to be the two-sided ideal of $\operatorname{End}_R(T)$ consisting of all morphisms factoring through $\operatorname{add} S$.

We use the functional convention for composing arrows, so $f \cdot g$ means g then f. With this convention, M is a $\operatorname{End}_R(M)^{\mathrm{op}}$-module. Furthermore, $\operatorname{Hom}_R(M, N)$ is a $\operatorname{End}_R(M)$-module and a $\operatorname{End}_R(N)^{\mathrm{op}}$-module, in fact a bimodule. Note also that $\operatorname{Hom}_R({}_S M_R, {}_T M_R)$ is an S–T bimodule and $\operatorname{Hom}_{R^{\mathrm{op}}}({}_R M_S, {}_R M_T)$ is a T–S bimodule.

If X is a scheme, $\mathcal{O}_{X,x}$ will denote the localization of the structure sheaf at the closed point $x \in X$, whereas $\mathcal{O}_x$ will always denote the skyscraper sheaf at x. We will write $\widehat{\mathcal{O}}_{X,x}$ for the completion of $\mathcal{O}_{X,x}$ at the unique maximal ideal. Throughout, *locally* will always mean Zariski locally, and when we discuss the completion, we will speak of working *complete locally*.

2. Flops Setting and Notation

In this section we fix notation, and provide the necessary preliminary results.

2.1. **Perverse Sheaves.** Consider a projective birational morphism $f\colon X \to X_{\mathrm{con}}$ between noetherian integral normal $\mathbb{C}$-schemes with $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_{X_{\mathrm{con}}}$, such that the fibres are at most one-dimensional.

Definition 2.1. *For such a morphism $f\colon X \to X_{\mathrm{con}}$, recall* [B, V] *that ${}^0\mathrm{Per}(X, X_{\mathrm{con}})$, the category of perverse sheaves on X, is defined*

$$ {}^0\mathrm{Per}(X, X_{\mathrm{con}}) = \left\{ a \in \mathrm{D^b}(\operatorname{coh} X) \;\middle|\; \begin{array}{c} H^i(a) = 0 \text{ if } i \neq 0, -1 \\ f_* H^{-1}(a) = 0,\ \mathbf{R}^1 f_* H^0(a) = 0 \\ \operatorname{Hom}(c, H^{-1}(a)) = 0 \text{ for all } c \in \mathcal{C}^0 \end{array} \right\}, $$

where

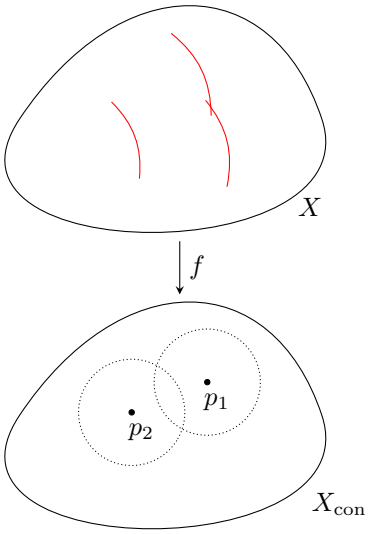
$$ \mathcal{C} := \{ c \in \mathrm{D^b}(\operatorname{coh} X) \mid \mathbf{R}f_* c = 0 \} $$

and $\mathcal{C}^0$ denotes the full subcategory of $\mathcal{C}$ whose object have cohomology in degree 0.

2.2. **Global and Local Flops Notation.** In this subsection, and for the remainder of this paper, we will make use of the following geometric setup.

Setup 2.2. (Global flops) We let $f\colon X \to X_{\mathrm{con}}$ be a flopping contraction, where X is a projective 3-fold with only Gorenstein terminal singularities. This means that f is crepant and an isomorphism in codimension two. We write $\mathrm{Ram}(X_{\mathrm{con}})$ for the (finite) set of points in X_{con} above which f is not an isomorphism.

We will not assume that X is $\mathbb{Q}$-factorial unless explicitly stated. With the assumptions in 2.2, around each point $p \in \mathrm{Ram}(X_{\mathrm{con}})$ we can find an affine open neighbourhood $\operatorname{Spec} R$ containing p but none of the other points in $\mathrm{Ram}(X_{\mathrm{con}})$, as shown below.



We set $U := f^{-1}(\operatorname{Spec} R)$ and thus consider the morphism

$$f|_U\colon U \to \operatorname{Spec} R.$$

By construction, this is an isomorphism away from a single point, and above that point is a connected chain of rational curves. Many of our global problems can be reduced to the following Zariski local setup.

Setup 2.3. (Zariski local flops, single chain) Suppose that $f\colon U \to \operatorname{Spec} R$ is a crepant projective birational contraction, with fibres at most one-dimensional, which is an isomorphism away from precisely one point $\mathfrak{m} \in \operatorname{Max} R$. We assume that U has only terminal Gorenstein singularities. As notation, above $\mathfrak{m}$ is a connected chain C of n curves with reduced scheme structure $C^{\mathrm{red}} = \bigcup_{j=1}^{n} C_j$ such that each $C_j \cong \mathbb{P}^1$.

Passing to the completion will bring technical advantages.

Setup 2.4. (Complete local flops) With notation and setup as in 2.3, we let $\mathfrak{R}$ denote the completion of R at the maximal ideal $\mathfrak{m}$. We let $\varphi\colon \mathfrak{U} \to \operatorname{Spec} \mathfrak{R}$ denote the formal fibre. Above the unique closed point is a connected chain C of n curves with $C^{\mathrm{red}} = \bigcup_{j=1}^{n} C_j$ such that each $C_j \cong \mathbb{P}^1$. Because R is terminal Gorenstein, necessarily $\mathfrak{R}$ is an isolated hypersurface singularity, by [R83, 0.6(I)].

2.3. **The Contraction and Fibre Algebras.** In the Zariski local flops setup in 2.3, it is well-known [V, 3.2.8] that there is a bundle $\mathcal{V} := \mathcal{O}_U \oplus \mathcal{N}$ inducing a derived equivalence

$$\begin{array}{ccc} \mathrm{D^b}(\operatorname{coh} U) & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)} & \mathrm{D^b}(\operatorname{mod}\operatorname{End}_U(\mathcal{V})) \\ \uparrow & & \uparrow \\ {}^0\mathrm{Per}(U,R) & \xrightarrow[\sim]{} & \operatorname{mod}\operatorname{End}_U(\mathcal{V}) \end{array} \tag{2.A}$$

In this Zariski local setting there is choice in the construction of $\mathcal{V}$, but in the setting of the formal fibre later in §2.4, there is a canonical choice.

Throughout we set

$$\Lambda := \operatorname{End}_U(\mathcal{V}) = \operatorname{End}_U(\mathcal{O}_U \oplus \mathcal{N}).$$

Recall that if $\mathcal{F}, \mathcal{G} \in \operatorname{coh} U$ where $\mathcal{F}$ is a summand of $\mathcal{G}$, then we define the ideal $[\mathcal{F}]$ to be the two-sided ideal of $\operatorname{End}_U(\mathcal{G})$ consisting of all morphisms factoring through $\operatorname{add}\mathcal{F}$.

Definition 2.5. *With notation as above, we define the* contraction algebra *associated to* Λ *to be* $\Lambda_{\mathrm{con}} := \operatorname{End}_U(\mathcal{O}_U \oplus \mathcal{N})/[\mathcal{O}_U]$.

We remark that Λ_{con} defined above depends on Λ and thus the choice of derived equivalence (2.A). In the complete local case there is a canonical choice for this; see §2.4 below.

Lemma 2.6. *Under the Zariski local setup of 2.3, the basic algebra morita equivalent to* Λ_{con} *has precisely* n *primitive idempotents.*

Proof. As in [DW1, 2.16], $\Lambda_{\mathrm{con}} \cong \widehat{\Lambda}_{\mathrm{con}}$, and on the completion the assertion is clear, for example using (2.F). □

Since f is a flopping contraction, it follows from [V, 4.2.1] that

$$\Lambda = \operatorname{End}_U(\mathcal{V}) \cong \operatorname{End}_R(R \oplus f_*\mathcal{N}). \tag{2.B}$$

We set $L := f_*\mathcal{N}$ so that $\Lambda \cong \operatorname{End}_R(R \oplus L)$. Translating 2.5 through this isomorphism, the contraction algebra associated to Λ becomes

$$\Lambda_{\mathrm{con}} \cong \operatorname{End}_R(R \oplus L)/[R] \cong \operatorname{End}_R(L)/[R].$$

There is a canonical ring homomorphism $\Lambda \to \Lambda_{\mathrm{con}}$, and we denote its kernel by I_{con}. Necessarily this is a two-sided ideal of Λ, so there is a short exact sequence

$$0 \to I_{\mathrm{con}} \to \Lambda \to \Lambda_{\mathrm{con}} \to 0 \tag{2.C}$$

of Λ-bimodules. For global twist functors later, we need a more refined version of this. Under the morita equivalence in 2.6, Λ_{con} inherits n primitive idempotents $e_1, \ldots, e_n$. We pick a subset $J \subseteq \{1, \ldots, n\}$ of these idempotents (equivalently, a subset of the exceptional curves in 2.3), and write

$$\Lambda_J := \frac{\Lambda_{\mathrm{con}}}{\Lambda_{\mathrm{con}}(1 - \sum_{j \in J} e_j)\Lambda_{\mathrm{con}}}.$$

The composition of the ring homomorphisms $\Lambda \to \Lambda_{\mathrm{con}} \to \Lambda_J$ is a surjective ring homomorphism. We denote the kernel by I_J, which is a two-sided ideal of Λ, and thus for each subset $J \subseteq \{1, \ldots, n\}$, there is a short exact sequence

$$0 \to I_J \to \Lambda \to \Lambda_J \to 0 \tag{2.D}$$

of Λ-bimodules. This exact sequence is needed later to extract a Zariski local noncommutative twist functor from the formal fibre.

Naively, we want to repeat the above analysis for $\Lambda/[L] = \operatorname{End}_R(R \oplus L)/[L]$ to obtain a similar exact sequence, as this later will give another, different, derived autoequivalence. However, by the failure of Krull–Schmidt it may happen that $R \in \operatorname{add} L$, in which case $\Lambda/[L] = 0$. We get round this problem by passing to the localization $\Lambda_{\mathfrak{m}}$, where we can

use lifting numbers. Indeed, localizing Λ with respect to the maximal ideal $\mathfrak{m}$ in setup 2.3 gives $\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \oplus L_{\mathfrak{m}})$, so we can then use the following well-known lemma.

Lemma 2.7. *Suppose that $(S,\mathfrak{m})$ is a commutative noetherian local ring, and suppose that $T \in \operatorname{mod} S$. Then we can write $T \cong S^a \oplus X$ for some $a \geq 0$ with $S \notin \operatorname{add} X$.*

Proof. Since $\widehat{S}$ is complete local, we can write $\widehat{T} \cong \widehat{S}^{\oplus a} \oplus X_1^{\oplus a_1} \oplus \cdots \oplus X_n^{\oplus a_n}$ as its Krull–Schmidt decomposition into pairwise non-isomorphic indecomposables, where $a \geq 0$ and all $a_i \geq 1$. Since $\widehat{S}^{\oplus a}$ is a summand of $\widehat{T}$, it follows that $S^{\oplus a}$ is a summand of T (see e.g. [LW, 1.15]), so we can write $T \cong S^{\oplus a} \oplus X$ for some X. Completing this decomposition, again by Krull–Schmidt $\widehat{X} \cong X_1^{\oplus a_1} \oplus \cdots \oplus X_n^{\oplus a_n}$ and so $\widehat{S} \notin \operatorname{add} \widehat{X}$. It follows that $S \notin \operatorname{add} X$. □

By 2.7, we may pull out all free summands from $L_{\mathfrak{m}}$, and write

$$\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}^{\oplus a} \oplus K)$$

for some $K \in \operatorname{mod} R_{\mathfrak{m}}$ with $R_{\mathfrak{m}} \notin \operatorname{add} K$.

Definition 2.8. *With the Zariski local flops setup in 2.3, we define the fibre algebra Λ_0 to be $\Lambda_{\mathfrak{m}}/[K]$.*

Since by construction $R_{\mathfrak{m}} \notin \operatorname{add} K$, the fibre algebra Λ_0 is non-zero. Furthermore, the composition of ring homomorphisms

$$\Lambda \xrightarrow{\psi_1} \Lambda_{\mathfrak{m}} \xrightarrow{\psi_2} \Lambda_{\mathfrak{m}}/[K] = \Lambda_0$$

is a ring homomorphism, and we define $I_0 := \operatorname{Ker}(\psi_2\psi_1)$.

Lemma 2.9. *With the Zariski local flops setup as in 2.3, as an R-module Λ_0 is supported only at $\mathfrak{m}$, and further there is a short exact sequence*

$$0 \to I_0 \to \Lambda \to \Lambda_0 \to 0 \tag{2.E}$$

of Λ-bimodules.

Proof. Since $R_{\mathfrak{m}}$ is an isolated singularity, $\Lambda_0 = \Lambda_{\mathfrak{m}}/[K]$ has finite length as an $R_{\mathfrak{m}}$-module by [IW10, 6.19(3)]. Thus, as an R-module, it is supported only at $\mathfrak{m}$. The only thing that remains to be proved is that $\psi_2\psi_1$ is surjective. If we let C denote the cokernel, then since Λ_0 is supported only at $\mathfrak{m}$, $(\Lambda_0)_{\mathfrak{n}} = 0$ for all $\mathfrak{n} \in \operatorname{Max} R$ with $\mathfrak{n} \neq \mathfrak{m}$, and so $C_{\mathfrak{n}} = 0$ for all $\mathfrak{n} \neq \mathfrak{m}$. But on the other hand $(\psi_1)_{\mathfrak{m}}$ is an isomorphism, and $(\psi_2)_{\mathfrak{m}}$ is clearly surjective, hence $(\psi_2\psi_1)_{\mathfrak{m}}$ is also surjective, so $C_{\mathfrak{m}} = 0$. Hence $C = 0$. □

2.4. **Complete Local Derived Category Notation.** In this subsection we consider the complete local flops setup in 2.4, and fix notation. Completing the base in 2.3 with respect to $\mathfrak{m}$ gives $\operatorname{Spec}\mathfrak{R}$ and we consider the formal fibre $\varphi\colon \mathfrak{U} \to \operatorname{Spec}\mathfrak{R}$. The above derived equivalence (2.A) induces an equivalence

$$\mathrm{D^b}(\operatorname{coh}\mathfrak{U}) \xrightarrow[\sim]{\mathbf{R}\operatorname{Hom}_{\mathfrak{U}}(\widehat{\mathcal{V}},-)} \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}).$$

This can be described much more explicitly. Let $C = \varphi^{-1}(\mathfrak{m})$ where $\mathfrak{m}$ is the unique closed point of $\operatorname{Spec}\mathfrak{R}$. Then the reduced scheme $C^{\mathrm{red}} = \bigcup_{j=1}^n C_j$ with $C_j \cong \mathbb{P}^1$. Let $\mathcal{L}_j$ denote a line bundle on $\mathfrak{U}$ such that $\mathcal{L}_j \cdot C_i = \delta_{ji}$. If the multiplicity of C_j is equal to one, set $\mathcal{M}_j := \mathcal{L}_j$, else define $\mathcal{M}_j$ to be given by the maximal extension

$$0 \to \mathcal{O}_{\mathfrak{U}}^{\oplus(r-1)} \to \mathcal{M}_j \to \mathcal{L}_j \to 0$$

associated to a minimal set of $r-1$ generators of $H^1(\mathfrak{U}, \mathcal{L}_j^*)$ [V, 3.5.4].

Notation 2.10. In the complete local flops setup of 2.4,

(1) Set $\mathcal{N}_j := \mathcal{M}_j^*$, $\mathcal{N}_0 := \mathcal{O}_{\mathfrak{U}}$ and $\mathcal{V}_{\mathfrak{U}} := \bigoplus_{j=0}^n \mathcal{N}_j$.
(2) Set $N_j := H^0(\mathcal{N}_j)$ and $N := H^0(\mathcal{V}_{\mathfrak{U}})$. Set $N_0 := \mathfrak{R}$, so that $N = \bigoplus_{j=0}^n N_j$.

(3) Put $\mathrm{A} := \mathrm{End}_{\mathfrak{R}}(N)$.

By [V, 3.5.5], $\mathcal{V}_{\mathfrak{U}}$ is a basic progenerator of ${}^0\mathrm{Per}(\mathfrak{U},\mathfrak{R})$, and furthermore is a tilting bundle on $\mathfrak{U}$. The rank of N_j as an $\mathfrak{R}$-module, $\mathrm{rank}_{\mathfrak{R}}\, N_j$, is equal to the scheme-theoretic multiplicity of the curve C_j [V, 3.5.4]. Further, by [V, 3.5.5] we can write

$$\widehat{\mathcal{V}} \cong \mathcal{O}_{\mathfrak{U}}^{\oplus a_0} \oplus \bigoplus_{j=1}^{n} \mathcal{N}_j^{\oplus a_j} \tag{2.F}$$

for some $a_j \in \mathbb{N}$ and so consequently $\widehat{\Lambda} \cong \mathrm{End}_{\mathfrak{R}}(\bigoplus_{j=0}^{n} N_j^{\oplus a_j})$, so that A is the basic algebra morita equivalent to $\widehat{\Lambda}$.

Definition 2.11. *For any $J \subseteq \{0,1,\ldots,n\}$, set*

(1) $\mathcal{N}_J := \bigoplus_{j\in J} \mathcal{N}_j$ *and* $\mathcal{N}_{J^c} := \bigoplus_{k\notin J} \mathcal{N}_k$, *so that* $\mathcal{V}_{\mathfrak{U}} = \mathcal{N}_J \oplus \mathcal{N}_{J^c}$.
(2) $N_J := \bigoplus_{j\in J} N_j$ *and* $N_{J^c} := \bigoplus_{k\notin J} N_k$, *so that* $N = N_J \oplus N_{J^c}$.

There are three important special cases, namely

(3) *When $J \subseteq \{1,\ldots,n\}$, where we write*

$$\mathrm{A}_J := \mathrm{End}_{\mathfrak{R}}(N)/[N_{J^c}],$$

and call A_J the contraction algebra associated to $\bigcup_{j\in J} C_j$.

(4) *When $J = \{1,\ldots,n\}$, where we call A_J the* contraction algebra associated to φ, *and denote it by*

$$\mathrm{A}_{\mathrm{con}} \cong \mathrm{End}_{\mathfrak{R}}(N)/[\mathfrak{R}] \cong \mathrm{End}_{\mathfrak{R}}\left(\oplus_{j=1}^{n} N_j\right)/[\mathfrak{R}].$$

(5) *When $J = \{0\}$, we put*

$$\mathrm{A}_0 := \mathrm{End}_{\mathfrak{R}}(N)/[\oplus_{j=1}^{n} N_j] \cong \mathrm{End}_{\mathfrak{R}}(\mathfrak{R})/[\oplus_{j=1}^{n} N_j] \cong \mathfrak{R}/[\oplus_{j=1}^{n} N_j],$$

and call A_0 the fibre algebra associated to φ.

It follows from the definition that A_0 is always commutative.

Remark 2.12. We conjectured in [DW1, 1.4] that $\mathrm{A}_{\mathrm{con}}$ distinguishes the analytic type of irreducible contractible flopping curves. We remark here that A_0 certainly does not, for example inspecting the Laufer flop in [DW1, 1.3] we see that $\mathrm{A}_0 = \mathbb{C}$, and it is also well-known that $\mathrm{A}_0 = \mathbb{C}$ for the Atiyah flop. This already demonstrates that $\mathrm{A}_{\mathrm{con}}$ gives more information than A_0, although both play a role in the derived symmetry group.

