A 4D-2D equivalence for large-N Yang-Mills theory

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General string-theoretic considerations suggest that four-dimensional large-N gauge theories should have dual descriptions in terms of two-dimensional conformal field theories. However, for non-supersymmetric confining theories such as pure Yang-Mills theory, a long-standing challenge has been to explicitly show that such dual descriptions actually exist. In this paper, we consider the large-N limit of four-dimensional pure Yang-Mills theory compactified on a three-sphere in the solvable limit where the sphere radius is small compared to the strong length scale, and demonstrate that the confined-phase spectrum of this gauge theory coincides with the spectrum of an irrational two-dimensional conformal field theory.

Introduction. Confining gauge theories in the large-N limit are believed to have dual descriptions as weakly-coupled string theories [1]. Since string theories have 2D worldsheet conformal field theory (CFT) descriptions, it is therefore expected that confining 4D gauge theories may have alternative descriptions based on 2D CFTs. However, for non-supersymmetric quantum field theories (QFTs) such as Yang-Mills (YM) theory, no concrete relation between large-N confining theories and 2D CFTs has ever been found.

In this paper we tackle this problem by studying the large-N limit of 4D pure $SU(N)$ YM theory, formulated at temperature $T = \beta^{-1}$ and compactified on a three-sphere S^3 of radius R. One can thus view the theory as living on $S^3_R \times S^1_\beta$ with Euclidean metric signature. The virtues of this setting are two-fold. First, thanks to asymptotic freedom, if we take $\Lambda R \ll 1$ where Λ is the YM strong scale, then the 't Hooft coupling $\lambda \equiv g^2 N$ becomes small — *i.e.* $\lambda(1/R) \to 0$. As a result, the theory becomes solvable for any temperature $\beta \sim N^0$. Second, it is known [2] that large-N YM theory stays in the confined phase when $\beta/R \gtrsim 1$, even when $\lambda \to 0$. In this context "confinement" means that the system has an unbroken center symmetry and that its free energy scales as N^0. As sketched in Fig. 1, it is plausible that the physics of YM theory is smooth as a function of ΛR. Thus, the $\Lambda R \ll 1$ regime of the large-N confined phase represents a particularly tractable 4D starting point in our search for a dual 2D description.

Rather than attempt a string-theory construction of a 2D dual for large-N YM theory, we shall instead analyze the confined-phase spectrum of YM theory in the solvable $\Lambda R \ll 1$ limit. Remarkably, we find that a surprisingly simple 2D CFT description emerges. Thus, in this limit, we conclude that the large-N confined-phase spectrum of 4D YM theory coincides with the spectrum of a 2D CFT.

Specifically, recall that the spectrum of a QFT is encoded in its thermal partition function. We take 4D YM theory to be minimally coupled to the S^3 metric, so that

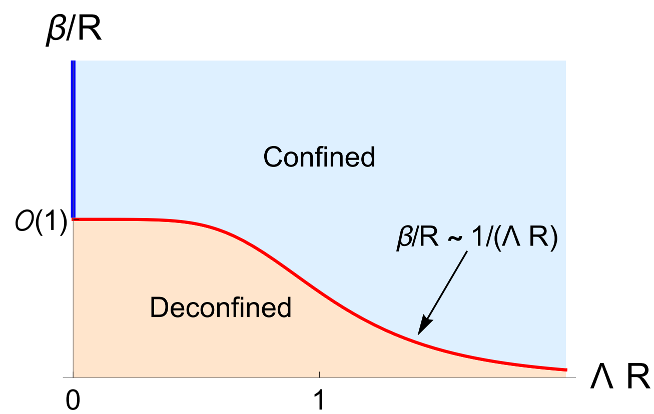

FIG. 1. A conjectured phase diagram for large-N YM theory on $S^3 \times S^1$. In the analytically tractable regime $\Lambda R \ll 1$, the deconfinement transition occurs at $\beta \sim R$, while for $\Lambda R \gg 1$, lattice studies have shown that it occurs at $\beta \sim 1/\Lambda$. This sketch illustrates the natural conjecture that these two limiting cases are smoothly connected. The results of this paper apply in the $\Lambda R \to 0$ region indicated by the blue line.

