ON RC VARIETIES WITHOUT SMOOTH RATIONAL CURVES

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Abstract. We construct normal rationally connected varieties (of arbitrarily large dimension) not containing any smooth rational curves.

1. Introduction

1.1. Let V be a rationally connected projective variety, not necessarily smooth or normal, defined over $\mathbb{C}$. Then any smooth V is covered by various images of morphisms $f : \mathbb{P}^1 \longrightarrow V$ such that the pullback f^*T_V is ample (see e.g. [11, Theorem 3.7]). One also observes that the image $f(\mathbb{P}^1)$ is *smooth* in this case for a generic choice of f (see [11, Theorem 3.14]).

The present paper grew out of an attempt to understand whether the preceding property holds for singular V as well (see [9], [10] for some related results). Namely, if V is *normal* and rationally connected, is there always at least one smooth rational curve on V? Or more generally, i.e. dropping the normality assumption, are there such rationally connected V that don't contain any smooth rational curves, with $\dim V$ being arbitrarily large?

In the latter case, the answer is evident when $\dim V = 1$ (normalization), although even for the product $V \times V$ we are not sure how to proceed (there might exist $\mathbb{P}^1 \subset V \times V$ with degree > 1 projection onto V). So the assumption $\dim V \to \infty$ makes the problem more interesting (we refer to Section 4 for further variations).

Our main result treats the normal case as follows:

Theorem 1.2. *There exists normal rationally connected V without smooth rational curves (we call such V* weird*) and with $\dim V$ arbitrarily large.*

Let us briefly outline the strategy of the proof of Theorem 1.2.

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1.3. To make the construction easier we'd like to consider those V that parameterize certain geometric objects. This should, in principle, allow one interpret smoothness of any rational curve $Z \subset V$ in terms of properties of the corresponding family of objects. We've given our preference to the moduli spaces $\mathcal{SU}_X(r)$ of rank $r > 1$ and det $= \mathcal{O}_X$ vector bundles over an algebraic curve X of genus $g > 1$ (compare with [5]).

Recall that $\mathcal{SU}_X(r)$ is a *Fano variety* (see Section 2 for its specific properties). Yet, unfortunately, it is too early to just set $V := \mathcal{SU}_X(r)$ and conclude the proof of Theorem 1.2:

Example 1.4. Given any stable vector bundle $E \in \mathcal{SU}_X(r)$, after twisting E by $\mathcal{O}_X(m)$ for some $m \in \mathbb{Z}$ (independent of E) one may identify $E \otimes \mathcal{O}_X(m)$ with an extension class from $\mathrm{Ext}^1(\mathcal{O}_X(rm), \mathcal{O}_X \otimes \mathbb{C}^{r-1}) =: L$. This provides a rational dominant map $\mathbb{P}(L) \dashrightarrow \mathcal{SU}_X(r)$ (see [7, Proposition 7.9] for instance) and shows that $\mathcal{SU}_X(r)$ is in fact *unirational*. Now comes the discouraging observation that $\mathcal{SU}_X(r)$ actually contains plenty of smooth rational curves. Namely, starting with any point $p \in X$, some generic vector bundle $\mathcal{E}$ with determinant $\det \mathcal{E} = \mathcal{O}_X(p)$, and a linear form l on the fiber $\mathcal{E}_p$, we consider the morphism of sheaves $\mathcal{E} \longrightarrow \mathcal{O}_p$ corresponding to l. The kernel of this morphism is again a rank r vector bundle; it is stable due to [12, Satz] and has trivial determinant. This shows that the whole $\mathbb{P}^{r-1} \ni l$ embeds into $\mathcal{SU}_X(r)$ (cf. [13, 5.9]).

Recall that all the (equivalent) constructions of $\mathcal{SU}_X(r)$ involve certain GIT quotient of a source space $\mathbb{S}$ by some group G. This $\mathbb{S}$ can be the space of relative Grassmannians over X (resp. $G = \mathrm{PGL}$), the space of $\mathrm{SU}_r(\mathbb{C})$-representations of $\pi_1(X)$ (resp. $G = \mathrm{SU}$), the space of flat connections on a fixed rank r topologically trivial vector bundle E over X (resp. $G =$ the gauge group), etc. Our idea then was, starting with a smooth rational curve $Z \subset \mathcal{SU}_X(r)$, lift it in a $1:1$ manner to $\mathbb{S}$ and obtain a contradiction with the fact that $\mathbb{S}$ is affine or something like this.