Remark 2.13. Even though A_0 does not distinguish analytic type, it still gives important information since it encodes deformations of the scheme-theoretic fibre C [DW2]. In the case of the higher order Laufer flop, from the presentation of A in [DW1, 3.14] it is easily seen that $\mathrm{A}_0 \cong \mathbb{C}[\varepsilon]/\varepsilon^n$, which corresponds to a single $(n-1)^{\mathrm{th}}$ order deformation. By contrast, the noncommutative deformations of the reduced curve C^{red} are much more complicated, with $\mathrm{A}_{\mathrm{con}} \cong \mathbb{C}\langle x,y\rangle/(xy=-yx,\, x^2=y^{2n+1})$.

The following is shown in a similar way to [DW1, 2.15].

Lemma 2.14. *With the complete local setup as above, for all $\mathfrak{p} \in \mathrm{Spec}\,\mathfrak{R}$,*

(1) $(\mathrm{A}_J)_{\mathfrak{p}} \cong (\mathrm{A}_{\mathfrak{p}})_J$ *and* $(\mathrm{A}_J)_{\mathfrak{p}} \otimes_{\mathfrak{R}_{\mathfrak{p}}} \widehat{\mathfrak{R}_{\mathfrak{p}}} \cong (\widehat{\mathrm{A}_{\mathfrak{p}}})_J$ *for all $J \subseteq \{1,\ldots,n\}$.*
(2) $(\mathrm{A}_0)_{\mathfrak{p}} \cong (\mathrm{A}_{\mathfrak{p}})_0$ *and* $(\mathrm{A}_0)_{\mathfrak{p}} \otimes_{\mathfrak{R}_{\mathfrak{p}}} \widehat{\mathfrak{R}_{\mathfrak{p}}} \cong (\widehat{\mathrm{A}_{\mathfrak{p}}})_0$.

Notation 2.15. For the Zariski local flops setup 2.3, and its formal fibre 2.4, as notation for the remainder of this paper,

(1) Write $\mathbb{F}\colon \mathrm{Mod}\,\mathrm{A} \to \mathrm{Mod}\,\widehat{\Lambda}$ for the morita equivalence induced from (2.F).
(2) For each $i \in \{1,\ldots,n\}$ set $E_i := \mathcal{O}_{C_i}(-1)$, considered as a complex in degree zero. Further, set $E_0 := \omega_C[1]$.

(3) For each $i \in \{0,1,\dots,n\}$, define the simple Λ-modules T_i to be the modules corresponding to the perverse sheaves E_i across the derived equivalence (2.A). Further, set $S_i := \mathbb{F}^{-1}\widehat{T}_i$, which are the corresponding simple A-modules.
(4) For any $J \subseteq \{1,\dots,n\}$, viewing $\mathbb{F}\mathrm{A}_J$ and $\mathbb{F}\mathrm{A}_0$ as Λ-modules via restriction of scalars, we put $\mathcal{E}_J := \mathbb{F}\mathrm{A}_J \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}$ and $\mathcal{E}_0 := \mathbb{F}\mathrm{A}_0 \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}$. As in [DW1, 3.7], both are perverse sheaves, with $\mathcal{E}_J$ concentrated in degree zero, and $\mathcal{E}_0$ concentrated in degree -1.

The above notation is summarised in the following diagram.

$$\begin{array}{ccccccc}
\mathrm{D^b}(\mathrm{Qcoh}\, U) & \underset{-\otimes^{\mathbf{L}}_{\Lambda}\mathcal{V}}{\overset{\mathbf{R}\mathrm{Hom}_U(\mathcal{V},-)}{\rightleftarrows}} & \mathrm{D^b}(\mathrm{Mod}\,\Lambda) & \underset{\mathrm{res}}{\overset{\otimes_{R_{\mathfrak{m}}}\widehat{R}}{\rightleftarrows}} & \mathrm{D^b}(\mathrm{Mod}\,\widehat{\Lambda}) & \xleftarrow{\ \mathbb{F}\ } & \mathrm{D^b}(\mathrm{Mod}\,\mathrm{A}) \\
E_i & \longleftrightarrow & T_i & \longmapsto & \widehat{T}_i & \longleftrightarrow & S_i \\
\mathcal{E}_J & \longleftrightarrow & \mathbb{F}\mathrm{A}_J & \longmapsfrom & \mathbb{F}\mathrm{A}_J & \longleftrightarrow & \mathrm{A}_J \\
\mathcal{E}_0 & \longleftrightarrow & \mathbb{F}\mathrm{A}_0 & \longmapsfrom & \mathbb{F}\mathrm{A}_0 & \longleftrightarrow & \mathrm{A}_0
\end{array}$$

2.5. **Maximal Modification Algebras.** Later the fibre twist autoequivalence requires a restriction to $\mathbb{Q}$-factorial singularities and some other technical results, which we review here. For a commutative noetherian local ring $(R,\mathfrak{m})$ and $M \in \mathrm{mod}\, R$ recall that the *depth* of M is defined to be

$$\mathrm{depth}_R\, M := \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\},$$

which coincides with the maximal length of an M-regular sequence. Keeping the assumption that $(R,\mathfrak{m})$ is local we say that $M \in \mathrm{mod}\, R$ is *maximal Cohen–Macaulay* (=CM) if $\mathrm{depth}_R\, M = \dim R$.

In the non-local setting, if R is an arbitrary commutative noetherian ring we say that $M \in \mathrm{mod}\, R$ is *CM* if $M_{\mathfrak{p}}$ is CM for all prime ideals $\mathfrak{p}$ in R, and we denote the category of CM R-modules by $\mathrm{CM}\, R$. We say that R is a *CM ring* if $R \in \mathrm{CM}\, R$, and if further $\mathrm{inj.dim}_R\, R < \infty$, we say that R is *Gorenstein*. We write $\mathrm{ref}\, R$ for the category of reflexive R-modules.

Recall the following [IW10].

Definition 2.16. *Suppose that R is a d-dimensional CM ring. We call $N \in \mathrm{ref}\, R$ a* maximal modifying (=MM) module *if*

$$\mathrm{add}\, N = \{X \in \mathrm{ref}\, R \mid \mathrm{End}_R(N \oplus X) \in \mathrm{CM}\, R\}.$$

If N is an MM module, we call $\mathrm{End}_R(N)$ a maximal modification algebra (=MMA).

Throughout, we say a normal scheme X is $\mathbb{Q}$-*factorial* if for every Weil divisor D, there exists $n \in \mathbb{N}$ for which nD is Cartier; this condition can be checked on the stalks $\mathcal{O}_{X,x}$ of the closed points $x \in X$. However this property is not complete local, so we say X is *complete locally* $\mathbb{Q}$-*factorial* if the completion $\widehat{\mathcal{O}}_{X,x}$ is $\mathbb{Q}$-factorial for all closed points $x \in X$.

Recall that if X and Y are varieties over $\mathbb{C}$, then a projective birational morphism $f\colon Y \to X$ is called *crepant* if $f^*\omega_X = \omega_Y$. A $\mathbb{Q}$-*factorial terminalization* of X is a crepant projective birational morphism $f\colon Y \to X$ such that Y has only $\mathbb{Q}$-factorial terminal singularities. When Y is furthermore smooth, we call f a *crepant resolution*.

In our geometric setup later, we will require that localizations of MMAs are MMAs. There is currently no known purely algebraic proof of this, but the following geometric proof suffices for our purposes.

Theorem 2.17. *Suppose that R is a three-dimensional Gorenstein normal domain over a field, and that Λ is derived equivalent to a $\mathbb{Q}$-factorial terminalization of* $\mathrm{Spec}\, R$. *Then for all $\mathfrak{m} \in \mathrm{Max}\, R$, $\Lambda_{\mathfrak{m}}$ is an MMA of $R_{\mathfrak{m}}$.*

Proof. For $\mathfrak{m} \in \operatorname{Max} R$, after base change

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R_{\mathfrak{m}} & \xrightarrow{j} & \operatorname{Spec} R \end{array}$$

$\Lambda_{\mathfrak{m}}$ is derived equivalent to X'. But the stalks of $\mathcal{O}_{X'}$ are isomorphic to stalks of $\mathcal{O}_X$, and so in particular all stalks of $\mathcal{O}_{X'}$ are isolated $\mathbb{Q}$-factorial hypersurfaces. By [IW14, 3.2(1)]

$$\mathrm{D_{sg}}(\Lambda_{\mathfrak{m}}) \hookrightarrow \bigoplus_{x \in \operatorname{Sing} X'} \underline{\mathrm{CM}}\, \mathcal{O}_{X',x}$$

and so $\mathrm{D_{sg}}(\Lambda_{\mathfrak{m}})$ is rigid-free since each $\underline{\mathrm{CM}}\, \mathcal{O}_{X',x}$ is [D10, 3.1(1)]. Since Λ has isolated singularities [IW14, 4.2(2)], this implies that $\Lambda_{\mathfrak{m}}$ is an MMA [IW14, 2.14]. □

2.6. **Mutation Notation.** Throughout this subsection we consider the complete local flops setting 2.4, and use notation from 2.10 and 2.11, so in particular $\mathfrak{R}$ is an isolated complete local Gorenstein 3-fold, and $\mathrm{A} := \operatorname{End}_{\mathfrak{R}}(N)$. We set $(-)^* := \operatorname{Hom}_{\mathfrak{R}}(-, \mathfrak{R})$.

Given our choice of summand N_J, we then mutate.

Setup 2.18. As in 2.11, write $N_J = \bigoplus_{j \in J} N_j$ as a direct sum of indecomposables. For each $j \in J$, consider a *minimal right* $(\operatorname{add} N_{J^c})$*-approximation*

$$V_j \xrightarrow{a_j} N_j$$

of N_j, which by definition means that

(1) $V_j \in \operatorname{add} N_{J^c}$ and $(a_j \cdot)\colon \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, V_j) \to \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, N_j)$ is surjective.
(2) If $g \in \operatorname{End}_{\mathfrak{R}}(V_j)$ satisfies $a_j = a_j g$, then g is an automorphism.

Since $\mathfrak{R}$ is complete, such an a_j exists and is unique up to isomorphism. Thus there are exact sequences

$$0 \to \operatorname{Ker} a_j \xrightarrow{c_j} V_j \xrightarrow{a_j} N_j \tag{2.G}$$
$$0 \to \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, \operatorname{Ker} a_j) \xrightarrow{c_j \cdot} \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, V_j) \xrightarrow{a_j \cdot} \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, N_j) \to 0$$

Summing the sequences (2.G) over all $j \in J$ gives exact sequences

$$0 \to \operatorname{Ker} a_J \xrightarrow{c_J} V_J \xrightarrow{a_J} N_J \tag{2.H}$$
$$0 \to \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, \operatorname{Ker} a_J) \xrightarrow{c_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, V_J) \xrightarrow{a_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N_{J^c}, N_J) \to 0$$

Note that applying $\operatorname{Hom}_{\mathfrak{R}}(N, -)$ to (2.H) yields an exact sequence

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(N, \operatorname{Ker} a_J) \xrightarrow{c_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N, V_J) \xrightarrow{a_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N, N_J) \to \mathrm{A}_J \to 0 \tag{2.I}$$

of A-modules.

Dually, for each $j \in J$, consider a minimal right $(\operatorname{add} N^*_{J^c})$-approximation

$$U^*_j \xrightarrow{b_j} N^*_j$$

of N^*_j, thus

$$0 \to \operatorname{Ker} b_j \xrightarrow{d_j} U^*_j \xrightarrow{b_j} N^*_j$$
$$0 \to \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, \operatorname{Ker} b_j) \xrightarrow{d_j \cdot} \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, U^*_j) \xrightarrow{b_j \cdot} \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, N^*_j) \to 0$$

are exact. Summing over all $j \in J$ gives exact sequences

$$0 \to \operatorname{Ker} b_J \xrightarrow{d_J} U^*_J \xrightarrow{b_J} N^*_J \tag{2.J}$$
$$0 \to \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, \operatorname{Ker} b_J) \xrightarrow{d_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, U^*_J) \xrightarrow{b_J \cdot} \operatorname{Hom}_{\mathfrak{R}}(N^*_{J^c}, N^*_J) \to 0.$$

Definition 2.19. *With notation as above, in particular* $\mathrm{A} := \mathrm{End}_{\mathfrak{R}}(N)$*, we define the* left mutation *of* N *at* N_J *as*

$$\nu_J N := N_{J^c} \oplus (\mathrm{Ker}\, b_J)^*,$$

and set $\nu_J \mathrm{A} := \mathrm{End}_{\mathfrak{R}}(\nu_J N)$.

One of the key properties of mutation is that it always gives rise to a derived equivalence. With the setup as above, the derived equivalence between A and $\nu_J \mathrm{A}$ is given by a tilting A-module T_J constructed as follows. There is an exact sequence

$$0 \to N_J \xrightarrow{b_J^*} U_J \xrightarrow{d_J^*} (\mathrm{Ker}\, b_J)^* \tag{2.K}$$

obtained by dualizing (2.J). Applying $\mathrm{Hom}_{\mathfrak{R}}(N, -)$ induces $(b_J^* \cdot)\colon \mathrm{Hom}_{\mathfrak{R}}(N, N_J) \to \mathrm{Hom}_{\mathfrak{R}}(N, U_J)$, so denoting the cokernel by C_J there is an exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{R}}(N, N_J) \xrightarrow{b_J^* \cdot} \mathrm{Hom}_{\mathfrak{R}}(N, U_J) \to C_J \to 0. \tag{2.L}$$

The tilting A-module T_J is defined to be $T_J := \mathrm{Hom}_{\mathfrak{R}}(N, N_{J^c}) \oplus C_J$. It turns out that $\mathrm{End}_{\mathrm{A}}(T_J) \cong \nu_J \mathrm{A}$ [IW10, 6.7, 6.8], and there is always an equivalence

$$\Phi_J := \mathbf{R}\mathrm{Hom}(T_J, -)\colon \mathrm{D^b}(\mathrm{mod}\, \mathrm{A}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \nu_J \mathrm{A}),$$

which is called the *mutation functor* [IW10, 6.8].

3. On the Braiding of Flops

In this section we establish the braiding of flop functors in dimension three, and describe the combinatorial objects that allow us to read off the length of the braid relations which appear. To account for the inconvenient fact that, algebraically, curves are often forced to flop together, we begin by establishing braiding in the complete local case, before addressing the Zariski local and global cases. Throughout this section, we will assume that the curves are all individually floppable, with the general situation being described later in §6.

3.1. Moduli Tracking.

Throughout this subsection we consider the complete local flops setup of 2.4, and use the notation from §2.4. Thus there is a flopping contraction $\varphi\colon \mathfrak{U} \to \mathrm{Spec}\, \mathfrak{R}$ of 3-folds, where $\mathfrak{U}$ has Gorenstein terminal singularities, and $\mathfrak{U}$ is derived equivalent to $\mathrm{A} := \mathrm{End}_{\mathfrak{R}}(N)$ from 2.10, where $N = \bigoplus_{i=0}^{n} N_i$. The N_i with $i > 0$ correspond to the n curves in the exceptional locus.

To prove the braiding of flops in this setting, we will heavily use the following moduli tracking result. As notation, we present A as a quiver with relations, and consider King stability; see for example [W, §5] for a brief introduction in this setting. In particular, we define the dimension vector $\mathsf{rk}\, \mathrm{A} := (\mathrm{rank}_{\mathfrak{R}}\, N_i)_{i=0}^{n}$, and denote the space of stability parameters by Θ.

Given $\vartheta \in \Theta$, denote by $\mathcal{S}_\vartheta(\mathrm{A})$ the full subcategory of $\mathsf{fl}\, \mathrm{A}$ which has as objects the ϑ-semistable objects, and denote by $\mathcal{S}_{\mathsf{rk},\vartheta}(\mathrm{A})$ the full subcategory of $\mathcal{S}_\vartheta(\mathrm{A})$ consisting of those objects with dimension vector $\mathsf{rk}\, \mathrm{A}$. We let $\mathcal{M}_{\mathsf{rk},\vartheta}(\mathrm{A})$ denote the moduli space of ϑ-stable A-representations of dimension vector $\mathsf{rk}\, \mathrm{A}$.

Since by definition any stability condition satisfies $\vartheta \cdot \mathsf{rk}\, \mathrm{A} = 0$, the fact that $N_0 = \mathfrak{R}$ has rank one then implies that

$$\vartheta_0 = -\sum_{i=1}^{n} (\mathrm{rank}_{\mathfrak{R}}\, N_i)\vartheta_i$$

and so $\Theta = \mathbb{Q}^n$, with co-ordinates ϑ_i for each $i = 1, \ldots, n$.

Theorem 3.1 (Moduli Tracking)**.** *In the complete local flops setup of 2.4, choose a subset of curves* J*, equivalently a subset* $J \subseteq \{1, \ldots, n\}$*. Applying the mutation setup of*

2.18 to $N = \bigoplus_{i=0}^{n} N_i$ *with summand* $N_J := \bigoplus_{j\in J} N_j$*, consider the mutation exchange sequence* (2.K)

$$0 \to N_j \to U_j \to (\operatorname{Ker} b_j)^*,$$

for each $j \in J$*. By Krull–Schmidt,* U_j *decomposes into*

$$U_j \cong \bigoplus_{i\notin J} N_i^{\oplus b_{j,i}}$$

for some collection of $b_{j,i} \geq 0$*. Write* $\mathbf{b_J}$ *for the data* $(b_{j,i})$ *with* $j \in J$*,* $i \notin J$*. Then for any stability parameter* $\vartheta \in \Theta$ *define the vector* $\nu_{\mathbf{b_J}}\vartheta$ *by*

$$(\nu_{\mathbf{b_J}}\vartheta)_i = \begin{cases} \vartheta_i + \sum_{j\in J} b_{j,i}\vartheta_j & \text{if } i \notin J \\ -\vartheta_i & \text{if } i \in J. \end{cases}$$

If $\vartheta_j > 0$ *for all* $j \in J$*, then*

(1) *The mutation functor* Φ_J *restricts to a categorical equivalence*

$$\mathcal{S}_{\mathsf{rk},\vartheta}(\mathrm{A}) \xrightarrow{\Phi_J} \mathcal{S}_{\mathsf{rk},\nu_{\mathbf{b_J}}\vartheta}(\nu_J \mathrm{A})$$

where the left-hand side has dimension vector $\mathsf{rk}\,\mathrm{A}$*, and the right-hand side dimension vector* $\mathsf{rk}\,\nu_J\mathrm{A}$*. This categorical equivalence preserves* S*-equivalence classes, and* ϑ*-stable modules correspond to* $\nu_{\mathbf{b_J}}\vartheta$*-stable modules. Further,* ϑ *is generic if and only if* $\nu_{\mathbf{b_J}}\vartheta$ *is generic.*

(2) *As schemes*

$$\mathcal{M}_{\mathsf{rk},\vartheta}(\mathrm{A}) \cong \mathcal{M}_{\mathsf{rk},\nu_{\mathbf{b_J}}\vartheta}(\nu_J \mathrm{A}).$$

Proof. (1) This is a special case of [W, 5.12], using [W, 2.25, 3.5] to see that the assumption (b) of [W, 5.12] is satisfied.
(2) This is [W, 5.13]. □

The stability parameter space Θ has a wall and chamber structure, and the combinatorics of this turns out to control the braiding. The strictly semi-stable parameters cut out codimension one walls, separating the generic stability conditions into chambers. Within a given chamber, the set of semi-stable representations does not vary. The following is known.