the Kaluza-Klein energies on the three-sphere are given by $E_n = n/R$ in the $\lambda \to 0$ limit [2]. The partition function then takes the form

$$Z_{\rm YM}(\beta/R) \;=\; \sum_{n=0}^{\infty} d_n e^{-\beta E_n} \;=\; \sum_{n=0}^{\infty} d_n q^n \qquad (1)$$

where $q = e^{-\beta/R}$ and d_n counts the number of states with energy E_n. Our main result, then, will be the demonstration that $Z_{\rm YM}$ coincides with a chiral partition function of a 2D CFT:

$$Z_{\rm YM}(\tau) \;=\; Z_{\rm 2D}(\tau)\;. \qquad (2)$$

In writing Eq. (2), we have analytically continued q to $e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the complex upper half-plane. Thus ${\rm Im}\,\tau = \beta/(2\pi R)$. It can also be shown that ${\rm Re}\,\tau = \mu_I \beta/(2\pi)$, where $\mu_I \in \mathbb{R}$ is an imaginary chemical potential for a combination of the S^3 angular momenta.

The 4D partition function. We begin by briefly explaining the computation of $Z_{\rm YM}$, leaving a more leisurely exposition to Ref. [3]. To calculate the 4D partition function $Z_{\rm YM}(\tau)$, we take the large-N limit with Λ held fixed, which means taking the continuum limit after the large-N limit. Working on $S^3\times S^1$, we allow an imaginary chemical potential μ_I for the Cartan $Q_L^3+Q_R^3$ charges associated to the two $SU(2)$ angular momenta on S^3, which has the isometry group $SO(4)\simeq SU(2)_L\times SU(2)_R$, and assume that $\beta, R, \mu_I\sim N^0$. We will not consider states with energies $\gtrsim N$ because they lie beyond our UV cutoff. As is typical in studies of large-N theories, we shall work with the $U(N)$ version of YM theory rather than the $SU(N)$ version [4]. When $\Lambda R\to 0$, the microscopic degrees of freedom of YM theory reduce to an infinite collection of color-adjoint-valued harmonic oscillators. These oscillators are counted by the massless-vector partition function, which can be written as $z_v(\tau)=(6q^2-2q^3)/(1-q)^3$. The physical states are then determined by imposing the color Gauss law. In the $\lambda=0$ confined phase, the physical single-particle states can be identified with single-trace operators, and their energies are proportional to their scaling dimensions. The counting problem for these states, and also for the multi-particle states, has been solved [2, 5], and the resulting grand-canonical confined-phase partition function is given by

$$Z_{\rm YM}(\tau)=\prod_{n=1}^{\infty}\frac{1}{1-z_v(q^n)}=\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{1-3q^n-3q^{2n}+q^{3n}} \\ =1+6q^2+16q^3+72q^4+\ldots \tag{3}$$

As expected in any confining large-N theory, we find that the d_n grow exponentially for large n. Thus, there are Hagedorn singularities in $Z_{\rm YM}(\tau)$. In Eq. (3), we find $d_n\sim e^{Cn}$ and $E_n\sim n$ for large n, with $C\equiv\log(2+\sqrt{3})\approx 1.317$. This contrasts with the behaviors $d_n\sim e^{\sqrt{n}}$ and $E_n\sim\sqrt{n}$ that would arise for a string theory with a flat target space. Of course, we are not in flat space: the spacetime curvature is $\sim 1/R$, which is of the same scale as the effective string tension $\alpha'\sim 1/R^2$ that follows from our spectrum. The scaling properties of d_n in Eq. (3) imply that the leading Hagedorn singularity of $Z_{\rm YM}(\beta,\mu_I)$ is at $\beta_H/R=C, \mu_I=0$. Consequently, at $\mu_I=0$, there must be a phase transition to a deconfined phase at β_H. This is discussed in detail in Refs. [2, 6].

