FOURIER-MUKAI TRANSFORMS AND STABILITY CONDITIONS ON ABELIAN VARIETIES

DULIP PIYARATNE

ABSTRACT. This article is based on a talk given at the Kinosaki Symposium on Algebraic Geometry in 2015, about a work in progress [Piy2]. We describe a polarization on a derived equivalent abelian variety by using Fourier-Mukai theory. We explicitly formulate a conjecture which says certain Fourier-Mukai transforms between derived categories give equivalences of some hearts of Bridgeland stability conditions. We establish it for abelian surfaces, which is already known due to D. Huybrechts, and for abelian 3-folds. This generalizes the author's previous joint work [MP1, MP2] with A. Maciocia on principally polarized abelian 3-folds with Picard rank one. Consequently, we see that the strong Bogomolov-Gieseker type inequalities hold for tilt stable objects on abelian 3-folds.

1. INTRODUCTION

The notion of Fourier-Mukai transform was introduced by Mukai in early 1980s (see [Muk2]). In particular, he showed that the Poincaré bundle induces a non-trivial equivalence between the derived categories of an abelian variety and its dual variety. Furthermore, he studied certain type of vector bundles on abelian varieties called semihomogeneous bundles, and moduli of them (see [Muk1]). In particular, the moduli space parametrizing simple semihomogeneous bundles on an abelian variety Y with a fixed Chern character is also an abelian variety, denoted by X. Moreover, the associated universal bundle $\mathcal{E}$ on $X \times Y$ induces a derived equivalence $\Phi_{\mathcal{E}}^{X \to Y}$ from X to Y, which is commonly known as the Fourier-Mukai transform. This transform induces a linear isomorphism $\Phi_{\mathcal{E}}^{H}$ from $H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$ to $H^{2*}_{\mathrm{alg}}(Y, \mathbb{Q})$, called the cohomological Fourier-Mukai transform. In this article, we realize this linear isomorphism in anti-diagonal form with respect to some twisted Chern characters (see Theorem 3.4). This generalizes similar work on principally polarized abelian varieties with Picard rank one case (see [MP2]). The following is one of the important consequences of this anti-diagonal representation, which generalizes the known result [BL] for the classical Fourier-Mukai transform with kernel the Poincaré bundle.

Theorem 1.1 ($= 3.2$). *For $\mathfrak{a} \in X$, $\mathfrak{b} \in Y$, let $\Xi : D^{\mathrm{b}}(X) \to D^{\mathrm{b}}(Y)$ be the Fourier-Mukai functor defined by*

$$\Xi = \mathcal{E}^{*}_{\{\mathfrak{a}\} \times Y} \circ \Phi_{\mathcal{E}}^{X \to Y} \circ \mathcal{E}^{*}_{X \times \{\mathfrak{b}\}},$$

where $\mathcal{E}^{*}_{\{\mathfrak{a}\} \times Y}$ denotes the functor $\mathcal{E}^{*}_{\{\mathfrak{a}\} \times Y} \otimes (-)$ and similar for $\mathcal{E}^{*}_{X \times \{\mathfrak{b}\}}$. If the ample line bundle L defines a polarization on X, then the line bundle $\det(\Xi(L))^{-1}$ is ample and so it defines a polarization on Y.

Motivated by Douglas's work on Π-stability for D-branes, Bridgeland introduced the notion of stability conditions on triangulated categories (see [Bri1]). This categorical stability notion can be interpreted essentially as an abstraction of the usual slope stability for sheaves on projective curves. The category of coherent sheaves does not arise as a heart of a Bridgeland stability condition for higher dimensional smooth projective varieties. So more work is needed to construct the hearts for stability conditions on projective varieties of dimension above one. In general, when $B + i\omega$ is a complexified ample class on a projective variety X, it is expected that $Z_{B+i\omega}^X(-) = -\int_X e^{-B-i\omega} \operatorname{ch}(-)$ defines a central charge function of some stability condition on X. In this article we conjecturally construct a heart for this central charge function by using the notion of very weak stability condition (see Conjecture 2.3). This essentially generalizes the single tilting construction of Bridgeland and Arcara-Bertram for surfaces, and the conjectural double tilting construction of Bayer-Macrì-Toda for 3-folds.

