Symmetry, Integrability and Geometry: Methods and Applications SIGMA * (201*), ***, 130 pages

On some quadratic algebras I $\frac{1}{2}$: Combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss-Catalan, universal Tutte and Reduced polynomials

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To the memory of Alain Lascoux 1944–2013, the great Mathematician, from whom I have learned a lot about the Schubert and Grothendieck polynomials.

Abstract. We study some combinatorial and algebraic properties of certain quadratic algebras related with dynamical classical and classical Yang– Baxter equations.

Key words: Dunkl an Gaudin elements, Dynamical Yang–Baxter relations; small quantum cohomology of flag varieties; Schubert, Grothendieck, Schröder, Ehrhart and Tutte polynomials; reduced polynomials; Chan–Robbins–Yuen polytope; k-dissections of a convex $(n+k+1)$-gon and Fuss–Catalan polynomials; VSASM and CSTCPP.

Extended Abstract

We introduce and study a certain class of quadratic algebras, which are nonhomogenious in general, together with the distinguish set of mutually commuting elements inside of each, the so-called *Dunkl elements*. We describe relations among the Dunkl elements in the case of a family of quadratic algebras corresponding to a certain splitting of the *universal classical Yang–Baxter relations* into two *three term relations*. This result is a further extension and generalization of analogous results obtained in [26],[76] and [51]. As an application we describe explicitly the set of relations among the Gaudin elements in the group ring of the symmetric group, cf [71]. We also study relations among the Dunkl elements in the case of (nonhomogeneous) quadratic algebras related with the *universal dynamical classical Yang–Baxter relations*. Some relations of results obtained in papers [26], [52], [47] with those obtained in [35] are pointed out. We also identify a subalgebra generated by the generators corresponding to the *simple roots* in the extended Fomin–Kirillov algebra with the *DAHA*, see Section 4.3.

The set of generators of algebras in question, naturally corresponds to the set of edges of the *complete graph* K_n (to the set of edges and loops of the complete graph with loops $\widetilde{K}_n$ in *dynamical* case). More generally, starting from any subgraph Γ of the complete graph with loops $\widetilde{K}_n$ we define a (graded) subalgebra $3T_n^{(0)}(\Gamma)$ of the (graded) algebra $3T_n^{(0)}(\widetilde{K}_n)$ [44]. In the case of loop-less graphs $\Gamma \subset K_n$ we state *Conjecture* which relates the *Hilbert polynomial* of the abelian quotient $3T_n^{(0)}(\Gamma)^{ab}$ of the algebra $3T_n^{(0)}(\Gamma)$ and the *chromatic polynomial* of the graph Γ we started with. We check our Conjecture for the complete graphs K_n and the complete bipartite graphs $K_{n,m}$. Besides, in the case of *complete multipartite graph* $K_{n_1,\ldots,n_r}$, we identify the commutative subalgebra in the algebra $3T_N^{(0)}(K_{n_1,\ldots,n_r})$, $N = n_1 + \cdots + n_r$, generated by elements

$$\theta_{j,k_j}^{(N)} := e_{k_j}(\theta_{N_{j-1}+1}^{(N)}, \ldots, \theta_{N_j}^{(N)}), \ \ 1 \le j \le r, \ \ 1 \le k_j \le n_j, \ N_j := n_1 + \ldots + n_j, \ N_0 = 0,$$

with the cohomology ring $H^*(\mathcal{F}l_{n_1,\ldots,n_r}, \mathbb{Z})$ of the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$. In other words, the set of (additive) Dunkl elements $\{\theta_{N_{j-1}+1}^{(N)}, \ldots, \theta_{N_j}^{(N)}\}$ plays a role of the *Chern roots* of the tautological vector bundles ξ_j, $j = 1, \ldots, r$, over the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$, see Section 4.1.2 for details. In a similar fashion, the set of *multiplicative* Dunkl elements $\{\Theta_{N_{j-1}+1}^{(N)}, \ldots, \Theta_{N_j}^{(N)}\}$ plays a role of the *K-theoretic version of Chern roots* of the tautological vector bundle ξ_j over the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$. As a byproduct for a given set of weights $\ell = \{\ell_{ij}\}_{1 \le i<j \le r}$ we compute the *Tutte polynomial* $T(K_{n_1,\ldots,n_k}^{(\ell)}, x, y)$ of the ℓ-weighted complete multipartite graph $K_{n_1,\ldots,n_k}^{(\ell)}$, see Section 4, Definition 4.1 and Theorem 4.2.
More generally, we introduce *universal Tutte polynomial*

$$T_n(\{q_{ij}\}, x, y) \in \mathbb{Z}[\{q_{ij}\}][x,y]$$

in such a way that for any collection of non-negative integers $\mathbf{m} = \{m_{ij}\}_{1 \le i<j \le n}$ and a subgraph $\Gamma \subset K_n^{(\mathbf{m})}$ of the weighted complete graph on n labeled vertices with each edge $(i,j) \in K_n^{(\mathbf{m})}$ appears with multiplicity m_{ij}, the specialization

$$q_{ij} \longrightarrow 0, \ if \ edge \ (i,j) \notin \Gamma, \ \ q_{ij} \longrightarrow [m_{ij}]_y := \frac{y^{m_{ij}}-1}{y-1}, \ if \ edge \ (i,j) \in \Gamma$$

of the universal Tutte polynomial is equal to the Tutte polynomial of graph Γ multiplied by $(x-1)^{\kappa(\Gamma)}$, see Section 4.1.2, Theorem 4.3, and *Comments and Examples*, for details.

We also introduce and study a family of *(super)* 6*-term relations* algebras, and suggest a definition of " multiparameter quantum deformation " of the algebra of the curvature of 2-forms of the Hermitian linear bundles over the complete flag variety $\mathcal{F}l_n$. This algebra can be treated as a natural generalization of the (multiparameter) quantum cohomology ring $QH^*(\mathcal{F}l_n)$, see Section 4.2.

Yet another objective of our paper is to describe several combinatorial properties of some special elements in the associative quasi-classical Yang–Baxter algebra [47], including among others the so-called *Coxeter element* and the *longest element*. In the case of *Coxeter element* we relate the corresponding reduced polynomials introduced in [90], with the β-Grothendieck polynomials [27] for some special permutations $\pi_k^{(n)}$. More generally, we identify the β-Grothendieck polynomial $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}(X_n)$ with a certain weighted sum running over the set of k-dissections of a convex $(n+k+1)$-gon. In particular we show that the specialization $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}(1)$ of the β-Grothendieck polynomial $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}(X_n)$ counts the number of *k-dissections* of a convex $(n+k+1)$-gon according to the number of diagonals involved. When the number of diagonals in a k-dissection is the maximal possible (equals to $n(2k-1)-1$), we recover the well-known fact that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the value of a certain Catalan-Hankel determinant, see e.g. [85].

We also show that for a certain 5-parameters family of vexillary permutations, the specialization $x_i = 1, \forall i \ge 1$, of the corresponding β-Schubert polynomials $\mathfrak{S}_w^{(\beta)}(X_n)$ turns out to be coincide either with the Fuss-Narayana polynomials and their generalizations, or with a (q,β)-deformation of $VSASM$ or that of $CSTCPP$ numbers, see Corollary 5.2, (**B**).. As examples we show that

(**a**) the reduced polynomial corresponding to a monomial $x_{12}^{n}\ x_{23}^{m}$ counts the number of (n,m)-*Delannoy paths* according to the number of NE-steps, see Lemma 5.2;

(**b**) if $\beta=0$, the reduced polynomial corresponding to monomial $(x_{12}\ x_{23})^{n}\ x_{34}^{k},\ n\geq k$, counts the number of of n up, n down permutations in the symmetric group $\mathbb{S}_{2n+k+1}$, see Proposition 5.9; see also Conjecture 18.

We also point out on a conjectural connection between the sets of *maximal* compatible sequences for the permutation $\sigma_{n,2n,2,0}$ and that $\sigma_{n,2n+1,2,0}$ from one side, and the set of $VSASM(n)$ and that of $CSTCPP(n)$ correspondingly, from the other, see Comments 5.7 for details. Finally, in Section 5.1.1 we introduce and study a multiparameter generalization of reduced polynomials introduced in [90], as well as that of the Catalan, Narayana and (small) Schröder numbers.

In the case of the *longest element* we relate the corresponding reduced polynomial with the Ehrhart polynomial of the Chan–Robbins–Yuen polytope, see Section 5.3. More generally, we relate the (t,β)-reduced polynomial corresponding to monomial

$$\prod_{J=1}^{n-1} x_{j,j+1}^{a_j} \prod_{j=2}^{n-2} \Big(\prod_{k=j+2}^{n} x_{jk} \Big), \quad a_j \in \mathbb{Z}_{\geq 0},\ \forall j,$$

with positive t-deformations of the Kostant partition function and that of the Ehrhart polynomial of some flow polytopes, see Section 5.3.

Contents

1 Introduction

The Dunkl operators have been introduced in the later part of 80's of the last century by Charles Dunkl [21], [22] as a powerful mean to study of harmonic and orthogonal polynomials related with finite Coxeter groups. In the present paper we don't need the definition of Dunkl operators for arbitrary (finite) Coxeter groups, see e.g. [21], but only for the special case of the symmetric group $\mathbb{S}_n$.

Definition 1.1. *Let $P_n = \mathbb{C}[x_1,\ldots,x_n]$ be the ring of polynomials in variables $x_1,\ldots,x_n$. The type A_{n-1} (additive) rational Dunkl operators $D_1,\ldots,D_n$ are the differential-difference operators of the following form*

$$D_i = \lambda \, \frac{\partial}{\partial x_i} + \sum_{j\neq i} \frac{1-s_{ij}}{x_i-x_j}, \tag{1.1}$$

Here s_{ij}, $1 \le i < j \le n$, denotes the exchange (or permutation) operator, namely,

$$s_{ij}(f)(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) = f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n);$$

$\frac{\partial}{\partial x_i}$ stands for the derivative w.r.t. the variable x_i; $\lambda \in \mathbb{C}$ is a parameter.

The key property of the Dunkl operators is the following result.

Theorem 1.1. *(C.Dunkl [21]) For any finite Coxeter group (W,S), where $S=\{s_1,\ldots,s_l\}$ denotes the set of simple reflections, the Dunkl operators $D_i := D_{s_i}$ and $D_j := D_{s_j}$ pairwise commute: $D_i \; D_j = D_j \; D_i$, $1 \le i,j \le l$.*

Another fundamental property of the Dunkl operators which finds a wide variety of applications in the theory of integrable systems, see e.g. [36], is the following statement:
the operator

$$\sum_{i=1}^{l} \, (D_i)^2$$

"essentially" coincides with the Hamiltonian of the rational Calogero–Moser model related to the finite Coxeter group (W,S).

Definition 1.2. *Truncated (additive) Dunkl operator (or the Dunkl operator at critical level), denoted by $\mathcal{D}_i$, $i=1,\ldots,l$, is an operator of the form (1.1) with parameter $\lambda = 0$.*

For example, the type A_{n-1} rational truncated Dunkl operator has the following form

$$\mathcal{D}_i = \sum_{j\neq i} \frac{1-s_{ij}}{x_i-x_j}.$$

Clearly the truncated Dunkl operators generate a commutative algebra.
The important property of the truncated Dunkl operators is the following result discovered and proved by C.Dunkl [22]; see also [4] for a more recent proof.

Theorem 1.2. *(C.Dunkl [22], Y.Bazlov [4]) For any finite Coxeter group (W,S) the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_1,\ldots,\mathcal{D}_l$ is canonically isomorphic to the coinvariant algebra $\mathcal{A}_W$ of the Coxeter group (W,S).*

Recall that for a finite *crystallographic* Coxeter group (W,S) the coinvariant algebra $\mathcal{A}_W$ is isomorphic to the cohomology ring $H^*(G/B,\mathbb{Q})$ of the flag variety G/B, where G stands for the Lie group corresponding to the crystallographic Coxeter group (W,S) we started with.

Example 1.1. In the case when $W = \mathbb{S}_n$ is the symmetric group, Theorem 1.2 states that the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_i = \sum_{j\neq i} \frac{1-s_{ij}}{x_i-x_j}$, $i=1,\ldots,n$, is canonically isomorphic to the cohomology ring of the full flag variety $\mathcal{F}l_n$ of type A_{n-1}

$$\mathbb{Q}[\mathcal{D}_1,\ldots,\mathcal{D}_n] \cong \mathbb{Q}[x_1,\ldots,x_n]/J_n, \tag{1.2}$$

where J_n denotes the ideal generated by the elementary symmetric polynomials $\{e_k(X_n), 1 \le k \le n\}$.

Recall that the elementary symmetric polynomials $e_i(X_n)$, $i = 1, \ldots, n$, are defined through the generating function

$$1 + \sum_{i=1}^{n} e_i(X_n)\ t^i = \prod_{i=1}^{n} (1 + t\ x_i),$$

where we set $X_n := (x_1, \ldots, x_n)$. It is well-known that in the case $W = \mathbb{S}_n$, the isomorphism (1.2) can be defined over the ring of integers $\mathbb{Z}$. ■

Theorem 1.2 by C.Dunkl has raised a number of natural questions:

(**A**) What is the algebra generated by the <u>truncated</u>

- trigonometric,
- elliptic,
- super, matrix, ...,

(a) additive Dunkl operators ?

(b) Ruijsenaars–Schneider–Macdonald operators ?

(c) Gaudin operators ?

(**B**) Describe commutative subalgebra generated by the Jucys–Murphy elements in

- the group ring of the symmetric group;
- the Hecke algebra ;
- the Brauer algebra, BMV algebra, ...

(**C**) Does there exist an analogue of Theorem 1.2 for

- Classical and quantum equivariant cohomology and equivariant K-theory rings of the partial flag varieties ?
- Cohomology and K-theory rings of affine flag varieties ?
- Diagonal coinvariant algebras of finite Coxeter groups ?
- Complex reflection groups ?

The present paper is an extended Introduction to a few items from Section 5 of [47].

The main purpose of my paper "On some quadratic algebras, II" is to give some partial answers on the above questions basically in the case of the symmetric group $\mathbb{S}_n$.

The purpose of the <u>present paper</u> is to draw attention to an interesting class of nonhomogeneous quadratic algebras closely connected (still mysteriously !) with different branches of Mathematics such as

Classical and Quantum Schubert and Grothendieck Calculi,

Low dimensional Topology,

Classical, Basic and Elliptic Hypergeometric functions,

Algebraic Combinatorics and Graph Theory,

Integrable Systems,

... .

What we try to explain in [47] is that upon passing to *a suitable representation* of the quadratic algebra in question, the subjects mentioned above, are a manifestation of certain general properties of that quadratic algebra.

From this point of view, we treat the commutative subalgebra generated by the additive (resp. multiplicative) truncated Dunkl elements in the algebra $3T_n(\beta)$, see Definition 3.1, as *universal cohomology* (resp. *universal K-theory*) ring of the complete flag variety $\mathcal{F}l_n$. The classical or quantum cohomology (resp. the classical or quantum K-theory) rings of the flag variety $\mathcal{F}l_n$ are certain quotients of that *universal ring.*

For example, in [50] we have computed relations among the (truncated) Dunkl elements $\{\theta_i,\ i = 1, \ldots, n\}$ in the *elliptic representation* of the algebra $3T_n(\beta = 0)$. We **expect** that the commutative subalgebra obtained is isomorphic to *elliptic cohomology ring* (not defined yet, but see [33] , [32]) of the flag variety $\mathcal{F}l_n$.

Another example from [47]. Consider the algebra $3T_n(\beta=0)$.
One can prove [47] the following *identities* in the algebra $3T_n(\beta=0)$

(A) Summation formula

$$\sum_{j=1}^{n-1}\Big(\prod_{b=j+1}^{n-1} u_{b,b+1}\Big)\, u_{1,n}\, \Big(\prod_{b=1}^{j-1} u_{b,b+1}\Big) = \prod_{a=1}^{n-1} u_{a,a+1}.$$

(B) Duality transformation formula Let $m \le n$, then

$$\sum_{j=m}^{n-1}\Big(\prod_{b=j+1}^{n-1} u_{b,b+1}\Big)\Big[\prod_{a=1}^{m-1} u_{a,a+n-1}\; u_{a,a+n}\Big]\, u_{m,m+n-1}\,\Big(\prod_{b=m}^{j-1} u_{b,b+1}\Big) +$$
$$\sum_{j=2}^{m}\Big[\prod_{a=j}^{m-1} u_{a,a+n-1}\; u_{a,a+n}\Big]\, u_{m,n+m-1}\,\Big(\prod_{b=m}^{n-1} u_{b,b+1}\Big)\, u_{1,n} \quad =$$
$$\sum_{j=1}^{m}\Big[\prod_{a=1}^{m-j} u_{a,a+n}\; u_{a+1,a+n}\Big]\,\Big(\prod_{b=m}^{n-1} u_{b,b+1}\Big)\,\Big[\prod_{a=1}^{j-1} u_{a,a+n-1}\; u_{a,a+n}\Big].$$

One can check that upon passing to the *elliptic representation* of the algebra $3T_n(\beta=0)$, see Section 3.1, or [47], Section 5.1.7, or [50], for the definition of *elliptic representation*, the above identities (**A**) and (**B**) finally end up correspondingly, to be the *Summation formula* and the $N=1$ case of the *Duality transformation formula* for multiple elliptic hypergeometric series (of type A_{n-1}), see e.g. [41] , or Appendix V, for the explicit forms of the latter. After passing to the so-called *Fay representation* [47], the identities (**A**) and (**B**) become correspondingly to be the Summation formula and Duality transformation formula for the Riemann theta functions of genus $g>0$, [47]. These formulas in the case $g \ge 2$ seems to be new.
Worthy to mention that the relation (**A**) above can be treated as a "non-commutative analogue" of the well-known recurrence relation among the *Catalan numbers*. The study of "descendent relations" in the quadratic algebras in question was originally motivated by the author attempts to construct a *monomial basis* in the algebra $3T_n^{(0)}$. This problem is still widely open, but gives rise the author to discovery of

several interesting connections with

- classical and quantum Schubert and Grothendieck Calculi,
- combinatorics of reduced decomposition of some special elements in the symmetric group,
- combinatorics of generalized *Chan–Robbins–Yuen* polytopes,
- relations among the Dunkl and Gaudin elements,
- computation of Tutte and chromatic polynomials of the weighted complete multipartite graphs, *etc.*

■

A few words about the content of the present paper.

Example 1.1 can be viewed as an illustration of the main problems we are treaded in Sections 2 and 3 of the present paper, namely the following ones.

- Let $\{u_{ij},\ 1 \le i,j \le n\}$ be a set of generators of a certain algebra over a commutative ring K. The first **problem** we are interested in is to describe "a natural set of relations" among the generators $\{u_{ij}\}_{1 \le i,j \le n}$ which implies the pair-wise commutativity of *dynamical Dunkl elements*

$$\theta_i = \theta_i^{(n)} =: \sum_{j=1}^{n} u_{ij}, \quad 1 \le i`len.$$

- Should this be the case then we are interested in to describe the algebra generated by "the integrals of motions", i.e. to describe the quotient of the algebra of polynomials $K[y_1,\ldots,y_n]$ by the two- sided ideal $\mathcal{J}_n$ generated by non-zero polynomials $F(y_1,\ldots,y_n)$ such that $F(\theta_1,\ldots,\theta_n)=0$ in the algebra over ring K generated by the elements $\{u_{ij}\}_{1\le i,j\le n}$.

- We are looking for a set of additional relations which imply that the values of elementary symmetric polynomials $e_k(y_1,\ldots,y_n)$, $1\le k\le n$, on the Dunkl elements $\theta_1^{(n)},\ldots,\theta_n^{(n)}$ **do not depend** on the variables $\{u_{ij},\ 1\le i\ne j\le n\}$. If so, one can defined *deformation* of elementary symmetric polynomials, and make use of it and the Jacobi–Trudi formula, to define deformed Schur functions, for example. We try to realize this program in Sections 2 and 3.

In Section 2, see Definition 2.2, we introduce the so-called *dynamical classical Yang–Baxter algebra* as "a natural quadratic algebra" in which the Dunkl elements form a pair-wise commuting family. It is the study of the algebra generated by the (truncated) Dunkl elements that is the main objective of our investigation in [47] and the present paper. In subsection 2.1 we describe few representations of the dynamical classical Yang–Baxter algebra $DCYB_n$ related with

- quantum cohomology $QH^*(\mathcal{F}l_n)$ of the complete flag variety $\mathcal{F}l_n$, cf [25];
- quantum equivariant cohomology $QH^*_{T^n\times C^*}(T^*\mathcal{F}l_n)$ of the cotangent bundle $T^*\mathcal{F}l_n$ to the complete flag variety, cf [35];
- Dunkl–Gaudin and Dunkl–Uglov representations, cf [71], [94].

In Section 3, see Definition 3.1, we introduce the algebra $3HT_n(\beta)$, which seems to be the most general (noncommutative) deformation of the (even) Orlik–Solomon algebra of type A_{n-1}, such that it's still possible to describe relations among the Dunkl elements, see Theorem 3.1. As an application we describe explicitly a set of relations among the (additive) Gaudin / Dunkl elements, cf [71].

►► It should be stressed at this place that we treat the Gaudin elements/operators (either additive or multiplicative) as *images* of the universal Dunkl elements/operators (additive or multiplicative) in the *Gaudin representation* of the algebra $3HT_n(0)$. There are several other important representations of that algebra, for example, the Calogero–Moser, Bruhat, Buchstaber–Felder–Veselov (elliptic), Fay trisecant (τ-functions), adjoint, and so on, considered (among others) in [47]. Specific properties of a representation chosen [3] (e.g. *Gaudin representation*) imply some additional relations among the images of the universal Dunkl elements (e.g. *Gaudin elements*) should to be unveiled. ◄◄

We start Section 3 with definition of algebra $3T_n(\beta)$ and its "Hecke" $3HT_n(\beta)$ and "elliptic" $3MT_n(\beta)$ quotients. In particular we define an elliptic representation of the algebra $3T_n(0)$, [50], and show how the well-known elliptic solutions of the quantum Yang–Baxter equation due to A. Belavin and V. Drinfeld, see e.g. [5], S. Shibukawa and K. Ueno [86], and G. Felder and V.Pasquier [24], can be plug in to our construction, see Section 3.1.

In Subsection 3.2 we introduce a *multiplicative* analogue of the the Dunkl elements $\{\Theta_j\in 3T_n(\beta),\ 1\le j\le n\}$ and describe the commutative subalgebra in the algebra $3T_n(\beta)$ generated by multiplicative Dunkl elements [51]. The latter commutative subalgebra turns out to be isomorphic to the quantum equivariant K-theory of the complete flag variety $\mathcal{F}l_n$ [51].

In Subsection 3.3 we describe relations among the truncated Dunkl–Gaudin elements. In this case the quantum parameters $q_{ij}=p_{ij}^2$, where parameters $\{p_{ij}=(z_i-z_j)^{-1},\ 1\le i<j\le n\}$ satisfy the both Arnold and Plücker relations. This observation has made it possible to describe a set of additional *rational relations* among the Dunkl–Gaudin elements, cf [71].

[3] For example, in the cases of either *Calogero–Moser* or *Bruhat* representations one has an additional constraint, namely, $u_{ij}^2=0$ for all $i\ne j$. In the case of *Gaudin* representation one has an additional constraint $u_{ij}^2=p_{ij}^2$, where the (quantum) parameters $\{p_{ij}=\frac{1}{x_i-x_j},\ i\ne j\}$, satisfy simultaneously the *Arnold* and *Plücker* relations, see Section 2, (**II**). Therefore, the (small) quantum cohomology ring of the type A_{n-1} full flag variety $\mathcal{F}l_n$ and the Bethe subalgebra(s) (i.e. the subalgebra generated by Gaudin elements in the algebra $3HT_n(0)$) correspond to *different specializations* of *" quantum parameters"* $\{q_{ij}:=u_{ij}^2\}$ of the *universal cohomology ring* (i.e. the subalgebra/ring in $3HT_n(0)$ generated by (universal) Dunkl elements). For more details and examples, see Section 2.1 and [47].

In Subsection 3.4 we introduce an equivariant version of multiplicative Dunkl elements, called *shifted Dunkl elements* in our paper, and describe (some) relations among the latter. This result is a generalization of that obtained in Section 3.1 and [51]. However we don't know any geometric interpretation of the commutative subalgebra generated by shifted Dunkl elements.

In Section 4.1 for any subgraph $\Gamma \subset K_n$ of the complete graph K_n we introduce [4] [47], [44], algebras $3T_n(\Gamma)$ and $3T_n^{(0)}(\Gamma)$ which can be seen as analogues of algebras $3T_n$ and $3T_n^{(0)}$ correspondingly [5].

▶▶ An analog of the algebras $3T_n$ and $3T_n^{(\beta)}$, $3HT_n$, etc treated in the present paper, can be defined for any (oriented or not) *matroid* $\mathcal{M}$. We denote these algebras as $3T(\mathcal{M})$ and $3T^{(\beta)}(\mathcal{M})$. One can show (A.K.) that the *abelianization* of the algebra $3T^{(\beta)}(\mathcal{M})$, denoted by $3T^{(\beta)}(\mathcal{M})^{ab}$, is isomorphic to the *Gelfand–Varchenko* algebra corresponding to a matroid $\mathcal{M}$, whereas the algebra $3T^{(\beta=0)}(\mathcal{M})^{ab}$ is isomorphic to the (even) *Orlik–Solomon* algebra $OS^{+}(\mathcal{M})$ of a matroid $\mathcal{M}$ [6]. We consider and treat the algebras $3T(\mathcal{M})$, $3HT(\mathcal{M})$,.... as *equivariant noncommutative* (or *quantum*) versions of the (even) Orlik–Solomon algebras associated with matroid (including hyperplane, graphic, ... arrangements). However a meaning of a quantum deformation of the (even or odd) Orlik–Solomon algebra suggested in the present paper, is missing, even for the braid arrangement of type A_n. Generalizations of the Gelfand–Varchenko algebra has been suggested and studied in[45], [47] and in the present paper under the name quasi-associative Yang–Baxter algebra, see Section 5.

◀◀

In the present paper we basically study the *abelian quotient* of the algebra $3T_n^{(0)}(\Gamma)$, where graph Γ has no loops and multiple edges, since we expect some applications of our approach to the theory of *chromatic polynomials* of planar graphs, in particular to the complete multipartite graphs $K_{n_1,\ldots,n_r}$ and the grid graphs $G_{m,n}$ [7]. Our main results hold for the complete multipartite, cyclic and line graphs. In particular we compute their *chromatic* and *Tutte* polynomials, see Proposition 4.2 and Theorem 4.3. As a byproduct we compute the Tutte polynomial of the ℓ-weighted complete multipartite graph $K_{n_1,\ldots,n_r}^{(\ell)}$ where $\ell = \{\ell_{ij}\}_{1 \le i < j \le r}$, is a collection of weights, i.e. a set of non-negative integers.

More generally, for a set of variables $\{\{q_{ij}\}_{1 \le i<j \le n},\ x,\ y\}$ we define *universal Tutte polynomial* $T_n(\{q_{ij}\},x,y) \in \mathbb{Z}[q_{ij}][x,y]$ such that for any collection on non-negative integers $\{m_{ij}\}_{1 \le i<j \le n}$ and a subgraph $\Gamma \subset K_n^{(\mathbf{m})}$ of the complete graph K_n with each edge (i,j) comes with multiplicity m_{ij}, the specialization

$$q_{ij} \longrightarrow 0, \quad if\ edge \quad (i,j) \notin \Gamma, \quad q_{ij} \longrightarrow [m_{ij}]_y := \frac{y^{m_{ij}}-1}{y-1} \quad if\ edge \quad (i,j) \in \Gamma$$

of the universal Tutte polynomial $T_n(\{q_{ij}\},x,y)$ is equal to the Tutte polynomial of graph Γ multiplied by the factor $(t-1)^{\kappa(\Gamma)}$:

$$(x-1)^{\kappa(\Gamma}\ Tutte(\Gamma,x,y) := T_n(\{q_{ij}\},x,y)\Bigg|_{\substack{q_{ij}=0,\ if\ (i,j)\notin\Gamma \\ q_{ij}=[m_{ij}]_y,\ if\ (i,j)\in\Gamma}}.$$

Here and after $\kappa(\Gamma)$ demotes the number of connected components of a graph Γ. In other words, one can treat the universal Tutte polynomial $T_n(\{q_{ij}\},x,y)$ as a "reproducing kernel" for

[4] Independently the algebra $3T_n^{(0)}(\Gamma)$ has been studied in [9], where the reader can find some examples and conjectures.

[5] To avoid confusions, it must be emphasized that the defining relations for algebras $3T_n(\Gamma)$ and $3T_n(\Gamma)^{(0)}$ may have more then three terms.

[6] For a definition and basic properties of the Orlik– Solomon algebra corresponding to a matroid see e.g, Y. Kawahara, *On Matroids and Orlik-Solomon Algebras* Annals of Combinatorics **8** (2004) 63-80.

[7] See e.g. *wolfram.com/GridGraph.htm* for a definition of *grid graph* $G_{m,n}$

the Tutte polynomials of all graphs with the number of vertices not exceeded n.
We also state Conjecture 4.2 that for any loopless graph Γ (possibly with multiple edges) the algebra $3T_{|\Gamma|}^{(0)}(\Gamma)^{ab}$ is isomorphic to the even Orlik–Solomom algebra $OS^{+}(\mathcal{A}_{\Gamma})$ of the *graphic arrangement* associated with graph Γ in question.
At the end we emphasize that the case of the complete graph $\Gamma = K_n$ reproduces the results of the present paper and those of [47], i.e. the case of the full flag variety $\mathcal{F}l_n$. The case of the *complete multipartite graph* $\Gamma = K_{n_1,\ldots,n_r}$ reproduces the analogue of results stated in the present paper for the case of full flag variety $\mathcal{F}l_n$, to the case of the partial flag variety $\mathcal{F}_{n_1,\ldots,n_r}$, see [47] for details.

In Section 4.1.3 we sketch how to generalize our constructions and some of our results to the case of the Lie algebras of **classical types** [8].

In Section 4. 2 we briefly overview our results concerning yet another interesting family of quadratic algebras, namely the *six-term relations algebras* $6T_n$, $6T_n^{(0)}$ and related ones. These algebras also contain a distinguished set of mutually commuting elements called *Dunkl elements* $\{\theta_i, i = 1,\ldots,n\}$ given by $\theta_i = \sum_{j \neq i} r_{ij}$, see Definition 4.10.

In Subsection 4.2.2 we introduce and study the algebra $6T_n^{\bigstar}$ in greater detail. In particular we introduce a "quantum deformation" of the algebra generated by the curvature of 2-forms of of the Hermitian linear bundles over the flag variety $\mathcal{F}l_n$, cf [78].

In Subsection 4.2.3 we state our results concerning the *classical Yang–Baxter algebra* CYB_n and the 6-term relation algebra $6T_n$. In particular we give formulas for the Hilbert series of these algebras. These formulas have been obtained independently in [3] The paper just mentioned, contains a description of a basis in the algebra $6T_n$, and much more.

In Subsection 4.2.4 we introduce a *super analog* of the algebra $6T_n$, denoted by $6T_{n,m}$, and compute its Hilbert series.

Finally, in Subsection 4.3 we introduce *extended nil-three term relations* algebra $3\mathfrak{T}_n$ and describe a subalgebra inside of it which is isomorphic to the double affine Hecke algebra of type A_{n-1}, cf [15].

In Section 5 we describe several combinatorial properties of some special elements in the associative quasi-classical Yang–Baxter algebra [9], denoted by $\widehat{ACYB}_n$. The main results in that direction were motivated and obtained as a by-product, in the process of the study of the *the structure* of the algebra $3HT_n(\beta)$. More specifically, the main results of Section 5 were obtained in the course of "hunting for descendant relations" in the algebra mentioned, which is an important problem to be solved to construct *a basis* in the nil-quotient algebra $3T_n^{(0)}$. This **problem** is still widely-open.

The results of Section 5.1, see Proposition 5.1, items (**1**)–(**5**), are more or less well-known among the specialists in the subject, while those of the item (**6**) seem to be new. Namely, we show that the polynomial $Q_n(x_{ij} = t_i)$ from [90], $(\mathbf{6}.\mathbf{C8}), (c)$, essentially coincides with the β-deformation [27] of the Lascoux-Schützenberger Grothendieck polynomial [57] for some particular permutation. The results of Proposition 5.1, (**6**), point out on a deep connection between reduced forms of monomials in the algebra $\widehat{ACYB}_n$ and the Schubert and Grothendieck Calculi. This observation was the starting point for the study of some combinatorial properties of certain specializations of the Schubert, the β-Grothendieck [28] and the double β- Grothendieck polynomials in Section 5.2 . One of the main results of Section 5.2 can be stated as follows.

Theorem 1.3.

[8]One can define an analogue of the algebra $3T_n^{(0)}$ for the root system of BC_n and $C_n^{\vee}C_n$ -types as well, but we are omitted these cases in the present paper

[9] The algebra $\widehat{ACYB}_n$ can be treated as "one-half" of the algebra $3T_n(\beta)$. It appears, see Lemma 5.1, that the basic relations among the Dunkl elements, which do **not** mutually commute anymore, are still valid, see Lemma 5.1.

(1) *Let* $w \in \mathbb{S}_n$ *be a permutation, consider the specialization* $x_1 := q, x_i = 1,\ \forall i \geq 2,$ *of the* β*-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_n)$. Then

$$\mathcal{R}_w(q,\beta+1) := \mathfrak{G}_w^{(\beta)}(x_1 = q, x_i = 1,\ \forall i \geq 2) \in \mathbb{N}[q, 1+\beta].$$

In other words, the polynomial $\mathcal{R}_w(q,\beta)$ has non-negative integer coefficients [10].
For late use we define *polynomials*

$$\mathfrak{R}_w(q,\beta) := q^{1-w(1)}\ \mathcal{R}_w(q,\beta).$$

(2) *Let* $w \in \mathbb{S}_n$ *be a permutation, consider the specialization* $x_i := q, y_i = t,\ \forall i \geq 1,$ *of the double* β*-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_n, Y_n)$. Then

$$\mathfrak{G}_w^{(\beta-1)}(x_i := q, y_i := t, \forall i \geq 1) \in \mathbb{N}[q,t,\beta].$$

(3) *Let* w *be a permutation, then*

$$\mathfrak{R}_w(1,\beta) = \mathfrak{R}_{1\times w}(0,\beta).$$

Note that $\mathcal{R}_w(1,\beta) = \mathcal{R}_{w^{-1}}(1,\beta)$, but $\mathcal{R}_w(t,\beta) \neq \mathcal{R}_{w^{-1}}(t,\beta)$, in general.

For the reader convenience we collect some basic definitions and results concerning the β-Grothendieck polynomials in Appendix I.

Let us observe that $\mathfrak{R}_w(1,1) = \mathfrak{S}_w(1)$, where $\mathfrak{S}_w(1)$ denotes the specialization $x_i := 1,\ \forall i \geq 1$, of the Schubert polynomial $\mathfrak{S}_w(X_n)$ corresponding to permutation w. Therefore, $\mathfrak{R}_w(1,1)$ is equal to the number of *compatible sequences* [8] (or *pipe dreams*, see e.g. [85]) corresponding to permutation w.

Problem 1.1.

Let $w \in \mathbb{S}_n$ *be a permutation and* $l := \ell(w)$ *be its length. Denote by* $CS(w) = \{\mathbf{a} = (a_1 \leq a_2 \leq \cdots \leq a_l) \in \mathbb{N}^l\ \}$ *the set of compatible sequences [8] corresponding to permutation* w.

- *Define statistics* $r(\mathbf{a})$ *on the set of all compatible sequences* $CS_n := \coprod_{w \in \mathbb{S}_n} CS(w)$ *in a such way that*

$$\sum_{\mathbf{a} \in CS(w)} q^{a_1}\ \beta^{r(\mathbf{a})} = \mathcal{R}_w(q,\beta).$$

- *Find a geometric interpretation, and investigate combinatorial and algebra-geometric properties of polynomials* $\mathfrak{S}_w^{(\beta)}(X_n)$,

where for a permutation $w \in \mathbb{S}_n$ *we denoted by* $\mathfrak{S}_w^{(\beta)}(X_n)$ *the* β*-Schubert polynomial defined as follows*

$$\mathfrak{S}_w^{(\beta)}(X_n) = \sum_{\mathbf{a} \in CS(w)} \beta^{r(\mathbf{a})} \prod_{i=1}^{l:=\ell(w)} x_{a_i}.$$

We **expect** that polynomial $\mathfrak{S}_w^{(\beta)}(1)$ coincides with the Hilbert polynomial of a certain graded commutative ring naturally associated to permutation w.

Remark 1.1. *It should be mentioned that, in general, the principal specialization*

$$\mathfrak{G}_w^{(\beta-1)}(x_i := q^{i-1},\ \forall i \geq 1)$$

of the $(\beta-1)$*-Grothendieck polynomial may have negative coefficients.* ■

[10] For a more general result see Appendix I, Corollary 6.2.

Our main objective in Section 5.2 is to study the polynomials $\mathfrak{R}_w(q,\beta)$ for a special class of permutations in the symmetric group $\mathbb{S}_\infty$. Namely, in Section 5.2 we study some combinatorial properties of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ for the five parameters family of *vexillary* permutations $\{\varpi_{\lambda,\phi}\}$ which have the shape

$\lambda := \lambda_{n,p,b} = (p(n-i+1)+b,\ i=1,\ldots,n+1)$ and flag

$\phi := \phi_{k,r} = (k+r(i-1),\ i=1,\ldots,n+1)$.

This class of permutations is notable for many reasons, including that the specialized value of the Schubert polynomial $\mathfrak{S}_{\varpi_{\lambda,\phi}}(1)$ admits a nice product formula [11], see Theorem 5.6. Moreover, we describe also some interesting connections of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ with plane partitions, the Fuss-Catalan numbers [12] and Fuss-Narayana polynomials, k-triangulations and k-dissections of a convex polygon, as well as a connection with two families of ASM. For example, let $\lambda=(b^n)$ and $\phi=(k^n)$ be rectangular shape partitions, then the polynomial $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ defines a (q,β)-deformation of the number of (ordinary) plane partitions [13] sitting in the box $b\times k\times n$. It seems an interesting **problem** to find an algebra-geometric interpretation of polynomials $\mathfrak{R}_w(q,\beta)$ in the general case.

Question Let a and b be mutually prime positive integers. Does there exist a family of permutations $w_{a,b}\in\mathbb{S}_{ab(a+b)}$ such that the specialization $x_i=1\ \ \forall i$ of the Schubert polynomial $\mathfrak{S}_{w_{a,b}}$ is equal o the rational Catalan number $C_{a/b}$? That is

$$\mathfrak{S}_{w_{a,b}}(1)=\frac{1}{a+b}\binom{a+b}{a}.$$

■

Many of the computations in Section 5.2 are based on the following determinantal formula for β-Grothendieck polynomials corresponding to grassmannian permutations, cf [59].

Theorem 1.4. *(see Comments 5.5)*

If $w=\sigma_\lambda$ is the grassmannian permutation with shape $\lambda=(\lambda_,\ldots,\lambda_n)$ and a unique descent at position n, then [14]

$$(\mathbf{A})\qquad \mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n)=DET|h^{(\beta)}_{\lambda_j+i,j}(X_n)|_{1\le i,j\le n}=\frac{DET\ |x_i^{\lambda_j+n-j}\ (1+\beta\ x_i)^{j-1}|_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)},$$

where $X_n=(x_i,x_1,\ldots,x_n)$, *and for any set of variables* X,

$$h^{(\beta)}_{n,k}(X)=\sum_{a=0}^{k-1}\binom{k-1}{a}h_{n-k+a}(X)\ \beta^a,$$

[11] One can prove a product formula for the principal specialization $\mathfrak{S}_{\varpi_{\lambda,\phi}}(x_i:=q^{i-1},\ \forall i\ge 1)$ of the corresponding Schubert polynomial. We don't need a such formula in the present paper.

[12] We define the (generalized) Fuss-Catalan numbers to be $FC_n^{(p)}(b):=\frac{1+b}{1+b+(n-1)p}\binom{np+b}{n}$. Connection of the Fuss-Catalan numbers with the *p-ballot numbers* $Bal_p(m,n):=\frac{n-mp+1}{n+m+1}\binom{n+m+1}{m}$ and the *Rothe numbers* $R_n(a,b):=\frac{a}{a+bn}\binom{a+bn}{n}$ can be described as follows

$$FC_n^{(p)}(b)=R_n(b+1,p)=Bal_{p-1}(n,(n-1)p+b).$$

[13] Let λ be a partition. An ordinary plane partition (plane partition for short)bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly decreasing.

A reverse plane partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly increasing.

[14] the equality

$$\mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n)=\frac{DET\ |x_i^{\lambda_j+n-j}\ (1+\beta\ x_i)^{j-1}|_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)},$$

has been proved independently in [70].

and $h_k(X)$ *denotes the complete symmetric polynomial of degree* k *in the variables from the set* X.

$$(\mathbf{B}) \qquad \mathfrak{G}_{\sigma_\lambda}(X,Y) = \frac{DET|\prod_{a=1}^{\lambda_j+n-j}(x_i+y_a+\beta\ x_i\ y_a)\ (1+\beta x_i)^{j-1}|_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}.$$

In Section 5.3 we give a partial answer on the question **6.C8**(d) by R.Stanley [90]. In particular, we relate the reduced polynomial corresponding to monomial

$$\left(x_{12}^{a_2}\cdots x_{n-1,n}{}^{a_n}\right)\prod_{j=2}^{n-2}\ \prod_{k=j+2}^{n} x_{jk}, \quad a_j\in\mathbb{Z}_{\ge 0}, \forall j,$$

with the Ehrhart polynomial of the generalized Chan–Robbins–Yuen polytope, if $a_2=\ldots=a_n=m+1$, cf [66], with a t-deformation of the Kostant partition function of type A_{n-1} and the Ehrhart polynomials of some flow polytopes, cf [67].

In Section 5.4 we investigate certain specializations of the reduced polynomials corresponding to monomials of the form

$$x_{12}^{m_1}\cdots x_{n-1,n}^{m_n}, \quad m_j\in\mathbb{Z}_{\ge 0}.\forall j.$$

First of all we observe that the corresponding specialized reduced polynomial appears to be a *piece-wise polynomial function* of parameters $\mathbf{m}=(m_1,\ldots,m_n)\in(\mathbb{R}_{\ge 0})^n$, denoted by $P_{\mathbf{m}}$. It is an interesting **problem** to compute the Laplas transform of that piece-wise polynomial function. In the present paper we compute the value of the function $P_{\mathbf{m}}$ in the dominant chamber $\mathcal{C}_n=(m_1\ge m_2\ge\ldots\ge m_n\ge 0)$, and give a combinatorial interpretation of the values of that function in points (n,m) and (n,m,k), $n\ge m\ge k$.

For the reader convenience, in Appendix I–V we collect some useful auxiliary information about the subjects we are treated in the present paper.

Almost all results in Section 5 state that some two specific sets have the same number of elements. Our proofs of these results are pure algebraic. It is an interesting **problem** to find *bijective proofs* of results from Section 5 which generalize and extend remarkable bijective proofs presented in [98], [85], [91], [67] to the cases of

- the β-Grothendieck polynomials,
- the (small) Schröder numbers,
- k-dissections of a convex $(n+k+1)$-gon,
- special values of reduced polynomials.

We are planning to treat and present these bijections in (a) separate publication(s).

■

We **expect** that the reduced polynomials corresponding to the higher-order powers of the Coxeter elements also admit an interesting combinatorial interpretation(s). Some preliminary results in this direction are discussed in Comments 5.8.

At the end of Introduction I want to add two remarks.

(**a**) After a suitable modification of the algebra $3HT_n$, see [52], and the case $\beta\ne 0$ in [47], one can compute the set of relations among the (additive) Dunkl elements (defined in Section 2, (2.1)). In the case $\beta=0$ and $q_{ij}=q_i\ \delta_{j-i,1}$, $1\le i<j\le n$, where $\delta_{a,b}$ is the Kronecker delta symbol, the commutative algebra generated by additive Dunkl elements (2.3) appears to be "almost" isomorphic to the equivariant quantum cohomology ring of the flag variety $\mathcal{F}l_n$, see [52] for details. Using the *multiplicative* version of Dunkl elements (3.14), one can extend the results from [52] to the case of equivariant quantum K-theory of the flag variety $\mathcal{F}l_n$, see [47].

(**b**) As it was pointed out previously, one can define an analogue of the algebra $3T_n^{(0)}$ for any (oriented) matroid $\mathcal{M}_n$, and state a conjecture which connects the Hilbert polynomial of the algebra $3T_n^{(0)}(\mathcal{M}_n)^{ab}, t)$ and the chromatic polynomial of matroid $\mathcal{M}_n$. We **expect** that algebra $3T_n^{(\beta=1)}(\mathcal{M}_n)^{ab}$ is isomorphic to the *Gelfand–Varchenko* algebra associated with matroid $\mathcal{M}$. It is an interesting **problem** to find a combinatorial meaning of the algebra $3T_n^{(\beta)}(\mathcal{M}_n)$ for $\beta = 0$ and $\beta \neq 0$.

Acknowledgments

I would like to express my deepest thanks to Professor Toshiaki Maeno for many years fruitful collaboration.

I'm also grateful to Professors Y. Bazlov, I. Burban, B. Feigin, S. Fomin, A. Isaev, M. Ishikawa, M. Noumi, B. Shapiro and Dr. Evgeny Smirnov for fruitful discussions on different stages of writing [47].

My special thanks are to Professor Anders Buch for sending me the programs for computation of the β-Grothendieck and double β-Grothendieck polynomials. Many results and examples in the present paper have been checked by using these programs, and

Professor Ole Warnaar (University of Queenslad) for a warm hospitality and a kind interest and fruitful discussions of some results from [47] concerning hypergeometric functions.

These notes represent an update version of Section 5 of my notes [47], and are based on my talks given at

- The Simons Center for Geometry and Physics, Stony Brook University, USA, January 2010;
- Department of Mathematical Sciences at the Indiana University– Purdue University Indianapolis (IUPUI), USA,*Departmental Colloquium*, January 2010;
- The Research School of Physics and Engineering, Australian National University (ANU), Canberra, ACT 0200, Australia, April 2010;
- The Institut de Mathématiques de Bourgogne, CNRS U.M.R. 5584, Université de Bourgogne, France, October 2010;
- The School of Mathematics and Statistics University of Sydney, NSW 2006, Australia, November 2010;
- The Institute of Advanced Studies at NTU, Singapore, *5th Asia– Pacific Workshop on Quantum Information Science in conjunction with the Festschrift in honor of Vladimir Korepin*, May 2011;
- The Center for Quantum Geometry of Moduli Spaces, Faculty of Science, Aarhus University, Denmark, August 2011;
- The Higher School of Economy (HES), and The Moscow State University, Russia, November 2011;
- The Research Institute for Mathematical Sciences (RIMS), the Conference *Combinatorial representation theory*, Japan, October 2011;
- The Kavli Institute for the Physics and Mathematics of the Universe (IPMU), Tokyo, August 2013;
- The University of Queensland, Brisbane, Australia, October–November 2013.

I would like to thank Professors Leon Takhtajan and Oleg Viro (Stony Brook), Jrgen E. Andersen (CGM, Aarhus University), Bumsig Kim (KIAS, Seoul), Vladimir Matveev (Université de Bourgogne), Vitaly Tarasov (IUPUI, USA), Vladimir Bazhanov (ANU), Alexander Molev (University of Sydney), Sergey Lando (HES, Moscow), Kyoji Saito (IPMU, Tokyo), Kazuhiro Hikami (Kyushu University), Reiho Sakamoto (Tokyo University of Science), Junichi Shiraishi (University of Tokyo) for invitations and hospitality during my visits of the Universities and the Institutes listed above.

Part of results stated in Section 3, **(II)** has been obtained during my visit of the University of

Sydney, Australia. I would like to thank Professors A. Molev and A. Isaev for the keen interest and useful comments on my paper

2 Dunkl elements

Let $\mathfrak{F}_n$ be the free associative algebra over $\mathbb{Z}$ with the set of generators $\{u_{ij},\ 1 \le i,j \le n\}$. In the subsequent text we will distinguish the set of generators $\{u_{ii}\}_{1\le i \le n}$ from that $\{u_{ij}\}_{1 \le i \ne j \le n}$, and set

$$x_i := u_{ii},\ i = 1,\ldots,n.$$

Definition 2.1. *(***Additive Dunkl elements***)*

The (additive) Dunkl elements $\theta_i, i = 1,\ldots,n$, in the algebra $\mathcal{F}_n$ are defined to be

$$\theta_i = x_i + \sum_{\substack{j=1 \\ j \ne i}}^{n} u_{ij}. \tag{2.1}$$

We are interested in to find "natural relations" among the generators $\{u_{ij}\}_{1 \le i,j \le n}$ such that the Dunkl elements (2.1) are pair-wise commute. One of the natural conditions which is the commonly accepted in the theory of integrable systems, is

- (Locality conditions)

$(a) \quad [x_i, x_j] = 0, \text{ if } i \ne j,$

$$(b) \quad u_{ij}\ u_{kl} = u_{kl}\ u_{ij}, \quad if \quad i \ne j,\ k \ne l \quad and \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{2.2}$$

Lemma 2.1.

Assume that elements $\{u_{ij}\}$ satisfy the locality condition (2.1). *If $i \ne j$, then*

$$[\theta_i, \theta_j] = \left[x_i + \sum_{k \ne i,j} u_{ik},\ u_{ij} + u_{ji}\right] + \left[u_{ij}, \sum_{k=1}^{n} x_k\right] + \sum_{k \ne i,j} w_{ijk},$$

where

$$w_{ijk} = [u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}]. \tag{2.3}$$

Therefore in order to ensure that the Dunkl elements form a pair-wise commuting family, it's natural to assume that the following conditions hold

- (Unitarity)

$$[u_{ij} + u_{ji}, u_{kl}] = 0 = [u_{ij} + u_{ji}, x_k] \quad for \ \ all \ \ distinct \ \ i,j,k,l, \tag{2.4}$$

i.e. the elements $u_{ij} + u_{ji}$ are central.

- ("Conservation laws")

$$[\sum_{k=1}^{n} x_k\ ,\ u_{ij}] = 0 \quad for \ \ all \ \ i,j, \tag{2.5}$$

i.e. the element $E := \sum_{k=1}^{n} x_k$ is central,

- **(Unitary dynamical classical Yang–Baxter relations)**

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}] = 0, \tag{2.6}$$

if i,j,k are pair-wise distinct.

Definition 2.2. *(***Dynamical six term relations algebra** $6DT_n$*)*
We denote by $6DT_n$ *the quotient of the algebra* $\mathcal{F}_n$ *by the two-sided ideal generated by relations* (2.2) − (2.6).

Clearly, the Dunkl elements (2.1) generate a commutative subalgebra inside of the algebra $6DT_n$, and the sum $\sum_{i=1}^n \theta_i = \sum_{i=1}^n x_i$ belongs to the center of the algebra $6DT_n$.

Remark Occasionally we will call the Dunkl elements of the form (2.1) by *dynamical Dunkl elements* to distinguish the latter from *truncated Dunkl elements*, corresponding to the case $x_i = 0, \ \forall i$.

2.1 Some representations of the algebra $6DT_n$

2.1.1 Dynamical Dunkl elements and equivariant quantum cohomology

(**I**) (cf [25]) Given a set $q_1, \ldots, q_{n-1}$ of mutually commuting parameters, define

$$q_{ij} = \prod_{a=i}^{j-1} q_a, \quad if \quad i < j,$$

and set $q_{ij} = q_{ji}$ in the case $i > j$. Clearly, that if $i < j < k$, then $q_{ij} q_{jk} = q_{ik}$.

Let $z_1, \ldots, z_n$ be a set of (mutually commuting) variables. Denote by $P_n := \mathbb{Z}[z_1, \ldots, z_n]$ the corresponding ring of polynomials. We consider the variable $z_i, \ i = 1, \ldots, n$, also as the operator acting on the ring of polynomials P_n by multiplication on the variable z_i.

Let $s_{ij} \in \mathbb{S}_n$ be the transposition that swaps the letters i and j and fixes the all other letters $k \neq i, j$. We consider the transposition s_{ij} also as the operator which acts on the ring P_n by interchanging z_i and z_j, and fixes all other variables. We denote by

$$\partial_{ij} = \frac{1 - s_{ij}}{z_i - z_j}, \qquad \partial_i := \partial_{i,i+1},$$

the divided difference operators corresponding to the transposition s_{ij} and the simple transposition $s_i := s_{i,i+1}$ correspondingly. Finally we define operator (cf [25])

$$\partial_{(ij)} := \partial_i \cdots \partial_{j-1} \partial_j \partial_{j-1} \cdots \partial_i, \quad if \quad i < j.$$

The operators $\partial_{(ij)}, \ 1 \le i < j \le n$, satisfy (among other things) the following set of relations (cf [25])

- $[z_j, \partial_{(ik)}] = 0, \quad if \quad j \notin [i, k], \qquad [\partial_{(ij)}, \sum_{a=i}^j z_a] = 0,$
- $[\partial_{(ij)}, \partial_{(kl)}] = \delta_{jk} \ [z_j, \partial_{(il)}] + \delta_{il} \ [\partial_{(kj)}, z_i], \ \text{ if } \quad i < j, \quad k < l.$

Therefore, if we set $u_{ij} = q_{ij} \ \partial_{(ij)}, \ \ if \ \ i < j,$ and $u_{(ij)} = -u_{(ji)}, \ \ if \ \ i > j$, then for a triple $i < j < k$ we will have

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [z_i, u_{jk}] + [u_{ik}, z_j] + [z_k, u_{jk}] = q_{ij} q_{jk} [\partial_{(ij)}, \partial_{(jk)}] + q_{ik} [\partial_{(ik)}, z_j] = 0.$$

Thus the elements $\{z_i, i = 1, \ldots, n\}$ and $\{u_{ij}, 1 \le i < j \le n\}$ define a representation of the algebra $DCYB_n$, and therefore the Dunkl elements

$$\theta_i := z_i + \sum_{j \neq i} u_{ij} = z_i - \sum_{j<i} q_{ji} \partial_{(ji)} + \sum_{j>i} q_{ij} \partial_{(ij)}$$

form a pairwise commuting family of operators acting on the ring of polynomials $\mathbb{Z}[q_1, \ldots, q_{n-1}][z_1, \ldots, z_n]$, cf [25]. This representation has been used in [25] to construct the small quantum cohomology ring of the complete flag variety of type A_{n-1}.

■

(II) Consider degenerate affine Hecke algebra $\mathfrak{H}_n$ generated by the central element h, the elements of the symmetric group $\mathbb{S}_n$, and the mutually commuting elements $y_1, \ldots, y_n$, subject to relations

$$s_i y_i - y_{i+1} s_i = h, \quad 1 \le i < n, \qquad s_i y_j = y_j s_i, \quad j \ne i, i+1,$$

where s_i stand for the simple transposition that swaps only indices i and $i+1$. For $i < j$, let $s_{ij} = s_i \cdots s_{j-1} s_j s_{j-1} \cdots s_i$ denotes the permutation that swaps only indices i and j. It is an easy exercise to show that

- $[y_j, s_{ik}] = h[s_{ij}, s_{jk}]$, if $i < j < k$,
- $y_i s_{ik} - s_{ik} y_k = h + h\ s_{ik} \sum_{i<j<k} s_{jk}$, if $i < k$.

Finally, consider a set of mutually commuting parameters $\{p_{ij},\ 1 \le i \ne j \le n,\ p_{ij} + p_{ji} = 0\}$, subject to the constraints

$$p_{ij} p_{jk} = p_{ik} p_{ij} + p_{jk} p_{ik} + p_{ik}, \quad i < j < k.$$

Comments 2.1. If parameters $\{p_{ij}\}$ are *invertible*, and satisfy relations

$$p_{ij} p_{jk} = p_{ik} p_{ij} + p_{jk} p_{ik} + \beta\ p_{ik}, \quad i < j < k,$$

then one can rewrite the above displayed relations in the following form:

$$1 + \frac{\beta}{p_{ik}} = \Big(1 + \frac{\beta}{p_{ij}}\Big)\Big(1 + \frac{\beta}{p_{jk}}\Big), \quad 1 \le i < j < k \le n.$$

Therefore there exist parameters $\{q_1, \ldots, q_n\}$ such that $1 + \beta/p_{ij} = q_i/q_j$, $1 \le i < j \le n$. In other words, $p_{ij} = \frac{\beta\ q_j}{q_j - q_j}$, $1 \le i < j \le n$. However in general, there are many other types of solutions, for example, solutions related to the Heaviside function [15] $H(x)$, namely, $p_{ij} = H(x_i - x_j)$, $x_i \in \mathbb{R}$, $\forall i$, and its discrete analogue, see Example **(III)** below. In the both cases $\beta = -1$; see also Comments 2.3 for other examples.

■

To continue presentation of Example **(II)**, define elements $u_{ij} = p_{ij} s_{ij}$, $1 \le i \ne j \le n$.

Lemma 2.2. *(Dynamical classical Yang–Baxter relations)*

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [u_{ik}, y_j] = 0, \quad 1 < i < j < k \le n. \tag{2.7}$$

Indeed,

$$u_{ij} u_{jk} = u_{ik} u_{ij} + u_{jk} u_{ik} + h\ p_{ik} s_{ij} s_{jk}, \quad u_{jk} u_{ij} = u_{ij} u_{ik} + u_{ik} u_{jk} + h\ p_{ik} s_{jk} s_{ik},$$

and moreover, $[y_j, u_{ik}] = h\ p_{ik} [s_{ij}, s_{jk}]$. ■

Therefore, the elements

$$\theta_i = y_i - h \sum_{j<i} u_{ij} + h \sum_{i<j} u_{ij}, \quad i = 1, \ldots, n,$$

form a mutually commuting set of elements in the algebra $\mathbb{Z}[\{p_{ij}\}] \otimes_{\mathbb{Z}} \mathfrak{H}_n$.

Theorem 2.1. *Define matrix $M_n = (m_{i,j})_{1 \le i,j \le n}$ as follows:*

$$m_{i,j}(u; z_1, \ldots, z_n) = \begin{cases} u - z_i & \text{if } i = j, \\ -h - p_{ij} & \text{if } i < j, \\ p_{ij} & \text{if } i > j. \end{cases}$$

[15] http://en.wikipedia.org/wiki/Heaviside step function

Then

$$DET\Big|M_n(u;\theta_1,\ldots,\theta_n)\Big| = \prod_{j=1}^{n} (u-y_j).$$

Moreover, let us set $q_{ij} := h^2(p_{ij}+p_{ij}^2) = h^2 q_i q_j (q_i-q_j)^{-2},\ i<j,$ *then*

$$e_k(\theta_1,\ldots,\theta_n) = e_k^{(\mathbf{q})}(y_1,\ldots,y_n), \quad 1\le k\le n,$$

where $e_k(x_1,\ldots,x_n)$ *and* $e_k^{(\mathbf{q})}(x_1,\ldots,x_n)$ *denote correspondingly the classical and multiparameter quantum [26] elementary polynomials* [16].

Let's stress that the elements y_i and θ_j do not commute in the algebra $\mathfrak{H}_n$, but the symmetric functions of $y_1,\ldots,y_n$, i.e. the center of the algebra $\mathfrak{H}_n$, do.

A few remarks in order. First of all, $u_{ij}^2 = p_{ij}^2$ are central elements. Secondly, in the case $h=0$ and $y_i=0,\ \forall i$, the equality

$$DET\Big|M_n(u;x_1,\ldots,x_n)\Big| = u^n$$

describes the set of *polynomial* relations among the Dunkl–Gaudin elements (with the following choice of parameters $p_{ij}=(q_i-q_j)^{-1}$ are taken). And our final remark is that according to [35], Section 8, the quotient ring

$$\mathcal{H}_n^{\mathbf{q}} := \mathbb{Q}[y_1,\ldots,y_n]^{\mathbb{S}_n}\otimes\mathbb{Q}[\theta_1,\ldots,\theta_n]\otimes\mathbb{Q}[h]\ /\ \Big\langle M_n(u;\theta_1,\ldots,\theta_n) = \prod_{j=1}^{n}(u-y_j)\Big\rangle$$

is isomorphic to the quantum equivariant cohomology ring of the cotangent bundle $T^*\mathcal{F}l_n$ of the complete flag variety of type A_{n-1}, namely,

$$\mathcal{H}_n^{\mathbf{q}} \cong QH^*_{T^n\times\mathbb{C}^*}(T^*\mathcal{F}l_n)$$

with the following choice of quantum parameters: $Q_i := h\ q_{i+1}/q_i,\ \ i=1,\ldots,n-1.$

On the other hand, in [52] we computed the so-called *multiparameter deformation* of the equivariant cohomology ring of the complete flag variety of type A_{n-1}.

A deformation defined in [52] depends on parameters $\{q_{ij},\ 1\le i<j\le n\}$ without any constraints are imposed. For the special choice of parameters

$$q_{ij} := h^2\frac{q_i\ q_j}{(q_i-q_j)^2}$$

the multiparameter deformation of the equivariant cohomology ring of the type A_{n-1} complete flag variety $\mathcal{F}l_n$ constructed in [52], is isomorphic to the ring $\mathcal{H}_n^{\mathbf{q}}$.

---

[16] For the reader convenience we remind [26] a definition of the quantum elementary polynomial $e_k^{\mathbf{q}}(x_1,\ldots,x_n)$. Let $\mathbf{q} := \{q_{ij}\}_{1\le i<j\le n}$ be a collection of "quantum parameters", then

$$e_k^{\mathbf{q}}(x_1,\ldots,x_n) = \sum_{\ell}\ \sum_{\substack{1\le i_1<\ldots<i_\ell\le n\\ j_1>i_1\ldots,j_\ell>i_\ell}} e_{k-2\ell}(X_{\overline{I\cup J}})\prod_{a=1}^{\ell} q_{i_a,j_a},$$

where $I=(i_1,\ldots i_\ell)$, $J=(j_1,\ldots,j_\ell)$ should be distinct elements of the set $\{1,\ldots,n\}$, and $X_{\overline{I\cup J}}$ denotes set of variables x_a for which the subscript a is neither one of i_m nor one of the j_m.

Comments 2.2. Let us fix a set of independent parameters $\{q_1,\ldots,q_n\}$ and define new parameters

$$\{q_{ij} := h\ p_{ij}(p_{ij}+h) = h^2\ \frac{q_i\ q_j}{(q_i-q_j)^2}\},\quad 1\le i<j\le n,\quad where\ \ p_{ij}=\frac{q_j}{q_i-q_j},\ i<j.$$

We set $deg(q_{ij})=2, deg(p_{ij})=1,\ \ deg(h)=1$.

The new parameters $\{q_{ij}\}_{1\le i<j\le n}$, do not free anymore, but satisfy rather complicated algebraic relations. We display some of these relations soon, having in mind a question:

is there some intrinsic meaning of the algebraic variety defined by the set of defining relations among the "quantum parameters" $\{q_{ij}\}$?

Let us denote by $\mathcal{A}_{n,h}$ the quotient ring of the ring of polynomials $\mathbb{Q}[h][x_{ij},\ 1\le i<j\le n]$ modulo the ideal generating by polynomials $f(x_{ij})$ such that the specialization $x_{ij}=q_{ij}$ of a polynomial $f(x_{ij})$, namely $f(q_{ij})$, is equal to zero. The algebra $\mathcal{A}_{n,h}$ has a natural filtration, and we denote by $\mathcal{A}_n = gr\mathcal{A}_{n,h}$ the corresponding associated graded algebra.