Modular symmetries. We now observe that the denominator in Eq. (3) can be factorized with roots that are inverses of each other:

$$1-3q^n-3q^{2n}+q^{3n} \;=\; (1+q^n)(1-q^n z)(1-q^n/z) \tag{4}$$

where $z=2+\sqrt{3}$. This pivotal algebraic observation was first made in Ref. [7] in the context of uncovering a subtle "temperature-reflection" symmetry for $Z_{\rm YM}$. For our purposes, however, the key point is that this allows $Z_{\rm YM}$ to be written as

$$Z_{\rm YM}=\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1+q^n)(1-q^n z)(1-q^n z^{-1})}\,. \tag{5}$$

This observation is very important because the structure of Eq. (5) matches the structure of the product representations of the Dedekind η-function and generalized Jacobi ϑ-functions. (In the related context of adjoint QCD, this was also noted in Ref. [8].) Specifically, the Dedekind η-function has the product representation $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, while the generalized ϑ-function $\vartheta\left[{\alpha \atop \beta}\right](\tau)\equiv\sum_{n\in\mathbb{Z}}q^{(n+\alpha)^2/2}e^{2\pi i n\beta}$ has a product representation of the form

$$\vartheta\left[{\alpha \atop \beta}\right](\tau)=q^{\alpha^2/2}\prod_{n=1}^{\infty}\Big[(1-q^n) \\ \times(1+q^{n-\frac{1}{2}+\alpha}e^{2i\pi\beta})(1+q^{n-\frac{1}{2}-\alpha}e^{-2i\pi\beta})\Big]\,. \tag{6}$$

Under the $S:\tau\to-1/\tau$ and $T:\tau\to\tau+1$ generators of the modular group $SL(2,\mathbb{Z})$, we find $\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau)$ and $\eta(\tau+1)=e^{i\pi/12}\,\eta(\tau)$, while

$$\begin{aligned} S:&\quad \vartheta\left[{\alpha \atop \beta}\right](-1/\tau)=\sqrt{-i\tau}\,e^{-2\pi i\alpha\beta}\vartheta\left[{-\beta \atop \alpha}\right](\tau)\,,\\ T:&\quad \vartheta\left[{\alpha \atop \beta}\right](\tau+1)=e^{i\pi\alpha^2}\vartheta\left[{\alpha \atop \beta+\alpha+1/2}\right](\tau)\,. \end{aligned} \tag{7}$$

Given these definitions, the structure of Eq. (5) allows us to rewrite the 4D partition function $Z_{\rm YM}$ as a finite product of Dedekind η-functions and Jacobi ϑ-functions:

$$Z_{\rm YM}(\tau) \;=\; \eta(\tau)^3\left(\frac{-\sqrt{2}e^{-i\pi b}\eta(\tau)}{\vartheta\left[{1/2 \atop b+1/2}\right](\tau)}\right)\sqrt{\frac{2\,\eta(\tau)}{\vartheta_2(\tau)}} \tag{8}$$

where $b=i\log(z)/2\pi\approx 0.21i$, where $\vartheta_2(\tau)\equiv\vartheta\left[{1/2 \atop 0}\right](\tau)$, and where the identity $2\eta(2\tau)^2=\eta(\tau)\vartheta_2(\tau)$ has been used in passing from Eq. (5) to Eq. (8). The fact that b is imaginary is the reason the degeneracy factors d_n in Eq. (3) grow as $d_n\sim e^{Cn}$. The expression in Eq. (8) — and our interpretation of this expression in terms of specific 2D CFTs, as discussed below — are the key results of our paper, with many striking consequences.