However, as Example 1.4 shows, this idea won't work directly. The reason behind is that the claimed "lifting to $\mathbb{S}$" doesn't exist: there's always an ambiguity in the choice of a point $\in \mathbb{S}$ to associate with any given point on Z. In order to circumvent this we use another construction of $\mathcal{SU}_X(r)$ (compare with [15, 5.1]). Namely, after some care (see Section 2), one may take $\mathbb{S} =$ the group of special $(r \times r)$-matrices with coefficients in a formal power series ring (resp. $G =$ some ind-group). Then for a particular locus $\mathcal{D} \subset \mathcal{SU}_X(r)$ (see **3.4**), with any point on

$\mathcal{D}$ one associates (canonically) a point in $\mathbb{S}$, modulo some moderate assumptions (cf. Proposition 3.2).

Remark 1.5. This $\mathcal{D}$ seems to be another natural object, in addition to the theta-divisor $\mathcal{L}$ (see **2.5** below), which comes for free with $\mathcal{SU}_X(r)$. It would be interesting to explore the relation between $\mathcal{D}$ and $\mathcal{L}$ further: whether, say, rationality (resp. precise dimension) statement for $\mathcal{L}$ (see e.g. [15, 3.1]) holds also for $\mathcal{D}$?

One observes next that $\mathcal{D}$ is normal, projective and rationally connected – the facts we derive from the ind-construction of $\mathcal{SU}_X(r)$. This readily shows that there is no smooth rational $Z \subset \mathcal{D}$, since otherwise it'll be lifted to $\mathbb{S}$ = direct limit of affine varieties, which is impossible (see Section 3 for details).

2. Preliminaries

2.1. Let the notations be as in **1.3**. We first briefly recall the ind-construction of the moduli spaces $\mathcal{SU}_X(r)$ (see [1], [2], [8] for a complete account). For this we fix a point $p \in X$, a small disk $\Delta \subset X$ around p and a local coordinate z on Δ such that $z(p) = 0$. We also put $K := \mathbb{C}((z))$ and $\mathcal{O} := \mathbb{C}[[z]]$.

Let E be a vector bundle on X of rank r and $\det E = \mathcal{O}_X$. One proves by induction on r that E is trivial over $X^* := X \setminus p$ (cf. [1, Lemma 3.5]). Then trivializing E also over Δ provides an element $\gamma \in \mathrm{SL}_r(K)$ whose class in the space $\mathcal{Q} := \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$ uniquely determines E up to the left action on $\mathrm{SL}_r(K)$ of the subgroup $\mathrm{SL}_r(\mathcal{O}_{X^*})$ (i.e. up to a choice of trivialization for $E\big|_{X^*}$).

Equivalently, one may associate with γ a *special lattice* $\Lambda \simeq \mathcal{O}^{\oplus r}$ (with $\mathrm{Vol}\,\Lambda := \det\gamma = 1$) generated as a $\mathcal{O}$-module by global sections of $E\big|_{X^*}$ and such that E (or γ) is reconstructed from Λ (see [1, Proposition 2.3]). In particular, one regards $\mathcal{Q}$ as a collection of all such Λ (with the obvious $\mathrm{SL}_r(\mathcal{O}_{X^*})$-action) and represents it as a direct limit of irreducible projective varieties $\mathcal{Q}^{(N)}, N \ge 0$, which consist of those Λ that have a basis $z^{d_i}e_i$ with $d_i \ge -N, \sum d_i = 0$, and $e_i \in \mathcal{O}^r$ for all $1 \le i \le r$ (see [1, Theorem 2.5, Proposition 2.6]).