To describe (a part of) relations among the parameters $\{q_{ij}\}$ let us observe that parameters $\{p_{ij}\}$ and $\{q_{ij}\}$ are related by the following identity

$$q_{ij}q_{jk}-q_{ik}(q_{ij}+q_{jk})+h^2q_{ik} = 2\ p_{ij}p_{ik}p_{jk}(p_{ik}+h),\quad if\ \ i<j<k.$$

Using this identity we can find the following relations among parameters in question

$$\begin{aligned}&q_{ij}^2q_{jk}^2+q_{ij}^2q_{ik}^2+h^4q_{ik}^2q_{jk}^2-2\ q_{ij}q_{ik}q_{jk}(q_{ij}+q_{jk}+q_{ik})-2\ h^2q_{ik}(q_{ij}q_{jk}+q_{ij}q_{ik}+q_{jk}q_{ik})\\&=8\ h\ q_{ij}\ q_{ik}\ q_{jk}\ \mathbf{p_{ik}},\end{aligned}\tag{2.8}$$

if $1\le i<j<k\le n$.

Finally, we come to a relation of degree 8 among the "quantum parameters" $\{q_{ij}\}$

$$\Big(LHS(2.8)\Big)^2 = 64\ h^2\ q_{ij}^2\ q_{ik}^3\ q_{jk}^2,\quad 1\le i<j<k\le n.$$

There are also higher degree relations among the parameters $\{q_{ij}\}$ some of whose in degree 16 follow from the deformed Plücker relation between parameters $\{p_{ij}\}$:

$$\frac{1}{p_{ik}p_{jl}}=\frac{1}{p_{ij}p_{kl}}+\frac{1}{p_{il}p_{jk}}+\frac{h}{p_{ij}p_{jk}p_{kl}},\quad i<j<k<l.$$

However, we don't know how to describe the algebra $\mathcal{A}_{n,h}$ generated by quantum parameters $\{q_{ij}\}_{1\le i<j\le n}$ even for n=4.

The algebra $\mathcal{A}_n = gr(\mathcal{A}_{n,h})$ is isomorphic to the quotient algebra of $\mathbb{Q}[x_{ij},\ 1\le i<j\le n]$ modulo the ideal generated by the set of relations between "quantum parameters"

$$\{\overline{q}_{ij} := \Big(\frac{1}{z_i-z_j}\Big)^2\}_{1\le i<j\le n},$$

which correspond to the Dunkl–Gaudin elements $\{\theta_i\}_{1\le i\le n}$, see Section 3.2 below for details. In this case the parameters $\{\overline{q}_{ij}\}$ satisfy the following relations

$$(\overline{q}_{ij}^2\overline{q}_{jk}^2+\overline{q}_{ij}^2\overline{q}_{ik}^2+\overline{q}_{jk}^2\overline{q}_{ik}^2 = 2\ \overline{q}_{ij}\overline{q}_{ik}\overline{q}_{jk}(\overline{q}_{ij}+\overline{q}_{jk}+\overline{q}_{jk})$$

which correspond to the relations (2.8) in the special case $h=0$. One can find a set of relations in degrees 6, 7 and 8, namely for a given pair-wise distinct integers $1\le i,j,k,l\le n$, one has

- one relation in degree 6

$$\overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{il}^2+\overline{q}_{ij}^2\overline{q}_{jk}^2\overline{q}_{jl}^2+\overline{q}_{ik}^2\overline{q}_{jk}^2\overline{q}_{kl}^2+\overline{q}_{il}^2\overline{q}_{jl}^2\overline{q}_{kl}^2-$$

$$2\ \overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl}\Big(\frac{\overline{q}_{ij}}{\overline{q}_{kl}}+\frac{\overline{q}_{kl}}{\overline{q}_{ij}}+\frac{\overline{q}_{ik}}{\overline{q}_{jl}}+\frac{\overline{q}_{jl}}{\overline{q}_{ik}}+\frac{\overline{q}_{il}}{\overline{q}_{jk}}+\frac{\overline{q}_{jk}}{\overline{q}_{il}}\Big)+8\ \overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl}=0;$$

- three relations in degree 7

$$\overline{q}_{ik}\quad\Big(\overline{q}_{ij}\overline{q}_{il}\overline{q}_{kl}-\overline{q}_{ij}\overline{q}_{il}\overline{q}_{jk}+\overline{q}_{ij}\overline{q}_{jk}\overline{q}_{kl}-\overline{q}_{il}\overline{q}_{jk}\overline{q}_{kl}\Big)^2=$$

$$8\ \overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{jk}\overline{q}_{kl}\Big(\overline{q}_{jk}+\overline{q}_{jl}+\overline{q}_{kl}\Big)-4\ \overline{q}_{ij}^2\overline{q}_{il}^2\overline{q}_{jl}\Big(\overline{q}_{jk}^2+\overline{q}_{kl}^2\Big),$$

- one relation in degree 8

$$\overline{q}_{ij}^2\overline{q}_{il}^2\overline{q}_{jk}^2\overline{q}_{kl}^2+\overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{jl}^2\overline{q}_{kl}^2+\overline{q}_{ik}^2\overline{q}_{il}^2\overline{q}_{jk}^2\overline{q}_{jl}^2=2\ \overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl}\Big(\overline{q}_{ij}\overline{q}_{kl}+\overline{q}_{ik}\overline{q}_{jl}+\overline{q}_{il}\overline{q}_{jk}\Big),$$

However we don't know does the list of relations displayed above, contains the all independent relations among the elements $\{\overline{q}_{ij}\}_{1\le i<j\le n}$ in degrees 6, 7 and 8, even for $n=4$. In degrees ≥ 9 and $n\ge 5$ some independent relations should appear.

Notice that the parameters $\{p_{ij}=\frac{h\ q_j}{q_i-q_j},\ i<j\}$ satisfy the so-called *Gelfand–Varchenko* relations, see e.g. [45]

$$p_{ij}p_{jk}=p_{ik}p_{ij}+p_{jk}p_{ik}+h\ p_{ik},\quad i<j<k,$$

whereas parameters $\{\overline{p}_{ij}=\frac{1}{q_i-q_j},\ i<j\}$ satisfy the so-called *Arnold* relations

$$\overline{p}_{ij}\overline{p}_{jk}=\overline{p}_{ik}\overline{p}_{ij}+\overline{p}_{jk}\overline{p}_{ik},\quad i<j<k.$$

Project 2.1. [17] *Find Hilbert series $Hilb(\mathcal{A}_n,t)$ for $n\ge 4$.*

For example, $Hilb(\mathcal{A}_3,t)=\frac{(1+t)(1+t^2)}{(1-t)^2}$.

Finally, if we set $q_i:=exp(h\ z_i)$ and take the limit $\lim_{h\to 0}\frac{h^2\ q_iq_j}{(q_i-q_j)^2}$, as a result we obtain the Dunkl–Gaudin parameter $\overline{q}_{ij}=\frac{1}{(z_i-z_j)^2}$.

■

(**III**) Consider the following representation of the degenerate affine Hecke algebra $\mathfrak{H}_n$ on the ring of polynomials $P_n=\mathbb{Q}[x_1,\ldots,x_n]$:

- the symmetric group $\mathbb{S}_n$ acts on P_n by means of operators

$$\overline{s}_i=1+(x_{i+1}-x_i-h)\partial_i, i=1,\ldots,n-1,$$

- y_i acts on the ring P_n by multiplication on the variable x_i: $y_i(f(x))=x_if(x),\ \ f\in P_n$. Clearly,

$$y_i\ \overline{s_i}-y_{i+1}\ \overline{s_i}=h,\quad and\quad y_i(\overline{s}_i-1)=(\overline{s}_i-1)y_{i+1}+x_{i+1}-x_i-h.$$

In the subsequent discussion we will identify the operator of multiplication by the variable x_i, namely the operator y_i, with x_i.

This time define $u_{ij}=p_{ij}(\overline{s}_i-1)$, if $i<j$ and set $u_{ij}=-u_{ji}\ if\ i>j$, where parameters $\{p_{ij}\}$ satisfy the same conditions as in the previous example.

[17] This is a particular case of more general problem we are interested in. Namely, let $\{f_\alpha\in\mathbb{R}[x_1,\ldots,x_n]\}_{1\le\alpha\le N}$ be a collection of linear forms, and $k\ge 2$ be an integer. Denote by $I(\{f_\alpha\})$ the ideal in the ring of polynomials $\mathbb{R}[z_1,\ldots,z_N]$ generated by polynomials $\Phi(z_1,\ldots,z_N)$ such that

$$\Phi(f_1^{-k},\ldots,f_N^{-k})=0.$$

Compute the Hilbert polynomial of the quotient algebra $\mathbb{R}[z_1,\ldots,z_N]/I(\{f_\alpha\})$.

Lemma 2.3. *The elements* $\{u_{ij}, \;\; 1 \le i < j \le n\}$, *satisfy the dynamical classical Yang–Baxter relations displayed in Lemma 2.2,* (2.7).

Therefore, the Dunkl elements

$$\overline{\theta}_i := \sum_{\substack{j \\ j \neq i}} u_{ij}, \quad i = 1, \ldots, n,$$

form a commutative set of elements.

Theorem 2.2. *([35]) Define matrix* $\overline{M}_n = (\overline{m}_{ij})_{1 \le i,j \le n}$ *as follows*

$$\overline{m}_{i,j}(u; z_1, \ldots, z_n) = \begin{cases} u - z_i + \sum_{j \neq i} \; h \; p_{ij} & \text{if } i = j, \\ -h - p_{ij} & \text{if } i < j, \\ p_{ij} & \text{if } i > j. \end{cases}$$

Then

$$DET \left| \overline{M}_n(u; \overline{\theta}_1, \ldots, \overline{\theta}_n) \right| = \prod_{j=1}^{n} \; (u - x_j).$$

■

Comments 2.3. Let us list a few more representations of the dynamical classical Yang–Baxter relations.

- (Trigonometric Calogero–Moser representation) Let $i < j$, define

$$u_{ij} = \frac{x_j}{x_i - x_j}(s_{ij} - \epsilon), \; \epsilon = 0 \; or \; 1; \quad s_{ij}(x_i) = x_j, \; s_{ij}(x_j) = x_i, \quad s_{ij}(x_k) = x_k, \; \forall k \neq i, j.$$

- (Mixed representation)

$$u_{ij} = (\frac{\lambda_j}{\lambda_i - \lambda_j} - \frac{x_j}{x_i - x_j})(s_{ij} - \epsilon), \quad \epsilon = 0 \; or \; 1; \quad s_{ij}(\lambda_k) = \lambda_k \;\; \forall k.$$

We set $u_{ij} = -u_{ji}, \; if \; i > j$. In all cases we define Dunkl elements to be $\theta_i = \sum_{j \neq i} u_{ij}$.

Note that operators

$$r_{ij} = (\frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} - \frac{x_i + x_j}{x_i - x_j})s_{ij}$$

satisfy the three term relations: $r_{ij}r_{jk} = r_{ik}r_{ij} + r_{jk}r_{ik}$, and $r_{jk}r_{ij} = r_{ij}r_{jk} + r_{ik}r_{jk}$, and thus satisfy the classical Yang–Baxter relations.

■

2.1.2 Dunkl–Uglov representation of degenerate affine Hecke algebra [94]

(*Step functions and the Dunkl–Uglov representations of the degenerate affine Hecke algebras*)

Consider step functions $\eta^{\pm} : \mathbb{R} \longrightarrow \{0, 1\}$

$$(Heaviside \; function) \quad \eta^{+}(x) = \begin{cases} 1, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0; \end{cases} \quad \eta^{-}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

For any two real numbers x_i and x_j set $\eta_{ij}^{\pm} = \eta^{\pm}(x_i - x_j)$.

Lemma 2.4. *The functions η_{ij} satisfy the following relations*

- $\eta^{\pm}_{ij}+\eta^{\pm}_{ji}=1+\delta_{x_i,x_j}, \quad (\eta^{\pm}_{ij})^2=\eta^{\pm}_{ij}$,
- $\eta^{\pm}_{ij}\eta^{\pm}_{jk}=\eta^{\pm}_{ik}\eta^{\pm}_{ij}+\eta^{\pm}_{jk}\eta^{\pm}_{ik}-\eta^{\pm}_{ik}$,

where $\delta_{x,y}$ denotes the Kronecker delta function.

To introduce the Dunkl–Uglov operators [94] we need a few more definitions and notation. To start with, denote by $\Delta_i^{\pm}$ the finite difference operators: $\Delta_i^{\pm}(f)(x_1,\ldots,x_n)=f(\ldots,x_i\pm 1,\ldots)$. Let as before, $\{s_{ij},\ 1\le i\ne j\le n,\ \ s_{ij}=s_{ji}\}$, denotes the set of transpositions in the symmetric group $\mathbb{S}_n$. Recall that $s_{ij}(x_i)=x_j\ \ s_{ij}(x_k)=x_k\ \forall k\ne i,j$. Finally define Dunkl–Uglov operators $d_i^{\pm}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ to be

$$d_i^{\pm}=\Delta_i^{\pm}+\sum_{j<i}\ \delta_{x_i,x_j}-\sum_{j<i}\ \eta^{\pm}_{ji}\ s_{ij}+\sum_{j>i}\eta^{\pm}_{ij}\ s_{ij}.$$

To simplify notation, set $u^{\pm}_{ij}:=\eta^{\pm}_{ij}s_{ij},\ if\ i<j$, and $\widetilde{\Delta}_i^{\pm}=\Delta_i^{\pm}+\sum_{j<i}\delta_{x_i,x_j}$.

Lemma 2.5. *The operators $\{u^{\pm}_{iu},\ 1\le i<j\le n\}$ satisfy the following relations*

$$[u^{\pm}_{ij},u^{\pm}_{ik}+u^{\pm}_{jk}]+[u^{\pm}_{ik},u^{\pm}_{jk}]+[u^{\pm}_{ik},\sum_{j<i}\delta_{x_i,x_j}]=0,\quad if\ \ i<j<k. \tag{2.9}$$

From now on we assume that $x_i\in\mathbb{Z},\ \ \forall i$, that is, we will work with the restriction of the all operators defined at beginning of Example (2.1 (**c**)), to the subset $\mathbb{Z}^n\subset\mathbb{R}^n$. It is easy to see that under the assumptions $x_i\in\mathbb{Z},\ \ \forall i$, we will have

$$\Delta_j^{\pm}\eta^{\pm}_{ij}=(\eta^{\pm}_{ij}\mp\delta_{x_i,x_j})\Delta_i^{\pm}. \tag{2.10}$$

Moreover, using relations (2.13), (2.14) one can prove that

Lemma 2.6.

- $[u^{\pm}_{ij},\widetilde{\Delta}_i^{\pm}+\widetilde{\Delta}_j^{\pm}]=0$,
- $[u^{\pm}_{ik},\widetilde{\Delta}_j^{\pm}]=[u^{\pm}_{ik},\sum_{j<i}\delta_{x_i,x_j}],\ \ i<j<k$.

Corollary 2.1.

- *The operators $\{u^{\pm}_{ij},\ 1\le i<j<k\le n,\}$ and $\widetilde{\Delta}_i^{\pm}, i=1,\ldots,n$ satisfy the dynamical classical Yang–Baxter relations*

$$[u^{\pm}_{ij},u^{\pm}_{ik}+u^{\pm}_{jk}]+[u^{\pm}_{ik},u^{\pm}_{jk}]+[u^{\pm}_{ik},\widetilde{\Delta}_j]]=0,\quad if\ \ i<j<k.$$

- *([94]) The operators $\{s_i:=s_{i,i+1},\ 1\le i<n,$ and $\widetilde{\Delta}_j^{\pm}, 1\le j\le n\}$ give rise to two representations of the degenerate affine Hecke algebra $\mathfrak{H}_n$. In particular, the Dunkl–Uglov operators are mutually commute: $[d_i^{\pm},d_j^{\pm}]=0$.*

■

2.1.3 Extended Kohno–Drinfeld algebra and Yangian Dunkl–Gaudin elements

Definition 2.3. *Extended Kohno–Drinfeld algebra is an associative algebra over $\mathbb{Q}[\beta]$ generated by the elements $\{z_1,\ldots,z_n\}$ and $\{y_{ij}\}_{1\le i\ne j\le n}$ subject to the set of relations*

(i) The elements $\{y_{ij}\{_{1\le i\ne j\le n}$ satisfy the Kohno–Drinfeld relations

- $y_{ij}=y_{ji},\ \ [y_{ij},y_{kl}]=0$, *if i,j,k,l are distinct.*
- $[y_{ij},y_{ik}+y_{jk}]=0=[y_{ij}+y_{ik},y_{jk}]$, *if $i<j<k$.*

(ii) The elements $z_1,\ldots,z_n$ generate the free associative algebra $\mathcal{F}_n$.

(iii) (Crossing relations)

- $[z_i,y_{jk}]=0,\ \ if\ \ i\ne j,k,\quad [z_i,z_j]=\beta\ [y_{ij},z_i],\ \ if\, i\ne j.$

To define the (yangian) Dunkl-Gaudin elements, cf [35], let us consider a set of elements $\{p_{ij}\}_{1\le i\ne j\le n}$ subject to relations

- $p_{ij}+p_{ji}=\beta,\ [p_{ij},y_{kl}]=0=[p_{ij},z_k]$ for all i,j,k.
- $p_{ij}\ p_{jk}=p_{ik}\big(p_{jk}-p_{ji}\big)$, if $i<j<k$.

Let us set $u_{ij}=p_{ij}\ y_{ij},\ i\ne j$, and define the (yangian) Dunkl–Gaudin elements as follows

$$\theta_i=z_i+\sum_{j\ne i}\ u_{ij},\quad i=1,\ldots,n.$$

Proposition 2.1. *(Cf [35], Lemma 3.5)*
The elements $\theta_1,\ldots,\theta_n$ form a mutually commuting family.

Indeed, let $i<j$, then $[\theta_i,\theta_j]=$

$$[z_i,z_j]+\beta[z_i,y_{ij}]+p_{ij}[y_{ij},z_i+z_j]+\sum_{k\ne i,j}\left(p_{ik}p_{jk}\Big[y_{ij}+y_{ik},y_{jk}\Big]+p_{ik}p_{ji}\Big[y_{ij},y_{ik}+y_{jk}\Big]\right)=0.$$

A representation of the extended Kohno–Drinfeld algebra has been constructed in [35], namely one can take

$$y_{ij}:=T_{ij}^{(1)}\ T_{ji}^{(1)}-T_{jj}^{(1)}=y_{ji},\quad z_i:=\beta\ T_{ii}^{(2)}-\frac{\beta}{2}\ T_{ii}^{(1)}\ (T_{ii}^{(1)}-1),\quad p_{ij}:=\frac{\beta\ q_j}{q_i-q_j},\ i\ne j,$$

where $q_1,\ldots,q_n$ stands for a set of mutually commuting *quantum* parameters, and $\{T_{ij}^{(s)}\}_{\substack{1\le i,j\le n\\ s\in\mathbb{Z}_{\ge0}}}$ denotes the set of generators of the Yangian $Y(\mathfrak{gl}_n)$, see e.g. [69].

A proof that the elements $\{z_i\}_{1\le i\le n}$ and $\{y_{ij}\}_{1\le i\ne j\le n}$ satisfy the extended Kohno–Drinfeld algebra relations is based on the following relations, see e.g. [35], Section 3

$$[T_{ij}^{(1)},T_{kl}^{(s)}]=\delta_{il}T_{kj}^{(s)}-\delta_{jk}T_{il}^{(s)},\quad i,j,k,l=1,\ldots,n,\quad s\in\mathbb{Z}_{\ge0}.$$

2.2 "Compatible" Dunkl elements and Manin matrices

("Compatible" Dunkl elements, Manin matrices and algebras related with weighted complete graphs rK_n)

Let us consider a collection of generators $\{u_{ij}^{(\alpha)},\ 1\le i,j\le n,\ \alpha=1,\ldots,r\}$, subject to the following relations

- either the unitarity (the case of sign "+"), or the symmetry relations (the case of sign " - ") [18]

$$:u_{ij}^{(\alpha)}\pm u_{ji}^{(\alpha)}=0,\forall,\alpha,i,j,\tag{2.11}$$

- (*local 3-term relations*)

$$u_{ij}^{(\alpha)}u_{jk}^{(\alpha)}+u_{jk}^{(\alpha)}u_{ki}^{\alpha)}+u_{ki}^{(\alpha)}u_{ij}^{(\alpha)}=0.\ \ i,j,k\ \ are\ \ distinct,\quad 1\le\alpha\le r.\tag{2.12}$$

[18] More generally one can impose the q-symmetry conditions

$$u_{ij}+qu_{ji}=0,\ \ 1\le i<j\le n$$

and ask about relations among the local Dunkl elements to ensure the commutativity of the global ones. As one might expect, the matrix $Q:=(\theta_j^{(a)})_{\substack{1\le a\le r\\ 1\le j\le n}}$ composed from the local Dunkl elements should be a q-Manin matrix. See e.g. [16], or *en.wikipedia.org/wiki/Manin.matrix* for a definition and basic properties of the latter.

We define *global* 3-term relations algebra $3T_{n,r}^{(\pm)}$ as " compatible product" of the local 3-term relations algebras. Namely, we require that the elements

$$U_{ij}^{(\lambda)} := \sum_{\alpha=1}^{r} \lambda_\alpha \, u_{ij}^{(\alpha)}, \quad 1 \le i,j \le n,$$

satisfy the 3-term relations (1.4) for all values of parameters $\{\lambda_i \in \mathbb{R}, \quad 1 \le \alpha \le r\}$.

It is easy to check that our request is equivalent to a validity of the following sets of relations among the generators $\{u_{ij}^{(\alpha)}\}$

(a) (*local 3-term relations*) $u_{ij}^{(\alpha)} \, u_{jk}^{\alpha)} + u_{jk}^{(\alpha)} \, u_{ki}^{(\alpha)} + u_{ki}^{\alpha)} \, u_{ij}^{(\alpha)} = 0$,

(b) (6-term crossing relations)

$$u_{ij}^{(\alpha)} \, u_{jk}^{(\beta)} + u_{ij}^{(\beta)} \, u_{jk}^{(\alpha)} + u_{k,i}^{(\alpha)} \, u_{ij}^{(\beta)} \, u_{ki}^{(\alpha)} + u_{jk}^{(\alpha)} \, u_{ki}^{(\beta)} + u_{jk}^{(\beta)} \, u_{ki}^{(\alpha)} = 0,$$

i,j,k are distinct, $\alpha \ne \beta$.

Now let us consider *local* Dunkl elements

$$\theta_i^{(\alpha)} := \sum_{j \ne i} u_{ij}^{(\alpha)}, \; j = 1,\ldots,n, \; \alpha = 1,\ldots,r.$$

It follows from the local 3-term relations $(\star)$ that for a fixed $\alpha \in [1,r]$ the local Dunkl elements $\{\theta_i^{(\alpha)}\}_{\substack{1 \le i \le n \\ 1 \le \alpha \le r}}$ either mutually commute (the sign "+"), or pairwise anticommute (the sign " - "). Similarly, the global 3-term relations imply that the global Dunkl elements

$$\theta_i^{(\lambda)} := \lambda_1 \theta_i^{(1)} + \cdots + \lambda_r \theta_i^{(r)} = \sum_{j \ne i} U_{ij}^{(\lambda)} \qquad i = 1,\ldots,n$$

also either mutually commute (the case " + ") or pairwise anticommute (the case " - ").

Now we are looking for a set of relations among the local Dunkl elements which is a consequence of the commutativity (anticommutativity) of the global Dunkl elements. It is quite clear that if $i < j$, then

$$[\theta_i^{(a)}, \theta_j^{(b)}]_\pm = \sum_{a=1}^{r} \lambda_a^2 \, [\theta_i^{(a)}, \theta_j^{(a)}]_\pm + \sum_{1 \le a < b \le r} \lambda_a \, \lambda_b \left([\theta_i^{(a)}, \theta_j^{(b)}]_\pm + [\theta_i^{(b)}, \theta_j^{(a)}]_\pm \right),$$

and the commutativity (or anticommutativity) of the global Dunkl elements for all $(\lambda_1,\ldots,\lambda_r) \in \mathbb{R}^r$ is equivalent to the following set of relations

- $[\theta_i,^{(a)}, \theta_j^{(a)}]_\pm = 0$,
- $[\theta_i^{(a)}, \theta_j^{(b)}]_\pm + [\theta_i^{(b)}, \theta_j^{(a)}]_\pm = 0, \;\; a < b$ and $i < j$,

where by definition we set $[a,b]_\pm := ab \mp ba$.

In other words , the matrix $\Theta_n := (\theta_i^{(a)})_{\substack{1 \le a \le r \\ 1 \le i \le n}}$ should be either a *Manin matrix* (the case " + "), or its super analogue (the case " - "). Clearly enough that a similar construction can be applied to the algebras studied in Section 2, **I-III**.,and thus it produces some interesting examples of the Manin matrices. It is an interesting **problem** to describe the algebra generated by the local Dunkl elements $\{\theta_i^{(a)}\}_{\substack{1 \le a \le r \\ 1 \le i \le n}}$ and a commutative subalgebra generated by the global Dunkl elements inside the former. It is also an interesting **question** whether or not the coefficients $C_1,\ldots,C_n$ of the column characteristic polynomial $Det^{col} \mid \Theta_n - t \; I_n \mid = \sum_{k=0}^{n} C_k \; t^{n-k}$ of the Manin matrix Θ_n generate a commutative subalgebra ? For a definition of the column determinant of a matrix, see e.g. [16].

However a close look at this problem and the question posed needs an additional treatment and has been omitted from the content of the present paper.

Here we are looking for a "natural conditions" to be imposed on the set of generators $\{u_{ij}^{\alpha}\}_{\substack{1\le\alpha\le r\\ 1\le i,j\le n}}$ in order to ensure that the local Dunkl elements satisfy the commutativity (or anticommutativity) relations:

$$[\theta_i^{(\alpha)},\theta_j^{(\beta)}]_{\pm}=0,\quad for\ \ all\ \ 1\le i<j\le n\ \ and\ \ 1\le\alpha,\beta\le r.$$

The "natural conditions" we have in mind are:

- (*locality relations*)

$$[u_{ij}^{(\alpha)},u_{kl}^{(\beta)}]_{\pm}=0, \tag{2.13}$$

- (*twisted classical Yang–Baxter relations*)

$$[u_{ij}^{(\alpha)},u_{jk}^{(\beta)}]_{\pm}+[u_{ik}^{(\alpha)},u_{ji}^{(\beta)}]_{\pm}+[u_{ik}^{(\alpha)},u_{jk}^{(\beta)}]_{\pm}=0, \tag{2.14}$$

if i,j,k,l are distinct and $1\le\alpha,\beta\le r$.

Finally we define a multiple analogue of the three term relations algebra, denoted by $3T^{\pm}(rK_n)$, to be the quotient of the global 3-term relations algebra $3T_{n,r}^{\pm}$ modulo the two-sided ideal generated by the left hand sides of relations (1.5), (1.6) and that of the following relations

- $\left(u_{ij}^{(\alpha)}\right)^2=0,\quad [u_{ij}^{(\alpha)},u_{ij}^{(\beta)}]_{\pm}=0$, for all $i\neq j,\ \alpha\neq\beta$.

The outputs of this construction are

- noncommutative quadratic algebra $3T^{(\pm)(rK_n)}$ generated by the elements $\{u_{ij}^{(\alpha)}\}_{\substack{1\le i<j\le n\\ \alpha=1,\ldots,r}}$,
- a family of nr either mutually commuting (the case "+"), or pairwise anticommuting (the case " - ") local Dunkl elements $\{\theta_i^{(\alpha)}\}_{\substack{i=1,\ldots,n\\ \alpha=1,\ldots,r}}$.

We **expect** that the subalgebra generated by local Dunkl elements in the algebra $3T^{+}(rK_n)$ is closely related (isomorphic for $r=2$) with the coinvariant algebra of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}[X_n^{(1)},\ldots,X_n^{(r)}]$, where $X_n^{(j)}$ stands for the set of variables $\{x_1^{(j)},\ldots,x_n^{(j)}\}$. The algebra $(3T^{-}(2K_n))^{(-)})^{anti}$ has been studied in [47], and [7]. In the present paper we state only our old conjecture.

Conjecture 2.1. *(A.N. Kirillov, 2000)*

$$Hilb((3T^{-}(3K_n))^{anti},t)=(1+t)^n(1+nt)^{n-2},$$

where for any algebra A we denote by A^{anti} the quotient of algebra A by the two-sided ideal generated by the set of anticommutators $\{ab+ba\ \mid\ (a,b)\in A\times A\}$.

According to observation of M. Haiman [37], the number $2^n\ (n+1)^{n-2}$ is thought of as being equal to to the dimension of the space of triple coinvariants of the symmetric group $\mathbb{S}_n$.

2.3 Miscellany

2.3.1 Non-unitary dynamical classical Yang–Baxter algebra $DCYB_n$

Let $\widetilde{\mathcal{A}_n}$ be the quotient of the algebra $\mathfrak{F}_n$ by the two-sided ideal generated by the relations (2.2), (2.5) and (2.6). Consider elements

$$\theta_i=x_i+\sum_{a\neq i}u_{ia},\quad and\quad \bar{\theta}_j=-x_j+\sum_{b\neq j}u_{bj},\quad 1\le i<j\le n.$$

Clearly, if $i<j$, then

$$[\theta_i,\bar{\theta}_j]+[x_i,x_j]=[\sum_{k=1}^{n}x_k\ ,u_{ij}]+\sum_{k\neq i,j}\ w_{ikj},$$

where the elements w_{ijk}, $i<j$, have been defined in Lemma 2.1, (2.3).

Therefore the elements θ_i and $\bar{\theta}_j$ commute in the algebra $\widetilde{A}_n$.

In the case when $x_i=0$ for all $i=1,\ldots,n$, the relations

$$w_{ijk}:=[u_{ij},u_{ik}+u_{jk}]+[u_{ik},u_{jk}]=0,\ \ if\ \ i,j,k\ \ are\ \ all\ \ distinct)$$

are well-known as the **non-unitary classical Yang-Baxter relations**. Note that for a given triple of pair-wise distinct (i,j,k) we have in fact 6 relations. These six relations imply that $[\theta_i,\bar{\theta_j}]=0$. However, in general,

$$[\theta_i,\theta_j]=\left[\sum_{k\neq i,j}\ u_{ik}\ ,\ u_{ij}+u_{ji}\right]\neq 0.$$

- (**Dynamical classical Yang–Baxter algebra** $DCYB_n$)

In order to ensure the commutativity relations among the Dunkl elements (2.1), i.e. $[\theta_i,\theta_j]=0$ for all i,j, let us remark that if $i\neq j$, then $[\theta_,\theta_j]=[x_i+u_{ij},x_j+\mathbf{u_{ji}}]+$

$$[x_i+x_j,u_{ij}]+[u_{ij},\sum_{k=1}^{n}\ x_k]+\sum_{\substack{k=1\\k\neq i,j}}^{n}\ [u_{ij}+u_{ik},u_{jk}]+[u_{ik},\mathbf{u_{ji}}]+[x_i,u_{jk}]+[u_{ik},x_j]+[x_k,u_{ij}].$$

Definition 2.4.

Define **dynamical non-unitary classical Yang–Baxter algebra** $DNUCYB_n$ *to be the quotient of the free associative algebra* $\mathbb{Q}\langle\{x_{1\leq i\leq n}\},\ \{u_{ij}\}_{1\leq i\neq j}\rangle$ *by the two-sided ideal generated by the following set of relations*

- *(Zero curvature conditions)*

$$[x_i+u_{ij},x_j+u_{ji}]=0,\quad 1\leq i\neq j\leq n, \tag{2.15}$$

- *(Conservation lows conditions)*

$$[u_{ij},\sum_{k=1}^{n}\ x_k]=0, for\ all\ \ i\neq j,\ \ and\ \ k.$$

- *(Crossing relations)*

$$[x_i+x_j,u_{ij}]=0,\quad i\neq j.$$

- *(Twisted dynamical classical Yang–Baxter relations)*

$$[u_{ij}+u_{ik},u_{jk}]+[u_{ik},\mathbf{u_{ji}}]+[x_i,u_{jk}]+[u_{ik},x_j]+[x_k,u_{ij}]=0,\ \ i,j,k\ \ are\ \ distinct, \tag{2.16}$$

It is easy to see that the twisted classical Yang–Baxter relations

$$[u_{ij}+u_{ik},u_{jk}]+[u_{ik},\mathbf{u_{ji}}]=0,\ \ i,j,k\ \ are\ \ distinct, \tag{2.17}$$

for a fixed triple of distinct indices i,j,k contain in fact 3 different relations whereas the non-unitary classical Yang–Baxter relations

$$[u_{ij}+u_{ik},u_{jk}]+[u_{ik},u_{ji}],\ \ i,j,k\ \ are\ \ distinct,$$

contain 6 different relations for a fixed triple of distinct indices i,j,k.

Definition 2.5.

• *Define* **dynamical classical Yang–Baxter algebra** $DCYB_n$ *to be the quotient of the algebra* $DNUCYB_n$ *by the two-sided ideal generated by the elements*

$$\sum_{k\neq i,j} [u_{ik}, u_{ij}+u_{ji}], \quad for \quad all \quad i\neq j.$$

• *Define* **classical Yang–Baxter algebra** CYB_n *to be the quotient of the dynamical classical Yang–Baxter algebra* $DCYB_n$ *by the set of relations*

$$x_i = 0 \quad for \quad i = 1,\cdots,n.$$

Examples 2.1.

(**a**) Define

$$p_{ij}(z_1,\ldots,z_n) = \begin{cases} \frac{z_i}{z_i-z_j}, & \text{if } 1\leq i<j\leq n,\\ -\frac{z_j}{z_j-z_i}, & \text{if } n\geq i>j\geq 1.\end{cases}$$

Clearly, $p_{ij}+p_{ji}=1$. Now define operators $u_{ij}=p_{ij}s_{ij}$, and the truncated Dunkl operators to be $\theta_i=\sum_{j\neq i} u_{ij}$, $i=1,\ldots,n$. All these operators act on the field of rational functions $\mathbb{Q}(z_1,\ldots,z_n)$; the operator $s_{ij}=s_{ji}$ acts as the exchange operator, namely, $s_{ij}(z_i)=z_j$, $s_{ij}(z_k)=z_k \ \forall k\neq i,j$, $s_{ij}(z_j)=z_i$.

Note that this time one has

$$p_{12}p_{23} = p_{13}p_{12} + p_{23}p_{13} - p_{13}.$$

It is easy to see that the operators $\{u_{ij},\ 1\leq i\neq j\leq n\}$ satisfy relations (3.1), Section 3, and therefore, satisfy the twisted classical Yang–Baxter relations (2.11). As a corollary we obtain that the truncated Dunkl operators $\{\theta_i,\ i=1,\ldots,n\}$ are pair-wise commute. Now consider the Dunkl operator $D_i=\partial_{z_i}+h\ \theta_i$, $i=1,\ldots,n$, where h is a parameter. Clearly that $[\partial_{z_i}+\partial_{z_j},u_{ij}]=0$, and therefore $[D_i,D_j]=0 \ \ \forall i,j$. It easy to see that

$$s_{i,i+1}D_i - D_{i+1}s_{i,i+1} = h, \quad [D_i, s_{j,j+1}] = 0, \quad if \quad j\neq i,i+1.$$

In such a manner we come to the well-known representation of the degenerate affine Hecke algebra $\mathfrak{H}_n$.

2.3.2 Equivariant multiparameter 3-term relations algebras

Let $\beta, \mathbf{h}=(h_2,\ldots,h_n)$, and $\mathbf{q}=\{q_{ij}\}_{1\leq i\neq j\leq n}$, $q_{ij}=q_{ji}$ be a collection of mutually commuting parameters.

Definition 2.6. *Denote by* $3QT_n(\beta,\mathbf{h})$ *an associative algebra generated over the ring* $\mathbb{Z}[\beta,h]\Big[\{q_{ij}\}_{1\leq i<j\leq n}\Big]$ *by the set of generators* $\{x_1,\ldots,x_n\}$ *and that* $\{u_{ij}\}_{1\leq i\neq j\leq n}\}$ *subject to the set of relations*

(1) *(Locality conditions)*
$[x_i,x_j]=0$, $[u_{uj},u_{kl}]=0$, $[x_k,u_{ij}]=0$, *if* i,j,k,l *are pairwise distinct,*

(2) *(Unitarity conditions)*
$u_{ij}+u_{ji}=\beta$,

(3) *(Hecke type conditions)*
$u_{ij}\ u_{ji}=-q_{ij}$, *if* $i\neq j$,

(4) *(Twisted* 3*-term relations)*
$u_{ij}\ u_{jk}=u_{jk}\ u_{ik}-u_{ik}\ u_{ji}$, *if* i,j,k *are distinct,*

(5) *(Crossing relations)*
$x_i\ u_{ji}=-u_{ij}\ x_j-h_{max(i,j)}$, *if* $i\neq j$.

As before we define the (additive) Dunkl elements to be

$$\theta_i = x_i + \sum_{j \neq i} u_{ij}, \quad i = 1, \ldots, n.$$

It is clearly seen from the defining relations listed in Definition 2.3 that for any triple of distinct indices (i, j, k) the elements $\{x_i, x_j, x_k, u_{ji}, u_{ik}, u_{jk}\}$ satisfy the twisted dynamical Yang–Baxter relations, and thus the Dunkl elements $\{\theta_i\}_{1 \le i \le n}$ generate a commutative subalgebra in the algebra $3QT_n(\beta, \mathbf{h})$.

Theorem 2.3. *(Cf Theorem 3.3, Section 3)*
Let $k \ge 1$ be an integer. There exist polynomials

$$R_k(\mathbf{q}, \mathbf{h}, z_1, \ldots, z_n) \in \mathbb{Z}[\beta, \mathbf{q}, \{h_j - h_i\}_{1 \le i < j \le n}][Z_n] \;\; \textit{and} \;\; T_k(\beta, \mathbf{h}, z_1, \ldots, z_n) \in \mathbb{Z}[\beta, \mathbf{h}][Z_n]^{\mathbb{S}_n}$$

such that
(1) $R_k(\mathbf{q}, \mathbf{h}, z_1, \ldots, z_n) =$
$e_k^{(\mathbf{q}+\mathbf{h})}(z_1, \ldots, z_n) +$ *monomials of total degree* $\le k-2$ *w.r.t. variables* $\{z_i\}_{1 \le i \le n}$,
(2) $T_k(\beta, \mathbf{h}, z_1, \ldots, z_n) = e_k(z_1, \ldots, z_n) + \sum_{j<k} c_{j,k}\, e_j(X_n), \quad c_{j,k} \in \mathbb{Z}[\beta, \mathbf{h}]$,
(3) $R_k(\theta_1, \ldots, \theta_n) = T_k(x_1, \ldots, x_n)$,
where $e_k^{(\mathbf{q}+\mathbf{h})}(z_1, \ldots, z_n)$ denotes the multiparameter quantum elementary polynomial corresponding to the set of parameters $\{(\mathbf{q}+\mathbf{h})\} = \{q_{ij} + h_j\}_{1 \le i < j \le n}$.

It is not difficult to see that the unitarity and crossing conditions imply the following relations

$$[x_i + x_j, u_{kl}] = 0 = [x_i\, x_j, u_{kl}], \quad \textit{and} \;\; [x_i^2, u_{kl}] = 0$$

are valid for all indices $i \neq j,\ k \neq l$. As a consequence of these relations one can deduce that the all symmetric polynomials $e_k(X_n) := e_k(x_1, \ldots, x_n)$, $k = 1, \ldots, n$ belong to the center of the algebra $3QT_n(\mathbf{q}, \mathbf{h})$, and therefore one has $[\theta_i, e_k(X_n)] = 0$ for all i and k. Let us denote by $QH(\beta, \mathbf{h})$ a commutative subalgebra in the algebra $3QT_n(\beta, \mathbf{h})$ generated by the elementary symmetric polynomials $\{e_k(X_n)\}_{1 \le k \le n}$ and the Dunkl elements $\{\theta_i\}_{1 \le i \le n}$. It is an interesting **problem** to give a geometric/cohomological interpretation of the commutative algebra $QH(\beta, \mathbf{h})$. We don't know any geometric interpretation of that commutative algebra, except the special case [52]

$$\beta = 0,\ h_j = 1,\ \forall j, \quad q_{ij} := q_i\ \delta_{i+1,j}. \tag{2.18}$$

Proposition 2.2. *([52])*
Under assumptions (2.12), *the algebra* $QH(0, \mathbf{0})$ *isomorphic to the equivariant quantum cohomology* $QH_T^*(\mathcal{F}l_n)$ *of the complete flag variety* $\mathcal{F}l_n$.

Examples 2.2. *Let us list the relations among the Dunkl elements in the algebra $3QT_n(\beta, \mathbf{h})$.*
(1) $e_1(\theta_1, \ldots, \theta_n) = e_1(X_n) + \binom{n}{2}\ \beta$,
(2) $e_2^{(\mathbf{q}+\mathbf{h})}(\theta_1, \ldots, \theta_n) = e_2(X_n) + (n-1)\ \beta\ e_1(X_n) + \frac{n(n-1)(n-2)(3\ n-1)}{24}\ \beta^2, \quad n \ge 3$,
(3) $e_3^{(\mathbf{q}+\mathbf{h})}(\theta_1, \theta_2, \theta_3) = e_3(X_3) + h_3\ \beta$,
$e_3^{(\mathbf{q}+\mathbf{h})}(\theta_1, \theta_2, \theta_3, \theta_4) =$
$e_3(X_4) + \beta\ e_2(X_4) + 2\ \beta^2\ e_1(X_4) + 6\ \beta^3 + \beta\ (h_3 + 3\ h_4)$,
(4) $e_4^{(\mathbf{q}+\mathbf{h})}(\theta_1, \theta_2, \theta_3, \theta_4) + \beta\ (h_4 - h_3)\ \theta_4 = e_4(X_4) + \beta\ h_4\ e_1(X_4) + 5\ \beta^2\ h_4$.
Note that $\frac{n(n-1)(n-2)(3\ n-1)}{24} = s(n-2, 2) = e_2(1, 2, \ldots, n-1)$ is equal to the Stirling number of the first kind.

Conjecture 2.2. *The polynomial $R_k(\mathbf{q}, \mathbf{h}, Z_n)$, see Theorem 2.3, can be written as a polynomial in the variables $\{h_{ij} := h_j - h_i,\ 1 \le i < j \le n,\ z_1, \ldots, z_n,\ \beta,\ q_{ij},\ 1 \le i < j \le n\}$ with nonnegative coefficients.*

Exercises 2.1.

(1) *(***Pieri formula in the algebra** $3T_n(0,h)$ **, [52]***)*

Assume that $\beta=0$ *and* $h_2=\ldots=h_n=h$, *and denote by* $\theta_i^{(n)}$, $i=1,\ldots,n$ *the Dunkl elements* (2.1) *in the algebra* $3T_n(0,h)$ *Show that*

$$e_k(\theta_1^{(n)},\ldots,\theta_m^{(n)})=\sum_{r\ge 0}(-h)^r\ N(m-k,2\ r)\ \left\{\sum_{\substack{S\subset[1,m]\\ I=\{i_a\},\ J=\{j_a\}}} X_S\ u_{i_1,j_1}\cdots u_{i_{|I|},j_{|J|}}\right\},$$

where

$$N(a,2b)=(2\ b-1)!!\ \binom{a+2\ b}{2\ b},$$

$X_S=\prod_{s\in S}\ x_s$, *and the second summation runs over triples of sets* $\{S,I,J\}$ *such that* $S\subset[1,m]$, $I\subset[1,m]\setminus S$, $|I|+|S|+2\ r=k$, $|I|=|J|$, $1\le i_a<m<j_a\le n$ *and* $j_1\le\ldots\le j_{|I|}$.

2.3.3 Algebra $3QL_n(\boldsymbol{\beta},\mathbf{h})$

Let $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_{n-1})$, $\mathbf{h}=(h_2,\ldots,h_n)$ and $\{q_{ij}\}_{1\le i<j\le n}$ be collections of mutually commuting parameters.

Definition 2.7.

Define the algebra $3QL_n(\boldsymbol{\beta},\mathbf{h})$ *as an associative algebra over the ring of polynomials* $\mathbb{Z}[\boldsymbol{\beta},\mathbf{h},\{q_{ij}\}]$ *generated by the set of generators* $\{x_i\}_{1\le i\le n}$ *and* $\{u_{ij}\}_{1\le\neq j\le n}$ *subject to the relations* (1), (3), (5) *displayed in Definition 2.3, and*

(2a) *("generalized unitarity conditions")*

$u_{ij}+u_{ji}=\beta_{max(i,j)-1}$,

(4a) *(associative twisted* 3*-term relations)*

$u_{ij}\ u_{jk}=u_{jk}\ u_{ik}-u_{ik}\ u_{ji}$, *if* $\underline{1\le i<j<k\le n}$.

We define the Dunkl elements θ_i, $i=1,\ldots,n$, by the formula (2.1). It is necessary to stress that the Dunkl elements $\{\theta\}_{1\le i\le n}$ do not commute in the algebra $3QL_n(\boldsymbol{\beta},\mathbf{h})$ but satisfy a noncommutative analogue of the relations displayed in Theorem 2.3. Namely, one needs to replace the both elementary polynomials $e_k(Z_n)$ and the quantum multiparameter elementary polynomials $e_k^{(\mathbf{q})}(Z_n)$ by its noncommutative versions. Recall that the noncommutative elementary polynomial $\underline{e}_k(Z_n)$ is equal to

$$\sum_{1\le j_1<j_2<\ldots<j_k\le n} z_{j_1}\ z_{j_2}\cdots z_{j_k}$$

and the noncommutative quantum multiparameters elementary polynomial $\underline{e}_k^{(\mathbf{q})}(Z_n)$ is equal to

$$\sum_{\ell}\ \sum_{\substack{1\le i_1<\ldots<j_\ell\le n\\ i_1<j_1,\ldots,i_\ell<j_\ell}} \underline{e}_{k-2\ell}(Z_{\overline{I\cup J}})\ \prod_{a=1}^{\ell}\ u_{i_a,j_a},$$

where $I=(i_1,\ldots.i_\ell)$, $J=(j_1,\ldots,j_\ell)$ should be distinct elements of the set $\{1,\ldots,n\}$, and $Z_{\overline{I\cup J}}$ denotes set of variables z_a for which the subscript a is neither one of i_m nor one of the j_m.

Example 2.1.

- $\underline{e}_2^{(\mathbf{q}+\mathbf{h})}(\theta_1,\ldots,\theta_n)=e_2(X_n)+(\sum_{j=1}^{n-1}\ \beta_j)\ e_1(X_n)+\sum_{1\le a<b\le n-1}\ a\ b\ \beta_a\ \beta_b.$

- $\underline{e}_3^{(\mathbf{q}+\mathbf{h})}(\theta_1,\theta_2,\theta_3,\theta_4)+(\beta_3-\beta_1)(\theta_3\ \theta_4+q_{34}+h_4+\beta_2(\theta_1+\theta_2))+(\beta_3-\beta_2)((\theta_1+\theta_2)\theta_4+q_{14}+q_{24}+2\ h_4+\beta_1\ \theta_3)=e_3(X_4)+\beta_3 e_2(X_4)+(\beta_1\beta_3+\beta_2\beta_3+\beta_3^2-\beta_1\beta_2)e_1(X_4)+(3\beta_3^2-\beta_1\beta_2)(\beta_1+2\beta_2)+\beta_1(h_3+h_4)+2\beta_2 h_4.$
- $\underline{e}_4^{(\mathbf{q}+\mathbf{h})}(\theta_1,\theta_2,\theta_3,\theta_4)+(\beta_2 h_4-\beta_1 h_3)\theta_4+h_4(\beta_2-\beta_1)\theta_3=e_4(X_4)+\beta_2 h_4 e_1(X_4)+\beta_2 h_4(2\beta_2+3\beta_3).$

Project 2.2. *(Noncommutative universal Schubert polynomials)*

Let $w\in\mathbb{S}_n$ be a permutation and $\mathfrak{S}_w(Z_n)$ be the corresponding Schubert polynomial.

(1) *There exists a (noncommutative) polynomial $\mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$ with non-negative integer coefficients such that the following identity*

$$\mathfrak{S}_w(\theta_1,\ldots,\theta_n)=\mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$$

holds in the algebra $3T_n^{(0)}$, where $\{\theta_j\}_{1\le j\le n}$ are the Dunkl elements in the algebra $3T_n^{(0)}$.

(2) *There exist polynomials $R_w(\beta,\mathbf{q},\mathbf{h},Z_n)\in\mathbb{N}[\beta,\mathbf{q},h_j-h_{i_{1\le i<j\le n}}][Z_n]$ and $T_w(\beta,\mathbf{h},Z_n)\in\mathbb{Z}[\beta,\mathbf{h}][Z_n]$ such that the following identity*

$$R_w(\beta,\mathbf{q},\mathbf{h},\theta_1,\ldots,\theta_n)=T_w(\beta,\mathbf{h},X_n)+\mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$$

holds in the algebra $3QT_n(\beta,\mathbf{h})$.

3) *Let $r\in\mathbb{Z}_{\ge 2}$ and $N=n_1+\cdots n_r,$, $n_j\in\mathbb{Z}_{\ge 1},\forall j$, be a composition of N, and set $N_j=n_1+\cdots+n_j$, $j\ge 1$, $N_0=0$, Eliminate the Dunkl elements $\theta_{N_{r-1}+1}^{(N)},\ldots\theta_N^{(N)}$ from the set of relations among the Dunkl elements $\theta_1^{(N)},\ldots,\theta_N^{(N)}$ in the algebra $3QT_n(\beta,\mathbf{h})$, by the use of the degree $1,\ldots,n_r$ relations among the former. As a result one obtains a set consisting of N_{r-1} relations among the N_{r-1} elements*

$$\theta_{j.k_j}^{(N)}:=e_{k_j}^{(\mathbf{q})}(\theta_{N_{j-1}+1}^{(N)},\ldots,\theta_{N_j}^{(N)}),\quad 1\le k_j\le n_j,\quad 1\le j\le r-1.$$

Give a geometric interpretation of the commutative subalgebra $QH_{n_1,\ldots,n_r}(\beta,\mathbf{h})\subset 3QT_n(\beta,\mathbf{h})$ generated by the set of elements $\theta_{j,k_j}^{(N)}$, $1\le k_j\le n_j, j=1,\ldots,r-1$.

2.3.4 Dunkl and Knizhnik–Zamolodchikov elements

- Assume that $\forall i,\ x_i=0$, and generators $\{u_{ij},1\le i<j\le n\}$ satisfy the locality conditions (2.2) and the classical Yang–Baxter relations

$$[u_{ij},u_{ik}+u_{jk}]+[u_{ik},u_{jk}]=0,\quad if\quad 1\le i<j<k\le n.$$

Let $y,z,t_1,\ldots,t_n$ be parameters, consider the rational function

$$F_{CYB}(z;\mathbf{t}):=F_{CYB}(z;t_1,\ldots,t_n)=\sum_{1\le i<j\le n}\frac{(t_i-t_j)u_{ij}}{(z-t_i)(z-t_j)}.$$

Then

$$[F_{CYB}(z;\mathbf{t}),F_{CYB}(y;\mathbf{t})]=0,\quad and\quad Res_{z=t_i}F_{CYB}(z;\mathbf{t})=\theta_i.$$

- Now assume that a set of generators $\{c_{ij},\ 1\le i\ne j\le n\}$ satisfy the locality and symmetry (i.e. $c_{ij}=c_{ji}$) conditions, and the Kohno–Drinfeld relations:

$$[c_{ij},c_{kl}]=0,\quad if\quad \{i,j\}\cap\{k,l\}=\emptyset,\quad [c_{ij},c_{jk}+c_{ik}]=0=[c_{ij}+c_{ik},c_{jk}],\quad i<j<k.$$

Let $y,z,t_1,\ldots,t_n$ be parameters, consider the rational function

$$F_{KD}(z;\mathbf{t}):=F_{KD}(z;t_1,\ldots,t_n)=\sum_{1\le i\ne j\le n}\frac{c_{ij}}{(z-t_i)(t_i-t_j)}=\sum_{1\le i<j\le n}\frac{c_{ij}}{(z-t_i)(z-t_j)}.$$

Then

$$[F_{KD}(z;\mathbf{t}), F_{KD}(y;\mathbf{t})] = 0, \quad and \quad Res_{z=t_i} F_{KD}(z;\mathbf{t}) = KZ_i,$$

where

$$KZ_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{c_{ij}}{t_i - t_j}$$

denotes the truncated *Knizhnik-Zamolodchikov* element.

2.3.5 Dunkl and Gaudin operators

(**a**) (**Rational Dunkl operators**) Consider the quotient of the algebra $DCYB_n$, see Definition 2.2, by the two-sided ideal generated by elements

$$\{[x_i + x_j, u_{ij}]\} \;\; and \;\; \{[x_k, u_{ij}], \; k \neq i, j\}.$$

Clearly the Dunkl elements (2.1) mutually commute. Now let us consider the so-called *Calogero–Moser* representation of the algebra $DCYB_n$ on the ring of polynomials $R_n := \mathbb{R}[z_1, \ldots, z_n]$ given by

$$x_i(p(z)) = \lambda \, \frac{\partial \, p(z)}{\partial z_i}, \quad u_{ij}(p(z)) = \frac{1}{z_i - z_j}(1 - s_{ij}) \; p(z), \quad p(z) \in R_n.$$

The symmetric group $\mathbb{S}_n$ acts on the ring R_n by means of transpositions $s_{ij} \in \mathbb{S}_n : \; s_{ij}(z_i) = z_j, \; s_{ij}(z_j) = z_i, \; s_{ij}(z_k) = z_k, \; if \; k \neq i, j,$

In the Calogero–Moser representation the Dunkl elements θ_i becomes the rational Dunkl operators [21], see Definition 1.1. Moreover, one has $[x_k, u_{ij}] = 0$, if $k \neq i, j$, and

$$x_i \; u_{ij} = u_{ij} \; x_j + \frac{1}{z_i - z_j} \; (x_i - x_j - u_{ij}), \quad x_j \; u_{ij} = u_{ij} \; x_i - \frac{1}{z_i - z_j} \; (x_i - x_j - u_{ij}).$$

(**b**) (**Gaudin operators**)

The Dunkl–Gaudin representation of the algebra $DCYB_n$ is defined on the field of rational functions $K_n := \mathbb{R}(q_1, \ldots, q_n)$ and given by

$$x_i(f(q)) := \lambda \frac{\partial \; f(q)}{\partial q_i}, \qquad u_{ij} = \frac{s_{ij}}{q_i - q_j}, \quad f(q) \in K_n,$$

but this time we <u>assume</u> that $w(q_i) = q_i, \forall i \in [1, n]$ and for all $w \in \mathbb{S}_n$. In the Dunkl–Gaudin representation the Dunkl elements becomes the rational Gaudin operators, see e.g. [71]. Moreover, one has $[x_k, u_{ij}] = 0$, if $k \neq i, j$, and

$$x_i \; u_{ij} = u_{ij} \; x_j - \frac{u_{ij}}{q_i - q_j}, \quad x_j \; u_{ij} = u_{ij} \; x_i + \frac{u_{ij}}{q_i - q_j}.$$

Comments 2.4.

It is easy to check that if $f \in \mathbb{R}[z_1, \ldots, z_n]$, then the following commutation relations are true

$$x_i \; f = f \; x_i + \frac{\partial}{\partial_{z_i}}(f), \quad u_{ij} \; f = s_{ij}(f) \; u_{ij} + \partial_{z_i, z_j}(f).$$

Using these relations it easy to check that in the both cases (**a**) and (**b**) the elementary symmetric polynomials $e_k(x_1, \ldots, x_n)$ commute with the all generators $\{u_{ij}\}_{1 \le i,j \le n}$, and therefore commute with the all Dunkl elements $\{\theta_i\}_{1 \le i \le n}$. Let us <u>stress</u> that $[\theta_i, x_k] \neq 0$ for all $1 \le i, k \le n$.

Project 2.3.

Describe a commutative algebra generated by the Dunkl elements $\{\theta_i\}_{1 \le i \le n}$ and the elementary symmetric polynomials $\{e_k(x_1, \ldots, x_n)\}_{1 \le k \le n}$.

2.3.6 Representation of the algebra $3T_n$ on the free algebra $\mathbb{Z}\langle t_1,\ldots,t_n\rangle$

.

Let $\mathcal{F}_n=\mathbb{Z}\langle t_1,\ldots,t_n\rangle$ be free associative algebra over the ring of integers $\mathbb{Z}$, equipped with the action of the symmetric group $\mathbb{S}_n$: $s_{ij}(t_i)=t_j,\ s_{ij}(t_k)=t_k, \forall k\neq i,j$.

Define the action of $u_{ij}\in 3T_n$ on the set of generators of the algebra $\mathcal{F}_n$ as follows

$$u_{ij}(t_k)=\delta_{i,k}\ t_i\ t_j-\delta_{j,k}\ t_j\ t_i. \tag{2.19}$$

The action of generator u_{ij} on the whole algebra $\mathcal{F}_n$ is defined by linearity and the twisted Leibniz rule:

$$u_{ij}(1)=0,\quad u_{ij}(a+b)=u_{ij}(a)+u_{ij}(b),\quad u_{ij}(a\ b)=u_{ij}(a)\ b+s_{ij}(a)\ u_{ij}(b).$$

It is easy to see from (2.15) that

$$s_{ij}\ u_{jk}=u_{ik}\ s_{ij},\quad s_{ij}\ u_{kl}=u_{kl}\ s_{ij},\quad if\quad \{i,j\}\cap\{k,l\}=\emptyset,\quad u_{ij}+u_{ji}=0. \tag{2.20}$$

Now let us consider operator

$$u_{ijk}:=u_{ij}\ u_{jk}-u_{jk}\ u_{ik}-u_{ik}\ u_{ij},\quad 1\le i<j<k\le n.$$

Lemma 2.7.

$$u_{ijk}(a\ b)=u_{ijk}(a)\ b+s_{ij}\ s_{jk}(a)\ u_{ijk}(b),\quad a,b\in\mathcal{F}_n$$

.

Lemma 2.8.

$$u_{ijk}(a)=0\quad \forall a\in\mathcal{F}_n.$$

Indeed,

$u_{ijk}(t_i)=-u_{jk}(u_{ij}(t_i))-u_{ik}(u_{ij}(t_i))=-t_i\ u_{jk}(t_k)-u_{ik}(t_i)\ t_j=t_i(t_k\ t_j)-(t_i\ t_k)\ t_j=0.$
$u_{ijk}(t_k)=u_{ij}(u_{jk}(t_k))-u_{jk}(u_{ik}(t_k))=-u_{ij}(t_k\ t_j)+u_{jk}(t_k\ t_i)=t_k\ (u_{ij}(t_j)+u_{jk}(t_k)\ t_i=0,$
$u_{ijk}(t_j)=u_{ij}(u_{jk}(t_j))-u_{ik}(u_{ij}(t_j))=-u_{ij}(t_j)\ t_k-t_j\ u_{ik}(t_i)=(t_j\ t_i)\ t_k-t_j\ (t_i\ t_k)=0.$
Therefore Lemma 2.8 follows from Lemma 2.7.

Let $\mathcal{F}_n^{\bullet}$ be the quotient of the free algebra $\mathcal{F}_n$ by the two-sided ideal generated by elements $t_i^2\ t_j-t_j\ t_i^2,\ \ 1\le i\neq j\le n$. Since $u_{i,j}^2(t_i)=t_i\ t_j^2-t_j^2\ t_i$, one can define a representation of the algebra $3T_n^{(0)}$ on that $\mathcal{F}_n^{\bullet}$. One can also define a representation of the algebra $3T_n^{(0)}$ on that $\mathcal{F}_n^{(0)}$, where $\mathcal{F}_n^{(0)}$ denotes the quotient of the algebra $\mathcal{F}_n$ by the two-sided ideal generated by elements $\{t_i^2,\ 1\le i\le n\}$. Note that $(u_{i,k}\ u_{j,k}\ u_{i,j})(t_k)=[t_i\ t_j\ t_i,t_k]\neq 0$ in the algebra $\mathcal{F}_n^{(0)}$, but the elements $u_{i,j}\ u_{i,k}\ u_{j,k}\ u_{i,j},\ \ 1\le i<j<k\le n$, from the kernel of the Calogero–Moser representation, act trivially both on the algebras $\mathcal{F}_n^{(0)}$ and that $\mathcal{F}_n^{\bullet}$.

Note finally that the algebra $\mathcal{F}_n^{(0}$ is <u>*Koszul*</u> and has Hilbert series $Hilb(\mathcal{F}_n^{(0)},t)=\frac{1+t}{1-(n-1)\ t}$, whereas the algebra $\mathcal{F}_n^{\bullet}$ is <u>*not*</u> Koszul for $n\ge 3$, and

$$Hilb(\mathcal{F}_n^{\bullet},t)=\frac{1}{(1-t)(1-(n-1)t)(1-t^2)^{n-1}}.$$

$\blacksquare$

2.3.7 Fulton universal ring, multiparameter quantum cohomology and $FKTL$

(*The Fulton universal ring [31], multiparameter quantum cohomology of flag varieties [26] and the full Kostant–Toda lattice [30]*)

Let $X_n = (x_1, \ldots, x_n)$ be be a set of variables, and

$$\mathbf{g} := \mathbf{g}^{(n)} = \{g_a[b] \mid a \geq 1,\ b \geq 1,\ a+b \leq n\}$$

be a set of parameters; we put $deg(x_i) = 1$ and $deg(g_a[b]) = b+1$ and $g_k[0] := x_k$, $k = 1, \ldots, n$. For a subset $S \subset [1,n]$ we denote by X_S the set of variables $\{x_i \mid i \in S\}$.
Let t be an auxiliary variable, denote by $M = (m_{ij})_{1 \leq i,j \leq n}$ the matrix of size n by n with the following elements:

$$m_{i,j} = \begin{cases} x_i + t, & \text{if } i = j, \\ g_i[j-i], & \text{if } j > i, \\ -1, & \text{if } i - j = 1, \\ 0, & \text{if } i - j > 1. \end{cases}$$

Let $P_n(X_n, t) = det|M|$.