Modularity versus dimensionality. The first interesting implication of Eq. (8) becomes apparent upon realizing that it is extremely unusual for the partition function of a 4D theory to be expressible as a finite product of modular functions, as in Eq. (8). The large-$|\tau|$ behavior of a modular function is tied, through the S modular transformation, to its behavior near $|\tau|=0$. For example, the Dedekind η-function has the large-$|\tau|$ expansion $\eta(\tau)=q^{1/24}(1-q+\ldots)$; the S transformation then requires this function to behave at small $|\tau|$ as $\eta(\tau)\sim\exp[-i\pi/(12\tau)]/\sqrt{-i\tau}$. Similar statements can be made for the ϑ-functions. Thus, if a partition function can be written as a finite product of modular η-functions and ϑ-functions, then it must have the leading behavior

$$\lim_{\arg\tau\to\pi/2}\left[\lim_{|\tau|\to 0}\log Z_{\rm modular}(\tau)\right] \;=\; \sigma R/\beta \qquad (9)$$

for a constant σ. This amounts to the statement that $\log Z_{\rm modular}\sim T$ as $T\to\infty$. This is indeed the expected behavior for a 2D QFT. However, it is certainly *not* the expected behavior for a 4D QFT, for which we generically expect

$$\log Z_{\rm 4D}\sim T^3 \quad \text{ as } T\to\infty\;. \qquad (10)$$

In this sense, 4D QFTs whose partition functions can be written in terms of modular functions behave as if they were 2D QFTs, since they follow Eq. (9) rather than Eq. (10).

In our case, $|\tau| = \frac{\beta}{2\pi R}\sqrt{1+(\mu_I R)^2}$ and $\arg\tau = \cot^{-1}(\mu_I R)$. If we were to reverse the order of limits on the left side of Eq. (9) and take the $T\to\infty$ limit with $\mu_I=0$, pure YM theory would follow the scaling in Eq. (10). Such a limit cannot be studied from Eq. (3) due to the Hagedorn singularities, and the physics is governed by the deconfined phase. But with the order of limits indicated in Eq. (9), which amount to taking $\beta/R\to 0$ before $\mu_I R\to 0$, Eq. (9) holds for pure YM theory. Note that in other theories such as adjoint QCD with periodic boundary conditions for fermions, the Hagedorn singularities do not lie along $\arg\tau=\pi/2$; the two limits then commute [8] and these theories exhibit 2D behavior in the sense of Eq. (9) irrespective of the order of limits.

Vacuum energy. Another major consequence of Eq. (8) is that the modular properties of the η- and ϑ-functions fix the vacuum energy $E_{\rm YM}$ of our large-N YM theory to be zero.

To see this, we first recall that if we write the q-series expansion of a modular function $f(\tau)$ in the form $f = q^\Delta\sum_{n=0}^\infty a_n q^n$, then Δ can be thought of as the 2D vacuum energy. Its value is fixed by the modular properties of f and tied to the values of a_n. Were one to abitrarily shift $\Delta\to\Delta+c$, the modular properties of $f(\tau)$ would be ruined because the S-transformation would map $q^c=e^{(2\pi i\tau)c}$ to $e^{(-2\pi i/\tau)c}$, thereby preventing $q^c f(\tau)$ from transforming as a modular function.

Next, we observe that the vacuum energy associated to the η-function is $1/24$, while $\vartheta\left[{a \atop b}\right]$ has vacuum energy $a^2/2$. Summing the vacuum energies of the individual modular functions in Eq. (8) we obtain a striking result:

$$E_{\rm YM} \;=\; 0\;. \qquad (11)$$

Indeed, this is the only value consistent with the q-expansion for $Z_{\rm YM}$ given in Eq. (3), provided that $E_{\rm YM}$ is calculated in a renormalization scheme which is consistent with the modular properties of $Z_{\rm YM}$ made evident in Eq. (8). This value, $E_{\rm YM}=0$, coincides with the result implied by T-reflection symmetry [7], and also agrees with a direct evaluation of the sum over the confined-phase spectrum of finite-temperature large-N YM theory compactified on S^3, as performed in Ref. [9].

CFT interpretation. The striking modular structure of Eq. (8) suggests that the spectrum of our 4D YM theory coincides with that of a chiral (*e.g.*, left- or right-moving) 2D CFT. This motivates the central question we shall now explore for the rest of this paper: what is the 2D CFT which gives rise to Eq. (8), and thus gives a 2D description of 4D YM theory in the large-N limit?