Proposition 3.2. *In the complete local flops setup of 2.4*

(1) *The region*

$$C_+ := \{\vartheta \in \Theta \mid \vartheta_i > 0 \text{ for all } i > 0\}$$

of Θ *is a chamber.*

(2) Θ *has a finite number of chambers, and the walls are given by a finite collection of hyperplanes containing the origin. The co-ordinate hyperplanes* $\vartheta_i = 0$ *are included in this collection.*

(3) *Considering iterated mutations at indecomposable summands, tracking the chamber* C_+ *from* $\nu_{j_1}\dots\nu_{j_t}\mathrm{A}$ *back to* Θ *gives all the chambers of* Θ*.*

Proof. This follows immediately from [W, 5.16, 5.23]. □

Remark 3.3. Moduli tracking works in both directions, and this is important for our application. First, 3.2(3) allows us to fix the input A, and track moduli from some iterated mutation $\nu_{j_1}\dots\nu_{j_t}\mathrm{A}$ back to A. This procedure computes the chamber structure for the fixed A, which gives the combinatorial data needed to state theorems on braiding later on, in §3.3. Second, 3.1 also allows us to track moduli starting from A to some iterated mutation $\nu_{j_1}\dots\nu_{j_t}\mathrm{A}$. This second direction is needed to prove the theorems, in particular to establish the braiding in 3.8.

3.2. **Chamber Structures.** We keep the notation and setting from above. In this subsection we give an example of a chamber structure arising in 3-fold flops. Although not strictly needed for the proof of the main theorem, this illustrates some new phenomena and subtleties in the combinatorics.

With input the flopping contraction $\mathfrak{U} \to \operatorname{Spec} \mathfrak{R}$ of 2.4, by Reid's general elephant principle [R83, 1.1, 1.14], cutting by a generic hyperplane section $g \in \mathfrak{R}$ gives

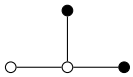
$$\begin{array}{ccc} \mathfrak{U}_g & \longrightarrow & \mathfrak{U} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \operatorname{Spec}(\mathfrak{R}/g) & \longrightarrow & \operatorname{Spec} \mathfrak{R} \end{array}$$

where $\mathfrak{R}/g$ is an ADE singularity and ϕ is a partial crepant resolution. By general theory, $\operatorname{End}_{\mathfrak{R}/g}(N/gN)$ is derived equivalent to $\mathfrak{U}_g$, and so the module N_i cuts to N_i/gN_i, which is precisely one of the CM modules corresponding to a vertex in an ADE Dynkin diagram via the McKay correspondence.

Following the notation from [K91], we encode $\mathfrak{U}_g$ pictorially by describing which curves are blown down from the minimal resolution. The diagrams

represent, respectively, the minimal resolution of the E_6 surface singularity, and the partial resolution obtained from it by contracting the curves corresponding to the black vertices.

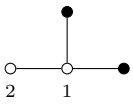
Example 3.4. There is an example [K91, 2.3] of a cD_4 singularity $\mathfrak{R}$ with crepant resolution $X \to \operatorname{Spec} \mathfrak{R}$ with two curves above the origin, that cuts under generic hyperplane section to give the configuration



(3.A)

By 3.2(3), tracking moduli from iterated mutations back to A computes the chamber structure of Θ_{A}. We illustrate this in the easiest non-Type A example, referring the reader to [W, 7.2] for more examples.

Example 3.5. This example computes the chamber structure for $\mathrm{A} = \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus N_1 \oplus N_2)$ in the situation of 3.4 above. As notation order the vertices



meaning that N_1 corresponds to the middle curve, and N_2 corresponds to the left-hand curve. The mutation exchange sequences are obtained by knitting (for details, see [W, 5.19, 5.24]), so that in this example the b's are determined by the data

$$U_1 \cong N_2 \tag{3.B}$$

$$U_2 \cong R^{\oplus 2} \oplus N_1^{\oplus 2}. \tag{3.C}$$

First, we track the C_+ chamber from $\nu_1\mathrm{A}$ to A. By 3.1,

$$\begin{array}{ccc} \phi_1 & \overset{(3.B)}{\mapsto} & -\phi_1 \\ \phi_2 & & \phi_1 + \phi_2 \end{array}$$

since from (3.B) the relevant b is one, so we negate ϕ_1 and add $1\phi_1$ to its neighbour. Thus the C_+ chamber (namely $\phi_1 > 0$, $\phi_2 > 0$) from $\nu_1\mathrm{A}$ corresponds to the region $\vartheta_1 < 0$ and $\vartheta_1 + \vartheta_2 > 0$ of Θ_{A}, and thus this gives a chamber for Θ_{A}.

Next, we track the C_+ chamber from $\nu_2\nu_1\mathrm{A}$ to $\nu_1\mathrm{A}$ to A. By the same logic

$$\begin{array}{cccccc} \phi_1 & \overset{(3.C)}{\mapsto} & \phi_1+2\phi_2 & \overset{(3.B)}{\mapsto} & -(\phi_1+2\phi_2) & = & -\phi_1-2\phi_2 \\ \phi_2 & & -\phi_2 & & -\phi_2+(\phi_1+2\phi_2) & & \phi_1+\phi_2 \end{array}$$

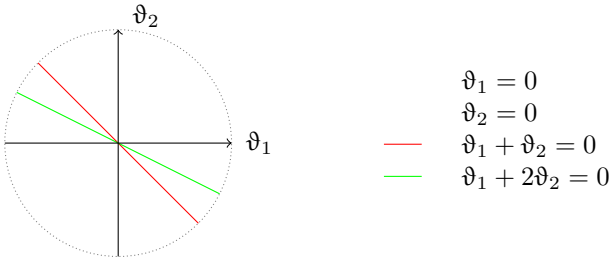
which gives the region $\vartheta_1+2\vartheta_2>0$ and $\vartheta_1+\vartheta_2<0$ of Θ_{A}, and so this too is a chamber.

Next, tracking C_+ from $\nu_1\nu_2\nu_1\mathrm{A}$ to $\nu_2\nu_1\mathrm{A}$ to $\nu_1\mathrm{A}$ to A gives

$$\begin{array}{ccccccc} \phi_1 & \overset{(3.B)}{\mapsto} & -\phi_1 & \overset{(3.C)}{\mapsto} & \phi_1+2\phi_2 & \overset{(3.B)}{\mapsto} & -\phi_1-2\phi_2 \\ \phi_2 & & \phi_1+\phi_2 & & -\phi_1-\phi_2 & & \phi_2 \end{array}$$

which gives the region $\vartheta_1+2\vartheta_2<0$ and $\vartheta_2>0$.

Continuing in this fashion, Θ_{A} has chamber structure as follows.



3.3. **Braiding: Two-Curve Case.** The aim of this subsection is to prove the following.

Theorem 3.6. *Suppose that $X\to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, contracting precisely two independently floppable irreducible curves. If X has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1\circ\cdots}_{n}\cong\mathsf{F}_{\{1,2\}}\cong\underbrace{\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\circ\cdots}_{n}$$

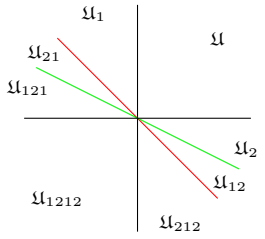
for some $n\geq 2$. Furthermore:

(1) *If the curves intersect, n is half the number of chambers in Θ_{A}, and $n\geq 3$.*
(2) *If the curves are disjoint, then $n=2$.*

The statement and proof of 3.6(1) is contained in 3.12, and similarly 3.6(2) is contained in 3.13.

We first prove 3.6(1). To do this, we give a complete local version of the result in 3.8. We then establish a Zariski local version in 3.10, before finally giving the result globally in 3.12. Before beginning the proof, which is notationally complicated, we first illustrate the strategy in an example.

Example 3.7. We continue the complete local example 3.5. Label the minimal models arising from the chambers by



By [W, 4.9], the above chamber structure implies that

$$\nu_2\nu_1\nu_2\nu_1 N\cong\nu_1\nu_2\nu_1\nu_2 N$$

since the chamber C_+ for both $\mathrm{End}_{\mathfrak{R}}(\nu_2\nu_1\nu_2\nu_1 N)$ and $\mathrm{End}_{\mathfrak{R}}(\nu_1\nu_2\nu_1\nu_2 N)$ gives, under moduli tracking, the same chamber on Θ_{A}. Hence there is a diagram of mutation functors

$$\begin{array}{ccccccc} & & \mathrm{D^b}(\nu_2\mathrm{A}) & \xrightarrow{\Phi_1} & \mathrm{D^b}(\nu_1\nu_2\mathrm{A}) & \xrightarrow{\Phi_2} & \mathrm{D^b}(\nu_2\nu_1\nu_2\mathrm{A}) \\ & \overset{\Phi_2}{\nearrow} & & & & & \searrow^{\Phi_1} \\ \mathrm{D^b}(\mathrm{A}) & & & & & & \mathrm{D^b}(\nu_1\nu_2\nu_1\nu_2\mathrm{A}) \\ & \underset{\Phi_1}{\searrow} & & & & & \nearrow_{\Phi_2} \\ & & \mathrm{D^b}(\nu_1\mathrm{A}) & \xrightarrow[\Phi_2]{} & \mathrm{D^b}(\nu_2\nu_1\mathrm{A}) & \xrightarrow[\Phi_1]{} & \mathrm{D^b}(\nu_1\nu_2\nu_1\mathrm{A}) \end{array} \tag{3.D}$$

Further, by [W, 4.2] the inverse of the flop functor is functorially isomorphic to mutation, so (3.D) is functorially isomorphic to the diagram of functors

$$\begin{array}{ccccccc} & & \mathrm{D^b}(\mathfrak{U}_2) & \xrightarrow{\mathsf{F}_1^{-1}} & \mathrm{D^b}(\mathfrak{U}_{12}) & \xrightarrow{\mathsf{F}_2^{-1}} & \mathrm{D^b}(\mathfrak{U}_{212}) \\ & \overset{\mathsf{F}_2^{-1}}{\nearrow} & & & & & \searrow^{\mathsf{F}_1^{-1}} \\ \mathrm{D^b}(\mathfrak{U}) & & & & & & \mathrm{D^b}(\mathfrak{U}_{1212}) \\ & \underset{\mathsf{F}_1^{-1}}{\searrow} & & & & & \nearrow_{\mathsf{F}_2^{-1}} \\ & & \mathrm{D^b}(\mathfrak{U}_1) & \xrightarrow[\mathsf{F}_2^{-1}]{} & \mathrm{D^b}(\mathfrak{U}_{21}) & \xrightarrow[\mathsf{F}_1^{-1}]{} & \mathrm{D^b}(\mathfrak{U}_{121}) \end{array} \tag{3.E}$$

Now choose a skyscraper $\mathcal{O}_x$ in $\mathrm{D^b}(\mathfrak{U})$. By [K14, §5.2], this corresponds to some ϑ-stable module M in $\mathrm{D^b}(\mathrm{A})$ for $\vartheta \in C_+$. Remarkably, under the mutation functors in (3.D), this module M is always sent to a module. This follows by using 3.1(1) repeatedly. Indeed the new module is stable for some stability parameter, which may be calculated using the formula given in 3.1. Tracking this data we see that under mutation the module M gets sent to modules stable for parameters as follows.

$$\begin{array}{ccccccc} & & \begin{matrix}\vartheta_1+2\vartheta_2\\ -\vartheta_2\end{matrix} & \xrightarrow{\Phi_1} & \begin{matrix}-\vartheta_1-2\vartheta_2\\ \vartheta_1+\vartheta_2\end{matrix} & \xrightarrow{\Phi_2} & \begin{matrix}\vartheta_1\\ -\vartheta_1-\vartheta_2\end{matrix} \\ & \overset{\Phi_2}{\nearrow} & & & & & \searrow^{\Phi_1} \\ \begin{matrix}\vartheta_1\\ \vartheta_2\end{matrix} & & & & & & \begin{matrix}-\vartheta_1\\ -\vartheta_2\end{matrix} \\ & \underset{\Phi_1^{-1}}{\swarrow} & & & & & \swarrow_{\Phi_2^{-1}} \\ & & \begin{matrix}-\vartheta_1\\ \vartheta_1+\vartheta_2\end{matrix} & \xleftarrow[\Phi_2^{-1}]{} & \begin{matrix}\vartheta_1+2\vartheta_2\\ -\vartheta_1-\vartheta_2\end{matrix} & \xleftarrow[\Phi_1^{-1}]{} & \begin{matrix}-\vartheta_1-2\vartheta_2\\ \vartheta_2\end{matrix} \end{array} \tag{3.F}$$

It follows immediately that the composition of mutations

$$\Phi_1^{-1}\circ\Phi_2^{-1}\circ\Phi_1^{-1}\circ\Phi_2^{-1}\circ\Phi_1\circ\Phi_2\circ\Phi_1\circ\Phi_2$$

sends M, which is ϑ-stable for $\vartheta=(\vartheta_1,\vartheta_2)\in C_+$, to a module which is also ϑ-stable. Since (3.D) is functorially isomorphic to (3.E), it follows that

$$\Psi := \mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1^{-1}\circ\mathsf{F}_2^{-1}\circ\mathsf{F}_1^{-1}\circ\mathsf{F}_2^{-1}$$

sends the skyscraper $\mathcal{O}_x$ to some object in $\mathrm{D^b}(\mathrm{coh}\,\mathfrak{U})$ corresponding to a ϑ-stable module. But again by [K14, §5.2] these are precisely the skyscrapers. Hence we obtain that Ψ is a Fourier–Mukai equivalence that takes skyscrapers to skyscrapers, fixes $\mathcal{O}_{\mathfrak{U}}$ and commutes with the pushdown $\mathbf{R}f_*$. It follows that $\Psi\cong\mathrm{Id}$ and so

$$\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\cong\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1.$$

The following proposition is a simple extension of the above example. Recall that $\Phi_{\{1,2\}}$ denotes the mutation functor for the summand $N_1\oplus N_2$.

Proposition 3.8. *Suppose that $\mathfrak{U}\to\mathrm{Spec}\,\mathfrak{R}$ is a complete local flopping contraction with precisely two floppable curves meeting at a point. If $\mathfrak{U}$ has at worst Gorenstein terminal*

singularities, then

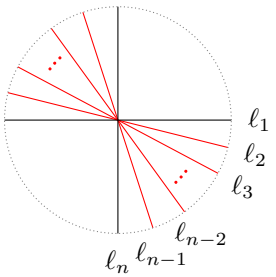
$$\underbrace{\Phi_1\circ\Phi_2\circ\Phi_1\circ\cdots}_{n}\cong\Phi_{\{1,2\}}\cong\underbrace{\Phi_2\circ\Phi_1\circ\Phi_2\circ\cdots}_{n}$$

and

$$\underbrace{\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1\circ\cdots}_{n}\cong\mathsf{F}_{\{1,2\}}\cong\underbrace{\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\circ\cdots}_{n}$$

for some $n\geq 3$, where n is half the number of chambers of Θ_{A}.

Proof. Consider $\mathrm{A}=\mathrm{End}_{\mathfrak{R}}(N)$, which is derived equivalent to $\mathfrak{U}$. By 3.2, Θ_{A} is a hyperplane arrangement in $\mathbb{Q}^2$, with C_+ being a chamber. This implies that the chamber structure for Θ_{A} is



(3.G)

for some lines given by $\ell_1=0$, $\ell_2=0$, …, $\ell_n=0$, where

$$\ell_1=\vartheta_2\quad\text{and}\quad\ell_n=\vartheta_1.$$

The fact that $n\geq 3$ follows immediately by knitting on the AR quivers of Kleinian singularities, as in [W, 5.23, 5.19].

We first claim that the above chamber structure (3.G) implies that

$$\underbrace{\ldots\nu_2\nu_1\nu_2}_{n}N\cong\nu_{\{1,2\}}N\cong\underbrace{\ldots\nu_1\nu_2\nu_1}_{n}N. \tag{3.H}$$

For notation, as in 3.7, we let $\mathfrak{U}_{\ldots kji}$ denote the scheme obtained from $\mathfrak{U}$ by first flopping curve i, then curve j, then curve k and so on (in that order). By [W, 4.2], we have

$$\nu_1N\cong H^0(\mathcal{V}_{\mathfrak{U}_1})\quad\text{and}\quad\nu_2N\cong H^0(\mathcal{V}_{\mathfrak{U}_2}).$$

Iterating, again by [W, 4.2] the left- and right-hand terms of (3.H) are $H^0(\mathcal{V}_{\mathfrak{U}_{\ldots 212}})$ and $H^0(\mathcal{V}_{\mathfrak{U}_{\ldots 121}})$, respectively. Further, by [K14, 5.2.5], the scheme $\mathfrak{U}_{\ldots 212}$ can be obtained as the moduli in the chamber C_+ for $\ldots\nu_2\nu_1\nu_2\mathrm{A}$, and $\mathfrak{U}_{\ldots 121}$ can be obtained as the moduli in the chamber C_+ for $\ldots\nu_1\nu_2\nu_1\mathrm{A}$. Tracking these chambers back to Θ_{A}, which we can do by 3.2(3) (see also the proof of [W, 5.22]), both give the region

$$C_-=\{\vartheta\mid\vartheta_i<0\text{ for }i=1,2\}\subset\Theta_{\mathrm{A}},$$

and so $\mathfrak{U}_{\ldots 212}\cong\mathfrak{U}_{\ldots 121}$ as schemes over $\operatorname{Spec}\mathfrak{R}$. In fact, again by moduli tracking, both $\mathfrak{U}_{\ldots 212}$ and $\mathfrak{U}_{\ldots 121}$ are isomorphic to the scheme obtained from $\mathfrak{U}$ by flopping $C_1\bigcup C_2$, which we denote by $\mathfrak{U}_{\{1,2\}}$. Using [W, 4.2] once again, the middle term of (3.H) is $H^0(\mathcal{V}_{\mathfrak{U}_{\{1,2\}}})$. Taking global sections of the progenerator of perverse sheaves on $\mathfrak{U}_{\{1,2\}}$ gives (3.H), as claimed.