We will need the following simple observation:

Lemma 2.2 (cf. [1, Lemma 4.5])**.** *With notations as above, every element from the* $\mathrm{SL}_r(\mathcal{O}_{X^*})$*-orbit of* Λ *is represented by some matrix in* $\mathrm{SL}_r(\mathbb{C}[[z^{-1}]])$ *(by definition* Λ *is the image of* γ *under the quotient morphism* $\mathrm{SL}_r(K) \longrightarrow \mathcal{Q}$*), so that* Λ *admits* *a basis consisting of vectors from* $\mathbb{C}[[z^{-1}]]^r$. *In particular, one may choose* $\gamma \in \mathrm{SL}_r(\mathbb{C}[[z^{-1}]])$ *to satisfy* $\gamma = I \mod z^{-1}$, *where* I *is the identity matrix.*

2.3. Further, recall that there's a *canonical* isomorphism in codimension 1 $\mathcal{SU}_X(r) \simeq \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$, where $\mathcal{SU}_X(r)$ (resp. $\mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$) is considered as an algebraic (resp. quotient) stack (see [1, Proposition 3.4, Lemma 8.2] and Remark 2.4 below). In addition, the group $\mathrm{SL}_r(\mathcal{O}_{X^*})$ also carries the structure of an integral ind-scheme, so that its action on $\mathcal{Q} = \varinjlim \mathcal{Q}^{(N)}$ is compatible with the underlying ind-structure (cf. [1, Proposition 6.4]).

Altogether this yields an ind-structure on $\mathcal{SU}_X(r)$, so that $\mathcal{SU}_X(r) \subset \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$ as an open ind-subscheme with codimension > 1 complement, and an ample line bundle $\mathcal{L}\big|_{\mathcal{SU}_X(r)}$ with the pullback $\pi^*\mathcal{L}$ under the natural projection $\pi : \mathcal{Q} \longrightarrow \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$ being some $\mathrm{SL}_r(\mathcal{O}_{X^*})$-invariant (determinant) line bundle (see [1, Sections 5, 7, 8]). The global sections from $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$ (*the space of conformal blocks*), $c \in \mathbb{Z}$, coincide by construction with the $\mathrm{SL}_r(\mathcal{O}_{X^*})$-invariant global sections of $\pi^*\mathcal{L}^c$ (see [1, Theorem 8.5]). We'll mention a few more properties of $\mathcal{L}$ in **2.5**.

Remark 2.4. Using the (mappings by) global sections of $\mathcal{L}^c, c \gg 1$, one may regard both $\mathcal{SU}_X(r)$ and $\mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$ as essentially the same *scheme* with two complementary structures: of a quasi-projective variety and an ind-scheme. In particular, any open (resp. closed) ind-subscheme of $\mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$ corresponds, tautologically, to an open (resp. closed) subscheme in $\mathcal{SU}_X(r)$, and vice versa (compare with the proof of Theorem 7.7 in [1]).

2.5. To conclude this section, let's fix a bit more of notation/conventions and recall some auxiliary facts, as those will be used later in Section 3.

Choose some semi-stable bundle $E \in \mathcal{SU}_X(r)$ (identified with a point in the corresponding moduli space) with a Hermitian structure specifying the gauge group action on the space of all connections on E. Using [15, 3.2] (cf. [7, Lemma 4.2]) and the main result of [6] (cf. [14]) one represents E as a sum of stable bundles.

Note that any two direct sum decompositions $\oplus E_i = E = \oplus E_i'$ differ only by the order of summands. Indeed, otherwise arguing by induction on r one obtains a surjection $E_i \longrightarrow E_i'$ between two stable bundles $E_i \not\simeq E_i'$ of degree 0, which is impossible. This leads to the following:

Theorem 2.6. *There exists a* unique *(up to* $\mathrm{Aut}\, E$*) flat unitary connection* ∇ *on* E.