Unfortunately, we will not be able to give a complete answer to this question. The reason ultimately has to do with the fact that many distinct CFTs can have coincident spectra without being equivalent. They may differ, for example, in their correlation functions. In general, the most important aspects of a given 2D CFT are governed by its central charge (conformal anomaly) c and its spectrum of operator conformal dimensions h_i, $i=1,...,n$, where n is the number of so-called "primary" fields in the CFT. Along with the explicit traces over states, knowledge of c and the h_i's goes a long way in nailing down relevant aspects of the CFT such as its selection rules and correlation functions. But partition functions are only sensitive to the combinations $h_i^{(\rm eff)}\equiv h_i - c/24$, rather than the values of c and h_i individually. Consequently, without additional assumptions about the CFT in question (such as the assumption of unitarity, which would additionally tell us that $\min\{h_i\}=0$), this represents a fundamental limitation on our ability to specify a unique CFT.

We will therefore answer a different but related question: do there exist any 2D CFTs to which our large-N YM theory is *isospectral*? Remarkably, we shall show that at least one such 2D CFT indeed exists. To see this, we first recall that a free $c=1$ scalar CFT has a left-moving spectrum whose trace is given by $1/\eta(\tau)$, while the $\mathbb{Z}_2$ orbifold of this CFT has a chiral sector whose trace is $(2\eta(\tau)/\vartheta_2(\tau))^{1/2}$. Furthermore, the direct product of two copies of the $c=-26$ bc ghost CFT has a left-moving spectrum whose trace is given by $\eta(\tau)^4$. Perhaps the most challenging to interpret is the remaining factor in Eq. (8), specifically

$$\frac{-\sqrt{2}e^{-i\pi b}\eta(\tau)}{\vartheta\left[{1/2 \atop b+1/2}\right](\tau)}\;. \qquad (12)$$

However, this can be identified as the trace of the chiral (*e.g.*, left-moving) states in the vacuum sector of the $c=2$ bosonic $\beta\gamma$ ghost CFT recently explored in Ref. [10]. This is a logarithmic CFT [11], and it has a $U(1)$ conserved charge. Thus the vacuum-sector chiral partition function of the $c=2$ $\beta\gamma$ CFT depends on the choice of a complex fugacity $z=e^{+\mu\beta}$. To match with our expressions for YM theory, we set $\mu\beta=2\pi i\, b=-\log(2+\sqrt{3})$.

Putting this together, we therefore conclude that the expression in Eq. (8) can be viewed as the trace over the chiral spectrum of a theory which is the direct product of five known CFTs, one of which is irrational. This then justifies the central claim of this paper in Eq. (2): there is indeed an irrational 2D CFT which is isospectral to the finite-temperature large-N 4D YM compactified on S^3 in the $\Lambda R \to 0$ limit. Aside from explaining our observations concerning $E_{\rm YM}$ and the small-$|\tau|$ behavior of $Z_{\rm YM}$, this fact has an intriguing further implication. Two-dimensional CFTs have infinite-dimensional symmetries, which always include the Virasoro symmetry. Eq. (2) then suggests that large-N YM theory has a hidden Virasoro symmetry. It would be very interesting to demonstrate this explicitly within YM theory.

Primary operator spectrum. We now collect information concerning the spectrum of conformal dimensions $h_i^{(\rm eff)}$ corresponding to the primary fields of this tensor-product CFT. Our approach proceeds by determining the diagonal modular-invariant associated with the expression in Eq. (8), and then computing the eigenvalues of the modular T operator to extract $h_i^{(\rm eff)}$.