Definition 2.8. *The Fulton universal ring $\mathcal{R}_{n-1}$ is defined to be the quotient* [19]

$$\mathcal{R}_{n-1} = \mathbb{Z}[\mathbf{g}^{(n)}][x_1, \ldots, x_n] \Big/ \Big\langle P_n(X_n, t) - t^n \Big\rangle.$$

Lemma 2.9. *Let $P_n(X_n, t) = \sum_{k=0}^{n} c_k(n) t^{n-k}$, $c_0(n) = 1$. Then*

$$c_k(n) := c_k(n; X_n, \mathbf{g}^{(n)}) = \sum_{\substack{1 \leq i_1 < i_2 < \ldots < i_s < n \\ j_1 \geq 1, \ldots, j_s \geq 1 \\ m := \sum (j_a + 1) \leq n}} \prod_{a=1}^{s} g_{i_a}[j_a]\ e_{k-m}(X_{[1,n]} \setminus \cup_{a=1}^{s}\ {[i_a, i_a + j_a]}), \quad (2.21)$$

where in the summation we assume additionally that the sets $[i_a, i_a + j_a] := \{i_a, i_a + 1, \ldots, i_a + j_a\}$, $a = 1, \ldots, s$, *are pairwise disjoint.*

It is clear that $\mathcal{R}_{n-1} = \mathbb{Z}[\mathbf{g}^{(n)}][x_1, \ldots, x_n] \Big/ \Big\langle c_n(1), \ldots, c_n(n) \Big\rangle$.
One can easily see that the coefficients $c_k(n)$ and $g_m[k]$ satisfy the following recurrence relations [31]:

$$c_k(n) = c_k(n-1) + \sum_{a=0}^{k-1} g_{n-a}[a] c_{k-a-1}(n-a-1), \quad c_0(n) = 1, \qquad (2.22)$$

$$g_m[k] = c_{k+1}(m+k) - c_{k+1}(m+k-1) - \sum_{a=0}^{k-1} g_{m+k-a}[a] c_{k-a}(m+k-a), \quad g_m[0] := x_m.$$

On the other hand, let $\{q_{ij}\}_{1 \leq i < j \leq n}$ be a set of (quantum) parameters, and $e_k^{(\mathbf{q})}(X_n)$ be the multiparameter quantum elementary polynomial introduced in [26]. We are interested in to

[19] If $P(t, X_n) = \sum_{k \geq 1} f_k(X_n)\ t^k$, $f_k(X_n) \in \mathbb{Q}[X_n]$ is a polynomial, we denote by

$$\Big\langle P(t, X_n) \Big\rangle$$

the ideal in the polynomial ring $\mathbb{Q}[X_n]$ generated by the coefficients $\{f_1, f_2, \ldots\}$.

describe a set of relations between the parameters $\{g_i[j]\}_{\substack{i\ge 1, j\ge 1\\ i+j\le n}}$ and the quantum parameters $\{q_{ij}\}_{1\le i<j\le n}$ which implies that

$$c_k(n) = e_k^{(\mathbf{q})}(X_n), \quad for \quad k=1,\ldots,n.$$

To start with, let us recall the recurrence relations among the quantum elementary polynomials, cf [76]. To do so, consider the generating function

$$E_n(X_n;\{q_{ij}\}_{1\le i<j\le n}) = \sum_{k=0}^{n} e_k^{(\mathbf{q})}(X_n)\ t^{n-k}.$$

Lemma 2.10. *([25],[76]) One has*

$$E_n(X_n;\{q_{ij}\}_{1\le i<j\le n}) = (t+x_n)\ E_{n-1}(X_{n-1};\{q_{ij}\}_{1\le i<j\le n-1}) +$$

$$\sum_{j=1}^{n-1} q_{jn}\ E_{n-2}(X_{[1,n-1]}\ \backslash \{j\}; \{q_{a,b}\}_{\substack{1\le a<b\le n-1\\ a\ne j, b\ne j}}.$$

Proposition 2.3.
Parameters $\{g_a[b]\}$ can be expressed polynomially in terms of quantum parameters $\{q_{ij}\}$ and variables $x_1,\ldots,x_n$, in a such way that

$$c_k(n) = e_k^{(\mathbf{q})}(X_n), \quad \forall k,n.$$

Moreover,

- $g_a[b] = \sum_{k=1}^{a} q_{k,a+b} \prod_{j=a+1}^{a+b-1} (x_j - x_k)$ *+ lower degree polynomials in* $x_1,\ldots,x_n$.

- *The quantum parameters $\{q_{ij}\}$ can be presented as rational functions in terms of variables $x_1,\ldots,x_n$ and polynomially in terms of parameters $\{g_a[b]\}$ such that the equality $c_k(n) = e_k^{(\mathbf{q})}(X_n)$ holds for all k,n.*

In other words, the transformation

$$\{q_{ij}\}_{1\le i<j\le n} \longleftrightarrow \{g_a[b]\}_{\substack{a+b\le n\\ a\ge 1,\ b\ge 1}}$$

defines a "birational transformation" between the algebra $\mathbb{Z}[\mathbf{g}^{(n)}][X_n]/\Big\langle P_n(X_n,t) - t^n\Big\rangle$ and multiparameter quantum deformation of the algebra $H^*(\mathcal{F}l_n,\mathbb{Z})$.

Example 2.2. *Clearly,*
$g_{n-1}[1] = \sum_{j=1}^{n-1} q_{j,n}, n\ge 2$ *and* $g_{n-2}[2] = \sum_{j=1}^{n-2} q_{jn}\ (x_{n-1}-x_j),\ n\ge 3.$ *Moreover*
$g_1[3] = q_{14}\Big((x_2-x_1)(x_3-x_1) + q_{23} - q_{12}\Big) + q_{24}\Big(q_{13}-q_{12}\Big),$
$g_2[3] = q_{15}\Big((x_3-x_1)(x_4-x_1) + q_{24} + q_{34} - q_{12} - q_{13}\Big) +$
$q_{25}\Big((x_3-x_2)(x_4-x_2) + q_{14} + q_{34} - q_{12} - q_{23}\Big) + q_{35}\Big(q_{14} + q_{24} - q_{13} - q_{23}\Big).$

Comments 2.5. The full Kostant–Toda lattice (FKTL for short) has been introduced in the end of $70's$ of the last century by B. Kostant and since that time has been extensively studied both in Mathematical and Physical literature. We refer the reader to the original paper by B.Kostant [30] (a), and [30] (b), for the definition of the $FKTL$ and its basic properties. In the present paper we just want to point out on a connection of the Fulton universal ring and hence the multiparameter deformation of the cohomology ring of complete flag varieties, and polynomial integral of motion of the FKTL. Namely,

Polynomials $c_k(n; X_n, \mathbf{g}^{(n)})$ defined by (2.17) coincide with the polynomial integrals of motion of the FKTL.

It seems an interesting task to clarify a meaning of the $FKTL$ rational integrals of motion in the context of the universal Schubert Calculus [31] and the algebra $3HT_n(0)$, as well as any meaning of universal Schubert or Grothendieck polynomials in the context of the Toda or full Kostant-Toda lattices.

3 Algebra $3HT_n$

Consider the twisted classical Yang–Baxter relation

$$[u_{ij} + u_{ia}, u_{ja}] + [u_{ia}, u_{ji}] = 0, \quad \textit{where} \;\; i, j, k \;\; \textit{are} \;\; \textit{distinct}.$$

Having in mind applications of the Dunkl elements to Combinatorics and Algebraic Geometry, we split the above relation into two relations

$$\underline{u_{ij}\ u_{jk} = u_{jk}\ u_{ik} - u_{ik}\ u_{ji}} \quad \textit{and} \quad \underline{u_{jk}\ u_{ij} = u_{ik}\ u_{jk} - u_{ji}\ u_{ik}} \tag{3.1}$$

and impose the following *unitarity* constraints

$$u_{ij} + u_{ji} = \beta,$$

where β is a central element. Summarizing, we come to the following definition.

Definition 3.1.
Define algebra $3T_n(\beta)$ to be the quotient of the free associative algebra

$$\mathbb{Z}[\beta] \left\langle \;\; u_{ij}, \;\; 1 \le i < j \le n \;\right\rangle$$

by the set of relations

- *(* **Locality** *)* $u_{ij}\ u_{kl} = u_{kl}\ u_{ij}$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$,
- *(* **3-term relations** *)*

$u_{ij}\ u_{jk} = u_{ik}\ u_{ij} + u_{jk}\ u_{ik} - \beta\ u_{ik}$, *and* $u_{jk}\ u_{ij} = u_{ij}\ u_{ik} + u_{ik}\ u_{jk} - \beta\ u_{ik}$,
if $1 \le i < j < k \le n$.

It is clear that the elements $\{u_{ij}, u_{jk}, u_{ik},\ 1 \le i < j < k \le n\}$ satisfy the classical Yang–Baxter relations, and therefore, the elements $\{\theta_i := \sum_{j \ne i}\ u_{ij},\ 1 = 1, \ldots, n\}$ form a mutually commuting set of elements in the algebra $3T_n(\beta)$.

Definition 3.2. *We will call* $\theta_1, \ldots, \theta_n$ *by the (universal) additive Dunkl elements.*

For each pair of indices $i < j$, we define element $q_{ij} := u_{ij}^2 - \beta\ u_{ij} \in 3T_n(\beta)$.

Lemma 3.1.

(1) *The elements* $\{q_{ij},\ 1 \le i < j \le n\}$ *satisfy the Kohno– Drinfeld relations (known also as the horizontal four term relations)*

$$q_{ij}\ q_{kl} = q_{kl}\ q_{ij}, \quad if \quad \{i,j\} \cap \{k,l\} = \emptyset,$$
$$[q_{ij}, q_{ik} + q_{jk}] = 0, \quad [q_{ij} + q_{ik}, q_{jk}] = 0, \quad if \quad i < j < k.$$

(2) *For a triple* $(i < j < k)$ *define* $u_{ijk} := u_{ij} - u_{ik} + u_{jk}$. *Then*

$$u_{ijk}^2 = \beta\ u_{ijk} + q_{ij} + q_{ik} + q_{jk}.$$

(3) *(Deviation from the Yang–Baxter and Coxeter relations)*
$u_{ij}\ u_{ik}\ u_{jk} - u_{jk}\ u_{ik}\ u_{ij} = [u_{ik}, q_{ij}] = [q_{jk}, u_{ik}]$,
$u_{ij}\ u_{jk}\ u_{ij} - u_{jk}\ u_{ij}\ u_{jk} = q_{ij}\ u_{ik} - u_{ik}\ q_{jk}$.

Comments 3.1. It is easy to see that the horizontal 4-term relations listed in Lemma 3.1, (1), are consequences of the locality conditions among the generators $\{q_{ij}\}$, together with the commutativity conditions among the Jucys–Murphy elements

$$d_i := \sum_{j=i+1}^{n} q_{ij}, \quad i = 2, \ldots, n,$$

namely, $[d_i, d_j] = 0$. In [47] we describe some properties of a commutative subalgebra generated by the Jucys-Murphy elements in the (nil) Kohno–Drinfeld algebra. It is well-known that the Jucys–Murphy elements generate a maximal commutative subalgebra in the group ring of the symmetric group $\mathbb{S}_n$. It is an open problem

describe defining relations among the Jucys–Murphy elements in the group ring $\mathbb{Z}[\mathbb{S}_n]$.

■

Finally we introduce the "Hecke quotient" of the algebra $3T_n(\beta)$, denoted by $3HT_n(\beta)$.

Definition 3.3. *Define algebra* $3HT_n(\beta)$ *to be the quotient of the algebra* $3T_n(\beta)$ *by the set of relations*

$$q_{ij}\ q_{kl} = q_{kl}\ q_{ij}, \quad for \quad all \quad i,j,k,l.$$

In other words we assume that the all elements $\{q_{ij},\quad 1 \le i < j \le n\}$ are central in the algebra $3T_n(\beta)$. From Lemma 3.1 follows immediately that in the algebra $3HT_n(\beta)$ the elements $\{u_{ij}\}$ satisfy the multiplicative (or quantum) Yang–Baxter relations

$$u_{ij}\ u_{ik}\ u_{jk} = u_{jk}\ u_{ik}\ u_{ij}, \quad if \quad i < j < k. \tag{3.2}$$

3.1 Modified three term relations algebra $3MT_n(\beta, \psi)$

Let $\beta, \{q_{ij} = q_{ji}, \psi_{ij} = \psi_{ji},\ 1 \le i,j \le n\}$, be a set of mutually commuting elements.

Definition 3.4. *Modified* 3*-term relation algebra* $3MT_n(\beta, \psi)$ *is an associative algebra over the ring of polynomials* $\mathbb{Z}[\beta, q_{ij}, \psi_{ij}]$ *with the set of generators* $\{u_{ij},\ 1 \le i,j \le n\}$ *subject to the set of relations*

- $u_{ij} + u_{ji} = 0,\ \ u_{ij}\ u_{kl} = u_{kl}\ u_{ij},\ \ if\ \{i,j\} \cap \{k,l\} = \emptyset;$
- *(* **three term relations** *)*

$$u_{ij}\ u_{jk} + u_{ki}\ u_{ij} + u_{jk}\ u_{ki} = 0, \quad if \quad i,j,k \quad are \quad distinct;$$

- $u_{ij}^2 = \beta\ u_{uj} + q_{ij} + \psi_{ij}$, *if* $i \ne j$;
- $u_{ij}\ \psi_{kl} = \psi_{kl}\ u_{ij}$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$;
- *(***exchange relations***)* $u_{ij}\ \psi_{jk} = \psi_{ik}\ u_{ij}$, *if* i,j,k *are distinct;*
- *elements* β, $\{q_{ij},\ 1 \le i,j \le n\}$ *are* *central.*

It is easy to see that in the algebra $3MT_n(\beta,\psi)$ the generators $\{u_{ij}\}$ satisfy the modified *Coxeter* and modified *quantum Yang–Baxter relations*, namely

- (**modified Coxeter relations**) $u_{ij}\ u_{jk}\ u_{ij} - u_{jk}\ u_{ij}\ u_{jk} = (q_{ij} - q_{jk})\ u_{ik}$,
- (**modified quantum Yang–Baxter relations**)

$$u_{ij}\ u_{ik}\ u_{jk} - u_{jk}\ u_{ik}\ u_{ij} = (\psi_{jk} - \psi_{ij})\ u_{ik},$$

if i,j,k are distinct

Clearly the additive Dunkl elements $\{\theta_i := \sum_{j \ne i}\ u_{ij},\ i = 1,\ldots,n\}$ generate a commutative subalgebra in $3MT_n(\beta,\psi)$.

It is still possible to describe relations among the additive Dunkl elements [47], cf [50]. However we don't know any geometric interpretation of the commutative algebra obtained. It is not unlikely that this commutative subalgebra is a common generalization of the small quantum cohomology and elliptic cohomology (remains to be defined !) of complete flag varieties.

The algebra $3MT_n(\beta = 0,\psi)$ has an elliptic representation [47], [50]. Namely,

$$u_{ij} := \sigma_{\lambda_i - \lambda_j}(z_i - z_j)\ s_{ij}, \quad q_{ij} = \wp(\lambda_i - \lambda_j), \quad \psi_{ij} = -\wp(z_i - z_j),$$

where $\{\lambda_i, i = 1,\ldots,n\}$ is a set of parameters (e.g. complex numbers), and $\{z_1,\ldots,z_n\}$ is a set of variables; $s_{ij},\ i < j$, denotes the transposition that swaps i on j and fixes all other variables;

$$\sigma_\lambda(z) := \frac{\theta(z-\lambda)\ \theta'(0)}{\theta(z)\theta(\lambda)}$$

denotes the *Kronecker sigma function*; $\wp(z)$ denotes the *Weierstrass P-function.*

►► ("Multiplicative" version of the elliptic representation)

Let q be parameter. In this place we will use the same symbol $\theta(x)$ to denote the "multiplicative" version of the Riemann theta function

$$\theta(x) := \theta(x;q) = (x;q)_\infty\ (q/x;q)_\infty,$$

where by definition $(x;q)_\infty = (x)_\infty = \prod_{k \ge 0}(1 - x\ q^k)$. Let us state some well-known properties of the Riemann theta function :

- | $\theta(qx;q) = \theta(1/x;q) = -x^{-1}\ \theta(x;q)$,
- (Functional equation)

$$x/y\ \theta(u\ x^{\pm 1})\ \theta(y\ v^{\pm 1}) + \theta(u\ v^{\pm 1})\ \theta(x\ y^{\pm 1}) = \theta(u\ y^{\pm 1})\ \theta(x\ v^{\pm 1}),$$

where by definition $\theta(x\ y^{\pm 1}) := \theta(x\ y)\ \theta(x\ y^{-1})$.

- (Jacobi triple product identity)

$(q;q)_\infty\ \theta(x;q) = \sum_{n \in \mathbb{Z}}\ (-x)^n\ q^{\binom{n}{2}}$.

One can easily check that after the change of variables

$$x := (\frac{z^2}{\lambda\ w})^{1/2}, \quad y := (\frac{w}{\lambda})^{1/2}, \quad u := (\frac{w}{\lambda\ \mu^2})^{1/2}, \quad v := (w\ \lambda)^{1/2},$$

the functional equation for the Riemann theta function $\theta(x)$ takes the following form

$$\sigma_\lambda(z)\ \sigma\mu(w) = \sigma_{\lambda\mu}(z)\sigma_\mu(w/z) + \sigma_{\lambda\mu}(w)\ \sigma_\lambda(z/w),$$

where

$$\sigma_\lambda(z) := \frac{\theta(z/\lambda)}{\theta(z)\ \theta(\lambda^{-1}}$$

denotes the *Kronecker sigma function.* Therefor, the operators

$$u_{ij}(f) := \sigma_{\lambda_i/\lambda_j}(z_i/z_j)\ s_{ij}(f),$$

where s_{ij} denotes the exchange operator which swaps the variables z_i and z_j, namely $s_{ij}(z_i) = z_j$, $s_{ij}(z_j) = z_i$, $s_{ij}(z_k) = z_k$, $\forall k \ne i,j$, and s_{ij} acts trivially on dynamical parameters λ_i, namely, $s_{ij}(\lambda_k) = \lambda_k$, $\forall k$, give rise to a representation of the algebra $3MT_n(0,\psi)$.

◄ ◄

The 3-term relations among the elements $\{u_{ij}\}$ are consequence (in fact equivalent) to the famous *Jacobi-Riemann* 3-term relation of degree 4 among the theta function $\theta(z)$, see e.g. [97], p.451, Example 5. In several cases, see Introduction, relations (**A**) and (**B**), identities among the Riemann theta functions can be rewritten in terms of the elliptic Kronecker sigma functions and turn out to be a consequence of certain relations in the algebra $3MT_n(0,\psi)$ for some integer n, and vice versa [20].

The algebra $3HT_n(\beta)$ is the quotient of algebra $3MT_n(\beta,\psi)$ by the two-sided ideal generated by the elements $\{\psi_{ij}\}$. Therefore the elements $\{u_{ij}\}$ of the algebra $3HT_n(\beta)$ satisfy the quantum Yang– Baxter relations $u_{ij}\ u_{ik}\ u_{jk} = u_{jk}\ u_{ik}\ u_{ij}$, $i<j<k$, and as a consequence, the multiplicative Dunkl elements

$$\Theta_i = \prod_{a=i-1}^{1} (1+h\ u_{a,i})^{-1} \prod_{a=i+1}^{n} (1+h\ u_{i,a}),\ i=1,\ldots,n,\quad u_{0,i} = u_{i,n+1} = 0$$

generate a commutative subalgebra in the algebra $3HT_n(\beta)$, see Section 3.1. We emphasize that the Dunkl elements $\Theta_j, j=1,\ldots,n$, do not pairwise commute in the algebra $3MT_n(\beta,\psi)$, if $\psi_{ij} \ne 0$ for some $i \ne j$. One way to construct a multiplicative analog of additive Dunkl elements $\theta_i := \sum_{j\ne i} u_{ij}$ is to add a new set of mutually commuting generators denoted by $\{\rho_{ij}, \rho_{ij}+\rho_{ji}=0,\ \ 1 \le i \ne j \le n\}$ subject to crossing relations

- ρ_{ij} commutes with β, q_{kl} and $\psi_{k,l}$ for all i,j,k,l,
- $\rho_{ij}\ u_{kl} = u_{kl}\ \rho_{ij}$, if $\{i,j\} \cap \{k,l\} = \emptyset$,

$\rho_{ij}\ u_{jk} = u_{jk}\ \rho_{ik}$, if i,j,k are distinct,

- $\rho_{ij}^2 - \beta\ \rho_{ij} + \psi_{ij} = \rho_{jk}^2 - \beta\ \rho_{jk} + \psi_{jk}$ for all triples $1 \le i < j < k \le n$.

Under these assumptions one can check that elements

$$R_{ij} := \rho_{ij} + u_{ij}, \quad 1 \le i < j \le n$$

satisfy the quantum Yang–Baxter relations

$$R_{ij}\ R_{ik}\ R_{jk} = R_{jk}\ R_{ik}\ R_{ij},\ \ i<j<k.$$

In the case of *elliptic representation* defined above, one can take

$$\rho_{ij} := \sigma_\mu(z_i - z_j),$$

where $\mu \in \mathbb{C}^*$ is a parameter. This solution to the quantum Yang– Baxter equation has been discovered in [86]. It can be seen as an operator form of the famous (finite dimensional) solution to $QYBE$ due to A. Belavin and V. Drinfeld [5]. One can go one step more and add to the algebra in question new set of generators corresponding to the shift operators $T_{i,q}\ : z_i \longrightarrow q\ z_i$, cf [24]. In this case one can define *multiplicative Dunkl elements* which are closely related with the elliptic Ruijsenaars–Schneider–Macdonald operators.

[20] It is commonly believed that any identity between the Riemann theta functions is a consequence of the Jacobi–Riemann three term relations among the former. However we do not expect that the all hypergeometric type identities among the Riemann theta functions can be obtained from certain relations in the algebra $3MT_n(0,\psi)$ after applying the *elliptic representation* of the latter.

3.2 Multiplicative Dunkl elements

Since the elements u_{ij}, u_{ik} and u_{jk}, $i<j<k$, satisfy the classical and quantum Yang–Baxter relations (3.1) and(3.2), one can define a multiplicative analogue denoted by Θ_i, $1 \le i \le n$, of the Dunkl elements θ_i. Namely, to start with, we define elements

$$h_{ij} := h_{ij}(t) = 1 + t\ u_{ij}, \quad i \ne j.$$

We consider $h_{ij}(t)$ as an element of the algebra $\widetilde{3HT_n} := 3HT_n(\beta) \otimes \mathbb{Z}[[q_{ij}^{\pm 1}, t, x, y, \ldots]]$, where we assume that the all parameters $\{q_{ij}, t, x, y, \ldots\}$ are central in the algebra $\widetilde{3HT_n}$.

Lemma 3.2.
(1a) $h_{ij}(x)\ h_{ij}(y) = h_{ij}(x+y+\beta\ xy) + q_{ij}\ xy$,
(1b) $h_{ij}(x)\ h_{ji}(y) = h_{ij}(x-y) + \beta\ y - q_{ij}\ x\ y$, *if* $i<j$.
It follows from (1b) that $h_{ij}(t)\ h_{ji}(t) = 1 + \beta\ t - t^2\ q_{ij}$, if $i<j$, and therefore the elements $\{h_{ij}\}$ are invertible in the algebra $\widetilde{3HT_n}$.
(2) $h_{ij}(x)\ h_{jk}(y) = h_{jk}(y)\ h_{ik}(x) + h_{ik}(y)\ h_{ij}(x) - h_{ik}(x+y+\beta\ xy)$.
(3) *(Multiplicative Yang–Baxter relations)*

$$h_{ij}\ h_{ik}\ h_{jk} = h_{jk}\ h_{ik}\ h_{ij}, \quad if \quad i<j<k.$$

(4) *Define multiplicative Dunkl elements (in the algebra $\widetilde{3HT_n}$) as follows*

$$\Theta_j := \Theta_j(t) = \Big(\prod_{a=j-1}^{1} h_{aj}^{-1} \Big) \ \Big(\prod_{a=n}^{j+1} h_{ja} \Big), \quad 1 \le j \le n. \tag{3.3}$$

Then the multiplicative Dunkl elements pair-wise commute.

Clearly

$$\prod_{j=1}^{n} \Theta_j = 1, \quad \Theta_j = 1 + t\ \theta_j + t^2(\ldots), \quad and \quad \Theta_I \prod_{\substack{i \notin I, j \in I \\ i<j}} (1 + t\beta - t^2\ q_{ij}) \in 3HT_n.$$

Here for a subset $I \subset [1,n]$ we use notation $\Theta_I = \prod_{a \in I}\ \Theta_a$,
Our main result of this Section is a description of relations among the multiplicative Dunkl elements.

Theorem 3.1. *(A.N. Kirillov and T.Maeno, [51])*
In the algebra $3HT_n(\beta)$ the following relations hold true

$$\sum_{\substack{I \subset [1,n] \\ |I|=k}} \Theta_I \prod_{\substack{i \notin I, j \in J \\ i<j}} (1 + t\ \beta - t^2\ q_{ij}) = \begin{bmatrix} n \\ k \end{bmatrix}_{1+t\beta}.$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-Gaussian polynomial.

Corollary 3.1.
Assume that $q_{ij} \ne 0$ for all $1 \le i < j \le n$. Then the all elements $\{u_{ij}\}$ are invertible and $u_{ij}^{-1} = q_{ij}^{-1}(u_{ij} - \beta)$ Now define elements $\Phi_i \in \widetilde{3HT_n}$ as follows

$$\Phi_i = \Big\{ \prod_{a=i-1}^{1} u_{ai}^{-1} \Big\} \ \Big\{ \prod_{a=n}^{i+1} u_{ia} \Big\}, \quad i = 1, \ldots, n.$$

Then we have

(1) *(Relationship among* Θ_j *and* Φ_j *)*

$$t^{n-2j+1}\ \Theta_j(t^{-1})\ |_{t=0} = (-1)^j\ \Phi_j.$$

(2) *The elements* $\{\Phi_i,\ 1 \le i \le n,\}$ *generate a commutative subalgebra in the algebra* $\widetilde{3HT_n}$.

(3) *For each* $k = 1,\ldots,n$, *the following relation in the algebra* $3HT_n$ *among the elements* $\{\Phi_i\}$ *holds*

$$\sum_{\substack{I\subset[1,n]\\ |I|=k}} \prod_{\substack{i\notin I,\ j\in I\\ i<j}} (-q_{ij})\ \Phi_I = \beta^{k(n-k)},$$

where $\Phi_I := \prod_{a\in I}\ \Phi_a$.

In fact the element Φ_i admits the following "reduced expression" (i.e. one with the minimal number of terms involved) which is useful for proofs and applications

$$\Phi_i = \Big\{\overrightarrow{\prod_{j\in I}}\Big\{\overrightarrow{\prod_{\substack{i\in I^c_+\\ i<j}}}\ u_{ij}^{-1}\Big\}\Big\}\ \ \Big\{\overrightarrow{\prod_{j\in I^c_+}}\Big\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}}\ u_{ij}\Big\}\Big\}. \tag{3.4}$$

Let us explain notations. For any (totally) ordered set $I = (i_1 < i_2 < \ldots <\ i_k)$ we denote by I_+ the set I with the opposite order, i.e. $I_+ = (i_k > i_{k-1} > \ldots > i_1)$;
if $I \subset [1,n]$, then we set $I^c := [1,n] \setminus I$. For any (totally) ordered set I we denote by $\overrightarrow{\prod_{i\in I}}$ the ordered product according to the order of the set I.

Note that the total number of terms in the RHS of (3.4) is equal to $i(n-i)$.

Finally, from the "reduced expression" (3.4) for the element Φ_i one can see that

$$\prod_{\substack{i\notin I, j\in I\\ i<j}} (-q_{ij})\ \Phi_I = \Big\{\overrightarrow{\prod_{j\in I}}\Big\{\overrightarrow{\prod_{\substack{i\in I^c_+\\ i<j}}}\ (\beta-u_{ij})\Big\}\Big\}\ \ \Big\{\overrightarrow{\prod_{j\in I^c_+}}\Big\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}}\ u_{ij}\Big\}\Big\} := \widetilde{\Phi_I} \in 3HT_n.$$

Therefore the identity

$$\sum_{\substack{I\subset[1,n]\\ |I|=k}} \widetilde{\Phi_I} = \beta^{k(n-k)}$$

is true in the algebra $3HT_n$ for any set of parameters $\{q_{ij}\}$.

Comments 3.2.

In fact from our proof of Theorem 3.1 we can deduce more general statement, namely, consider integers m and k such that $1 \le k \le m \le n$. Then

$$\sum_{\substack{I\subset[1,m]\\ |I|=k}} \Theta_I \prod_{\substack{i\in[1,m]\setminus I, j\in J\\ i<j}} (1 + t\ \beta - t^2\ q_{ij}) = \begin{bmatrix} m\\ k\end{bmatrix}_{1+t\beta} + \sum_{\substack{A\subset[1,n], B\subset[1,n]\\ |A|=|B|=r}} u_{A,B}, \tag{3.5}$$

where , by definition, for two sets $A = (i_1,\ldots,i_r)$ and $B = (j_1,\ldots,j_r)$ the symbol $u_{A,B}$ is equal to the (ordered) product $\prod_{a=1}^r\ u_{i_a,j_a}$. Moreover, the elements of the sets A and B have to satisfy the following conditions:

- for each $a = 1,\ldots,r$ one has $1 \le i_a \le m < j_a \le n$, and $k \le r \le k(n-k)$.

Even more, if $r = k$, then sets A and B have to satisfy the following additional conditions:

- $B = (j_1 \le j_2 \le \ldots \le j_k)$, and the elements of the set A are pair-wise distinct.

In the case $\beta = 0$ and $r = k$, i.e. in the case of additive (truncated) Dunkl elements, the above statement, also known as the quantum Pieri formula, has been stated as Conjecture in [26], and has been proved later in [76].

Corollary 3.2. *([51])*

In the case when $\beta = 0$ *and* $q_{ij} = q_i \ \delta_{j-i,1}$*, the algebra over* $\mathbb{Z}[q_1, \ldots, q_{n-1}]$ *generated by the multiplicative Dunkl elements* $\{\Theta_i$ *and* $\Theta_i^{-1}, \quad 1 \le i \le n\}$ *is canonically isomorphic to the quantum K-theory of the complete flag variety* $\mathcal{F}l_n$ *of type* A_{n-1}*.*

It is still an open **problem** to describe explicitly the set of monomials $\{u_{A,B}\}$ which appear in the RHS of (3.5) when $r > k$.

3.3 Truncated Gaudin operators

Let $\{p_{ij} \ 1 \le i \ne j \le n\}$ be a set of mutually commuting parameters. We assume that parameters $\{p_{ij}\}_{1 \le i < j \le n}$ are invertible and satisfy the Arnold relations

$$\frac{1}{p_{ik}} = \frac{1}{p_{ij}} + \frac{1}{p_{jk}}, \quad i < j, k.$$

For example one can take $p_{ij} = (z_i - z_j)^{-1}$, where $z = (z_1, \ldots, z_n) \in (\mathbb{C} \backslash 0)^n$.

Definition 3.5. *Truncated (rational) Gaudin operator corresponding to the set of parameters* $\{p_{ij}\}$*, is defined to be*

$$G_i = \sum_{j \ne i} p_{ij}^{-1} \ s_{ij}, \quad 1 \le i \le n,$$

where s_{ij} *denotes the exchange operator which switches variables* x_i *and* x_j*, and fixes parameters* $\{p_{ij}\}$*.*

We consider the Gaudin operator G_i *as an element of the group ring* $\mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$*, call this element* $G_i \in \mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$*,* $i = 1, \ldots, n$*, by Gaudin element and denoted it by* $\theta_i^{(n)}$*.*

It is easy to see that the elements $u_{ij} := p_{ij}^{-1} \ s_{ij}$, $1 \le i \ne j \le n$, define a representation of the algebra $3HT_n(\beta)$ with parameters $\beta = 0$ and $q_{ij} = u_{ij}^2 = p_{ij}^2$.

Therefore one can consider the (truncated) Gaudin elements as a special case of the (truncated) Dunkl elements. Now one can rewrite the relations among the Dunkl elements, as well as the quantum Pieri formula [26] , [76], in terms of the Gaudin elements.

The key observation which allows to rewrite the quantum Pieri formula as a certain relation among the Gaudin elements, is the following one:

parameters $\{p_{ij}^{-1}\}$ satisfy the *Plücker* relations

$$\frac{1}{p_{ik} \ p_{jl}} = \frac{1}{p_{ij} \ p_{kl}} + \frac{1}{p_{il} \ p_{jk}}, \quad if \quad i < j < k < l.$$

To describe relations among the Gaudin elements $\theta_i^{(n)}$, $i = 1, \ldots, n$, we need a bit of notation. Let $\{p_{ij}\}$ be a set of invertible parameters as before. $i_a < j_a, \quad a = 1, \ldots, r$. Define polynomials in the variables $\mathbf{h} = (h_1, \ldots, h_n)$

$$G_{m,k,r}^{(n)}(\mathbf{h}, \{p_{ij}\}) = \sum_{\substack{I \subset [1,n-1] \\ |I|=r}} \frac{1}{\prod_{i \in I} p_{in}} \sum_{\substack{J \subset [1,n] \\ |I|+m=|J|+k}} \binom{n - |I \bigcup J|}{n - m - |I|} \tilde{h}_J, \tag{3.6}$$

where

$$\tilde{h}_J = \sum_{\substack{K \subset J, \ L \subset J, \\ |K|=|L|, \ \ K \bigcap L = \emptyset}} \prod_{j \in J \backslash (K \bigcup L)} h_j \prod_{k_a \in K, \ l_a \in L} p_{k_a, l_a}^2,$$

and summation runs over subsets $K = \{k_1 < k_2 < \ldots < k_r\}$ and $L = \{l_1 < l_2 < \ldots < l_r\} \subset J\}$, such that $k_a < l_a \quad a = 1, \ldots, r$.

Theorem 3.2. *(Relations among the* **Gaudin** *elements, [47], cf [71])*
Under the assumption that elements $\{p_{ij},\ 1 \le i < j \le n\}$ are invertible, mutually commute and satisfy the Arnold relations, one has

- $$G_{m,k,r}^{(n)}(\theta_1^{(n)},\ldots,\theta_n^{(n)},\{p_{ij}\}) = 0, \quad if \quad m > k, \tag{3.7}$$

- $$G_{0,0,r}^{(n)}(\theta_1^{(n)},\ldots,\theta_n^{(n)},\{p_{ij}\}) = e_r(d_2,\ldots,d_n),$$

where $d_2,\ldots,d_n$ denote the Jucys–Murphy elements in the group ring $\mathbb{Z}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$, see Comments 3.1 for a definition of the Jucys–Murphy elements.

- *Let $J = \{j_1 < j_2 \ldots < j_r\} \subset [1,n]$, define matrix $M_J := (m_{a,b})_{1 \le a,b \le r}$, where*

$$m_{a,b} := m_{a,b}(\mathbf{h};\{p_{ij}\}) = \begin{cases} h_{j_a}, & if\ a = b, \\ p_{j_a,j_b}, & if\ a < b, \\ -p_{j_b,j_a} & if\ a > b. \end{cases}$$

Then

$$\tilde{h}_J = DET\ |M_J|.$$

Examples 3.1. (1) Let us display the polynomials $G_{m,k,r}^{(n)}(\mathbf{h},\{p_{ij}\})$ a few cases.

- $$G_{m,0,r}^{(n)}(\mathbf{h},\{p_{ij}\}) = \sum_{\substack{I \subset [1,n-1] \\ |I|=r}} \prod_{i \in I} p_{in}^{-1} \Big(\sum_{\substack{J \subset [1,n] \\ |J|=m+r, I \subset J}} \tilde{h}_J \Big).$$

- $$G_{m,k,0}^{(n)}(\mathbf{h},\{p_{ij}\}) = \binom{n-m+k}{k}\, e_{m-k}^{\mathbf{q}}(h_1,\ldots,h_n).$$

- $$G_{m,1,r}^{(n)}(\mathbf{h},\{p_{ij}\}) = \sum_{\substack{I \subset [1,n-1] \\ |I|=r}} \prod_{i \in I} p_{in}^{-1} \Big(\sum_{\substack{J \subset [1,n] \\ I \subset J,\ |J|=m+r}} (n-m-r+1)\ \tilde{h}_J + \sum_{\substack{J \subset [1,n] \\ |J|=m+r-1,\ \ |I \cup J|=m+r}} \tilde{h}_J \Big).$$

(2) Let us list the relations (3.6) among the Gaudin elements in the case $n = 3$. First of all, the Gaudin elements satisfy the "standard" relations among the Dunkl elements $\theta_1+\theta_2+\theta_3 = 0$, $\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3+q_{12}+q_{13}+q_{23} = 0$,
$\theta_1\theta_2\theta_3+q_{12}\ \theta_3+q_{13}\ \theta_2+q_{23}\ \theta_1 = 0$. Moreover, we have additional relations which are specific for the Gaudin elements

$$G_{2,0,1}^{(3)} = \frac{1}{p_{13}}\big(\theta_1\theta_2+\theta_1\theta_3+q_{12}+q_{13}\big) + \frac{1}{p_{23}}\big(\theta_1\theta_2+\theta_2\theta_3+q_{12}+q_{23}\big) = 0,$$

the elements $p_{23}\ \theta_1 + p_{13}\ \theta_2$ and $\theta_1\ \theta_2$ are central. ■

It is well-known that the elementary symmetric polynomials $e_r(d_2,\ldots,d_n) := C_r$, $r = 1,\ldots,n-1$, generate the center of the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$, whereas the Gaudin elements $\{\theta_i^{(n)},\ i = 1,\ldots,n\}$, generate a maximal commutative subalgebra $\mathcal{B}(p_{ij})$, the so-called Bethe subalgebra, in $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. It is well-known, see e.g. [71], that $\mathcal{B}(p_{ij}) = \bigoplus_{\lambda \vdash n} \mathcal{B}_\lambda(p_{ij})$, where $\mathcal{B}_\lambda(p_{ij})$ is the $\lambda-$isotypic component of $\mathcal{B}(p_{ij})$. On each $\lambda-$isotypic component the value of the central element

C_k is the explicitly known constant $c_k(\lambda)$.
It follows from [71] that the relations (3.6) together with relations

$$G_{0,0,r}(\theta_1^{(n)},\ldots,\theta_n^{(n)},\{p_{ij}\})=c_r(\lambda),$$

are the defining relations for the algebra $\mathcal{B}_\lambda(p_{ij})$.

Let us remark that in the definition of the Gaudin elements we can use *any* set of mutually commuting, invertible elements $\{p_{ij}\}$ which satisfies the Arnold conditions. For example, we can take

$$p_{ij}:=\frac{q^{j-2}(1-q)}{1-q^{j-i}},\quad 1\le i<j\le n.$$

It is not difficult to see that in this case

$$\lim_{q\to 0}\frac{\theta_J^{(n)}}{p_{1j}}=-d_j=-\sum_{a=1}^{j-1}s_{aj},$$

where as before, d_j denotes the Jucys–Murphy element in the group ring $\mathbb{Z}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$. Basically from relations (2.15) one can deduce the relations among the Jucys–Murphy elements $d_2,\ldots,d_n$ after plugging in (3.6) the values $p_{ij}:=\frac{q^{j-2}(1-q)}{1-q^{j-i}}$ and passing to the limit $q\to 0$. However the real computations are rather involved.

Finally we note that the multiplicative Dunkl / Gaudin elements $\{\Theta_i,\ 1,\ldots,n\}$ also generate a maximal commutative subalgebra in the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. Some relations among the elements $\{\Theta_l\}$ follow from Theorem 3.2, but we don't know an analogue of relations (3.6) for the multiplicative Gaudin elements, but see [71].

Exercises 3.1.

Let $A=(a_{i,j})$ be a $2m\times 2m$ **skew-symmetric** *matrix. The* **Pfaffian** *and* **Hafnian** *of A are defined correspondingly by the equations*

$$\mathrm{Pf}(A)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\mathrm{sgn}(\sigma)\prod_{i=1}^m a_{\sigma(2i-1),\sigma(2i)},\quad \mathrm{Hf}(A)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\prod_{i=1}^m a_{\sigma(2i-1),\sigma(2i)}\quad (3.8)$$

where $\mathbb{S}_{2m}$ is the symmetric group and $sgn(\sigma)$ is the signature of a permutation $\sigma\in\mathbb{S}_{2m}$, see e.g. http://en.wikipedia.org/wiki/Pfaffian.

Now let n be a positive integer, and $\{p_{ij},\ \ 1\le i\ne j\le n,\ p_{ij}+p_{ji}=0\}$ be a set of skew-symmetric, invertible and mutually commuting elements. We set $p_{ii}=0$ for all i, and $\mathbf{q}:=\{p_{ij}^2\}_{1\le i<j\le n}$.

Now let us assume that the elements $\{p_{ij}\}_{1\le i<j\le n}$ satisfy the Plüker relations for the elements $\{p_{ij}^{-1}\}_{1\le i<j\le n}$, namely,

$$\frac{1}{p_{ik}\ p_{jl}}=\frac{1}{p_{ij}\ p_{kl}}+\frac{1}{p_{il}\ p_{jk}}\qquad for\ all\ 1\le i<j<k<l\le n,$$

(a) Let n be an even positive integer. Let us define $A_n(p_{ij}):=(p_{ij})_{1\le i,j\le n}$ to be the $n\times n$ skew-symmetric matrix corresponding to the family $\{p_{ij}\}_{1\le i<j\le n}$.

Show that

$$DET\mid A_n(p_{ij})\mid=\mathrm{Hf}(A_n(p_{ij}^2)).$$

(b) Let n be a positive integer, and $z_1,\ldots,z_n$ be a set of mutually commuting variables, define polynomials $H_i(z_1,\ldots,z_n\mid\{p_{ij}\})$, $i=1,\ldots,n$ from the equation

$$DET\mid diag(t+z_1,\ldots,t+z_n)+A_n(p_{ij})\mid=t^n+\sum_{i=1}^n H_i(z_1,\ldots,z_n\mid\{p_{ij}\})\ t^{n-i},$$

where $diag(t+z_1,\ldots,t+z_n)$ means the diagonal matrix.

Show that

For $k=1,\ldots,n$ the polynomial $H_k(z_1,\ldots,z_n \mid \{p_{ij}\})$ is equal to the multiparameter quantum elementary polynomial $e_k^{(\mathbf{q})}(z_1,\ldots,z_n)$, see e.g. [26], or Theorem 2.1.

For example, take $n=4$, then $DET \mid A(p_{ij}) \mid = (p_{12}\ p_{34} - p_{13}\ p_{24} + p_{14}\ p_{23})^2 = p_{12}^2\ p_{34}^2 + p_{13}^2\ p_{24}^2 + p_{14}^2\ p_{23}^2 - 2\ p_{12}p_{13}\ p_{23}\ p_{14}\ p_{24}\ p_{34}\left(\frac{1}{p_{12}\ p_{34}} - \frac{1}{p_{13}\ p_{24}} + \frac{1}{p_{14}\ p_{23}}\right) =$
$p_{12}^2\ p_{34}^2 + p_{13}^2\ p_{24}^2 + p_{14}^2\ p_{23}^2 = Hf(A_4(\{p_{ij}\})$.

On the other hand, if one assumes that a set of skew symmetric parameters $\{r_{ij}\}_{1\le i<j\le n}$, $r_{ij}+r_{ji}=0$, satisfies the "standard" Plüker relations, namely

$$r_{ik}\ r_{jl} = r_{ij}\ r_{kl} + r_{il}\ r_{jk}, \quad i<j<k<l,$$

then $DET \mid A_n(r_{ij}) \mid = 0$.

3.4 Shifted Dunkl elements $\mathfrak{d}_i$ and $\mathfrak{D}_i$

As it was stated in Corollary 3.2, the truncated additive and multiplicative Dunkl elements in the algebra $3HT_n(0)$ generate over the ring of polynomials $\mathbb{Z}[q_1,\ldots,q_{n-1}]$ correspondingly the quantum cohomology and quantum $K-theory$ rings of the full flag variety $\mathcal{F}l_n$. In order to describe the corresponding *equivariant* theories, we will introduce the *shifted* additive and multiplicative Dunkl elements. To start with we need at first to introduce an extension of the algebra $3HT_n(\beta)$.

Let $\{z_1,\ldots,z_n\}$ be a set of mutually commuting elements and $\{\beta,\mathbf{h}=(h_1,\ldots,h_{n-1}),t,q_{ij}=q_{ji},\ 1\le i,j\le n\}$ be a set of parameters. We set $h_n:=0$.

Definition 3.6. *Cf Definition 2.4)*

Define algebra $\overline{3TH_n(\beta,\mathbf{h})}$ to be the semi-direct product of the algebra $3TH_n(\beta)$ and the ring of polynomials $\mathbb{Z}[\mathbf{h},t][z_1,\ldots,z_n]$ with respect to the crossing relations

(1) $z_i\ u_{kl} = u_{kl}\ z_i$ *if* $i \notin \{k,l\}$,

(2) $z_i\ u_{ij} = u_{ij}\ z_j + \beta\ z_i + h_j,\ \ z_j\ u_{ij} = u_{ij}\ z_i - \beta\ z_i - h_{j-1},$ *if* $1\le i<j<k\le n$.

Now we set as before $h_{ij} := h_{ij}(t) = 1 + t\ u_{ij}$.

Definition 3.7.

- *Define shifted additive Dunkl elements to be*

$$\mathfrak{d}_i = z_i - \sum_{i<j} u_{ij} + \sum_{i<j} u_{ji}.$$

- *Define shifted multiplicative Dunkl elements to be*

$$\mathfrak{D}_i = \Big(\prod_{a=i-1}^{1} h_{ai}^{-1}\Big)\ (1+z_i)\ \Big(\prod_{a=n}^{i+1} h_{ia}\Big).$$

Lemma 3.3.

$$[\mathfrak{d}_i,\mathfrak{d}_j]=0, \quad [\mathfrak{D}_i,\mathfrak{D}_j]=0 \quad for \quad all \quad i,j.$$

Now we stated an analogue of Theorem 3.1. for shifted multiplicative Dunkl elements.
As a preliminary step, for any subset $I\subset[1,n]$ let us set $\mathfrak{D}_I=\prod_{a\in I}\mathfrak{D}_a$. It is clear that

$$\mathfrak{D}_I \prod_{\substack{i\notin I,\ j\in I\\ i<j}} (1+t\ \beta - t^2\ q_{ij}) \in \overline{3HT_n(\beta,\mathbf{h})}.$$

Theorem 3.3.
In the algebra $\overline{3HT_n(\beta,\mathbf{h})}$ *the following relations hold true*

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \mathfrak{D}_I \prod_{\substack{i\notin I, j\in J\\ i<j}} (1+t\ \beta - t^2\ q_{ij}) =$$

$$\sum_{\substack{I\subset[1,n]\\ I=\{1\le i_1<\ldots<i_k\le n\}}} \prod_{a=1}^{k} (1+t\ \beta)^{n-k-i_a+a} \left(z_{i_a}(1+t\ \beta)^{i_a-a}+1+h_{i_a}\ \frac{(1+t\ \beta)^{i_a-a}-1}{\beta}\right).$$

In particular, if $\beta=0$, we will have

Corollary 3.3. *In the algebra* $\overline{3HT_n(0,\mathbf{h})}$ *the following relations hold*

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \mathfrak{D}_I \prod_{\substack{i\notin I, j\in J\\ i<j}} (1-t^2\ q_{ij}) = \sum_{\substack{I\subset[1,n]\\ I=\{1\le i_1,\ldots,i_k\le n\}}} \prod_{a=1}^{k} \Big(z_{i_a}+1+t\ h_{i_a}\ (i_a-a)\Big). \tag{3.9}$$

Conjecture 3.1. *If* $h_1=\cdots=h_{n-1}=1$, $t=1$ *and* $q_{ij}=\delta_{i,j+1}$, *then the subalgebra generated by multiplicative Dunkl elements* $\mathfrak{D}_i$, $i=1,\ldots,n$, *in the algebra* $\overline{3HT_n(0,\mathbf{h}=\mathbf{1})}$ *(and* $t=1$*), / is isomorphic to the equivariant quantum K-theory of the complete flag variety* $\mathcal{F}l_n$.

Our proof is based on induction on k and the following relations in the algebra $\overline{3HT_n(\beta,\mathbf{h})}$

$$h_{ji}\centerdot(1+x_j)=h_{j-1}+\beta\ x_j-x_i+(1+x_i)\centerdot h_{ji},\quad h_{ji}h_{jk}=h_{jk}h_{ki}+h_{ik}h_{ji}-1-\beta,$$

if $i<j<k$, and we set $h_{ij}:=h_{ij}(1)$. These relations allow to reduce the left hand side of the relations listed in Theorem 3.3 to the case when $z_i=0$, $h_i=0$, $\forall i$. Under these assumptions one needs to proof the following relations in the algebra $3HT_n(\beta)$, see Theorem 3.1,

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \mathfrak{D}_I \prod_{\substack{i\notin I, j\in J\\ i<j}} (1+t\ \beta-t^2\ q_{ij}) = \begin{bmatrix} n\\ k\end{bmatrix}_{1+t\beta}. \tag{3.10}$$

In the case $\beta=0$ the identity (3.9) has been proved in [51]

One of the main steps in our proof of Theorem 3.1. is the following explicit formula for the elements $\mathfrak{D}_I$.

Lemma 3.4. *One has*

$$\widetilde{\mathfrak{D}_I}:=\mathfrak{D}_I \prod_{\substack{i\notin I,\ j\in I\\ i<j}} (1+t\ \beta-t^2\ q_{ij}) = \prod_{b\in I}^{\nearrow} \Big(\prod_{\substack{a\notin I\\ a<b}}^{\searrow} h_{ba}\Big) \prod_{a\in I}^{\nearrow} \Big((1+z_a) \prod_{\substack{b\notin I\\ a<b}}^{\searrow} h_{ab}\Big).$$

Note that if $a<b$, then $h_{ba}=1+\beta t-u_{ab}$. Here we have used the symbol

$$\prod_{b\in I}^{\nearrow} \Big(\prod_{\substack{a\notin I\\ a<b}}^{\searrow} h_{ba}\Big)$$

to denote the following product. At first, for a given element $b\in I$ let us define the set $I(b):=\{a\in[1,n]\backslash I,\ a<b\}:=(a_1^{(b)}<\ldots<a_p^{(b)})$ for some p (depending on b). If $I=(b_1<b_2\ldots<b_k)$ i.e. $b_i=a_i^{(b)}$, then we set

$$\prod_{b\in I}^{\nearrow} \Big(\prod_{\substack{a\notin I\\ a<b}}^{\searrow} h_{ba}\Big) = \prod_{j=1}^{k}\Big(u_{b_j,a_s}\ u_{b_j,a_{s-1}}\cdots u_{b_j,a_1}\Big).$$

For example, let us take $n=6$ and $I=(1,3,5)$, then

$\widetilde{\mathfrak{D}}_I = h_{32}h_{54}h_{52}(1+z_1)h_{16}h_{14}h_{12}(1+z_3)h_{36}h_{34}(1+z_5)h_{56}$.

Let us stress that the element $\widetilde{\mathfrak{D}}_I \in \overline{3HT_n(\beta)}$ is a linear combination of square free monomials and therefore, a computation of the left hand side of the equality stated in Theorem 3.3 can be performed in the "classical case" that is in the case $q_{ij}=0, \forall i<j$. This case corresponds to the computation of the classical equivariant cohomology of the type A_{n-1} complete flag variety $\mathcal{F}l_n$, if $h=1$.

A proof of the $\beta=0$ case given in [51], Theorem 1, can be immediately extended to the case $\beta \neq 0$.

Exercises 3.2.

(1) *Show that*

$$\sum_{1 \le i_1 < \ldots < i_k \le n} \prod_{a=1}^{k} (1+\beta)^{n-k-i_a+a} = \begin{bmatrix} n \\ k \end{bmatrix}_{1+t\beta}.$$

(2) *((β,h)-Stirling polynomials of the second type)*

Define polynomials $S_{n.k}(\beta,h)$ as follows

$$S_{n.k}(\beta,h) = \sum_{\substack{I \subset [1,n] \\ I=\{1 \le i_1,\ldots,i_k \le n\}}} \prod_{a=1}^{k} \Big(\beta^{n-k-i_a+a} \ + \ h\ \frac{\beta^{n-k-i_a+a}-1}{\beta-1}\Big).$$

Show that

$$S_{n,k}(1,1) = \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}, \quad S_{n,k}(\beta,0) = \begin{bmatrix} n \\ k \end{bmatrix}_{\beta}.$$

4 Algebra $3T_n^{(0)}(\Gamma)$ and Tutte polynomial of graphs

4.1 Graph and nil-graph subalgebras, and partial flag varieties

Let's consider the set $R_n := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \ \mid 1 \le i < j \le n\}$ as the set of edges of the complete graph K_n on n labeled vertices $v_1,\ldots,v_n$. Any subset $S \subset R_n$ is the set of edges of a unique subgraph $\Gamma := \Gamma_S$ of the complete graph K_n.

Definition 4.1. *(***Graph and nil-graph subalgebras***)*

The graph subalgebra $3T_n(\Gamma)$ (resp. nil-graph subalgebra $3T_n^{(0)}(\Gamma)$) corresponding to a subgraph $\Gamma \subset K_n$ of the complete graph K_n, is defined to be the subalgebra in the algebra $3T_n$ (resp.$3T_n^{(0)}$) generated by the elements $\{u_{ij} \mid (i,j) \in \Gamma\}$.

In subsequent Subsections 4.1.1 and 4.1.2 we will study some examples of graph subalgebras corresponding to the complete multipartite graphs, cycle graphs and linear graphs.

4.1.1 NilCoxeter and affine nilCoxeter subalgebras in $3T_n^{(0)}$

Our first example is concerned with the case when the graph Γ corresponds to either the set $S := \{(i,i+1) \mid i=1,\ldots,n-1\}$ of *simple roots* of type A_{n-1}, or the set $S^{aff} := S \bigcup \{(1,n)\}$ of *affine simple roots* of type $A_{n-1}^{(1)}$.

Definition 4.2. (**a**) *Denote by $\widetilde{NC}_n$ subalgebra in the algebra $3T_n^{(0)}$ generated by the elements $u_{i,i+1}$, $1 \le i \le n-1$.*

(**b**) *Denote by $\widetilde{ANC}_n$ subalgebra in the algebra $3T_n^{(0)}$ generated by the elements $u_{i,i+1}$, $1 \le i \le n-1$ and $-u_{1,n}$.*

Theorem 4.1.

(**A**) *(cf [4]) The subalgebra* $\widetilde{NC}_n$ *is canonically isomorphic to the NilCoxeter algebra* NC_n*. In particular,* $Hilb(\widetilde{NC}_n, t) = [n]_t!$.

(**B**) *The subalgebra* $\widetilde{ANC}_n$ *has finite dimension and its Hilbert polynomial is equal to*

$Hilb(\widetilde{ANC}_n, t) = [n]_t \prod_{1 \le j \le n-1} [j(n-j)]_t = [n]_t! \prod_{1 \le j \le n-1} [j]_{t^{n-j}}$.

In particular, $\dim \widetilde{ANC}_n = (n-1)!\ n!$, $\deg_t\ Hilb(\widetilde{ANC}_n, t) = \binom{n+1}{3.}$

(**C**) *The kernel of the map* $\pi : \widetilde{ANC}_n \longrightarrow \widetilde{NC}_n$, $\pi(u_{1,n}) = 0, \pi(u_{i,i+1}) = u_{i,i+1},\ 1 \le i \le n-1$, *is generated by the following elements:*

$$f_n^{(k)} = \prod_{j=k}^{1} \prod_{a=j}^{n-k+j-1} u_{a,a+1}, \quad 1 \le k \le n-1.$$

Note that $\deg\ f_n^{(k)} = k(n-k)$.

The statement (C) of Theorem 4.1 means that the element $f_n^{(k)}$ which does not contain the generator $u_{1,n}$, can be written as a linear combination of degree $k(n-k)$ monomials in the algebra $\widetilde{ANC}_n$, each contains the generator $u_{1,n}$ at least once. By this means we obtain a set of all extra relations (i.e. additional to those in the algebra $\widetilde{NC}_n$) in the algebra $\widetilde{ANC}_n$. Moreover, each monomial M in all linear combinations mentioned above, appears with coefficient $(-1)^{\#|u_{1,n} \in M|+1}$. For example,

$f_4^{(1)} := u_{1,2}u_{2,3}u_{3,4} = u_{2,3}u_{3,4}u_{1,4} + u_{3,4}u_{1,4}u_{1,2} + u_{1,4}u_{1,2}u_{2,3}$; $f_4^{(2)} := u_{2,3}u_{3,4}u_{1,2}u_{2,3} = u_{1,2}u_{3,4}u_{2,3}u_{1,4} + u_{1,2}u_{2,3}u_{1,4}u_{1,2} + u_{2,3}u_{1,4}u_{1,2}u_{3,4} + u_{3,4}u_{2,3}u_{1,4}u_{3,4} - u_{1,4}u_{1,2}u_{3,4}u_{1,4}$.

Remark 4.1. More generally, let (W, S) be a finite crystallographic Coxeter group of rank l with the set of exponents $1 = m_1 \le m_2 \le \cdots \le m_l$.

Let $\mathcal{B}_W$ be the corresponding Nichols–Woronowicz algebra, see e.g. [4]. Follow [4], denote by $\widetilde{NC}_W$ the subalgebra in $\mathcal{B}_W$ generated by the elements $[\alpha_s] \in \mathcal{B}_W$ corresponding to simple roots $s \in S$. Denote by $\widetilde{ANWC}_W$ the subalgebra in $\mathcal{B}_W$ generated by $\widetilde{NC}_W$ and the element $[a_\theta]$, where $[a_\theta]$ stands for the element in $\mathcal{B}_W$ corresponding to the highest root θ for W. In other words, $\widetilde{ANWC}_W$ is the image of the algebra $\widetilde{ANC}_W$ under the natural map $B\mathcal{E}(W) \longrightarrow \mathcal{B}_W$, see e.g. [4], [49]. It follows from [4], Section 6, that $Hilb(\widetilde{NC}_W, t) = \prod_{i=1}^{l} [m_i + 1]_t$.

Conjecture 4.1. *(Y. Bazlov and A.N. Kirillov, 2002)*

$$Hilb(\widetilde{ANWC}_W, t) = \prod_{i=1}^{l} \frac{1 - t^{m_i+1}}{1 - t^{m_i}} \prod_{i=1}^{l} \frac{1 - t^{a_i}}{1 - t} = P_{aff}(W, t)\ \prod_{i=1}^{l} (1 - t^{a_i}),$$

where

$$P_{aff}(W, t) := \sum_{w \in W_{aff}} t^{l(w)} = \prod_{i=1}^{l} \frac{(1 + t + \cdots + t^{m_i})}{1 - t^{m_i}}$$

denotes the Poincaré polynomial corresponding to the affine Weyl group W_{aff}*, see [12], p.245;* $a_i := (2\rho, \alpha_i^{\vee})$, $1 \le i \le l$, *denote the coefficients of the decomposition of the sum of positive roots* 2ρ *in terms of the simple roots* α_i.

In particular, $\dim \widetilde{ANWC}_W = |W| \frac{\prod_{i=1}^{l} a_i}{\prod_{i=1}^{l} m_i}$ and $\deg Hilb(\widetilde{ANWC}_W, t) = \sum_{1=1}^{l} a_i$.

It is well-known that the product $\prod_{i=1}^{l} \frac{1-t^{a_i}}{1-t^{m_i}}$ is a symmetric (and *unimodal* ?) polynomial with non–negative integer coefficients.

Example 4.1. (a)

$$Hilb(\widetilde{ANC}_3,t)=[2]_t^2[3]_t,Hilb(\widetilde{ANC}_4,t)=[3]_t^2[4]_t^2,Hilb(\widetilde{ANC}_5,t)=[4]_t^2[5]_t[6]_t^2.$$

(b) $\quad Hilb(B\mathcal{E}_2,t)=(1+t)^4(1+t^2)^2,$

$$Hilb(\widetilde{ANC}_{B_2},t)=(1+t)^3(1+t^2)^2=P_{aff}(B_2,t)(1-t^3)(1-t^4).$$

(c) $\quad Hilb(\widetilde{ANC}_{B_3},t)=$

$$(1+t)^3(1+t^2)^2(1+t^3)(1+t^4)(1+t+t^2)(1+t^3+t^6)=P_{aff}(B_3,t)(1-t^5)(1-t^8)(1-t^9).$$

Indeed, $m_{B_3}=(1,3,5), \quad a_{B_3}=(5,8,9).$

■

Definition 4.3. *Let* $\langle\widetilde{ANC}_n\rangle$ *denote the two-sided ideal in* $3T_n^{(0)}$ *generated by the elements* $\{u_{i,i+1}\}, \quad 1\le i\le n-1,$ *and* $u_{1,n}$. *Denote by* U_n *the quotient* $U_n=3T_n^0/\langle\widetilde{ANC}_n\rangle$.

Proposition 4.1.

$$U_4\cong\langle u_{1,3},u_{2,4}\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2;\quad U_5\cong\langle u_{1,4},u_{2,4},u_{2,5},u_{3,5},u_{1,3}\rangle\cong\widetilde{ANC}_5.$$

In particular, $Hilb(3T_5^{(0)},t)=\Big[Hilb(\widetilde{ANC}_5,t)\Big]^2.$

4.1.2 Parabolic 3-term relations algebras and partial flag varieties

In fact one can construct an analogue of the algebra $3HT_n$ and a commutative subalgebra inside it, for <u>any</u> graph $\Gamma=(V,E)$ on n vertices, possibly with loops and multiple edges, [47]. We denote this algebra by $3T_n(\Gamma)$, and denote by $3T_n^{(0)}(\Gamma)$ its *nil-quotient*, which may be considered as a "classical limit of the algebra $3T_n(\Gamma)$".
The case of the complete graph $\Gamma=K_n$ reproduces the results of the present paper and those of [47], i.e. the case of the full flag variety $\mathcal{F}l_n$. The case of the *complete multipartite graph* $\Gamma=K_{n_1,\ldots,n_r}$ reproduces the analogue of results stated in the present paper for the full flag variety $\mathcal{F}l_n$, to the case of the <u>partial flag</u> variety $\mathcal{F}_{n_1,\ldots,n_r}$, see [47] for details.
We **expect** that in the case of the complete graph with all edges having the same multiplicity m, denoted by $\Gamma=K_n^{(m)}$, or mK_n in the present paper, the commutative subalgebra generated by the Dunkl elements in the algebra $3T_n^{(0)}(\Gamma)$ is related to the algebra of coinvariants of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}[X_n^{(1)},\ldots,X_n^{(m)}]$, where we set $X_n^{(i)}=\{x_1^{(i)},\ldots,x_n^{(i)}\}$.

Example 4.2. *Take* $\Gamma=K_{2,2}$. *The algebra* $3T^{(0)}(\Gamma)$ *is generated by four elements* $\{a=u_{13},b=u_{14},c=u_{23},d=u_{24}\}$ *subject to the following set of (defining) relations*

- $a^2=b^2=c^2=d^2=0, \quad c\ b=b\ c, \quad a\ d=d\ a,$
- $a\ b\ a+b\ a\ b=0=a\ c\ a+c\ a\ c, \quad b\ d\ b+d\ b\ d=0=c\ d\ c+d\ c\ d,$
  $a\ b\ d-b\ d\ c-c\ a\ b+d\ c\ a=0=a\ c\ d-b\ a\ c-c\ d\ b+d\ b\ a,$
- $a\ b\ c\ a+a\ d\ b\ c+b\ a\ d\ b+b\ c\ a\ d+c\ a\ d\ c+d\ b\ c\ d=0.$

It is not difficult to see that [21]

$$Hilb(3T^{(0)}(K_{2,2}),t)=[3]_t^2\ [4]_t^2,\quad Hilb(3T^{(0)}(K_{2,2})^{ab},t)=(1,4,6,3).$$

Here for any algebra A *we denote by* A^{ab} *its* <u>*abelianization*</u> [22].

[21] Hereinafter we shell use notation
$$(a_0,a_1,\ldots,a_k)_t:=a_0+a_1t+\cdots+a_kt^k.$$
[22] See groupprops.subwiki.org/wiki/Abelianization

The commutative subalgebra in $3T^{(0)}(K_{2,2})$, which corresponds to the intersection $3T^{(0)}(K_{2,2}) \bigcap \mathbb{Z}[\theta_1,\theta_2,\theta_3,\theta_4]$, is generated by the elements $c_1 := \theta_1+\theta_2 = (a+b+c+d)$ and $c_2 := \theta_1\ \theta_2 = (ac+ca+bd+db+ad+bc)$. The elements c_1 and c_2 commute and satisfy the following relations

$$c_1^3 - 2\ c_1\ c_2 = 0, \quad c_2^2 - c_1^2\ c_2 = 0.$$

The ring of polynomials $\mathbb{Z}[c_1,c_2]$ is isomorphic to the cohomology ring $H^*(Gr(2,4),\mathbb{Z})$ of the Grassmannian variety $Gr(2,4)$.