Finally, although we'll not quite need this, let us mention for consistency some other facts about $\mathcal{SU}_X(r)$ (see also Section 4). First of all, variety $\mathcal{SU}_X(r)$ is *locally factorial*, with Picard group generated by $\mathcal{L}$ (see [7, Theorems A, B]). One may equivalently interpret $\mathcal{L}$ as the so-called *theta-divisor*. The latter consists of all $E \in \mathcal{SU}_X(r)$ for which $H^0(X, E \otimes \xi) \neq 0$ with respect to some fixed cycle ξ on X of degree $g-1$. Furthermore, the canonical class $K_{\mathcal{SU}_X(r)}$ equals $-2r\mathcal{L}$, so that $\mathcal{SU}_X(r)$ is a Fano variety.

3. Proof of Theorem 1.2

3.1. We retain the notations of Section 2. Put $t := z^{-1}$ and let γ (resp. Λ) be as in Lemma 2.2. Recall that $E\big|_{\Delta} = \mathbb{C}^r \times \Delta$ with constant basis, while the $\mathcal{O}_{X^*}$-module $E\big|_{X^*}$ is generated by Λ, so that the two data are glued over $\Delta \cap X^*$ via γ. We will additionally assume that $\gamma = I \mod t^2$.

To apply the strategy outlined in **1.3** one should have (at least) the following:

Proposition 3.2. *In the previous setting, Λ carries a* unique *(Bohr-Sommerfeld) basis, which does not depend on the choice of γ. In fact, there is only one γ associated with Λ (cf.* **2.1***), satisfying $\gamma = I \mod t^2$.*

Proof. According to our assumption there exists a collection of vectors $\{e_1, \ldots, e_r\} \subset \mathbb{C}[[t]]^r$ that generates $E\big|_{X^*}$ and coincides with the standard basis of $\mathbb{C}^r$ modulo t^2. Using this, the flat unitary connection ∇ on E (see Theorem 2.6), plus the preceding description of E in terms of Δ, X^* and γ, we construct (via the parallel transport starting with $e_i(0)$) global C^∞-sections ε_i of $E, 1 \leq i \leq r$, which generate $E\big|_{X^*}$ and satisfy the equation $\nabla \varepsilon_i = 0$ in a small disk $\Delta_0 \subset X^*$ around $t = 0$.

More precisely, since ∇ is flat and $de_i(0) = 0$,[1)] equation $\nabla \varepsilon_i = 0$ and its solutions $\{\varepsilon_1, \ldots, \varepsilon_r\}$ do not depend on the choice of e_i (aka γ). In particular, there is a correctly defined (canonical) extension of every ε_i over the entire X, as claimed. We also observe that by uniqueness of the solutions all entries in ε_i are some elements from $\mathbb{C}[[t]]$ because $\nabla^{0,1} = \bar{\partial}$ locally on Δ_0. This gives the desired basis for Λ.

[1)] Here d is the usual Kählher differential of the ring $\mathbb{C}[[t]]$.

Recall next that ∇ is unique up to $\operatorname{Aut} E$. Now the last claim of proposition follows from the fact that one may take $\{\varepsilon_1, \ldots, \varepsilon_r\}$ to be the columns of γ and that $g\gamma g^{-1} = \gamma$ for any $g \in \operatorname{Aut} E$ by definition. $\square$

Remark 3.3. (Notations as in the proof of Proposition 3.2) The sections ε_i provide an isomorphism $E \simeq \mathcal{O}_X^{\oplus r}$ of C^∞-bundles.[2)] Let $\mathcal{D} \subset \mathcal{SU}_X(r)$ be the moduli space of all *completely reducible* flat connections on $\mathcal{O}_X^{\oplus r}$. Then conversely, for any C^∞-trivial $E \in \mathcal{D}$ the corresponding Λ, γ can be chosen to satisfy $\gamma = I \mod t^2$. Indeed, any such E is generated by r ∇-flat C^∞-sections ε_i, obtained from some constant sections ε_i^0 of $\mathcal{O}_X^{\oplus r}$ via fiberwise (gauge) transformation. Locally on Δ_0 one has $\varepsilon_i = \varepsilon_i^0$ modulo $(t, \bar{t})^2$ by construction, which implies that $d\varepsilon_i(0) = 0 = \nabla\varepsilon_i$. Then all ε_i depend only on t due to $\nabla^{0,1} = \bar{\partial}$ and uniqueness of the solutions. Note that this construction of ε_i requires $\Delta_0 \ni 0$ to be *fixed*. It also shows that any other lattice $g \cdot \Lambda$, for $g \in \mathrm{SL}_r(\mathcal{O}_{X^*})$, has the corresponding $g\gamma g^{-1}$ equal to $I \mod t^2$ as well. Indeed, recall that the condition for ε_i to be ∇-flat translates into $\varepsilon_i = \varepsilon_i^0$ modulo $(t, \bar{t})^2$, which must be gauge invariant (aka preserved under the parallel transports).