We begin by defining the quantities

$$T_{m,n} \;\equiv\; \frac{-\sqrt{2}\, e^{-i\pi b n}\,\eta(\tau)^4}{\vartheta\left[{mb\,+\,1/2 \atop nb\,+\,1/2}\right](\tau)} \left(\frac{\eta(\tau)}{\vartheta\left[{P(m)/2 \atop P(n)/2}\right](\tau)}\right)^{1/2} , \quad (13)$$

where $\{m,n\}$ are relatively prime integers (a relationship which we shall henceforth denote $m \perp n$), and $P(k) \equiv \frac{1}{2}(1+(-1)^k), k \in \mathbb{Z}$. Thus $P(k) = 0, 1$ for odd or even k, respectively. The set $\{T_{m,n}\}$ is a basis for a vector space over the field $\mathbb{C}$ with two key properties: it contains the "seed term" in Eq. (8), and it is the minimal set which is closed under the action of the $SL(2,\mathbb{Z})$ modular group.

The first property follows by noting that $T_{0,1}(\tau)$ coincides with Eq. (8). The verification of the second property proceeds in two steps. First, it can be shown that, up to overall phases and extraneous factors of $\sqrt{-i\tau}$, the S and T modular transformations map $T_{m,n}$ to $T_{-n,m}$ and $T_{m,n+m}$, respectively. Second, we observe that if $\{m,n\}$ are relatively prime, then $\{-n,m\}$ and $\{m,n+m\}$ are also relatively prime. Since all modular transformations can be generated by sequences of S and T, it then follows that the full modular "orbit" of our seed term $T_{0,1}$ is contained within the set of coprime integers $\{m,n\}$. Indeed, it is also possible to demonstrate [3] that the modular orbit actually covers *all* coprimes.

As a result, the minimal "diagonal" modular-invariant generated from Eq. (8) is given by

$$Z_{\rm diagonal} \;=\; ({\rm Im}\,\tau)^{3/2} \sum_{m\perp n} \left|T_{m,n}\right|^2 . \quad (14)$$

The appearance of the factor of $({\rm Im}\,\tau)^{3/2}$ is standard when combining holomorphic and anti-holomorphic components, such as our $T_{m,n}$ factors, each of which has modular weight $k = 3/2$. It also ensures that $Z_{\rm diagonal}$ is fully modular-invariant. Moreover, it can be verified numerically that the infinite sum in Eq. (14) converges except for an isolated set of points corresponding to the Hagedorn singularities. The numerical values of $Z_{\rm diagonal}$ on the interior of the unit-q disk are shown in Fig. 2.

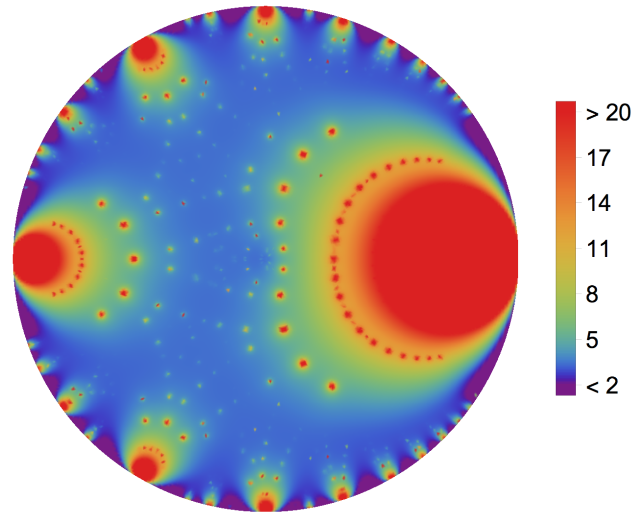

FIG. 2. The numerical values of Eq. (14) with $|m|, |n| \leq 10$, plotted within the unit-q disk.

