Acyclicity of non-linearizable line bundles on fake projective planes

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Abstract. On the projective plane there is a unique cubic root of the canonical bundle and this root is acyclic. On fake projective planes such root exists and is unique if there are no 3-torsion divisors (and usually exists but not unique otherwise). Earlier we conjectured that any such cubic root (assuming it exists) must be acyclic. In the present note we give a new short proof of this statement and show acyclicity of some other line bundles on those fake projective planes with at least 9 automorphisms. Similarly to our earlier work we employ simple representation theory for non-abelian finite groups. The novelty stems from the idea that if some line bundle is non-linearizable with respect to a finite abelian group, then it should be linearized by a finite (non-abelian) Heisenberg group. Our argument also exploits J. Rogawski's vanishing theorem and the linearization of an auxiliary line bundle.

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1. Introduction

1.1. In [10] Mumford gave an ingenious construction of *a smooth complex algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$*. All such surfaces are now known under the name of *fake projective planes*. They have been recently classified into 100 isomorphism classes by Prasad – Yeung [11] and Cartwright – Steger [4]. Universal cover of any fake projective plane is the complex ball and papers [11, 4, 5] describe explicitly all subgroups in the automorphism group of the ball which are fundamental groups of the fake projective planes.

However, these surfaces are poorly understood from the algebro-geometric perspective, since the uniformization maps (both complex and, as in Mumford's case, 2-adic) are highly transcendental, and so far not a single fake projective plane has been constructed geometrically. Most notably the Bloch's conjecture on zero-cycles (see [3]) for the fake projective planes is not established yet.

Earlier we have initiated the study of fake projective planes from the homological algebra perspective (see [6]). Namely, for $\mathbb{P}^2$ the corresponding bounded derived category of coherent sheaves has a semiorthogonal decomposition, $D^b(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$, as was shown by A. Beilinson in [2]. The "easy" part of his argument was in checking that the line bundles $\mathcal{O}(1)$ and $\mathcal{O}(2)$ are acyclic, and that any line bundle on $\mathbb{P}^2$ is exceptional. All these results follow from Serre's computation. In turn, the "hard" part consisted of checking that $\mathcal{O}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$ actually generate $D^b(\mathbb{P}^2)$. But for fake projective planes, according to [6], one can not construct a full exceptional collection this way. Anyhow, one still can define an analogue of $\mathcal{O}(1), \mathcal{O}(2)$ for some of these planes (cf. **1.2** below) and try to establish the "easy part" for them. Then exceptionality of line bundles is equivalent to the vanishing of $h^{0,1}$ and $h^{0,2}$, which is clear, while acyclicity is not at all obvious.

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We are going to treat acyclicity problem, more generally, in the context of ball quotients and modular forms. We hope that our argument might be useful for proving the absence of modular forms of small weights on complex balls (compare with [13]).

1.2. In the present paper we study those fake projective planes S whose group of automorphisms A_S has order at least 9. All these surfaces fall into the six cases represented in Table A below (cf. [5] and [6, Section 6]). There one denotes by Π the fundamental group of S, so that $S = \mathbb{B}/\Pi$ for the unit ball $\mathbb{B} \subset \mathbb{CP}^2$, and $N(\Pi)$ denotes the normalizer of Π in $PU(2,1)$.

One of the principle observations is that the group Π lifts to $SU(2,1)$. The lifting produces a line bundle $\mathcal{O}_S(1) \in \operatorname{Pic} S$ such that $\mathcal{O}_S(3) := \mathcal{O}_S(1)^{\otimes 3} \simeq \omega_S$, the canonical sheaf of S. Moreover, the preimage $\widetilde{N(\Pi)} \subset SU(2,1)$ of $N(\Pi) \subset PU(2,1)$ acts fiberwise-linearly on the total space $\operatorname{Tot} \mathcal{O}_S(1) \longrightarrow S$, which provides a natural linearization for $\mathcal{O}_S(1)$ (and consequently for all $\mathcal{O}_S(k)$). Furthermore, the action of the group $\Pi \subset \widetilde{N(\Pi)}$ is trivial. Thus all vector spaces $H^0(S, \mathcal{O}_S(k)), k \in \mathbb{Z}$, are endowed with the structure of G-modules, where $G := \widetilde{N(\Pi)}/\Pi$. In the same way one obtains the structure of G-module on $H^0(S, \mathcal{O}_S(k) \otimes \varepsilon)$ for any A_S-invariant torsion line bundle $\varepsilon \in \operatorname{Pic} S$ (cf. **2.1** below).