3.4. The locus $\mathcal{D} \subset \mathcal{SU}_X(r)$ from Remark 3.3 is our candidate for the weird rationally connected variety V. Let's study the geometry of $\mathcal{D}$ more closely by employing its description in terms of the lattices Λ. But first we make the following:

Assumption. Fix $r \geq 3$ and choose the initial curve X to be generic of genus ≥ 3. This gives $\operatorname{Aut} E = \mathbb{C}^*$ for generic $E \in \mathcal{D}$ (take for instance $E =$ the direct sum of different line bundles $\delta_i \neq \mathcal{O}_X$ satisfying $\deg \delta_i = 0$ and $\delta_i \neq -\delta_j$ for all i, j).

We will write $\mathcal{SU}_X(r) = \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{Q}$ in what follows (cf. **2.3** and Remark 2.4). This aims to just simplify the notation and won't cause any loss of generality.

Consider the subset $\mathcal{D}_0 \subset \mathcal{Q}$ of all lattices Λ satisfying $\gamma = I \mod t^2$ as in **3.1**. Let also $\mathcal{M} \subset \mathrm{SL}_r(\mathbb{C}[[t]])$ be the locus of all matrices $I + \sum_{i=2}^{\infty} t^i \Theta_i$ for various $\Theta_i \in \mathrm{M}_r(\mathbb{C})$. It follows from the previous constructions that $\mathcal{M}$ maps onto $\mathcal{D}_0$ under the quotient morphism $\mathrm{SL}_r(K) \longrightarrow \mathcal{Q} = \mathrm{SL}_r(K)/\mathrm{SL}_r(\mathcal{O})$.

Lemma 3.5. *$\mathcal{D}_0 \subset \mathcal{Q}$ is a $\mathrm{SL}_r(\mathcal{O}_{X^*})$-invariant closed ind-subscheme.*

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2)The bundle $\mathcal{O}_X^{\oplus r}$ carries a natural Hermitian structure which varies together with E.

Proof. Consider some $\Lambda \in \mathcal{D}_0$. Recall that Λ uniquely determines the corresponding $\gamma \in \mathcal{M}$ (see Proposition 3.2). Then, since the group $\mathrm{SL}_r(\mathcal{O})$ acts freely on $\mathrm{SL}_r(K) \supset \mathcal{M}$, the set $\mathcal{D}_0$ may be identified with $\mathcal{M}$ near Λ.

In particular, any (bounded in analytic topology) Cauchy sequence of lattices $\Lambda_i \in \mathcal{D}_0$ yields the corresponding sequence of $\gamma_i \in \mathcal{M}$, as follows from the definition of $\mathrm{SL}_r(K) \longrightarrow \mathcal{Q}$. One then (obviously) has a limit $\lim \gamma_i \in \mathcal{M}$ that maps to $\lim \Lambda_i \in \mathcal{D}_0$. This shows that $\mathcal{D}_0$ is closed in analytic topology. Now using the representation $\mathcal{Q} = \varinjlim \mathcal{Q}^{(N)}$ from **2.3** we obtain that $\mathcal{D}_0 \subset \mathcal{Q}$ is a closed ind-subscheme.

Finally, the $\mathrm{SL}_r(\mathcal{O}_{X^*})$-invariance of $\mathcal{D}_0$ follows from Remark 3.3, which concludes the proof. $\square$

From Lemma 3.5 we obtain

$$\mathcal{D} = \pi(\mathcal{D}_0) = \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \mathcal{D}_0$$

for the natural $\mathrm{SL}_r(\mathcal{O}_{X^*})$-equivariant morphism $\pi : \mathcal{Q} \longrightarrow \mathcal{SU}_X(r)$ of ind-schemes.