■

To continue exposition, let us take $m \le n$, and consider the complete multipartite graph $K_{n,m}$ which corresponds to the grassman variety $Gr(n,m+n.)$ One can show

$$Hilb(3T^{(0)}_{n+m}(K_{n,m})^{ab},t) = \sum_{k=0}^{n-1} (-1)^k\ (1+(n-k)\ t)^{m-1} \prod_{j=1}^{n-k} (1+j\ t) \left\{ {n \atop n-k} \right\}$$

$$= t^{n+m-1}\ Tutte(K_{n,m},1+t^{-1},0),$$

where $\left\{ {n \atop k} \right\} := S(n,k)$ denotes the Stirling numbers of the second kind, that is the number of ways to partition a set of n labeled objects into k nonempty unlabeled subsets, and for any graph Γ, $Tutte(\Gamma,x,y)$ denotes the **Tutte polynomial** [23] corresponding to graph Γ.

It is well-known that the Stirling numbers $S(n,k)$ satisfy the following identities

$$\sum_{k=0}^{n-1} (-1)^k\ S(n,n-k) \prod_{j=1}^{n-k} (1+j\ t) = (1+t)^n, \qquad \sum_{n \ge k} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{e^x-1)^k}{k!}.$$

Let us observe that $\dim(3T^{(0)}(K_{n,n})^{ab} =$

$$\sum_{k=0}^{n-1} (-1)^k\ (n+1-k)^{n-1}\ (n+1-k)! \left\{ {n \atop n-k} \right\} = A048163,\ [87].$$

Moreover, if $m \ge 0$, then

$$\sum_{n \ge 1} \dim(3T^{(0)}(K_{n,n+m})^{ab})\ t^n \ = \ \sum_{k \ge 1} \frac{k^{k+m-1}\ (k-1)!\ t^k}{\prod_{j=1}^{k-1}(1+k\ j\ t)},$$

$$\sum_{n \ge 1} Hilb(3T^{(0)}(K_{n,m})^{ab},t)\ z^{n-1} = \sum_{k \ge 0} (1+k\ t)^{m-1} \prod_{j=1}^{k} \frac{z\ (1+j\ t)}{1+j\ z}.$$

Comments 4.1. Poly-Bernoulli numbers

Based on listed above identities involving the Stirling numbers $S(n,k)$, one can prove the following *combinatorial* formula

$$\dim(3T^{(0)}(K_{n,m})^{ab}) = \sum_{j=1}^{min(n,m)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {m+1 \atop j+1} \right\} = B_n^{(-m)} = B_m^{(-n)}, \tag{4.1}$$

[23] See e.g. http://en.wikipedia.org/wiki/Tutte.polynomial. It is well-known that

$$Tutte(\Gamma,1+t,0) = (-1)^{|\Gamma|}\ t^{-\kappa(\Gamma)}\ Chrom(\Gamma,-t),$$

where for any graph Γ, $|\Gamma|$ is equal to the number of vertices and $\kappa(\Gamma)$ is equal to the number of connected components of Γ. Finally $Chrom(\Gamma,t)$ denotes the *chromatic polynomial* corresponding to graph Γ, see e.g., [96], or http://en.wikipedia/wiki/Chromatic.polynomial.

where $B_n^{(k)}$ denotes the *poly-Bernoulli number* introduced by M. Kaneko [42].

For the reader's convenient, we recall below a definition of *poly-Bernoulli numbers*. To start with, let k be an integer, consider the formal power series

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

If $k \geq 1$, $Li_k(z)$ is the k-th *polylogarithm*, and if $k \leq 0$, then $Li_k(z)$ is a rational function. Clearly $Li_1(z) = -ln(1-z)$. Now define *poly-Bernoulli numbers* through the generating function

$$\frac{Li_k(1-e^{-z})}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{z^n}{n!}.$$

Note that a combinatorial formula for the numbers $B_n^{(-k)}$ stated in (4.1) is a consequence of the following identity [42]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!}\frac{z^k}{k!} = \frac{e^{x+z}}{1-(1-e^x)(1-e^z)}.$$

■

Now let $\theta_i^{(n+m)} = \sum_{j\neq i} u_{ij}$, $1 \leq i \leq n+m$, be the Dunkl elements in the algebra $3T^{(0)}(K_{n+m})$, define the following elements the in the algebra $3T^{(0)}(K_{n,m})$

$$c_k := e_k(\theta_1^{(n+m)},\ldots,\theta_n^{(n+m)}), \quad 1 \leq k \leq n, \quad \overline{c}_r := e_r(\theta_{n+1}^{(n+m)},\ldots,\theta_{n+m}^{(n+m)}, \quad 1 \leq r \leq m.$$

Clearly,

$$(1+\sum_{k=1}^{n} c_k\, t^k)(1+\sum_{r=1}^{m} \overline{c}_r\, t^r) = \prod_{j=1}^{n+m}(1+\theta_j^{(n+m)}) = 1.$$

Moreover, there exist the natural isomorphisms of algebras

$$H^*(Gr(n,n+m),\mathbb{Z}) \cong \mathbb{Z}[c_1,\ldots,c_n]/\Big\langle (1+\sum_{k=1}^{n} c_k\, t^k)(1+\sum_{r=1}^{m} \overline{c}_r\, t^r) - 1\Big\rangle,$$

$$QH^*(Gr(n,n+m)) \cong \mathbb{Z}[q][c_1,\ldots,c_n]/\Big\langle (1+\sum_{k=1}^{n} c_k\, t^k)(1+\sum_{r=1}^{m} \overline{c}_r\, t^r) - 1 - q\, t^{n+m}\Big\rangle.$$

Let us recall, see Section 2, footnote 16, that for a commutative ring R and a polynomial $p(t) = \sum_{j=1}^{s} g_j\, t^j \in R[t]$, we denote by $\langle p(t)\rangle$ the ideal in the ring R generated by the coefficients $g_1,\ldots,g_s$.

These examples are illustrative of the similar results valid for the **general complete multipartite graphs** $K_{n_1,\ldots,n_r}$, i.e. for the partial flag varieties [47].

To state our results for *partial flag varieties* we need a bit of notation. Let $N := n_1+\ldots+n_r$, $n_j > 0$, $\forall j$, be a composition of size N. We set $N_j := n_1+\cdots+n_j$, $j = 1,\ldots,r$, *and* $N_0 = 0$, Now, consider the commutative subalgebra in the algebra $3T_N^{(0)}(K_N)$ generated by the set of Dunkl elements $\{\theta_1^{(N)},\ldots,\theta_N^{(N)}\}$, and define elements $\{c_{k_j}^{(j,N)} \in 3T_N^{(0)}(K_{n_1,\ldots,n_r})\}$ to be the degree k_j elementary symmetric polynomials of the Dunkl elements $\theta_{N_{j-1}+1}^{(N)},\ldots,\theta_{N_j}^{(N)}$, namely

$$c_k^{(j)} := c_{k_j}^{(j,N)} = e_k(\theta_{N_{j-1}+1}^{(N)},\ldots,\theta_{N_j}^{(N)}), \quad 1 \leq k_j \leq n_j, \quad j = 1,\ldots,r, \quad c_0^{(j)} = 1, \ \forall j.$$

Clearly

$$\prod_{j=1}^{r} (\sum_{a=0}^{n_j} c_a^{(j)} \ t^a) = \prod_{j=1}^{N} (1+\theta_j^{(N)} t^j) = 1.$$

Theorem 4.2.
The commutative subalgebra generated by the elements $\{c_{k_j}^{(j)}, \ 1 \le k_j \le n_j, \ 1 \le j \le r-1\}$, *in the algebra* $3T_N^{(0)}(K_{n_1,\ldots,n_r})$ *is isomorphic to the cohomology ring* $H^*(\mathcal{F}l_{n_1,\ldots,n_r}, \mathbb{Z})$ *of the partial flag variety* $\mathcal{F}l_{n_1,\ldots,n_r}$.

► In other words, we treat the Dunkl elements $\{\theta_{N_{j-1}+a}^{(N)}, \ 1 \le a \le n_j\}$, $j = 1, \ldots, r$, as the *Chern roots* of the vector bundles $\{\xi_j := \mathcal{F}_j/\mathcal{F}_{j-1}\}$, $j = 1, \ldots, r$, over the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$.

Recall that a point $\mathbf{F}$ of the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$, $n_1 + \cdots + n_r = N$, is a sequence of embedded subspaces

$$\mathbf{F} = \{0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_r = \mathbb{C}^N\} \ \ \textit{such that} \ \ \dim(F_i/F_{i-1}) = n_i, \ \ i = 1, \ldots, r.$$

By definition, the fiber of the vector bundle ξ_i over a point $\mathbf{F} \in \mathcal{F}l_{n_1,\ldots,n_r}$ is the n_i-dimensional vector space F_i/F_{i-1}.

◄

A meaning of the algebra $3T_n^{(0)}(\Gamma)$ and the corresponding commutative subalgebra inside it for a general graph Γ, is still unclear.

Conjecture 4.2.

(1) *Let* $\Gamma = (V, E)$ *be a connected subgraph of the complete graph* K_n *on* n *vertices.* <u>*Then*</u>

$$Hilb(3T_n^{(0)}(\Gamma)^{ab}, t) = t^{|V|-1} \ \ Tutte(\Gamma; 1+t^{-1}, 0).$$

(2) *Let* $\Gamma = (V, E, \{m_{ij}), (ij) \in E\})$ *be a connected subgraph of the complete graph* $K_n^{\mathbf{m}}$ *with multiple edges such that an edge* $(ij) \in K_n$ *has the multiplicity* m_{ij}. *Let* $3T_n^{(0)}(\Gamma, \mathbf{m})$ *denotes the subalgebra in the algebra* $3T_n^{(0)}(\mathbf{m})$ *generated by elements* $\{u_{ij}^{(\alpha_{(ij)})}, \ (ij) \in E, \ ! \le \alpha_{(ij)} \le m_{ij}\}$, *see Section 4.2.5. Let* $\mathcal{A}(\Gamma, \{m_{ij}\})$ *denotes the graphic arrangement corresponding to the graph* $(\Gamma, \{m_{ij}\})$, *that is the set of hyperplanes* $\{H_{(ij),a} = (x_i - x_j = a), \ 0 \le a \le m_{ij} - 1, \ (ij) \in E\}$. <u>*Then*</u>

$$3T_n^{(0)}(\Gamma, \mathbf{m})^{ab} = OS^+(\mathcal{A}(\Gamma, \{m_{ij}\})),$$

where for any arrangements of hyperplanes $\mathcal{A}$, $OS^+(\mathcal{A})$ *denotes the* <u>*even*</u> *Orlik–Salamon algebra of the arrangement* $\mathcal{A}$, *[75].*
In the case when $m_{ij} = 1, \ \forall \ 1 \le i < j \le n$, $\quad 3T_n^{(0)}(\Gamma)^{anti} = OS(\mathcal{A}(\Gamma))$.

Examples 4.1.

(1) *Let* $G = K_{2,2}$ *be complete bipartite graph of type* $(2,2)$. *Then,*
$Hilb(3T_4^0(2,2)^{ab}, t) = (1,4,6,3) = t^2 \ (1+t) + t \ (1+t)^2 + (1+t)^3$,
and the Tutte polynomial for the graph $K_{2,2}$ *is equal to* $x + x^2 + x^3 + y$.

(2) *Let* $G = K_{3,2}$ *be complete bipartite graph of type* $(3,2)$. *Then,*
$Hilb(3T_5^0(3,2)^{ab}, t) = (1,6,15,17,7) = t^3 \ (1+t) + 3 \ t^2 \ (1+t)^2 + 2t \ (1+t)^3 + (1+t)^4$,
and the Tutte polynomial for the graph $K_{3,2}$ *is equal to* $x + 3 \ x^2 + 2 \ x^3 + x^4 + y + 3 \ x \ y + y^2$.

(3) *Let* $G = K_{3,3}$ *be complete bipartite graph of type* $(3,3)$. *Then*
$Hilb(3T_6^0(3,3)^{ab}, t) = (1,9,36,75,78,31) =$
$(1+t)^5 + 4t(1+t)^4 + 10t^2(1+t)^3 + 11t^3(1+t)^2 + 5t^4(1+t)$,
and the Tutte polynomial of the bipartite graph $K_{3,3}$ *is equal to*
$5x + 11x^2 + 10x^3 + 4x^4 + x^5 + 15xy + 9x^2y + 6xy^2 + 5y + 9y^2 + 5y^3 + y^4$.

(4) *Consider complete multipartite graph $K_{2,2,2}$. One can show that*

$$Hilb(3T_6^{(0)}(K_{2,2,2})^{ab},t) = (1,12,58,137,154,64) =$$

$$11\ t^4(1+t)+25\ t^3(1+t)^2+20\ t^2(1+t)^3+7\ t(1+t)^4+(1+t)^5,$$

and $Tutte(K_{2,2,2},x,y) = x(11,25,20,7,1)_x + y\ (11,46,39,8)_x + y^2(32,52,12)_x + y^3(40,24)_x + y^4(29,6)_x + 15y^5 + 5y^6 + y^7.$

■

The above examples show that the Hilbert polynomial $Hilb(3T_n^0(G)^{ab},t)$ appears to be a certain specialization of the Tutte polynomial of the corresponding graph G. Instead of using the Hilbert polynomial of the algebra $3T_n^0(G)^{ab}$ one can consider the graded Betti numbers polynomial $Betti(3T_n^0(G)^{ab},x,y)$. For example,

$$Betti(3T_3^0(K_3)^{ab},x,y) = 1+4\ x\ y+x^2\ (2\ y+3\ y^2)+2\ x^3\ y^2,$$

$$Betti(3T_4^0(K_{2,2})^{ab},x,y) = 1+x\ (4\ y+y^2)+x^2\ (9\ y^2+y^3)+x^3\ (3\ y^2+6\ y^3)+3\ x^4\ y^3,$$

$Betti(3T_4^0(K_4)^{ab},x,y) =$

$$1+10\ x\ y+x^2\ (10\ y+24\ y^2)+x^3\ (46\ y^2+15\ y^3)+x^4\ (25\ y^2+36\ y^3)+x^5\ (6\ y^2+25\ y^3)+6\ x^6\ y^3.$$

Claim Let $G=(V,E)$ be a connected graph without loops. Then ($n=|V(G)|=$ *number of vertices*, $e=|E(G)|=$ *number of edges*)

$$Betti(3T_n^0(G)^{ab},-x,x) = (1-x)^e\ Hilb(3T_n^0(G)^{ab},x),$$

Question Let G be a connected subgraph of the complete graph K_n. Does the graded Betti polynomial $Betti(3T_n^0(G)^{ab},x,y)$ is a certain specialization of the Tutte polynomial $T(G,x,y)$?

Conjecture 4.3. *Let $\mathbf{n}=(n_1,\ldots,n_r)$ be a composition of $n\in\mathbb{Z}_{\geq 1}$, then*

$$Hilb(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab},t) = \sum_{\substack{\mathbf{k}=(k_1,\ldots,k_r)\\ 0<k_j\leq n_j}} (-t)^{|\mathbf{n}|-|\mathbf{k}|}\prod_{j=1}^{r}\left\{\begin{matrix} n_j\\ k_j\end{matrix}\right\}\prod_{j=1}^{|\mathbf{k}|-1}(1+jt),$$

where we set $|\mathbf{k}|:=k_1+\ldots+k_r$.

Corollary 4.1. *If Conjecture (4.3) is true, then*

$$(a)\quad 1+t(t-1)\sum_{(n_1,\ldots,n_r)\in\mathbb{Z}_{\geq 0}^r\ \backslash 0^r} Hilb(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab},t)\ \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_r^{n_r}}{n_r!} =$$

$$\Big(1+t\sum_{j=1}^{r}\ (e^{-x_j}-1)\Big)^{1-t}.$$

$$(b)\qquad \sum_{(n_1,n_2,\ldots,n_r)\in\mathbb{Z}_{\geq 0}\backslash 0^r}\dim(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab}\ \frac{x^{n_1}}{n_1!}\cdots\frac{x^{n_r}}{n_r!} = -log\Big(1-r+\sum_{j=1}^{r}\ e^{-x_j}\Big).$$

$$(c)\quad Hilb(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab},t) = (-t)^{|\mathbf{n}|}\ Chrom(K_{n_1,\ldots,n_r},-t^{-1}),$$

where for any graph Γ we denote by $Chrom(\Gamma,x)$ the chromatic polynomial of that graph.

Indeed, one can show [24]

Proposition 4.2. *If $r \in \mathbb{Z}_{\geq 1}$, then*

$$Chrom(K_{n_1,\ldots,n_r},t) = \sum_{\mathbf{k}=(k_1,\ldots,k_r)} \prod_{j=1}^{r} \left\{ {n_j \atop k_j} \right\} (t)_{|\mathbf{k}|},$$

where by definition $(t)_m := \prod_{j=1}^{m-1} \ (t-j)$, $\quad (t)_0 = 1, \ \ (t)_m = 0, \ if \ m < 0.$

Finally we describe explicitly the exponential generating function for the *Tutte polynomials* of the weighted complete multipartite graphs. We refer the reader to [68] for a definition and a list of basic properties of the Tutte polynomial of a graph.

Definition 4.4. *Let $r \geq 2$ be a positive integer and $\{S_1,\ldots,S_r\}$ be a collection of sets of cardinalities $\#|S_j| = n_j, \ j = 1,\ldots,r$. Let $\ell := \{\ell_{ij}\}_{1 \leq i < j \leq n}$ be a collection of non- negative integers.*

The ℓ-weighted complete multipartite graph $K^{(\ell)}_{n_1,\ldots,n_r}$ is a graph with the set of vertices equals to the disjoint union $\coprod_{j=1}^{r} \ S_i$ of the sets $S_1,\ldots,S_r$, and the set of edges $\{(\alpha_i,\beta_j), \alpha_i \in S_i, \ \beta_j \in S_j\}_{1 \leq i < j \leq r}$ of multiplicity ℓ_{ij} each edge $9\alpha,\beta_j)$.

Theorem 4.3. *Let us fix an integer $r \geq 2$ and a collection of non-negative integers $\ell := \{\ell_{ij}\}_{1 \leq i < j \leq r}$. Then*

$$1 \ + \sum_{\substack{\mathbf{n}=(n_1,\ldots,n_r) \in \mathbb{Z}^r_{\geq 0} \\ \mathbf{n} \neq \mathbf{0}}} (x-1)^{\kappa(\ell,\mathbf{n})} \quad Tutte(K^{(\ell)}_{n_1,\ldots,n_r},x,y) \ \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!} =$$

$$\Bigg(\sum_{\mathbf{m}=(m_1,\ldots,m_r) \in \mathbb{Z}^r_{\geq 0}} y^{\sum_{1 \leq i < j \leq r} \ell_{ij} \ m_i m_j} \ (y-1)^{-|\mathbf{m}|} \ \frac{t_1^{m_1}}{m_1!} \cdots \frac{t_r^{m_r}}{m_r!} \Bigg)^{(x-1)(y-1)},$$

where $\kappa(\ell,\mathbf{n})$ denotes the number of connected components of the graph $K^{(\ell)}_{n_1,\ldots,n_r}$.

- (**Comments and Examples**)

(a) Clearly the condition $\ell_{ij} = 0$ means that there are no edges between vertices from the sets S_i and S_j. Therefore Theorem 4.3 allows to compute the Tutte polynomial of any (finite) graph. For example,
$Tutte(K^{(1^6)}_{2,2,2,2},x,y) = \{(0,362,927,911,451,121,17,1)_x, (362,2154,2928,1584,374,32)_x,$
$(1589,4731,3744,1072,96)_x, (3376,6096,2928,448,16)_x, (4828,5736,1764,152)_x,$
$(5404,4464,900,32)_x, (5140,3040,380)_x, (4340,1840,124)_x, (3325,984,24)_x, (2331,448)_x,$
$(1492,168)_x, (868,48)_x, (454,8)_x, 210,84,28,7,1\}_y.$

(b) One can show that a formula for the chromatic polynomials from Proposition 4.2 corresponds to the specialization $y = 0$ (but not direct substitution !) of the formula for generating function for the Tutte polynomials stated in Theorem 4.3.

(c) The Tutte polynomial $Tutte(K^{(\ell)}_{n_1,\ldots},x,y)$ does not symmetric with respect to parameters $\{\ell_{ij}\}_{1 \leq i < j \leq n}$. For example, let us write $\ell = (\ell_{12},\ell_{23},\ell_{13},\ell_{14},\ell_{24},\ell_{34})$, then $Tutte(K^{(6,3,4,5,2,4)}_{2,2,2,2},1,1) =$

[24] If $r = 1$, the complete unipartite graph $K_{(n)}$ consists of n distinct points, and

$$Chrom(K_{(n)},x) = x^n = \sum_{k=0}^{n-1} \left\{ {n \atop k} \right\} (x)_k.$$

Let us stress that to abuse of notation the complete unipartite graph $K_{(n)}$ consists of n disjoint points with the Tutte polynomial equals to 1 for all $n \geq 1$, whereas the complete graph K_n is equal to the complete multipartite graph $K_{(1^n)}$.

$2^8 \cdot 3 \cdot 5 \cdot 11^3 \cdot 241 = 1231760640$. On the other hand, $Tutte(K_{2,2,2,2}^{(6,4,3,5,2,4)}, 1, 1) = 2^{13} \cdot 3 \cdot 7 \cdot 11^2 \cdot 61 = 1269768192$.

$\Rrightarrow$ (d) (**Universal Tutte polynomials**)

Let $\mathbf{m} = (m_{ij},\ 1 \le i < j \le n)$ be a collection of non-negative integers. Define *generalized Tutte polynomial* $\widetilde{T}_n(\mathbf{m}, x, y)$ as follows : $\widetilde{T}_n(\mathbf{m}, x, y) =$

$$Coeff_{[t_1 \cdots t_n]} \Big(\sum_{\substack{\ell_1,\ldots,\ell_n \\ \ell_i \in \{0,1\}, \forall i}} y^{\sum_{1 \le i < j \le n} m_{ij}\ \ell_i \ell_j}\ (y-1)^{-\sum_J\ \ell_j}\ \frac{t_1^{\ell_1}}{\ell_1!} \cdots \frac{t_n^{\ell_n}}{\ell_n!} \Big)^{(x-1)(y-1)}.$$

Clearly that if $\Gamma \subset K_n^{(\ell)}$ is a subgraph of the weighted complete graph $K_n^{(\ell)} \stackrel{def}{:=} K_{1^n}^{(\ell)}$, then the Tutte polynomial of graph Γ multiplied by $(x-1)^{\kappa(\Gamma)}$ is equal to the following specialization

$$m_{ij} = 0, \quad if \quad edge\ (i,j) \notin \Gamma, \quad m_{ij} = \ell_{ij},\ if \quad edge \quad (i,j) \in \Gamma$$

of the generalized Tutte polynomial

$$(x-1)^{\kappa(\Gamma)}\ Tutte(\Gamma, x, y) = \widetilde{T}_n(\mathbf{m}, x, y) \Big|_{\substack{m_{ij}=0,\ if\ (i,j) \notin \Gamma \\ m_{ij}=\ell_{ij}\ if\ (i,j) \in \Gamma}}.$$

For example,

(a) Take $n = 6$ and $\Gamma = K_6 \setminus \{15, 16, 24, 25, 34, 36\}$, then $Tutte(\Gamma, x, y) = \{(0,4,9,8,4,1)_x, (4,13,9)_x, (8,7)_x, 5, 1\}_y$.

(b) Take $n = 6$ and $\Gamma = K_6 \setminus \{15, 26, 34\}$, then $Tutte(\Gamma, x, y) =$
$\{(0, 11, 25, 20, 7, 1)_x, (11, 46, 39, 8)_x, (32, 52, 12)_x, (40, 24)_x, (29, 6)_x, 15, 5, 1\}_y$.

(c) Take $n = 6$ and $\Gamma = K_6 \setminus \{12.34.56\} = K_{2,2,2}$. As a result one obtains an expression for the Tutte polynomial of the graph $K_{2,2,2}$ displayed in Example 4.1.

Now set us set

$$q_{ij} := \frac{y^{m_{ij}} - 1}{y - 1}.$$

Lemma 4.1. *The generalized Tutte polynomial* $\widetilde{T}_n(\mathbf{m}, x, y)$ *is a polynomial in the variables* $\{q_{ij}\}_{1 \le i < j \le n}$, x *and* y.

Definition 4.5. *The universal Tutte polynomial* $T_n(\{q_{ij}\}, x, y)$ *is defined to be the polynomial in the variables* $\{q_{ij}\}$, x, *and* y *defined in Lemma 4.1.*

Explicitly, $T_n(\{q_{ij}\}, x, y) =$

$$Coeff_{[t_1 \cdots t_n]} \Big(\sum_{\substack{\ell_1,\ldots,\ell_n \\ \ell_i \in \{0,1\}, \forall i}} \prod_{1 \le i < j \le n} (q_{ij}\ (y-1) + 1)^{\ell_i \ell_j}\ (y-1)^{-\sum_J\ \ell_j}\ \frac{t_1^{\ell_1}}{\ell_1!} \cdots \frac{t_n^{\ell_n}}{\ell_n!} \Big)^{(x-1)(y-1)}.$$

Corollary 4.2. *Let* $\{m_{ij}\}_{1 \le i < j \le n}$ *be a collection of positive integers. Then the specialization*

$$q_{ij} \longrightarrow [m_{ij}]_y := \frac{y^{m_{ij}} - 1}{y - 1}$$

of the universal Tutte polynomial $T_n(\{q_{ij}\}, x, y)$ *is equal to the Tutte polynomial of the complete graph* K_n *with each edge* (i,j) *of the multiplicity* m_{ij}.

Further specialization $q_{ij} \longrightarrow 0,\ if\ edge(i,j) \notin \Gamma$ allows to compute the Tutte polynomial for any graph. ■

Exercises 4.1.

(1) *Assume that* $\ell_{ij} = \ell$ *for all* $1 \le i < j \le r$. *Based on the above formula for the exponential generating function for the Tutte polynomials of the complete multipartite graphs* $K_{n_1,\ldots,n_r}$, *deduce the following well-known formula*

$$Tutte(K^{(\ell)}_{n_1,\ldots,n_r}, 1, 1) = \ell^{N-1}\ N^{r-2} \prod_{j=1}^{r} (N - n_j)^{n_j - 1},$$

where $N := n_1 + \cdots + n_r$. *It is well-known that the number* $Tutte(\Gamma, 1, 1)$ *is equal to the number of spanning trees of a connected graph* Γ.

(2) *Take* $r = 3$ *and let* n_1, n_2, n_3 *and* $\ell_{12}, \ell_{13}, \ell_{23}$ *be positive integers. Set* $N := \ell_{12}\ell_{13}n_1 + \ell_{12}\ell_{23}n_2 + \ell_{13}\ell_{23}n_3$ *Show that*

$$Tutte(K^{\ell_1,\ell_2,\ell_3}_{n_1,n_2,n_3}, 1, 1) = N\ (\ell_{12}n_2 + \ell_{13}n_3)^{n_1-1}(\ell_{12}n_1 + \ell_{13}n_3)^{n_2-1})(\ell_{13}n_1 + \ell_{23}n_2)^{n_3-1}.$$

(3) *Let* $r \ge 2$, *consider weighted complete multipartite graph* $K^{(\ell)}_{\underbrace{n,\ldots,n}_{r}}$, *where* $\ell = (\ell_{ij})$ *such that* $\ell_{1,j} = \ell,\ j = 1, \ldots, r$ *and* $\ell_{ij} = k,\ 2 \le i < j \le r$. *Show that*

$$Tutte(K^{(\ell)}_{\underbrace{n,\ldots,n}_{r}}, 1, 1) = k^n\ (r-1)^{n-1}\ \Big((r-1)\ell + k\Big)^{r-2}\ \Big((r-2)\ell + k\Big)^{(r-1)(n-1)}\ n^{nr-1}.$$

■

Let $\Gamma_n(*)$ be a spanning star subgraph of the complete graph K_n. For example, one can take for a graph $\Gamma_n(*)$ the subgraph $K_{1,n-1}$ with the set of vertices $V := \{1, 2, \ldots, n\}$ and that of edges $E := \{(i, n),\ i = 1, \ldots, n-1\}$. The algebra $3T_n^{(0)}(K_{1,n-1})$ can be treated as a "noncommutative analog" of the projective space $\mathbb{P}^{n-1}$.
We have $\theta_1 = u_{12} + u_{13} + \ldots + u_{1n}$. It is not difficult to see that
$Hilb(3T_n^{(0)}(K_{1,n-1})^{ab}, t) = (1+t)^{n-1}$, and $\theta_1^n = 0$.
Let us observe that $Chrom(\Gamma_n(\star), t) = t(t-1)^{n-1}$.

Problem 4.1. *Compute the Hilbert series of the algebra* $3T_n^{(0)}(K_{n_1,\ldots,n_r})$.

The first non-trivial case is that of *projective space*, i.e. the case $r = 2, n_1 = 1,\ n_2 = 5$.

On the other hand, if $\Gamma_n = \{(1,2) \to (2,3) \to \ldots \to (n-1,n)\}$ is the Dynkin graph of type A_{n-1}, then the algebra $3T_n^{(0)}(\Gamma_n)$ is isomorphic to the nil-Coxeter algebra of type A_{n-1}, and if $\Gamma_n^{(aff)} = \{(1,2) \to (2,3) \to \ldots \to (n-1,n) \to -(1,n)\}$ is the Dynkin graph of type $A^{(1)}_{n-1}$, i.e. *a cycle*, then the algebra $3T_n^{(0)}(\Gamma_n^{(aff)})$ is isomorphic to a certain quotient of the affine nil-Coxeter algebra of type $A^{(1)}_{n-1}$ by the two-sided ideal which can be described explicitly [47]. Moreover, *ibid*,

$$Hilb(3T_n^{0)}(\Gamma^{(aff)}), t) = [n]_t \prod_{j=1}^{n-1} [j(n-j)]_t,$$

see Theorem 4.1. Therefore, the dimension $dim(3T^{(0)}(\Gamma^{aff}))$ is equal to $n!\ (n-1)!$ and is equal also to the number of (directed) Hamiltonian cycles in the complete bipartite graph $K_{n,n}$, see [87], $A010790$.
It is not difficult to see that

$$Hilb(3T_n^{(0)}(\Gamma_n)^{ab}, t) = (t+1)^{n-1}, \quad Hilb(3T^{(0)}(\Gamma_n^{aff})^{ab}, t) = t^{-1}\ ((t+1)^n - t - 1),$$

whereas

$$Chrom(\Gamma_n, t) = t(t-1)^{n-1}, \quad Chrom(\Gamma_n^{aff}, t) = (t-1)^n + (-1)^n\ (t-1).$$

Exercises 4.2. *Let $K_{n_1,\ldots,n_r}$ be complete multipartite graph, $N := n_1 + \cdots + n_r$.*
Show that [25]

$$Hilb(3T_N(K_{n_1,\ldots,n_r}), t) = \frac{\prod_{j=1}^{r}\ \prod_{a=1}^{n_j-1}\ (1-a\ t)}{\prod_{j=1}^{N-1}\ (1-j\ t)}.$$

4.1.3 Quasi-classical and associative classical Yang–Baxter algebras of type B_n.

In this Section we introduce an analogue of the algebra $3T_n(\beta)$ for the classical root systems.

Definition 4.6.

(**A**) **The quasi-classical Yang–Baxter algebra** $\widehat{ACYB(B_n)}$ **of type** B_n *is an associative algebra with the set of generators* $\{x_{ij}, y_{ij}, z_i,\ 1 \le i \ne j \le n\}$ *subject to the set of defining relations*

(1) $x_{ij} + x_{ij} = 0, \quad y_{ij} = y_{ji}, \quad if \quad i \ne j,$

(2) $z_i\ z_j = z_j\ z_i,$

(3) $x_{ij}\ x_{kl} = x_{kl}\ x_{ij}, \quad x_{ij}\ y_{kl} = y_{kl}\ x_{ij}, \quad y_{ij}\ y_{kl} = y_{kl}\ y_{ij},$ *if i, j, k, l are distinct,*

(4) $z_i\ x_{kl} = x_{kl}\ z_i, \quad z_i\ y_{kl} = y_{kl}\ z_i,$ *if $i \ne k, l$,*

(5) *(Three term relations)*

$x_{ij}\ x_{jk} = x_{ik}\ x_{ij} + x_{jk}\ x_{ik} - \beta\ x_{ik}, \quad x_{ij}\ y_{jk} = y_{ik}\ x_{ij} + y_{jk}\ y_{ik} - \beta\ y_{ik},$

$x_{ik}\ y_{jk} = y_{jk}\ y_{ij} + y_{ij}\ x_{ik} + \beta\ y_{ij}, \quad y_{ik}\ x_{jk} = x_{jk}\ y_{ij} + y_{ij}\ y_{ik} + \beta\ y_{ij},$

if $1 \le i < j < k \le n$,

(6) *(Four term relations)*

$x_{ij}\ z_j = z_i\ x_{ij} + y_{ij}\ z_i + z_j\ y_{ij} - \beta\ z_i,$

if $i < j$.

(**B**) **The associative classical Yang–Baxter algebra** $ACYB(B_n)$ **of type** B_n *is the special case $\beta = 0$ of the algebra $\widehat{ACYB(B_n)}$.*

Comments 4.2.

- In the case $\beta = 0$ the algebra $ACYB(B_n)$ has a rational representation

$$x_{ij} \longrightarrow (x_i - x_j)^{-1}, \quad y_{ij} \longrightarrow (x_i + x_j)^{-1}, \quad z_i \longrightarrow x_i^{-1}.$$

- In the case $\beta = 1$ the algebra $\widehat{ACYB(B_n)}$ has a "trigonometric" representation

$$x_{ij} \longrightarrow (1 - q^{x_i - x_j})^{-1}, \quad y_{ij} \longrightarrow (1 - q^{x_i + x_j})^{-1}, \quad z_i \longrightarrow (1 + q^{x_i})(1 - q^{x_i})^{-1}.$$

Definition 4.7. The bracket algebra $\mathcal{E}(B_n)$ of type B_n *is an associative algebra with the set of generators $\{x_{ij}, y_{ij}, z_i,\ 1 \le i \ne j \le n\}$ subject to the set of relations* $(1)-(6)$ *listed in Definition 4.6, and the additional relations*

(5a) $x_{jk}\ x_{ij} = x_{ij}\ x_{ik} + x_{ik}\ x_{jk} - \beta\ x_{ik}, \quad y_{jk}\ x_{ij} = x_{ij}\ y_{ik} + y_{ik}\ y_{jk} - \beta\ y_{ik},$

$y_{jk}\ x_{ik} = y_{ij}\ y_{jk} + x_{ik}\ y_{ij} + \beta\ y_{ij}, \quad x_{jk}\ y_{ik} = y_{ij}\ x_{jk} + y_{ik}\ y_{ij} + \beta\ y_{ij},$

if $1 \le i < j < k \le n$,

(6a) $z_j\ x_{ij} = x_{ij}\ z_i + z_i\ y_{ij} + y_{ij}\ z_j - \beta\ z_i,$

if $i < j$.

[25] It should be remembered that to abuse of notation, the complete graph K_n, by definition, is equal to the complete multipartite graph $K(\underbrace{(1,\ldots,1)}_{n})$, whereas the graph $K_{(n)}$ is a collection of n distinct points.

Definition 4.8. *The quasi-classical Yang–Baxter algebra $\widehat{ACYB(D_n)}$ of type D_n, as well as the algebras $ACYB(D_n)$ and $\mathcal{E}(D_n)$, are defined by putting $z_i = 0,\ i = 1, \ldots, n$, in the corresponding B_n-versions of algebras in question.*

Conjecture 4.4. *The both algebras $\mathcal{E}(B_n)$ and $\mathcal{E}(D_n)$ are* **Koszul**, *and*

$$Hilb(\mathcal{E}(B_n), t) = \Big(\prod_{j=1}^{n}(1-(2j-1)t)\Big)^{-1}; \quad if \quad n \geq 4, \quad Hilb(\mathcal{E}(D_n), t) = \Big(\prod_{j=1}^{n-1}(1-2j\ t)\Big)^{-1}.$$

Example 4.3. $Hilb(ACYB(B_2), t) = (1-4t+2t^2)^{-1}$,
$Hilb(ACYB(B_3), t) = (1-9t+16t^2-4t^3)^{-1}$,
$Hilb(ACYB(B_4), t) = (1-16t+64t^2-60t^3+9t^4)^{-1}$,
$Hilb(ACYB(D_4), t) = (1-12t+18t^2-4t^3)^{-1}$.
However, $Hilb(ACYB(B_5), t) = (1-25t+180t^2-400t^3+221t^4-31t^5)^{-1}$.

■

Let us introduce the following Coxeter type elements:

$$h_{B_n} := \prod_{a=1}^{n-1} x_{a,a+1}\ z_n \in \mathcal{E}(B_n), \quad and \quad h_{D_n} := \prod_{a=1}^{n-1} x_{a,a+1}\ y_{n-1,n} \in \mathcal{E}(D_n). \tag{4.2}$$

Let us bring the element h_{B_n} (resp. h_{D_n}) to the reduced form in the algebra $\mathcal{E}(B_n)$ that is, let us consecutively apply the defining relations $(1)-(6)$, $(5a, 6a)$ to the element h_{B_n} (resp. apply to h_{D_n} the defining relations for algebra $\mathcal{E}(D_n)$) in any order until unable to do so. Denote the the resulting (noncommutative) polynomial by $P_{B_n}(x_{ij}, y_{ij}, z)$ (resp. $P_{D_n}(x_{ij}, y_{ij})$). In principal, this polynomial itself can depend on the order in which the relations $(1)-(6)$, $(5a, 6a)$ are applied.

Conjecture 4.5. *(Cf [90],* **6.C5**, *(c))*

(1) *Apart from applying the commutativity relations* $(1)-(4)$ *, the polynomial* $P_{B_n}(x_{ij}, y_{ij}, z)$ *(resp.* $P_{D_n}(x_{ij}, y_{ij})$*) does not depend on the order in which the defining relations have been applied.*

(2) *Define polynomial* $P_{B_n}(s, r, t)$ *(resp.* $P_{D_n}(s, r)$*) to be the the image of that* $P_{B_n}(x_{ij}, y_{ij}, z)$ *(resp.* $P_{D_n}(x_{ij}, y_{ij})$*) under the specialization*

$$x_{ij} \longrightarrow s, \quad y_{ij} \longrightarrow r, \quad z_i \longrightarrow t.$$

Then
$P_{B_n}(1,1,1) = \frac{1}{2}\binom{2n}{n} = \frac{1}{2}\ Cat_{B_n}$.

Note that $P_{B_n}(1,0,1) = Cat_{A_{n-1}}$.

Problem 4.2. *Investigate the B_n and D_n types reduced polynomials corresponding to the Coxeter elements* (4.2), *and the reduced polynomials corresponding to the longest elements*

$$w_{B_n} := \prod_{J=1}^{n} z_j \left(\prod_{1 \leq i<j \leq n} x_{ij}\ y_{ij}\right), \quad w_{D_n} = \prod_{1 \leq i<j \leq n} x_{ij}\ y_{ij}.$$

4.2 Super analogue of 6-term relations and classical Yang–Baxter algebras

4.2.1 Six term relations algebra $6T_n$, its quadratic dual $(6T_n)^!$, and algebra $6HT_n$

Definition 4.9. **The 6 term relations algebra** $6T_n$ *is an associative algebra (say over $\mathbb{Q}$) with the set of generators $\{r_{i,j}, 1 \leq i \neq j < n\}$, subject to the following relations:*

1) $r_{i,j}$ *and* $r_{k,l}$ *commute, if* $\{i,j\} \cap \{k,l\} = \emptyset$,
*2) (***unitarity condition***)* $r_{ij} + r_{ji} = 0$,
*3) (***Classical Yang–Baxter relations***)*
$[r_{ij}, r_{ik} + r_{jk}] + [r_{ik}, r_{jk}] = 0$, *if* i,j,k *are distinct.*

We denote by CYB_n, named by *classical Yang–Baxter algebra*, an associative algebra over $\mathbb{Q}$ generated by elements $\{r_{ij},\ 1 \le i \ne j \le n\}$ subject to relations 1) and 3).

Note that the algebra $6T_n$ is given by $\binom{n}{2}$ generators and $\binom{n}{3} + 3\ \binom{n}{4}$ quadratic relations.

Definition 4.10. *Define Dunkl elements in the algebra* $6T_n$ *to be*

$$\theta_i = \sum_{j \ne i} r_{ij}, \quad i = 1, \ldots, n.$$

It easy to see that the Dunkl elements $\{\theta_i\}_{1 \le i \le n}$ generate a commutative subalgebra in the algebra $6T_n$.

Example 4.4. *(***Some "rational and trigonometric" representations of the algebra** $6T_n$*)*

Let $A = U(sl(2))$ be the universal enveloping algebra of the Lie algebra $sl(2)$. Recall that the algebra $sl(2)$ is spanned by the elements e, f, h, such that $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

Let's search for solutions to the $CYBE$ in the form

$$r_{i,j} = a(u_i,u_j)\ h \otimes h + b(u_i,u_j)\ e \otimes f + c(u_i,u_j)\ f \otimes e,$$

where $a(u,v), b(u,v) \ne 0, c(u,v) \ne 0$ are meromorphic functions of the variables $(u,v) \in \mathbb{C}^2$, defined in a neighborhood of $(0,0)$, taking values in $A \otimes A$. Let $a_{ij} := a(u_i,u_j)$ (resp. $b_{ij} := b(u_i,u_j), c_{ij} := c(u_i,u_j)$).

Lemma 4.2. *The elements* $r_{i,j} := a_{ij}\ h \otimes h + b_{ij}\ e \otimes f + c_{ij}\ f \otimes e$ *satisfy CYBE iff*
$b_{ij}\ b_{jk}\ c_{ik} = c_{ij}\ c_{jk}\ b_{ik}$ *and* $4\ a_{ik} = b_{ij}\ b_{jk}/b_{ik} - b_{ik}\ c_{jk}/b_{ij} - b_{ik}\ c_{ij}/b_{jk}$,
for $1 \le i < j < k \le n$.

It is not hard to see that

- there are three rational solutions:

$$r_1(u,v) = \frac{1/2\ h \otimes h + \ e \otimes f + f \otimes e}{u-v}, \quad r_2(u,v) = \frac{u+v}{4(u-v)}\ h \otimes h + \frac{u}{u-v}\ e \otimes f + \frac{v}{u-v} f \otimes e,$$

and $r_3(u,v) := -r_2(v,u)$.

- there is a trigonometric solution

$$r_{trig}(u,v) = \frac{1}{4}\ \frac{q^{2u}+q^{2v}}{q^{2u}-q^{2v}}\ h \otimes h + \frac{q^{u+v}}{q^{2u}-q^{2v}}\ \Bigl(e \otimes f + f \otimes e\Bigr).$$

Notice that the **Dunkl element** $\theta_j := \sum_{a \ne j} r_{trig}(u_a,u_j)$ corresponds to the truncated (or level 0) **trigonometric Knizhnik–Zamolodchikov operator**.

In fact, the "sl_n-Casimir element" $\Omega = \frac{1}{2}\Bigl(\sum_{i=1}^n\ E_{ii} \otimes E_{ii}\Bigr) + \sum_{1 \le i < j \le n}\ E_{ij} \otimes E_{ji}$ satisfies the 4-term relations

$$[\Omega_{12}, \Omega_{13} + \Omega_{23}] = 0 = [\Omega_{12} + \Omega_{13}, \Omega_{23}],$$

and the elements $r_{ij} := \frac{\Omega_{ij}}{u_i - u_j}$, $1 \le i < j \le n$, satisfy the classical Yang–Baxter relations.

Recall that the set $\{E_{ij} := (\delta_{ik}\ \delta_{jl})_{1 \le k,l \le n},\ \ 1 \le i,j \le n\}$, stands for the standard basis of the algebra $Mat(n,\mathbb{R})$. ■

Definition 4.11. *Denote by* $6T_n^{(0)}$ *the quotient of the algebra* $6T_n$ *by the (two-sided) ideal generated by the set of elements* $\{r_{i,j}^2,\ 1 \le i < j \le n\}$.

More generally, let $\{\beta,\ q_{ij},\ 1 \le i < j \le n\}$ be a set of parameters. Let $R := \mathbb{Q}[\beta][q_{ij}^{\pm 1}]$.

Definition 4.12. *Denote by* $6HT_n$ *the quotient of the algebra* $6T_n \otimes R$ *by the (two-sided) ideal generated by the set of elements* $\{r_{i,j}^2 - \beta\ r_{i,j} - q_{ij},\ 1 \le i < j \le n\}$.

All these algebras are naturally graded, with $deg(r_{i,j}) = 1$, $deg(\beta) = 1$, $deg(q_{ij}) = 2$. It is clear that the algebra $6T_n^{(0)}$ can be considered as the infinitesimal deformation $R_{i,j} := 1 + \epsilon\ r_{i,j}$, $\epsilon \longrightarrow 0$, of the Yang-Baxter group [26] YB_n.

Corollary 4.3. *Define* $h_{ij} = 1 + r_{ij} \in 6HT_n$. *Then the following relations in the algebra* $6HT_n$ *are satisfied:*

(1) $r_{ij}\ r_{ik}\ r_{jk} = r_{jk}\ r_{ik}\ r_{ij}$ *for all pairwise distinct* i, j *and* k;

(2) *(Yang-Baxter relations)* $h_{ij}\ h_{ik}\ h_{jk} = h_{jk}\ h_{ik}\ h_{ij}$, *if* $1 \le i < j < k \le n$.

Note, the item (1) includes three relations in fact.

Proposition 4.3.

(1) **The quadratic dual** $(6T_n)^!$ *of the algebra* $6T_n$ *is a quadratic algebra generated by the elements* $\{t_{i,j},\ 1 \le i < j \le n\}$ *subject to the set of relations*

(i) $t_{i,j}^2 = 0$ *for all* $i \ne j$;

(ii) (Anticommutativity) $t_{ij}\ t_{k,l} + t_{k,l}\ t_{i,j} = 0$ *for all* $i \ne j$ *and* $k \ne l$;

(iii) $t_{i,j}\ t_{i,k} = t_{i,k}\ t_{j,k} = t_{i,j}\ t_{j,k}$, *if* i, j, k *are distinct.*

(2) **The quadratic dual** $(6T_n^{(0)})^!$ **of the algebra** $6T_n^{(0)}$ *is a quadratic algebra with generators* $\{t_{i,j},\ 1 \le i < j \le n\}$ *subject to the relations (ii)-(iii) above only.*

4.2.2 Algebras $6T_n^{(0)}$ and $6T_n^{\bigstar}$

We are reminded that the algebra $6T_n^{(0)}$ is the quotient of the six term relation algebra $6T_n$ by the two-sided ideal generated by the elements $\{r_{ij}\}_{1 \le i < j \le n}$. Important consequence of the classical Yang–Baxter relations and relations $r_{ij}^2 = 0, \forall i \ne j$, is that the both additive Dunkl elements $\{\theta_i\}_{1 \le i \le n}$ and multiplicative ones $\{\Theta_i = \prod_{a=i-1}^{1} h_{ai}^{-1} \prod_{a=i+1}^{n} h_{ia}\}_{1 \le i \le n}$ generate commutative subalgebras in the algebra $6T_n^{(0)}$ (and in the algebra $6T_n$ as well), see Corollary 4.3. The problem we are interested in, is to describe commutative subalgebras generated by additive (resp. multiplicative) Dunkl elements in the algebra $6T_n^{(0)}$. Notice that the subalgebra generated by additive Dunkl elements in the abelianization [27] of the algebra $6T_n(0)$ has been studied in [85],[78]. In order to state the result from [78] we need, let us introduce a bit of notation. As before, let $\mathcal{F}l_n$ denotes the complete flag variety, and denote by $\mathcal{A}_n$ the algebra generated by the curvature of 2-forms of the standard Hermitian linear bundles over the flag variety $\mathcal{F}l_n$, see e.g [78]. Finally, denote by I_n the ideal in the ring of polynomials $\mathbb{Z}[t_1, \ldots, t_n]$ generated by the set of elements

$$(t_{i_1} + \cdots + t_{i_k})^{k(n-k)+1},$$

[26] For the reader convenience we recall the definition of the *Yang–Baxter* group

Definition 4.13. *The Yang–Baxter group* YB_n *is a group generated by elements* $\{R_{ij}^{\pm 1},\ 1 \le i < j \le n\}$, *subject to the set of defining relations*

- $R_{ij} R_{kl} = R_{kl} R_{ij}$, *if* i, j, k, l, *are distinct,*
- *(***Quantum Yang–Baxter relations***)*

$$R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij}, \quad if \quad 1 \le i < j < k \le n.$$

[27] See e.g. http://mathworld.wolfram.com/Abelianization.html

for all sequences of indices $1 \le i_1 < i_2 < \ldots < i_k \le n, \quad k = 1, \ldots, n.$

Theorem 4.4. *([85],[78])*
(**A**) *There exists a natural isomorphism*

$$\mathcal{A}_n \longrightarrow \mathbb{Z}[t_1, \ldots, t_n]/I_n,$$

(**B**) $Hilb(\mathcal{A}_n, t) = t^{\binom{n}{2}} \; Tutte(K_n, 1+t, t^{-1}).$

Therefore the dimension of $\mathcal{A}_n$ (as a $\mathbb{Z}$-vector space) is equal to the number $\mathcal{F}(n)$ of forests on n labeled vertices. It is well-known that

$$\sum_{n \ge 1} \mathcal{F}(n) \frac{x^n}{n!} = exp\Big(\sum_{n \ge 1} n^{n-1} \frac{x^n}{n!}\Big) - 1.$$

For example, $Hilb(\mathcal{A}_3, t) = (1,2,3,1)$, $Hilb(\mathcal{A}_4, t) = (1,3,6,10,11,6,1)$,
$Hilb(\mathcal{A}_5, t) = (1,4,10,20,35,51,64,60,35,10,1)$,
$Hilb(\mathcal{A}_6, t) = (1,5,15,35,70,126,204,300,405,490,511,424,245,85,15,1)$.

Problem 4.3. *Describe subalgebra in* $(6T_n^{(0)})^{ab}$ *generated by the multiplicative Dunkl elements* $\{\Theta_i\}_{1 \le i \le n}$.

On the other hand, the commutative subalgebra $\mathcal{B}_n$ generated by the additive Dunkl elements in the algebra $6T_n^{(0)}$, $n \ge 3$, has *infinite* dimension. For example,

$$\mathcal{B}_3 \cong \mathbb{Z}[x,y]/\langle \; xy(x+y) \; \rangle,$$

and the Dunkl elements $\theta_j^{(3)}, j = 1,2,3$, have infinite order.

Definition 4.14. *Define algebra* $6T_n^{\bigstar}$ *to be the quotient of that* $6T_n^{(0)}$ *by the two-sided ideal generated by the set of "cyclic relations"*

$$\sum_{j=2}^{m} \prod_{a=j}^{m} r_{i_1,i_a} \prod_{a=2}^{j} r_{i_1,i_a} = 0$$

for all sequences $\{1 \le i_1, i_2, \ldots, i_m \le n\}$ *of pairwise distinct integers, and all integers* $2 \le m \le n$.

For example,

- $Hilb(6T_3^{\bigstar}, t) = (1,3,5,4,1) = (1+t)(1,2,3,1)$.
- Subalgebra (over $\mathbb{Z}$) in the algebra $6T_3^{\bigstar}$ generated by Dunkl elements θ_1 and θ_2 has the Hilbert polynomial equal to (1,2,3,1), and the following presentation: $\mathbb{Z}\,[x,y]/I_3$, where I_3 denotes the ideal in $\mathbb{Z}[x,y]$ generated by x^3, y^3, and $(x+y)^3$.
- $Hilb(6T_4^{\bigstar}, t) = (1,6,23,65,134,164,111,43,11,1)_t$.

As a consequence of the cyclic relations, one can check that for any integer $n \ge 2$ the n-th power of the additive Dunkl element θ_i is equal to zero in the algebra $6T_n^{\bigstar}$ for all $i = 1, \ldots, n$. Therefore, the Dunkl elements generate a finite dimensional commutative subalgebra in the algebra $6T_n^{\bigstar}$. There exist natural homomorphisms

$$6T_n^{\bigstar} \longrightarrow 3T_n^{(0)}, \qquad \mathcal{B}_n \xrightarrow{\;\tilde{\pi}\;} \mathcal{A}_n \longrightarrow H^*(\mathcal{F}l_n, \mathbb{Z}) \tag{4.3}$$

The first and third arrows in (4.19) are epimorphism. We expect that the map $\tilde{\pi}$ is also epimorphism [28], and looking for a description of the kernel $ker(\tilde{\pi})$.

[28] Contrary to the case of the map $pr_n : \mathbb{Z}[\theta_1, \ldots, \theta_n] \longrightarrow (3T_n(0))^{ab}$, where the image $Im(pr_n)$ has dimension equals to the number of permutations in $\mathbb{S}_n$ with (n-1) inversions see [87],$A001892$.

Comments 4.3.

• Let us denote by $\mathcal{B}_n^{mult}$ and $\mathcal{A}_n^{mult}$ the subalgebras generated by **multiplicative** Dunkl elements in the algebras $6T_n^{(0)}$ and $(6T_n^{(0)})^{ab}$ correspondingly. One can define a sequence of maps

$$\mathcal{B}_n^{mult} \longrightarrow \mathcal{A}_n^{mult} \xrightarrow{\tilde{\phi}} K^*(\mathcal{F}l_n), \tag{4.4}$$

which is a K-theoretic analog of that (4.3). It is an interesting problem to find a geometric interpretation of the algebra $\mathcal{A}_n^{mult}$ and the map $\tilde{\phi}$.

• (**"Quantization"**) Let β and $\{q_{ij} = q_{ji}, 1 \le i, j \le n\}$ be parameters.

Definition 4.15. *Define algebra $6HT_n$ to be the quotient of the algebra $6T_n$ by the two sided ideal generated by the elements $\{r_{ij}^2 - \beta\ r_{ij} - q_{ij}\}_{1 \le i,j \le n}$.*

Lemma 4.3. *The both additive $\{\theta_i\}_{1 \le i \le n}$ and multiplicative $\{\Theta_i\}_{1 \le i \le n}$ Dunkl elements generate commutative subalgebras in the algebra $6HT_n$.*

Therefore one can define algebras $6\mathcal{HB}_n$ and $6\mathcal{HA}_n$ which are a "quantum deformation" of algebras $\mathcal{B}_n$ and $\mathcal{A}_n$ respectively. We **expect** that in the case $\beta = 0$ and a special choice of "arithmetic parameters" $\{q_{ij}\}$, the algebra $\mathcal{HA}_n$ is connected with the Arithmetic Schubert and Grothendieck Calculi, cf [93], [85]. Moreover, for a "general"set of parameters $\{q_{ij}\}_{1 \le i,j \le n}$ and $\beta = 0$, we **expect** an existence of a natural homomorphism

$$\mathcal{HA}_n^{mult} \longrightarrow \mathcal{QK}^*(\mathcal{F}l_n),$$

where $\mathcal{QK}^*(\mathcal{F}l_n)$ denotes a *multiparameter quantum deformation* of the K-theory ring $K^*(\mathcal{F}l_n)$, [47], [51]; see also Section 3.1. Thus, we treat the algebra $\mathcal{HA}_n^{mult}$ as the K-theory version of a multiparameter quantum deformation of the algebra $\mathcal{A}_n^{mult}$ which is generated by the curvature of 2-forms of the Hermitian linear bundles over the flag variety $\mathcal{F}l_n$.

• One can define an analogue of the algebras $6T_n^{(0)}$, $6HT_n$ etc, denoted by $6T(\Gamma)$, etc, for any subgraph $\Gamma \subset K_n$ of the complete graph K_n, and in fact for any oriented matroid. It is known that $Hilb((6T_n(\Gamma)^{ab}, t) = t^{e(\Gamma)}\ Tutte(\Gamma, 1+t, t^{-1})$, see e.g. [6] and the literature quoted therein.

4.2.3 Hilbert series of algebras CYB_n and $6T_n$ [29]

Examples 4.2. $Hilb(6T_3, t) = (1 - 3t + t^2)^{-1}$,
$Hilb(6T_4, t) = (1 - 6t + 7t^2 - t^3)^{-1}$, $Hilb(6T_5, t) = (1 - 10t + 25t^2 - 15t^3 + t^4)^{-1}$,
$Hilb(6T_6, t) = (1 - 15t + 65t^2 - 90t^3 + 31t^4 - t^5)^{-1}$.
$Hilb(6T_3^{(0)}, t) = [2][3](1-t)^{-1}$, $Hilb(6T_4^{(0)}, t) = [4](1-t)^{-2}(1-3t+t^2)^{-1}$.

In fact, the following statements are true.

Proposition 4.4. *(Cf [3]) Let $n \ge 2$, then*

• *The algebras $6T_n$ and CYB_n are* **Koszul***;*

• *We have*

$$Hilb(6T_n, t) = \left(\sum_{k=0}^{n-1} (-1)^k \begin{Bmatrix} n \\ n-k \end{Bmatrix} t^k \right)^{-1},$$

where $\left\{ {n \atop k} \right\}$ stands for the Stirling numbers of the second kind, i.e. the number of ways to partition a set of n things into k nonempty subsets.

[29] Results of this Subsection have been obtained independently in [3]. This paper contains, among other things, a description of a basis in the algebra $6T_n$, and much more.

•

$$Hilb(CYB_n,t) = \Big(\sum_{k=0}^{n-1} (-1)^k \ (k+1)! \ N(k,n) \ t^k \Big)^{-1},$$

where $N(k,n) = \frac{1}{n}\binom{n}{k}\ \binom{n}{k+1}$ *denotes the Narayana number, i.e the number of Dyck n-paths with exactly k peaks.*

Corollary 4.4.

(**A**) *The Hilbert polynomial of the quadratic dual of the algebra $6T_n$ is equal to*

$$Hilb(6T_n^!,t) = \sum_{k=0}^{n-1} \ \left\{ {n \atop n-k} \right\} \ t^k.$$

It is well-known that

$$\sum_{n \ge 0} \Big(\sum_{k=0}^{n-1} \ \left\{ {n \atop n-k} \right\} \ t^k \Big) \frac{z^n}{n!} = \exp\Big(\frac{\exp(zt)-1}{t}\Big).$$

Therefore,

$$\dim(6T_n)^! = Bell_n,$$

where $Bell_n$ denotes the n-th Bell number, i.e. the number of ways to partition n things into subsets, see [87]

Recall, that $\sum_{n \ge 0} Bell_n \ \frac{z^n}{n!} = \exp(\exp(z)-1))$.

(**B**) *The Hilbert polynomial of the quadratic dual of the algebra CYB_n is equal to*

$$Hilb((CYB_n)^!,t) = \sum_{k=0}^{n-1} (k+1)! \ N(k,n) \ t^k = (n-1)! \ L_{n-1}^{(\alpha=1)}(-t^{-1}) \ t^{n-1},$$

where $L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x}x^{n+\alpha}\right)$ *denotes the generalized Laguerre polynomial.*

It is well-known that

$$\sum_{n \ge 0} \Big(\sum_{k \ge 0}^{n-1} (k+1)! N(k,n) \ t^k \Big) \frac{z^n}{n!} = \exp\Big(z(1-zt)^{-1}\Big).$$

Comments 4.4. Let $\mathcal{E}_n(u)$, $u \ne 0,1$, be the **Yokonuma-Hecke** algebra, see e.g. [83] and the literature quoted therein. It is known that the dimension of the Yokonuma–Hecke algebra $\mathcal{E}_n(u)$ is equal to $n! \ B_n$, where B_n denotes as before the n-th Bell number. Therefore, $dim(\mathcal{E}_n(u)) = dim((6T_n)^! \rtimes \mathbb{S}_n)$, where $(6T_n)^! \rtimes \mathbb{S}_n$ denotes the semi-direct product of the algebra $(6T_n)^!$ and the symmetric group $\mathbb{S}_n$. It seems an interesting task to check whether or not the algebras $(6T_n)^! \rtimes \mathbb{S}_n$ and $\mathcal{E}_n(u)$ are isomorphic. ■

Remark 4.2. Denote by $\mathcal{M}YB_n$ the group algebra over $\mathbb{Q}$ of the **monoid** corresponding to the Yang–Baxter group YB_n, see e.g. Definition 4.10. Let $P(\mathcal{M}YB_n,s,t)$ denotes the Poincare polynomial of the algebra $\mathcal{M}YB_n$. One can show that

$$Hilb(6T_n,s) = P(\mathcal{M}YB_n,-s,1)^{-1}.$$

For example,

$P(\mathcal{M}YB_3,s,t) = 1+3s \ t + s^2 \ t^3$, $P(\mathcal{M}YB_4,s,t) = 1+6s \ t+s^2 \ (3t^2+4t^3)+s^3 \ t^6$, $P(\mathcal{M}YB_5,s,t) = 1+10s \ t+s^2 \ (15t^2+10t^3)+s^3 \ (10t^4+5t^6)+s^4 \ t^{10}$.

Note that $Hilb(\mathcal{M}YB_n,t) = P(\mathcal{M}YB_n,-1,t)^{-1}$ and $P(\mathcal{M}YB_n,1,1) = Bell_n$, the n-th Bell number.

Conjecture 4.6.

$$P(\mathcal{M}YB_n, s, t) = \sum_{\pi} s^{\#(\pi)}\ t^{n(\pi)},$$

where the sum runs over all partitions $\pi = (I_1, \ldots, I_k)$ *of the set* $[n] := [1, \ldots, n]$ *into nonempty subsets* $I_1, \ldots, I_k$, *and we set by definition,* $\#(\pi) := n - k$, $n(\pi) := \sum_{a=1}^{k} \binom{|I_a|}{2}$.

∎

Remark 4.3. For any finite Coxeter group (W, S) one can define the algebra $CYB(W) := CYB(W, S)$ which is an analog of the algebra $CYB_n = CYB(A_{n-1})$ for other root systems.

Conjecture 4.7. *(A.N. Kirillov, Y. Bazlov) Let* (W, S) *be a finite Coxeter group with the root system* Φ. *Then*

- *the algebra* $CYB(W)$ *is* **Koszul**;
- $Hilb(CYB(W), t) = \left\{ \sum_{k=0}^{|S|} r_k(\Phi)\ (-t)^k \right\}^{-1}$,

where $r_k(\Phi)$ *is equal to the number of subsets in* Φ^+ *which constitute the positive part of a root subsystem of rank* k. *For example,* $r_1(\Phi) = |\Phi^+|$, *and* $r_2(\Phi)$ *is equal to the number of defining relations in a representation of the algebra* $CYB(W)$.

Example 4.5. $Hilb(CYB(B_2)^!, t) = (1, 4, 3)$, $\quad Hilb(CYB(B_3)^!, t) = (1, 9, 13, 2)$,
$Hilb(CYB(B_4)^!, t) = (1, 16, 46, 28, 5)$, $Hilb(CYB(B_5)^!, t) = (1, 25, 130, 200, 101, 12)$;
$Hilb(CYB(D_4)^!, t) = (1, 12, 34, 24, 4)$, $\quad Hilb(CYB(D_5)^!, t) = (1, 20, 110, 190, 96, 11)$,

Exercises 4.3.