Another observation is the use of vanishing result due to J. Rogawski (see Theorem 3.2 below), which we apply to show that $h^0(S, \omega_S \otimes \varepsilon) = 1$ whenever $\varepsilon \neq 0$. Twisting $H^0(S, \mathcal{O}_S(2) \otimes \varepsilon)$ (resp. $H^0(S, \mathcal{O}_S(2))$) by the global section of $\omega_S \otimes \varepsilon$ allows us to prove our main result:

Theorem 1.3. *Let S be a fake projective plane with at least nine automorphisms. Then $H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) = H^0(S, \mathcal{O}_S(2)) = 0$ for any* non-trivial *A_S-invariant torsion line bundle $\varepsilon \in \operatorname{Pic} S$.*

Let us also point out that one can prove Theorem 1.3 directly by employing the arguments from [6] and the only fact that the group G is non-abelian (see **4.2** for details).

l or $\mathcal{C}$	p	$\mathcal{T}_1$	N	$\#\Pi$	A_S	$H_1(S,\mathbb{Z})$	Π lifts?	$H_1(S/A_S,\mathbb{Z})$	$N(\Pi)$ lifts?
$\mathbb{Q}(\sqrt{-7})$	2	$\emptyset$	21	3	G_{21}	$(\mathbb{Z}/2)^4$	yes	$\mathbb{Z}/2$	yes
		$\{7\}$	21	4	G_{21}	$(\mathbb{Z}/2)^3$	yes	0	yes
$\mathcal{C}_{20}$	2	$\emptyset$	21	1	G_{21}	$(\mathbb{Z}/2)^6$	yes	0	yes
$\mathcal{C}_2$	2	$\emptyset$	9	6	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/14$	yes	$\mathbb{Z}/2$	no
		$\{3\}$	9	1	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/7$	yes	0	no
$\mathcal{C}_{18}$	3	$\emptyset$	9	1	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/26 \times \mathbb{Z}/2$	yes	0	no

Table A

2. Preliminaries

2.1. We begin by recalling the next

Lemma 2.2 (see [6, Lemma 2.1])**.** *Let S be a fake projective plane with no 3-torsion in $H_1(S,\mathbb{Z})$. Then there exists a* unique *(ample) line bundle $\mathcal{O}_S(1)$ such that $\omega_S \cong \mathcal{O}_S(3)$.*

Let S and $\mathcal{O}_S(1)$ be as in Lemma 2.2. We will assume for what follows that $A_S = G_{21}$ or $(\mathbb{Z}/3)^2$. This implies that one has a lifting of the fundamental group $\Pi \subset PU(2,1)$ to $SU(2,1)$ (see Table A). Fix a lifting $r : \Pi \hookrightarrow SU(2,1)$ and consider the central extension

$$1 \to \mu_3 \to SU(2,1) \to PU(2,1) \to 1 \tag{2.3}$$

(here μ_3 denotes the cyclic group of order 3). Note that since $H_1(S,\mathbb{Z}) = \Pi/[\Pi,\Pi]$ does not contain a 3-torsion in our case, the embedding r is *unique*.