Proposition 3.6. *$\mathcal{D}$ is a normal, projective and rationally connected variety.*

Proof. It follows from the above *Assumption* that the group $\mathrm{SL}_r(\mathcal{O}_{X^*})$ acts freely at the general point on $\mathcal{D}_0$. This gives

Lemma 3.7. *$\mathcal{D}$ is irreducible and reduced (i.e. integral).*

Proof. Recall that $\mathcal{D}_0$ and $\mathcal{M}$ are isomorphic at the general point (cf. the proof of Lemma 3.5). Note also that $\mathcal{M}$ is integral (compare with the proof of [1, Proposition 2.6]). Hence $\mathcal{D}_0$ is integral as well. Then, since $\mathcal{D}_0 \simeq \mathcal{D} \times \mathrm{SL}_r(\mathcal{O}_{X^*})$ (equivariantly) at the general point, the claim follows from [1, Proposition 6.4, Lemma 6.3, d)]. $\square$

From the definition of $\pi : \mathcal{D}_0 \longrightarrow \mathcal{D}$ we deduce that the locus $\mathcal{D} \subset \mathcal{SU}_X(r)$ is closed in analytic topology (cf. Lemma 3.5 and Remark 2.4). Hence $\mathcal{D}$ is a projective integral variety.

Further, consider some lattice $\Lambda \in \mathcal{D}_0$, identified with the matrix $\gamma \in \mathcal{M}$ as in the proof of Lemma 3.5. Choose another lattice $\Lambda' \in \mathcal{D}_0$, with associated matrix γ', and define the curve

$$C := \{(1-x)\gamma + x\gamma' \mid x \in \mathbb{C}\} \subset \mathcal{D}_0.$$

Lemma 3.8. *$\pi(C)$ is a rational curve.*

Proof. Indeed, there's an obvious birational map $\mathbb{P}^1 \dashrightarrow C$ of ind-schemes (for $\mathbb{P}^1 = \varinjlim \mathbb{P}^1$ tautologically), which yields a rational dominant map $\mathbb{P}^1 \dashrightarrow \pi(C)$. $\square$

It follows from Lemmas 3.7 and 3.8 that $\mathcal{D}$ is rationally connected (cf. [11, Ch. IV, Proposition 3.6]). Thus it remains to prove normality.

Firstly, the proof of [1, Proposition 6.1] shows that $\mathcal{M}$ is smooth in codimension 1, which implies that $\mathcal{M}$ is normal because it is a complete intersection on $\mathrm{SL}_r(\mathbb{C}[[t]]) \subset \mathrm{SL}_r(K)$ (more precisely, $\mathcal{M}$ is a direct limit of finite-dimensional complete intersections, which are smooth in codimension 1, hence normal by Serre's criterion). Then $\mathcal{D}_0 \subset \mathcal{Q}$ is also normal for $\mathrm{SL}_r(\mathcal{O})$ acting freely on $\mathrm{SL}_r(K) \supset \mathcal{M}$.

Secondly, since $\mathrm{rk}\,\Lambda > 1$ for any lattice $\Lambda \in \mathcal{D}_0$, the condition $\Lambda = g \cdot \Lambda$ for some $g \in \mathrm{SL}_r(\mathcal{O}_{X^*}) \setminus \{\mathrm{id}\}$ is of codimension > 1 on $\mathcal{D}_0$. Thus we get $\mathcal{D}_0 = \mathcal{D} \times \mathrm{SL}_r(\mathcal{O}_{X^*})$ and $\mathcal{D}$ is normal – all in codimension 1.

Finally, if $\mathcal{L}$ is the ample generator of $\mathrm{Pic}\,\mathcal{SU}_X(r)$ (see **2.5**), it follows from the quotient construction of $\mathcal{D}$ that any function in $\mathbb{C}(\mathcal{D})$ can be represented as a ratio of global sections of various restrictions $\mathcal{L}^c\big|_{\mathcal{D}}, c \gg 1$. In particular, the S_2 property is satisfied for all the rational functions on $\mathcal{D}$, and so $\mathcal{D}$ is normal by Serre's criterion.