Because of (3.H), there exists a diagram of mutation functors

$$\begin{array}{ccccc}
 & & \mathrm{D^b}(\nu_2\mathrm{A}) & \xrightarrow{\Phi_1}\cdots & \mathrm{D^b}(\ldots\nu_1\nu_2\mathrm{A}) \\
 & \overset{\Phi_2}{\nearrow} & & & \searrow \\
\mathrm{D^b}(\mathrm{A}) & & \xrightarrow{\qquad\Phi_{\{1,2\}}\qquad} & & \mathrm{D^b}(\ldots\nu_2\nu_1\nu_2\mathrm{A}) \\
 & \underset{\Phi_1}{\searrow} & & & \nearrow \\
 & & \mathrm{D^b}(\nu_1\mathrm{A}) & \xrightarrow[\Phi_2]{}\cdots & \mathrm{D^b}(\ldots\nu_2\nu_1\mathrm{A})
\end{array} \tag{3.I}$$

where the functors on the right-hand side depend on whether n is odd or even; respectively they are

$$\begin{array}{ccc} \mathrm{D^b}(\dots \nu_1\nu_2 \mathrm{A}) & & \\ & \searrow^{\Phi_2} & \\ & & \mathrm{D^b}(\dots \nu_2\nu_1\nu_2 \mathrm{A}) \\ & \nearrow_{\Phi_1} & \\ \mathrm{D^b}(\dots \nu_2\nu_1 \mathrm{A}) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{D^b}(\dots \nu_1\nu_2 \mathrm{A}) & & \\ & \searrow^{\Phi_1} & \\ & & \mathrm{D^b}(\dots \nu_2\nu_1\nu_2 \mathrm{A}). \\ & \nearrow_{\Phi_2} & \\ \mathrm{D^b}(\dots \nu_2\nu_1 \mathrm{A}) & & \end{array}$$

We next chase moduli around (3.I), repeatedly applying 3.1 using the characterisation of the chamber structure (3.G). Consider a ϑ-stable A-module M, for $\vartheta \in C_+$. Tracking M around (3.I), we find that it is sent to a module which is stable for the following parameters: when n is even

$$\begin{array}{ccccccccccccc} & & \ell_2 & \xrightarrow{\Phi_1} & -\ell_2 & \cdots & -\ell_{n-2} & \xrightarrow{\Phi_2} & \ell_n & & & & \\ & \nearrow^{\Phi_2} & -\ell_1 & & \ell_3 & & \ell_{n-1} & & -\ell_{n-1} & \searrow^{\Phi_1} & & & \\ \vartheta_1 = & \ell_n & & & & & & & & & -\ell_n & = & -\vartheta_1 \\ \vartheta_2 = & \ell_1 & & & & & & & & & -\ell_1 & = & -\vartheta_2 \\ & \searrow^{\Phi_1} & -\ell_n & & \ell_{n-2} & \cdots & \ell_2 & & -\ell_2 & \nearrow_{\Phi_2} & & & \\ & & \ell_{n-1} & \xrightarrow{\Phi_2} & -\ell_{n-1} & & -\ell_3 & \xrightarrow{\Phi_1} & \ell_1 & & & & \end{array}$$

and when n is odd

$$\begin{array}{ccccccccccccc} & & \ell_2 & \xrightarrow{\Phi_1} & -\ell_2 & \cdots & -\ell_{n-1} & & & & & & \\ & \nearrow^{\Phi_2} & -\ell_1 & & \ell_3 & & \ell_n & & \searrow^{\Phi_2} & & & & \\ \vartheta_1 = & \ell_n & & & & & & & & -\ell_1 & = & -\vartheta_2 & \\ \vartheta_2 = & \ell_1 & & & & & & & & -\ell_n & = & -\vartheta_1 & \\ & \searrow^{\Phi_1} & -\ell_n & & \ell_{n-2} & \cdots & \ell_1 & & \nearrow_{\Phi_1} & & & & \\ & & \ell_{n-1} & \xrightarrow{\Phi_2} & -\ell_{n-1} & & -\ell_2 & & & & & & \end{array}$$

In either case,

$$(\Phi_1^{-1} \circ \Phi_2^{-1} \circ \Phi_1^{-1} \circ \cdots) \circ (\cdots \circ \Phi_2 \circ \Phi_1 \circ \Phi_2) \tag{3.J}$$

sends a ϑ-stable module to a ϑ-stable module. Similarly, since $\Phi_{\{1,2\}}$ negates both parameters,

$$\Phi_{\{1,2\}}^{-1} \circ (\cdots \circ \Phi_2 \circ \Phi_1 \circ \Phi_2) \tag{3.K}$$

sends a ϑ-stable module to a module stable for some parameter in C_+.

Now by [W, 4.2] mutation is functorially isomorphic to the inverse of the flop functor, and by [K14, §5.2] under the derived equivalence skyscrapers correspond precisely to ϑ-stable modules, for $\vartheta \in C_+$. Hence (3.J) and (3.K) are functorially isomorphic to the corresponding chain of inverse flop functors, and it follows that

$$\begin{gathered} (\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots) \circ (\cdots \circ \mathsf{F}_2^{-1} \circ \mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1}) \\ \text{and} \quad \mathsf{F}_{\{1,2\}} \circ (\cdots \circ \mathsf{F}_2^{-1} \circ \mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1}) \end{gathered}$$

are equivalences that take skyscrapers to skyscrapers. Since they also fix the structure sheaf $\mathcal{O}_{\mathcal{U}}$, and commute with the pushdown $\mathbf{R}f_*$ as in [DW1, 7.16(1)], it follows that both are naturally isomorphic to the identity. Finally, we deduce that (3.J) and (3.K) are also naturally isomorphic to the identity, and the result follows. □

We next lift the above into the algebraic setting. To do this, we use the Zariski local tilting bundle $\mathcal{V}$ in §2.3.

Lemma 3.9. *Suppose that $U \to \operatorname{Spec} R$ is an algebraic flopping contraction with precisely two independently floppable curves meeting at a point. Denote by U^+ the flop of U at one of the curves, and consider the tilting bundles $\mathcal{V}$ and $\mathcal{V}^+$ from §2.3. Set $\Lambda := \operatorname{End}_U(\mathcal{V})$ and $\Lambda^+ := \operatorname{End}_{U^+}(\mathcal{V}^+)$, then*

(1) *$Z := \operatorname{Hom}_R(H^0(\mathcal{V}), H^0(\mathcal{V}^+))$ is a tilting Λ-module with $\operatorname{End}_\Lambda(Z) \cong \Lambda^+$.*
(2) *The following is a commutative diagram of equivalences.*

$$\begin{array}{ccc} \mathrm{D^b}(\operatorname{coh} U) & \xrightarrow{\mathbf{R}\operatorname{Hom}_U(\mathcal{V},-)} & \mathrm{D^b}(\operatorname{mod}\Lambda) \\ {\scriptstyle \mathsf{F}^{-1}}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\operatorname{Hom}_\Lambda(Z,-)} \\ \mathrm{D^b}(\operatorname{coh} U^+) & \xrightarrow{\mathbf{R}\operatorname{Hom}_{U^+}(\mathcal{V}^+,-)} & \mathrm{D^b}(\operatorname{mod}\Lambda^+) \end{array} \tag{3.L}$$

Proof. (1) For the statement on endomorphism rings, by repeated use of (2.B) we see that

$$\operatorname{End}_\Lambda(Z) \cong \operatorname{End}_{\operatorname{End}_R(H^0(\mathcal{V}))}\big(\operatorname{Hom}_R(H^0(\mathcal{V}), H^0(\mathcal{V}^+))\big) \cong \operatorname{End}_R(H^0(\mathcal{V}^+)) \cong \Lambda^+,$$

where the second isomorphism is reflexive equivalence. Now the property of being a tilting module can be checked complete locally, and certainly $\widehat{Z}_{\mathfrak{n}}$ is a tilting $\widehat{\Lambda}_{\mathfrak{n}}$-module for all $\mathfrak{n} \in \operatorname{Max} R$ with $\mathfrak{n} \neq \mathfrak{m}$. Further, for the point $\mathfrak{m}$, by (2.F) we know that $\operatorname{add}(H^0(\mathcal{V}) \otimes_R \mathfrak{R}) = \operatorname{add} N$ and similarly $\operatorname{add}(H^0(\mathcal{V}^+) \otimes_R \mathfrak{R}) = \operatorname{add} N^+$. But exactly as in [DW1, 5.8] (or in 4.4 below), there is a commutative diagram

$$\begin{array}{ccc} \mathrm{D^b}(\operatorname{mod} \mathrm{A}) & \xrightarrow{morita} & \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}) \\ {\scriptstyle \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(N,N^+),-)}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{Z},-)} \\ \mathrm{D^b}(\operatorname{mod} \mathrm{A}^+) & \xrightarrow{morita} & \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}^+). \end{array} \tag{3.M}$$

Say curve i has been flopped, then by [W, 4.17(1)] $N^+ = \nu_i N$, and so the left-hand functor is the mutation functor (see e.g. [W, 2.22(1)] or 4.2(2)), which is an equivalence. Hence the right-hand functor is also an equivalence, and the statement follows.
(2) For each $\mathfrak{n} \in \operatorname{Max} R$ consider the formal fibre version of the diagram. If $\mathfrak{n} \neq \mathfrak{m}$ then the diagram clearly commutes, since $f\colon U \to \operatorname{Spec} R$ is an isomorphism away from $\mathfrak{m}$. If $\mathfrak{n} = \mathfrak{m}$ then the diagram commutes by combining (3.M) with [W, 4.2].

Now write Ψ for a composition of the four equivalences in the square (3.L) to give an autoequivalence of $\mathrm{D^b}(\operatorname{coh} U)$. Consider a skyscraper $\mathcal{O}_u \in \mathrm{D^b}(\operatorname{coh} U)$. Since the formal fibre versions of the diagram (3.L) commute, it follows that Ψ fixes $\mathcal{O}_u$. Furthermore Ψ fixes $\mathcal{O}_U$, and commutes with the pushdown $\mathbf{R}f_*$ by the compatibility result of [W, 2.14(2)]. We conclude that Ψ is functorially isomorphic to the identity, and thence that (3.L) is functorially commutative. □

Using the above, we can now extend 3.8 into the Zariski local setting:

Theorem 3.10. *Suppose that $U \to \operatorname{Spec} R$ is an algebraic flopping contraction with precisely two independently floppable curves meeting at a point. If U has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots}_{n} \cong \mathsf{F}_{\{1,2\}} \cong \underbrace{\mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \cdots}_{n}$$

for some $n \geq 3$, where n is half the number of chambers in Θ_{A}.

Proof. Since each algebraic flop, after passing to the formal fibre, is still a flop, iteratively flopping all possible combinations of all possible subsets of the two curves (which, since the two curves are individually floppable, we may do) gives the same number of schemes in both cases, and the combinatorics are the same for both. In particular $n \geq 3$ by 3.8.

To show braiding for a chain of algebraic flops, again we track skyscrapers $\mathcal{O}_u$ under the chain

$$(\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots) \circ (\cdots \circ \mathsf{F}_2^{-1} \circ \mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1}). \tag{3.N}$$

Using 3.9(2), we can reduce the problem to tracking ϑ-stable Λ-modules M for $\vartheta \in C_+$. Since M is supported only at a single point $\mathfrak{n}$ as an R-module,

$$\mathbf{R}\mathrm{Hom}_\Lambda(Z, M) \otimes_R \widehat{R}_\mathfrak{n} \cong \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}_\mathfrak{n}}(\widehat{Z}_\mathfrak{n}, M)$$

and so it suffices to track M complete locally. If $\mathfrak{n} \neq \mathfrak{m}$ it is clear that M tracks to itself. If $\mathfrak{n} = \mathfrak{m}$ then by (3.M) $\mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{Z}, -)$ is naturally isomorphic to the mutation functor. Hence by 3.8 all skyscrapers track to skyscrapers under (3.N), so again since the structure sheaf is fixed and the functors all commute with the pushdown, (3.N) is functorially isomorphic to the identity. A similar argument shows that

$$\mathsf{F}_{\{1,2\}} \circ (\cdots \circ \mathsf{F}_2^{-1} \circ \mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1})$$

is functorially isomorphic to the identity, and the result follows. □

Next we focus on the global setting of 3.6, for the case of two intersecting curves. Recall that we have a contraction $f\colon X \to X_{\mathrm{con}}$ mapping these curves to a point p. As in §2.2, put $C = f^{-1}(p)$, so that $C^{\mathrm{red}} = C_1 \cup C_2$ with $C_i \cong \mathbb{P}^1$. Then we choose an affine open neighbourhood $U_{\mathrm{con}} \cong \operatorname{Spec} R$ around p, so that setting $U := f^{-1}(U_{\mathrm{con}})$, we have the commutative diagram

$$\begin{array}{ccccc} C & \xhookrightarrow{e} & U & \xhookrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ p & \in & U_{\mathrm{con}} & \hookrightarrow & X_{\mathrm{con}} \end{array}$$

where e is a closed embedding, and i is an open embedding. Choose one of the curves C_i, and write U^+ and X^+ for the schemes obtained by flopping C_i in U and X respectively. The following is similar to [DW1, 7.8], and is well-known to experts.

Lemma 3.11. *In the setting of 3.6, and with notation as above, the following diagram is naturally commutative:*

$$\begin{array}{ccc} \mathrm{D}(\mathrm{Qcoh}\, U) & \xrightarrow{\mathbf{R}i_*} & \mathrm{D}(\mathrm{Qcoh}\, X) \\ {\scriptstyle \mathsf{F}_i}\downarrow & & \downarrow{\scriptstyle \mathsf{F}_i} \\ \mathrm{D}(\mathrm{Qcoh}\, U^+) & \xrightarrow{\mathbf{R}i^+_*} & \mathrm{D}(\mathrm{Qcoh}\, X^+). \end{array}$$

Proof. Write Γ_U (respectively Γ_X) for the fibre product of U (respectively X) with its flop, over the contracted base U_{con} (respectively X_{con}). Then there are maps

$$g_U\colon \Gamma_U \to U \times U^+ \quad \text{and} \quad g_X\colon \Gamma_X \to X \times X^+,$$

and a natural inclusion $\iota\colon \Gamma_U \to \Gamma_X$. Using this notation to translate the claim into the language of Fourier–Mukai kernels [H, 5.12], we require that

$$\mathbf{L}(i \times \mathrm{Id})^* g_{X*}\mathcal{O} \cong \mathbf{R}(\mathrm{Id} \times i^+)_* g_{U*}\mathcal{O}.$$

This follows by considering the diagram

$$\begin{array}{ccc} \Gamma_U & \xrightarrow{h} & U \times X^+ \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle i \times \mathrm{Id}} \\ \Gamma_X & \xrightarrow{g_X} & X \times X^+ \end{array}$$

where h is the natural map $(\mathrm{Id}\times i^+)\circ g_U$. Under the birational correspondence between X and X^+, points of U correspond to points of U^+ and vice versa, so this square is cartesian. The right-hand map is flat, and so base change gives $\mathbf{L}(i\times\mathrm{Id})^*\mathbf{R}g_{X*}\cong\mathbf{R}h_*\mathbf{L}\iota^*$. The result follows by applying this isomorphism to $\mathcal{O}$ on Γ_X. □

Using 3.10 and 3.11, we can extend 3.10 to the global setting:

Corollary 3.12. *Suppose that $X\to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, with precisely two independently floppable curves intersecting at a point. If X has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1\circ\mathsf{F}_2\circ\mathsf{F}_1\circ\cdots}_{n}\cong\mathsf{F}_{\{1,2\}}\cong\underbrace{\mathsf{F}_2\circ\mathsf{F}_1\circ\mathsf{F}_2\circ\cdots}_{n}$$

for some $n\geq 3$, where n is half the number of chambers in Θ_{A}.

Proof. Once again, we track skyscrapers $\mathcal{O}_x$ under the chain

$$\mathsf{F}_{\{1,2\}}\circ(\cdots\circ\mathsf{F}_2^{-1}\circ\mathsf{F}_1^{-1}\circ\mathsf{F}_2^{-1}). \tag{3.O}$$

If $x\notin U$, then certainly $x\notin C$, so all the flop functors take $\mathcal{O}_x$ to $\mathcal{O}_x$, and hence the composition does. On the other hand, if $x\in U$ then we can combine 3.10 and 3.11 to conclude that the chain of functors in (3.O) takes $\mathcal{O}_x$ to $\mathcal{O}_x$. Thus, either way, (3.O) preserves skyscrapers, so again since the structure sheaf is fixed and the functors commute with the pushdown, (3.N) is functorially isomorphic to the identity. This gives one of the isomorphisms in the statement, and the other follows by symmetry. □

In contrast, the disjoint curves case is easy, and is well-known.

Lemma 3.13. *Suppose that $X\to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, with precisely two independently floppable disjoint curves. If X has at worst Gorenstein terminal singularities, then*

$$\mathsf{F}_1\circ\mathsf{F}_2\cong\mathsf{F}_{\{1,2\}}\cong\mathsf{F}_2\circ\mathsf{F}_1.$$

Proof. Since the curves are disjoint, from the definition of flop it is immediate that the order of flops does not matter. Further, since flop functors are local over the common singular base, chasing skyscrapers and using 3.9 the result is immediate, using the same argument as in 3.12. □

3.4. **Braiding: Multiple Curves.** The combinatorics of braiding of multiple curves requires knowledge of the structure of the hyperplane arrangements Θ_{A}. Recall that a real hyperplane arrangement is *simplicial* if the intersection of all the hyperplanes is $\{0\}$, and furthermore every chamber is an open simplicial cone. The following result, which is an immediate consequence of the Homological MMP, will be used heavily.

Lemma 3.14. *With notation as in §3.1, Θ_{A} is a simplicial hyperplane arrangement.*

Proof. By 3.2(2), the codimension-one walls of Θ_{A} are given by a finite collection of hyperplanes, all of which contain the origin, and furthermore the co-ordinate hyperplanes $x_i=0$ are included in this collection. It follows that the intersection of all the hyperplanes is the origin. Further, since C_+ is clearly an open simplicial cone, and by 3.2(3) every chamber of Θ_{A} is given by tracking the chambers C_+ under moduli tracking, it follows that all chambers are open simplicial cones. □

Simplicial hyperplane arrangements were studied in the seminal work of Deligne [D72], and the resulting *Deligne groupoid* has many equivalent definitions in the literature. Here for convenience we follow Paris [P93, §2.A, §2.B].

Definition 3.15. *Given the real hyperplane arrangement Θ_{A}, we first associate the oriented graph X_1 which has vertices v_i corresponding to the chambers of Θ_{A}, with an arrow $a\colon v_i \to v_j$ between pairs of vertices corresponding to adjacent chambers. A* path of length n *in X_1 is defined to be a formal symbol*

$$a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n}$$

with each $\varepsilon_i \in \{\pm 1\}$, whenever there exists a sequence of vertices $v_0, \dots, v_n$ of X_1 such that $a_i\colon v_{i-1} \to v_i$ if $\varepsilon_i = 1$, and $a_i\colon v_{i-1} \leftarrow v_i$ if $\varepsilon_i = -1$. A path is said to be positive *if each $\varepsilon_i = 1$. Such a path is called* minimal *if there is no positive path in X_1 of smaller length, and with the same endpoints.*

Let $\sim$ be the smallest identification *(an equivalence relation satisfying appropriate properties, see* [P93, p152]*) on the set of paths of X_1 such that if f and g are both positive minimal paths with the same endpoints then $f \sim g$. Then the pair $(X_1, \sim)$ determines a groupoid, the* Deligne groupoid $\mathbb{G}_{\mathrm{A}}$*, where the objects are the vertices, and the morphisms are the equivalence classes of paths.*

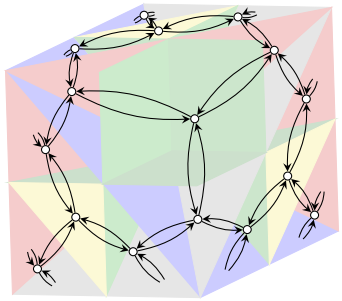


FIGURE 3. The front part of the oriented graph X_1 for Θ_{A} in Figure 1.

Remark 3.16. By [P93, S87] (see [P00, 2.1]), any vertex group of the groupoid $\mathbb{G}_{\mathrm{A}}$ defined above is isomorphic to the fundamental group of the complexified complement of the real hyperplane arrangement Θ_{A}. We thus let $\pi_1(\mathbb{G}_{\mathrm{A}})$ denote a vertex group of the Deligne groupoid.

By [D72, 1.10, 1.12], to produce a representation of the Deligne groupoid, it is sufficient to check certain codimension-two relations. As in the previous subsection, we first consider this problem in the formal fibre setting.