(1) <u>*Show*</u> *that*

$$exp(z\ (1 - zt)^{-q}) = 1 + \sum_{n \geq 1} \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \prod_{a=0}^{k-1} (a + (n-k)\ q)\ t^k \right) \frac{z^n}{n!}.$$

(2) **The even generic Orlik–Solomon algebra**

Definition 4.16. *The even generic Orlik–Solomon algebra* $OS^{+}(\Gamma_n)$ *is defined to be an associative algebra (say over* $\mathbb{Z}$*) generated by the set of* **mutually commuting** *elements* $y_{i,j}$, $1 \leq i \neq j \leq n$, *subject to the set of cyclic relations*

$$y_{i,j} = y_{j,i}, \quad y_{i_1,i_2}\ y_{i_2,i_3} \cdots y_{i_{k-1},i_k}\ y_{i_1,i_k} = 0, \quad for \quad k = 2, \ldots, n,$$

and all sequences of pairwise distinct integers $1 \leq i_1, \ldots, i_k \leq n$.

- <u>Show</u> that the number of degree k, $k \geq 3$, relations in the definition of the Orlik–Solomon algebra $OS^{`+}(\Gamma_n)$ is equal to $\frac{1}{2}\ (k-1)!\ \binom{n}{k}$ and also is equal to the maximal number of k-cycles in the complete graph K_n.

Note that if one replaces the commutativity condition in the above Definition on the condition that $y_{i,j}{}'s$ pairwise **anticommute**, then the resulting algebra appears to be isomorphic to the Orlik–Solomon algebra $OS(\Gamma_n)$ corresponding to the generic hyperplane arrangement Γ_n, see [79]. It is known, *ibid*, Corollary 5.3, that

$$Hilb(OS(\Gamma_n), t) = \sum_{F} t^{|F|},$$

where the sum runs over all forests F on the vertices $1, \ldots, n$, and $|F|$ denotes the number of edges in a forest F.

It follows from Corollary 4.4, that

$$\sum_{n\geq 1} Hilb(OS(\Gamma_n),t)\ \frac{z^n}{n!} = exp\Big(\sum_{n\geq 1} n^{n-2}\ t^{n-1}\ \frac{z^n}{n!}\Big).$$

It is not difficult to see that $Hilb(OS^{+}(\Gamma_n),t) = Hilb(OS(\Gamma_n),t)$. In particular, $dim\, OS^{+}(\Gamma_n) = \mathcal{F}(n)$. Note also that a sequence $\{Hilb(OS(\Gamma_n),-1)\}_{n\geq 2}$ appears in [87], A057817. The polynomials $Hilb(\mathcal{A}_n,t)$, $F_n(x,t)$ and $Hilb(OS^{+}(\Gamma_n),t)$ can be expressed, see e.g. [78], as certain *specializations* of the Tutte polynomial $T(G;x,y)$ corresponding to the complete graph $G := K_n$. Namely,

$$Hilb(\mathcal{A}_n,t) = t^{\binom{n}{2}} T(K_n;1+t,t^{-1}),\quad Hilb(OS^{+}(\Gamma_n),t) = t^{n-1}T(K_n;1+t^{-1},1).$$

4.2.4 Super analogue of 6-term relations algebra

Let n, m be non-negative integers.

Definition 4.17. **The super 6-term relations algebra** $6T_{n,m}$ *is an associative algebra over* $\mathbb{Q}$ *generated by the elements* $\{x_{i,j},\ 1\leq i\neq j\leq n\}$ *and* $\{\ y_{\alpha,\beta},\ 1\leq \alpha\neq\beta\leq m\}$ *subject to the set of relations*

(0) $x_{i,j}+x_{j,i}=0,\quad y_{\alpha,\beta}=y_{\beta,\alpha}$;

(1) $x_{i,j}\ x_{k,l}=x_{k,l}\ x_{i,j},\ \ x_{i,j}\ y_{\alpha,\beta}=y_{\alpha,\beta}\ x_{i,j},\ \ y_{\alpha,\beta}\ y_{\gamma,\delta}+y_{\gamma,\delta}\ y_{\alpha,\beta}=0,$
if tuples (i,j,k,l), (i,j,α,β), *as well as* $(\alpha,\beta,\gamma,\delta)$ *consist of pair-wise distinct integers;*

(2) *(Classical Yang–Baxter relations and theirs super analogue)*
$[x_{i,k},x_{j,i}+x_{j,k}]+[x_{i,j},x_{j,k}]=0,$
if $1\leq i,j,k\leq n$ *are distinct,*
$[x_{i,k},y_{j,i}+y_{j,k}]+[x_{i,j},y_{j,k}]=0,$
if $1\leq i,j,k\leq min(n,m)$ *are distinct,*
$[y_{\alpha,\gamma},y_{\beta,\alpha}+y_{\beta,\gamma}]_{+}+[y_{\alpha,\beta},y_{\beta,\gamma}]_{+}=0,$
if $1\leq\alpha,\beta,\gamma\leq m$ *are distinct.*

Recall that $[a,b]_{+} := a\ b+b\ a$ *denotes the anticommutator of elements* a *and* b.

Conjecture 4.8.

- *The algebra* $6T_{n,m}$ *is* **Koszul**.

Theorem 4.5. *Let* $n,m\in\mathbb{Z}_{\geq 1}$, *one has*

- $Hilb((6T_n)^{!},t)\ Hilb((6T_m)^{!},t) =$

$$\sum_{k=0}^{min(n,m)-1} \left\{ {min(n,m) \atop min(n,m)-k} \right\} Hilb((6T_{n-k,m-k})^{!},t)\ t^{2k},$$

where as before $\left\{ {n \atop n-k} \right\}$ *denotes the Stirling numbers of the second kind, see for e.g. [87], A008278.*

Corollary 4.5. *Let* $n,m\in\mathbb{Z}_{\geq 1}$. *One has*

(a) *(Symmetry)* $Hilb(6T_{n,m},t)=Hilb(6T_{m,n},t)$.

(b) *Let* $n\leq m$, *then* $Hilb((6T_{n,m})^{!},t) =$

$$\sum_{k=0}^{n-1} s(n-1,n-k)\ Hilb((6T_{n-k})^{!},t)\ Hilb((6T_{m-k})^{!},t)\ t^{2k},$$

where $s(n-1,n-k)$ denotes the Stirling numbers of the first kind, i.e.

$$\sum_{k=0}^{n-1} s(n-1,n-k)\ t^k = \prod_{j=1}^{n-1} (1-j\ t).$$

(c) $dim(6T_{n,n})^!$ is equal to the number of pairs of partitions of the set $\{1,2,\ldots,n\}$ whose meet is the partition $\{\{1\},\{2\},\ldots,\{n\}\}$, see e.g. [87], A059849.

Example 4.6. $Hilb((6T_{3,2})^!,t) = Hilb((6T_{2,3})^!,t) = (1,4,3)$,
$Hilb((6T_{2,4})^!,t) = Hilb((6T_{4,2})^!,t) = (1,7,12,5)$, $Hilb((6T_{3,3})^!,t) = (1,6,8)$,
$Hilb((6T_{2,5})^!,t) = Hilb((6T_{5,2})^!,t) = (1,11,34,34,9)$,
$Hilb((6T_{3,4})^!,t) = Hilb((6T_{4,3})^!,t) = (1,9,23,16)$, $Hilb((6T_{4,4})^!,t) = (1,12,44,50,6)$,
$Hilb((6T_{3,5})^!,t) = Hilb((6T_{5,3})^!,t) = (1,13,53,79,34)$,
$Hilb((6T_{4,5})^!,t) = Hilb((6T_{5,4})^!,t) = (1,16,86,182,131,12)$,
$Hilb((6T_{5,5})^!,t) = (1,20,140,410,462,120)$.

Now let us define in the algebra $6T_{n,m}$ the Dunkl elements $\theta_i := \sum_{j\neq i} x_{i,j}$, $1\le i\le n$, and $\bar\theta_\alpha := \sum_{\beta\neq\alpha} y_{\alpha,\beta}$, $1\le\alpha\le m$.

Lemma 4.4. *One has*
- $[\theta_i,\theta_j]=0$,
- $[\theta_i,\bar\theta_\alpha]=[x_{i,\alpha},y_{i,\alpha}]$,
- $[\bar\theta_\alpha,\bar\theta_\beta]_+ = 2\ y^2_{\alpha,\beta}$, *if* $\alpha\neq\beta$.

Remark 4.4. (**"Odd" six-term relations algebra**) In particular, one can define an "odd" analog $6T_n^{(-)} = 6T_{0,n}$ of the six term relations algebra $6T_n$. Namely, the algebra $6T_n^{(-)}$ is given by the set of generators $\{y_{ij},\ 1\le i<j\le n\}$, and that of relations:

1) $y_{i,j}$ and $y_{k,l}$ **anti**commute if i,j,k,l are pairwise distinct;

2) $[y_{i,j},y_{i,k}+y_{j,k}]_+ + [y_{i,k},y_{j,k}]_+ = 0$, if $1\le i<j\le k\le n$, where $[x,y]_+ = xy+yx$ denotes the anticommutator of x and y.

The "odd" three term relations algebra $3T_n^-$ can be obtained as the quotient of the algebra $6T_n^-$ by the two-sided ideal generated by the three term relations

$y_{ij}\ y_{jk} + y_{jk}\ y_{ki} + y_{ki}\ y_{ij} = 0$, if i,j,k are pairwise distinct.

One can show that the Dunkl elements θ_i and θ_j, $i\neq j$, given by formula

$$\theta_i = \sum_{j\neq i} y_{ij},\quad i=1,\ldots,n,$$

form an **anticommutative** family of elements in the algebra $6T_n^{(-)}$.

In a similar fashion one can define an "odd" analogue of the dynamical six term relations algebra $6DT_n$, see Definition 2.2 and Section 2.2, as well as define an "odd' analogues of the algebra $3HQ_n(\beta,\mathbf{0})$, see Definition 2.6, the Kohno–Drinfeld algebra, the Hecke algebra and few others considered in the present paper. Details are omitted in the present paper.

More generally, one can ask what are natural q-analogues of the six term and three term relations algebras ? In other words to describe relations which ensure the q-commutativity of Dunkl elements defined above. First of all it would appear natural that the "q-locality and q-symmetry conditions" hold among the set of generators $\{y_{ij},\ 1\le i\neq j\le n\}$, that is

$y_{ij}+q\ y_{ji}=0$, $y_{ij}\ y_{kl} = q\ y_{kl}\ y_{ij}$ if $i<j,k<l$, and $\{i,j\}\cap\{k,l\}=\emptyset$.

Another natural condition is the fulfillment of q-analogue of the classical Yang–Baxter relations, namely

$[y_{ik}, y_{jk}]_q + [y_{ik}, y_{ji}]_q + [Y_{ij}, y_{jk}]_q = 0$, if $i < j < k$, where $[x,y]_q := x\ y - q\ y\ x$ denotes the q-commutator. However we are not able to find the q-analogue of the classical Yang– Baxter relation listed above in the Mathematical and Physical literature yet. Only cases $q = 1$ and $q = -1$ have been extensively studied.

■

4.2.5 Compatible Dunkl elements and Manin matrices

(**Compatible Dunkl elements, Manin matrices and algebras related with weighted complete graphs** rK_n)

Let us consider a collection of generators $\{u_{ij}^{(\alpha)},\ 1 \le i,j \le n,\ \alpha = 1,\ldots,r\}$, subject to the following relations

- either the unitarity (the case of sign "+"), or the symmetry relations (the case of sign " - ") [30]

$$u_{ij}^{(\alpha)} \pm u_{ji}^{(\alpha)} = 0, \forall, \alpha, i, j, \tag{4.5}$$

- (*local 3-term relations*)

$$u_{ij}^{(\alpha)} u_{jk}^{(\alpha)} + u_{jk}^{(\alpha)} u_{ki}^{(\alpha)} + u_{ki}^{(\alpha)} u_{ij}^{(\alpha)} = 0.\ \ i,j,k \ \ are \ \ distinct, \ \ 1 \le \alpha \le r. \tag{4.6}$$

We define *global* 3-term relations algebra $3T_{n,r}^{(\pm)}$ as " compatible product" of the local 3-term relations algebras. Namely, we require that the elements

$$U_{ij}^{(\lambda)} := \sum_{\alpha=1}^{r} \lambda_\alpha\ u_{ij}^{(\alpha)}, \ \ 1 \le i,j \le n,$$

satisfy the 3-term relations (1.4) for all values of parameters $\{\lambda_i \in \mathbb{R},\ \ 1 \le \alpha \le r\}$.

It is easy to check that our request is equivalent to a validity of the following sets of relations among the generators $\{u_{ij}^{(\alpha)}\}$

(a) (*local 3-term relations*) $u_{ij}^{(\alpha)}\ u_{jk}^{(\alpha)} + u_{jk}^{(\alpha)}\ u_{ki}^{(\alpha)} + u_{ki}^{(\alpha)}\ u_{ij}^{(\alpha)} = 0,$

(b) (6-term crossing relations)

$$u_{ij}^{(\alpha)}\ u_{jk}^{(\beta)} + u_{ij}^{(\beta)}\ u_{jk}^{(\alpha)} + u_{k,i}^{(\alpha)}\ u_{ij}^{(\beta)}\ u_{ki}^{(\alpha)} + u_{jk}^{(\alpha)}\ u_{ki}^{(\beta)} + u_{jk}^{(\beta)}\ u_{ki}^{(\alpha)} = 0,$$

i,j,k are distinct, $\alpha \ne \beta$.

Now let us consider *local* Dunkl elements

$$\theta_i^{(\alpha)} := \sum_{j \ne i} u_{ij}^{(\alpha)},\ j = 1,\ldots,n,\ \alpha = 1,\ldots,r.$$

It follows from the local 3-term relations $(\star)$ that for a fixed $\alpha \in [1,r]$ the local Dunkl elements $\{\theta_i^{(\alpha)}\}_{\substack{1 \le i \le n \\ 1 \le \alpha \le r}}$ either mutually commute (the sign "+"), or pairwise anticommute (the sign " - "). Similarly, the global 3-term relations imply that the global Dunkl elements

$$\theta_i^{(\lambda)} := \lambda_1 \theta_i^{(1)} + \cdots + \lambda_r \theta_i^{(r)} = \sum_{j \ne i} U_{ij}^{(\lambda)} \qquad i = 1,\ldots,n$$

[30] More generally one can impose the q-symmetry conditions

$$u_{ij} + q u_{ji} = 0, \ \ 1 \le i < j \le n$$

and ask about relations among the local Dunkl elements to ensure the commutativity of the global ones. As one might expect, the matrix $Q := (\theta_j^{(a)})_{\substack{1 \le a \le r \\ 1 \le j \le n}}$ composed from the local Dunkl elements should be a q-Manin matrix. See e.g. [16], or *en.wikipedia.org/wiki/Manin.matrix* for a definition and basic properties of the latter.

also either mutually commute (the case " $+$ ") or pairwise anticommute (the case " - ").

Now we are looking for a set of relations among the local Dunkl elements which is a consequence of the commutativity (anticommutativity) of the global Dunkl elements. It is quite clear that if $i<j$, then

$$[\theta_i^{(a)},\theta_j^{(b)}]_{\pm}=\sum_{a=1}^{r}\ \lambda_a^2\ [\theta_i^{(a)},\theta_j^{(a)}]_{\pm}+\sum_{1\le a<b\le r}\lambda_a\ \lambda_b\ \left([\theta_i^{(a)},\theta_j^{(b)}]_{\pm}+[\theta_i^{(b)},\theta_j^{(a)}]_{\pm}\right),$$

and the commutativity (or anticommutativity) of the global Dunkl elements for all $(\lambda_1,\ldots,\lambda_r)\in\mathbb{R}^r$ is equivalent to the following set of relations

- $[\theta_i,^{(a)},\theta_j^{(a)}]_{\pm}=0,$
- $[\theta_i^{(a)},\theta_j^{(b)}]_{\pm}+[\theta_i^{(b)},\theta_j^{(a)}]_{\pm}=0,\ \ a<b\ \text{ and }\ i<j,$

where by definition we set $[a,b]_{\pm}:=ab\mp ba$.

In other words , the matrix $\Theta_n:=(\theta_i^{(a)})_{\substack{1\le a\le r\\1\le i\le n}}$ should be either a *Manin matrix* (the case " $+$ "), or its super analogue (the case " - "). Clearly enough that a similar construction can be applied to the algebras studied in Section 2, **I-III**.,and thus it produces some interesting examples of the Manin matrices. It is an interesting **problem** to describe the algebra generated by the local Dunkl elements $\{\theta_i^{(a)}\}_{\substack{1\le a\le r\\1\le i\le n}}$ and a commutative subalgebra generated by the global Dunkl elements inside the former. It is also an interesting **question** whether or not the coefficients $C_1,\ldots,C_n$ of the column characteristic polynomial $Det^{col}\mid\Theta_n-t\ I_n\mid=\sum_{k=0}^{n}C_k\ t^{n-k}$ of the Manin matrix Θ_n generate a commutative subalgebra ? For a definition of the column determinant of a matrix, see e.g. [16].

However a close look at this problem and the question posed needs an additional treatment and has been omitted from the content of the present paper.

Here we are looking for a "natural conditions" to be imposed on the set of generators $\{u_{ij}^{\alpha}\}_{\substack{1\le\alpha\le r\\1\le i,j\le n}}$ in order to ensure that the local Dunkl elements satisfy the commutativity (or anticommutativity) relations:

$$[\theta_i^{(\alpha)},\theta_j^{(\beta)}]_{\pm}=0,\quad for\ \ all\ \ 1\le i<j\le n\ \ and\ \ 1\le\alpha,\beta\le r.$$

The "natural conditions" we have in mind are:

- (*locality relations*)

$$[u_{ij}^{(\alpha)},u_{kl}^{\beta)}]_{\pm}=0, \tag{4.7}$$

- (*twisted classical Yang–Baxter relations*)

$$[u_{ij}^{(\alpha)},u_{jk}^{(\beta)}]_{\pm}+[u_{ik}^{(\alpha)},u_{ji}^{(\beta)}]_{\pm}+[u_{ik}^{(\alpha)},u_{jk}^{(\beta)}]_{\pm}=0, \tag{4.8}$$

if i,j,k,l are distinct and $1\le\alpha,\beta\le r$.

Finally we define a multiple analogue of the three term relations algebra, denoted by $3T^{\pm}(rK_n)$, to be the quotient of the global 3-term relations algebra $3T_{n,r}^{\pm}$ modulo the two-sided ideal generated by the left hand sides of relations (4.7), (4.8) and that of the following relations

- $\left(u_{ij}^{(\alpha)}\right)^2=0,\quad [u_{ij}^{(\alpha)},u_{ij}^{(\beta)}]_{\pm}=0$, for all $i\ne j,\ \alpha\ne\beta$.

The outputs of this construction are

- noncommutative quadratic algebra $3T^{(\pm)(rK_n)}$ generated by the elements $\{u_{ij}^{(\alpha)}\}_{\substack{1\le i<j\le n\\ \alpha=1,\ldots,r}}$,
- a family of nr either mutually commuting (the case "$+$"), or pairwise anticommuting (the case " - ") local Dunkl elements $\{\theta_i^{(\alpha)}\}_{\substack{i=1,\ldots,n\\ \alpha=1,\ldots,r}}$.

We **expect** that the subalgebra generated by local Dunkl elements in the algebra $3T^{+}(rK_n)$ is closely related (isomorphic for $r=2$) with the coinvariant algebra of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}[X_n^{(1)},\ldots,X_n^{(r)}]$, where $X_n^{(j)}$ stands for the set of variables $\{x_1^{(j)},\ldots,x_n^{(j)}\}$. The algebra $(3T^{-}(2K_n))^{(-)})^{anti}$ has been studied in [47], and [7]. In the present paper we state only our old conjecture.

Conjecture 4.9. *(A.N. Kirillov, 2000)*

$$Hilb((3T^{-}(3K_n))^{anti},t)=(1+t)^n(1+nt)^{n-2},$$

where for any algebra A we denote by A^{anti} the quotient of algebra A by the two-sided ideal generated by the set of anticommutators $\{ab+ba \mid (a,b)\in A\times A\}$.

According to observation of M. Haiman [37], the number $2^n\ (n+1)^{n-2}$ is thought of as being equal to to the dimension of the space of triple coinvariants of the symmetric group $\mathbb{S}_n$.

4.3 Four term relations algebras / Kohno–Drinfeld algebras

4.3.1 Kohno–Drinfeld algebra $4T_n$ and that CYB_n

Definition 4.18. The 4 term relations algebra *(or the Kohno–Drinfeld algebra, or infinitesimal pure braids algebra) $4T_n$ is an associative algebra (say over $\mathbb{Q}$) with the set of generators $y_{i,j}, 1\le i<j\le n$, subject to the following relations*

1) $y_{i,j}$ and $y_{k,l}$ are commute, if i,j,k,l are all distinct;

2) $[y_{i,j},y_{i,k}+y_{j,k}]=0,\quad [y_{i,j}+y_{i,k},y_{j,k}]=0$, if $1\le i<j\le k\le n$.

Note that the algebra $4T_n$ is given by $\binom{n}{2}$ generators and $2\ \binom{n}{3}+3\ \binom{n}{4}$ quadratic relations, and the element

$$c:=\sum_{1\le i<j\le n} y_{i,j}$$

belongs to the center of the Kohno–Drinfeld algebra.

Definition 4.19.

Denote by $4T_n^0$ the quotient of the algebra $4T_n$ by the (two-sided) ideal generated by by the set of elements $\{y_{i,j}^2, 1\le i<j\le n\}$.

More generally, let β, $\{q_{ij},\ 1\le i<j\le n\}$ be the set of parameters, denote by $4HT_n$ the quotient of the algebra $4T_n$ by the two-sided ideal generated by the set of elements $\{y_{ij}^2-\beta y_{ij}-q_{ij},\ 1\le i<j\le n\}$.

These algebras are naturally graded, with $deg(y_{i,j})=1, deg(\beta)=1, deg(q_{ij})=2$, as well as each of that algebras has a natural filtration by setting $deg(y_{i,j})=1$, $deg(\beta)=0$, $deg(q_{ij})=0,\ \forall i\ne j$.

It is clear that the algebra $4T_n$ can be considered as the infinitesimal deformation $g_{i,j}:=1+\epsilon\ y_{i,j}$, $\epsilon\longrightarrow 0$, of the pure braid group P_n.

There is a natural action of the symmetric group $\mathbb{S}_n$ on the algebra $4T_n$ (and also on $4T_n^0$) which preserves the grading: it is defined by $w\cdot y_{i,j}=y_{w(i),w(j)}$ for $w\in\mathbb{S}_n$. The semi-direct product $\mathbb{Q}\mathbb{S}_n\ltimes 4T_n$ (and also that $\mathbb{Q}\mathbb{S}_n\ltimes 4T_n^0$) is a Hopf algebra denoted by $\mathcal{B}_n$ (respectively $\mathcal{B}_n^0$).

Remark 4.5. There exists the natural map

$$CYB_n\longrightarrow 4T_n,\quad given\ \ by\ \ y_{i,j}:=u_{i,j}+u_{j,i}.$$

Indeed, one can easily check that

$$[y_{ij},y_{ik}+y_{jk}]=w_{ijk}+w_{jik}-w_{kij}-w_{kji},$$

see Section 2.3.1, Definition 2.5 for a definition of the classical Yang–Baxter algebra CYB_n, and Section 2, (2.3), for a definition of the element w_{ijk}.

Remark 4.6. It follows from the classical 3-term identity ("Jacobi identity")

$$\frac{1}{(a-b)(a-c)} - \frac{1}{(a-b)(b-c)} + \frac{1}{(a-c)(b-c)} = 0, \tag{4.9}$$

that if elements $\{y_{i,j} \mid 1 \le i < j \le n\}$ satisfy the 4-term algebra relations, see Definition 4.18, and $t_1, \cdots, t_n$, a set of (pairwise) commuting parameters, then the elements

$$r_{i,j} := \frac{y_{i,j}}{t_i - t_j}$$

satisfy the 6-term relations algebra $6T_n$, see Section 4.2.1,, Definition 4.9. In particular, the Knizhnik–Zamolodchikov elements

$$KZ_j := \sum_{i \ne j} \frac{y_{i,j}}{t_i - t_j}, \quad 1 \le j \le n,$$

form a pairwise commuting family (by definition, we put $y_{i,j} = y_{j,i}$, if $i > j$).

Example 4.7. (**1**) *(Cf Subsection 4.2.1, Example 4.4)*
*(***Yang representation of the** $4T_n$*).*

Let $\mathbb{S}_n$ be the symmetric group acting identically on the set of variables $\{x_1, \ldots, x_n\}$. Clearly that the elements $\{y_{i,j} := s_{ij}\}_{1 \le i < j \le n}, \; y_{i,j} := y_{j,i}, \; if \; i > j$, satisfy the Kohno-Drinfeld relations listed in Definition 4.18. Therefore the operators u_{ij} defined by

$$u_{ij} = (x_i - x_j)^{-1} \; s_{ij}$$

give rise to a representation of the algebra $3T_n$ on the field of rational functions $\mathbb{Q}(x_1, \ldots, x_n)$. The Dunkl-Gaudin elements

$$\theta_i = \sum_{j, j \ne i} y_{ij}, \quad i = 1, \ldots, n$$

correspond to the truncated Gaudin operators acting in the tensor space $(\mathbb{C}^n)^{\otimes n}$.

(**2**) Let $A = U(sl(2))$ be the universal enveloping algebra of the Lie algebra $sl(2)$. Recall that the algebra $sl(2)$ is spanned by the elements e, f, h, so that $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. Consider the element $\Omega = \frac{1}{2}\, h \otimes h + e \otimes f + f \otimes e$. Then the map $y_{i,j} \longrightarrow \Omega_{i,j} \in A^{\otimes n}$ defines a representation of the Kohno–Drinfeld algebra $4T_n$ on that $A^{\otimes n}$. The element KZ_j defined above, corresponds to the truncated (or at critical level) rational Knizhnik–Zamolodchikov operator.

Proposition 4.5. *(T. Kohno, V. Drinfeld)*

$$Hilb(4T_n, t) = \prod_{j=1}^{n-1} (1 - jt)^{-1} = \sum_{k \ge 0} \left\{ {n+k-1 \atop n-1} \right\} t^k,$$

where $\left\{ {n \atop k} \right\}$ *stands for the Stirling numbers of the second kind, i.e. the number of ways to partition a set of* n *things into* k *nonempty subsets.*

Remark 4.7. It follows from [2] that $Hilb(4T_n, t)$ is equal to the generating function

$$1 + \sum_{d \ge 1} v_d^{(n)} \; t^d$$

for the number $v_d^{(n)}$ of **Vassiliev invariants of order** d for <u>n-strand braids</u>. Therefore, one has the following equality:

$$v_d^{(n)} = \left\{ {n+d-1 \atop n-1} \right\},$$

i.e. the number of Vassiliev invariants of order d for n-strand braids is equal to the Stirling number of the second kind $\left\{ {n+d-1 \atop n-1} \right\}$.

We **expect** that the generating function

$$1+\sum_{d \geq 1} \widehat{v}_d^{(n)}\ t^d$$

for the number $\widehat{v}_d^{(n)}$ of **Vassiliev invariants of order** d for n-strand virtual braids is equal to the Hilbert series $Hilb(4NT_n,t)$ of the nonsymmetric Kohno-Drinfeld algebra $4NT_n$, see Section 4.3.2.

Proposition 4.6. *(Cf [3]) The algebra $4NT_n,t)$ is* **Koszul***, and*

$$Hilb(4NT_n,t) = \left(\sum_{k=0}^{n-1} (k+1)!\ N(k,n)\ (-t)^k \right)^{-1},\ \ Hilb((4NT_n)^!,t) = (n-1)!\ L_{n-1}^{(\alpha=1)}(-t^{-1})\ t^{n-1},$$

where $N(k,n) := \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$ denotes the Narayana number, i.e. the number of Dyck n-paths with exactly k peaks;

$$L_n^{(\alpha)}(x) = \frac{x^{\alpha}\ e^x}{n!}\ \frac{d^n}{dx^n} \left(e_x\ x^{n+\alpha} \right)$$

denotes the generalized Laguerre polynomial.

See also Theorem 4.6 below.

It is well-known that the quadratic dual $4T_n^!$ of the Kohno–Drinfeld algebra $4T_n$ is isomorphic to the Orlik–Solomon algebra of type A_{n-1}, as well as the algebra $3T_n^{anti}$. However the algebra $4T_n^0$ is failed to be Koszul.

Examples 4.3.
$Hilb(4T_3^0,t) = [2]^2[3]$, $Hilb(4T_4^0,t) = (1,6,19,42,70,90,87,57,23,6,1)$.
$Hilb((4T_3^0)^!,t)(1-t) = (1,2,2,1)$, $\quad Hilb((4T_4^0)^!,t)(1-t)^2 = (1,4,6,2,-4,-3)$,
$Hilb((4T_5^0)^!,t)(1-t)^2 = (1,8,26,40,24,-3,-6)$.

We **expect** that $Hilb((4T_n^0)^!,t)$ is a rational function with the only pole at $t=1$ of order $[n/2]$, cf. Examples 4.1.

Remark 4.8. One can show that if $n \geq 4$, then $Hilb(4T_n^0,t) < Hilb(3T_n^0,t)$ **contrary** to the statement of Conjecture 9.6 from [45].

4.3.2 Nonsymmetric Kohno–Drinfeld algebra $4NT_n$, and McCool algebra $\mathcal{P}\Sigma_n$

(Nonsymmetric Kohno–Drinfeld algebra $4NT_n$, and McCool algebras $\mathcal{P}\Sigma_n$ and $\mathcal{P}\Sigma_n^+$)

Definition 4.20. The nonsymmetric 4 term relations algebra *(or the nonsymmetric Kohno–Drinfeld algebra) $4NT_n$ is an associative algebra (say over $\mathbb{Q}$) with the set of generators $y_{i,j}, 1 \leq i \neq j \leq n$, subject to the following relations*
1) $y_{i,j}$ and $y_{k,l}$ are commute, if i,j,k,l are all distinct;
2) $[y_{i,j}, y_{i,k}+y_{j,k}] = 0$, $\quad$ if i,j,k are all distinct.

We denote by $4NT_n^+$ the quotient of the algebra $4NT_n$ by the two- sided ideal generated by the elements $\{y_{ij}+y_{ji}=0, 1 \leq i \neq j \leq n\}$.

Theorem 4.6. *One has*

$$Hilb(4NT_n, t) = Hilb(CYB_n, t), \qquad Hilb(4NT_n^+, t) = Hilb(6T_n, t)$$

for all $n \geq 2$.

We expect that the both algebras $4NT_n$ and $4NT_n^+$ are **Koszul**.

Definition 4.21.

(1) *Define* **the McCool algebra** $\mathcal{P}\Sigma_n$ *to be the quotient of the nonsymmetric Kohno–Drinfeld algebra* $4NT_n$ *by the two-sided ideal generated by the elements*

$$\{y_{ik}\ y_{jk} - y_{jk}\ y_{ik}\}$$

for all pairwise distinct i, j *and* k.

(2) *Define* **the upper triangular McCool algebra** $\mathcal{P}\Sigma_n^+$ *to be the quotient of the McCool algebra* $\mathcal{P}\Sigma_n$ *by the two-sided ideal generated by the elements*

$$\{y_{ij} + y_{ji}\},$$

$1 \leq i \neq j \leq n$.

Theorem 4.7. *The quadratic duals of the algebras* $\mathcal{P}\Sigma_n$ *and* $\mathcal{P}\Sigma_n^+$ *have the following Hilbert polynomials*

$$Hilb(\mathcal{P}\Sigma_n^!, t) = (1+nt)^{n-1}, \qquad Hilb((\mathcal{P}\Sigma_n^+)^!, t) = \prod_{j=1}^{n-1} (1+jt).$$

Proposition 4.7.

(1) **The quadratic dual** $\mathcal{P}\Sigma_n^!$ *of the algebra* $\mathcal{P}\Sigma_n$ *admits the following description. It is generated over* $\mathbb{Z}$ *by the set of pairwise* **anticommuting** *elements*

$$\{y_{ij},\ 1 \leq i \neq j \leq n\},$$

subject to the set of relations

(a) $y_{ij}^2 = 0$, $y_{ij}\ y_{ji} = 0$, $1 \leq i \neq j \leq n$,

(b) $y_{ik}\ y_{jk} = 0$, *for all distinct* i, j, k,

(c) $y_{ij}\ y_{jk} + y_{ik}\ y_{ij} + y_{kj}\ y_{ik} = 0$, *for all distinct* i, j, k.

(2) **The quadratic dual** $(\mathcal{P}\Sigma_n^+)^!$ *of the algebra* $\mathcal{P}\Sigma_n^+$ *admits the following description. It is generated over* $\mathbb{Z}$ *by the set of pairwise* **anticommuting** *elements* $\{z_{ij},\ 1 \leq i < j \leq n\}$, *subject to the set of relations*

(a) $z_{ij}^2 = 0$ *for all* $i < j$,

(b) $z_{ij}\ z_{jk} = z_{ij}\ z_{ik}$ *for all* $1 \leq i < j < k \leq n$.

Comments 4.5. The McCool groups and algebras

The McCool group $P\Sigma_n$ is by definition, the group of pure symmetric automorphisms of the free group F_n consisting of all automorphism that, for a fixed basis $\{x_1, \ldots, x_n\}$, send each x_i to a conjugate of itself. This group is generated by automorphisms α_{ij}, $1 \leq i \neq j \leq n$, defined by

$$\alpha_{ij}(x_k) = \begin{cases} x_j\ x_i\ x_j^{-1}, & k = i; \\ x_k, & k \neq i. \end{cases}$$

McCool have proved that the relations

$$\begin{cases} [\alpha_{ij}, \alpha_{kl}] = 1, & i,j,k,l \text{ are distinct}, \\ [\alpha_{ij}, \alpha_{ji}] = 1, & i \neq j, \\ [\alpha_{ij}, \alpha_{ik}\ \alpha_{jk}] = 1, & i,j,k \text{ are distinct.} \end{cases}$$

form the set of defining relations for the group $P\Sigma_n$ The subgroup of $P\Sigma_n$ generated by the α_{ij} for $1 \le i < j \le n$ is denoted by $P\Sigma_n^+$ and is called by **upper triangular McCool group.** It is easy to see that the McCool algebras $\mathcal{P}\Sigma_n$ and $\mathcal{P}\Sigma_n^+$ are the " infinitesimal deformations " of the McCool groups $P\Sigma_n$ and $P\Sigma_n^+$ respectively.

Theorem 4.8.
(1) *([39]) There exists a natural isomorphism*

$$H^*(P\Sigma_n, \mathbb{Z}) \simeq \mathcal{P}\Sigma_n^!$$

of the quadratic dual $\mathcal{P}\Sigma_n^!$ of the McCool algebra $\mathcal{P}\Sigma_n$ and the cohomology ring $H^*(P\Sigma_n, \mathbb{Z})$ of the McCool group $P\Sigma_n$.
(2) *([17]) There exists a natural isomorphism*

$$H^*(P\Sigma_n^+, \mathbb{Z}) \simeq (\mathcal{P}\Sigma_n^+)^!$$

of the quadratic dual $(\mathcal{P}\Sigma_n^+)^!$ of the upper triangular McCool algebra $\mathcal{P}\Sigma_n^+$ and the cohomology ring $H^*(P\Sigma_n^+, \mathbb{Z})$ of the upper triangular McCool group $P\Sigma_n^+$.

■

4.3.3 Algebras $4TT_n$ and $4ST_n$

Definition 4.22.
(**I**) **Algebra** $4TT_n$ *is generated over $\mathbb{Z}$ by the set of elements $\{x_{ij}, 1 \le i \neq j \le n\}$, subject to the set of relations*
(1) $x_{ij}\ x_{kl} = x_{kl}\ x_{ij}$, *if all* i,j,k,l *are distinct,*
(2) $[x_{ij} + x_{jk}, x_{ik}] = 0$, $[x_{ji} + x_{kj}, x_{ki}] = 0$, *if* $i < j < k$.
(**II**) **Algebra** $4ST_n$ *is generated over $\mathbb{Z}$ by the set of elements $\{x_{ij},\ 1 \le i \neq j \le n\}$, subject to the set of relations*
(1) $[x_{ij}, x_{kl}] = 0$, $[x_{ij}, x_{ji}] = 0$, *if* i,j,k,l *are distinct;*
(2) $[x_{ij}, x_{ik}] = [x_{ik}, x_{jk}] = [x_{jk}, x_{ij}]$, $[x_{ji}, x_{ki}] = [x_{ki}, x_{kj}] = [x_{kj}, x_{ii}]$,
(3) $[x_{ij}, x_{ki}] = [x_{kj}, x_{ij}] = [x_{ji}, x_{ik}] = [x_{ik}, x_{kj}] = [x_{ki}, x_{jk}] = [x_{jk}, x_{ji}]$,
if $i < j < k$.

Proposition 4.8. *One has*

$$t \sum_{n \ge 2} Hilb((4TT_n)^!, t)\ \frac{z^n}{n!} = \frac{exp(-tz)}{(1-z)^{2t}} - 1 - tz.$$

Therefore, $dim(4TT_n)^!$ is equal to the number of permutations of the set $[1, ..., n+1]$ having no substring $[k, k+1]$; also, for $n \ge 1$ equals to the maximal <u>*permanent*</u> *of a nonsingular $n \times n$ $(0,1)$-matrix, see [87], A000255* [31]. *Moreover, one has*

$$Hilb((4ST_n)^!, t) = (1+t)^n\ (1+nt)^{n-2},$$

cf. Conjecture 4.9.

[31] See also a paper by F. Hivert, J-C. Novelli and J-Y. Thibon *Commutative combinatorial Hopf algebras,* J. Algebraic Combin. **28** (2008), no. 1, 65–95, Section 3.8.4, for yet another combinatorial interpretation of the dimension of the algebra $(4TT_n)^!$.

We expect that The both algebras $4TT_n$ and $4ST_n$ are **Koszul**.

Problem. Give a combinatorial interpretation of polynomials $Hilb((4TT_n)^!, t)$ and construct a monomial basis in the algebras $(4TT_n)^!$ and $4ST_n$.

4.4 Subalgebra generated by Jucys–Murphy elements in $4T_n^0$

Definition 4.23. The Jucys–Murphy elements d_j, $2 \le j \le n$, *in the quadratic algebra* $4T_n$ *are defined as follows*

$$d_j = \sum_{1 \le i < j} y_{i,j}, \quad j = 2, ..., n. \tag{4.10}$$

It is clear that Jucys–Murphy's elements d_j are the infinitesimal deformation of the elements $D_{1,j} \in P_n$.

Theorem 4.9.

1^0 *The Jucys–Murphy elements* d_j, $\quad 2 \le j \le n$, *commute pairwise in the algebra* $4T_n$.

2^0 *In the algebra* $4T_n^0$ *the Jucys–Murphy elements* d_j, $2 \le j \le n$, *satisfy the following relations*

$$(d_2 + \cdots + d_j)\ d_j^{2j-3} = 0, \quad 2 \le j \le n.$$

3^0 *Subalgebra (over* $\mathbb{Z}$*) in* $4T_n^0$ *generated by the Jucys–Murphy elements* $d_2, \cdots, d_n$ *has the following Hilbert polynomial* $\quad \prod_{j=1}^{n-1}\ [2j]$.

4^0 *There exists an (birational) isomorphism* $\mathbb{Z}\ [x_1, \dots, x_{n-1}]/J_{n-1} \longrightarrow \mathbb{Z}\ [d_2, \dots, d_n]$ *defined by* $d_j := \prod_{i=1}^{n-j}\ x_i$, $\quad 2 \le j \le n$, *where* J_{n-1} *is a (two-sided) ideal generated by* $e_i(x_1^2, \dots, x_{n-1}^2)$, $1 \le i \le n-1$, *and* $e_i(x_1, \dots, x_{n-1})$ *stands for the* $i-th$ *elementary symmetric polynomial in the variables* $x_1, \dots, x_{n-1}$.

Remark 4.9.

(1) It is clearly seen that the commutativity of the Jucys–Murphy elements is equivalent to the validity of the Kohno-Drinfeld relations and the locality relations among the generators $\{y_{i,j}\}_{1 \le i < j \le n}$.

(2) Let's stress that $d_j^{2j-2} \ne 0$ in the algebra $4T_n^0$, for $j = 3, \dots, n$. For example, $d_3^4 = y_{13}\ y_{23}\ y_{13}\ y_{23} + y_{23}\ y_{13}\ y_{23}\ y_{13} \ne 0$ since $dim(4T_3^0)_4 = 1$ and it is generated by the element d_3^4.

(3) The map $\iota : y_{i,j} \longrightarrow y_{n+1-j,n+1-i}$ preserves the relations 1) and 2) in the definition of the algebra $4T_n$, and therefore defines an involution of the Kohno–Drinfeld algebra. Hence the elements

$$\widehat{d_j} := \sum_{k=j+1}^{n} y_{j,k} = \iota(d_{n+1-j}), \qquad 1 \le j \le n-1$$

also form a pairwise commuting family.

■

Problems 4.1. (a) *Compute Hilbert series of the algebra* $4T_n^0$ *and its quadratic dual algebra* $(4T_n^0)^!$.

(b) *Describe subalgebra in the algebra* $4HT_n$ *generated by the Jucys–Murphy elements* d_j, $\ 2 \le j \le n$.

It is well-known that the Kohno–Drinfeld algebra $4T_n$ is *Koszul*, and its quadratic dual $4T_n^!$ is isomorphic to the anticommutative quotient $3T_n^{0,anti}$ of the algebra $3T_n^{(-),0}$.

On the other hand, if $n \ge 3$ the algebra $4T_n^0$ is not *Koszul*, and its quadratic dual is isomorphic to the quotient of the ring of polynomials in the set of anticommutative variables $\{t_{i,j} \mid 1 \le i < j \le n\}$, where *we do not impose conditions* $t_{ij}^2 = 0$, modulo the ideal generated by Arnold's relations $\{t_{i,j}\ t_{j,k} + t_{i,k}\ (t_{i,j} - t_{j,k}) = 0\}$ for all pairwise distinct i, j and k.

4.5 Nonlocal Kohno–Drinfeld algebra $NL4T_n$

Definition 4.24. Nonlocal Kohno–Drinfeld algebra $NL4T_n$ *is an associative algebra over* $\mathbb{Z}$ *with the set of generators* $\{y_{ij},\ 1 \le i < j \le n\}$ *subject to the set of relations*

(1) $y_{ij}\ y_{kl} = y_{kl}\ y_{ij}$ *if* $(i-k)(i-l)(j-k)(j-l) > 0$,

(2) $[y_{ij}, \sum_{a=i}^{j}\ y_{ak}] = 0,$ *if* $i < j < k$,

(3) $[y_{jk}, \sum_{a=j}^{k}\ y_{ia}] = 0,$ *if* $i < j < k$.

It's not difficult to see that relations $(1)-(3)$ imply the following relations

(4) $[x_{ij}, \sum_{a=i+1}^{j-1}\ (y_{ia}+y_{aj})] = 0$, if $i < j$.

Let's introduce in the nonlocal Kohno–Drinfeld algebra $NL4T_n$ the Jucys–Murphy elements (JM-elements for short) d_j and the dual JM-elements $\hat{d}_j$ as follows

$$d_j = \sum_{a=1}^{j-1}\ y_{aj}, \quad \hat{d}_j = \sum_{a=n-j+2}^{n},\ y_{n-j+1,a} \quad j = 2,\ldots,n. \tag{4.11}$$

It follows from relations (1) and (2) (resp. (1) and (3)) that the Jucys–Murphy elements $d_2,\ldots,d_n$ (resp. $\hat{d}_2,\ldots,\hat{d}_n$) form a commutative subalgebra in the algebra $NL4T_n$.. Moreover, it follows from relations $(1)-(3)$ that the element $c_1 := \sum_{j=2}^{n}\ d_j = \ \sum_{j=2}^{n}\ \hat{d}_j$ belongs to the center of the algebra $NL4T_n$.

Theorem 4.10.

(1) *The algebra* $NL4T_n$ *is* **Koszul**, *and*

$$Hilb((NL4T_n)^{!}, t) = \sum_{k=0}^{n-1}\ C_k\ \binom{n+k-1}{2k}\ t^k,$$

where $C_k = \frac{1}{k+1}\ \binom{2k}{k}$ *stands for the k-th Catalan number.*

(2) *The quadratic dual* $(NL4T_n)^{!}$ *of the nonlocal Kohno - Drinfeld algebra* $NL4T_n$ *is an associative algebra generated by the set of mutually* **anticommuting** *elements* $\{t_{ij}\ 1 \le i < j \le n\}$ *subject to the set of relations*

- $t_{ij}^2 = 0,$ *if* $1 \le i < j \le n$,
- *(Arnold's relations)* $t_{ij}\ t_{jk} + t_{ik}\ t_{ij} + t_{jk}\ t_{ik} = 0,$ *if* $i < j < k$,
- *(Disentanglement relations)* $t_{ik}\ t_{jl} + t_{il}\ t_{ik} + t_{jl}\ t_{il} = 0,$ *if* $i < j < k < l$.

Therefore the algebra $(NL4T_n)^{!}$ is the quotient of the the Orlik-Solomon algebra OS_n by the ideal generated by Disentanglement relations, and $dim((NL4T_{n+1})^{!})$ is equal to the number of Schroeder paths , i.e. paths from $(0,0)$ to $(2n,0)$ consisting of steps $U=(1,1)$, $D=(1,-1)$, $H=(2,0)$ and never going below the $x-axis$. The Hilbert polynomial $Hilb((NL4T_n)^{!},t)$ is the generating function of such paths with respect to the number of $U's$, see [87], A088617.

Remark 4.10.

Denote by $H_n(q)$ "the normalized" Hecke algebra of type A_n, i.e. an associative algebra generated over $\mathbb{Z}[q,q^{-1}]$ by elements $T_1,\ldots,T_{n-1}$ subject to the set of relations

(a) $T_i\ T_j = T_j\ T_i$, if $|i-j| > 1$, $T_i\ T_j\ T_i = T_j\ T_i\ T_j$, if $|i-j| = 1$,

(b) $T_i^2 = (q-q^{-1})\ T_i + 1$ for $i = 1,\ldots,n-1$.

If $1 \le i < j \le n-1$, let's consider elements $T_{(ij)} := T_i\ T_{i+1} \cdots T_{j-1}\ T_j\ T_{j-1} \cdots T_{i+1}\ T_i$.

Lemma 4.5. *The elements* $\{T_{(ij)},\quad 1 \le i < j < n-1\}$ *satisfy the defining relations of the non-local Kohno-Drinfeld algebra* $NL4T_{n-1}$, *see Definition 4.23.*

Therefore the map $y_{ij} \to H_{(ij)}$ *defines a epimorphism* $\iota_n : NL4T_n \longrightarrow H_{n+1}(q)$.

Definition 4.25. *Denote by $\mathcal{NL}4T_n$ the quotient of the non-local Kohno-Drinfeld algebra $NL4T_n$ by the two-sided ideal $\mathcal{I}_n$ generated by the following set of degree three elements:*

(1) $z_{ij} := y_{i,j+1}\ y_{ij}\ y_{j,j+1} - y_{j,j+1}\ y_{ij}\ y_{i,j+1}$, *if* $1 \le i < j \le n$,

$$(2)\quad u_i := y_{i,i+1}\left(\sum_{a=1}^{i-1}\ \sum_{b=1,\ b\neq a}^{i-1} y_{ai}\ y_{b,i+1}\right) - \left(\sum_{a=1}^{i-1}\ \sum_{b=1,\ b\neq a}^{i-1} y_{b,i+1}\ y_{ai}\right) y_{i,i+1}, \quad if \quad 1 \le i \le n-1,$$

$$(3)\quad v_i := y_{i,i+1}\left(\sum_{a=i+1}^{n}\ \sum_{b=i+1\ b\neq a}^{n} y_{i+1,a}\ y_{i,b}\right) - \left(\sum_{a=i+1}^{n}\ \sum_{b=i+1\ b\neq a}^{n} y_{i+1,a} y_{i,b}\right) y_{i,i+1},$$

if $1 \le i \le n-1$.

Proposition 4.9.

(1) *The ideal $\mathcal{T}_n$ belongs to the kernel of the epimorphism ι_n: $\mathcal{I}_n \subset Ker(\iota_n)$,*

(2) *Let $d_2,\ldots,d_n$ (resp. $\hat{d}_2,\ldots,\hat{d}_n$) be the Jucys-Murphy elements (resp. dual JM-elements) in the algebra $\mathcal{NL}4T_n$ given by the formula* (4.11).

Then the all elementary symmetric polynomials $e_k(d_2,\ldots,d_n)$ (resp. $e_k(\hat{d}_2,\ldots,\hat{d}_n)$) of degree k, $1 \le k < n$, in the Jucys–Murphy elements $d_2,\ldots,d_n$, (resp. in the dual JM-elements $\hat{d}_2,\ldots,\hat{d}_n$,) <u>commute</u> in the algebra $\mathcal{NL}4T_n$ with the all elements $y_{i,i+1}$, $i = 1,\ldots,n-1$.

Therefore, there exists an epimorphism of algebras $\mathcal{NL}4T_n \longrightarrow H_n(q)$, and images of the elements $e_k(d_2,\ldots,d_n)$, (resp. $e_k(\hat{d}_2,\ldots,\hat{d}_n)$ $1 \le k < n$, belongs to the center of the "normalized" Hecke algebra $H_n(q)$, and in fact <u>*generate*</u> the center of algebra $H_n(q)$.

Few comments in order:

(**A**) Let $N\ell 4T_n$ be an associative algebra over $\mathbb{Z}$ with the set of generators $\{y_{ij},\ 1 \le i < j \le n\}$ subject to the set of relations

(1) $y_{ij}\ y_{kl} = y_{kl}\ y_{ij}$, if $(i-k)(i-l)(j-k)(j-l) > 0$,

(2) $[y_{ij}, \sum_{a=i}^{j}\ y_{ak}] = 0$, if $i < j < k$.

Proposition 4.10.

(1) *The algebra $N\ell 4T_n$ is* **Koszul** *and has the Hilbert series equals to*

$$Hilb(N\ell 4T_n, t) = (\sum_{k=0}^{n-1}(-1)^k\ N(k,n)\ t^k)^{-1},$$

where $N(k,n) := \frac{1}{n}\binom{n}{k}\ \binom{n}{k+1}$ denotes the Narayana number, i.e. the number of Dyck n-paths with exactly k peaks, see e.g. [87], A001263.

Therefore, $dim(N\ell 4T_n)^! = \frac{1}{n+1}\binom{2\,n}{n}$, the n-th Catalan number.

(2) *Elementary symmetric polynomials $e_k(d_2,\ldots,d_n)$ of degree k, $1 \le k < n$, in the Jucys–Murphy elements $d_2,\ldots,d_n$, <u>commute</u> in the algebra $N\ell 4T_n$ with the all elements $y_{i,i+1}$, $i = 1,\ldots,n-1$.*

(**B**) The kernel of the epimorphism $\mathcal{NL}4T_n \longrightarrow H_n(q)$ contains the elements

$\{y_{i,i+1}\ y_{i+1,i+2}\ y_{i,i+1} - y_{i+1,i+2}\ y_{i,i+1}\ y_{i+1,i+2},\ i = 1,\ldots,n-2\}$, $\{T_{i,i+1}^2 - (q-q^{-1})\ T_{i,i+1} - 1\}$,

as well as the following set of commutators

$$[y_{ij}, e_k(d_i,\ldots,d_j)], \quad 1 \le k \le j-i+1.$$

It is an interesting **task** to find defining relations among the Jucys– Murphy elements $\{d_j,\ j = 2,\ldots,n\}$ in the algebra $NL4T_n$ or that $N\ell 4T_n$. We **expect** that the Jucys–Murphy element d_k satisfies the following relation (= minimal polynomial) in the *Hecke algebra* $H_n(q)$, $n \ge k$,

$$\prod_{a=1}^{k-1}(d_k - \frac{q-q^{2a+1}}{1-q^2})\ (d_k + \frac{q^{-1}-q^{-2a-1}}{1-q^{-2}}) = 0. \tag{4.12}$$

■

4.5.1 On relations among JM-elements in Hecke algebras

Let $H_n(q)$ be the "normalized" Hecke algebra of type A_n, see Remark 4.10. Let $\lambda \vdash n$ be a partition of n. For a box $x=(i,j) \in \lambda$ define

$$c_{\lambda}(x;q) := q\ \frac{1-q^{2\ (j-i)}}{1-q^2} \tag{4.13}$$

It is clear that if $q=1$, $c_{q=1}(x)$ is equal to the content $c(x)$ of a box $x \in \lambda$. Denote by

$$\Lambda_q^{(n)} = \mathbb{Z}[q,q^{-1}]\ [z_1,\ldots,z_n]^{\mathbb{S}_n}$$

the space of symmetric polynomials over the ring $\mathbb{Z}[q,q^{-1}]$ in variables $\{z_1,\ldots,z_n\}$.

Definition 4.26. *Denote by $J_q^{(n)}$ the set of symmetric polynomials $f \in \Lambda_q^{(n)}$ such that for any partition $\lambda \vdash n$ one has*

$$f(c_{\lambda}(x;q) \mid x \in \lambda) = 0.$$

For example, one can check that symmetric polynomial

$$e_1^2 - (q^2+1+q^{-2})\ e_2 - 2\ (q-q^{-1})\ e_1 - 3$$

belongs to the set $J_q^{(3)}$.

Finally, denote by $\mathbb{J}_q^{(n)}$ the ideal in the ring $\mathbb{Z}[q,q^{-1}]\ [z_1,\ldots,z_n]$ generated by the set $J_q^{(n)}$.

Conjecture 4.10. *The algebra over $\mathbb{Z}[q,q^{-1}]$ generated by the Jucys–Murphy elements $d_2,\ldots,d_n$ corresponding to the the Hecke algebra $H_n(q)$ of type A_{n-1}, is isomorphic to the quotient of the algebra $\mathbb{Z}[q,q^{-1}]\ [z_1,\ldots,z_n]$ by the ideal $\mathbb{J}_q^{(n)}$.*

It seems an interesting **problem** to find a minimal set of generators for the ideal $\mathbb{J}_q^{(n)}$.

Comments 4.6. Denote by $JM(n)$ the algebra over $\mathbb{Z}$ generated by the JM-elements $d_2,\ldots,d_n$, $deg(d_i=1, \forall i$, corresponding to the symmetric group $\mathbb{S}_n$. In this case one can check Conjecture 8 for $n<8$, and compute the Hilbert polynomial(s) of the associated graded algebra(s) $gr(JM(n))$. For example [32]
$Hilb(gr(JM(2),t)=(1,1)$, $Hilb(gr(JM(3),t)=(1,2,1),$, $Hilb(gr(JM(4),t)=(1,3,4,2)$,
$Hilb(gr(JM(5),t)=(1,4,8,9,4)$, $Hilb(gr(JM(6),t)=(1,5,13,21,21,12,3)$,
$Hilb(gr(JM(7),t)=(1,6,19,40,59,60,37,10)$.

It seems an interesting **task** to find a combinatorial interpretation of the polynomials $Hilb(gr(JM(n)),t)$ in terms of standard Young tableaux of size n.

■

Let $\{\chi^{\lambda},\ \lambda \vdash n\}$ be the characters of the irreducible representations of the symmetric group $\mathbb{S}_n$, which form a basis of the center $\mathcal{Z}_n$ of the group ring $\mathbb{Z}[\mathbb{S}_n]$. The famous result by A. Jucys [40] states that for any symmetric polynomial $f(z_1,\ldots,z_n)$ the character expansion of $f(d_2,\ldots,d_n,0) \in \mathcal{Z}_n$ is

$$f(d_2,\ldots,d_n,0) = \sum_{\lambda \vdash n} \frac{f(C_{\lambda})}{H_{\lambda}}\ \chi^{\lambda}, \tag{4.14}$$

where $H_{\lambda} = \prod_{x \in \lambda} h_x$ denotes the product of all hook-lengths of λ, and $C_{\lambda} := \{c(x)\}_{x \in \lambda}$ denotes the set of contents of all boxes of λ.

[32] I would like to thank DR. S. Tsuchioka for computation the Hilbert polynomials $Hilb(JM(n),t)$, as well as the sets of defining relations among the Jucys–Murphy elements in the symmetric group $\mathbb{S}_n$ for $n \leq 7$.

Recall that the Jucys–Murphy elements $\{d_j^H\}_{2\le j\le n}$ in the (normalized) Hecke algebra $H_n(q)$ are defined as follows: $d_j^H := \sum_{i<j} T_{(ij)}$, where $T_{(ij)} := T_i \cdots T_{j-1}\, T_j\, T_{j-1} \cdots T_i$. Finally denote by $H_\lambda(q)$ and $C_\lambda^{(q)}$ the hook polynomial and the set $\{c_\lambda x; q)\}_x \in \lambda$. Then for any symmetric polynomial $f(z_1,\ldots,z_n)$ one has

$$f(d_2^H,\ldots,d_n^H,0) = \sum_{\lambda\vdash n} \frac{f(C_\lambda^{(q)})}{H_\lambda(q)}\ \chi_q^\lambda, \tag{4.15}$$

where chi_q^λ denotes the q-character of the algebra $H_{n(q)}$.

Therefore, if $f \in J_q^{(n)}$, then $f(d_2^H,\ldots,d_n^H,0)=0$. It is an open **problem** to prove/disprove that if $f(d_2^H,\ldots,d_n^H,0)=0$, then $f(C_\lambda^{(q)})=0$ for all partitions of size n (even in the case $q=1$).

4.6 Extended nil-three term relations algebra and DAHA, cf [15]

Let $A := \{q,t,a,b,c,h,e,f,\ldots\}$ be a set of parameters.

Definition 4.27. *Extended nil-three term relations algebra $3\mathfrak{T}_n$ is an associative algebra over $\mathbb{Z}[q^{\pm1},t^{\pm1},a,b,c,h,e,\ldots]$ with the set of generators $\{u_{i,j},\ 1\le i\ne j\le n,\ \ x_i,\ 1\le i\le n,\ \ \pi\}$ subject to the set of relations*

(0) $u_{i,j}+u_{j,i}=0,\quad u_{i,j}^2=0,$
(1) $x_i\ x_j = x_j\ x_i,\ \ u_{i,j}\ u_{k,l} = u_{k,l}\ u_{i,j}$, *if i,j,k,l are distinct,*
(2) $x_i\ u_{kl} = u_{k,l}\ x_i$, *if $i\ne k,l$,*
(3) $x_i\ u_{i,j} = u_{i,j}\ x_j+1,\qquad x_j\ u_{i,j} = u_{i,j}\ x_i-1,$
(4) $u_{i,j}\ u_{j,k}+u_{k,i}\ u_{i,j}+u_{j,k}\ u_{k,i}=0$, *if i,j,k are distinct,*
(5) $\pi\ x_i = x_{i+1}\ \pi$, *if $1\le i<n$,* $\pi\ x_n = t^{-1}\ x_1\ \pi,$
(6) $\pi\ u_{ij} = u_{i+1,j+1}$, *if $1\le i<j<n$,* $\pi^j\ u_{n-j+1,n} = t\ u_{1,j}\ \pi^j,\ 2\le j\le n.$

Note that the algebra $3\mathfrak{T}_n$ contains also the set of elements $\{\pi^a\ u_{jn},\ \ 1\le a\le n-j\}$.

Definition 4.28. *(Cf. [58]) Let $1\le i<j\le n$, define*

$$T_{i,j} = a + (b\ x_i + c\ x_j + h + e\ x_i\ x_j)\ u_{i,j}.$$

Lemma 4.6.
(1) $T_{i,j}^2 = (2a+b-c)\ T_{i,j} - a(a+b-c)$, *if $a=0$, then $T_{ij}^2 = (b-c)\ T_{ij}$.*
(2) *(Coxeter relations) Relations*

$$T_{i,j}\ T_{j,k}\ T_{i,j} = T_{j,k}\ T_{i,j}\ T_{j,k},$$

are valid, if and only if the following relation holds

$$(a+b)(a-c)+h\ e=0. \tag{4.16}$$

(3) *(Yang–Baxter relations) Relations*

$$T_{i,j}\ T_{i,k}\ T_{j.k} = T_{j,k}\ T_{i,k}\ T_{i,j}$$

are valid if and only if $b=c=e=0$, i.e. $T_{ij}=a+d\ u_{ij}$.
(4) $T_{ij}^2=1$ *if and only if $a=\pm1, c=b\pm2,\ he=(b\pm1)^2$.*
(5) *Assume that parameters a,b,c,h,e satisfy the conditions* (4.16) *and that $b\ c+1=h\ e$. Then*

$$T_{ij}\ x_i\ T_{ij} = x_j + (h+(a+b)(x_i+x_j)+e\ x_i\ x_j)\ T_{ij}.$$

(6) *(Quantum Yang–Baxterization) Assume that parameters a,b,c,h,e satisfy the conditions* (4.5) *and that $\beta := 2a+b-c\ne0$. Then (cf [60], [38] and the literature quoted therein)*

the elements $R_{ij}(u,v) := 1 + \frac{\lambda-\mu}{\beta\mu}\, T_{ij}$ *satisfy the* twisted *quantum Yang–Baxter relations*

$$R_{ij}(\lambda_i,\mu_j)R_{jk}(\lambda_i,\nu_k)R_{ij}(\mu_j,\nu_k) = R_{jk}(\mu_j,\nu_k)R_{ij}(\lambda_i,\nu_k)R_{jk}(\lambda_i,\mu_j), \quad i<j<k,$$

where $\{\lambda_i,\mu_i,\nu_i\}_{1\le i\le n}$ *are parameters. .*

Corollary 4.6. *If* $(a+b)(a-c)+he=0$, *then for any permutation* $w \in \mathbb{S}_n$ *the element*

$$T_w := T_{i_1}\cdots T_{i_l} \in 3\mathfrak{T}_n,$$

where $w = s_{i_1}\cdots s_{i_l}$ *is any reduced decomposition of* w, *is well-defined.*

Example 4.8.
Each of the set of elements

$$s_i^{(h)} = 1 + (x_{i+1} - x_i + h)\ u_{i,i+1} \quad and$$

$$t_i^{(h)} = -1 + (x_i - x_{i+1} + h(1+x_i)(1+x_{i+1})u_{ij}, \quad i=1,\ldots,n-1,$$

by itself generate the symmetric group $\mathbb{S}_n$.

Comments 4.7. Let $A=(a,b,c,h,e)$ be a sequence of integers satisfying the conditions (4.5). Denote by ∂_i^A the divided difference operator

$$\partial_i^A = (a + (b\ x_i + c\ x_{i+1} + h + e\ x_i\ x_{i+1})\ \partial_i, \quad i=1,\ldots,n-1.$$

It follows from Lemma 4.5 that the operators $\{\partial_i^A\}_{1\le i\le n}$ satisfy the Coxeter relations

$$\partial_i^A\ \partial_{i+1}^A\ \partial_i^A = \partial_{i+1}^A\ \partial_i^A\ \partial_{i+1}^A, \quad i=1,\ldots,n-1.$$

Definition 4.29.
(1) *Let* $w \in \mathbb{S}_n$ *be a permutation. Define the generalized Schubert polynomial corresponding to permutation* w *as follows*

$$\mathfrak{S}_w^A(X_n) = \partial_{w^{-1}\ w_0}^A\ x^{\delta_n}, \quad where \quad x^{\delta_n} := x_1^{n-1}\ x_2^{n-2}\cdots x_{n-1},$$

and w_0 *denotes the longest element in the symmetric group* $\mathbb{S}_n$.
(2) *Let* α *be a composition with at most* n *parts, denote by* $w_\alpha \in \mathbb{S}_n$ *the permutation such that* $w_\alpha(\alpha) = \overline{\alpha}$, *where* $\overline{\alpha}$ *denotes a unique partition corresponding to composition* α.