Proposition 3.6 is completely proved. $\square$

From (the proof of) Proposition 3.6 we deduce

Corollary 3.9. *$\mathcal{D} \subset \mathcal{SU}_X(r)$ has codimension $\leq r^2$ (hence in particular $\dim \mathcal{D} \geq (r^2-1)(g-1) - r^2$ can be made arbitrarily large).*

Proof. This follows from the fact that the locus $\mathcal{M} \subset \mathrm{SL}_r(\mathbb{C}[[t]])$ is defined by equation $\Theta_1 = 0$ for generic matrix $I + \sum_{i=1}^{\infty} t^i \Theta_i \in \mathrm{SL}_r(\mathbb{C}[[t]])$ (cf. **3.4** and Lemma 2.2). $\square$

3.10. Now let's turn to the proof of Theorem 1.2. Suppose that $Z \subset \mathcal{D}$ is a smooth rational curve. Associating with any point $E \in Z$ the lattice Λ as in **2.1** yields an affine bundle $\mathcal{V}$ over Z. More specifically, since every Λ carries the canonical basis of Proposition 3.2, using it one identifies Λ with the affine space $\mathbb{C}^r$.

Further, since $Z = \mathbb{P}^1$ and the bundle $\mathcal{V}$ is locally analytically trivial by construction, we get $\mathcal{V} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i)$ for some $d_i \in \mathbb{Z}$. Note that $\sum d_i = \deg \mathcal{V} = 0$ for $\mathrm{Vol}\,\Lambda = 1$. Then taking a global section of $\mathcal{V}$ gives an embedding $Z \subset \mathcal{D}_0$.

Thus we obtain a non-constant family of matrices $\gamma \in \mathrm{SL}_r(K)$ *algebraically* parameterized by Z. In particular, there exists a rational function $f \in \mathbb{C}(Z)$ such that $f(z) \neq \infty$ for all $z \in Z$, which is absurd. The proof of Theorem 1.2 is complete (cf. Proposition 3.6 and Corollary 3.9).

4. Some questions and comments

We'd like to conclude the paper by asking the following questions:

- Is there a weird V over a field of positive characteristic (cf. [3], [4])? Similarly, modifying the notion of rational connectivity accordingly, is there a non-Kähler compact complex (maybe even smooth) weird V (cf. [16])?
- Is the locus $\mathcal{D} \subset \mathcal{SU}_X(r)$ a Fano variety (and how to express it in terms of theta-divisors)? Does it have locally factorial singularities (resp. what is its Picard group)? Same questions for any weird V.
- By applying the weak factorization theorem it would be interesting to find out whether being weird provides an obstruction for variety to be rational. What about the case of $\mathcal{D}$ again (cf. Remark 1.5)?
- Note that the locus $\mathcal{D}$ consists entirely of *strictly* semi-stable bundles (cf. Remark 3.3). In particular, when $r = 2$, writing any $E \in \mathcal{D}$ as a sum of two line bundles one finds that there is a $2:1$-cover $\mathrm{Pic}^0(X) \longrightarrow \mathcal{D}$ (i.e. $\mathcal{D}$ is the *Kummer variety*).[3)] Is the same true for an arbitrary $r > 2$ (with $\mathrm{Pic}^0(X)$ replaced by an appropriate Abelian variety and "$2:1$" by "generically Galois")? Similar question for any weird V.
- Is it possible to find weird V in *any* given dimension (cf. Corollary 3.9)?

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[3)] This example shows that the *Assumption* made in **3.4** is crucial for $\mathcal{D}$ to be rationally connected. Indeed, for $r = 2$ the locus $\mathcal{D}$ is not rationally connected (as $h^0(\mathcal{D}, K_{\mathcal{D}}^m) = 1, m \gg 1$), and the reason is that one lacks the canonical correspondence $\Lambda \leftrightarrow \gamma$ here (cf. the proof of Lemmas 3.7 and 3.8).

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