Lemma 3.17. *With notation as in the complete local setup of §3.1, suppose that c is a chamber in Θ_{A} with a codimension-two wall w. Then*

1. *From c, crashing through w corresponds to flopping a pair of curves $C_{i_1} \bigcup C_{i_2}$ in $\mathcal{M}_{\mathsf{rk},c}(\mathrm{A})$, where $\mathcal{M}_{\mathsf{rk},c}(\mathrm{A})$ is the scheme of c-stable A-modules of dimension vector rk corresponding to the chamber c in Θ_{A}.*
2. *Iterating $\cdots \circ \mathsf{F}_{i_1} \circ \mathsf{F}_{i_2} \circ \mathsf{F}_{i_1}$ traverses one direction around the codimension two wall w, whilst $\cdots \circ \mathsf{F}_{i_2} \circ \mathsf{F}_{i_1} \circ \mathsf{F}_{i_2}$ traverses the other direction.*

Proof. Viewing $\mathcal{M}_{\mathsf{rk},c}(\mathrm{A})$ abstractly, we associate an algebra B to it using the procedure in §2.4. In turn, this algebra has a chamber structure, which we denote Θ_{B}. By 3.2(3), the chamber c in Θ_{A} is the tracking, under moduli tracking, of the chamber C_+ in Θ_{B}. Furthermore, since under moduli tracking walls get sent to walls, the codimension-two wall w corresponds to one of the codimension-two walls of C_+ in Θ_{B}, which without loss of generality we can assume is $x_1 = x_2 = 0$. Thus, since as schemes $\mathcal{M}_{\mathsf{rk},c}(\mathrm{A}) = \mathcal{M}_{\mathsf{rk},C_+}(\mathrm{B})$, to prove (1) it suffices to prove that, from C_+ in Θ_{B}, crashing through the codimension-two wall $x_1 = x_2 = 0$ corresponds to flopping $C_1 \bigcup C_2$. But this is immediate by [W, 4.16] and moduli tracking.

To prove (2), since in all the chamber structures crossing codimension-one walls corresponds to a flop [W, §5], it suffices to show that iterating $\cdots \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1$ traverses one direction around the codimension two wall $x_1 = x_2 = 0$ in Θ_{B}, whilst $\cdots \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2$ traverses the other direction.

Beginning in the chamber C_+ of Θ_{B}, flopping F_1 corresponds to crashing through the single codimension one-wall $\vartheta_1 = 0$ [W, 5.21], and produces a new chamber that by the moduli tracking formula has a codimension two wall $x_1 = x_2 = 0$. This new chamber can then be viewed as C_+ on another algebra. Repeating the argument for that chamber then tracking back to Θ_{B}, the iterate $\mathsf{F}_2 \circ \mathsf{F}_1$ corresponds to crashing through two consecutive walls of Θ_{B}. Further, by the moduli tracking formula, both of the obtained chambers share the codimension-two wall $x_1 = x_2 = 0$. By induction, iterating $\cdots \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1$ produces a series of chambers, at each stage crossing a single codimension-one wall, and each chamber has a codimension-two wall $x_1 = x_2 = 0$. It follows that iterating $\cdots \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1$ traverses one direction around the codimension two wall $x_1 = x_2 = 0$. By symmetry of the argument, necessarily $\cdots \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2$ traverses the other direction. □

Definition 3.18. *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, with precisely one connected chain of n independently floppable curves. Given this data, the derived flops groupoid $\mathbb{D}_{\mathsf{Flop}}$ is defined by the following generating set. It has vertices $\mathrm{D^b}(\mathrm{coh}\, X)$, running over all varieties obtained from X by iteratively flopping the n curves, and as arrows we connect vertices by the Bridgeland–Chen flop functors, running through all possible combinations of single flopping curves.*

The following is the main result of this section.

Theorem 3.19. *Suppose that $X \to X_{\mathrm{con}}$ is an algebraic flopping contraction with precisely one connected chain of curves contracting to a point, where X has terminal Gorenstein singularities and each of the curves is individually (algebraically) floppable. Then:*

(1) *There is a homomorphism of groupoids $\mathbb{G}_{\mathrm{A}} \to \mathbb{D}_{\mathsf{Flop}}$.*
(2) *The group $\pi_1(\mathbb{G}_{\mathrm{A}})$ acts on $\mathrm{D^b}(\mathrm{coh}\, X)$.*

Proof. As in 3.10, since each algebraic flop, after passing to the formal fibre, is still a flop, iteratively flopping all possible single curves gives the same number of schemes in both cases, and the combinatorics are the same for both. Hence the braiding of the algebraic flop functors is governed by the same simplicial hyperplane arrangement Θ_{A} which we considered in the complete local setup.

By [D72, 1.10, 1.12] (see also [CM]), to prove (1) we only need to check that the relations on the flop functors in 3.18 arising from each codimension-two wall are satisfied by the algebraic flop functors. But this follows immediately from 3.6 and 3.17. Part (2) follows directly from (1). □

Remark 3.20. The above proof does not require a presentation of $\pi_1(\mathbb{G}_{\mathrm{A}})$, which is convenient since we do not know one in general. It is known how to obtain a presentation given the explicit hyperplanes [A, R82, S87], but all the possible simplicial hyperplane arrangements arising from flops have not yet been fully classified.

4. Mutation in the Flops Setting

We now work towards dropping the assumption that all the curves are individually floppable. The aim of this section is to apply the mutation in §2.6 to the setting of flops to obtain on the formal fibre various intrinsic derived autoequivalences. These will then be made algebraic in §5, and will give intrinsic algebraic autoequivalences regardless of whether the curves flop individually.

The results related to the fibre twist in §4.3 and §4.4 will require an additional assumption on the singularities.

4.1. **Mutation for Flops.** We keep the complete local flops setup of §2.4, and in particular the notation of 2.10 where $\mathrm{A} := \mathrm{End}_{\mathfrak{R}}(N)$. As in 2.11, for any $J \subseteq \{0,1,\dots,n\}$ we set $N_J := \bigoplus_{j\in J} N_j$ and $N_{J^c} := \bigoplus_{i\notin J} N_i$, so that $N = N_J \oplus N_{J^c}$.

Remark 4.1. In later sections we will be interested in two special cases. The first is $J = \{0\}$, which will give the 'fibre twist' corresponding to deformations of the scheme-theoretic fibre $\mathcal{O}_C$. The second will be when $J \subseteq \{1,\dots,n\}$, as this will give the J-twist corresponding to the noncommutative deformations of the family $\{E_j \mid j \in J\}$.

The following is elementary.

Lemma 4.2. *In the complete local flops setup of 2.4, for any $J \subseteq \{0,1,\dots,n\}$*

(1) A_J *is a finite dimensional algebra.*
(2) $T_J \cong \mathrm{Hom}_{\mathfrak{R}}(N, \nu_J N)$*, and is a tilting* A*-module of projective dimension one.*

Proof. Since $\mathfrak{R}$ is an isolated singularity, (1) is [IW10, 6.19(3)]. Part (2) then follows by [IW10, 6.14]. □

As in §2.6, the mutation functor gives an equivalence

$$\Phi_J := \mathbf{R}\mathrm{Hom}_{\mathrm{A}}(T_J, -)\colon \mathrm{D^b}(\mathrm{mod}\,\mathrm{A}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\nu_J\mathrm{A}).$$

If further $\nu_J\nu_J N \cong N$ (see 4.6 and 4.10 later) then we can mutate $\mathrm{End}_{\mathfrak{R}}(\nu_J N)$ back to obtain $\mathrm{End}_{\mathfrak{R}}(N) \cong \mathrm{A}$. Applying 4.2 to $\nu_J N$, $W_J := \mathrm{Hom}_{\mathfrak{R}}(\nu_J N, N)$ is a tilting $\nu_J\mathrm{A}$-module, giving rise to an equivalence which by abuse of notation we also denote

$$\Phi_J := \mathbf{R}\mathrm{Hom}_{\nu_J\mathrm{A}}(W_J, -)\colon \mathrm{D^b}(\mathrm{mod}\,\nu_J\mathrm{A}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\mathrm{A}).$$

The following is an easy generalisation of [DW1, 5.9, 5.10, 5.11].

Proposition 4.3. *In the complete local flops setup of 2.4, for any $J \subseteq \{0,1,\dots,n\}$ such that $\nu_J\nu_J N \cong N$, the following statements hold.*

(1) $\Phi_J \circ \Phi_J \cong \mathbf{R}\mathrm{Hom}_{\mathrm{A}}([N_{J^c}], -)$*, where $[N_{J^c}]$ is the two-sided ideal defined in §1.4.*
(2) $\Phi_J \circ \Phi_J(\mathrm{A}_J) \cong \mathrm{A}_J[-2]$.
(3) $\Phi_J \circ \Phi_J(S) \cong S[-2]$ *for all simple A_J-modules S.*

Proof. (1) By the assumption $\nu_J\nu_J N \cong N$ it follows that $(\mathrm{Ker}\, b_J)^* \cong \mathrm{Ker}\, a_J$. From here, the proof is then identical to [DW1, 5.10].
(2) Since $(\mathrm{Ker}\, b_J)^* \cong \mathrm{Ker}\, a_J$, combining (2.H) and (2.K) gives us a complex of R-modules

$$0 \to (\mathrm{Ker}\, b_J)^* \xrightarrow{c_J} V_J \xrightarrow{b_J^* \cdot a_J} U_J \xrightarrow{d_J^*} (\mathrm{Ker}\, b_J)^* \to 0.$$

From here the proof of (2) is word-for-word identical to [DW1, 5.9(1–2)], since although the above complex need not be exact, whereas it was in [DW1, 5.9], this does not affect anything.
(3) Since by (1) $\Phi_J \circ \Phi_J(-) \cong \mathbf{R}\mathrm{Hom}_{\mathrm{A}}([N_{J^c}], -)$, part (3) follows by tensoring both sides of (2) by $\mathrm{A}_J/\mathrm{Rad}(\mathrm{A}_J)$, just as in [DW1, 5.11], then applying idempotents. □

Since by (2.F) $\widehat{\Lambda} \cong \mathrm{End}_{\mathfrak{R}}(\bigoplus_{i=0}^n N_i^{\oplus a_i})$, which is only morita equivalent to A, we need to describe the compatibility between mutation and morita equivalence. For the positive integers a_i from (2.F), we set $Z := \bigoplus_{i=0}^n N_i^{\oplus a_i}$ so that $\widehat{\Lambda} = \mathrm{End}_{\mathfrak{R}}(Z)$. For a choice of $J \subseteq \{0,1,\dots,n\}$, consider the summand $Z_J = \bigoplus_{j\in J} N_j^{\oplus a_j}$ of Z and set $Z_{J^c} := \bigoplus_{i\notin J} N_i^{\oplus a_i}$. In an identical way to the above, there is a mutation functor

$$\Phi'_J := \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\mathrm{Hom}_{\mathfrak{R}}(Z, \nu_J Z), -)\colon \mathrm{D^b}(\mathrm{mod}\,\widehat{\Lambda}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\nu_J\widehat{\Lambda}).$$

where $\nu_J\widehat{\Lambda} := \mathrm{End}_{\mathfrak{R}}\Big(\big(\bigoplus_{j\in J}(\mathrm{Ker}\, b_j)^{*\oplus a_j}\big) \oplus \big(\bigoplus_{i\notin J} N_i^{\oplus a_i}\big)\Big)$. The following is elementary.

Lemma 4.4. *The following diagram commutes.*

$$\begin{array}{ccc} \mathrm{D^b}(\operatorname{mod}\mathrm{A}) & \xrightarrow{\ morita\ } & \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}) \\ {\scriptstyle \Phi_J}\downarrow & & \downarrow{\scriptstyle \Phi'_J} \\ \mathrm{D^b}(\operatorname{mod}\nu_J\mathrm{A}) & \xrightarrow{\ morita\ } & \mathrm{D^b}(\operatorname{mod}\nu_J\widehat{\Lambda}) \end{array}$$

Proof. This was stated in [DW1, 5.8], but since we are working more generally we give the proof here. For simplicity, we drop all J from the notation. As in 2.15 we denote the top morita functor by $\mathbb{F}$, and we also denote the bottom by $\mathbb{G}$. Since $P := \operatorname{Hom}_{\mathfrak{R}}(N, Z)$ is a progenerator, it gives a morita context $(\mathrm{A}, \operatorname{End}_{\mathrm{A}}(P), P_{\mathrm{A}}, \operatorname{Hom}_{\mathrm{A}}(P, \mathrm{A}))$ which by reflexive equivalence is $(\mathrm{A}, \widehat{\Lambda}, P_{\mathrm{A}}, {}_{\mathrm{A}}Q)$ where $Q = \operatorname{Hom}_{\mathfrak{R}}(Z, N)$. Standard morita theory gives an equivalence of categories, and natural isomorphisms

$$\operatorname{mod}\mathrm{A} \underset{\operatorname{Hom}_{\widehat{\Lambda}}(Q,-)\cong -\otimes_{\widehat{\Lambda}}P}{\overset{\mathbb{F}:=\operatorname{Hom}_{\mathrm{A}}(P,-)\cong -\otimes_{\mathrm{A}}Q}{\rightleftarrows}} \operatorname{mod}\widehat{\Lambda}. \tag{4.A}$$

There is a similar left version, namely

$$\operatorname{mod}\mathrm{A}^{\mathrm{op}} \underset{\operatorname{Hom}_{\widehat{\Lambda}^{\mathrm{op}}}(P,-)\cong Q\otimes_{\widehat{\Lambda}}-}{\overset{\operatorname{Hom}_{\mathrm{A}^{\mathrm{op}}}(Q,-)\cong P\otimes_{\mathrm{A}}-}{\rightleftarrows}} \operatorname{mod}\widehat{\Lambda}^{\mathrm{op}}. \tag{4.B}$$

Now on one hand

$$\begin{aligned} \Phi'\circ\mathbb{F} &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(Z,\nu Z)\otimes_{\widehat{\Lambda}}P,-) && \text{(by adjunction)} \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\widehat{\Lambda}}(Q,\operatorname{Hom}_{\mathfrak{R}}(Z,\nu Z)),-) && \text{(by (4.A))} \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(N,\nu Z),-). && \text{(by reflexive equivalence)} \end{aligned}$$

On the other hand $\mathbb{G} = \operatorname{Hom}_{\nu\mathrm{A}}(P',-)$ for $P' := \operatorname{Hom}_{\mathfrak{R}}(\nu N, \nu Z)$, with inverse given by $Q' = \operatorname{Hom}_{\mathfrak{R}}(\nu Z, \nu N)$. Thus

$$\begin{aligned} \mathbb{G}\circ\Phi &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(P'\otimes_{\nu\mathrm{A}}\operatorname{Hom}_{\mathfrak{R}}(N,\nu N),-) && \text{(by adjunction)} \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{(\nu\mathrm{A})^{\mathrm{op}}}(Q',\operatorname{Hom}_{\mathfrak{R}}(N,\nu N)),-) && \text{(by (4.B)$'$)} \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\nu\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(N,\nu N)^*,(Q')^*),-) && ((-)^* \text{ duality on first term}) \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\nu\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(\nu N,N),\operatorname{Hom}_{\mathfrak{R}}(\nu N,\nu Z)),-) \\ &\cong \mathbf{R}\operatorname{Hom}_{\mathrm{A}}(\operatorname{Hom}_{\mathfrak{R}}(N,\nu Z),-) && \text{(by reflexive equivalence)} \end{aligned}$$

and so $\Phi'\circ\mathbb{F}\cong\mathbb{G}\circ\Phi$, as required. □

Corollary 4.5. *In the complete local flops setup of 2.4, and with notation as in 2.15, for any $J\subseteq\{0,1,\dots,n\}$ such that $\nu_J\nu_J N\cong N$, the following statements hold.*

(1) $\Phi'_J\Phi'_J(\mathbb{F}\mathrm{A}_J)\cong\mathbb{F}\mathrm{A}_J[-2]$.
(2) $\Phi'_J\Phi'_J(\mathbb{F}S)\cong\mathbb{F}S[-2]$ *for all simple* A_J*-modules* S.

Proof. Since minimal approximations sum, it follows that $\nu_J\nu_J Z\cong Z$. Thus applying 4.4 twice gives a commutative diagram

$$\begin{array}{ccc} \mathrm{D^b}(\operatorname{mod}\mathrm{A}) & \xrightarrow{\ \mathbb{F}\ } & \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}) \\ {\scriptstyle \Phi_J\Phi_J}\downarrow & & \downarrow{\scriptstyle \Phi'_J\Phi'_J} \\ \mathrm{D^b}(\operatorname{mod}\mathrm{A}) & \xrightarrow{\ \mathbb{F}\ } & \mathrm{D^b}(\operatorname{mod}\widehat{\Lambda}). \end{array}$$

Hence the result follows from 4.3(2) and 4.3(3). □

4.2. **Application 1: The J-Twists.** In this subsection we exclude 0 and consider the special case $J \subseteq \{1,\dots,n\}$ of §4.1. The situation $n=1$ was considered in [DW1].

Proposition 4.6. *In the complete local flops setup of 2.4, for any $J \subseteq \{1,\dots,n\}$,*

(1) $\nu_J \nu_J N \cong N$.
(2) *The simple A_J-modules are precisely S_j for $j \in J$.*

Proof. Contract the curves in J to obtain $\mathfrak{U} \to \mathfrak{U}_{\mathrm{con}} \to \operatorname{Spec} \mathfrak{R}$. Since $\mathfrak{U}$ has only Gorenstein terminal singularities, and this is a flopping contraction, $\mathfrak{U}_{\mathrm{con}}$ has only Gorenstein terminal singularities. Locally, it follows that $\mathfrak{U}_{\mathrm{con}}$ has only hypersurface singularities, so part (1) follows from [W, 2.25]. Part (2) is clear. □

The following is then the multi-curve analogue of [DW1, 5.6, 5.7].

Proposition 4.7. *In the complete local flops setup of 2.4, for any $J \subseteq \{1,\dots,n\}$,*

(1) *The minimal projective resolution of A_J as an A-module has the form*

$$0 \to P \to Q_1 \to Q_0 \to P \to \mathrm{A}_J \to 0$$

where $P := \operatorname{Hom}_{\mathfrak{R}}(N, N_J)$, and $Q_i \in \operatorname{add} Q$ for $Q := \operatorname{Hom}_{\mathfrak{R}}(N, N_{J^c})$.
(2) $\operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_J = 3$ *and* $\operatorname{pd}_{\widehat{\Lambda}} \widehat{I}_J = 2$.
(3) *We have*

$$\operatorname{Ext}^t_{\Lambda}(\mathbb{F}\mathrm{A}_J, T_j) \cong \operatorname{Ext}^t_{\widehat{\Lambda}}(\mathbb{F}\mathrm{A}_J, \widehat{T}_j) \cong \operatorname{Ext}^t_{\mathrm{A}}(\mathrm{A}_J, S_j) \cong \begin{cases} \mathbb{C} & \text{if } t = 0,3 \\ 0 & \text{else,} \end{cases}$$

for all $j \in J$, and further A_J is a self-injective algebra.

Proof. (1) This is [W, A.7(3)].
(2) Since projective dimension is preserved across morita equivalence, using (1) it follows that $\operatorname{pd}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_J = 3$. But $\operatorname{add}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_J = \operatorname{add}_{\widehat{\Lambda}} \widehat{\Lambda}_J$, so $\operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_J = \operatorname{pd}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_J = 3$. The statement for $\widehat{I}_J$ is then obvious from the completion of (2.D).
(3) The first two isomorphisms are consequences of the fact that the Ext groups are supported only at $\mathfrak{m}$, and the third isomorphism is a consequence of (1). Since $\operatorname{mod} \mathrm{A}_J$ is extension-closed in $\operatorname{mod} \mathrm{A}$ we have

$$\operatorname{Ext}^1_{\mathrm{A}_J}(S_j, \mathrm{A}_J) = \operatorname{Ext}^1_{\mathrm{A}}(S_j, \mathrm{A}_J) \cong D \operatorname{Ext}^2_{\mathrm{A}}(\mathrm{A}_J, S_j) \tag{4.C}$$

where the last isomorphism holds since A is 3-sCY [IW10, 2.22(2)], $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_J < \infty$ and S_j has finite length. Thus (4.C) shows that $\operatorname{Ext}^1_{\mathrm{A}_J}(S_j, \mathrm{A}_J) = 0$ for all $j \in J$. Since A_J is finite dimensional, every finitely generated module is filtered by simples, so it follows that A_J is self-injective. □

4.3. **Application 2: The Fibre Twist.** This subsection considers the special case $J = \{0\}$ of §4.1, in which case A_J is the fibre algebra A_0 of 2.11. Since $J = \{0\}$, this involves mutating the summand $\mathfrak{R}$, which results in reflexive modules that are not Cohen–Macaulay. Consequently, there is no easy reason for the assumption in 4.3 to be satisfied, and so this subsection is technically much harder than the previous §4.2. As a result, this subsection requires additional assumptions.