Proposition 4.11. *([46]) Let* $w \in \mathbb{S}_n$ *be a permutation.*

- *If* $A=(0,0,0,1,0)$, *then* $\mathfrak{S}_w^A(X_n)$ *is equal to the Schubert polynomial* $\mathfrak{S}_w(X_n)$.
- *If* $A=(-\beta,\beta,0,1,0)$, *then* $\mathfrak{S}_w^A(X_n)$ *is equal to the* β*-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_n)$ *introduced in [27].*
- *If* $A=(0,1,0,1,0)$ *then* $\mathfrak{S}_w^A(X_n)$ *is equal to the dual Grothendieck polynomial, [59], [46].*
- *If* $A=(-1,2,0,1,1)$, *then* $\mathfrak{S}_w^A(X_n)$ *is equal to the Di-Francesco–Zinn-Justin polynomials introduced in [18] and [46].*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ *have non-negative integer coefficients.*

- *If* $A=(1,-1,1,-h,0)$, *then* $\mathfrak{S}_w^A(X_n)$ *is equal to the* h*-Schubert polynomials introduced in [46].*

Define the generalized key or Demazure polynomial corresponding to a composition α *as follows*

$$K_\alpha^A(X_n) = \partial_{w_\alpha}\ x^{\overline{\alpha}}.$$

• *If $A = (1,0,1,0,0)$, then $K_\alpha^A(X_n)$ is equal to key (or Demazure) polynomial corresponding to α.*

• *If $A = (0,0,1,0,0)$, then $K_\alpha^A(X_n)$ is equal to the reduced key polynomial introduced in [46].*

• *If $A = (1,0,1,0,\beta)$, then $K_\alpha^A(X_n)$ is equal to the key Grothendieck polynomial $KG_\alpha(X_n)$ introduced in [46].*

• *If $A = (0,0,1,0,\beta)$, then $K_\alpha^A(X_n)$ is equal to the reduced key Grothendieck polynomial, [46].*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients.

Exercises 4.4.

(1) *Let b, c, h, e be a collection of integers, define elements $P_{ij} := f_{ij}u_{ij} \in 3\mathfrak{T}$, where $f_{ij} := bx_i + cx_j + h + ex_i\, x_j$.*

Show that

• $P_{ij}^2 = (b-c)P_{ij}$,

• $P_{ij}P_{jk}P_{ij} = f_{ij}f_{ik}f_{jk}\ u_{ij}u_{jk}u_{ij} + (bc-eh)P_{ij}$,

$P_{jk}P_{ij}P_{jk} = f_{ij}f_{ik}f_{jk}\ u_{ij}u_{jk}u_{ij} - (bc-eh)P_{jk}$.

(2) *Assume that $a = q,\ b = -q,\ c = q^{-1}, h = e = 0$, and introduce elements*

$$e_{ij} := (q\ x_i - q^{-1}x_j)\ u_{ij}, \qquad 1 \le i < j < k \le n.$$

(a) *Show that if i, j, k are distinct, then*

$$e_{ij}e_{jk}e_{ij} = e_{ij} + (qx_i - q^{-1}x_j)(q\ x_i - q^{-1}x_k)(q\ x_j - q^{-1}x_k)\ u_{ij}u_{jk}\ u_{ij}, \quad e_{ij}^2 = (q+q^{-1})\ e_{ij}.$$

(b) *Assume additionally that*

$$u_{ij}u_{jk}u_{ij} = 0, \quad if \quad i,j,k \quad are \;\; distinct.$$

Show that the elements $\{e_i := e_{i,i+1},\ i = 1,\ldots,n-1\}$, generate a subalgebra in $3\mathfrak{L}_n$ which is isomorphic to the Temperly–Lieb algebra $TL_n(q+q^{-1})$.

(3) *Let us set $T_i := T_{i,i+1}$, $i = 1,\ldots,n-1$, and define*

$$T_0 := \pi T_{n-1}\pi^{-1}.$$

Show that if $(a+b)(a-c)+eh = 0$, then

$$T_1T_0T_1 = T_1T_0T_1, \quad T_{n-1}T_0T_{n-1} = T_0T_{n-1}T_0,$$

Recall that $T_i^2 = (2a+b-c)T_i - a(a+b-c)$, $0 \le i \le n-1$.

■

In what follows we take $a = q, b = -q, c = q^{-1}, h = e = 0$. Therefore, $T_{i,j}^2 = (q-q^{-1})T_{i,j}+1$. We denote by $\mathcal{H}_n(q)$ a subalgebra in $3\mathfrak{T}_n$ generated by the elements $T_i := T_{i,i+1},\ i = 1,\ldots,n-1$.

Remark 4.11. Let us stress on a difference between elements T_{ij} as a part of generators of the algebra $3\mathfrak{T}_n$, and the elements

$$T_{(ij)} := T_i \cdots T_{j-1}T_jT_{j-1}\cdots T_i \in \mathcal{H}_n(q).$$

Whereas one has $[T_{ij}, T_{kl}] = 0$, if i,j,k,l are distinct, the relation $[T_{(ij)}, T_{(kl)}] = 0$ in the algebra $\mathcal{H}_n(q)$ holds (for general q and $i \le k$) if and only if either one has $i < j < k < l$, or $i < k < l < j$. ■

Lemma 4.7.

(1) $T_{ij}\ T_{kl} = T_{kl}\ T_{ij}$, *if* i,j,k,l *are distinct,*
(2) $T_{i,j}\ x_i\ T_{i,j} = x_j$, *if* $1 \le i < j \le n$,
(3) $\pi T_{i,j} = T_{i+1,j+1}$, *if* $1 \le i < j < n$, $\pi^j\ T_{n-j+1,n} = T_{1,j}\ \pi^j$.

Definition 4.30. *Let* $1 \le i < j \le n$, *set*

$$Y_{i,j} = T_{i-1,j-1}^{-1} T_{i-2,j-2}^{-1} \cdots T_{1,j-i+1}^{-1}\ \pi^{j-i}\ T_{n-j+i,n} \cdots T_{i+1,j+1} T_{i,j}, \quad 1 \le i < j \le n,$$

and $Y_n = T_{n-1,n}^{-1} \cdots T_{1,2}^{-1} \pi$.

For example, $Y_{1,j} = \pi^{j-1}\ T_{n-j+1,n} \cdots T_{1,j}$, $j \ge 2$,
$Y_{2,j} = T_{1,j-1}^{-1} \pi^{j-2}\ T_{n-j+2,n} \cdots T_{2,j}$, and so on,
$Y_{j-1,j} = T_{j-2,j-1}^{-1} \cdots T_{1,2}^{-1} \pi\ T_{n-1,n} \cdots T_{j-1,j}$.

Proposition 4.12.

(1) $x_j x_j\ T_{ij} = T_{ij}\ x_i x_j$,
(2) $Y_{i,j} = T_{i,j}\ Y_{i+1,j+1}\ T_{i,j}$, *if* $1 \le i < j < n$,
(3) $Y_{i,j}\ Y_{i+k,j+k} = Y_{i+k,j+k}\ Y_{i,}$, *if* $1 \le i < j \le n-k$,
(4) *One has*

$$x_{i-1}\ Y_{i,j}^{-1} = Y_{i,j}^{-1}\ x_{i-1}\ T_{i-1,j-1}^2, \quad 2 \le i < j \le n,$$

(5) $Y_{i,j}\ x_1\ x_2\ \cdots x_n = t\ x_1 x_2 \cdots x_n\ Y_{i,j}$,
(6) $x_i\ Y_1\ Y_2 \cdots Y_n = t^{-1}\ Y_1\ Y_2 \cdots Y_n\ x_i$,
where we set $Y_i := Y_{i,i+1}$, $1 \le i < j < n$.

Conjecture 4.11.

Subalgebra of $3\mathfrak{T}_n$ *generated by the elements* $\{T_i := T_{i,i+1},\ 1 \le i < n,\ Y_1,\ldots,Y_n,$ *and* $x_1,\ldots,x_n\}$, *is isomorphic to the double affine Hecke algebra* $DAHA_{q,t}(n)$.

Note that the algebra $3\mathfrak{T}_n$ contains also two additional commutative subalgebras generated by additive $\{\theta_i = \sum_{j \ne i} u_{ij}\}_{1 \le i \le n}$ and multiplicative

$$\{\Theta_i = \prod_{a=1}^{i-1} (1 - u_{ai}) \prod_{a=i+1}^{n} (1 + u_{ia})\}_{1 \le i \le n}$$

Dunkl elements correspondingly.

Finally we introduce (cf [15]) a (projective) representation of the modular group $SL(2,\mathbb{Z})$ on the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_n$ over the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ generated by elements

$$\{T_1,\ldots,T_{n-1}\}, \quad \pi, \quad and \quad \{x_1,\ldots,x_n\}.$$

It is well-known that the group $SL(2,\mathbb{Z})$ can be generated by two matrices

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

which satisfy the following relations

$$\tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}, \qquad (\tau_+ \tau_-^{-1} \tau_+)^6 = I_{2\times 2}.$$

Let us introduce operators τ_+ and τ_- acting on the extended affine algebra $\widehat{\mathcal{H}}_n$. Namely,

$$\tau_+(\pi) = x_1 \pi, \quad \tau_+(T_i) = T_i, \quad \tau_+(x_i) = x_i, \quad \forall\ \ i,$$

$$\tau_-(\pi) = \pi, \quad \tau_-(T_i) = T_i, \quad \tau_-(x_i) = \Big(\prod_{a=i-1}^{1} T_a \Big)\ \pi\ \Big(\prod_{a=n}^{i} T_a \Big)\ x_i.$$

Lemma 4.8.

- $\tau_+(Y_i) = \left(\prod_{a=i-1}^{1} T_a^{-1}\right) \left(\prod_{a=1}^{i-1} T_a^{-1}\right) x_i\ Y_i$,
- $\tau_-(x_i) = \left(\prod_{a=i-1}^{1} T_a\right) \left(\prod_{a=1}^{i-1} T_a\right) Y_i\ x_i$,
- $(\tau_+\tau_-^{-1}\tau_+)(x_i) = Y_i^{-1} = (\tau_-^{-1}\tau_+\tau_-^{-1})(x_i)$,
- $(\tau_+\tau_-^{-1}\tau_+)(Y_i) = t\ x_i\ \left(\prod_{a=i-1}^{1} T_a\right)\left(T_1 \cdots T_{n-1}\right)\ \left(\prod_{a=n-1}^{i} T_a\right)$,

$i = 1, \ldots, n$.

In the last formula we set $T_n = 1$ for convenience.

5 Combinatorics of associative Yang–Baxter algebras

Let α and β be parameters.

Definition 5.1 ([47]).
(1) *The associative quasi-classical Yang–Baxter algebra of weight* (α, β), *denoted by* $\widehat{ACYB}_n(\alpha,\beta)$, *is an associative algebra, over the ring of polynomials* $\mathbb{Z}[\alpha,\beta]$, *generated by the set of elements* $\{x_{ij},\ 1 \le i < j \le n\}$, *subject to the set of relations*
(*a*) $x_{ij}\ x_{kl} = x_{kl}\ x_{ij}$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$,
(*b*) $x_{ij}\ x_{jk} = x_{ik}\ x_{ij} + x_{jk}\ x_{ik} + \beta\ x_{ik} + \alpha$, *if* $1 \le 1 < i < j \le n$.
(2) *Define* **associative quasi-classical Yang–Baxter algebra of weight** β *,denoted by* $\widehat{ACYB}_n(\beta)$, *to be* $\widehat{ACYB}_n(0,\beta)$.

Comments 5.1.
The algebra $3T_n(\beta)$, see Definition 3.1, is the quotient of the algebra $\widehat{ACYB}_n(-\beta)$, by the "dual relations"

$$x_{jk}x_{ij} - x_{ij}\ x_{ik} - x_{ik}\ x_{jk} + \beta\ x_{ik} = 0, \quad i < j < k.$$

The (truncated) Dunkl elements $\theta_i = \sum_{j \ne i} x_{ij}$, $i = 1, \ldots, n$, do not commute in the algebra $\widehat{ACYB}_n(\beta)$. However a certain version of noncommutative elementary polynomial of degree $k \ge 1$, still is equal to zero after the substitution of Dunkl elements instead of variables, [47]. We state here the corresponding result only "in classical case", i.e. if $\beta = 0$ and $q_{ij} = 0$ *for all* i,j.

Lemma 5.1. *([47]) Define noncommutative elementary polynomial* $L_k(x_1, \ldots, x_n)$ *as follows*

$$L_k(x_1, \ldots, x_n) = \sum_{I=(i_1<i_2<\ldots<i_k)\subset[1,n]} x_{i_1}\ x_{i_2} \cdots x_{i_k}.$$

Then $L_k(\theta_1, \theta_2, \ldots, \theta_n) = 0$.
Moreover, if $1 \le k \le m \le n$, *then one can show that the value of the noncommutative polynomial* $L_k(\theta_1^{(n)}, \ldots, \theta_m^{(n)})$ *in the algebra* $\widehat{ACYB}_n(\beta)$ *is given by the Pieri formula, see [26], [76].*

5.1 Combinatorics of Coxeter element

Consider the "Coxeter element" $w \in \widehat{ACYB}_n(\alpha,\beta)$ which is equal to the ordered product of "simple generators":

$$w := w_n = \prod_{a=1}^{n-1} x_{a,a+1}.$$

Let us bring the element w to the reduced form in the algebra $\widehat{ACYB}_n(\alpha,\beta)$, that is, let us consecutively apply the defining relations (a) and (b) to the element w in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P_n(x_{ij};\alpha,\beta)$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied. We set $P_n(x_{ij};\beta) := P_n(x_{ij};0,\beta)$.

Proposition 5.1. *(Cf [90],* **8.C5***, (c); [65])*

(**1**) *Apart from applying the relation (a) (commutativity), the polynomial* $P_n(x_{ij};\beta)$ *does not depend on the order in which relations (a) and (b) have been applied, and can be written in a unique way as a linear combination:*

$$P_n(x_{ij};\beta) = \sum_{s=1}^{n-1} \beta^{n-s-1} \sum_{\{i_a\}} \prod_{a=1}^{s} x_{i_a,j_a},$$

where the second summation runs over all sequences of integers $\{i_a\}_{a=1}^{s}$ *such that* $n-1 \ge i_1 \ge i_2 \ge \ldots \ge i_s = 1$, *and* $i_a \le n-a$ *for* $a = 1,\ldots,s-1$; *moreover, the corresponding sequence* $\{j_a\}_{a=1}^{n-1}$ *can be defined uniquely by that* $\{i_a\}_{a=1}^{n-1}$.

- It is clear that the polynomial $P(x_{ij};\beta)$ also can be written in a unique way as a linear combination of monomials $\prod_{a=1}^{s} x_{i_a,j_a}$ such that $j_1 \ge j_2 \ldots \ge j_s$.

(**2**) *Let us set* $deg(x_{ij}) = 1$, $deg(\beta) = 0$. *Denote by* $T_n(k,r)$ *the number of degree* k *monomials in the polynomial* $P(x_{ij};\beta)$ *which contain exactly* r *factors of the form* $x_{*,n}$. *(Note that* $1 \le r \le k \le n-1$*). Then*

$$T_n(k,r) = \frac{r}{k} \binom{n+k-r-2}{n-2} \binom{n-2}{k-1}.$$

In other words,

$$P_n(t,\beta) = \sum_{1 \le r \le k < n} T_n(k,r)\ t^r\ \beta^{n-1-k},$$

where $P_n(t,\beta)$ *denotes the following specialization*

$$x_{ij} \longrightarrow 1, \quad if \quad j < n, \quad x_{in} \longrightarrow t, \quad \forall\ i = 1,\ldots,n-1$$

of the polynomial $P_n(x_{ij};\beta)$.

In particular, $T_n(k,k) = \binom{n-2}{k-1}$, *and* $T_n(k,1) = T(n-2,k-1)$, *where*

$$T(n,k) := \frac{1}{k+1} \binom{n+k}{k} \binom{n}{k}$$

is equal to the number of Schröder paths (i.e. consisting of steps $U=(1,1)$, $D=(1,-1), H = (2,0)$ *and never going below the x-axis) from* $(0,0)$ *to* $(2n,0)$, *having* k U*'s, see [87], A088617.*

Moreover, $T_n(n-1,r) = Tab(n-2,r-1)$, *where*

$$Tab(n,k) := \frac{k+1}{n+1} \binom{2n-k}{n} = F_{n-k}^{(2)}(k)$$

is equal to the number of standard Young tableaux of the shape $(n,n-k)$, *see [87], A009766. Recall that* $F_n^{(p)}(b) = \frac{1+b}{n}\binom{np+b}{n-1}$ *stands for the generalized Fuss–Catalan number.*

(**3**) *After the specialization* $x_{ij} \longrightarrow 1$ *the polynomial* $P(x_{ij})$ *is transformed to the polynomial*

$$P_n(\beta) := \sum_{k=0}^{n-1} N(n,k)\ (1+\beta)^k,$$

where $N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, $k = 0,\ldots,n-1$, *stand for the Narayana numbers. Furthermore,* $P_n(\beta) = \sum_{d=0}^{n-1} s_n(d)\ \beta^d$, *where*

$$s_n(d) = \frac{1}{n+1}\binom{2n-d}{n}\binom{n-1}{d}$$

is the number of ways to draw $n-1-d$ *diagonals in a convex* $(n+2)$*-gon, such that* no *two diagonals intersect their interior.*

Therefore, the number of (nonzero) terms in the polynomial $P(x_{ij};\beta)$ *is equal to the* n*-th little Schröder number* $s_n := \sum_{d=0}^{n-1} s_n(d)$*, also known as the* n*-th super-Catalan number, see e.g. [87], A001003.*

(**4**) *Upon the specialization* $x_{1j} \longrightarrow t$, $1 \le j \le n$, *and that* $x_{ij} \longrightarrow 1$, *if* $2 \le i < j \le n$, *the polynomial* $P(x_{ij};\beta)$ *is transformed to the polynomial*

$$P_n(\beta,t) = t\sum_{k=1}^{n} (1+\beta)^{n-k} \sum_{\pi} t^{p(\pi)},$$

where the second summation runs over the set of Dick paths π *of length* $2n$ *with exactly* k *picks (UD-steps), and* $p(\pi)$ *denotes the number of valleys (DU-steps) that touch upon the line* $x = 0$.

(**5**) *The polynomial* $P(x_{ij};\beta)$ *is invariant under the action of anti-involution* $\phi \circ \tau$*, see Section 5.1.1 [47] for definitions of* ϕ *and* τ.

(**6**) *Follow [90],* **6.C8**, (*c*)*, consider the specialization*

$$x_{ij} \longrightarrow t_i, \quad 1 \le i < j \le n,$$

and define $P_n(t_1,\ldots,t_{n-1};\beta) = P_n(x_{ij} = t_i;\beta)$.

One can show, ibid , that

$$P_n(t_1,\ldots,t_{n-1};\beta) = \sum \beta^{n-k}\ \ t_{i_1}\cdots t_{i_k}, \tag{5.1}$$

where the sum runs over all pairs $\{(a_1,\ldots,a_k),(i_1,\ldots,i_k) \in \mathbb{Z}_{\ge 1}\times\mathbb{Z}_{\ge 1}\}$ *such that* $1 \le a_1 < a_2 < \ldots < a_k$, $1 \le i_1 \le i_2 \ldots \le i_k \le n$ *and* $i_j \le a_j$ *for all* j.

Now we are ready to state our main result about polynomials $P_n(t_1,\ldots,t_n;\beta)$.

Let $\pi := \pi_n \in \mathbb{S}_n$ *be the permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & n & n-1 & \ldots & 2 \end{pmatrix}$. *Then*

$$P_n(t_1,\ldots,t_{n-1};\beta) = \Big(\prod_{i=1}^{n-1} t_i^{n-i}\Big)\ \mathfrak{G}_{\pi}^{(\beta)}(t_1^{-1},\ldots,t_{n-1}^{-1}),$$

where $\mathfrak{G}_w^{(\beta)}(x_1,\ldots,x_{n-1})$ *denotes the* β*-Grothendieck polynomial corresponding to a permutation* $w \in \mathbb{S}_n$*, [27], or Appendix I.*

In particular,

$$\mathfrak{G}_{\pi}^{(\beta)}(x_1 = 1,\ldots,x_{n-1} = 1) = \sum_{k=0}^{n-1} N(n,k)\ (1+\beta)^k,$$

where $N(n,k)$ *denotes the Narayana numbers, see item* (**3**) *of Proposition 5.1.*

More generally, write $P_n(t,\beta) = \sum_k\ P_n^{(k)}(\beta)\ t^k$. *Then*

$$\mathfrak{G}_{\pi}^{(\beta)}(x_1 = t, x_i = 1, \forall i \ge 2) = \sum_{k=0}^{n-1} P_{n-1}^{(k)}(\beta^{-1})\beta^k\ t^{n-1-k}.$$

Comments 5.2.

• Note that if $\beta = 0$, then one has $\mathfrak{G}_w^{(\beta=0)}(x_1,\ldots,x_{n-1}) = \mathfrak{S}_w(x_1,\ldots,x_{n-1})$, that is the β-Grothendieck polynomial at $\beta=0$, is equal to the Schubert polynomial corresponding to the same permutation w. Therefore, if $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & n & n-1 & \ldots & 2 \end{pmatrix}$, then

$$\mathfrak{S}_\pi(x_1 = 1,\ldots,t_{n-1} = 1) = C_{n-1}, \tag{5.2}$$

where C_m denotes the m-th Catalan number. Using the formula (5.20) it is not difficult to check that the following formula for the principal specialization of the Schubert polynomial $\mathfrak{S}_\pi(X_n)$ is true

$$\mathfrak{S}_\pi(1,q,\ldots,q^{n-1}) = q^{\binom{n-1}{3}}\ C_{n-1}(q), \tag{5.3}$$

where $C_m(q)$ denotes the Carlitz - Riordan q-analogue of the Catalan numbers, see e.g. [88]. The formula (5.20) has been proved in [29] using the observation that π is a *vexillary* permutation, see [61] for the a definition of the latter. A combinatorial/bijective proof of the formula (5.20) is is due to A.Woo [98].

• The Grothendieck polynomials defined by A. Lascoux and M.-P. Schützenberger, see e.g. [57], correspond to the case $\beta = -1$. In this case $P_n(-1) = 1$, if $n \ge 0$, and therefore the specialization $\mathfrak{G}_w^{(-1)}(x_1 = 1,\ldots,x_{n-1} = 1) = 1$ for all $w \in \mathbb{S}_n$. ■

Exercises 5.1.

(**1**) *Let as before,* $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & n & n-1 & \ldots & 2 \end{pmatrix}$. *Show that*

$$\mathfrak{S}_\pi(x_1 = q, x_j = 1, \forall j \ne i) = \sum_{a=0}^{n-2} \frac{n-a-1}{n-1} \binom{n+a-2}{a} q^a.$$

Note that the number

$$\frac{n-k+1}{n+1}\binom{n+k}{k}$$

is equal to the dimension of irreducible representation of the symmetric group $\mathbb{S}_{n+k}$ that corresponds to partition $(n+k,k)$.

(2) *Consider the commutative quotient $\widetilde{ACYB}_n^{ab}(\alpha,\beta)$ of the algebra $\widetilde{ACYB}_n(\alpha,\beta)$, i.e. assume that the all generators $\{x_{ij}|\ 1 \le i < j \le n$ are mutually commute. Denote by $\overline{P}_n(x_{ij};\alpha,\beta)$ the image of polynomial the $P_n(x_{ij};\alpha,\beta) \in \widetilde{ACYB}_n(\alpha,\beta)$ in the algebra $\widetilde{ACYB}_n^{ab}(\alpha,\beta)$. Finally, define polynomials $P_n(t,\alpha,\beta)$ to be the specialization*

$$x_{ij} \longrightarrow 1, \quad if \quad j < n, \qquad x_{in} \longrightarrow t, \quad if \quad 1 \le i < n.$$

Show that

(*a*) *Polynomial $P_n(t,\alpha,\beta)$ does not depend on on order in which relations (a) and (b), see Definition 5.1, have been applied.*

(*b*)

$$P_n(1,\alpha=1,\beta=0) = \sum_{k\ge 0} \frac{(2n-2k)!}{k!\ (n+1-k)\ !\ (n-2k)!},$$

see [87], A052709(n), for combinatorial interpretations of these numbers.

For example,

$P_7(t,\alpha,\beta) = t^7 + 6(1+\beta)\ t^6 + \Big[(20,35,15)_\beta + 6\ \alpha\Big]\ t^5 + \Big[(48,112,84,20)_\beta +$

$\alpha(34,29)_\beta\Big]\ t^4+\Big[(90,252,252,105,15)_\beta+\alpha(104,155,55)_\beta+14\alpha^2\Big]\ t^3+$
$\Big[(132,420,504,280,70,6)_\beta+\alpha(216,428,265,50)_\beta+\alpha^2(70,49)_\beta\Big]\ t^2+$
$\Big[(132,462,630,420,140,21,1)_\beta+\alpha(300,708,580,190,20)_\beta+\alpha^2(168,203,56)_\beta+$
$14\alpha^3\Big]\ t+\alpha(132,330,300,120,20,1)_\beta+\alpha^2(168,252,112,14)_\beta+\alpha^3(42,21)_\beta.$

(*c*) *Show that in fact*

$$P_n(1,\alpha,0)=\sum_{k\geq 0}\ \frac{1}{n+1}\binom{2n-2k}{n}\binom{n+1}{k}\ \alpha^k=\sum_{k\geq 0}\frac{T_{n+2}(n-k,k+1)}{2n-1-2k}\ \ \alpha^k,$$

see Proposition 5.1,(2), for definition of numbers $T_n(k,r)$. As for a combinatorial interpretation of the polynomials $P_n(1,\alpha,0)$, see [87], A117434, A085880.

(3) *Consider polynomials $P_n(t,\beta)$ as it has been defined in Proposition 5. 1,* (**2**).
Show that

$$P_n(t,\beta)=1+\sum_{r=1}^{n}\ t^r\ \Big(\sum_{k=0}^{n-1-r}\ \frac{r}{n}\binom{n}{k+r}\binom{n-r-1}{k}\ (1+\beta)^{n-r-k}\Big),$$

cf, e.g., [87], A033877.

A few comments in order. Several combinatorial interpretations of the integer numbers

$$U_n(r,k):=\frac{r}{n+1}\binom{n+1}{k+r}\ \binom{n-r}{k}$$

are well-known. For example,

if $r=1$, the numbers $U_n(1,k)=\frac{1}{n}\ \binom{n}{k+1}\binom{n}{k}$ are equal to the Narayana numbers, see e.g. [87], $A001263$;

if $r=2$, the number $U_n(2,k)$ counts the number of Dyck $(n+1)$-paths whose last descent has length 2 and which contain $n-k$ peaks, see [87] , $A108838$ for details.

Finally, it's easily seen, that $P_n(1,\beta)=A127529(n)$, and $P_n(t,1)=A033184(n)$, see [87].

5.1.1 Multiparameter deformation of Catalan, Narayana and Schröder numbers

Let $\mathfrak{b}=(\beta_1,\ldots,\beta_{n-1})$ be a set of mutually commuting parameters. We define a multiparameter analogue of the associative quasi-classical Yang–Baxter algebra $\widehat{MACYB}_n(\mathfrak{b})$ as follows.

Definition 5.2. *(Cf Definition 2.4) The multiparameter associative quasi- classical Yang–Baxter algebra of weight* $\mathfrak{b}$, *denoted by* $\widehat{MACYB}_n(\mathfrak{b})$, *is an associative algebra, over the ring of polynomials* $\mathbb{Z}[\beta_1,\ldots,\beta_{n-1}]$, *generated by the set of elements* $\{x_{ij},\ 1\leq i<j\leq n\}$, *subject to the set of relations*

(*a*) $x_{ij}\ x_{kl}=x_{kl}\ x_{ij},\quad$ *if* $\{i,j\}\cap\{k,l\}=\emptyset$,

(*b*) $x_{ij}\ x_{jk}=x_{ik}\ x_{ij}+x_{jk}\ x_{ik}+\beta_i\ x_{ik},\quad$ *if* $1\leq 1<i<j\leq n.$

Consider the "Coxeter element" $w_n\in\widehat{MACYB}_n(\mathfrak{b})$ which is equal to the ordered product of "simple generators":

$$w_n:=\prod_{a=1}^{n-1}\ x_{a,a+1}.$$

Now we can use the same method as in [90], **8.C5**, (c) , see Section 5.1, to define the **reduced form** of the Coxeter element w_n. Namely, let us bring the element w_n to the reduced form in the algebra $\widehat{MACYB}_n(\mathfrak{b})$, that is, let us consecutively apply the defining relations (a) and (b)

to the element w_n in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P(x_{ij};\mathfrak{b})$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied.

Proposition 5.2 (Cf [90], **8.C5**, (c); [65]). *Apart from applying the relation (a) (commutativity), the polynomial $P(x_{ij};\mathfrak{b})$ does not depend on the order in which relations (a) and (b) have been applied.*

To state our main result of this Subsection, let us define polynomials

$$Q(\beta_1,\ldots,\beta_{n-1}) := P(x_{ij}=1,\forall i,j\ ;\beta_1-1,\beta_2-1,\ldots,\beta_{n-1}-1).$$

Example 5.1.
$Q(\beta_1,\beta_2) = 1+2\ \beta_1+\beta_2+\beta_1^2$,
$Q(\beta_1,\beta_2,\beta_3) = 1+3\beta_1+2\beta_2+\beta_3+3\beta_1^2+\beta_1\beta_2+\beta_1\beta_3+\beta_2^2+\beta_1^3$,
$Q(\beta_1,\beta_2,\beta_3,\beta_4) = 1+4\beta_1+3\beta_2+2\beta_3+\beta_4+\beta_1(6\beta_1+3\beta_2+3\beta_3+2\beta_4)+\beta_2(3\beta_2+\beta_3+\beta_4)+\beta_3^2+\beta_1^2\ (4\beta_1+\beta_2+\beta_3+\beta_4)+\beta_1(\beta_2^2+\beta_3^2)+\beta_2^3+\beta_1^4$.

Theorem 5.1.
Polynomial $Q(\beta_1,\ldots,\beta_{n-1})$ has non-negative integer coefficients.

It follows from [90] and Proposition 5.1, that

$$Q(\beta_1,\ldots,\beta_{n-1})\Big|_{\beta_1=1,\ldots,\beta_{n-1}=1} = Cat_n.$$

Polynomials $Q(\beta_1,\ldots,\beta_{n-1})$ and $Q(\beta_1+1,\ldots,\beta_{n-1}+1)$ can be considered as a multiparameter deformation of the Catalan and (small) Schröder numbers correspondingly, and the homogeneous degree k part of $Q(\beta_1,\ldots,\beta_{n-1})$ as a multiparameter analogue of Narayana numbers.

5.2 Grothendieck and q-Schröder polynomials

5.2.1 Schröder paths and polynomials

Definition 5.3. *A Schröder path of the length n is an over diagonal path from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and steps $D=(1,1)$ without steps of type D on the diagonal $x=y$.*

If p is a Schröder path, we denote by $d(p)$ the number of the diagonal steps resting on the path p, and by $a(p)$ the number of unit squares located between the path p and the diagonal $x=y$. For each (unit) diagonal step D of a path p we denote by $i(D)$ the x-coordinate of the column which contains the diagonal step D. Finally, define the index $i(p)$ of a path p as the some of the numbers $i(D)$ for all diagonal steps of the path p.

Definition 5.4. *Define q-Schröder polynomial $S_n(q;\beta)$ as follows*

$$S_n(q;\beta) = \sum_p q^{a(p)+i(p)}\ \beta^{d(p)}, \tag{5.4}$$

where the sum runs over the set of all Schröder paths of length n.

Example 5.2.
$S_1(q;\beta)=1$, $S_2(q;\beta)=1+q+\beta\ q$, $S_3(q;\beta)=1+2\ q+q^2+q^3+\beta\ (q+2q^2+2q^3)+\beta^2\ q^3$,
$S_4(q;\beta)=1+3q+3q^2+3q^3+2q^4+q^5+q^6+\beta(q+3q^2+5q^3+6q^4+3q^5+3q^6)+\beta^2(q^3+2q^4+3q^5+3q^6)+\beta^3\ q^6$.

Comments 5.3.

The q-Schröder polynomials defined by the formula (5.22) are <u>different</u> from the q-analogue of Schröder polynomials which has been considered in [11]. It seems that there are no simple connections between the both.

Proposition 5.3. *(Recurrence relations for q-Schröder polynomials)*
The q-Schröder polynomials satisfy the following relations

$$S_{n+1}(q;\beta)=(1+q^n+\beta\ q^n)\ S_n(q;\beta)+\sum_{k=1}^{k=n-1}\ (q^k+\beta\ q^{n-k})\ S_k(q;q^{n-k}\ \beta)\ S_{n-k}(q;\beta), \tag{5.5}$$

and the initial condition $S_1(q;\beta)=1$.

Note that $P_n(\beta)=S_n(1;\beta)$ and in particular, the polynomials $P_n(\beta)$ satisfy the following recurrence relations

$$P_{n+1}(\beta)=(2+\beta)\ P_n(\beta)+(1+\beta)\ \sum_{k=1}^{n-1}\ P_k(\beta)\ P_{n-k}(\beta). \tag{5.6}$$

Theorem 5.2. *(Evaluation of the Schröder – Hankel Determinant)*
Consider permutation

$$\pi_k^{(n)}=\begin{pmatrix}1 & 2 & \dots & k & k+1 & k+2 & \dots & n\\ 1 & 2 & \dots & k & n & n-1 & \dots & k+1\end{pmatrix}.$$

Let as before

$$P_n(\beta)=\sum_{j=0}^{n-1}\ N(n,j)\ (1+\beta)^j, \quad n\ge 1, \tag{5.7}$$

be Schröder polynomials. <u>*Then*</u>

$$(1+\beta)^{\binom{k}{2}}\ \mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1=1,\dots,x_{n-k}=1)=Det\ |P_{n+k-i-j}(\beta)\ |_{1\le i,j\le k}. \tag{5.8}$$

Proof is based on an observation that the permutation $\pi_k^{(n)}$ is a *vexillary* one and the recurrence relations (5.5).

Comments 5.4.

(**1**) In the case $\beta=0$, i.e. in the case of <u>Schubert polynomials</u>, Theorem 5.1 has been proved in [29].

(**2**) In the cases when $\beta=1$ and $0\le n-k\le 2$, the value of the determinant in the RHS(5.8) is known, see e.g. [11], or M. Ichikawa talk *Hankel determinants of Catalan, Motzkin and Schr̈der numbers and its q-analogue*, http:/denjoy.ms.u-tokyo.ac.jp. One can check that in the all cases mentioned above, the formula (5.8) gives the same results.

(**3**) **Grothendieck and Narayana polynomials**

It follows from the expression (5.7) for the Narayana-Schröder polynomials that $P_n(\beta-1)=\mathfrak{N}_n(\beta)$, where

$$\mathfrak{N}_n(\beta):=\sum_{j=0}^{n-1}\ \frac{1}{n}\ \binom{n}{j}\ \binom{n}{j+1}\ \beta^j,$$

denotes the n-th Narayana polynomial. Therefore, $P_n(\beta-1)=\mathfrak{N}_n(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients. Moreover, the value of the polynomial $P_n(\beta-1)$ at $\beta=1$ is equal to the n-th Catalan number $C_n:=\frac{1}{n+1}\ \binom{2n}{n}$.

It is well-known, see e.g. [92], that the Narayana polynomial $\mathfrak{N}_n(\beta)$ is equal to the generating function of the statistics $\pi(\mathfrak{p}) = (\textit{number of } \underline{\textit{peaks}} \textit{ of a Dick path } \mathfrak{p}) - 1$ on the set $Dick_n$ of Dick paths of the length $2n$

$$\mathfrak{N}_n(\beta) = \sum_{\mathfrak{p}} \beta^{\pi(\mathfrak{p})}.$$

Moreover, using the Lindstr$\ddot{o}$m–Gessel–Viennot lemma, see e.g.,
http://en.wikipedia.org/wiki/Lindström–Gessel–Viennot lemma,
one can see that

$$DET|\mathfrak{N}_{n+k-i-j}(\beta)|_{1\le i,j\le k} = \beta^{\binom{k}{2}} \sum_{(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)} \beta^{\pi(\mathfrak{p}_1)+\ldots+\pi(\mathfrak{p}_k)}, \tag{5.9}$$

where the sum runs over k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$ such that the path $\mathfrak{p}_i$ starts from the point $(i-1,0)$ and has length $2(n-i+1)$, $i=1,\ldots,k$.

We <u>denote</u> the sum in the $RHS(5.9)$ by $\mathfrak{N}_n^{(k)}(\beta)$. Note that $\mathfrak{N}_{k-1}^{(k)}(\beta)=1$ for all $k\ge 2$.

Thus, $\mathfrak{N}_n^{(k)}(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients, and

$$\mathfrak{N}_n^{(k)}(\beta=1) = C_n^{(k)} = \prod_{1\le i\le j\le n-k} \frac{2k+i+j}{i+j} = \prod_{2\ a\ \le n-k-1} \frac{\binom{2n-2a}{2k}}{\binom{2k+2a+1}{2k}}.$$

As a corollary we obtain the following statement

Proposition 5.4. *Let $n \ge k$, then*

$$\mathfrak{G}_{\pi_k^{(n)}}^{(\beta-1)}(x_1=1,\ldots,x_n=1) = \mathfrak{N}_n^{(k)}(\beta).$$

Summarizing, the specialization $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta-1)}(x_1=1,\ldots,x_n=1)$ is a symmetric polynomial in β with non-negative integer coefficients, and coincides with the generating function of the statistics $\sum_{i=1}^k \pi(\mathfrak{p}_i)$ on the set k-$Dick_n$ of k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$. ■

Example 5.3. *Take $n=5$, $k=1$. Then $\pi_1^{(5)}=(15432)$ and one has*

$$\mathfrak{G}_{\pi_1^{(5)}}^{(\beta)}(1,q,q^2,q^3) = q^4(1,3,3,3,2,1,1) + q^5\ (1,3,5,6,3,3)\ \beta + q^7(1,2,3,3)\beta^2 + q^{10}\beta^3.$$

It is easy to compute the Carlitz-Riordan q-analogue of the Catalan number C_5, namely,
$C_5(q) = (1,3,3,3,2,1,1)$.

Remark 5.1. The value $\mathfrak{N}_n(4)$ of the Narayana polynomial at $\beta=4$ has the following combinatorial interpretation :

$\mathfrak{N}_n(4)$ is equal to the number of different lattice paths from the point $(0,0)$ to that $(n,0)$ using steps from the set $\Sigma = \{(k,k)\ or\ (k,-k),\ k\in\mathbb{Z}_{>0}\}$, that never go below the x-axis, see [87], $A059231$.

Exercises 5.2. (a) <u>*Show*</u> *that*

$$\gamma_{k,n} := \frac{C_n^{(k+1)}}{C_n^{(k)}} = \frac{(2n-2k)!\ (2k+1)\ !}{(n-k)\ !\ (n+k+1)\ !}.$$

(b) <u>*Show*</u> *that*
$\gamma_{k,n}\le 1$, *if* $k\le n\le 3k+1$, *and* $\gamma_{k,n}\ge 2^{n-3k-1}$, *if* $n>3k+1$.

■

(4) **Polynomials** $\mathfrak{F}_w(\beta)$, $\mathfrak{H}_w(\beta)$, $\mathfrak{H}_w(q,t;\beta)$ **and** $\mathfrak{R}_w(q;\beta)$

Let $w \in \mathbb{S}_n$ be a permutation and $\mathfrak{G}_w^{(\beta)}(X_n)$ and $\mathfrak{G}_w^{(\beta)}(X_n,Y_n)$ be the corresponding β-Grothendieck and double β-Grothendieck polynomials. We denote by $\mathfrak{G}_w^{(\beta)}(1)$ and by $\mathfrak{G}_w^{(\beta)}(1;1)$ the specializations $X_n := (x_1 = 1,\ldots,x_n = 1)$, $Y_n := (y_1 = 1,\ldots,y_n = 1)$ of the β-Grothendieck polynomials introduced above.

Theorem 5.3. *Let $w \in \mathbb{S}_n$ be a permutation. Then*

(*i*) *The polynomials* $\mathfrak{F}_w(\beta) := \mathfrak{G}_w^{(\beta-1)}(1)$ *and* $\mathfrak{H}_w(\beta) := \mathfrak{G}_w^{(\beta-1)}(1;1)$ *have both non-negative integer coefficients.*

(*ii*) *One has*

$$\mathfrak{H}_w(\beta) = (1+\beta)^{\ell(w)}\ \mathfrak{F}_w(\beta^2).$$

(*iii*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomials*

$$\mathfrak{H}_w(q,t;\beta) := \mathfrak{G}_w^{(\beta)}(x_1=q,x_2=q,\ldots,x_n=q,y_1=t,y_2=t,\ldots,y_n=t)$$

to be the specialization $\{x_i = q, y_i = t,\ \forall i\}$, *of the double β-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_n,Y_n)$. *Then*

$$\mathfrak{H}_w(q,t;\beta) = (q+t+\beta\ q\ t)^{\ell(w)}\ \mathfrak{F}_w((1+\beta\ q)(1+\beta\ t)).$$

In particular, $\mathfrak{H}_w(1,1;\beta) = (2+\beta)^{\ell(w)}\ \mathfrak{F}_w((1+\beta)^2)$.

(*iv*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomial*

$$\mathcal{R}_w(q;\beta) := \mathfrak{G}_w^{(\beta-1)}(x_1=q,x_2=1,x_3=1,\ldots)$$

to be the specialization $\{x_1 = q, x_i = 1,\ \forall i \geq 2\}$, *of the $(\beta-1)$-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta-1)}(X_n)$. *Then*

$$\mathcal{R}_w(q;\beta) = q^{w(1)-1}\ \mathfrak{R}_w(q;\beta),$$

where $\mathfrak{R}_w(q;\beta)$ is a polynomial in q and β with non-negative integer coefficients, and $\mathfrak{R}_w(0;\beta=0) = 1$.

(*v*) *Consider permutation* $w_n^{(1)} := [1,n,n-1,n-2,\cdots,3,2] \in \mathbb{S}_n$.

Then $\mathfrak{H}_{w_n^{(1)}}(1,1;1) = 3^{\binom{n-1}{2}}\ \mathfrak{N}_n(4)$.

In particular, if $w_n^{(k)} = (1,2,\ldots,k,n,n-1,\ldots,k+1) \in \mathbb{S}_n$, then

$$\mathfrak{G}_{w_n^{(k)}}^{(\beta-1)}(1;1) = (1+\beta)^{\binom{n-k}{2}}\ \mathfrak{G}_{w_n^{(k)}}^{(\beta-1)}(\beta^2).$$

See Remark 5.1 for a combinatorial interpretation of the number $\mathfrak{N}_n(4)$.

Example 5.4.

Consider permutation $v = [2,3,5,6,8,9,1,4,7] \in \mathbb{S}_9$ of the length 12, and set $x := (1+\beta q)(1+\beta t)$. One can check that

$$\mathfrak{H}_v(q,t;\beta) = x^{12}\ (1+2\ x)(1+6x+19x^2+24x^3+13x^4),$$

and $\mathfrak{F}_v(\beta) = (1+2\beta)(1+6\beta+19\beta^2+24\beta^3+13\beta^4)$.

Note that $\mathfrak{F}_v(\beta=1) = 27 \times 7$, and $7 = AMS(3)$, $26 = CSTCTPP(3)$, cf Conjecture 5.4, Section 5.2.4. ■

Remark 5.2.

One can show, cf [61], p. 89, that if $w \in \mathbb{S}_n$, then $\mathcal{R}_w(1,\beta) = \mathcal{R}_{w^{-1}}(1,\beta)$. However, the equality $\mathfrak{R}_w(q,\beta) = \mathfrak{R}_{w^{-1}}(q,\beta)$ can be violated, and it seems that in general, there are no simple connections between polynomials $\mathfrak{R}_w(q,\beta)$ and $\mathfrak{R}_{w^{-1}}(q,\beta)$, if so.

From this point we shell use the notation $(a_0,a_1,\ldots,a_r)_\beta := \sum_{j=0}^r\ a_j\ \beta^j$, *etc.*

Example 5.5. Let us take $w = [1,3,4,6,7,9,10,2,5,8]$. Then $\mathfrak{R}_w(q,\beta) =$
$(1,6,21,36,51,48,26)_\beta + q\beta\ (6,36,126,216,306,288,156)_\beta +$
$q^2\beta^3\ (20,125,242,403,460,289)_\beta + q^3\beta^5\ (6,46,114,204,170)_\beta$. Moreover,
$\mathfrak{R}_w(q,1) = (189,1134,1539,540)_q$. On the other hand,
$w^{-1} = [1,8,2,3,9,4,5,10,6,7]$, and $\mathfrak{R}_{w^{-1}}(q,\beta) = (1,6,21,36,51,48,26)_\beta +$
$q\beta\ (1,6,31,56,96,110,78)_\beta + q^2\beta\ (1,6,27,58,92,122,120,78)_\beta +$
$q^3\beta\ (1,6,24,58,92,126,132,102,26)_\beta + q^4\beta\ (1,6,22,57,92,127,134,105,44)_\beta +$
$q^5\beta\ (1,6,21,56,91,126,133,104,50)_\beta + q^6\beta\ (1,6,21,56,91,126,133,104,50)_\beta$.
Moreover, $\mathfrak{R}_{w^{-1}}(q,1) = (189,378,504,567,588,588,588)_q$.
Notice that $w = 1 \times u$, where $u = [2,3,5,6,8,9,1,4,7]$. One can show that
$\mathfrak{R}_u(q,\beta) = (1,6,11,16,11)_\beta + q\beta^2\ (10,20,35,34)_\beta + q^2\beta^4\ (5,14,26)_\beta$. On the other hand,
$u^{-1} = [7,1,2,8,3,4,9,5,6]$ and $\mathfrak{R}_{u^{-1}}(1,\beta) = (1,6,21,36,51,48,26)_\beta = \mathfrak{R}_u(1,\beta)$.

[Recall that by our definition $(a_0,a_1,\ldots,a_r)_\beta := \sum_{j=0}^{r}\ a_j\ \beta^j$.]

5.2.2 Grothendieck polynomials and k-dissections

Let $k \in \mathbb{N}$ and $n \ge k-1$, be a integer, define *a k-dissection* of a convex $(n+k+1)$-gon to be a collection $\mathcal{E}$ of diagonals in $(n+k+1)$-gon not containing $(k+1)$-subset of pairwise crossing diagonals and such that at least $2(k-1)$ diagonals are coming from each vertex of the $(n+k+1)$-gon in question. One can show that the number of diagonals in any k-dissection $\mathcal{E}$ of a convex $(n+k+1)$-gon contains at least $(n+k+1)(k-1)$ and at most $n(2k-1)-1$ diagonals. We define the *index* of a k-dissection $\mathcal{E}$ to be $i(\mathcal{E}) = n(2k-1)-1-\#|\mathcal{E}|$. Dnote by

$$\mathcal{T}_n^{(k)}(\beta) = \sum_{\mathcal{E}} \beta^{i(\mathcal{E})}$$

the generating function for the number of k-dissections with a fixed index, where the above sum runs over the set of all k-dissections of a convex $(n+k+1)$-gon.

Theorem 5.4.

$$\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1 = 1,\ldots,x_n = 1) = \mathcal{T}_n^{(k)}(\beta).$$

Mopre generally, let $n \ge k > 0$ be integers, consider a convex (n+k+1)-gon P_{n+k+1} and a vertex $v_0 \in P_{n+k+1}$. Let us label clockwise the vertices of P_{n+k+1} by the numbers $1,2,\ldots,n+k+1$ starting from the vertex v_0. Let $Dis(P_{n+k+1})$ denotes the set of all k-dissections of the $(n+k+1)$-gon P_{n+k+1}. We denote by $D_0 := Dis_0(P_{n+k+1}$ the "minimal" k-dissection of the $(n+k+1)$-gon P_{n+k+1} in question cosisting of the set of diagonals connecting vertices v_a and $v_{\overline{a+r}}$, where $2 \le r \le k$, $1 \le a \le n+k+1$, and for any positive integer a we denote by $\overline{a}$ a unique integer such that $1 \le \overline{a} \le n+k+1$ and $a \equiv \overline{a}\ (mod\ (n+k+1))$. For examle, if $k=1$, then $Dis_0(P_{n+2}) = \emptyset$; if $k=3$ and $n=4$, in other words, P_8 is a octagon, the minimal 3-dissection consists of 16 diagonals connecting vertices with the folloing labels
$1 \to 3 \to 5 \to 7 \to \overline{9} = 1$; $2 \to 4 \to 6 \to 8 \to \overline{10} = 2$;
$1 \to 4 \to 7 \to \overline{10} = 2 \to 5 \to 8 \to \overline{11} = 3 \to 6 \to \overline{9} = 1$.

Now let $D \in Dis(P_{n+k+1})$ be a dissection. Consider a diagonal $d_{ij} \in (D \setminus D_0)$, $i < j$ which connects vertex v_i with that v_j. We attach variable x_i to the diagonal d_{ij} in question and consider the folloeing expression

$$\mathcal{T}_{P_{n+k+1}}(X_{n+k+1}) = \sum_{D \in Diss(P_{n+k+1})} \beta^{\#|D\setminus D_0|} \sum_{\substack{d_{ij}\in(D\setminus D_0)\\ i<j}} \prod x_i.$$

Theorem 5.5. *One has*

$$\mathcal{T}_{P_{n+n+1}}(X_{n+k+1}) = \beta^{k(n-k)} \prod_{a=1}^{n} x_a^{min(n-a+1,n-k)} \quad \mathfrak{G}^{\beta^{-1}}_{w_k^n}(x_1^{-1},\ldots,x_n^{-1}).$$

=

Exercises 5.3. *It is not difficult to check that*
$\mathfrak{G}^{\beta}_{15432}(X_5) = \beta^3\, x_1^3\, x_2^3\, x_3^2\, x_4 + \beta^2\, (x_1^3\, x_2^3\, x_3 + 2\, x_1^3\, x_2^3\, x_3\, x_4 + 3\, x_1^3\, x_2^2\, x_3^2\, x_4 + 3\, x_1^2\, x_2^3\, x_3^2\, x_4) +$ $+\beta\, (x_1^3\, x_2^3\, x_3 + x_1^3\, x_2^3\, x_4 + 2\, x_1^3\, x_2^2\, x_3 + 2\, x_1^2\, x_2^3\, x_3^2 + 3\, x_1^3\, x_2^2\, x_3\, x_4 + 3\, x_1^3\, x_2\, x_3^2\, x_4 + 3\, x_1^2\, x_2^3\, x_3\, x_4 +$ $3\, x_1^2\, x_2^2\, x_3^2\, x_4 + 3\, x_1\, x_2^3\, x_3^2\, x_4) + x_1^3\, x_2^2\, x_3 + x_1^3\, x_2^2\, x_4 + x_1^3\, x_2\, x_3^2 + x_1^3\, x_2\, x_3\, x_4 + x_1^3\, x_3^2\, x_4 + x_1^2\, x_2^3\, x_3 +$ $x_1^2\, x_2^3\, x_4 + x_1^2\, x_2^2\, x_3^2 + x_1^2\, x_2^2\, x_3\, x_4 + x_1^2\, x_2\, x_3^2\, x_4 + x_1\, x_2^3\, x_3^2 + x_1\, x_2^3\, x_3\, x_4 + x_1\, x_2^2\, x_3^2\, x_4 + x_2^3\, x_3^2\, x_4.$

Describe

bijection between dissections of hexagon P_6 (the case k=1, n=4) and the above listed monomials involved in the β-Grothendieck polynomial $\mathfrak{G}^{\beta}_{15432}(x_1,x_2,x_3,x_4)$.

■

A k-dissection of a convex $(n+k+1)$-gon with the maximal number of diagonals (which is equal to $n(2k-1)-1$), is called **k-triangulation**. It is well-known that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the Catalan-Hankel number $C_{n-1}^{(k)}$. Explicit bijection between the set of k-triangulations of a convex $(n+k+1)$-gon and the set of k-tuple of non-crossing Dick paths $(\gamma_1,\ldots,\gamma_k)$ such that the Dick path γ_i connects points $(i-1,0)$ and $(2n-i-1,0)$, has been constructed in [85], [91].

5.2.3 Grothendieck polynomials and q-Schröder polynomials

Let $\pi_k^{(n)} = 1^k \times w_0^{(n-k)} \in \mathbb{S}_n$ be the vexillary permutation as before, see Theorem 5.1. Recall that

$$\pi_k^{(n)} = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & k+2 & \ldots & n \\ 1 & 2 & \ldots & k & n & n-1 & \ldots & k+1 \end{pmatrix}.$$

(A) Principal specialization of the Schubert polynomial $\mathfrak{S}_{\pi_k^{(n)}}$

Note that $\pi_k^{(n)}$ is a vexillary permutation of the staircase shape $\lambda = (n-k-1,\ldots,2,1)$ and has the staircase flag $\phi = (k+1,k+2,\ldots,n-1)$. It is known, see e.g. [95], [61], that for a vexillary permutation $w \in \mathbb{S}_n$ of the shape λ and flag $\phi = (\phi_1,\ldots,\phi_r)$, $r = \ell(\lambda)$, the corresponding Schubert polynomial $\mathfrak{S}_w(X_n)$ is equal to the multi-Schur polynomial $s_\lambda(X_\phi)$, where X_ϕ denotes the flagged set of variables , namely, $X_\phi = (X_{\phi_1},\ldots,X_{\phi_r})$ and $X_m = (x_1,\ldots,x_m)$. Therefore we can write the following determinantal formula for the principal specialization of the Schubert polynomial corresponding to the vexillary permutation $\pi_k^{(n)}$

$$\mathfrak{S}_{\pi_k^{(n)}}(1,q,,q^2,\ldots) = DET\Big(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q\Big)_{1\le i,j\le n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient.

Let us observe that the Carlitz–Riordan q-analogue $C_n(q)$ of the Catalan number C_n is equal to the value of the q-Schröder polynomial at $\beta = 0$, namely, $C_n(q) = S_n(q,0)$.

Lemma 5.2. *Let k,n be integers and $n > k$, then*

$$(1) \qquad DET\Big(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q\Big)_{1\le i,j\le n-k} = q^{\binom{n-k}{3}}\ C_n^{(k)}(q),$$

$$(2) \qquad DET\Big(C_{n+k-i-j}(q)\Big)_{1\le i,j\le k} = q^{k(k-1)(6n-2k-5)/6}\ C_n^{(k)}(q).$$

(B) Principal specialization of the Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}$

Theorem 5.6.

$$q^{\binom{n-k+1}{3}-(k-1)\binom{n-k}{2}}\ DET|S_{n+k-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le k} =$$

$$q^{k(k-1)(4k+1)/6}\ \prod_{a=1}^{k-1}(1+q^{a-1}\beta)\ \ \mathfrak{G}_{\pi_k^{(n)}}(1,q,q^2,\ldots).$$

Corollary 5.1. (1) *If $k=n-1$, then*

$$DET|S_{2n-1-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le n-1} = q^{(n-1)(n-2)(4n-3)/6}\ \prod_{a=1}^{n-2}(1+q^{a-1}\beta)^{n-a-1},$$

(2) *If $k=n-2$, then*

$$q^{n-2}\ DET|S_{2n-2-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le n-2} =$$

$$q^{(n-2)(n-3)(4n-7)/6}\ \prod_{a=1}^{n-3}(1+q^{a-1}\beta)^{n-a-2}\ \Big\{\frac{(1+\beta)^{n-1}-1}{\beta}\Big\}.$$

- **Generalization**

Let $\mathbf{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$ be a composition of n so that $n=n_1+\cdots+n_p$. We set $n^{(j)}=n_1+\cdots+n_j$, $j=1,\ldots,p$, $n^{(0)}=0$.

Now consider the permutation $w^{(\mathbf{n})}=w_0^{(n_1)}\times w_0^{(n_2)}\times\cdots\times w_0^{(n_p)}\in\mathbb{S}_n$,

where $w_0^{(m)}\in\mathbb{S}_m$ denotes the longest permutation in the symmetric group $\mathbb{S}_m$. In other words,

$$w^{(\mathbf{n})}=\begin{pmatrix} 1 & 2 & \ldots & n_1 & n^{(2)} & \ldots & n_1+1 & \ldots & n^{(p-1)} & \ldots n \\ n_1 & n_1-1 & \ldots & 1 & n_1+1 & \ldots & n^{(2)} & \ldots & n & \ldots n^{(p-1)+1}\end{pmatrix}.$$

For the permutation $w^{(\mathbf{n})}$ defined above, one has the following factorization formula for the Grothendieck polynomial corresponding to $w^{(\mathbf{n})}$, [61],

$$\mathfrak{G}^{(\beta)}_{w^{(\mathbf{n})}}=\mathfrak{G}^{(\beta)}_{w_0^{(n_1)}}\times\mathfrak{G}^{(\beta)}_{1^{n_1}\times w_0^{(n_2)}}\times\mathfrak{G}^{(\beta)}_{1^{n_1+n_2}\times w_0^{(n_3)}}\times\cdots\times\mathfrak{G}^{(\beta)}_{1^{n_1+\cdots n_{p-1}}\times w_0^{(n_p)}}.$$

In particular, if

$$w^{(\mathbf{n})}=w_0^{(n_1)}\times w_0^{(n_2)}\times\cdots\times w_0^{(n_p)}\in\mathbb{S}_n, \tag{5.10}$$

then the principal specialization $\mathfrak{G}^{(\beta)}_{w^{(\mathbf{n})}}$ of the Grothendieck polynomial corresponding to the permutation w, is the product of q-Schröder–Hankel polynomials. Finally, we observe that from discussions in Section 5.2,1, **Grothendieck & Narayana polynomials**, one can deduce that

$$\mathfrak{G}^{(\beta-1)}_{w^{(\mathbf{n})}}(x_1=1,\ldots,x_n=1)=\prod_{j=1}^{p-1}\ \mathfrak{N}^{(n^{(j)})}_{n^{(j+1)}}\ (\beta).$$

In particular, the polynomial $\mathfrak{G}^{(\beta-1)}_{w^{(\mathbf{n})}}(x_1,\ldots,x_n)$ is a symmetric polynomial in β with non-negative integer coefficients.

Example 5.6.

(1) Let us take (non vexillary) permutation $w = 2143 = s_1\ s_3$. One can check that $\mathfrak{G}_w^{(\beta)}(1,1,1,1) = 3 + 3\ \beta + \beta^2 = 1 + (\beta+1) + (\beta+1)^2$, and $\mathfrak{N}_4(\beta) = (1,6,6,1)$, $\mathfrak{N}_3(\beta) = (1,3,1)$, $\mathfrak{N}_2(\beta) = (1,1)$. It is easy to see that
$\beta\ \mathfrak{G}_w^{(\beta)}(1,1,1,1) = DET \begin{vmatrix} \mathfrak{N}_4(\beta) & \mathfrak{N}_3(\beta) \\ \mathfrak{N}_3(\beta) & \mathfrak{N}_2(\beta) \end{vmatrix}$. On the other hand,
$DET \begin{vmatrix} P_4(\beta) & P_3(\beta) \\ P_3(\beta) & P_2(\beta) \end{vmatrix} = (3,6,4,1) = \underline{(3+3\beta+\beta^2)}\ (1+\beta)$. It is more involved to check that

$$q^5(1+\beta)\ \mathfrak{G}_w^{(\beta)}(1,q,q^2,q^3) = DET \begin{vmatrix} S_4(q;\beta) & S_3(q;\beta) \\ S_3(q;q\beta) & S_2(q;q\beta) \end{vmatrix}.$$

(2) Let us illustrate Theorem 5.5 by a few examples. For the sake of simplicity, we consider the case $\beta = 0$, i.e. the case of Schubert polynomials. In this case $P_n(q;\beta=0) = C_n(q)$ is equal to the Carlitz– Riordan q-analogue of Catalan numbers. We are reminded that the q-Catalan–Hankel polynomials are defined as follows

$$C_n^{(k)}(q) = q^{k(1-k)(4k-1)/6}\ DET|C_{n+k-i-j}(q)|_{1 \le i,j \le n}.$$

In the case $\beta = 0$ the Theorem 5.5 states that if $\mathbf{n} = (n_1,\ldots,n_p) \in \mathbb{N}^p$ and the permutation $w_{(\mathbf{n})} \in \mathbb{S}_n$ is defined by the use of (5.10), then

$$\mathfrak{S}_{w^{(\mathbf{n})}}(1,q,q^2,\ldots) = q^{\sum \binom{n_i}{3}}\ C_{n_1+n_2}^{(n_1)}(q) \times C_{n_1+n_2+n_3}^{(n_1+n_2)}(q) \times C_n^{(n-n_p)}(q).$$

Now let us consider a few examples for $n = 6$.

- $\mathbf{n} = (1,5), \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^{10}\ C_6^{(1)}(q) = C_5(q)$.
- $\mathbf{n} = (2,4), \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^4\ C_6^{(2)}(q) = DET \begin{vmatrix} C_6(q) & C_5(q) \\ C_5(q) & C_4(q) \end{vmatrix}$.

  Note that $\mathfrak{S}_{w^{(2,4)}}(1,q,\ldots) = \mathfrak{S}_{w^{(1,1,4)}}(1,q,\ldots)$.
- $\mathbf{n} = (2,2,2) \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = C_4^{(2)}(q)\ C_6^{(4)}(q)$.
- $\mathbf{n} = (1,1,4) \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^4\ C_2^{(1)}(q)\ C_4^{(2)}(q) = q^4\ C_4^{(2)}(q)$,

  the last equality follows from that $C_{k+1}^{(k)}(q) = 1$ for all $k \ge 1$.
- $\mathbf{n} = (1,2,3) \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q\ C_3^{(1)}(q)\ C_6^{(3)}(q)$. On the other hand,
- $\mathbf{n} = (3,2,1) \implies \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q\ C_5^{(3)}(q)\ C_6^{(5)}(q) = q\ C_5^{(3)}(q) = q(1,1,1,1)$.

  Note that $C_{k+2}^{(k)}(q) = \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$.

Exercises 5.4.

Let $1 \le k \le m \le n$ be integers, $n \ge 2k+1$. Consider permutation

$$w = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & & n \\ m & m-1 & \ldots & m-k+1 & n & \ldots & \ldots & 1 \end{pmatrix} \in \mathbb{S}_n.$$

Show that

$$\mathfrak{S}_w(1,q,\ldots) = q^{n(D(w))}\ C_{n-m+k}^{(m)}(q),$$

where for any permutation w, $n(D(w)) = \sum \binom{d_i(w)}{2}$ and $d_i(w)$ denotes the number of boxes in the i-th column of the (Rothe) diagram $D(w)$ of the permutation w, see [61]. p.8.

■

(**C**) **A determinantal formula for the Grothendieck polynomials** $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}$

Define polynomials

$$\Phi_n^{(m)}(X_n) = \sum_{a=m}^{n} e_a(X_n)\ \beta^{a-m},$$

$$A_{i,j}(X_{n+k-1}) = \frac{1}{(i-j)!}\Big(\frac{\partial}{\partial \beta}\Big)^{j-1}\ \Phi_{k+n-i}^{(n+1-i)}(X_{k+n-i}),\quad if\quad 1 \le i \le j \le n,$$

and

$$A_{i,j}(X_{k+n-1}) = \sum_{a=0}^{i-j-1} e_{n-i-a}(X_{n+k-i})\ \binom{i-j-1}{a},\quad if\quad 1 \le j < i \le n.$$

Theorem 5.7.

$$DET|A_{i,j}|_{1 \le i,j \le n} = \mathfrak{G}_{\pi_{k+n}^{(k)}}^{(\beta)}(X_{k+n-1}).$$

Comments 5.5.