We thus get a linear action of $r(\Pi)$ on $\mathbb{C}^3$. In particular, both $\mathrm{Bl}_0\,\mathbb{C}^3 = \mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^2}(-1)$ and its restriction to the ball $\mathbb{B} \subset \mathbb{P}^2$ are preserved by $r(\Pi)$, so that we get the equality

$$\mathrm{Tot}\,\mathcal{O}_S(1) = (\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^2}(-1)\big|_{\mathbb{B}})/r(\Pi)$$

(cf. [9, 8.9]). Further, since there is a natural identification

$$\mathrm{Pic}^0\ S = \mathrm{Hom}\,(\Pi,\mathbb{C}^*) = \mathrm{Hom}\,(H_1(S,\mathbb{Z}),\mathbb{C}^*),$$

every torsion line bundle $\varepsilon \in \mathrm{Pic}\ S$ corresponds to a character $\chi_\varepsilon : \Pi \to \mathbb{C}^*$. One may twist the fiberwise $r(\Pi)$-action on $\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^2}(-1)$ by χ_ε (we will refer to this modified action as $r(\Pi)_{\chi_\varepsilon}$) and obtain

$$\mathrm{Tot}\,\mathcal{O}_S(1) \otimes \varepsilon = (\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^2}(-1)\big|_{\mathbb{B}})/r(\Pi)_{\chi_\varepsilon}.^{1)}$$

Observe that according to Table A there always exists such $\varepsilon \neq 0$. This table also shows that in two cases one can choose ε to be A_S-invariant.

2.4. Note that $A_S = N(\Pi)/\Pi$ for the normalizer $N(\Pi) \subset PU(2,1)$ of Π. Then (2.3) yields a central extension

$$1 \to \mu_3 \to G \to A_S \to 1$$

for $G := \widetilde{N(\Pi)}/r(\Pi)$ and the preimage $\widetilde{N(\Pi)} \subset SU(2,1)$ of $N(\Pi)$. Further, the previous construction of $\mathcal{O}_S(1)$ is $\widetilde{N(\Pi)}$-equivariant by the A_S-invariance of $\mathcal{O}_S(1)$, which gives a *linear* G-action on all the spaces $H^0(S,\mathcal{O}_S(k)), k \in \mathbb{Z}$. Similarly, if the torsion bundle ε is A_S-invariant, we get a linear G-action on all $H^0(S,\mathcal{O}_S(k)\otimes\varepsilon)$.

Recall next that when $A_S = G_{21}$, the bundle $\mathcal{O}_S(1)$ is A_S-linearizable (i. e. the group A_S lifts to G and the corresponding extension splits), and so the spaces $H^0(S,\mathcal{O}_S(k))$ are some linear A_S-representations in this case (see Table A or [6, Lemma 2.2]). The same holds for all $H^0(S,\mathcal{O}_S(k)\otimes\varepsilon)$ and any A_S-invariant ε.

In turn, if $A_S = (\mathbb{Z}/3)^2$, then extension G of this A_S does not split (see Table A), i. e. G coincides with the *Heisenberg group* H_3 of order 27. Again the G-action on $H^0(S,\mathcal{O}_S(k))$ (resp. on $H^0(S,\mathcal{O}_S(k)\otimes\varepsilon)$) is linear here.

Remark 2.5. Let $\xi,\eta \in G = H_3$ be two elements that map to order 3 generators of $A_S = (\mathbb{Z}/3)^2$. Then their commutator $[\xi,\eta]$ generates the center $\mu_3 \subset G$ and one obtains the following irreducible 3-dimensional (Schrödinger) representation of G:

$$\xi : x_i \mapsto \omega^{-i}x_i, \qquad \eta : x_i \mapsto x_{i+1} \quad (i \in \mathbb{Z}/3,\ \omega := e^{\frac{2\pi\sqrt{-1}}{3}}),$$

with x_i forming a basis in $\mathbb{C}^3$. (This representation, together with its complex conjugate, are the only 3-dimensional irreducible representations of H_3.) Observe at this point that according to the discussion above, $[\xi,\eta]$ acts on S

1) All these matters are extensively treated in [9, Ch. 8]

trivially and on $\mathcal{O}_S(1)$ via fiberwise-scaling by ω. Furthermore, given any A_S-invariant $\varepsilon \in \mathrm{Pic}^0\, S$, the commutator $[\xi, \eta]$ acts *non-trivially* on $\mathcal{O}_S(k) \otimes \varepsilon$ whenever k is coprime to 3.