In the Zariski local setup in 2.3, of which the complete local flops setup 2.4 is the formal fibre, $f\colon U \to \operatorname{Spec} R$ is a Zariski local crepant contraction, where U has only terminal Gorenstein singularities, contracting precisely one connected chain C of curves with $C^{\mathrm{red}} = \bigcup_{j=1}^n \mathbb{P}^1$ to a point $\mathfrak{m}$ of $\operatorname{Spec} R$. We have that $\Lambda := \operatorname{End}_R(R \oplus L)$ is derived equivalent to U and recall from §2.3 that we write $\Lambda_{\mathfrak{m}} \cong \operatorname{End}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}^{\oplus a} \oplus K)$. Since f is crepant $\Lambda_{\mathfrak{m}} \in \operatorname{CM} R_{\mathfrak{m}}$, so necessarily $K \in \operatorname{CM} R_{\mathfrak{m}}$.

In this subsection, we make the additional assumption that U (not $\mathfrak{U}$) is $\mathbb{Q}$-factorial with only Gorenstein terminal singularities, so that $\Lambda_{\mathfrak{m}}$ is an MMA by 2.17.

Remark 4.8. If we assume in addition that U is complete locally $\mathbb{Q}$-factorial, equivalently $\mathfrak{U}$ is $\mathbb{Q}$-factorial (which happens for example if U is smooth), then $\widehat{\Lambda}$ and A are MMAs and so all the results in this subsection follow immediately from [IW10, §6.3]. However, later we will be working algebraically and it is well-known that the property of being $\mathbb{Q}$-factorial does not pass to the completion. Thus, since we are working with algebraic assumptions, this subsection requires some delicate global–local arguments.

Theorem 4.9. *Consider the formal fibre setup 2.4 where in the Zariski local flops setting 2.3, U is in addition $\mathbb{Q}$-factorial. Then*

(1) $\operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_0 = 3$, $\operatorname{pd}_{\widehat{\Lambda}} \widehat{I}_0 = 2$ *and* $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_0 = 3$.
(2) *The minimal projective resolution of the* A*-module* A_0 *has the form*

$$0 \to Q \to P_1 \to P_0 \to Q \to \mathrm{A}_0 \to 0$$

where $Q := \operatorname{Hom}_{\mathfrak{R}}(N, \mathfrak{R})$, *and* $P_i \in \operatorname{add} P$ *for* $P := \operatorname{Hom}_{\mathfrak{R}}(N, \bigoplus_{j=1}^{n} N_j)$.

Proof. (1) Since $\Lambda_{\mathfrak{m}}$ is an MMA of $R_{\mathfrak{m}}$ by 2.17, and by definition $\Lambda_0 = \Lambda_{\mathfrak{m}}/[K]$, it follows that $\operatorname{pd}_{\Lambda_{\mathfrak{m}}} \Lambda_0 < \infty$ by [IW14, 4.16]. By 2.14, $\Lambda_0 \otimes_{R_{\mathfrak{m}}} \mathfrak{R} \cong \widehat{\Lambda}_0$. Since completion preserves finite projective dimension, it follows that $\operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_0 \leq 3$. Since $\widehat{\Lambda}_0$ is finite dimensional, a projective dimension strictly less than three would contradict the depth lemma, so $\operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_0 = 3$. The statement $\operatorname{pd}_{\widehat{\Lambda}} \widehat{I}_0 = 2$ follows by completing (2.E). Finally, $\operatorname{add}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_0 = \operatorname{add}_{\widehat{\Lambda}} \widehat{\Lambda}_0$, so $\operatorname{pd}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_0 = \operatorname{pd}_{\widehat{\Lambda}} \widehat{\Lambda}_0 = 3$. Since projective dimension is preserved across morita equivalence, $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_0 = 3$ follows.
(2) This is very similar to the arguments in [IR, 4.3, 5.6] and [IW10, 6.23], but our assumptions here are weaker, so we include the proof. Since by (1) A_0 has finite projective dimension as an A-module, it follows that $\operatorname{inj.dim}_{\mathrm{A}_0} \mathrm{A}_0 \leq 1$ by e.g. [IW10, 6.19(4)], in particular the injective dimension is finite. This being the case, since A_0 is local we deduce that

$$\operatorname{depth}_{\mathfrak{R}} \mathrm{A}_0 = \dim_{\mathfrak{R}} \mathrm{A}_0 = \operatorname{inj.dim}_{\mathrm{A}_0} \mathrm{A}_0$$

by Ramras [R69, 2.15]. Since A_0 is finite dimensional, this number is zero and in particular A_0 is a Cohen–Macaulay $\mathfrak{R}$-module of dimension zero.

Let e be the idempotent in A corresponding to the summand $\mathfrak{R}$, so that $Q = e\mathrm{A}$, $P = (1-e)\mathrm{A}$ and $\mathrm{A}_0 = \mathrm{A}/\mathrm{A}(1-e)\mathrm{A}$. By (1) we have a minimal projective resolution

$$0 \to P_3 \to P_2 \to P_1 \xrightarrow{f} e\mathrm{A} \to \mathrm{A}_0 \to 0 \tag{4.D}$$

with f a minimal right $(\operatorname{add}(1-e)\mathrm{A})$-approximation since it is a projective cover of $e\mathrm{A}(1-e)\mathrm{A}$. In particular, $P_1 \in \operatorname{add} P$. Now A_0 is finite dimensional and A is perfect, so since A is 3-sCY we have

$$\operatorname{Ext}^t_{\mathrm{A}}(\mathrm{A}_0, \mathrm{A}) \cong D \operatorname{Ext}^{3-t}_{\mathrm{A}}(\mathrm{A}, \mathrm{A}_0)$$

which is zero for $t \neq 3$. Hence applying $(-)^\vee := \operatorname{Hom}_{\mathrm{A}}(-, \mathrm{A})$ to (4.D) we obtain an exact sequence

$$0 \to (e\mathrm{A})^\vee \to (P_1)^\vee \to (P_2)^\vee \to (P_3)^\vee \to \operatorname{Ext}^3_{\mathrm{A}}(\mathrm{A}_0, \mathrm{A}) \to 0 \tag{4.E}$$

which is the minimal projective resolution of $\operatorname{Ext}^3_{\mathrm{A}}(\mathrm{A}_0, \mathrm{A})$ as an A^{op}-module. But we have

$$\operatorname{Ext}^3_{\mathrm{A}}(\mathrm{A}_0, \mathrm{A}) \cong \operatorname{Ext}^3_{\mathfrak{R}}(\mathrm{A}_0, \mathfrak{R})$$

by [IR, 3.4(5)], and this is a projective $\mathrm{A}_0^{\mathrm{op}}$-module by [GN, 1.1(3)]. Since A_0 is a local ring, it is a free $\mathrm{A}_0^{\mathrm{op}}$-module. Further, since A_0 is a Cohen–Macaulay $\mathfrak{R}$-module of dimension zero, we have

$$\operatorname{Ext}^3_{\mathfrak{R}}(\operatorname{Ext}^3_{\mathfrak{R}}(\mathrm{A}_0, \mathfrak{R}), \mathfrak{R}) \cong \mathrm{A}_0.$$

and so the rank has to be one, forcing $\operatorname{Ext}^3_{\mathfrak{R}}(\mathrm{A}_0, \mathfrak{R}) \cong \mathrm{A}_0$ as $\mathrm{A}_0^{\mathrm{op}}$-modules. Hence (4.E) is the minimal projective resolution

$$0 \to \operatorname{Hom}_{\mathrm{A}}(e\mathrm{A}, \mathrm{A}) \to \operatorname{Hom}_{\mathrm{A}}(P_1, \mathrm{A}) \to \operatorname{Hom}_{\mathrm{A}}(P_2, \mathrm{A}) \to \operatorname{Hom}_{\mathrm{A}}(P_3, \mathrm{A}) \to \mathrm{A}_0 \to 0$$

of A_0 as an A^{op}-module. Thus we have $\mathrm{Hom}_{\mathrm{A}}(P_3,\mathrm{A})\cong \mathrm{A}e$ and so $P_3\cong e\mathrm{A}=Q$. Similarly $\mathrm{Hom}_{\mathrm{A}}(P_2,\mathrm{A})\in \mathrm{add}\,\mathrm{A}(1-e)$ forces $P_2\in \mathrm{add}\,(1-e)\mathrm{A}=\mathrm{add}\,P$. □

The following corollary ensures that the assumption in 4.3 is satisfied.

Corollary 4.10. *Consider the formal fibre setup 2.4 where in the Zariski local flops setting 2.3, U is in addition $\mathbb{Q}$-factorial. Then for $I=\{0\}$, $\boldsymbol{\nu}_I\boldsymbol{\nu}_I N\cong N$.*

Proof. Set $M:=\bigoplus_{i=1}^n N_i$ so that $N=N_0\oplus M=\mathfrak{R}\oplus M$. By 4.2,

$$T_I\cong \mathrm{Hom}_{\mathfrak{R}}(N,(\mathrm{Ker}\,b_0)^*\oplus M)$$

is a tilting A-module with $\mathrm{pd}_{\mathrm{A}}T_I=1$. We claim that

$$T:=\mathrm{Hom}_{\mathfrak{R}}(N,(\mathrm{Ker}\,a_0)\oplus M)$$

is also a tilting A-module with $\mathrm{pd}_{\mathrm{A}}T_I=1$. Then, since the projective dimension of both being one implies that both are not isomorphic to the ring A, and further as A-modules T_I and T share all summands except possibly one, by a Riedtmann–Schofield type theorem [IR, 4.2] they must coincide, i.e. $\mathrm{Hom}_{\mathfrak{R}}(N,(\mathrm{Ker}\,b_0)^*\oplus M)\cong \mathrm{Hom}_{\mathfrak{R}}(N,(\mathrm{Ker}\,a_0)\oplus M)$. By reflexive equivalence and Krull–Schmidt it then follows that $(\mathrm{Ker}\,b_0)^*\cong \mathrm{Ker}\,a_0$, proving the statement.

Now $T=\mathrm{Hom}_{\mathfrak{R}}(N,M)\oplus \mathrm{Hom}_{\mathfrak{R}}(N,\mathrm{Ker}\,a_0)$, and the first summand is projective. Further the sequence (2.I) becomes

$$0\to \mathrm{Hom}_{\mathfrak{R}}(N,\mathrm{Ker}\,a_0)\xrightarrow{c\cdot}\mathrm{Hom}_{\mathfrak{R}}(N,V_0)\xrightarrow{a\cdot}\mathrm{Hom}_{\mathfrak{R}}(N,\mathfrak{R})\to \mathrm{A}_0\to 0$$

which is the beginning of the minimal projective resolution of A_0. Since $\mathrm{pd}_{\mathrm{A}}\mathrm{A}_0=3$ by 4.9 it follows that $\mathrm{pd}_{\mathrm{A}}\mathrm{Hom}_{\mathfrak{R}}(N,\mathrm{Ker}\,a_0)=1$ and so $\mathrm{pd}_{\mathrm{A}}T=1$.

Now by reflexive equivalence $\mathrm{End}_{\mathrm{A}}(T)\cong \mathrm{End}_{\mathfrak{R}}(M\oplus(\mathrm{Ker}\,a_0))$, and this is a CM $\mathfrak{R}$-module by [IW10, 6.10]. Thus it follows that $\mathrm{depth}_{\mathfrak{R}}\mathrm{Ext}^1_{\mathrm{A}}(T,T)>0$ by the depth lemma (see e.g. [IW10, 2.7]). But on the other hand for any prime $\mathfrak{p}\in\mathrm{Spec}\,\mathfrak{R}$ of height two, $T_{\mathfrak{p}}\in \mathrm{CM}\,\mathrm{A}_{\mathfrak{p}}$ since reflexive modules are CM for two-dimensional rings. Since $\mathrm{pd}_{\mathrm{A}_{\mathfrak{p}}}T_{\mathfrak{p}}\leq 1$, Auslander–Buchsbaum then implies that $T_{\mathfrak{p}}$ is projective, and so $\mathrm{Ext}^1_{\mathrm{A}}(T,T)_{\mathfrak{p}}=\mathrm{Ext}^1_{\mathrm{A}_{\mathfrak{p}}}(T_{\mathfrak{p}},T_{\mathfrak{p}})=0$, implying that the Ext group has finite length. Combining, it follows that $\mathrm{Ext}^1_{\mathrm{A}}(T,T)=0$, and since $\mathrm{pd}_{\mathrm{A}}T=1$ we deduce that $\mathrm{Ext}^i_{\mathrm{A}}(T,T)=0$ for all $i>0$. Hence T is a partial tilting A-module with exactly n non-isomorphic summands. By the Bongartz completion and Krull–Schmidt, it follows that T is a tilting A-module with projective dimension one. □

4.4. **Summary of Complete Local Twists.** The following corollary summarises the main results in the previous two subsections.

Corollary 4.11. *With the Zariski local setup in 2.3, and its formal fibre setup in 2.4, then with notation as in (2.D) and (2.E),*

(1) *For any $J\subseteq\{1,\dots,n\}$,*

$$\Phi'_J\circ\Phi'_J\cong \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_J,-)$$

is an autoequivalence of $\mathrm{D^b}(\mathrm{mod}\,\widehat{\Lambda})$, and $\widehat{I}_J$ is a tilting $\widehat{\Lambda}$-module of projective dimension two. This autoequivalence sends
   (a) $\mathbb{F}\mathrm{A}_J\mapsto \mathbb{F}\mathrm{A}_J[-2]$.
   (b) $\mathbb{F}S_j\mapsto \mathbb{F}S_j[-2]$ *for all* $j\in J$.

(2) *If further U is $\mathbb{Q}$-factorial then*

$$\Phi'_0\circ\Phi'_0\cong \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_0,-)$$

is an autoequivalence of $\mathrm{D^b}(\mathrm{mod}\,\widehat{\Lambda})$, and $\widehat{I}_0$ is a tilting $\widehat{\Lambda}$-module of projective dimension two. This autoequivalence sends
   (a) $\mathbb{F}\mathrm{A}_0\mapsto \mathbb{F}\mathrm{A}_0[-2]$.
   (b) $\mathbb{F}S_0\mapsto \mathbb{F}S_0[-2]$.

Proof. (1) The two-sided ideal $[Z_{J^c}]$ of $\widehat{\Lambda} = \mathrm{End}_{\mathfrak{R}}(Z)$ is $\widehat{I}_J$, so the first statement $\Phi'_J\Phi'_J \cong \mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_J, -)$ is identical to 4.3(1). Since mutation is an equivalence, this means that $\widehat{I}_J$ must be a tilting module, and it has projective dimension two by 4.7(2). Statements (a) and (b) are special cases of 4.5.
(2) The two-sided ideal $[\widehat{K}]$ of $\mathrm{End}_{\mathfrak{R}}(Z)$ is $\widehat{I}_0$, so the proof is identical to (1), except now 4.9(1) is used to establish that the projective dimension of $\widehat{I}_0$ is two. □

5. Global and Zariski Local Twists

5.1. **Zariski Local Twists.** We revert to the Zariski local setup in 2.3, namely $f\colon U \to \operatorname{Spec} R$ is a Zariski local crepant contraction, where U has only terminal Gorenstein singularities, contracting precisely one connected chain C of curves with $C^{\mathrm{red}} = \bigcup_{j=1}^{n} C_j$ such that each $C_j \cong \mathbb{P}^1$. We have that U is derived equivalent to $\Lambda := \mathrm{End}_R(R \oplus L)$, with the basic complete local version $\mathrm{A} = \mathrm{End}_{\mathfrak{R}}(\mathfrak{R} \oplus N)$.

The aim of this subsection is to produce two new types of derived autoequivalences on U. The first type depends on a choice of subset of the reduced curves C_j, whereas the second type is associated to deformations of the whole scheme-theoretic fibre C, and requires the additional assumption that U is $\mathbb{Q}$-factorial.

Given the work in §4, leading up to the key 4.11, together with the short exact sequences of bimodules (2.C) and (2.E), the construction of these new autoequivalences follows using the same strategy as for [DW1, §5–7].

Corollary 5.1. *With the Zariski local flops setup in 2.3,*

(1) *For any $J \subseteq \{1, \dots, n\}$, I_J is a tilting Λ-module of projective dimension two.*
(2) *If U is $\mathbb{Q}$-factorial, then I_0 is a tilting Λ-module of projective dimension two.*

In either case, if we let I denote either I_J or I_0, viewing I as a right Λ-module we have $\Lambda \cong \mathrm{End}_{\Lambda}(I)$, and under this isomorphism the bimodule ${}_{\mathrm{End}_\Lambda(I)}I_\Lambda$ coincides with the natural bimodule structure ${}_\Lambda I_\Lambda$.

Proof. (1) The statement is local, and since Λ_J is supported only at $\mathfrak{m}$, it is clear that $(I_J)_{\mathfrak{n}}$ is free for all $\mathfrak{n} \in \operatorname{Max} R$ with $\mathfrak{n} \neq \mathfrak{m}$. Thus it suffices to check that $(I_J)_{\mathfrak{m}}$ is a tilting $\Lambda_{\mathfrak{m}}$-module of projective dimension two. Since $R_{\mathfrak{m}}$ is local, and a module being zero can be detected on the completion, the statement is equivalent to $\widehat{I}_J = I_J \otimes_R \mathfrak{R}$ being a tilting $\widehat{\Lambda}$-module of projective dimension two. But this is just 4.11(1).
(2) By the same logic as in (1), the statement is equivalent to $\widehat{I}_0$ being a tilting $\widehat{\Lambda}$-module of projective dimension two. Again, since U is now $\mathbb{Q}$-factorial, this is just 4.11(2).
The final statements follow immediately by repeating the argument in [DW1, 6.1]. □

The following is the extension of 4.11 to the Zariski local setting.

Proposition 5.2. *With the Zariski local setup in 2.3,*

(1) *For all $J \subseteq \{1, \dots, n\}$,*

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(I_J, -)\colon \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod}\Lambda), \tag{5.A}$$

and further this autoequivalence sends
(a) $\mathbb{F}\mathrm{A}_J \mapsto \mathbb{F}\mathrm{A}_J[-2]$,
(b) $T_j \mapsto T_j[-2]$ *for all* $j \in J$.

(2) *If further U is $\mathbb{Q}$-factorial,*

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(I_0, -)\colon \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod}\Lambda), \tag{5.B}$$

and further this autoequivalence sends
(a) $\mathbb{F}\mathrm{A}_0 \mapsto \mathbb{F}\mathrm{A}_0[-2]$,
(b) $T_0 \mapsto T_0[-2]$.