In order to extract the spectrum of effective conformal dimensions $h_i^{(\rm eff)}$, we now rewrite $Z_{\rm diagonal}$ in a basis of eigenfunctions of the modular $T: \tau \to \tau+1$ operator. We do this because such eigenfunctions $\chi(\tau)$ will have eigenvalues $\exp[2\pi i h_i^{(\rm eff)}]$ under T, allowing us to read off the values of $h_i^{(\rm eff)}$ (mod 1). Fortunately, constructing eigenfunctions of the T-operator from linear combinations of the $T_{m,n}$'s in Eq. (13) is relatively straightforward. Since

$$T_{m,n}(\tau+1) = e^{\pi i \left\{[1-P(m)]/8 + m^2|b|^2\right\}}\, T_{m,n+m}(\tau) , \quad (15)$$

we see that any linear combination which includes $T_{m,n}$ must also include $T_{m,n+m}$, $T_{m,n+2m}$, and indeed all $T_{m,n+km}$ where $k \in \mathbb{Z}$. Our T-invariant linear combinations can therefore be indexed by an arbitrary integer m and a second integer $\ell \perp m$ obeying $0 \leq \ell < m$. Hence T-eigenfunctions can be constructed analogously to Bloch eigenfunctions, by summing over all components $T_{m,\ell+km}$ with $k \in \mathbb{Z}$ with a Bloch phase $\alpha \in [0,1) \subset \mathbb{R}$:

$$\chi_{m,\ell,\alpha} \;=\; \sum_{k\in\mathbb{Z}} e^{2\pi i \alpha k}\, T_{m,\ell+mk} . \quad (16)$$

It then follows that

$$\chi_{m,\ell,\alpha}(\tau+1) \;=\; e^{2\pi i h^{(\rm eff)}_{m,\ell,\alpha}}\, \chi_{m,\ell,\alpha}(\tau) , \quad (17)$$

where

$$h^{(\rm eff)}_{m,\ell,\alpha} \;=\; \frac{1}{2}\left[\frac{1-P(m)}{8} + m^2|b|^2\right] - \alpha . \quad (18)$$

One might wonder whether $\{\chi_{m,\ell,\alpha}\}$ is the complete set of T-eigenfunctions. However, we have verified this by checking that summing over $\chi_{m,\ell,\alpha}$ reproduces Eq. (14):

$$Z_{\text{diagonal}} = (\text{Im}\,\tau)^{3/2} \sum_{m\in\mathbb{Z}} \sum_{\substack{0\leq\ell<m \\ \ell\perp m}} \int_0^1 d\alpha\, |\chi_{m,\ell,\alpha}|^2. \quad (19)$$

This confirms that Eq. (18) is the desired set of effective conformal dimensions (mod 1) of the primary operators in our CFT. The fact that these dimensions depend on α — a continuous real variable — confirms that we are dealing with an *irrational* CFT [12]. Our observations are consistent with the 2D logarithmic CFT interpretation discussed above, since it is known that logarithmic CFTs typically have a continuously infinite number of primary operators [13].

Outlook. We have presented evidence that the confined phase of finite-temperature 4D non-supersymmetric large-N pure Yang-Mills theory compactified on a three-sphere of radius R is isospectral to an irrational 2D CFT in the $\Lambda R \to 0$ limit. This gives credence to the hope and expectation that non-supersymmetric large-N confining gauge theories are dual to 2D CFTs. Moreover, as we shall demonstrate in a separate paper [3], modularity in the sense of Eq. (8) turns out to be a generic property of large-N confined-phase gauge theories with adjoint massless matter in the $\lambda \to 0$ limit. In Ref. [3] we shall also show that this structure is present in the large-N limit of the $\mathcal{N} = 4$ superconformal index.

Our results suggest a large number of interesting topics for future research. For example, it is important to understand whether our large-N 4D-2D spectral equivalence extends to correlation functions, and to explore how it is related to other known 4D-2D relations, such as those discussed in Refs. [14]. It would also be interesting to understand the origin of Eq. (8) within string theory, perhaps by making contact with the ideas in Refs. [15]. Given recent progress in the understanding of the bulk duals of 2D CFTs (see, *e.g.*, Ref. [16]), it is tempting to wonder whether our results may help to uncover the bulk dual of YM theory and of other non-supersymmetric 4D adjoint-matter theories. It would also be interesting to understand whether the continuous spectrum of primary operators in the 2D theory suggested by our analysis has an interpretation in 4D YM theory. Finally, there remains the very important question of determining how our 4D-2D relation might evolve for $\lambda > 0$.

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