2.6. Let us conclude this section by recalling two supplementary results.

Firstly, it follows from the Riemann – Roch formula that

$$\chi(\mathcal{O}_S(k) \otimes \varepsilon) = \frac{(k-1)(k-2)}{2}$$

for all $k \in \mathbb{Z}$ and $\varepsilon \in \mathrm{Pic}^0\, S$. In particular, we get $H^0(S, \mathcal{O}_S(k) \otimes \varepsilon) \simeq \mathbb{C}^{(k-1)(k-2)/2}$ when $k \geqslant 4$ because

$$H^i(S, \mathcal{O}_S(k) \otimes \varepsilon) = H^i(S, \omega_S \otimes \mathcal{O}_S(k-3) \otimes \varepsilon) = 0$$

for all $i > 0$ by Kodaira vanishing.

Secondly, for every $\tau \in A_S$ with only isolated set of fixed points $P_1, \ldots, P_N$ and any line bundle $L \in \mathrm{Pic}\, S$ satisfying $\tau^* L \cong L$, one has the *Holomorphic Lefschetz Fixed Point Formula* (see [1, Theorem 2]):

$$\sum_{i=0}^{2} (-1)^i \operatorname{Tr} \tau\big|_{H^i(S,L)} = \sum_{i=1}^{N} \frac{\operatorname{Tr} \tau|_{L_{P_i}}}{(1-\alpha_1(P_i))(1-\alpha_2(P_i))},$$

where $\alpha_1(P_i)$, $\alpha_2(P_i)$ are the eigenvalues of τ acting on the tangent spaces T_{S,P_i} to S at P_i (resp. L_{P_i} are the closed fibers of L at P_i).

3. Automorphic forms of higher weight

3.1. We retain the notation of Section 2. Fix some A_S-invariant torsion line bundle ε.

Recall that fundamental group Π of S is a torsion-free *type II arithmetic subgroup* in $PU(2,1)$ (see [8, 14]). The same applies to any finite index subgroup of Π. In particular, for the unramified cyclic covering $\phi : S' \longrightarrow S$ associated with ε one has $[\Pi : \pi_1(S')] < \infty$ and the following important result takes place (without any assumption on A_S and invariance of ε):

Theorem 3.2 (see [12, Theorem 15. 3. 1])**.** *The surface S' is regular. That is* $\operatorname{rank} H_1(S', \mathbb{Q}) := q(S') = 0$.

We have

$$S' = \mathrm{Spec}_S \left(\bigoplus_{i=0}^{m-1} \varepsilon^i \right),^{2)}$$

where m is the order of $\varepsilon \in \mathrm{Pic}^0\, S$, and hence

$$\phi_* \mathcal{O}_{S'} = \bigoplus_{i=0}^{m-1} \varepsilon^i.$$

From Leray spectral sequence and Theorem 3.2 we deduce

$$0 = q(S') = h^1(S, \mathcal{O}_{S'}) = h^1(S, \phi_* \mathcal{O}_{S'}) = \bigoplus_{i=0}^{m-1} h^1(S, \varepsilon^i).$$

Thus $h^1(S, \varepsilon^i) = 0$ for all $i \in \mathbb{Z}$.

Lemma 3.3. $h^0(S, \omega_S \otimes \varepsilon^i) = 1$ *for all i not divisible by m.*

2) We are using the notation $\varepsilon^i := \varepsilon^{\otimes i}$.

Proof. Indeed, we have $\chi(S, \omega_S \otimes \varepsilon^i) = 1$ (see **2.6**) and also

$$h^2(S, \omega_S \otimes \varepsilon^i) = h^0(S, \varepsilon^i) = 0 = h^1(S, \varepsilon^i) = h^1(S, \varepsilon^{-i}) = h^1(S, \omega_S \otimes \varepsilon^i)$$

by Serre duality, which gives $h^0(S, \omega_S \otimes \varepsilon^i) = \chi(S, \omega_S \otimes \varepsilon^i) = 1$. $\Box$

3.4. Let $\tau \in A_S$ be any element of order 3 with a *faithful* action on S. Choose any linearization of the action of τ on $\operatorname{Tot} \mathcal{O}_S(1)$.