Proof. (1) By 4.11(1), $\mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_J, \mathbb{F}\mathrm{A}_J) \cong \mathbb{F}\mathrm{A}_J[-2]$. Since $\mathbb{F}\mathrm{A}_J$ is supported only at the point $\mathfrak{m}$, $\mathbf{R}\mathrm{Hom}_{\Lambda}(I_J, \mathbb{F}\mathrm{A}_J) \cong \mathbb{F}\mathrm{A}_J[-2]$ is a formal consequence, as in [DW1, 6.3]. Since $\mathbb{F}S_j = \widehat{T}_j$ by 2.15, the second statement is also a formal consequence of 4.11(1).
(2) Similarly $\mathbf{R}\mathrm{Hom}_{\widehat{\Lambda}}(\widehat{I}_0, \mathbb{F}\mathrm{A}_0) \cong \mathbb{F}\mathrm{A}_0[-2]$ by 4.11(2), and again $\mathbb{F}\mathrm{A}_0$ is supported only at the point $\mathfrak{m}$. □

By passing across the equivalence $\mathrm{D^b}(\mathrm{mod}\,\Lambda) \cong \mathrm{D^b}(\mathrm{coh}\,U)$, (5.A) and (5.B) give autoequivalences on U. To describe these, consider the projections $p_1, p_2 \colon U \times U \to U$, and set $\mathcal{V}^\vee \boxtimes \mathcal{V} = p_1^*\mathcal{V}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{U\times U}} p_2^*\mathcal{V}$. Then there is an induced derived equivalence

$$\mathrm{D^b}(\mathrm{coh}\,U \times U) \underset{-\otimes^{\mathbf{L}}_{\Lambda^{\mathrm{e}}}(\mathcal{V}^\vee\boxtimes\mathcal{V})}{\overset{\mathbf{R}\mathrm{Hom}_{U\times U}(\mathcal{V}^\vee\boxtimes\mathcal{V},-)}{\rightleftarrows}} \mathrm{D^b}(\mathrm{mod}\,\Lambda \otimes_{\mathbb{C}} \Lambda^{\mathrm{op}}) \tag{5.C}$$

as in [BH], where we denote the enveloping algebra by $\Lambda^{\mathrm{e}} := \Lambda \otimes_{\mathbb{C}} \Lambda^{\mathrm{op}}$. Applying the lower functor in (5.C) to (2.D) and (2.E), we obtain exact triangles of Fourier–Mukai kernels on $U \times U$ denoted

$$\mathcal{W}'_J \to \mathcal{O}_{\Delta,U} \xrightarrow{\phi'_J} \mathcal{Q}'_J \to \tag{5.D}$$

$$\mathcal{W}'_0 \to \mathcal{O}_{\Delta,U} \xrightarrow{\phi'_0} \mathcal{Q}'_0 \to . \tag{5.E}$$

Dualizing (2.D) to obtain an exact triangle

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(\Lambda_J, \Lambda) \to \Lambda \to \mathbf{R}\mathrm{Hom}_{\Lambda}(I_J, \Lambda) \to$$

then applying the lower functor in (5.C), and similarly for (2.E), we obtain exact triangles of Fourier–Mukai kernels on $U \times U$ denoted

$$\mathcal{Q}_J \xrightarrow{\phi_J} \mathcal{O}_{\Delta,U} \to \mathcal{W}_J \to \tag{5.F}$$

$$\mathcal{Q}_0 \xrightarrow{\phi_0} \mathcal{O}_{\Delta,U} \to \mathcal{W}_0 \to . \tag{5.G}$$

Then, using the obvious adjunctions given by restriction and extension of scalars from the ring homomorphisms $\Lambda \to \Lambda_J$ and $\Lambda \to \Lambda_0$, passing through the above derived equivalence and the morita equivalence $\mathbb{F}$ from 2.15, exactly as in [DW1, 6.10, 6.11, 6.16], (5.F) and (5.G) yield functorial triangles

$$\mathbf{R}\mathrm{Hom}_U(\mathcal{E}_J, -) \otimes^{\mathbf{L}}_{\mathrm{A}_J} \mathcal{E}_J \to \mathrm{Id} \to \mathsf{FM}(\mathcal{W}_J) \to$$
$$\mathbf{R}\mathrm{Hom}_U(\mathcal{E}_0, -) \otimes^{\mathbf{L}}_{\mathrm{A}_0} \mathcal{E}_0 \to \mathrm{Id} \to \mathsf{FM}(\mathcal{W}_0) \to,$$

where $\mathsf{FM}(\mathcal{W}_J)$ is the autoequivalence on U corresponding to (5.A), and $\mathsf{FM}(\mathcal{W}_0)$ is the autoequivalence on U corresponding to (5.B).

Translating 5.2 through the derived equivalences immediately gives the following.

Corollary 5.3. *With the Zariski local setup in 2.3,*

(1) $\mathsf{FM}(\mathcal{W}_J) \colon \mathrm{D^b}(\mathrm{coh}\,U) \to \mathrm{D^b}(\mathrm{coh}\,U)$ *is an autoequivalence, sending*
   (a) $\mathcal{E}_J \mapsto \mathcal{E}_J[-2]$,
   (b) $E_j \mapsto E_j[-2]$ *for all* $j \in J$.

(2) *If further* U *is* $\mathbb{Q}$*-factorial then* $\mathsf{FM}(\mathcal{W}_0) \colon \mathrm{D^b}(\mathrm{coh}\,U) \to \mathrm{D^b}(\mathrm{coh}\,U)$ *is an autoequivalence, sending*
   (a) $\mathcal{E}_0 \mapsto \mathcal{E}_0[-2]$,
   (b) $E_0 \mapsto E_0[-2]$.

5.2. **Global Twists.** We finally consider the global projective flops setup $f \colon X \to X_{\mathrm{con}}$ of 2.2. As explained there, for each point $p \in \mathrm{Ram}(X_{\mathrm{con}})$ we choose an affine open neighbourhood containing p but none of the other points in $\mathrm{Ram}(X_{\mathrm{con}})$, and consider the open set in X given by the inverse image of this affine open, under f, which we denote by U_p. As further notation, for each point $p \in \mathrm{Ram}(X_{\mathrm{con}})$ we let n_p denote the number of curves above p.

We now fix $p \in \operatorname{Ram}(X_{\mathrm{con}})$ and from it produce autoequivalences on X. To ease notation, since U_p is fixed, throughout this subsection we will denote it simply by U, and write the inclusion as $i\colon U \hookrightarrow X$. We will compose these (over all $p \in \operatorname{Ram}(X_{\mathrm{con}})$) to get the overall twists in §5.3. The following result is one of the key technical statements in this paper, and is a consequence of our local study of mutation. Here $\operatorname{per}(X)$ denotes the category of perfect complexes.

Theorem 5.4. *Consider the global projective flops setup $f\colon X \to X_{\mathrm{con}}$ of 2.2. For a fixed $p \in \operatorname{Ram}(X_{\mathrm{con}})$,*

1. *For any $J \subseteq \{1,\dots,n_p\}$, $\mathbf{R}i_*\mathcal{E}_J \cong i_*\mathcal{E}_J \in \operatorname{per}(X)$, and by abuse of notation we denote this simply by $\mathcal{E}_J$. Furthermore, $\mathcal{E}_J \otimes \omega_X \cong \mathcal{E}_J$.*
2. *If further U is $\mathbb{Q}$-factorial then $\mathbf{R}i_*\mathcal{E}_0 \cong i_*\mathcal{E}_0 \in \operatorname{per}(X)$, and by abuse of notation we denote this simply by $\mathcal{E}_0$. Furthermore, $\mathcal{E}_0 \otimes \omega_X \cong \mathcal{E}_0$.*

Proof. (1) By 4.7 $\operatorname{pd}_{\widehat{\Lambda}} \mathbb{F}\mathrm{A}_J = 3$, so since $\mathbb{F}\mathrm{A}_J$ is supported only at $\mathfrak{m}$, it follows that $\mathbb{F}\mathrm{A}_J$ is a perfect Λ-module. Since by 2.15 $\mathcal{E}_J$ corresponds to $\mathbb{F}\mathrm{A}_J$ across the derived equivalence, it follows that $\mathcal{E}_J$ is perfect on U. This in turn implies that $i_*\mathcal{E}_J$ is a perfect complex on X, since we can check this locally. Since $\mathcal{E}_J$ has a finite filtration by objects $\mathcal{O}_{C_j}(-1)$ for $j \in J$, it follows that $\mathbf{R}i_*\mathcal{E}_J = i_*\mathcal{E}_J$. Lastly, $\mathbb{F}\mathrm{A}_J$ is supported only at $\mathfrak{m}$, so just as in [DW1, 6.6] the Serre functor on Λ acts trivially on $\mathbb{F}\mathrm{A}_J$. Across the equivalence, by uniqueness of Serre functor this means that the Serre functor on U acts trivially on $\mathcal{E}_J$. This then implies the last statement, since

$$\mathbf{R}i_*\mathcal{E}_J \otimes^{\mathbf{L}}_X \omega_X \cong \mathbf{R}i_*(\mathcal{E}_J \otimes^{\mathbf{L}}_U i^*\omega_X) \cong \mathbf{R}i_*(\mathcal{E}_J \otimes^{\mathbf{L}}_U \omega_U) \cong \mathbf{R}i_*\mathcal{E}_J.$$

(2) The proof is identical to (1), using 4.9 instead to give $\operatorname{pd}_{\widehat{\Lambda}} \mathrm{A}_0 = 3$. □

This subsection now builds towards showing 5.10, that the twist functors $J\mathsf{Twist}_p$ and $F\mathsf{Twist}_p$ defined below are autoequivalences. This follows a parallel argument to [DW1, §7], using the Fourier–Mukai kernels $\mathcal{Q}'_J$, $\mathcal{W}'_J$, $\mathcal{Q}'_0$, and $\mathcal{W}'_0$ from the previous subsection §5.1. We outline this argument now for the convenience of the reader, and to fix notation.

As in [DW1, §7], it is technically easier to construct the inverse twist functors $J\mathsf{Twist}^*_p$ and $F\mathsf{Twist}^*_p$, and this proceeds by gluing in the Zariski local construction of §5.1.

Definition 5.5. *With the global projective flops setup $f\colon X \to X_{\mathrm{con}}$ of 2.2,*

1. *For $J \subseteq \{1,\dots,n_p\}$, the inverse J-twist functor $J\mathsf{Twist}^*_p$ is the Fourier–Mukai functor $\mathsf{FM}(\operatorname{Cone}\psi'_J)$ where*

$$\psi'_J\colon \mathcal{O}_\Delta \xrightarrow{\eta_\Delta} \mathbf{R}(i\times i)_*\mathcal{O}_{\Delta,U} \xrightarrow{\mathbf{R}(i\times i)_*\phi'_J} \mathbf{R}(i\times i)_*\mathcal{Q}'_J. \tag{5.H}$$

   The notation here comes from (5.D), and the natural morphism η_Δ is described explicitly in [DW1, 7.3]. The J-twist functor $J\mathsf{Twist}_p$ is defined

$$J\mathsf{Twist}_p := \mathsf{FM}(D(\operatorname{Cone}\psi'_J))$$

   where $D := (-)^\vee \otimes p_1^*\omega_X[\dim X]$.
2. *If U is $\mathbb{Q}$-factorial, the inverse fibre twist $F\mathsf{Twist}^*_p$ and fibre twist $F\mathsf{Twist}_p$ are defined in an identical way, using instead ϕ'_0 from (5.E).*

A formal consequence of this definition is the following intertwinement of $J\mathsf{Twist}^*_p$ and $F\mathsf{Twist}^*_p$ with their counterpart functors acting on $\mathrm{D}(\operatorname{Qcoh} U)$.

Proposition 5.6. *For $\mathbf{R}i_* : \mathrm{D}(\operatorname{Qcoh} U) \to \mathrm{D}(\operatorname{Qcoh} X)$,*

$$J\mathsf{Twist}^*_p \circ \mathbf{R}i_* \cong \mathbf{R}i_* \circ \mathsf{FM}(\mathcal{W}'_J)$$
$$F\mathsf{Twist}^*_p \circ \mathbf{R}i_* \cong \mathbf{R}i_* \circ \mathsf{FM}(\mathcal{W}'_0),$$

where the latter only holds when U is in addition $\mathbb{Q}$-factorial.

Proof. This is a standard application of Fourier–Mukai techniques, following [DW1, 7.8] line for line. □

The next result describes $J\mathsf{Twist}_p^*$ and $F\mathsf{Twist}_p^*$ as 'twists' by fitting them into certain functorial triangles, and shows that they preserve the bounded derived categories, and possess adjoints.

Proposition 5.7. *With the global projective flops setup of 2.2,*

(1) *For each $J \subseteq \{1,\dots,n_p\}$, there exist adjunctions*

$$\mathrm{D}(\operatorname{Mod} \mathrm{A}_J) \underset{G_J^{\mathrm{RA}}}{\overset{G_J^{\mathrm{LA}}}{\rightleftarrows}} \mathrm{D}(\operatorname{Qcoh} X) \qquad (G_J : \mathrm{D}(\operatorname{Mod} \mathrm{A}_J) \to \mathrm{D}(\operatorname{Qcoh} X))$$

where $G_J := \mathbf{R}i_*(- \otimes^{\mathbf{L}}_{\mathrm{A}_J} \mathcal{E}_J) \cong - \otimes^{\mathbf{L}}_{\mathrm{A}_J} \mathcal{E}_J$. Furthermore

(a) *For all $x \in \mathrm{D}(\operatorname{Qcoh} X)$,*

$$J\mathsf{Twist}_p^*(x) \to x \to G_J \circ G_J^{\mathrm{LA}}(x) \to$$

is a triangle in $\mathrm{D}(\operatorname{Qcoh} X)$.

(b) *$J\mathsf{Twist}_p^*$ preserves the bounded derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, and has left and right adjoints with the same property.*

(2) *If further U is $\mathbb{Q}$-factorial then there exist adjunctions*

$$\mathrm{D}(\operatorname{Mod} \mathrm{A}_0) \underset{G_0^{\mathrm{RA}}}{\overset{G_0^{\mathrm{LA}}}{\rightleftarrows}} \mathrm{D}(\operatorname{Qcoh} X) \qquad (G_0 : \mathrm{D}(\operatorname{Mod} \mathrm{A}_0) \to \mathrm{D}(\operatorname{Qcoh} X))$$

where $G_0 := \mathbf{R}i_*(- \otimes^{\mathbf{L}}_{\mathrm{A}_0} \mathcal{E}_0) \cong - \otimes^{\mathbf{L}}_{\mathrm{A}_0} \mathcal{E}_0$. Furthermore

(a) *For all $x \in \mathrm{D}(\operatorname{Qcoh} X)$,*

$$F\mathsf{Twist}_p^*(x) \to x \to G_0 \circ G_0^{\mathrm{LA}}(x) \to$$

is a triangle in $\mathrm{D}(\operatorname{Qcoh} X)$.

(b) *$F\mathsf{Twist}_p^*$ preserves the bounded derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, and has left and right adjoints with the same property.*

Proof. Part (1) follows as in [DW1, 7.4, 7.5, 7.6], with the statement on adjoints following using the projective Gorenstein assumption of 2.2. The proof of part (2) is very similar. □

We continue to build towards showing that $J\mathsf{Twist}_p^*$ and $F\mathsf{Twist}_p^*$ are autoequivalences. The following lemma constructs spanning classes that will be used in the proof. The key result for this is 5.4, which shows that $\mathcal{E}_J$ and $\mathcal{E}_0$ are perfect. Then, we use 5.6 and 5.7 to describe the action of the functors $J\mathsf{Twist}_p^*$ and $F\mathsf{Twist}_p^*$ on parts of the classes.

Lemma 5.8. *With the global projective flops setup of 2.2,*

(1) *For any $J \subseteq \{1,\dots,n_p\}$, $\Omega_J := \mathcal{E}_J \cup \mathcal{E}_J^{\perp}$ is a spanning class in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.*

(2) *$J\mathsf{Twist}_p^*$ sends $\mathcal{E}_J$ to $\mathcal{E}_J[2]$, E_j to $E_j[2]$ for all $j \in J$, and is functorially isomorphic to the identity on $\mathcal{E}_J^{\perp}$.*

If further U is $\mathbb{Q}$-factorial then we also have

(3) *$\Omega_0 := \mathcal{E}_0 \cup \mathcal{E}_0^{\perp}$ is a spanning class in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.*

(4) *$F\mathsf{Twist}_p^*$ sends $\mathcal{E}_0$ to $\mathcal{E}_0[2]$, E_0 to $E_0[2]$, and is functorially isomorphic to the identity on $\mathcal{E}_0^{\perp}$.*

Proof. Part (1) is a formal consequence of the definitions, as in [DW1, 7.9], since by 5.4(1) $\mathcal{E}_J$ is perfect and is fixed by the canonical. Part (3) is similar, using instead 5.4(2).

The last statements in parts (2) and (4) are consequences of the functorial triangles in 5.7. The other statements in (2) and (4) follow by combining 5.3 with 5.6, after inverting the local twists appearing there. □

As in [DW1], we need the following lemma, which gives an appropriate sufficient condition for a fully faithful functor to be an equivalence.

Lemma 5.9 ([DW1, 7.11])**.** *Let $\mathcal{C}$ be a triangulated category, and $F\colon \mathcal{C} \to \mathcal{C}$ an exact fully faithful functor with right adjoint F^{RA}. Suppose that there exists an object $c \in \mathcal{C}$ such that $F(c) \cong c[i]$ for some i, and further $F(x) \cong x$ for all $x \in c^{\perp}$. Then F is an equivalence.*

Combining the above gives the following.

Theorem 5.10. *With the global projective flops setup of 2.2,*

(1) *For any $J \subseteq \{1, \dots, n_p\}$, the inverse J-twist*

$$J\mathsf{Twist}_p^*\colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is an equivalence.

(2) *If furthermore U is $\mathbb{Q}$-factorial then the inverse fibre twist*

$$F\mathsf{Twist}_p^*\colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is an equivalence.

Proof. Putting together 5.7, 5.8 and 5.9, we find that $J\mathsf{Twist}_p^*$ and $F\mathsf{Twist}_p^*$ are equivalences provided that they are fully faithful. This fact follows by a simple calculation on the spanning classes of 5.8, as given in [DW1, 7.12]. In this calculation, the crucial point is 5.6, which allows us to prove the hard case by using the fact that locally the twist is an autoequivalence. □

Corollary 5.11. *With the global projective flops setup of 2.2,*

(1) *For any $J \subseteq \{1, \dots, n_p\}$, the J-twist*

$$J\mathsf{Twist}_p\colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is an equivalence. If furthermore U is $\mathbb{Q}$-factorial then the fibre twist

$$F\mathsf{Twist}_p\colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is an equivalence.

(2) *When the twists are defined, there are functorial triangles*

$$G_J \circ G_J^{\mathrm{RA}}(x) \to x \to J\mathsf{Twist}_p(x) \to$$

where $G_J \circ G_J^{\mathrm{RA}}(x) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{E}_J, x) \otimes^{\mathbf{L}}_{\mathrm{A}_J} \mathcal{E}_J$, and

$$G_0 \circ G_0^{\mathrm{RA}}(x) \to x \to F\mathsf{Twist}_p(x) \to$$

where $G_0 \circ G_0^{\mathrm{RA}}(x) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{E}_0, x) \otimes^{\mathbf{L}}_{\mathrm{A}_0} \mathcal{E}_0$.

Proof. This follows by standard facts on adjunctions, as in [DW1, 7.14]. □

The following result is used in the next subsection to compare autoequivalences.

Corollary 5.12. *The J-twist $J\mathsf{Twist}_p$ commutes with the pushdown $\mathbf{R}f_*$, i.e.*

$$\mathbf{R}f_* \circ J\mathsf{Twist}_p \cong \mathbf{R}f_*.$$

Proof. This is shown by the method of [DW1, 7.15]. The key point is that $\mathcal{E}_J$ is filtered by the E_j, and so $\mathbf{R}f_*\mathcal{E}_J = 0$. □

5.3. **Comparison of Autoequivalences.** The input to this subsection is a flopping contraction in the global projective flops setup $f\colon X \to X_{\mathrm{con}}$ of 2.2.