(**a**) One can compute the Grothendieck polynomials for yet another interesting family of permutations. namely, *grassmannian* permutations

$$\sigma_k^{(n)} = \begin{pmatrix} 1 & 2 & \ldots & k-1 & k & k+1 & k+2 & \ldots & n+k \\ 1 & 2 & \ldots & k-1 & n+k & k & k+1\ldots & & n+k-1 \end{pmatrix} =$$

$$s_k s_{k+1} \ldots s_{n+k-1} \in \mathbb{S}_{n+k}.$$

Then

$$\mathfrak{G}_{\sigma_k{}^{(n)}}^{(\beta)}(x_1, \ldots, x_{n+k}) = \sum_{j=0}^{k-1} s_{(n,1^j)}(X_k)\ \beta^j,$$

where $s_{(n,1^j)}(X_k)$ denotes the Schur polynomial corresponding to the hook shape partition $(n,1^j)$ and the set of variables $X_k := (x_1, \ldots, x_k)$. In particular,

$$\mathfrak{G}_{\sigma_k{}^{(n)}}^{(\beta)}(x_j = 1, \forall j) = \binom{n+k-1}{k}\Big(\sum_{j=0}^{k-1} \frac{k}{n+j}\binom{k-1}{j}\ \beta^j\Big) = \sum_{j=0}^{k-1}\binom{n+j-1}{j}\ (1+\beta)^j.$$

(**b**) **Grothendieck polynomials for grassmannian permutations**

In the case of a grassmannian permutation $w := \sigma_\lambda \in \mathbb{S}_\infty$ of the shape $\lambda = (\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n)$ where n is a unique descent of w, one can prove the following formulas for the β-Grothendieck polynomial

$$\mathfrak{G}_{\sigma_\lambda}^{(\beta)}(X_n) = \frac{DET\ |x_i^{\lambda_j+n-j}\ (1+\beta\ x_i)^{j-1}|_{1 \le i,j \le n}}{\prod_{1 \le i < j \le n}(x_i - x_j)} = \tag{5.11}$$

$$DET|h_{\lambda_j+i,j}^{(\beta)}(X_{[i,n]})|_{1 \le i,j \le n} = DET|h_{\lambda_j+i,j}^{(\beta)}(X_n)|_{1 \le i,j \le n}, \tag{5.12}$$

where $X_{[i,n]} = (x_i, x_{i+1}, \ldots, x_n)$, and for any set of variables X,

$$h_{n,k}^{(\beta)}(X) = \sum_{a=0}^{k-1}\binom{k-1}{a}\ h_{n-k+a}(X)\ \beta^a,$$

and $h_k(X)$ denotes the complete symmetric polynomial of degree k in the variables from the set X.

A proof is a straightforward adaptation of the proof of special case $\beta=0$ (the case of Schur polynomials) given by I. Macdonald [61], Section 2, (2.10) and Section 4, (4.8).

Indeed, consider β-divided difference operators $\pi_j^{(\beta)}$, $j=1,\ldots,n-1$, and $\pi_w^{(\beta)}$, $w\in\mathbb{S}_n$, introduced in [27]. For example,
$\pi_j^{(\beta)}(f)=\frac{1}{x_j-x_{j+1}}\Big((1+\beta x_{j+1})f(X_n)-(1+\beta x_j)f(s_j(X_n))\Big)$.

Now let $w_0:=w_0^{(n)}$ be the longest element in the symmetric group $\mathbb{S}_n$. The same proves of the statements (2.10), (2.16) from [61] show that

$$\pi_{w_0}^{(\beta)}=a_\delta^{-1}w_0\Big(\sum_{\sigma\in\mathbb{S}_n}(-1)^{\ell(\sigma)}\prod_{j=1}^{n-1}(1+\beta x_j)^{n-j}\ \sigma\Big),$$

where $a_\delta=\prod_{1\le i<j\le n}(x_i-x_j)$.

On the other hand, the same arguments as in the proof of statement (4.8) from [61] show that

$$\mathfrak{G}_{\sigma_\lambda}^{(\beta)}(X_n)=\pi_{w^{(0)}}^{(\beta)}(x^{\lambda+\delta_n}).$$

Application of the formula for operator $\pi_{w_n^{(0)}}^{(\beta)}$ displayed above to the monomial $x^{\lambda+\delta_n}$ finishes the proof of the first equality in (5.11). The statement that the right hand side of the equality (5.12) coincides with determinants displayed in the identity (5.12) can be checked by means of simple transformations..

■

Problems 5.1.

(1) *Give a bijective prove of Theorem 3.3, i.e. construct a bijection between*

• *the set of k-tuple of mutually non-crossing Schröder paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$ of lengths $(n,n-1,\ldots,n-k+1)$ correspondingly, and*

• *the set of pairs $(\mathfrak{m},\mathcal{T})$, where $\mathcal{T}$ is a k-dissection of a convex $(n+k+1)$-gon, and $\mathfrak{m}$ is a upper triangle $(0,1)$-matrix of size $(k-1)\times(k-1)$,*

which is compatible with natural statistics on the both sets.

(2) *Let $w\in\mathbb{S}_n$ be a permutation, and $CS(w)$ be the set of compatible sequences corresponding to w, see e.g. [8].*

Define statistics $c(\bullet)$ on the set $CS(w)$ such that

$$\mathfrak{G}_w^{(\beta-1)}(x_1=1,x_2=1,\ldots)=\sum_{a\in CS(w)}\beta^{c(a)}.$$

(3) *Let w be a vexillary permutation.*

Find a determinantal formula for the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X)$.

(4) *Let w be a permutation*

Find a geometric interpretation of coefficients of the polynomials $\mathfrak{S}_w^{(\beta)}(x_i=1)$ and $\mathfrak{S}_w^{(\beta)}(x_i=q,x_j=1,\forall j\ne i)$.

For example, let $w\in\mathbb{S}_n$ be an involution, i.e. $w^2=1$, and $w'\in\mathbb{S}_{n+1}$ be the image of w under the natural embedding $\mathbb{S}_n\hookrightarrow\mathbb{S}_{n+1}$ given by $w\in\mathbb{S}_n\longrightarrow(w,n+1)\in\mathbb{S}_{n+1}$.

It is well-known, see e.g. [53], [98], that the multiplicity $m_{e,w}$ of the 0-dimensional Schubert cell $\{pt\}=Y_{w_0^{(n+1)}}$ in the Schubert variety $\overline{Y}_{w'}$ is equal to the specialization $\mathfrak{S}_w(x_i=1)$ of the Schubert polynomial $\mathfrak{S}_w(X_n)$. Therefore one can consider the polynomial $\mathfrak{S}_w^{(\beta)}(x_i=1)$ as a β-deformation of the multiplicity $m_{e,w}$.

Question What is a geometrical meaning of the coefficients of the polynomial $\mathfrak{S}_w^{(\beta)}(x_i=1)\in\mathbb{N}[\beta]$?

Conjecture 5.1. *The polynomial $\mathfrak{S}_w^{(\beta)}(x_i=1)$ is a unimodal polynomial for any permutation w.*

5.2.4 Specialization of Schubert polynomials

Let $n,\ k,\ r$ be positive integers and $p,\ b$ be non-negative integers such that $r \le p+1$. It is well-known [61] that in this case there exists a unique vexillary permutation $\varpi := \varpi_{\lambda,\phi} \in \mathbb{S}_{\infty}$ which has the shape $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ and the flag $\phi = (\phi_1, \ldots, \phi_{n+1})$, where

$$\lambda_i = (n-i+1)\ p+b, \qquad \phi_i = k+1+r\ (i-1), \quad 1 \le i \le n+1-\delta_{b,0}.$$

According to a theorem by M.Wachs [95], the Schubert polynomial $\mathfrak{S}_{\varpi}(X)$ admits the following determinantal representation

$$\mathfrak{S}_{\varpi}(X) = DET\Bigg(h_{\lambda_i-i+j}(X_{\phi_i}) \Bigg)_{1 \le i,j \le n+1}.$$

Therefore we have $\mathfrak{S}_{\varpi}(1) := \mathfrak{S}_{\varpi}(x_1 = 1, x_2 = 1, \ldots) =$

$$DET\left(\binom{(n-i+1)p+b-i+j+k+(i-1)r}{k+(i-1)r} \right)_{1 \le i,j \le n+1}.$$

We denote the above determinant by $D(n,k,r,b,p)$.

Theorem 5.8. $D(n,k,r,b,p) =$

$$\prod_{(i,j) \in \mathcal{A}_{n,k,r}} \frac{i+b+jp}{i} \prod_{(i,j) \in \mathcal{B}_{n,k,r}} \frac{(k-i+1)(p+1)+(i+j-1)r+r(b+np)}{k-i+1+(i+j-1)r},$$

where

$$\mathcal{A}_{n,k,r} = \Big\{ (i,j) \in \mathbb{Z}^2_{\ge 0} \quad | \quad j \le n, \quad j < i \le k+(r-1)(n-j) \Big\},$$

$$\mathcal{B}_{n,k,r} = \Big\{ (i,j) \in \mathbb{Z}^2_{\ge 1} \quad | \quad i+j \le n+1, \quad i \ne k+1+r\ s, \quad s \in \mathbb{Z}_{\ge 0} \Big\}.$$

■

It is convenient to re-wright the above formula for $D(n,k,r,b,p)$ in the following form

$$D(n,k,r,b,p) =$$

$$\prod_{j=1}^{n+1} \frac{\Big((n-j+1)p+b+k+(j-1)(r-1)\Big)!\ (n-j+1)!}{\Big(k+(j-1)r\Big)!\ \Big((n-j+1)(p+1)+b\Big)!} \times$$

$$\prod_{1 \le i \le j \le n} \Big((k-i+1)(p+1)+jr+(np+b)r\Big).$$

Corollary 5.2. *(***Some special cases***)*

(**A**) **The case** $r=1$

We consider below some special cases of Theorem 5.7 in the case $r=1$. To simplify notation, we set $D(n,k,b,p) := D(n,k,r=1,b,p)$. Then we can rewrite the above formula for $D(n,k,r,b,p)$ as follows $D(n,k,b,p) =$

$$\prod_{j=1}^{n+1} \frac{\Big((n+k-j+1)(p+1)+b\Big)!\ \Big((n-j+1)p+b+k\Big)!\ (j-1)!}{\Big((n-j+1)(p+1)+b\Big)!\ \Big((k+n-j+1)p+b+k\Big)!\ (k+j-1)!}.$$

(1) *If* $k \le n+1$, *then* $D(n,k,b,p) =$

$$\prod_{j=1}^{k} \binom{(n+k+1-j)(p+1)+b}{n-j+1} \binom{(k-j)p+b+k}{j} \frac{j!\ (k-j)!\ (n-j+1)!}{(n+k-j+1)!}.$$

In particular,

- *If* $k=1$, *then*

$$D(n,1,b,p) = \frac{1+b}{1+b+(n+1)p} \binom{(p+1)(n+1)+b}{n+1} := F_{n+1}^{(p+1)}(b),$$

where $F_n^p(b) := \frac{1+b}{1+b+(p-1)n} \binom{pn+b}{n}$ *denotes the generalized Fuss-Catalan number.*

- *if* $k=2$, *then*

$$D(n,2,b,p) = \frac{(2+b)(2+b+p)}{(1+b)(2+b+(n+1)p)(2+b+(n+2)p)} F_{n+1}^{(p+1)}(b)\ F_{n+2}^{(p+1)}(b).$$

In particular,

$$D(n,2,0,1) = \frac{6}{(n+3)(n+4)} Cat_{n+1}\ Cat_{n+2}.$$

See [87], A005700 for several combinatorial interpretations of these numbers.

(2) *(R.A. Proctor [82]) Consider the Young diagram*

$$\lambda := \lambda_{n,p,b} = \{(i,j) \in \mathbb{Z}_{\ge 1} \times \mathbb{Z}_{\ge 1} \mid 1 \le i \le n+1, 1 \le j \le (n+1-i)p+b\}.$$

For each box $(i,j) \in \lambda$ *define the numbers* $c(i,j) := n+1-i+j$, *and*

$$l_{(i,j)}(k) = \begin{cases} \frac{k+c(p,j)}{c(i,j)}, & \text{if} \quad j \le (n+1-i)(p-1)+b, \\ \frac{(p+1)k+c(i,j)}{c(i,j)}, & \text{if} \quad (n+1-i)(p-1) < j-b \le (n+1-i)p. \end{cases}$$

Then

$$D(n,k,b,p) = \prod_{(i,j)\in\lambda} l_{(i,j)}(k). \tag{5.13}$$

Therefore, $D(n,k,b,p)$ *is a polynomial in* k *with rational coefficients.*

(3) *If* $p=0$, *then*

$$D(n,k,b,0) = dim\ V_{(n+1)^k}^{\mathfrak{gl}(b+k)} = \prod_{j=1}^{n+k} (\frac{j+b}{j})^{min(j,n+k+1-j)},$$

where for any partition μ, $\ell(\mu) \le m$, $V_{\mu}^{\mathfrak{gl}(m)}$ *denotes the irreducible* $\mathfrak{gl}(m)$*-module with the highest weight* μ. In particular,

- $$D(n,2,b,0) = \frac{1}{n+2+b}\binom{n+2+b}{b}\binom{n+2+b}{b+1}$$

is equal to the Narayana number $N(n+b+2,b)$;

- $$D(1,k,b,0) = \frac{(b+k)!\ (b+k+1)!}{k!b!(k+1)!(b+1)!} := N(b+k+1,k),$$

and therefore the number $D(1,k,b,0)$ counts the number of pairs of non-crossing lattice paths inside a rectangular of size $(b+1)\times(k+1)$, which go from the point $(1,0)$ (resp. from that $(0,1)$) to the point $(b+1,k)$ (resp. to that $(b,k+1)$), consisting of steps $U=(1,0)$ and $R=(0,1)$, see [87], A001263, for some list of combinatorial interpretations of the Narayana numbers.

(4) *If $p=b=1$, then*

$$D(n,k,1,1)=C_{n+k+1}^{(k)}:=\prod_{1\le i\le j\le n+1}\frac{2k+i+j}{i+j}.$$

(5) *(R.A. Proctor [80],[81]) If $p=1$ and b is odd integer, then $D(n,k,b,1)$ is equal to the dimension of the irreducible representation of the symplectic Lie algebra $Sp(b+2n+1)$ with the highest wright $k\omega_{n+1}$.*

(6) *If $p=1$ and $b=0$, then*

$$D(n,k,1,0)=D(n-1,k,1,1)=\prod_{1\le i\le j\le n}\frac{2k+i+j}{i+j}=C_{n+k}^{(k)},$$

see subsection Grothendieck and Narayana polynomials.

(7) *(Cf [29]) Let ϖ_λ be a unique dominant permutation of shape $\lambda:=\lambda_{n,p,b}$ and $\ell:=\ell_{n,p,b}=\frac{1}{2}(n+1)(np+2b)$ be its length. Then*

$$\sum_{\mathbf{a}\in R(\varpi_\lambda)}\prod_{i=1}^{\ell}(x+a_i)=\ell!\ B(n,x,p,b).$$

Here for any permutation w of length l, we denote by $R(w)$ the set $\{\mathbf{a}=(a_1,\ldots,a_l)\}$ of all reduced decompositions of w.

Exercises 5.5.

Show that

- $$DET\left|F_{n+i+j-2}^{(2)}(b)\right|_{1\le i,j\le k}=\prod_{j=1}^{k}F_{n+k-1}^{(2)}(b)\ \frac{\binom{k+1}{2}!}{\prod_{\substack{1\le i\le k-1\\1\le j\le k}}(n+i+j)}.$$

- $$D(n,k,b,1)=\prod_{j=1}^{k}F_{n+j}^{(2)}\ \frac{\prod_{1\le i\le j\le k}(b+i+j-1)}{\prod_{\substack{1\le i\le k-1\\1\le j\le k}}(n+b+i+j+1)}.$$

Clearly that if $b=0$, then $F_n^{(2)}(0)=C_n$, and $D(n,k,0,1)$ is equal to the Catalan–Hankel determinant $C_n^{(k)}$.
Finally we recall that the generalized Fuss-Catalan number $F_{n+1}^{(p+1)}(b)$ counts the number of lattice paths from $(0,0)$ to $(b+np,n)$ that do not go above the line $x=py$, see e.g. [55].

Comments 5.6.

It is well-known, see e.g. [82], or [88], vol.2, Exercise **7.101.b**, that the number $D(n,k,b,p)$ is equal to the total number $pp^{\lambda_{n,p,b}}(k)$ of plane partitions [33] bounded by k and contained in the shape $\lambda_{n,b,p}$.

More generally, see e.g. [29], for any partition λ denote by $w_\lambda\in\mathfrak{S}_\infty$ a unique *dominant* permutation of shape λ, that is a unique permutation with the code $c(w)=\lambda$. Now for any

[33] Let λ be a partition. A plane (ordinary) partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly decreasing.
A reverse plane partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly increasing.

non-negative integer k consider the so-called *shifted dominant* permutation $w_{\lambda}^{(k)}$ which has the shape λ and the flag $\phi = (\phi_i = k+i-1, i = 1, \ldots, \ell(\lambda))$. Then

$$\mathfrak{S}_{w_{\lambda}^{(k)}}(1) = pp^{\lambda}(\le k),$$

where $pp^{\lambda}(\le k)$ denotes the number of all plane partitions bounded by k and contained in λ. Moreover,

$$\sum_{\pi \in PP^{\lambda}(\le k)} q^{|\pi|} = q^{n(\lambda)} \ \mathfrak{S}_{w_{\lambda}^{(k)}}(1, q^{-1}, q^{-2}, \ldots),$$

where $PP^{\lambda}(\le k)$ denotes the set of all plane partitions bounded by k and contained in λ.

Exercises 5.6.

(1) *Show that*

$$\lim_{k \to \infty} \ \mathfrak{S}_{w_{\lambda}^{(k)}}(1, q, q^2, \ldots) = \frac{q^{n(\lambda)}}{H_{\lambda}(q)},$$

where $H_{\lambda}(q) = \prod_{x \in \lambda} \ (1 - q^{h(x)})$ *denotes the hook polynomial corresponding to a given partition* λ.

(2) *Let* $\lambda = ((n+\ell)^{\ell}, \ell^{n})$ *be a fat hook.*
Show that

$$\lim_{k \to \infty} \ q^{n(\lambda)} \ \mathfrak{S}_{w_{\lambda}^{(k)}}(1, q^{-1}, q^{-2}, \ldots) = q^{s(\ell,n)} \ \ \frac{K_{\lambda}(q)}{M_{\ell}(2n+2\ell-1; q)},$$

where $a(\ell, n)$ *is a certain integer we don't need to specify in what follows;*

$$M_{\ell}(N; q) = \prod_{j=1}^{N} \left(\frac{1}{1-q^j} \right)^{min(j, N+1-j, \ell)}$$

denotes the MacMahon generating function for the number of plane partitions fit inside the box $N \times N \times \ell$ *;* $K_{\lambda}(q)$ *is a polynomial in* q *such that* $K_{\lambda}(0) = 1$.

(a) *Show that*

$$(1-q)^{|\lambda|} \ \frac{K_{\lambda}(q)}{M_{\ell}(2n+2\ell-1; q)} \ \Bigg|_{q=1} = \frac{1}{\prod_{x \in \lambda} \ h(x)}.$$

(b) *Show that*

- $K_{\lambda}(q) \in \mathbb{N}[q]$ *and* $K_{\lambda}(1) = M(n, n, \ell)$,

where $M(a,b,c)$ *denotes the number of plane partitions fit inside the box* $a \times b \times c$. *It is well-known, see e.g.[62], p. 81, that*

$$M(a,b,c) = \prod_{\substack{1 \le i \le a, \\ 1 \le j \le b, \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2} = \prod_{i=1}^{c} \frac{(a+b+i-1)! \ (i-1)!}{(a+i-1)! \ (b+1-1)!} = dim \ V_{(a^c)}^{\mathfrak{gl}_{b+c}}.$$

- $K_{\lambda}(q) = \sum_{\pi \in B_{n,n,\ell}} q^{wt_{\ell}(\pi)},$

where the sum runs over the set of plane partitions $\pi = (\pi_{ij})_{1 \le i,j \le n}$ *fit inside the box* $B_{n,n,\ell} := n \times n \times \ell$, *and*

$$wt_{\ell}(\pi) = \sum_{i,j} \ \pi_{ij} + \ell \sum_{i} \ \pi_{ii}.$$

(c) *Assume as before that* $\lambda := ((n+\ell)^{\ell}, \ell^{n})$.
Show that

$$\lim_{n \to \infty} K_{\lambda}(q) = M_{\ell}(q) \sum_{\substack{\mu \\ \ell(\mu) \le \ell}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod_{x \in \mu}(1-q^{h(x)})} \right)^2,$$

where the sum runs over the set of partitions μ *with the number of parts at most* ℓ, *and* $n(\mu) = \sum_i (i-1)\, \mu_i$;

$$M_{\ell}(q) := \prod_{j \ge 1} (1-q^j)^{min(j,\ell)}.$$

Therefore the generating function $PP^{(\ell,0)}(q) := \sum_{\pi \in PP^{(\ell,0)}} q^{|\pi|}$ *is equal to*

$$\sum_{\substack{\mu \\ \ell(\mu) \le \ell}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod_{x \in \mu}(1-q^{h(x)})} \right)^2,$$

where $PP^{(\ell,k)} := \{\pi = (\pi_{ij})_{i,j \ge 1} \mid \pi_{ij} \ge 0,\ \ \pi_{\ell+1,\ell+1} \le k\}, \quad |\pi| = \sum_{i,j} \pi_{ij}$.
(d) *Show that*

$$PP^{(\ell,0)}(q) = \frac{1}{M_{\ell}(q)^2} \sum_{\substack{\mu, \\ \ell(\mu) \le \ell}} (-q)^{|\mu|}\, q^{n(\mu)+n(\mu')} \left(\dim_q V_{\mu}^{\mathfrak{gl}(\ell)}\right)^2, \tag{5.14}$$

where μ' *denotes the conjugate partition of* μ, *therefore* $n(\mu') = \sum_{i \ge 1} \binom{\mu_i}{2}$.
The formula (5.14) *is the special case* $n = m$ *of Theorem 1.2, [72]. In particular, if* $\ell = 1$ *then one come to following identity*

$$\frac{1}{(q;q)_{\infty}^2} \sum_{k \ge 0} (-1)^k\, q^{\binom{k+1}{2}} = \sum_{k \ge 0} q^k \left(\frac{1}{(q;q)_k} \right)^2.$$

(e) *Let* $k \ge 0, \ell \ge 1$ *be integers.*
Show that the (fermionic) generating function for the number of plane partitions $\pi = (\pi_{ij}) \in PP^{(\ell,k)}$ *is equal to*

$$\sum_{\pi \in PP^{(\ell,k)}} q^{|\pi|} = \sum_{\substack{\mu \\ \mu_{\ell+1} \le k}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod_{x \in \mu}(1-q^{h(x)})} \right)^2.$$

■

(**B**) **The case** $k = 0$
(1) $D(n,0,1,p,b) = 1$ for all nonnegative n, p, b.
(2) $D(n,0,2,2,2) = VSASM(n)$, i.e. the number of alternating sign $2n+1 \times 2n+1$ matrices symmetric about the vertical axis, see e.g. [87], $A005156$.
(3) $D(n,0,2,1,2) = CSTCPP(n)$, i.e. the number of cyclically symmetric transpose complement plane partitions, see e.g. [87], $A051255$.

Theorem 5.9. *Let* $\varpi_{n,k,p}$ *be a unique vexillary permutation of the shape* $\lambda_{n.p} := (n, n-1, \ldots, 2, 1)p$ *and flag* $\phi_{n,k} := (k+1, k+2, \ldots, k+n-1, k+n)$. *Then*

- $$\mathfrak{G}_{\varpi_{n,1,p}}^{(\beta-1)}(1) = \sum_{j=1}^{n+1} \frac{1}{n+1} \binom{n+1}{j} \binom{(n+1)p}{j-1} \beta^{j-1}.$$
- *If* $k \ge 2$, *then* $G_{n,k,p}(\beta) := \mathfrak{G}_{\varpi_{n,k,p}}^{(\beta-1)}(1)$ *is a polynomial of degree* nk *in* β, *and* $Coeff_{[\beta^{nk}]}(G_{n,k,p}(\beta)) = D(n,k,1,p-1,0)$.

The polynomial

$$\sum_{j=1}^{n} \frac{1}{n} \binom{n}{j} \binom{pn}{j-1} t^{j-1} := \mathfrak{F}\mathfrak{N}_n(t)$$

is known as the Fuss-Narayana polynomial and can be considered as a t-deformation of the Fuss-Catalan number $FC_n^p(0)$.

Recall that the number $\frac{1}{n}\binom{n}{j}\binom{pn}{j-1}$ counts paths from $(0,0)$ to $(np,0)$ in the first quadrant, consisting of steps $U=(1,1)$ and $D=(1,-p)$ and have j peaks (i.e. UD's), cf. [87], A108767.

For example, take $n=3, k=2, p=3, r=1, b=0$. Then
$\varpi_{3,2,3} = [1,2,12,9,6,3,4,5,7,8,10,11] \in \mathbb{S}_{12}$, and $G_{3,2,3}(\beta) =$
$(1,18,171,747,1767,1995,1001)$. Therefore, $G_{3,2,3}(1) = 5700 = D(3,2,3,0)$ and
$Coeff_{[\beta^6]}(G_{3,2,3}(\beta)) = 1001 = D(3,2,2,0)$.

Proposition 5.5. *([73]) The value of the Fuss–Catalan polynomial at* $t=2$, *that is the number*

$$\sum_{j=1}^{n} \frac{1}{n} \binom{n}{j} \binom{pn}{j-1} 2^{j-1}$$

is equal to the number of hyperplactic classes of p*-parking functions of length* n*, see [73] for definition of* p*-parking functions, its properties and connections with some combinatorial Hopf algebras.*

Therefore, the value of the Grothendieck polynomial $\mathfrak{G}_{\varpi_{n,1,p}}^{(\beta=1)}(1)$ at $\beta=1$ and $x_i=1$, $\forall i$, is equal to the number of p-parking functions of length $n+1$. It is an open problem to find combinatorial interpretations of the polynomials $\mathfrak{G}_{\varpi_{n,k,p}}^{(\beta)}(1)$ in the case $k \geq 2$. Note finally, that in the case $p=2$, $k=1$ the values of the Fuss–Catalan polynomials at $t=2$ one can find in [87], $A034015$. ■

Comments 5.7. $(\Longrightarrow)$ **The case r=0**

It follows from Theorem 5.7 that in the case $r=0$ and $k \geq n$, one has

$$D(n,k,0,p,b) = dim\ V_{\lambda_{n,p,b}}^{\mathfrak{gl}(k+1)} = (1+p)^{\binom{n+1}{2}} \prod_{j=1}^{n+1} \frac{\binom{(n-j+1)p+b+k-j+1}{k-j+1}}{\binom{(n-j+1)(p+1)+b}{n-j+1}}.$$

Now consider the conjugate $\nu := \nu_{n,p,b} := ((n+1)^b, n^p, (n-1)^p, \ldots, 1^p)$ of the partition $\lambda_{n,p,b}$, and a rectangular shape partition $\psi = (\underbrace{k,\ldots,k}_{np+b})$. If $k \geq np+b$, then there exists a unique grassmannian permutation $\sigma := \sigma_{n,k,p,b}$ of the shape ν and the flag ψ, [61]. It is easy to see from the above formula for $D(n,k,0,p,b)$, that

$$\mathfrak{S}_{\sigma_{n,k,p,b}}(1) = dim\ V_{\nu_{n,p,b}}^{\mathfrak{gl}(k-1)} =$$

$$(1+p)^{\binom{n}{2}} \binom{k+n-1}{b} \prod_{j=1}^{n} \frac{(p+1)(n-j+1)}{(n-j+1)(p+1)+b} \prod_{j=1}^{n} \frac{\binom{k+j-2}{(n-j+1)p+b}}{\binom{(n-j+1)(p+1)+b-1}{n-j}}.$$

After the substitution $k := np+b+1$ in the above formula we will have

$$\mathfrak{S}_{\sigma_{n,np+b+1,p,b}}(1) = (1+p)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\binom{np+b+j-1}{(n-j+1)p}}{\binom{j(p+1)-1}{j-1}}.$$

In the case $b=0$ some simplifications are happened, namely

$$\mathfrak{S}_{\sigma_{n,k,p,0}}(1)=(1+p)^{\binom{n}{2}}\prod_{j=1}^{n}\frac{\binom{k+j-2}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}}.$$

Finally we observe that if $k=np+1$, then

$$\prod_{j=1}^{n}\frac{\binom{np+j-1}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}}=\prod_{j=2}^{n}\frac{\binom{np+j-1}{(p+1)(j-1)}}{\binom{j(p+1)-1}{j-1}}=\prod_{j=1}^{n-1}\frac{j!\ (n(p+1)-j-1)!}{((n-j)(p+1))!\ ((n-j)(p+1)-1)!}:=A_n^{(p)},$$

where the numbers $A_n^{(p)}$ are <u>integers</u> that generalize the numbers of **alternating sign matrices** (ASM) of size $n\times n$, recovered in the case $p=2$, see [74], [19] for details.

Examples 5.1.

(1) Let us consider polynomials $\mathfrak{G}_n(\beta):=\mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1)$.

• If $n=2$, then $\sigma_{2,4,2,0}=235614\in\mathbb{S}_6$, and $\mathfrak{G}_2(\beta)=(1,2,\mathbf{3}):=1+2\beta+\mathbf{3}\beta^2$.
Moreover, $\mathfrak{R}_{\sigma_{2,4,2,0}}(q;\beta)=(1,\mathbf{2})_\beta+\mathbf{3}\ q\beta^2$.

• If $n=3$, then $\sigma_{3,6,2,0}=235689147\in\mathbb{S}_9$, and $\mathfrak{G}_3(\beta)=(1,6,21,36,51,48,\mathbf{26})$.
Moreover, $\mathfrak{R}_{\sigma_{3,6,2,0}}(q;\beta)=(1,6,11,16,\mathbf{11})_\beta+q\ \beta^2(10,20,35,34)_\beta+q^2\beta^4(5,14,\mathbf{26})_\beta$;
$\mathfrak{R}_{\sigma_{3,6,2,0}}(q;1)=(45,99,45)_q$.

• If $n=4$, then $\sigma_{4,8,2,0}=[2,3,5,6,8,9,11,12,1,4,7,10]\in\mathbb{S}_{12}$, and $\mathfrak{G}_4(\beta)=$
$(1,12,78,308,903,2016,3528,4944,5886,5696,4320,2280,\mathbf{646})$.
Moreover, $\mathfrak{R}_{\sigma_{4,8,2,0}}(q;\beta)=(1,12,57,182,392,602,763,730,493,\mathbf{170})_\beta+$
$q\beta^2(21,126,476,1190,1925,2626,2713,2026,804)_\beta+$
$q^2\beta^4(35,224,833,1534,2446,2974,2607,1254)_\beta+q^3\beta^6(7,54,234,526,909,1026,\mathbf{646})_\beta$;
$\mathfrak{R}_{\sigma_{4,8,2,0}}(q;1)=(3402,11907,11907,3402)_q=1701\ (2,7,7,2)_q$.

• If $n=5$, then $\sigma_{5,10,2}=[2,3,5,6,8,9,11,12,14,15,1,4,7,10,13]\in\mathbb{S}_{15}$, and $\mathfrak{G}_5(\beta)=$
$(1,20,210,1420,7085,27636,87430,230240,516375,997790,1676587,2466840,$
$3204065,3695650,3778095,3371612,2569795,1610910,782175,262200,\mathbf{45885})$.
Moreover, $\mathfrak{R}_{\sigma_{5,10,2,0}}(q;\beta)=(1,20,174,988,4025,12516,31402,64760,111510,162170,$
$202957,220200,202493,153106,89355,35972,\mathbf{7429})_\beta+$
$q\beta^2(36,432,2934,13608,45990,123516,269703,487908,738927,956430,1076265,$
$1028808,813177,499374,213597,47538)_\beta+$
$q^2\beta^4(126,1512,9954,40860,127359,314172,627831,1029726,1421253,1711728,$
$1753893,1492974,991809,461322,112860)_\beta+$
$q^3\beta^6(84,1104,7794,33408,105840,255492,486324,753984,1019538,1169520,1112340,$
$825930,428895,117990)_\beta+$
$q^4\beta^8(9,132,1032,4992,17730,48024,102132,173772,244620,276120,240420,144210,$
$\mathbf{45885})_\beta$.
$\mathfrak{R}_{\sigma_{5,10,2,0}}(q;1)=(1299078,6318243,10097379,6318243,1299078)_q=$
$59049(22,107,171,107,22)_q$.

$\Big[$ We are reminded that over the paper we have used the notation $(a_0,a_1,\ldots,a_r)_\beta:=\sum_{j=0}^{r}\ a_j\ \beta^j,\ etc\Big]$.

One can show that $deg_{[\beta]}\mathfrak{G}_n(\beta)=n(n-1)$, $deg_{[q]}\mathfrak{R}_{\sigma_{n,2n,2,0}}(q,1)=n-1$, and looking on the numbers $3,26,646,45885$ we made

Conjecture 5.2. Let $a(n):=Coeff[\beta^{n(n-1)}]\Big(\mathfrak{G}_n(\beta)\Big)$. <u>Then</u>

$$a(n)=VSASM(n)=OSASM(n)=\prod_{j=1}^{n-1}\frac{(3j+2)(6j+3)!\ (2j+1)!}{(4j+2)!\ (4j+3)!},$$

where

$VSASM(n)$ is the number of alternating sign $2n+1 \times 2n+1$ matrices symmetric about the vertical axis;

$OSASM(n)$ is the number of $2n \times 2n$ off-diagonal symmetric alternating sign matrices.

See [87], A005156, [74] and references therein, for details.

Conjecture 5.3.

Polynomial $\mathfrak{R}_{\sigma_{n,2n,2,0}}(q;1)$ *is symmetric and* $\mathfrak{R}_{\sigma_{n,2n,2,0}}(0;1) = A20342(2n-1)$, *see [87].*

(2) Let us consider polynomials $\mathfrak{F}_n(\beta) := \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1)$.

- If $n=1$, then $\sigma_{1,3,2,0} = 1342 \in \mathbb{S}_4$, and $\mathfrak{F}_2(\beta) = (1,\mathbf{2}) := 1 + \mathbf{2}\beta$.
- If $n=2$, then $\sigma_{2,5,2,0} = 1346725 \in \mathbb{S}_7$, and $\mathfrak{F}_3(\beta) = (1,6,11,16,\mathbf{11})$.

Moreover, $\mathfrak{R}_{\sigma_{2,5,2,0}}(q;\beta) = (1,2,\mathbf{3})_\beta + q\beta(4,8,12)_\beta + q^2\beta^3(4,\mathbf{11})_\beta$.

- If $n=3$, then $\sigma_{3,7,2,0} = [1,3,4,6,7,9,10,2,5,8] \in \mathbb{S}_{10}$, and $\mathfrak{F}_4(\beta) =$ $(1,12,57,182,392,602,763,730,493,\mathbf{170})$.

Moreover,

$\mathfrak{R}_{\sigma_{3,7,2,0}}(q;\beta) = (1,6,21,36,51,48,\mathbf{26})_\beta +\ q\ \beta\ (6,36,126,216,306,288,156)_\beta$
$+\ q^2\beta^3(20,125,242,403,460,289)_\beta + q^3\beta^5(6,46,114,204,\mathbf{170})_\beta$;

$\mathfrak{R}_{\sigma_{3,7,2,0}}(q;1) = (189,1134,1539,540)_q = 27\ (7,42,57,20)_q$.

- If $n=4$, then $\sigma_{4,9,2,0} = [1,3,4,6,7,9,10,12,13,2,5,8,11] \in \mathbb{S}_{13}$, and $\mathfrak{F}_5(\beta) =$ $(1,20,174,988,4025,12516,31402,64760,111510,162170,202957,220200,202493,$ $153106,89355,35972,\mathbf{7429})$.

Moreover,

$\mathfrak{R}_{\sigma_{4,9,2,0}}(q;\beta) = (1,12,78,308,903,2016,3528,4944,5886,5696,4320,2280,\mathbf{646})_\beta +$
$q\beta\ (8,96,624,2464,7224,16128,28224,39552,47088,45568,34560,18240,5168)_\beta +$
$q^2\beta^3(56,658,3220,11018,27848,53135,78902,100109,103436,84201,47830,14467)_\beta +$
$q^3\beta^5(56,728,3736,12820,29788,50236,72652,85444,78868,50876,17204)_\beta +$
$q^4\beta^7(8,117,696,2724,7272,13962,21240,24012,18768,\mathbf{7429})_\beta$;

$\mathfrak{R}_{\sigma_{4,9,2,0}}(q;1) = (30618,244944,524880,402408,96228)_q = 4374\ (7,56,120,92,22)_q$.

One can show that $\mathfrak{F}_n(\beta)$ is a polynomial in β of degree n^2, and looking on the numbers $2,11,170,7429$ we made

Conjecture 5.4. Let $b(n) := Coeff_{[\beta^{(n-1)^2}]}\Big(\mathfrak{F}_n(\beta)\Big)$. Then

$b(n) = CSTCPP(n)$. In other words, $b(n)$ is equal to the number of cyclically symmetric transpose complement plane partitions in an $2n \times 2n \times 2n$ box. This number is known to be

$$\prod_{j}^{n-1} \frac{(3j+1)(6j)!\ (2j)!}{(4j+1)!\ (4j)!},$$

see [87], A051255, [10], p.199.

It ease to see that polynomial $\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q;1)$ has degree n.

Conjecture 5.5.

- $$Coeff_{[\beta^n]}\Bigg(\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q;1)\Bigg) = A20342(2n),$$

see [87];

- $$\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(0;1) = A^{(1)}_{QT}(4n;3) = 3^{n(n-1)/2}\ ASM(n),$$

see [56], Theorem 5, or [87], A059491.

Proposition 5.6. *One has*

$$\mathfrak{R}_{\sigma_{4,2n+1,2,0}}(0;\beta)=\mathfrak{G}_n(\beta)=\mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1),\quad \mathfrak{R}_{\sigma_{n,2n,2,0}}(0,\beta)=\mathfrak{F}_n(\beta)=\mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1).$$

Finally we define (β,q)-deformations of the numbers $VSASM(n)$ and $CSCTPP(n)$. To accomplish these ends, let us consider permutations

$$w_k^{-}=(2,4,\ldots,2k,2k-1,2k-3,\ldots,3,1) \quad \textit{and} \quad w_k^{+}=(2,4,\ldots,2k,2k+1,2k-1,\ldots,3,1).$$

Proposition 5.7. *One has*

$$\mathfrak{S}_{w_k^-}(1)=VSAM(k),\quad \mathfrak{S}_{w_k^+}(1)=CSTCPP(k).$$

Therefore the polynomials $\mathfrak{G}^{(\beta-1)}_{w_k^-}(x{=}q,x_j=1,\forall j\ge 2)$ and $\mathfrak{G}^{(\beta-1)}_{w_k^+}(x{=}q,x_j=1,\forall j\ge 2)$ define (β,q)-deformations of the numbers $VSAM(k)$ and $CSTCPP(k)$ respectively. Note that the inverse permutations $(w_k^-)^{-1}=(\underbrace{2k,1},\ldots,\underbrace{2k+1-i,i},\ldots,\underbrace{k+1,k})$ and $(w_k^+)^{-1}=(\underbrace{2k+1,1},\ldots,\underbrace{2k+2-j,j},\ldots,\underbrace{k+2,k},k+1)$ also define a (β,q)-deformation of the numbers considered above.

Problem 5.1.

It is well-known, see e.g. [23], p.43, that the set $\mathcal{VSASM}(n)$ of alternating sign $(2n+1)\times(2n+1)$ matrices symmetric about the vertical axis has the same cardinality as the set $SYT_2(\lambda(n),\le n)$ of semistandard Young tableaux of the shape $\lambda(n):=(2n-1,2n-3,\ldots,3,1)$ filled by the numbers from the set $\{1,2,\ldots,n\}$, and such that the entries are weakly increasing down the anti-diagonals.

On the other hand, consider the set $\mathcal{CS}(w_k^-)$ of compatible sequences, see e.g. [8], [27], corresponding to the permutation $w_k^-\in\mathbb{S}_{2k}$.

Challenge *Construct bijections between the sets $\mathcal{CS}(w_k^-)$, $SYT_2(\lambda(k),\le k)$ and $\mathcal{VSASM}(k)$.* ■

Remarks 5.1. One can compute the principal specialization of the Schubert polynomial corresponding to the transposition $t_{k,n}:=(k,n-k)\in\mathbb{S}_n$ that interchanges k and $n-k$, and fixes all other elements of $[1,n]$.

Proposition 5.8. $q^{(n-1)(k-1)}\ \mathfrak{S}_{t_{k,n-k}}(1,q^{-1},q^{-2},q^{-3},\ldots)=$

$$\sum_{j=1}^{k}(-1)^{j-1}\ q^{\binom{j}{2}}\ \begin{bmatrix}n-1\\k-j\end{bmatrix}_q\ \begin{bmatrix}n-2+j\\k+j-1\end{bmatrix}_q=\sum_{j=1}^{n-2}\ q^j\ \left(\begin{bmatrix}j+k-2\\k-1\end{bmatrix}_q\right)^2.$$

Exercises 5.7. (1) <u>Show</u> that if $k\ge 1$, then

$$Coeff_{[q^k\beta^{2k}]}\Big(\mathfrak{R}_{\sigma_{n,2n,2,0}}(q;t)\Big)=\binom{2n-1}{2k},\quad Coeff_{[q^k\beta^{2k-1}]}\Big(\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q;t)\Big)=\binom{2n}{2k-1}.$$

(2) Let $n\ge 1$ be a positive integer, consider "zig-zag" permutation

$$w=\begin{pmatrix}1&2&3&4&\ldots&2k+1&2k+2&\ldots&2n-1&2n\\2&1&4&3&\ldots&2k+2&2k+1&\ldots&2n&2n-1\end{pmatrix}\in\mathbb{S}_{2n}.$$

<u>Show</u> that

$$\mathfrak{R}_w(q,\beta)=\prod_{k=0}^{n-1}\left(\frac{1-\beta^{2k}}{1-\beta}+q\beta^{2k}\right).$$

(3) Let $\sigma_{k,n,m}$ be grassmannian permutation with shape $\lambda = (n^m)$ and flag $\phi = (k+1)^m$, i.e.

$$\sigma_{k,n,m} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & k+n & k+n+1 & \dots & k+n+m \\ 1 & 2 & \dots & k & k+m+1 & \dots & k+m+n & k+1 & \dots & k+m \end{pmatrix}.$$

Clearly $\sigma_{k+1,n,m} = 1 \times \sigma_{k,n,m}$.

Show that

the coefficient $\quad Coeff_{\beta^m}\left(\mathfrak{R}_{\sigma_{k,n,m}}(1,\beta)\right)\quad$ is equal to the Narayana number $N(k+n+m,k)$.

(4) Consider permutation $w := w^{(n)} = (w_1,\ldots,w_{2n+1})$, where $w_{2k-1} = 2k+1$ for $k = 1,\ldots,n$, $w_{2n+1} = 2n$, $w_2 = 1$ and $w_{2k} = 2k-2$ for $k = 2,\ldots,n$. For example, $w^{(3)} = (3152746)$. We set $w^{(0)} = 1$.

Show that

the polynomial $\mathfrak{S}_w^{(\beta)}(x_i = 1, \forall i)$ has degree $n(n-1)$ and the coefficient

$Coeff_{\beta^{n(n-1)}}\left(\mathfrak{S}_w^{(\beta)}(x_i = 1, \forall i)\right)$ is equal to the n-th Catalan number C_n.

Note that the specialization $\mathfrak{S}_w^{(\beta)}(x_i = 1)|_{\beta=1}$ is equal to the $2n$-th Euler (or up/down) number, see [87], $A000111$.

More generally, consider permutation $w_k^{(n)} := 1^k \times w^{(n)} \in \mathbb{S}_{k+2n+1}$, and polynomials

$$P_k(z) = \sum_{j \ge 0} (-1)^j \ \mathfrak{S}_{w_{k-2j}^{(j)}}(x_i = 1)\ z^{k-2j}, \quad k \ge 0.$$

Show that

$$\sum_{k \ge 0} \ P_k(z)\ \frac{t^k}{k!} = \exp(tz)\ sech(t).$$

The polynomials $P_k(z)$ are well-known as *Swiss-Knife* polynomials, see [87], $A153641$, where one can find an overview of some properties of the Swiss-Knife polynomials.

(5) Assume that $n = 2k+3$. $k \ge 1$, and consider permutation $v_n = (v_1,\ldots,v_n) \in \mathbb{S}_n$, where $v_{2a+1} = 2a+3$, $a = 0,\ldots,n-1$, $w_2 = 1$ and $w_{2a} = 2a-2$, $a = 2,\ldots,k+1$. For example, $v_4 = [31527496, 11, 8, 10]$ and $\mathfrak{S}_{v_4}(1) = 50521 = E_{10}$.

Show that

$$\mathfrak{S}_{v_n}(q, x_i = 1, \forall i \ge 2) = (n-2)\ E_{n-3}\ q^2 + \cdots + q^{k-1}\ (k-1)!\ q^{k+2}, \quad \mathfrak{S}_{v_n}(x_i = 1, \forall i \ge 1) = E_{n-1}.$$

(6) Consider permutation $u := u_n = (u_1,\ldots,u_{2n}) \in \mathbb{S}_{2n}$, $n \ge 2$, where $u_1 = 2$, $u_{2k+1} = 2k-1$, $k = 1,\ldots,n$, $u_{2k} = 2k+2$, $k = 1,\ldots,n-1$, $u_{2n} = 2n-1$. For example, $u_4 = (24163857)$.

Now consider polynomial

$$R_n^{(k)}(q) = \mathfrak{S}_{1^k \times u_n}(x_1 = q, x_i = 1, \forall i \ge 2).$$

Show that

- $R_n^{(k)}(1) = \binom{2n+k-1}{k} E_{2n-1}$, where $E_{2k-1}, k \ge 1$, denotes the Euler number, see [87], $A00111$.

In particular, $R_n^{(1)}(1) = 2^{2n-1}\ G_n$, where G_n denotes the *unsigned Genocchi* number, see [87], $A110501$.

- $deg_q R_n^{(k)}(q) = n$ and $Coeff_{q^n}\left(R_n^{(0)}(q)\right) = (2n-3)!!$.

(7) Consider permutation $w_k := (2k+1, 2k-1, \ldots, 3, 1, 2k, 2k-2, \ldots, 4, 2) \in \mathbb{S}_{2k+1}$, Show that

$$\mathfrak{S}_{w_k}^{(\beta-1)}(x_1 = q, x_j = 1, \forall j \ge 2) = q^{2k}\ (1+\beta)^{\binom{n}{2}}.$$

(8) Consider permutations $\sigma_k^+ = (1,3,5,\ldots,2k{+}1,2k{+}2,2k,\ldots,4,2)$ and $\sigma_k^- = (1,3,5,\ldots,2k{+}1,2k,2k-2,\ldots,4,2)$, and define polynomials

$$S_k^{\pm}(q) = \mathfrak{S}_{\sigma_k^{\pm}}(x_1 = q, x_j = 1, \forall j \geq 2).$$

Show that $S_k^+(0) = VSASM(k)$, $S_k^+(1) = VSASM(k+1)$,
$\frac{\partial}{\partial q} S_k^+(q)|_{q=0} = 2k\ S_k^+(0)$ $Coeff_{q^k}\Big(S_k^+(q)\Big) = CSTCPP(k+1)$.
$S_k^-(0) = CSTCPP(k)$, $S_k^-(1) = CSTCPP(k+1)$,
$\frac{\partial}{\partial q} S_k^-(q)|_{q=0} = (2k-1)\ S_k^-(0)$, $Coeff_{q^k}\Big(S_k^-(q)\Big) = VSASM(k)$.

Let's observe that $\sigma_k^{\pm} = 1 \times \tau_{k-1}^{\pm}$, where $\tau_k^+ = (2,4,\ldots,2k,2k+1,2k-1,\ldots,3,1)$ and $\tau_k^- = (2,4,\ldots,2k,2k-1,2k-3,\ldots,3,1)$. Therefore,

$$\mathfrak{S}_{\tau_k^{\pm}}(x_1 = q, x_j = 1,\ \forall j \geq 2) = q\ S_{k-1}^{\pm}(q).$$

Recall that $CSTCPP(n)$ denotes the number of cyclically symmetric transpose compliment plane partitions in a $2\,n \times 2\,n$ box, see e.g. [87], $A051255$, and $VSASM(n)$ denotes the number of alternating sign $2\,n{+}1 \times 2\,n{+}1$ matrices symmetric t6he vertical axis, see e.g. [87], $A005156$.

- It might be well to point out that

$\mathfrak{S}_{\sigma_{n-1}^+}(x_1 = x, x_i = 1,\ \forall i \geq 2) = G_{2n-1,n-1}(x, y = 1)$,
$\mathfrak{S}_{\sigma_n^-}(x_1 = x, x_i = 1,\ \forall i \geq 2) = F_{2n,n-1}(x, y = 1)$,
where (homogeneous) polynomials $G_{m,n}(x,y)$ and $F_{m.n}(x,y)$ are defined in [77], and related with integral solutions to Pascal's hexagon relations

$$f_{m-1,n}\ f_{m+1,n} + f_{m,n-1}\ f_{m,n+1} = f_{m-1,n-1}\ f_{m+1,n+1}, \quad (m,n) \in \mathbb{Z}^2.$$

(9) Consider permutation

$$u_n = \begin{pmatrix} 1 & 2 & \ldots & n & n+1 & n+2 & n+3 & \ldots 2n & \\ 2 & 4 & \ldots & 2n & 1 & 3 & 5 & \ldots & 2n-1 \end{pmatrix},$$

and set $u_n^{(k)} := 1^{2k+1} \times u_n$.

Show that

$$\mathfrak{G}_{u_n^{(k)}}^{(\beta-1)}(x_i = 1, \forall i \geq 1) = (1+\beta)^{\binom{n+1}{2}}\ \mathfrak{G}_{1^k \times w_0^{(n+1)}}^{((\beta)^2-1)}(x_i = 1, \forall i \geq 1),$$

where $w_0^{(n+)}$ denotes the permutation $(n+1, n, n-1, \ldots, 2, 1)$.

(10) Let $n \geq 0$ be an integer.

- Conceder permutation $u_n = 1^n \times 321 \in \mathbb{S}_{3+n}$. Show That

$$\mathfrak{S}_{u_n}(x_1 = t, x_i = 1,\ \forall i \geq 2) = \frac{1}{4}\binom{2\ n+2}{3} + \frac{n}{2}\binom{2n+2}{1}\ t + \frac{1}{2}\binom{2n+2}{1}\ t^2.$$

- Consider permutation $v_n := 1^n \times 4321 \in \mathbb{S}_{n+4}$.

Show that $\mathfrak{S}_{v_n}(x_1 = t, x_i = 1,\ \forall i \geq 2) =$

$$\frac{1}{24}\binom{2n+4}{5}\binom{2n+2}{1} + \frac{1}{2}\binom{2n+4}{5}\ t + \frac{n}{4}\binom{2n+4}{3}\ t^2 + \frac{1}{4}\binom{2n+4}{3}\ t^3.$$

(11) Show that

$$\sum_{(a,b,c) \in (\mathbb{Z}_{\geq 0})^3} q^{a+b+c} \begin{bmatrix} a+b \\ b \end{bmatrix}_q \begin{bmatrix} a+c \\ c \end{bmatrix}_q \begin{bmatrix} b+c \\ b \end{bmatrix}_q = \frac{1}{(q;q)_\infty^3}\left(\sum_{k \geq 2} (-1)^k \binom{k}{2} q^{\binom{k}{2}-1}\right).$$

It is not difficult to see that the left hand side sum of the above identity counts the weighted number of plane partitions $\pi = (\pi_{ij})$ such that

$$\pi_{i,j} \geq 0, \quad \pi_{ij} \geq max(\pi_{i+1,j}, \pi_{i,j+1}), \quad \pi_{ij} \leq 1, \quad if \quad i \geq 2 \quad ana \quad j \geq 2,$$

and the weight $wt(\pi) := \sum_{i,j} \ \pi_{ij}$.

(12) Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0)$ be a partition of size n. For an integer k such that $1 \leq k \leq n-p$ define a grassmannian permutation

$$w_\lambda^{(k)} = [1, \ldots, k, \lambda_p + k + 1, \lambda_{p-1} + k + 2, \ldots, \lambda_1 + k + p, a_1, \ldots, a_{n-p-k}],$$

where we denote by $(a_1 < a_2 < \ldots < a_{n-k-p})$ the complement $[1,n] \backslash (1, \ldots, k, \lambda_p + k + 1, \lambda_{p-1} + k + 2, \ldots, \lambda_1 + k + p)]$.

- Show that the Grothendieck polynomial

$$G_\lambda(\beta) := \mathfrak{G}_{{w_\lambda}^k}^{\beta-1}(1^n)$$

is a polynomial of β with nonnegative coefficients.

Clearly, $G_\lambda(1) = dim V_\lambda^{\mathfrak{Gl}(k+\ell(\lambda))}$.

- Find a combinatorial interpretations of polynomial $G_\lambda(\beta)$. ■

Final remark, it follows from the seventh exercise listed above, that the polynomials $\mathfrak{G}_{\sigma_k^{\pm}}^{(\beta)}(x_1 = q, x_j = 1, \forall j \geq 2)$ define a (q,β)-deformation of the number $VSASM(k)$ (the case σ_k^+) and the number $CSTCPP(k)$ (the case σ_k^-), respectively.

5.2.5 Specialization of Grothendieck polynomials

Let p, b, n and i, $2i < n$ be positive integers. Denote by $\mathcal{T}_{p,b,n}^{(i)}$ the *trapezoid*, i.e. a convex quadrangle having vertices at the points

$$(ip, i), \quad (ip, n-i), \quad (b+ip, i) \quad and \quad (b+(n-i)p, n-i).$$

Definition 5.5. *Denote by $FC_{b,p,n}^{(i)}$ the set of lattice path from the point (ip, i) to that $(b+(n-i)p, n-i)$ with east steps $E = (0,1)$ and north steps $N = (1,0)$, which are located inside of the trapezoid $\mathcal{T}_{p,b,n}^{(i)}$.*

If $\mathfrak{p} \in FC_{b,p,n}^{(i)}$ is a path, we denote by $p(\mathfrak{p})$ the number of peaks, i.e.

$$p(\mathfrak{p}) = NE(\mathfrak{p}) + E_{in}(\mathfrak{p}) + N_{end}(\mathfrak{p}),$$

where $NE(\mathfrak{p})$ is equal to the number of steps NE resting on path $\mathfrak{p}$; $E_{in}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ starts with step E and 0 otherwise; $N_{end}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ ends by the step N and 0 otherwise.

Note that the equality $N_{end}(\mathfrak{p}) = 1$ may happened only in the case $b = 0$.

Definition 5.6. *Denote by $FC_{b,p,n}^{(k)}$ the set of k-tuples $\mathfrak{P} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ of non-crossing lattice paths, where for each $i = 1, \ldots, k$, $\mathfrak{p}_i \in FC_{b,p,n}^{(i)}$.*

Let

$$FC_{b,p,n}^{(k)}(\beta) := \sum_{\mathfrak{P} \in FC_{b,p,n}^{(k)}} \beta^{p(\mathfrak{P})}$$

denotes the generating function of the statistics $p(\mathfrak{P}) := \sum_{i=1}^k \ p(\mathfrak{p}_i) \ - k$.

Theorem 5.10. *The following equality holds*

$$\mathfrak{G}^{(\beta)}_{\sigma_{n,k,p,b}}(x_1=1,x_2=1,\ldots)=FC^{(k)}_{p,b,n+k}(\beta+1),$$

where $\sigma_{n,k,p,b}$ *is a unique grassmannian permutation with shape* $((n+1)^b,n^p,(n-1)^p,\ldots,1^p)$ *and flag* $\underbrace{(k,\ldots,k)}_{np+b}$.

5.3 The "longest element" and Chan–Robbins–Yuen polytope

5.3.1 The Chan–Robbins–Yuen polytope $\mathcal{CRY}_n$

Assume additionally, cf [90], **6**.**C8**, (d), that the condition (a) in Definition 5.1 is replaced by that

(a') : x_{ij} and x_{kl} commute for all i,j,k and l.

Consider the element $w_0^{(n)}:=\prod_{1\le i<j\le n} x_{ij}$. Let us bring the element $w_0^{(n)}$ to the reduced form, that is, let us consecutively apply the defining relations (a') and (b) to the element $w_0^{(n)}$ in any order until unable to do so. Denote the resulting polynomial by $Q_n(x_{ij};\alpha,\beta)$. Note that the polynomial itself **depends** on the order in which the relations (a') and (b) are applied.

We denote by $Q_n(\beta)$ the specialization $x_{ij}=1$ for all i and j, of the polynomial $Q_n(x_{ij};\alpha=0,\beta)$.

Example 5.7.

$Q_3(\beta)=(2,1)=1+(\beta+1)$, $Q_4(\beta)=(10,13,4)=1+5(\beta+1)+4(\beta+1)^2$,
$Q_5(\beta)=(140,336,280,92,9)=1+16(\beta+1)+58(\beta+1)^2+56(\beta+1)^3+9(\beta+1)^4$,
$Q_6(\beta)=1+42(\beta+1)+448(\beta+1)^2+1674(\beta+1)^3+2364(\beta+1)^4+1182(\beta+1)^5+169(\beta+1)^6$.
$Q_7(\beta)=(1,99,2569,25587,114005,242415,248817,118587,22924,1156)_{\beta+1}$
$Q_8(\beta)=(1,219,\ 12444,\ 279616,\ 2990335,\ 16804401,\ 52421688,\ 93221276,\ 94803125,\ 53910939,\ 16163947,\ 2255749,\ 108900)_{\beta+1}$.

What one can say about the polynomial $Q_n(\beta):=Q_n(x_{ij};\beta)|_{x_{ij}=1,\forall i,j}$?

It is known, [90], **6**.**C8**, (d), that the constant term of the polynomial $Q_n(\beta)$ is equal to the product of Catalan numbers $\prod_{j=1}^{n-1} C_j$. It is not difficult to see that if $n\ge 3$, then $Coeff_{[\beta+1]}(Q_n(\beta))=2^n-1-\binom{n+1}{2}$.

Theorem 5.11. One has

$$Q_n(\beta-1)=\left(\sum_{m\ge 0}\iota(\mathcal{CRY}_{n+1},m)\ \beta^m\right)(1-\beta)^{\binom{n+1}{2}+1},$$

where $\mathcal{CRY}_m$ denotes the Chan–Robbins-Yuen polytope [13], [14], i.e. the convex polytope given by the following conditions :

$\mathcal{CRY}_m=\left\{(a_{ij})\in Mat_{m\times m}(\mathbb{Z}_{\ge 0})\right\}$ such that
(1) $\sum_i a_{ij}=1$, $\sum_j a_{ij}=1$,
(2) $a_{ij}=0$, if $j>i+1$.

Here for any integral convex polytope $\mathcal{P}\subset\mathbb{Z}^d$, $\iota(\mathcal{P},n)$ denotes the number of integer points in the set $n\mathcal{P}\cap\mathbb{Z}^d$.

In particular, the polynomial $Q_n(\beta)$ does not depend on the order in which the relations (a') and (b) have been applied.

Now let us denote by $Q_n(t;\alpha,\beta)$ the specialization

$$x_{ij}=1,\ \ i<j<n,\ \ and\ \ x_{i,n}=t,\ \ if\ \ i=1,\ldots,n-1,$$

of the (reduced) polynomial $Q_n(x_{ij};\alpha,\beta)$ obtained by applying the relations (a') and (b) in a certain order. The polynomial itself **depends** on the order selected.

Conjecture 5.6. (**A**) *Let $n \ge 4$ and write*

$$Q_n(t=1;\alpha,\beta) := \sum_{k\ge 0}(1+\beta)^k \; c_{k,n}(\alpha), \quad \underline{then} \quad c_{k,n}(\alpha) \in \mathbb{Z}_{\ge 0}[\alpha].$$

(**B**)

- *The polynomial* $Q_n(t,\beta)$ *has degree* $d_n := [\frac{(n-1)^2}{4}]$.
- *Write*

$$Q_n(t,\beta) = t^{n-2} \sum_{k=0}^{d_n} c_n^{(k)}(t).$$

Then

$$c_n^{(d_n)}(1) = a_n^2 \quad for \quad some \quad non-negative \quad integer \quad a_n.$$

Moreover, there exists a polynomial $a_n(t) \in \mathbb{N}[t]$ such that

$$c_n^{(d_n)}(t) = a_n(1)\; a_n(t), \qquad a_n(0) = a_{n-1}.$$

(**C**) *The all roots of the polynomial* $Q_n(\beta)$ *belong to the set* $\mathbb{R}_{<-1}$.

For example,
(*a*) $Q_4(t=1;\alpha,\beta) = (1,5,4)_{\beta+1} + \alpha\;(5,7)_{\beta+1} + 3\;\alpha^2$, $Q_5(t=1;\alpha,\beta) = (1,16,58,56,9)_{\beta+1} + \alpha\;(16,109,146,29)_{\beta+1} + \alpha^2\;(51,125,34)_{\beta+1} + \alpha^3\;(35,17)_{\beta+1}$.
(*b*) $c_6^{(6)} = 13\;(2,3,3,3,2)$, $c_7^{(9)}(t) = 34\;(3,5,6,6,6,5,3)$, $c_8^{(12)}(t) = 330\;(13,27,37,43,45,45,43,37,27,13)$.

Comments 5.8.
(**1**) We **expect** that for each integer $n \ge 2$ the set

$$\Psi_{n+1} := \{w \in \mathbb{S}_{2n-1} \mid \mathfrak{S}_w(1) = \prod_{j=1}^{n} Cat_j\}$$

is non empty, whereas the set $\{w \in \mathbb{S}_{2n-2} \mid \mathfrak{S}_w(1) = \prod_{j=1}^{n} Cat_j\}$ is empty. For example, $\Psi_4 = \{\;[1,5,3,4,2]\;\}$, $\Psi_5 = \{\;[1,5,7,3,2,6,4],\;[1,5,4,7,2,6,3]\;\}$,
$\Psi_6 = \{\; w := [1,3,2,8,6,9,4,5,7],\; w^{-1}, \dots\;\}$, $\Psi_7 = \{???\}$, but one can check that for $w = [2358,10,549,12,11] \in \mathbb{S}_{12}$, $\mathfrak{S}_w(1) = 776160 = \prod_{j=2}^{6}\; Cat_j$.
More generally, for any positive integer N define

$$\kappa(N) = \min\{n \mid \exists w \in \mathbb{S}_n \;\; such \;\; that \;\; \mathfrak{S}_w(1) = N\}.$$

It is clear that $\kappa(N) \le N+1$.
Problem Compute the following numbers

$$\kappa(n!), \quad \kappa(\prod_{j=1}^{n} Cat_j), \quad \kappa(ASM(n)), \quad \kappa((n+1)^{n-1})$$

For example, $10 \le \kappa(ASM(6) = 7436) \le 12$. Indeed, take $w = [716983254,10,12,11] \in \mathbb{S}_{12}$. One can show that

$$\mathfrak{S}_w(x_1 = t, x_i = 1,\; \forall i \ge 2) = 13t^6(t+10)(15t+37),$$

so that $\mathfrak{S}_w(1) = ASM(6)$; $\kappa(6^4) = 9$, namely, one can take $w = [157364298]$.