Lemma 3.5. *The automorphism τ has only three fixed points, and in the holomorphic fixed point formula all denominators coincide (and are equal to 3) and all numerators are three distinct 3rd roots of unity. In other words, one can choose a numbering of the fixed points P_i and weights $\alpha_j(P_i)$ in such a way that the following holds:*

- $\alpha_j(P_i) = \omega^j$ *for all* i, j;
- $w_i := \operatorname{Tr} \tau|_{\mathcal{O}_S(1)_{P_i}} = \omega^i$ *for all* i.

Moreover, for any τ-invariant $\varepsilon \in \operatorname{Pic}^0 S$ of order m coprime to 3 we can choose a linearization such that $\operatorname{Tr} \tau|_{\varepsilon_{P_i}} = 1$. In particular, $\operatorname{Tr} \tau|_{(\mathcal{O}_S(k) \otimes \varepsilon^l)_{P_i}} = \omega^{ik}$ for all i, k, l.

Proof. The claim about P_i and $\alpha_j(P_i)$ follows from [7, Proposition 3.1].

Further, we have $V := H^0(S, \mathcal{O}_S(4)) \cong \mathbb{C}^3$ (see **2.6**). Let v_1, v_2, v_3 be the eigen values of τ acting on V. Then $v_i^3 = 1$ for all i and $\operatorname{Tr} \tau\big|_V = v_1 + v_2 + v_3$. At the same time, as follows from definitions and **2.6** (with $N = 3, L = \mathcal{O}_S(4)$), we have

$$\operatorname{Tr} \tau\big|_V = \frac{w_1^4 + w_2^4 + w_3^4}{(1-\omega)(1-\omega^2)} = \frac{w_1 + w_2 + w_3}{3}$$

The latter can be equal to $v_1 + v_2 + v_3$ only when all v_i (resp. all w_i) are pairwise distinct (so that both sums are zero). This is due to the fact that the sum of three 3rd roots of unity has the norm $\in \{0, \sqrt{3}, 3\}$ and is zero iff all roots are distinct.

Finally, since $(m, 3) = 1$, we may replace ε by ε^3, so that the action of τ on the closed fibers ε_{P_i} is trivial. The last assertion about $\mathcal{O}_S(k) \otimes \varepsilon^l$ is evident. $\Box$

Recall that the group G from **2.4** acts linearly on all spaces $V := H^0(S, \mathcal{O}_S(k) \otimes \varepsilon)$.

Proposition 3.6. *Let $G = H_3$. Then for $k \geqslant 4$ the following holds (cf. Table B below):*

- *for $k \equiv 0 \mod 3$ we have $V = V_0 \oplus \mathbb{C}[(\mathbb{Z}/3)^2]^a$ as G-representations, where $a := \frac{k}{6}(\frac{k}{3} - 1)$ and $V_0 \simeq \mathbb{C}$ (resp. $\mathbb{C}[(\mathbb{Z}/3)^2]$) is the trivial (resp. regular) representation;*
- *for $k \equiv 1 \mod 3$ we have $V = \mathcal{V}_3^{\oplus(k-1)(k-2)/6}$ as G-representations, where $\mathcal{V}_3$ is an irreducible 3-dimensional representation of H_3;*
- *for $k \equiv 2 \mod 3$ we have $V = \overline{\mathcal{V}}_3^{\oplus(k-1)(k-2)/6}$ as G-representations, where $\overline{\mathcal{V}}_3$ is the complex conjugate to $\mathcal{V}_3$ above.*

$4 \leqslant k \mod 3$	0	1	2
$H^0(S, \mathcal{O}_S(k) \otimes \varepsilon)$	$V_0 \oplus \mathbb{C}[(\mathbb{Z}/3)^2]^a$	$\mathcal{V}_3^{\oplus(k-1)(k-2)/6}$	$\overline{\mathcal{V}}_3^{\oplus(k-1)(k-2)/6}$

Table B

Proof. Suppose that $k \equiv 0 \mod 3$. Then, since every element in H_3 has order 3, applying Lemma 3.5 to any non-central $\tau \in G$ we obtain $(1 - \alpha_1(P_i))(1 - \alpha_2(P_i)) = 3$ for all i and $\operatorname{Tr} \tau\big|_V = 1$ (cf. **2.6**).