Definition 5.13. *With the global projective flops setup of 2.2, we enumerate the points of the non-isomorphism locus* $\mathrm{Ram}(X_{\mathrm{con}})$ *as* $\{p_1,\dots,p_t\}$*, and let* J *denote a collection* $(J_1,\dots,J_t)$ *of subsets* $J_i \subseteq \{1,\dots,n_{p_i}\}$*. We then define the* J-twist *to be*

$$J\mathsf{Twist} := J_1\mathsf{Twist}_{p_1} \circ \cdots \circ J_t\mathsf{Twist}_{p_t}.$$

The twists $J_i\mathsf{Twist}_{p_i}$ *are provided by 5.11. Similarly, when* X *is* $\mathbb{Q}$*-factorial we set*

$$F\mathsf{Twist} := F\mathsf{Twist}_{p_1} \circ \cdots \circ F\mathsf{Twist}_{p_t}.$$

and call it the fibre twist.

Remark 5.14. The order of the compositions in the definitions does not matter, since the individual functors all commute. This may be shown, for instance, by tracking skyscraper sheaves as in 3.13, or using 6.1(1) later.

Being the composition of equivalences by 5.11, these functors are all equivalences regardless of whether or not the curves in J algebraically flop. Being able to vary J without worrying about whether the curves in J flop is the key later to obtaining group actions on derived categories without strong assumptions on the flopping contraction.

In the case when the curves in J do algebraically flop, $J\mathsf{Twist}$ is inverse to the Bridgeland–Chen flop–flop functor.

Proposition 5.15. *With the global projective flops setup of 2.2, suppose that a subset* J *of the exceptional curves flops algebraically (e.g. when* J *equals all exceptional curves). Then* $J\mathsf{Twist} \circ (\mathsf{F}_J \circ \mathsf{F}_J) \cong \mathrm{Id}$.

Proof. Since the curves in J flop algebraically, there exists some factorisation of f into

$$X \xrightarrow{g} X_J \to X_{\mathrm{con}}$$

where g is a flopping contraction. First, note that $\{E_j \mid j \in J\}$ generates

$$\mathcal{C}_g := \{c \in \mathrm{D^b}(\mathrm{coh}\, X) \mid \mathbf{R}g_*c = 0\},$$

since $\mathcal{C}$ splits as a direct sum over the points p of the non-isomorphism locus of g, so we may follow the argument of [KIWY, 5.3].

We next argue that $\Psi := J\mathsf{Twist}\circ(\mathsf{F}_J\circ\mathsf{F}_J)$ preserves ${}^0\mathrm{Per}(X, X_J)$, since the remainder of the proof then follows exactly as in [DW1, 7.18]. For this, note that

$$J\mathsf{Twist}(E_j) = E_j[-2] \tag{5.I}$$

for all $j \in J$, since

$$\begin{array}{ll} J\mathsf{Twist}_p(E_j) = E_j[-2] & \text{if } C_j \in g^{-1}(p) \\ J\mathsf{Twist}_p(E_j) = E_j & \text{if } C_j \notin g^{-1}(p) \end{array}$$

where the top line holds by 5.8, and the bottom also by 5.8 since if $C_j \notin g^{-1}(p)$, then $E_j \in (i_{p_*}\mathcal{E}_{J_p})^{\perp}$. On the other hand

$$(\mathsf{F}_J \circ \mathsf{F}_J)(E_j) = E_j[2] \tag{5.J}$$

for all $j \in J$, using the description in [T08, §3(i)] of the action of the flop on the sheaves $E_j = \mathcal{O}_{C_j}(-1)$, noting a correction given in [T14, Appendix B] to the sign of the shift in [B]. Combining (5.I) and (5.J), we find that Ψ fixes the set $\{E_j \mid j \in J\}$ and thus preserves $\mathcal{C}_g$.

Next we note that Ψ commutes with the pushdown $\mathbf{R}g_*$. This follows because $J\mathsf{Twist}_p$ commutes with the pushdown by 5.12, and the flop–flop commutes with the pushdown exactly as in [DW1, 7.16(1)]. It then follows by the argument of [DW1, 7.17] that Ψ preserves ${}^0\mathrm{Per}(X, X_J)$.

Finally, by combining as in [DW1, 7.16(2)], we have that $\Psi(\mathcal{O}_X) \cong \mathcal{O}_X$. Using this, and the fact that Ψ commutes with the pushdown $\mathbf{R}g_*$ as noted above, the proof now follows just as in [DW1, 7.18]. □

The final result of this section compares the functors $F\mathsf{Twist}$ and $J\mathsf{Twist}$ for flopping contractions of smooth rational curves. In the simplest case, when the curve is of type A, these functors may be conjugate by a certain line bundle. When the curve is not of type A, however, we next show that the two functors are never related in this way. As stated in the introduction, the functor $F\mathsf{Twist}$ is not expected to be related to the flop–flop functor in general, instead it is expected to be the affine element in some (pure) braid action. Because of the intricacy of the combinatorics, we will return to this more general affine action in the future.

Theorem 5.16. *In the global projective flops setup of 2.2, for a contraction of a single irreducible curve to a point p, where in addition X is $\mathbb{Q}$-factorial, there exists a functorial isomorphism*

$$F\mathsf{Twist}(x \otimes \mathcal{F}) \cong J\mathsf{Twist}(x) \otimes \mathcal{F}$$

for some line bundle $\mathcal{F}$ on X, if and only if the following conditions hold.

(1) *The point p is cDV of Type A.*
(2) *There exists a line bundle $\mathcal{F}$ on X such that* $\deg(\mathcal{F}|_{f^{-1}(p)}) = -1$.

Proof. By assumption $J = \{1\}$, corresponding to the single irreducible curve in the fibre $C = f^{-1}(p)$.

($\Leftarrow$) We require a natural isomorphism of functors

$$F\mathsf{Twist} \cong (- \otimes \mathcal{F}) \circ J\mathsf{Twist} \circ (- \otimes \mathcal{F}^{-1}) =: J\mathsf{Twist}_{\mathcal{F}}. \tag{5.K}$$

Observe that, using 5.11(2), there are functorial triangles for all $x \in \mathrm{D}(\operatorname{Qcoh} X)$,

$$\begin{aligned} \mathbf{R}\operatorname{Hom}_X(\mathcal{E}_0, x) \otimes^{\mathbf{L}}_{\mathrm{A}_0} \quad \mathcal{E}_0 \quad &\to x \to F\mathsf{Twist}(x) \to \\ \mathbf{R}\operatorname{Hom}_X(\mathcal{E}_J \otimes \mathcal{F}, x) \otimes^{\mathbf{L}}_{\mathrm{A}_J} (\mathcal{E}_J \otimes \mathcal{F}) &\to x \to J\mathsf{Twist}_{\mathcal{F}}(x) \to. \end{aligned}$$

By the arguments of [DW1, §3], the objects $\mathcal{E}_0$ and $\mathcal{E}_J$ are universal families, corresponding to the representing objects A_0 and A_J respectively, for the noncommutative deformations of

$$E_0 = \omega_C[1] \quad \text{and} \quad E_1 = \mathcal{O}_{C^{\mathrm{red}}}(-1).$$

Under our assumption (1), we have that $C = C^{\mathrm{red}} \cong \mathbb{P}^1$, and so $\omega_C \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and thence using (2)

$$E_0 \cong E_1 \otimes \mathcal{F}[1].$$

The functor $(-\otimes\mathcal{F})[1]$ is an equivalence, and so we obtain an isomorphism of representing objects $\mathrm{A}_0 \cong \mathrm{A}_J$, and an isomorphism of universal families

$$\mathcal{E}_0 \cong \mathcal{E}_J \otimes \mathcal{F}[1], \tag{5.L}$$

respecting the module structures over the algebras A_0 and A_J. We thus obtain isomorphisms between the objects in the functorial triangles above, and the natural isomorphism (5.K) follows.

($\Rightarrow$) Suppose as in the statement that the twists are conjugate by some line bundle $\mathcal{F}$. It will be more convenient to work with perverse sheaves in the category ${}^{-1}\mathrm{Per}$, and so we conjugate by the dualizing functor $\mathbb{D}$ to find

$$(\mathbb{D}\, J\mathsf{Twist}\, \mathbb{D}^{-1}) \circ (- \otimes \mathcal{F}) \cong (- \otimes \mathcal{F}) \circ (\mathbb{D}\, F\mathsf{Twist}\, \mathbb{D}^{-1}).$$

Recall that $\mathbb{D}$ exchanges $E_0 = \omega_C[1]$ and $\mathcal{O}_C$ [V, 3.5.8], so that by 5.8(4) we have

$$(\mathbb{D}\, F\mathsf{Twist}\, \mathbb{D}^{-1})(\mathcal{O}_C) \cong \mathcal{O}_C[2].$$

It follows immediately that

$$(\mathbb{D}\, J\mathsf{Twist}\, \mathbb{D}^{-1})(\mathcal{O}_C \otimes \mathcal{F}) \cong \mathcal{O}_C \otimes \mathcal{F}[2].$$

The functor $\mathbb{D}\, J\mathsf{Twist}\, \mathbb{D}^{-1}$ commutes with the pushdown $\mathbf{R}f_*$, by the argument of 5.12. The key point there was that $\mathbf{R}f_*\mathcal{E}_J = 0$, whereas here we use $\mathbf{R}f_*(\mathbb{D}\,\mathcal{E}_J) = 0$, which is obtained from relative Serre duality, as stated for instance in [DW1, 6.17]. We

then deduce that $\mathbf{R}f_*(\mathcal{O}_C \otimes \mathcal{F}) = 0$ by boundedness. As in §2.4 we may work complete locally around p, on a formal fibre $\mathfrak{U}$. It follows by base change that

$$\mathbf{R}f_*(\mathcal{O}_C \otimes \mathcal{F}|_{\mathfrak{U}}) = 0. \tag{5.M}$$

We now analyse this pushdown via the tilting equivalence on $\mathfrak{U}$, and use its vanishing to control the degree of $\mathcal{F}$ on the contracted curve C.

From §2.4, we have that $\mathcal{O}_{\mathfrak{U}} \oplus \mathcal{N}_1$ is a tilting bundle on $\mathfrak{U}$, so its dual $\mathcal{W} := \mathcal{O}_{\mathfrak{U}} \oplus \mathcal{M}_1$ is also a tilting bundle, and so gives an equivalence

$$\mathrm{D^b}(\operatorname{coh}\mathfrak{U}) \underset{\Psi}{\overset{\Phi=\mathbf{R}\mathrm{Hom}_{\mathfrak{U}}(\mathcal{W},-)}{\underset{\sim}{\rightleftarrows}}} \mathrm{D^b}(\operatorname{mod}\mathrm{A}^{\mathrm{op}}).$$

We will place the natural surjection $\mathcal{O}_C \to \mathcal{O}_{C^{\mathrm{red}}}$ into a distinguished triangle, by passing through the equivalence Φ.

Recall that there is an exact sequence

$$0 \to \mathcal{O}_{\mathfrak{U}}^{\oplus(r-1)} \to \mathcal{M}_1 \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is a line bundle having degree 1 on C^{red}, and $r \geq 1$, with $r = 1$ occurring precisely when p is cDV of Type A. We see immediately from the standard calculation of the cohomology of line bundles on $C^{\mathrm{red}} \cong \mathbb{P}^1$ that $\mathbf{R}\mathrm{Hom}_{\mathfrak{U}}(\mathcal{L}, \mathcal{O}_{C^{\mathrm{red}}}) = 0$, and we thence obtain that

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{\mathfrak{U}}(\mathcal{O}_{\mathfrak{U}}, \mathcal{O}_{C^{\mathrm{red}}}) &= \mathbb{C} \\ \mathbf{R}\mathrm{Hom}_{\mathfrak{U}}(\mathcal{M}_1, \mathcal{O}_{C^{\mathrm{red}}}) &= \mathbb{C}^{r-1}. \end{aligned}$$

In particular, $\Phi(\mathcal{O}_{C^{\mathrm{red}}})$ is a module in degree zero, which when viewed as a quiver representation has $\mathbb{C}$ at the vertex 0, and $\mathbb{C}^{r-1}$ at the vertex 1.

Since Φ is an equivalence, $\Phi(\mathcal{O}_C \to \mathcal{O}_{C^{\mathrm{red}}})$ is a non-zero morphism. But $\Phi(\mathcal{O}_C) = S_0$, which is just $\mathbb{C}$ at the vertex 0, and so this is simply a non-zero morphism between representations. Since $\Phi(\mathcal{O}_C)$ is one-dimensional, necessarily this is injective, so we set $\mathcal{G}$ to be the cokernel and so obtain an exact sequence

$$0 \to \Phi(\mathcal{O}_C) \to \Phi(\mathcal{O}_{C^{\mathrm{red}}}) \to \mathcal{G} \to 0.$$

By the above, $\mathcal{G}$ is a module with a filtration consisting of $r-1$ copies of S_1. Applying Ψ we get a triangle

$$\mathcal{H} := \Psi(\mathcal{G})[-1] \to \mathcal{O}_C \to \mathcal{O}_{C^{\mathrm{red}}} \to \tag{5.N}$$

where $\mathcal{H}$ is a sheaf with a filtration consisting of $r-1$ copies of $\Psi(S)[-1] \cong \mathcal{O}_{C^{\mathrm{red}}}(-1)$.

Tensoring (5.N) by $\mathcal{F}|_{\mathfrak{U}}$, and writing $d = \deg(\mathcal{F}|_{C^{\mathrm{red}}})$ for convenience, we have a short exact sequence

$$0 \to \mathcal{H}(d) \to \mathcal{O}_C \otimes \mathcal{F}|_{\mathfrak{U}} \to \mathcal{O}_{C^{\mathrm{red}}}(d) \to 0$$

where as above $\mathcal{H}$ is filtered by objects $\mathcal{O}_{C^{\mathrm{red}}}(-1)$. Applying $\mathbf{R}f_*$, and using the cohomology vanishing (5.M), we find

$$\mathbf{R}f_*(\mathcal{O}_{C^{\mathrm{red}}}(d)) \cong \mathbf{R}f_*(\mathcal{H}(d))[1].$$

We now make further use of the cohomology of line bundles on $C^{\mathrm{red}} \cong \mathbb{P}^1$ to deduce our result. The fibres of f are at most one-dimensional, so $\mathbf{R}^{>1}f_*(\mathcal{H}(d)) = 0$, and therefore $\mathbf{R}^1 f_*(\mathcal{O}_{C^{\mathrm{red}}}(d)) = 0$ which implies $d \geq -1$. Suppose, for a contradiction, that furthermore $d \geq 0$. Then, using the filtration of $\mathcal{H}$, $\mathbf{R}^{>0}f_*(\mathcal{H}(d)) = 0$ whereas $f_*(\mathcal{O}_{C^{\mathrm{red}}}(d)) \neq 0$. It follows that $d = -1$, yielding (2). We then have $\mathbf{R}f_*(\mathcal{H}(-1)) = 0$, forcing $\mathcal{H} = 0$, so that $r = 1$, giving (1). □

6. Pure Braid-Type Actions

In this section we piece together the previous sections, and give a group action for any algebraic flopping contraction in the global setup of 2.2.

Proposition 6.1. *Under the Zariski local setup 2.3, or the global setup of 2.2,*

(1) $J\mathsf{Twist}(\mathcal{O}_x) \cong \mathcal{O}_x$ *for all* $x \notin \bigcup_{j\in J} C_j$.
(2) *Choose* $p \in \mathrm{Ram}(X_{\mathrm{con}})$ *and a collection* $J_1, \dots, J_m$ *of subsets of* $\{1, \dots, n_p\}$. *If the twists* $J_1\mathsf{Twist}_p, \dots, J_m\mathsf{Twist}_p$ *satisfy some relation on the formal fibre above* p, *then they satisfy the same relation Zariski locally.*

Proof. We give the proof in the Zariski local setup 2.3, since the proof of the global setup is obtained by simply replacing U by X throughout.
(1) Since $\mathcal{E}_J$ is filtered by $\mathcal{O}_{C_j}(-1)$ with $j \in J$, it follows that $\mathbf{R}\mathrm{Hom}_U(\mathcal{E}_J, \mathcal{O}_x) = 0$ for all $x \notin \bigcup_{j\in J} C_j$, since $x \notin \operatorname{Supp} \mathcal{E}_J$ [BM, 5.3]. Thus, the triangle

$$\mathbf{R}\mathrm{Hom}_U(\mathcal{E}_J, \mathcal{O}_x) \otimes^{\mathbf{L}}_{\mathrm{A}_J} \mathcal{E}_J \to \mathcal{O}_x \to J\mathsf{Twist}(\mathcal{O}_x) \to$$

implies that $J\mathsf{Twist}(\mathcal{O}_x) \cong \mathcal{O}_x$.
(2) Suppose that the relation on the formal fibre above p can be written as

$$(J_{i_1}\mathsf{Twist}_p)^{a_1} \dots (J_{i_\ell}\mathsf{Twist}_p)^{a_\ell} \cong \mathrm{Id} \tag{6.A}$$

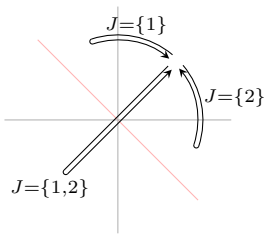
for some $a_1, \dots, a_\ell \in \mathbb{Z}$ and some $i_1, \dots, i_\ell \in \{1, \dots, m\}$. We again track skyscrapers. By (1), certainly the skyscrapers not supported on $\bigcup_{i=1}^{n_p} C_i$ are fixed under the left-hand side of (6.A), so we need only track the skyscrapers on $\bigcup_{i=1}^{n_p} C_i$. As in the latter stages of the proof of 3.10, we can do this by passing to the formal fibre, and by assumption we know that the relation (6.A) holds there. Hence on the formal fibre these skyscrapers are fixed under the left-hand side of (6.A). Hence overall every skyscraper $\mathcal{O}_x$ gets sent to some skyscraper under the left-hand side of (6.A), so since the twists commute with pushdown, as before it follows that the relation holds Zariski locally. □

Combining 5.14 with 6.1(2) shows that globally we can still view the J-twists as a subgroup of $\prod_i \pi_1(\mathbb{G}_{\mathrm{A}_i})$. Since the J-twists are equivalences by 5.10, this then immediately gives the following, which is our main result.

Corollary 6.2. *Suppose that* $X \to X_{\mathrm{con}}$ *is a flopping contraction, where* X *is projective and has only Gorenstein terminal singularities. The subgroup* K *of* $\prod_i \pi_1(\mathbb{G}_{\mathrm{A}_i})$ *generated by the* J*-twists, as* J *ranges over all subsets of curves, acts on* $\mathrm{D^b}(\mathrm{coh}\, X)$.

We remark that the subgroup K can equal $\prod_i \pi_1(\mathbb{G}_{\mathrm{A}_i})$, as the following example illustrates. As stated in the introduction, it is unclear in what level of generality this holds, and indeed this seems to be an interesting problem, both geometrically and group-theoretically.

Example 6.3. Consider an algebraic flopping contraction $X \to X_{\mathrm{con}}$ of two intersecting curves contracting to a cA_n singularity. In this situation the chamber structure and subgroup K are illustrated as follows:



Thus inside $\pi_1(\mathbb{G})$, which is the pure braid group, $K = \langle a^2, b^2, (aba)^2 \rangle$ where a and b are the standard braids in the classical presentation of the braid group on three strands. Now

$$f := a^2ba^2b = ababab = (aba)^2 \in K,$$

and hence also $a^{-2}fb^{-2} = ba^2b^{-1} \in K$. But it is well-known that the pure braid group is generated by a^2, b^2, and ba^2b^{-1} (see e.g. [BB, p5]), and hence $K = \pi_1(\mathbb{G})$ in this case.

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