Question Let N be a positive integer. Does there exist a vexillary (grassmannian ?) permutation $w \in \mathbb{S}_n$ such that $n \le 2\kappa(N)$ and $\mathfrak{S}_w(1) = N$?

For example, $w = [1,4,5,6,8,3,5,7] \in \mathbb{S}_8$ is a grassmannian permutation such that $\mathfrak{S}_w(1) = 140$, and $\mathfrak{R}_w(1,\beta) = (1,9,27,43,38,18,4)$.

Remark 5.3. *We expect that for $n \ge 5$ there are <u>no</u> permutations $w \in \mathbb{S}_\infty$ such that $Q_n(\beta) = \mathfrak{S}_w^{(\beta)}(1)$.*

(**3**) The numbers $\mathfrak{C}_n := \prod_{j=1}^{n} Cat_j$ appear also as the values of the *Kostant partition function* of the type A_{n-1} on some special vectors. Namely,

$$\mathfrak{C}_n = K_{\Phi(1^n)}(\gamma_n), \quad where \quad \gamma_n = (1,2,3,\ldots,n-1,-\binom{n}{2}),$$

see e.g. [90], **6.C10**, and [43], 173–178. More generally [43], (7,18), (7.25),one has

$$K_{\Phi(1^n)}(\gamma_{n,d}) = pp^{\delta_n}(d)\ \mathfrak{C}_{n-1} = \prod_{j=d}^{n+d-2} \frac{1}{2j+1} \binom{n+d+j}{2j},$$

where $\gamma_{n,d} = (d+1, d+2, \ldots, d+n-1, -n(2d+n-1)/2)$, $pp^{\delta_n}(d)$ denotes the set of reversed (weak) plane partitions bounded by d and contained in the shape $\delta_n = (n-1, n-2, \ldots, 1)$. Clearly, $pp^{\delta_n}(1) = \prod_{1 \le i < j \le n} \frac{i+j+1}{i+j-1} = C_n$, where C_n is the n-th Catalan number [34].

Conjecture 5.7.

For any permutation $w \in \mathbb{S}_n$ there exists a graph $\Gamma_w = (V,E)$, possibly with multiple edges, such that the reduced volume $\widetilde{vol}(\mathcal{F}_{\Gamma_w})$ of the *flow polytope* $\mathcal{F}_{\Gamma_w}$, see e.g. [89] for a definition of the former, is <u>equal</u> to $\mathfrak{S}_w(1)$. ■

For a family of vexillary permutations $w_{n,p}$ of the shape $\lambda = p\delta_{n+1}$ and flag $\phi = (1,2,\ldots,n-1,n)$ the corresponding graphs $\Gamma_{n,p}$ have been constructed in [66], Section 6. In this case the reduced volume of the flow polytope $\mathcal{F}_{\Gamma_{n,p}}$ is equal to the Fuss-Catalan number $\frac{1}{1+(n+1)p}\binom{(n+1)(p+1)}{n+1} = \mathfrak{S}_{w_{n,p}}(1)$, cf Corollary 5.2

Exercises 5.8.

(*a*) *<u>Show</u> that*
the polynomial $R_n(t) := t^{1-n}\, Q_n(t;0,0)$ is symmetric (*unimodal ?*), and $R_n(0) = \prod_{k=1}^{n-2} Cat_k$.

For example, $R_4(t) = (1+t)(2+t+2\,t^2)$, $R_5(t) = 2\ (5,10,13,14,13,10,5)_t$.
$R_6(t) = 10\ (2,3,2)_t\ (7,7,10,13,10,13,10,7,7)_t$.
Note that $R_n(1) = \prod_{k=1}^{n-1} Cat_k$.
(*b*) More generally, write $R_n(t,\beta) := Q_n(t;0,\beta) = \sum_{k \ge 0} R_n^{(k)}(t)\ \beta^k$.
<u>Show</u> that the polynomials $R_n^{(k)}(t)$ are symmetric for all k.
(*c*) *Consider a reduced polynomial* $\overline{R}_n(\{x_{ij}\})$ *of the element*

$$\prod_{\substack{1 \le i < j \le n \\ (i,j) \ne (n-1,n)}} x_{ij} \in \widehat{ACYB}(\alpha = \beta = 0)^{ab},$$

[34] For example, if $n = 3$, there exist 5 reverse (weak) plane partitions of shape $\delta_3 = (2,1)$ bounded by 1, namely reverse plane partitions $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} \right\}$.

see Definition 5.1. Here we assume additionally, that all elements $\{x_{ij}\}$ are mutually commute. Define polynomial $\widetilde{R}_n(q,t)$ to be the following specialization

$$x_{ij} \longrightarrow 1, \quad if \quad i<j<n-1, \quad x_{i,n-1} \longrightarrow q, \quad x_{i,n} \longrightarrow t, \quad \forall i$$

of the polynomial $\overline{R}_n(\{x_{ij}\})$ in question.

Show that polynomials $\widetilde{R}_n(q,t)$ are well-defined, and

$$\widetilde{R}_n(q,t) = \widetilde{R}_n(t,q).$$

Examples 5.2.

$R_4(t,\beta) = (2,3,3,2)_t + (4,5,4)_t\ \beta + (2,2)_t\ \beta^2, \qquad R_5(t,\beta) =$
$(10,20,26,28,26,20,10)_t + (33,61,74,74,61,33)_t\ \beta + (39,65,72,65,39)_t\ \beta^2 +$
$(19,27,27,19)_t\ \beta^3 + (3,3,3)_t\ \beta^4, \qquad R_6(t,\beta) =$
$(140,350,550,700,790,820,790,700,550,350,140)_t +$
$(686,1640,2478,3044,3322,3322,3044,2478,1640,686)_t\ \beta +$
$(1370,3106,4480,5280,5537,5280,4480,3106,1370)_t\ \beta^2 +$
$(1420,3017,4113,4615,4615,4113,3017,1420)_t\ \beta^3 +\ ,$
$(800,1565,1987,2105,1987,1565,800)_t\ \beta^4 +$
$(230,403,465,465,403,230)_t\ \beta^5 +$
$(26,39,39,39,26)_t\ \beta^6.$
$R_6(1,\beta) = (5880,22340,34009,26330,10809,2196,169)_\beta.$

$R_7(t,\beta) = (5880,17640,32340,47040,59790,69630,76230,79530,79530,76230,$
$69630,59790,47040,32340,17640,5880)_t\ +$
$(39980,116510,208196,295954,368410,420850,452226,462648,452226,420850,368410,$
$295954,208196,116510,39980)_t\ \beta\ +$
$(118179,333345,578812,802004,975555,1090913,1147982,1147982,1090913,975555,$
$802004,578812,333345,118179)_t\ \beta^2\ +$
$(198519,539551,906940,1221060,1447565,1580835,1624550,1580835,1447565,1221060,$
$906940,539551,198519)_t\ \beta^3\ +$
$(207712,540840,875969,1141589,1314942,1398556,1398556,1314942,1141589,875969,$
$540840,207712)_t\ \beta^4\ +$
$(139320,344910,535107,671897,749338,773900,749338,671897,535107,344910,$
$139320)_t\ \beta^5$
$+(59235,137985,203527,244815,263389,263389,244815,203527,137985,59235)_t\ \beta^6\ +$
$(15119,32635,45333,51865,53691,51865,45333,32635,15119)_t\ \beta^7\ +$
$(2034,3966,5132,5532,5532,5132,3966,2034)\ \beta^8\ + (102,170,204,204,204,170,102)_t\ \beta^9.$

$R_7(1,\beta) =$
$(776160,4266900,10093580,13413490,10959216,5655044,1817902,343595,33328,1156)_\beta.$

$\blacksquare$

5.3.2 The Chan–Robbins–Mészáros polytope $\mathcal{P}_{n,m}$

Let $m \geq 0$ and $n \geq 2$ be integers, consider the reduced polynomial $Q_{n,m}(t,\beta)$ corresponding to the element

$$M_{n.m} := \Big(\prod_{j=2}^{n} x_{1j}\Big)^{m+1} \prod_{j=2}^{n-2} \prod_{k=j+2}^{n} x_{jk}.$$

For example $Q_{2,4}(t,\beta)=(4,7,9,10,10,9,7,4)_t+(10,17,21,22,21,17,10)_t\ \beta$ $+(8,13,15,15,13,8)_t\beta^2+(2,3,3,3,2)_t\ \beta^3$, $Q_{2,4}(1,\beta)=(60,118,72,13)_{\beta}$.

$Q_{2,5}(t,\beta)=(60,144,228,298,348,378,388,378,348,298,228,144,60)_t$
$+(262,614,948,1208,1378,1462,1462,1378,1208,948,614,262)_t\ \beta$
$+(458,1042,1560,1930,2142,2211,2142,1930,1560,1042,458)_t\ \beta^2$
$+(405,887,1278,1526,1640,1640,1526,1278,887,405)_t\ \beta^4$
$+(187,389,534,610,632,610,534,389,187)_t\ \beta^4$
$+\ (41,79,102,110,110,102,79,41)_t\ \beta^5+(3,5,6,6,6,5,3)_t\ \beta^6$,
$Q_{2,5}(1,\beta)=(3300,11744,16475,11472,4072,664,34)_{\beta}$,
$Q_{2,6}(1,\beta)=(660660,3626584,8574762,11407812,9355194,4866708,1589799,$ $310172,32182,1320)_{\beta}$, $Q_{2,7}(\beta)=(1,213,12145,\ 279189,\ 3102220,\ 18400252,$ $61726264,\ 120846096,\ 139463706,\ 93866194,\ 5567810,\ 7053370,\ 626730,\ 16290)_{\beta+1}$.

Theorem 5.12. *One has*

$$(\mathbf{a})\quad Q_{m,n}(1,1)=\prod_{k=1}^{n-2} Cat_k \prod_{1\le i<j\le n-1}\frac{2(m+1)+i+j-1}{i+j-1}$$

$$(\mathbf{b})\quad \sum_{k\ge 0}\iota(\mathcal{P}_{n,m};k)\beta^k=\frac{Q_{m,n}(1,\beta-1)}{(1-\beta)^{\binom{n+1}{2}+1}},$$

where $\mathcal{P}_{n,m}$ denotes the generalized Chan-Robbins–Yuen polytope defined in [66], and for any integral convex polytope $\mathcal{P}$, $\iota(\mathcal{P},k)$ denotes the Ehrhart polynomial of polytope $\mathcal{P}$.

Conjecture 5.8. *Let $n\ge 3, m\ge 0$ be integers, , write*

$$Q_{m,n}(t,\beta)=\sum_{k\ge 0} c_{m,n}^{(k)}(t)\ \beta^k,\quad and\quad set\quad b(m,n):=max(k\mid c_{m,n}^{(k)}(t)\ne 0).$$

Denote by $\tilde{c}_{m,n}(t)$ the polynomial obtained from that $c_{m,n}^{(b(m,n))}(t)$ by dividing the all coefficients of the latter on their GCD. <u>*Then*</u>

$$\tilde{c}_{n,m}(t)=a_{n+m}(t),$$

where the polynomials $a_n(t):=c_{0,n}(t)$ have been defined in Conjecture 16, (**B**.

For example, $c_{2,5}(t)=4\ a_7(t)$, $c_{2,6}(t)=10\ a_8(t)$, $c_{3,5}(t)=a_8(t)$,
$c_{2,7}(t)=10\ (34,78,118,148,168,178,181,178,168,148,118,78,34)\ \overset{?}{=}\ 10\ a_9(t)$.

■

It is known [43], [65] that

$$\prod_{k=1}^{n-2} Cat_k \prod_{1\le i<j\le n-1}\frac{2(m+1)+i+j-1}{i+j-1}=\prod_{j=m+1}^{m+n-2}\frac{1}{2j+1}\binom{n+m+j}{2\ j}=$$

$$K_{A_{n=1}}(m+1,m+2,\ldots,n+m,-mn-\binom{n}{2}).$$

Conjecture 5.9.
Let $\mathbf{a}=(a_2,a_3,\ldots,a_n)$ be a sequence of non-negative integers, consider the following element

$$M_{(\mathbf{a})}=\Big(\prod_{j=2}^{n} x_{1j}^{a_j}\Big)\prod_{j=2}^{n-2}\Big(\prod_{k=j+2}^{n} x_{jk}\Big).$$

<u>*Then*</u>

(1) *Let* $R_{\mathbf{a}}(t_1,\ldots,t_{n-1},\ \alpha,\beta)$ *be the following specialization*

$$x_{ij} \longrightarrow t_{j-1} \quad for \quad all \quad 1 \le i < j \le n$$

of the reduced polynomial $R_{\mathbf{a}}(x_{ij})$ *of monomial* $M_{\mathbf{a}} \in \widehat{ACYB}_n(\alpha,\beta)$.

Then the polynomial $R_{\mathbf{a}}(t_1,\ldots,t_{n-1},\ \alpha,\beta)$ *is well-defined, i.e. does not depend on an order in which relations* (a') *and* (b) *, Definition 5.1, have been applied.*

$$(2) \qquad Q_{M_{\mathbf{a}}}(1,1) = K_{A_{n+1}}(a_2+1, a_3+2,\ldots,a_n+n-1, -\binom{n}{2} - \sum_{j=2}^{n} a_j).$$

(3) *Write*

$$Q_{M_{\mathbf{a}}}(t,\beta) = \sum_{k\ge 0} c_{\mathbf{a}}^{(k)}(t)\ \beta^k.$$

The polynomials $c_{\mathbf{a}}^{(k)}(t)$ *are symmetric (unimodal ?) for all* k.

Example 5.8. *Let's take* $n=5, \mathbf{a}=(2,1,1,0)$. *One can show that the value of the Kostant partition function* $K_{A_5}(3,3,4,4,-14)$ *is equal to* 1967. *On the other hand, one has*

$Q_{(2,1,1,0)}(t,\beta)\ t^{-3} = (50,118,183,233,263,273,263,233,183,118,50)_t +$
$(214,491,738,908,992,992,908,738,491,214)_t\ \beta + (365,808,1167,1379,1448,1379,$
$1167,808,365)_t\ \beta^2 + (313,661,906,1020,1020,906,661,313)_t\ \beta^3 +$
$(139,275,351,373,351,275,139)_t\ \beta^4 + (29,52,60,60,52,29)_t\ \beta^5 + (2,3,3,3,2)_t\ \beta^6.$
$Q_{(2,1,1,0)}(1,\beta) = (1967,6686,8886,5800,1903,282,13) = (1,34,279,748,688,204,13)_{\beta+1}.$

Exercises 5.9.

(1) *Show that*

$$R_n(t,-1) = t^{2(n-2)}\ R_{n-1}(-t^{-1},1).$$

(2) *Show that the ratio*

$$\frac{R_n(0,\beta)}{(1+\beta)^{n-2}}$$

is a polynomial in $(\beta+1)$ *with non-negative coefficients.*

(3) *Show that polynomial* $R_n(t,1)$ *has degree* $e_n := (n+1)(n-2)/2$, *and*

$$Coeff[t^{e_n}]\ \ R_n(t,1) = \prod_{k=1}^{n-1} Cat_k.$$

Problems 5.2.

(1) *Assume additionally to the conditions* (a') *and* (b) *above that*

$$x_{ij}^2 = \beta\ x_{ij} + 1, \quad if \quad 1 \le i < j \le n.$$

What one can say about a reduced form of the element w_0 *in this case ?*

(2) *According to a result by S. Matsumoto and J. Novak [64], if* $\pi \in \mathbb{S}_n$ *is a permutation of the cyclic type* $\lambda \vdash n$, *then the total number of primitive factorizations (see definition in [64]) of* π *into product of* $n-\ell(\lambda)$ *transpositions, denoted by* $Prim_{n-\ell(\lambda)}(\lambda)$, *is equal to the product of Catalan numbers:*

$$Prim_{n-\ell(\lambda)}(\lambda) = \prod_{i=1}^{\ell(\lambda)} Cat_{\lambda_i - 1}.$$

Recall that the Catalan number $Cat_n := C_n = \frac{1}{n}\binom{2n}{n}$. *Now take* $\lambda = (2,3,\ldots,n+1)$. *Then*

$$Q_n(1) = \prod_{a=1}^{n} Cat_a = Prim_{\binom{n}{2}}(\lambda).$$

Does there exist "a natural" bijection between the primitive factorizations and monomials which appear in the polynomial $Q_n(x_{ij};\beta)$ *?*

(3) *Compute in the algebra* $\widehat{ACYB}_n(\alpha,\beta)$ *the specialization*

$$x_{ij} \longrightarrow 1, \quad if \quad j<n, \ x_{ij} \longrightarrow t, \quad 1 \le i < n,$$

denoted by $P_{w_n}(t,\alpha,\beta)$, *of the reduced polynomial* $P_{s_{ij}}(\{x_{ij}\},\alpha,\beta)$ *corresponding to the transposition* $s_{ij} := \Big(\prod_{k=i}^{j-2} \ x_{k,k+1}\Big)\ x_{j-1,j}\Big(\prod_{k=j-2}^{i} \ x_{k,k+1}\Big) \in \widehat{ACYB}_n(\alpha,\beta)$.

For example, $P_{s_{14}}(t,\alpha,\beta) = t^5 + 3(1+\beta)t^4 + ((3,5,2)_\beta + 3\alpha)t^3 + (2(1+\beta)^2 + \alpha(5+4\beta))t^2$ $((1+\beta((1+3\alpha)+2\alpha^2)t + \alpha + \alpha^2$.

5.4 Reduced polynomials of certain monomials

In this subsection we compute the reduced polynomials corresponding to *dominant* monomials of the form

$$x_{\mathbf{m}} := x_{1,2}^{m_1}\ x_{23}^{m_2} \cdots x_{n-1,n}^{m_{n-1}} \in (\widehat{ACYB}_n(\beta))^{ab},$$

where $\mathbf{m} = (m_1 \ge m_2 \ge \ldots \ge m_{n-1} \ge 0)$ is a partition, and we apply the relations (a') and (b) in the algebra $(\widehat{ACYB}_n(\beta))^{ab}$, see Definition 5.1, and Section 5.3.1, successively, starting from $x_{12}^{m_1}\ x_{23}$.

Proposition 5.9. *The function*

$$\mathbb{Z}_{\ge 0}^{n-1} \longrightarrow \mathbb{Z}_{\ge 0}^{n-1}, \quad \mathbf{m} \longrightarrow P_{\mathbf{m}}(t=1;\beta=1)$$

can be extended to a piece-wise polynomial function on the space $\mathbb{R}_{\ge 0}^{n-1}$.

We start with the study of powers of Coxeter elements. Namely, for powers of Coxeter elements, one has [35]

$P_{(x_{12}\ x_{23})^2}(\beta) = (6,6,1)$, $P_{(x_{12}\ x_{23}\ x_{34})^2}(\beta) = (71,142,91,20,1) = (1,16,37,16,1)_{\beta+1}$,
$P_{(x_{12}x_{23}x_{34})^3}(\beta) = (1301,3903,4407,2309,555,51,1) = (1,45,315,579,315,45,1)_{\beta+1}$,
$P_{(x_{12}\ x_{23}\ x_{34}\ x_{45})^2}(\beta) = (1266,3798,4289,2248,541,50,1) = (1,44,306,564,306,44,1)_{\beta+1}$,
$P_{(x_{12}x_{23}x_{34})^3}(\beta=1) = 12527$, $P_{(x_{12}x_{23}x_{34})^4}(\beta=0) = 26599$,
$P_{(x_{12}x_{23}x_{34})^4}(\beta=1) = 539601$, $P_{(x_{12}\ x_{23}\ x_{34}\ x_{45})^2}(\beta=1) = 12193$,
$P_{(x_{12}\ x_{23}\ x_{34}\ x_{45})^3}(\beta=0) = 50000$, $P_{(x_{12}\ x_{23}\ x_{34}\ x_{45})^3}(\beta=1) = 1090199$.

Lemma 5.3. *One has*

$$P_{x_{12}^n\ x_{23}^m}(\beta) = \sum_{k=0}^{min(n,m)} \binom{n+m-k}{m}\binom{m}{k}\beta^k = \sum_{k=0}^{min(n,m)} \binom{n}{k}\binom{m}{k}(1+\beta)^k.$$

Moreover,

- *polynomial* $P_{(x_{12}x_{23}\cdots x_{n-1,n})^m}(\beta-1)$ *is a symmetric polynomial in* β *with non-negative coefficients.*
- *polynomial* $P_{x_{12}^n\ x_{23}^m}(\beta)$ *counts the number of* (n,m)*-Delannoy paths according to the number of* NE *steps* [36].

[35] To simplify notation we set $P_w(\beta) := P_w(x_{ij}=1;\beta)$.

[36] Recall that a (n,m)-Delannoy path is a lattice paths from $(0,0)$ to (n,m) with steps $E=(1,0)$, $N=(0,1)$ and $NE=(1,1)$ only.

For the definition and examples of the Delannoy paths and numbers, see [87], $A001850$, $A008288$, and http://mathworld.wolfram.com/DelannoyNumber.html.

Proposition 5.10. *Let n and k, $0 \le k \le n$, be integers. The number*

$$P_{(x_{12}x_{23})^n\ (x_{34})^k}(\beta=0)$$

is equal to the number of n up, n down permutations in the symmetric group $\mathbb{S}_{2n+k+1}$, see [87], A229892 and Exercises 5.3, (2).

Conjecture 5.10. *Let n,m,k be nonnegative integers. Then the number*

$$P_{x_{12}^n\ x_{23}^m\ x_{34}^k}(\beta=0)$$

is equal to the number of n up, m down and k up permutations in the symmetric group $\mathbb{S}_{n+m+k+1}$.

For example,

- Take $n=2, k=0$, the six permutations in $\mathbb{S}_5$ with 2 up, 2 down are **12543, 13542, 14532, 23541, 24531, 34521**.
- Take $n=3, k=1$, the twenty permutations in $\mathbb{S}_7$ with 3 up, 3 down are **1237654, 1247653, 1257643, 1267543, 1347652, 1357642, 1367542, 1457632, 1467532, 1567432, 2347651, 2357641, 2367541, 2457631, 2467531, 2567431, 3457621, 3467521, 3567421, 4567321**, see [87], *A229892*,
- Take $n=3, m=2, k=1$, the number of 3 up, 2 down and 1 up permutations in $\mathbb{S}_7$ is equal to $50=P_{321}(0)$: **1237645, 1237546, ..., 4567312**.
- Take $n=1, m=3, k=2$, the number of 1 up, 3 down and 2 up permutations in $\mathbb{S}_7$ is equal to $55=P_{132}(0)$, as it can be easily checked.

On the other hand, $P_{x_{12}^4\ x_{23}^3\ x_{34}^2 x_{45}}(\beta=0)=7203<7910$, where 7910 is the number of 4 up, 3 down, 2 up and 1 down permutations in the symmetric group $\mathbb{S}_{11}$.

Conjecture 5.11. *Let $k_1,\ldots,k_{n-1}$ be a sequence of non-negative integer numbers, consider monomial $M := x_{12}^{k_1}x_{23}^{k_2}\cdots x_{n-1,n}^{k_{n-1}}$. Then*

- *reduced polynomial $P_M(\beta-1)$ is a unimodal polynomial in β with non-negative coefficients.*

Example 5.9.

$P_{3,2,1}(\beta)=(1,14,27,8)_{\beta+1}=P_{1,2,3}(\beta)$, $P_{2,3,1}(\beta)=(1,15,30,9)_{\beta+1}=P_{1,3,2}(\beta)$, $P_{3,1,2}(\beta)=(1,11,18,4)_{\beta+1}=P_{2,1,3}(\beta)$, $P_{4,3,2,1}(\beta)=(1,74,837,2630,2708,885,68)_{\beta+1}$, $P_{4,3,2,1}(0)=7203=3\times 7^4$, $P_{5,4,3,2,1}(\beta)=(1,394,19177,270210,1485163,3638790, 4198361,2282942,553828,51945,1300)_{\beta+1}$, $P_{5,4,3,2,1}(0)=12502111=1019\times 12269$.

Exercises 5.10.

(1) *Show that if $n \ge m$, then*

$$x_{ij}^n\ x_{jk}^m\Big|_{x_{ij}=1=x_{jk}}=\sum_{a=0}^{n}\binom{m+a-1}{a}\left(\sum_{p=0}^{n-a}\binom{m}{p}\beta^p\right)x_{ik}^{m+a}.$$

(2) *Show that if $n\ge m\ge k$, then* $P_{x_{12}^n\ x_{23}^m\ x_{34}^k}(\beta)=P_{x_{12}^n\ x_{23}^m}(\beta)+$

$$\sum_{\substack{a\ge 1\\ b,p\ge 0}}\binom{m}{p}\binom{k}{a}\binom{a-1}{b}\binom{n+1}{p+a-b}\binom{m+a-1-b}{a}(\beta+1)^{p+a}.$$

In particular, if $n\ge m\ge k$, then

$$P_{x_{12}^n\ x_{23}^m\ x_{34}^k}(0)=\binom{m+n}{n}+\sum_{a\ge 1}\binom{k}{a}\Big(\sum_{b=1}^{a}\binom{m+n+1}{m+b}\binom{a-1}{b-1}\binom{m+b-1}{a}\Big).$$

Note that the set of relations from the item (1) allows to give an explicit formula for the polynomial $P_M(\beta)$ for any *dominant* sequence $M=(m_1 \ge m_2 \ge \ldots \ge m_k) \in (\mathbb{Z}_{>0})^k$. Namely, $P_M(\beta+1)=$

$$\sum_{\mathbf{a}} \prod_{j=2}^{k} \binom{m_j+a_{j-1}-1}{a_{j-1}} \Big(\sum_{\mathbf{b}} \prod_{j=1}^{k-1} \binom{m_{j+1}}{b_j} \beta^{b_j} \Big),$$

where the first sum runs over the following set $\mathcal{A}(M)$ of integer sequences $\mathbf{a}=(a_1,\ldots,a_{k-1})$

$$\mathcal{A}(M) := \{0 \le a_j \le m_j + a_{j-1}, \quad j=1,\ldots,k-1\}, \quad a_0=0,$$

and the second sum runs over the set $\mathcal{B}(M)$ of all integer sequences $\mathbf{b}=(b_1,\ldots,b_{k-1})$

$$\mathcal{B}(M) := \bigcup_{\mathbf{a}\in\mathcal{A}(M)} \{0 \ \le \ b_j \le \ \min(m_{j+1}, m_j - a_j + a_{j-1})\}, \quad j=1,\ldots,k-1.$$

(3) Show that

$$\#|\mathcal{A}(n,1^{k-1})| = \frac{n+1}{k}\binom{2k+n}{k-1} = f^{(n+k,k)},$$

where $f^{(n+k,k)}$ denotes the number of standard Young tableaux of shape $(n+k,k)$. In particular, $\#|\mathcal{A}(1^k)| = C_{k+1}$.

(4) Let $n \ge m \ge 1$ be integers and set $M=(n,m,1^k)$. Show that

$$P_M(x_{ij}=1;\beta=0) = \sum_{p=0}^{n} \frac{m+p+1}{k} \binom{m+p-1}{p}\binom{m+2k+p}{k-1} := P_k(n,m).$$

In particular, $P_1(n,m) = \binom{n+m}{n} + m\ \binom{n+m+1}{n}$,

$$P_k(n,1) = \frac{n+1}{k+1}\binom{2k+2+n}{k}, \quad P_k(2,2) = (79k^2+341k+360)\frac{(2k+2)!}{k!\ (k+5)!}.$$

(5) Let $T \in STY((n+k,k))$ be a standard Young tableau of shape $(n+k,k)$. Denote by $r(T)$ the number of integers $j \in [1,n+k]$ such that the integer j belongs to the second row of tableau T, whereas the number $j+1$ belongs to the first row of T. Show that

$$P_{x_{12}^n x_{23} \cdots x_{k+1,k+2}}(\beta-1) = \sum_{T \in STY((n+k,k))} \beta^{r(T)}.$$

(6) Let $M=(m_1,m_2,\ldots,m_{k-1}) \in \mathbb{Z}_{>0}^{k-1}$ be a composition. Denote by $\overleftarrow{M}$ the composition $(m_{k-1},m_{k-2},\ldots,m_2,m_1)$, and set for short $P_M(\beta) := P_{\prod_{i=1}^{k-1} x_{i,i+1}^{m_i}}(x_{ij}=1;\beta)$.

Show that $$P_M(\beta) = P_{\overleftarrow{M}}(\beta).$$

Note that in general, $P_{\prod_{i=1}^{k-1}\ x_{i,i+1}^{m_i}}(x_{ij};\beta) \ne P_{\prod_{i=1}^{k-1}\ x_{i,i+1}^{m_{k-i}}}(x_{ij};\beta)$.

(7) Define polynomial $P_M(t,\beta)$ to be the following specialization

$$x_{ij} \longrightarrow 1, \ \ if \ \ i<j<n, \ \ and \ \ x_{in} \longrightarrow t, \ \ if \ \ i=1,\ldots,n-1$$

of a polynomial $P_{\prod_{i=1}^{k-1} x_{i,i+1}^{m_i}}(x_{ij};\beta)$.

Show that if $n \ge m$, then

$$P_{x_{12}^n\ x_{23}^m}(t,\beta) = \sum_{j=0}^{m} \binom{m}{j} \Big(\sum_{k=m-1}^{n+m-j-1} \binom{k}{m-1} t^{k-m+1} \Big) \beta^j.$$

See Lemma 5.2 for the case $t=1$.

(8) Define polynomials $\widetilde{R}_n(t)$ as follows

$$\widetilde{R}_n(t) := P_{(x_{12}x_{23}x_{34})^n}(-t^{-1}, \beta=-1)\ (-t)^{3n}.$$

Show that polynomials $\widetilde{R}_n(t)$ have non-negative coefficients, and

$$\widetilde{R}_n(0) = \frac{(3n)\ !}{6\ (n\ !)^3}.$$

(9) Consider reduced polynomial $P_{n,2,2}(\beta)$ corresponding to monomial $x_{12}^n\ (x_{23}x_{34})^2$ and set $\tilde{P}_{n,2,2}(\beta) := P_{n,2,2}(\beta-1)$. Show that

$$\tilde{P}_{n,2,2}(\beta) \in \mathbb{N}[\beta] \quad and \quad \tilde{P}_{n,2,2}(1) = T(n+5,3),$$

where the numbers $T(n,k)$ are defined in [87],$A110952$, $A001701$.

Conjecture 5.12. Let λ be a partition. The element $s_\lambda(\theta_1^{(n)},\ldots,\theta_m^{(n)})$ of the algebra $3T_n^{(0)}$ can be written in this algebra as a sum of

$$\Big(\prod_{x\in\lambda} h(x)\Big) \times dim\ V_{\lambda'}{}^{(\mathfrak{gl}(n-m))} \times dim\ V_\lambda{}^{(\mathfrak{gl}(m))}$$

monomials with all coefficients are equal to 1.

Here $s_\lambda(x_1,\ldots,x_m)$ denotes the Schur function corresponding to the partition λ and the set of variables $\{x_1,\ldots,x_m\}$; for $x\in\lambda$, $h(x)$ denotes the *hook length* corresponding to a box x; $V_\lambda^{(\mathfrak{gl}(n))}$ denotes the highest weight λ irreducible representation of the Lie algebra $\mathfrak{gl}(n)$.

Problems 5.3.

(1) *Define a bijection between monomials of the form* $\prod_{a=1}^s x_{i_a,j_a}$ *involved in the polynomial* $P(x_{ij};\beta)$*, and dissections of a convex* $(n+2)$*-gon by* s *diagonals, such that no two diagonals intersect their interior.*

(2) *Describe permutations* $w\in\mathbb{S}_n$ *such that the Grothendieck polynomial* $\mathfrak{G}_w(t_1,\ldots,t_n)$ *is equal to the "reduced polynomial" for a some monomial in the associative quasi-classical Yang–Baxter algebra* $\widehat{ACYB_n}(\beta)$*. ?*

(3) *Study "reduced polynomials" corresponding to the monomials*

- *(transposition)* $s_{1n} := (x_{12}x_{23}\cdots x_{n-2,n-1})^2\ x_{n-1,n}$,
- *(powers of the Coxeter element)* $(x_{12}x_{23}\cdots x_{n-1,n})^k$.

in the algebra $\widehat{ACYB}_n(\alpha,\beta)^{ab}$.

(4) Construct *a bijection between the set of* k*-dissections of a convex* $(n+k+1)$*-gon and " pipe dreams" corresponding to the Grothendieck polynomial* $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1,\ldots,x_n)$*. As for a definition of "pipe dreams" for Grothendieck polynomials, see [54]; see also [27].*

Comments 5.9. We don't know any "good" combinatorial interpretation of polynomials which appear in Problem 5.3, (3) for general n and k. For example,
$P_{s_{13}}(x_{ij}=1;\beta) = (3,2)_\beta$, $P_{s_{14}}(x_{ij}=1;\beta) = (26,42,19,2)_\beta$,
$P_{s_{15}}(x_{ij}=1;\beta) = (381,988,917,362,55,2)_\beta$ and $P_{s_{15}}(x_{ij}=1;1) = 2705$. On the other hand, $P_{(x_{12}x_{23})^2\ x_{34}\ (x_{45})^2}(x_{ij}=1;\beta) = (252,633,565,212,30,1)$, that is in deciding on different reduced decompositions of the transposition s_{1n}. one obtains in general different reduced polynomials.

One can compare these formulas for polynomials $P_{s_{ab}}(x_{ij}=1;\beta)$ with those for the β-Grothendieck polynomials corresponding to transpositions (a,b), see Comments 5.5.

6 Appendixes

6.1 Appendix I Grothendieck polynomials

Definition 6.1. *Let β be a parameter. The Id-Coxeter algebra $IdC_n(\beta)$ is an associative algebra over the ring of polynomials $\mathbb{Z}[\beta]$ generated by elements $\Big\langle e_1, \ldots, e_{n-1} \Big\rangle$ subject to the set of relations*

- $e_i e_j = e_j e_i$, *if* $\Big| i-j \Big| \geq 2$,
- $e_i e_j e_i = e_j e_i e_j$, *if* $\Big| i-j \Big| = 1$,
- $e_i^2 = \beta\ e_i$, $\quad 1 \leq i \leq n-1$.

It is well-known that the elements $\{e_w,\ w \in \mathbb{S}_n\}$ form a $\mathbb{Z}[\beta]$-linear basis of the algebra $IdC_n(\beta)$. Here for a permutation $w \in \mathbb{S}_n$ we denoted by e_w the product $e_{i_1} e_{i_2} \cdots e_{i_\ell} \in IdC_n(\beta)$, where $(i_1, i_2, \ldots, i_\ell)$ is any *reduced* word for a permutation w, i.e. $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $\ell = \ell(w)$ is the length of w.

Let $x_1, x_2, \ldots, x_{n-1}, x_n = y, x_{n+1} = z, \ldots$ be a set of mutually commuting variables. We assume that x_i and e_j commute for all values of i and j. Let us define

$$h_i(x) = 1 + x e_i, \quad \textit{and} \quad A_i(x) = \prod_{a=n-1}^{i} h_a(x), \quad i = 1, \ldots, n-1.$$

Lemma 6.1. *One has*

(1) *(Addition formula)*

$$h_i(x)\ h_i(y) = h_i(x \oplus y),$$

where we set $(x \oplus y) := x + y + \beta xy$;

(2) *(Yang–Baxter relation)*

$$h_i(x) h_{i+1}(x \oplus y) h_i(y) = h_{i+1}(y) h_i(x \oplus y) h_{i+1}(x).$$

Corollary 6.1.

(1) $[h_{i+1}(x) h_i(x), h_{i+1}(y) h_i(y)] = 0$.

(2) $[A_i(x), A_i(y)] = 0$, $\ i = 1, 2, \ldots, n-1$.

The second equality follows from the first one by induction using the Addition formula, whereas the fist equality follows directly from the Yang–Baxter relation.

Definition 6.2. *(Grothendieck expression)*

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) := A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}).$$

Theorem 6.1. *([27]) The following identity*

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) = \sum_{w \in \mathbb{S}_n} \mathfrak{G}_w^{(\beta)}(X_{n-1})\ e_w$$

holds in the algebra $IdC_n \otimes \mathbb{Z}[x_1, \ldots, x_{n-1}]$.

Definition 6.3. *We will call polynomial $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ as the β-Grothendieck polynomial corresponding to a permutation w.*

Corollary 6.2.

(1) *If* $\beta=-1$, *the polynomials* $\mathfrak{G}_w^{(-1)}(X_{n-1})$ *coincide with the Grothendieck polynomials introduced by Lascoux and M.-P. Schützenberger [57].*

(2) *The* β*-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ *is divisible by* $x_1^{w(1)-1}$.

(3) *For any integer* $k\in[1,n-1]$ *the polynomial* $\mathfrak{G}_w^{(\beta-1)}(x_k=q, x_a=1, \forall a\neq k)$ *is a polynomial in the variables* q *and* β *with non-negative integer coefficients.*

Proof (Sketch) It is enough to show that the specialized Grothendieck expression $\mathfrak{G}_n(x_k=q, x_a=1, \forall a\neq k)$ can be written in the algebra $IdC_n(\beta-1)\otimes\mathbb{Z}[q,\beta]$ as a linear combination of elements $\{e_w\}_{w\in\mathbb{S}_n}$ with coefficients which are polynomials in the variables q and β with non-negative coefficients. Observe that one can rewrite the relation $e_k^2=(\beta-1)e_k$ in the following form $e_k(e_k+1)=\beta\ e_k$. Now, all possible negative contributions to the expression $\mathfrak{G}_n(x_k=q, x_a=1, \forall a\neq k)$ can appear only from products of a form $c_a(q):=(1+qe_k)(1+e_k)^a$. But using the Addition formula one can see that $(1+qe_k)(1+e_k)=1+(1+q\beta)e_k$. It follows by induction on a that $c_a(q)$ is a polynomial in the variables q and β with non-negative coefficients. ■

Definition 6.4.

- *The double* β*-Grothendieck expression* $\mathfrak{G}_n(X_n,Y_n)$ *is defined as follows*

$$\mathfrak{G}_n(X_n,Y_n)=\mathfrak{G}_n(X_n)\ \mathfrak{G}_n(-Y_n)^{-1}\in IdC_n(\beta)\otimes\mathbb{Z}[X_n,Y_n].$$

- *The double* β*-Grothendieck polynomials* $\{\mathfrak{G}_w(X_n,Y_n)\}_{w\in\mathbb{S}_n}$ *are defined from the decomposition*

$$\mathfrak{G}_n(X_n,Y_n)=\sum_{w\in\mathbb{S}_n}\mathfrak{G}_w(X_n,Y_n)\ e_w$$

of the double β*-Grothendieck expression in the algebra* $IdC_n(\beta)$.

More details about β-Grothendieck and related polynomials can be found in [59], [48]. ■

6.2 Appendix II Cohomology of partial flag varieties

Let $n=n_1+\cdots+n_k$, $n_i\in\mathbb{Z}_{\geq 1}\ \forall i$, be a composition of n, $k\geq 2$. For each $j=1,\ldots,k$ define the numbers $N_j=n_1+\cdots+n_j, N_0=0$, and $M_j=n_j+\cdots+n_k$. Denote by $\mathbf{X}:=\mathbf{X}_{n_1,\ldots,n_k}=\{x_a^{(i)}\mid i=1,\ldots,k,\ 1\leq a\leq n_i\}$ (resp. $\mathbf{Y}$, ...) a set of variables of the cardinality n. We set $deg(x_a^{(i)})=a,\ i=1,\ldots,k$. For each $i=1,\ldots,k$ define quasihomogeneous polynomial of degree n_i in variables $\mathbf{X}^{(i)}=\{x_a^{(i)}\mid 1\leq a\leq n_i\}$

$$p_{n_i}(\mathbf{X}^{(i)},t)=t^{n_i}+\sum_{a=1}^{n_i}x_a^{(i)}\ t^{n_i-a},$$

and put $p_{n_1,\ldots,n_k}(\mathbf{X},t)=\prod_{i=1}^k p_{n_i}(\mathbf{X}^{(i)},t)$. We summarize in the theorem below some well–known results about the classical and quantum cohomology and K-theory rings of type A_{n-1} partial flag varieties $\mathcal{F}l_{n_1,\ldots,n_k}$. Let $q_1,\ldots,q_{k-1}$, $deg(q_i)=n_i+n_{i+1}$, $i=1,\ldots,k-1$, be a set of "quantum parameters."

Theorem 6.2. *There are canonical isomorphisms*

$$H^*(\mathcal{F}l_{n_1,\ldots,n_k},\mathbb{Z})\cong\mathbb{Z}[\mathbf{X}_{n_1,\ldots,n_k}]/\Big\langle p_{n_1,\ldots,n_k}(\mathbf{X},t)-t^n\Big\rangle;$$

$$K^{\bullet}(\mathcal{F}l_{n_1,\ldots,n_k},\mathbb{Z}) \cong \mathbb{Z}[\mathbf{Y}^{\pm 1}]/\Big\langle p_{n_1,\ldots,n_k}(\mathbf{Y},t)-(1+t)^n\Big\rangle;$$

$$H_T^*(\mathcal{F}l_{n_1,\ldots,n_k},\mathbb{Z}) \cong \mathbb{Z}[\mathbf{X},\mathbf{Y}]/\Big\langle \prod_{i=1}^{k}\ \prod_{a=1}^{n_i}\ (x_a^{(i)}+t)-p_{n_1,\ldots,n_k}(\mathbf{Y},t)\Big\rangle;$$

$$(Cf.\quad [1]\ )\qquad QH^*(\mathcal{F}l_{n_1,\ldots,n_k}) \cong \mathbb{Z}[\mathbf{X}_{n_1,\ldots,n_k},q_1,\ldots,q_{k-1}]/\Big\langle \Delta_{n_1,\ldots,n_k}(\mathbf{X},t)-t^n\Big\rangle,$$

$$(Cf.\quad [1]\ )\qquad QH_T^*(\mathcal{F}l_{n_1,\ldots,n_k}) \cong \mathbb{Z}[\mathbf{X},\mathbf{Y},q_1,\ldots,q_{k-1}]/\Big\langle \Delta_{n_1,\ldots,n_k}(\mathbf{X},t)-p_{n_1,\ldots,n_k}(\mathbf{Y},t)\Big\rangle,$$

where [37] $\Delta_{n_1,\ldots,n_k}(\mathbf{X},t)=$

$$det\begin{vmatrix} p_{n_1}(\mathbf{X}^{(1)},t) & q_1 & 0 & \cdots & \cdots & \cdots & 0\\ -1 & p_{n_2}(\mathbf{X}^{(2)},t) & q_2 & 0 & \cdots & \cdots & 0\\ 0 & -1 & p_{n_3}(\mathbf{X}^{(3)},t) & q_3 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & 0 & -1 & p_{n_{k-1}}(\mathbf{X}^{(k-1)},t) & q_{k-1}\\ 0 & \cdots & \cdots & \cdots & 0 & -1 & p_{n_k}(\mathbf{X}^{(k)},t)\end{vmatrix}.$$

Here for any polynomial $P(\mathbf{x},t)=\sum_{j=0}^{r} b_j(\mathbf{x})t^{r-j}$ in variables $\mathbf{x}=(x_1,x_2,\ldots)$, we denote by $\Big\langle P(\mathbf{x},t)\Big\rangle$ the ideal in the ring $\mathbb{Z}[\mathbf{x}]$ generated by the coefficients $b_0(\mathbf{x}),\ldots,b_r(\mathbf{x})$. A similar meaning have the symbols $\Big\langle \prod_{i=1}^{k}\ \prod_{a=1}^{n_i}(x_a^{(i)}+t)-p_{n_1,\ldots,n_k}(\mathbf{y},t)\Big\rangle$, $\Big\langle \Delta_{n_1,\ldots,n_k}(\mathbf{x},t)-t^n\Big\rangle$ and so on.

Note that $dim(\mathcal{F}_{n_1,\cdots,n_k})=\sum_{i<j} n_i\ n_j$ and the Hilbert polynomial $Hilb(\mathcal{F}_{n_1,\cdots,n_k},q)$ of the partial flag variety $\mathcal{F}_{n_1,\ldots,n_k}$ is equal to the q-multinomial coefficient $\left[{n \atop n_1,\ldots,n_k}\right]_q$, and also is equal to the q-dimension of the weight $(n_1,\ldots,n_k)$ subspace of the n-th tensor power $(\mathbb{C}^n)^{\otimes n}$ of the fundamental representation of the Lie algebra $\mathfrak{gl}(n)$.

Comments 6.1. The cohomology and (small) quantum cohomology rings $H^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z})$ and $QH^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z})$, of the partial flag variety $\mathcal{F}_{n_1,\cdots,n_k}$ admit yet another representations we are going to present. To start with, let as before $n=n_1+\ldots+n_k$, $n_i\in\mathbb{Z}_{\geq 1}\ \forall i$, be a composition. Consider the set of variables $\widehat{\mathbf{X}}=X_{n_1,\ldots,n_{k-1}}:=\{x_a^{(i)}\ \mid\ 1\leq i\leq n_a, a=1,\ldots,k-1\}$, and set as before $deg\ x_a^{(i)}=a$. Note that the number of variables $\widehat{\mathbf{X}}$ is equal to $n-n_k$. To continue, let's define *elementary quasihomogeneous polynomials of degree r*

$$e_r(\widehat{\mathbf{X}})=\sum_{I,A}\ x_{a_1}^{(i_1)}\cdots x_{a_s}^{(i_s)},\quad e_0(\widehat{\mathbf{X}})=1,\quad e_{-r}(\widehat{\mathbf{X}})=0,\quad if\quad r>0,$$

where the sum runs over sequences of integers $I=(i_1,\ldots,i_s)$ and $A=(a_1,\ldots,a_s)$ such that

- $1\leq i_1<\ldots i_s\leq k-1$,
- $1\leq a_j\leq n_{i_j}$, $j=1,\ldots,s$, and $r=a_1+\cdots,a_s$,

and *complete homogeneous polynomials* of degree p

$$h_p(\widehat{\mathbf{X}})=det|e_{j-i+1}(\widehat{\mathbf{X}})|_{1\leq i,j\leq p}.$$

Finally, let's define the ideal $J_{n_1,\ldots,n_k}$ in the ring of polynomials $\mathbb{Z}[X_{n_1,\ldots,n_{k-1}}]$ generated by polynomials

$$h_{n_k+1}(\widehat{\mathbf{X}}),\ldots,h_n(\widehat{\mathbf{X}}).$$

Note that the ideal $J_{n_1,\ldots,n_k}$ is generated by $n-n_k=\#(X_{n_1,\ldots,n_{k-1}})$ elements.

[37] We prefer to use quantum parameters $\{q_i\ \mid\ 1\leq i\leq k-1\}$ instead of the parameters $\{(-1)^{n_i}q_i\ \mid\ 1\leq i\leq k-1\}$ have been used in [1].

Proposition 6.1. *There exists an isomorphism of rings*

$$H^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z}) \cong \mathbb{Z}[X_{n_1,\ldots,n_{k-1}}]/J_{n_1,\ldots,n_k}.$$

In a similar way one can describe relations in the (small) quantum cohomology ring of the partial flag variety $\mathcal{F}_{n_1,\cdots,n_k}$. To accomplish this let's introduce *quantum quasihomogeneous elementary polynomials* of degree j, $e_j^{(\mathbf{q})}(\mathbf{X}_{n_1,\ldots,n_r})$ through the decomposition

$$\Delta_{n_1,\ldots,n_r}(\mathbf{X}_{n_1,\ldots,n_r}) = \sum_{j=0}^{N_r} e_j^{(\mathbf{q})}(\mathbf{X}_{n_1,\ldots,n_r})\ t^{N_r-j}, \quad e_0^{(\mathbf{q})}(\mathbf{x}) = 1, \quad e_{-p}^{(\mathbf{q})}(\mathbf{x}) = 0, \quad if \quad p > 0.$$

To exclude redundant variables $\{x_a^{(k)},\ \ 1 \le a \le n_k\}$,, let us define *quantum quasihomogeneous Schur polynomials* $s_\alpha^{(\mathbf{q})}(\mathbf{X}_{n_1,\ldots,n_r})$ corresponding to a composition $\alpha = (\alpha_1 \le \alpha_2 \le \ldots \le \alpha_p)$ as follows

$$s_\alpha^{(\mathbf{q})}(\mathbf{X}_{n_1,\ldots,n_r}) = det|e_{j-i+\alpha_i}^{(\mathbf{q})}(\mathbf{X}_{n_1,\ldots,n_r})|_{1\le i,j\le p}.$$

Proposition 6.2. *The (small) quantum cohomology ring $QH^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z})$ is isomorphic to the quotient of the ring of polynomials $\mathbb{Z}[q_1,\ldots,q_{k-1}]$ $[\mathbf{X}_{n_1,\ldots,n_{k-1}}]$ by the ideal $I_{n_1,\ldots,n_{k-1}}$ generated by the elements*

$$g_r(\mathbf{X}_{n_1,\ldots,n_{k-1}}) := s_{(1^{n_k},\ r)}^{(q_1,\ldots,q_{k-1})}(\mathbf{X}_{n_1,\ldots,n_{k-1}}) - q_{k-1}\ e_{r-n_{k-1}}^{(q_1,\ldots,q_{k-2})}(\mathbf{X}_{n_1,\ldots,n_{k-2}}),$$

where $n_k + 1 \le r \le n$.

It is easy to see that the Jacobi matrix

$$\Big(\frac{\partial}{\partial x_a^{(i)}}\ g_r(\mathbf{X}_{n_1,\ldots,n_{k-1}}\Big)_{\substack{\{a=1,\ldots,k-1,\ 1\le i\le n_a \\ n_k+1\le r\le n\}}}$$

corresponding to the set of polynomials $g_r(\mathbf{X}_{n_1,\ldots,n_{k-1}})$ $n_k \le r \le n$, has nonzero determinant, and the component of maximal degree $n_{max} := \sum_{l<j} n_i\ n_j$ in the ring $QH^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z})$ is a $\mathbb{Z}[q_1,\ldots,q_{k-1}]-$module of rank one with generator

$$\Lambda = \prod_{i=1}^{k-1}\ \prod_{a=1}^{n_a}\ \left(x_a^{(i)}\right)^{M_i}.$$

Therefore, one can define a scalar product (the Grothendieck residue)

$$\langle \bullet, \bullet \rangle :\ HQ^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z}) \times HQ^*(\mathcal{F}_{n_1,\cdots,n_k},\mathbb{Z}) \longrightarrow \mathbb{Z}[a_1,\ldots,q_{k-1}]$$

setting for elements f and g of degrees a and b, $\langle f,h \rangle = 0$, if $a+b \ne n_{max}$, and $\langle f,h \rangle = \lambda(q)$, if $a+b = n_{max}$ and $f\ h = \lambda(q)\ \Lambda$. It is well known that the Grothendieck pairing $\langle \bullet, \bullet \rangle$ is nondegenerate (for any choice of parameters $q_1,\ldots,q_{k-1}$).

Finally we state "a mirror presentation" of the small quantum cohomology ring of partial flag varieties. To start with, let $n = n_1 + \ldots + n_k$, $k \in \mathbb{Z}_{ge2}$ be a composition of size n, and consider the set

$$\Sigma(\mathbf{n}) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z}\ |1 \le i \le N_a, \quad M_{a+1} + 1 \le j \le M_a, \quad a = 1,\ldots,k-1\},$$

where $N_a = n_1 + \ldots + n_a, N_0 = 0, N_k = n \ \ M_a = n_{a+1} + \ldots + n_k,\ M_0 = n, M_k = 0$.

With these data given, let us introduce the set of variables

$$Z_{\mathbf{n}} = \{z_{i,j} \mid (i,j) \in \Sigma(\mathbf{n})\},$$

and define "boundary conditions" as follows

- $z_{i,M_a+1}=0$, if $N_{a-1}+2 \le i \le N_a$, $a=1,\dots,k-1$,
- $z_{N_a+1,j}=\infty$, if $M_{a+1}+2 \le j \le M_a$, $a=1,\dots,k-1$,
- $z_{N_{a-1}+1,M_a+1}=q_a$, $a=1,\dots,k$, where $q_1,\dots,q_k$ are "quantum parameters.

Now we are ready, follow [34], to define **superpotential**

$$W_{q,\mathbf{n}} = \sum_{(p,j)\in\Sigma(\mathbf{n})} \Big(\frac{z_{i,j+1}}{z_{i,j}} + \frac{z_{i,j}}{z_{i+1,j}}\Big).$$

Conjecture 6.1. *(Cf. [34]) There exists an isomorphism of rings*

$$QH^*_{[2]}(\mathcal{F}l_{n_1,\dots,n_k},\mathbb{Z}) \cong \mathbb{Z}[q_1^{\pm 1},\dots,q_k^{\pm 1}][Z_{\mathbf{n}}^{\pm 1}]/J(W_{q,\mathbf{n}}),$$

where $QH^*_{[2]}(\mathcal{F}l_{n_1,\dots,n_k},\mathbb{Z})$ denotes the subring of the ring $QH^*(\mathcal{F}l_{n_1,\dots,n_k},\mathbb{Z})$ generated by the elements from $H^2(\mathcal{F}l_{n_1,\dots,n_k},\mathbb{Z})$;

$J(W_{q,\mathbf{n}})$ stands for the ideal generated by the partial derivatives of the **superpotential** $W_{q,\mathbf{n}}$:

$$J(W_{q,\mathbf{n}}) = \langle \frac{\partial W_q}{\partial z_{i,j}} \rangle,\ (i,j)\in\Sigma(\mathbf{n})\rangle.$$

Note that variables $\{z_{i,j} \in \Sigma(\mathbf{n}),\ i \ne N_a+1,\ \ a=0,\dots,k-2\}$ are redundant, whereas the variables $\{z_{a,j} := z_{N_a+1,j}^{-1},\ \ j=1,\dots,n_a,\ a=0,\dots,k-2\}$ satisfy the system of algebraic equations.

In the case of complete flag variety $\mathcal{F}l_n$ corresponds to partition $\mathbf{n}=(1^n)$ and the superpotential $W_{q,1^n}$ is equal to

$$W_{q,1^n} = \sum_{1\le i<j\le n-1} \Big(\frac{z_{i,j+1}}{z_{i,j}} + \frac{z_{i,j}}{z_{i-1,j+1}}\Big),$$

where we set $z_{i,n} := q_i$, $i=1,\dots,n$. The ideal $J(W_{q,1^n})$ is generated by elements

$$\frac{\partial W_{q,1^n}}{z_{i,j}} = \frac{1}{z_{i,j-1}} + \frac{1}{z_{i-1,j+1}} - \frac{z_{i,j+1}+z_{i-1,j-1}}{z_{i,j}^2}.$$

One can check that the ideal $J(W_{q,1^n})$ can be also generated by elements of the form

$$\sum_{j=0}^{i} A_j^{(i)}(q_1,\dots,q_{n-i+1},z_{n-1},\dots,z_{n-i+1})\ z_{n-i}^{j-i-1} = 1,\quad A_0^{(i)} = q_1\cdots q_{n-i+1},$$

where $z_i := z_{1,i}^{-1}$, $i=1,\dots n-1$. For example,

$$z_1^n\ q_1 \dots q_n = 1,\quad q_1\ q_2\ z_{n-1}^2 - q_2\ z_{n-2} = 1,$$

$$q_1\ q_2\ q_3\ z_{n-2}^3 - 2\ q_1\ q_2\ q_3\ z_{n-1}\ z_{n-2}\ z_{n-3} + q_2\ q_3\ z_{n-3}^2 + q_3\ z_{n-4} = 1.$$

Therefore the number of critical points of the superpotential W_q is equal to $n! = dim H^*(\mathcal{F}l_n,\mathbb{Z})$, as it should be. Note also that $QH^*(\mathcal{F}l_n,\mathbb{Z}) = QH^*_{[2]}(\mathcal{F}l_n,\mathbb{Z})$.

6.3 Appendix III Koszul dual of quadratic algebras and Betti numbers

Let k be a field of zero characteristic, $F^{(n)} := k < x_1, \ldots, x_n >= \bigoplus_{j\geq 0} F_j^{(n)}$ be the free associative algebra generated by $\{x_i,\ 1 \leq i \leq n\}$. Let $A = F^{(n)}/I$ be a quadratic algebra, i.e. the ideal of relations I is generated by the elements of degree 2, $I \subset F_2^{(n)}$. Let $F^{(n)*} = Hom(F_n, k) = \bigoplus_{j\geq 0} F_j^{(n)*}$ with a multiplication induced by the rule $fg(ab) = f(a)g(b)$, $f \in F_i^{(n)*},\ g \in F_j^{(n)*},\ a \in F_i^{(n)}, b \in F_j^{(n)}$. Let $I_2^{\perp} = \{f \in F_2^{(n)*}, f(I_2) = 0\}$, and denote by $I^{\perp}$ the two-sided ideal in $F^{(n)*}$ generated by the set $I_2^{\perp}$.

Definition 6.5. The Koszul (or quadratic) dual $A^!$ *of a quadratic algebra A is defined to be* $A^! := F^{(n)*}/I^{\perp}$.

The Koszul dual of a quadratic algebra A is a quadratic algebra and $(A^!)^! = A$.

Examples 6.1. (1) *Let $A = F^{(n)}$ be the free associative algebra, then the quadratic duel* $A^! = k < y_1, \ldots, y_n > /(y_i y_j, 1 \leq i, j \leq n)$.

(2) *If $A = k[x_1, \ldots, x_n]$ is the ring of polynomials, then*

$$A^! = k[y_1, \ldots, y_n]/([y_i, y_j]_-, 1 \leq i, j \leq n),$$

where we put by definition $[y_i, y_j]_- = y_i y_j + y_j y_i$, if $i \neq j$, and $[y_i, y_i]_- = y_i^2$.

(3) *(cf [63], (b), Chapter 5) Let $A = F^{(n)}/(f_1, \ldots, f_r)$, where $f_i = \sum_{1\leq j,k\leq n} a_{ijk} x_j\ x_k$, $i = 1, \ldots, r$ are linear independent elements of degree 2 in $F^{(n)}$. Then the quadratic dual of A is equal to the quotient algebra $A^! = k < y_1, \cdots, y_n > /J$, where the ideal $J =< g_1, \ldots, g_s >$, $s = n^2 - r$, is generated by elements $g_m = \sum_{1\leq j,k\leq n} b_{mjk}\ y_j\ y_k$. The coefficients $b_{mjk}, m = l, \ldots, s, 1 \leq j, k \leq n$, can be defined from the system of linear equations $\sum_{1\leq j,k\leq n} a_{ijk}\ b_{mjk} = 0,\ i = 1, \ldots, r,\ \ m = 1, \ldots, s$.*

■

Let $A = \bigoplus_{j\geq 0} A_j$ be a graded finitely generated algebra over field k.

Definition 6.6. The Hilbert series *of a graded algebra A is defined to be the generating function of dimensions of its homogeneous components: $Hilb(A, t) = \sum_{k\geq 0} dim A_k\ t^k$.*

The Betti numbers $B_A(n, m)$ *of a graded algebra A are defined to be* $B_A(i, j) := dim Tor_i^A(k, k)_j$.

The Poincarè *series of algebra A is defined to be the generating function for the Betti numbers: $P_A(s, t) := \sum_{i\geq 0, j\geq 0} B_A(i, j) s^i t^j$.*

Definition 6.7. *A quadratic algebra A is called* **Koszul** *iff the Betti numbers $B_A(i, j)$ are equal to zero unless $i = j$.*

(♣) It is well-known that $Hilb(A, t) P_A(-1, t) = 1$, and a quadratic algebra A is **Koszul**, if and only if $B_A(i, j) = 0$ for all $i \neq j$. In this case $Hilb(A, t)\ Hilb(A^!, -t) = 1$.

Example 6.1. *Let $F_n^{(0)}$ be a quotient of the free associative algebra F_n over field k with the set of generators $\{x_1, \ldots, x_n\}$ by the two-sided ideal generated by the set of elements $\{x_1^2, \ldots, x_n^2\}$. Then the algebra $F_f^{(0)} n$ is* **Koszul**, *and* $Hilb(F_n^{(0)}, t) = \frac{1+t}{1-(n-1)t}$.

6.4 Appendix IV Hilbert series $Hilb(3T_n^0, t)$ and $Hilb((3T_n^0)^!, t)$: Examples [38]

[38] All computations in this Section were performed by using the computer system **Bergman**, except computations of $Hilb(3T_6^0, t)$ in degrees from twelfth till fifteenth. The last computations were made by J. Backelin, S. Lundqvist and J.-E. Roos from Stockholm University, using the computer algebra system **aalg** mainly developed by S. Lundqvist.

Examples 6.2. $Hilb(3T_3^0,t) = [2]^2[3]$, $Hilb(3T_4^0,t) = [2]^2[3]^2[4]^2$,
$Hilb(3T_5^0,t) = [4]^4[5]^2[6]^4$, $Hilb(3T_6^0,t)$
$= (1, 15, 125, 765, 3831, 16605, 64432, 228855, 755777, 2347365, 6916867, 19468980, 52632322, 137268120, 346652740, 850296030, \cdots)$.
$= Hilb(3T_5^0,t)(1, 5, 20, 70, 220, 640, 1751, 4560, 11386, 27425, 64015, 145330, 321843, 696960, 1478887, 3080190, \cdots)$.
$Hilb(3T_7^0,t) = Hilb(3T_6^0,t)(1, 6, 30, 135, 560, 2190, 8181, 29472, 103032, 351192, 1170377, \cdots)$.
$Hilb(3T_8^0,t) = Hilb(3T_7^0,t)(1, 7, 42, 231, 1190, 5845, 27671, 127239, 571299, 2514463,$

$Hilb((3T_3^0)^!,t)(1-t) = (1,2,2,1)$, $Hilb((3T_4^0)^!,t)(1-t)^2 = (1,4,6,2,-5,-4,-1)$,
$Hilb((3T_5^0)^!,t)(1-t)^2 = (1,8,26,40,19,-18,-22,-8,-1)$,
$Hilb((3T_6^0)^!,t)(1-t)^3 = (1,12,58,134,109,-112,-245,-73,68,50,12,1)$,
$Hilb((3T_7^0)^!,t)(1-t)^3 = (1,18,136,545,1169,1022,-624,-1838,-837,312,374,123,18,1)$.

We **expect** *that $Hilb((3T_n^0)^!,t)$ is a rational function with the only pole at $t=1$ of order $[n/2]$, and the polynomial $Hilb((3T_n^0)^!,t)(1-t)^{[n/2]}$ has degree equals to $[5n/2]-4$, if $n \geq 2$.*

6.5 Appendix V Summation and Duality transformation formulas [41]

.

Summation Formula Let $a_1 + \cdots + a_m = b$. Then

$$\sum_{i=1}^{m} [a_i] \Big(\prod_{j \neq i} \frac{[x_i - x_j + a_j]}{[x_i - x_j]} \Big) \frac{[x_i + y - b]}{[x_i + y]} = [b] \prod_{1 \leq i \leq m} \frac{[y + x_i - a_i]}{[y + x_i]}.$$

Duality transformation, case $N = 1$ Let $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. Then

$$\sum_{i=1}^{m} [a_i] \prod_{j \neq i} \frac{[x_i - x_j + a_j]}{[x_i - x_j]} \prod_{1 \leq k \leq n} \frac{[x_i + y_k - b_k]}{[x_i + y_k]} =$$

$$\sum_{k=1}^{n} [b_k] \prod_{l \neq k} \frac{[y_k - y_l + b_l]}{[y_k - y_l]} \prod_{1 \leq i \leq m} \frac{[y_k + x_i - a_i]}{[y_k + x_i]}.$$

■

7 References

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