Further, the element $[\xi, \eta] \in G$ from Remark 2.5 acts trivially on $\mathcal{O}_S(k)$ (via scaling by $\omega^k = 1$). Also, since the order of any $\varepsilon \neq 0$ is coprime to 3 (see Table A) and ε is flat (cf. its construction in **2.1**), from Lemma 3.5 we find that $[\xi, \eta]$ acts trivially on $\mathcal{O}_S(k) \otimes \varepsilon$, hence on V as well. This implies that the G-action on V factors through that of its quotient $(\mathbb{Z}/3)^2$. Then the claimed decomposition $V = V_0 \oplus \mathbb{C}[(\mathbb{Z}/3)^2]^a$ follows from the fact that $1 + 9a = \dim V = \operatorname{Tr} [\xi, \eta]\big|_V$ and that $\operatorname{Tr} \tau\big|_V = 1$ for all non-central $\tau \in G$.

Let now $k \equiv 1 \mod 3$ (resp. $k \equiv 2 \mod 3$). Then it follows from Remark 2.5 that $[\xi, \eta]$ scales all vectors in $V = H^0(S, \mathcal{O}_S(k) \otimes \varepsilon) \cong \mathbb{C}^{(k-1)(k-2)/2}$ by $\omega^k \neq 1$. Furthermore, since $\operatorname{Tr} \xi\big|_V = 0 = \operatorname{Tr} \eta\big|_V$ according to Lemma 3.5 and **2.6**, all irreducible summands of V are faithful G-representations, hence isomorphic to $\mathcal{V}_3$ (resp. to $\overline{\mathcal{V}}_3$). This concludes the proof. □

Proposition 3.7. *Let $G \supset A_S = G_{21}$. Then for $k \geqslant 4$ and $V = H^0(S, \mathcal{O}_S(k) \otimes \varepsilon)$ we have the following equality of (virtual) G-representations*

$$V = \mathbb{C}[G]^{\oplus a_k} \oplus U_k$$

for some $a_k \in \mathbb{Z}$ expressed in terms of $\dim V$*, where U_k depends only on* $k \mod 21$ *and is explicitly given in the table below, with rows (resp. columns) being enumerated by* $k \mod 3$ *(resp.* $k \mod 7$*)*

	0	1	2	3	4	5	6
0	$\mathbb{C}$	$\mathcal{V}_3 \oplus \overline{\mathcal{V}}_3 \oplus \mathbb{C}$	$\mathcal{V}_3 \oplus \overline{\mathcal{V}}_3 \oplus \mathbb{C}$	$\mathbb{C}$	$\mathcal{V}_3^{\oplus 2} \oplus \overline{\mathcal{V}}_3 \oplus \mathbb{C}$	$(\mathcal{V}_3 \oplus \overline{\mathcal{V}}_3)^{\oplus 2} \oplus \mathbb{C}$	$\mathcal{V}_3 \oplus \overline{\mathcal{V}}_3^{\oplus 2} \oplus \mathbb{C}$
1 *or* 2	$(-\mathcal{V}_3) \oplus (-\overline{\mathcal{V}}_3)$	0	0	$(-\mathcal{V}_3) \oplus (-\overline{\mathcal{V}}_3)$	$\overline{\mathcal{V}}_3$	$\mathcal{V}_3 \oplus \overline{\mathcal{V}}_3$	$\mathcal{V}_3$

Proof. From **2.6** we obtain

$$\operatorname{Tr} \sigma\big|_V = \frac{\zeta^{6k}}{(1-\zeta)(1-\zeta^3)} + \frac{\zeta^{5k}}{(1-\zeta^2)(1-\zeta^6)} + \frac{\zeta^{3k}}{(1-\zeta^4)(1-\zeta^5)}$$

Here $\zeta := e^{\frac{2\pi\sqrt{-1}}{7}}$ and $\sigma \in G_{21}$ is an element of order 7. The value $\operatorname{Tr} \sigma\big|_V$ depends only on $k \mod 7$ and by direct computation we obtain the following table:

k	0	1	2	3	4	5	6
$\operatorname{Tr} \sigma\big|_V$	1	0	0	1	$\bar{b}$	-1	b

(Here $b := \zeta + \zeta^2 + \zeta^4$ and $\bar{b} = -1 - b = \zeta^3 + \zeta^5 + \zeta^6$.)

Let $\tau \in G_{21}$ be an element of order 3 such that $G_{21} = \langle \sigma, \tau \rangle$. Recall the character table for the group G_{21} (see e. g. the proof of Lemma 4.2 in [6]):

	1	Tr σ	Tr σ^3	Tr τ	Tr τ^2
$\mathbb{C}$	1	1	1	1	1
$\mathcal{V}_1$	1	1	1	ω	$\overline{\omega}$
$\overline{\mathcal{V}}_1$	1	1	1	$\overline{\omega}$	ω
$\mathcal{V}_3$	3	b	$\overline{b}$	0	0
$\overline{\mathcal{V}}_3$	3	$\overline{b}$	b	0	0

(Here "—" signifies, as usual, the complex conjugation and $\mathcal{V}_i$ are irreducible i-dimensional representations of G_{21}.)

Now from Lemma 3.5 (cf. **2.6**) and the previous tables we get the claimed options for V. This concludes the proof. □

4. Proof of Theorem 1.3

4.1. We retain the earlier notation.

Suppose that $G = H_3$ and $H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \neq 0$. Let also $\varepsilon \neq 0$ (cf. the end of **2.1**). Consider the natural homomorphism of G-modules

$$H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \otimes H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \to H^0(S, \mathcal{O}_S(4) \otimes \varepsilon^2).$$

Since $h^0(S, \mathcal{O}_S(4) \otimes \varepsilon^2) = \chi(S, \mathcal{O}_S(4) \otimes \varepsilon^2) = 3$ (see **2.6**), from [9, Lemma 15.6.2] we obtain $h^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \leqslant 2$ (cf. the proof of Lemma 4.2 in [6]).

On the other hand, there is natural homomorphism of G-modules

$$H^0(S, \omega_S \otimes \varepsilon^{-1}) \otimes H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \to H^0(S, \mathcal{O}_S(5)),$$

and so Lemma 3.3 implies that $H^0(S, \mathcal{O}_S(2) \otimes \varepsilon)$ is a *non-trivial* subrepresentation in $H^0(S, \mathcal{O}_S(5))$ of dimension $\leqslant 2$. But the latter contradicts Proposition 3.6 and Theorem 1.3 follows in this case.

Similarly, consider the natural homomorphism of G-modules

$$H^0(S, \mathcal{O}_S(2)) \otimes H^0(S, \mathcal{O}_S(2)) \to H^0(S, \mathcal{O}_S(4)),$$

where again $h^0(S, \mathcal{O}_S(2)) \leqslant 2$. Then the G-homomorphism

$$H^0(S, \omega_S \otimes \varepsilon) \otimes H^0(S, \mathcal{O}_S(2)) \to H^0(S, \mathcal{O}_S(5) \otimes \varepsilon)$$

gives contradiction with Proposition 3.6 whenever $H^0(S, \mathcal{O}_S(2)) \neq 0$. This concludes the proof of Theorem 1.3 for $A_S = (\mathbb{Z}/3)^2$. Finally, the case of $A_S = G_{21}$ is literally the same, with Proposition 3.7 used instead.

4.2. Alternatively, consider the natural homomorphism of G-modules for any G as in **2.4**:

$$S^2 H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \to H^0(S, \mathcal{O}_S(4)),$$

where ε is 2-torsion (cf. Table A) and the case $\varepsilon = 0$ is also allowed. Again we have $h^0(S, \mathcal{O}_S(2) \otimes \varepsilon) \leqslant 2$. Then, applying Propositions 3.6 and 3.7 we conclude that $H^0(S, \mathcal{O}_S(2) \otimes \varepsilon) = 0$, exactly as in the proof of [6, Theorem 1.3]. This is another proof of Theorem 1.3.

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