Action of the Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X \to Y}$ induces stability conditions on $D^b(Y)$ from the ones on $D^b(X)$. This can be defined via the induced the map on $\operatorname{Hom}(K(Y), \mathbb{C})$ from $\operatorname{Hom}(K(X), \mathbb{C})$ by the transform. For abelian varieties we view this as $\Phi_{\mathcal{E}}^{X \to Y} \cdot Z_{\Omega}^X = \zeta Z_{\Omega'}^Y$, for some $\zeta \in \mathbb{C}$, where Ω, Ω' are complexified ample classes on X, Y respectively. When ζ is real one can expect that the Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X \to Y}$ gives an equivalence of some hearts of particular stability conditions on X and Y, whose Ω and Ω' are determined by $\operatorname{Im} \zeta = 0$. More precisely we formulate the following:

Conjecture 1.2 (= 4.2). *The Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ between the abelian varieties gives the equivalence of stability condition hearts each of which are g-fold tilts as in Conjecture 2.3:*

$$\Phi_{\mathcal{E}}^{X \to Y}[k] \left(\mathcal{A}_{\Omega_k}^X \right) = \mathcal{A}_{\Omega_k'}^Y.$$

Here Ω_k, Ω_k' are some complexified ample classes on X, Y respectively, $1 \leqslant k \leqslant (g-1)$, and $g = \dim X = \dim Y$.

In the last section we establish the above conjecture for abelian surfaces and abelian 3-folds. In particular, we explicitly give the details for the surface case and our proof is essentially a generalization of Yoshioka's in [Yos], and closely follow the one in the author's PhD thesis [Piy1, Chapter 6]. The equivalences of such stability condition hearts are already known for a principally polarized abelian 3-fold with Picard rank one. Since we have the anti-diagonal representation of cohomological Fourier-Mukai transform as in Theorem 3.4, essentially the same arguments work in the general case. In particular, we deduce the following:

Theorem 1.3 (= 5.2). *Conjecture 1.2 is true for abelian 3-folds. Moreover, the strong Bogomolov-Gieseker type inequality introduced in [BMT] holds for abelian 3-folds.*

Notation

- Unless otherwise stated, throughout this paper, all the varieties are smooth projective and defined over $\mathbb{C}$. For a variety X, by $\operatorname{Coh}(X)$ we denote the category of coherent sheaves on X, and by $D^b(X)$ we denote the bounded derived category of $\operatorname{Coh}(X)$.
- For $0 \leqslant i \leqslant \dim X$, $\operatorname{Coh}_{\leqslant i}(X) := \{E \in \operatorname{Coh}(X) : \dim \operatorname{Supp}(E) \leqslant i\}$, $\operatorname{Coh}_{\geqslant i}(X) := \{E \in \operatorname{Coh}(X) : \text{for } 0 \neq F \subset E, \dim \operatorname{Supp}(F) \geqslant i\}$ and $\operatorname{Coh}_i(X) := \operatorname{Coh}_{\leqslant i}(X) \cap \operatorname{Coh}_{\geqslant i}(X)$.
- For $E \in D^b(X)$, $E^\vee = \mathbf{R} \mathcal{H}om(E, \mathcal{O}_X)$. When E is a torsion free sheaf we write its dual by E^*; so for a locally free sheaf E, $E^\vee = E^*$.
- The structure sheaf of a closed subscheme $Z \subset X$ as an object in $\operatorname{Coh}(X)$ is denoted by $\mathcal{O}_Z$ and when $Z = \{x\}$ for some closed point $x \in X$ it is simply denoted by $\mathcal{O}_x$.

- When $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, by $H_{\mathcal{A}}^i(-)$ we denote the corresponding i-th cohomology functor.
- For a set of objects $\mathcal{S} \subset \mathcal{D}$ in a triangulated category $\mathcal{D}$, by $\langle \mathcal{S} \rangle \subset \mathcal{D}$ we denote its extension closure, that is the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{S}$.
- We denote the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by $\mathbb{H}$.
- We will denote an $m \times m$ anti-diagonal matrix with entries a_k, $k = 1, \ldots, m$ by

$$\operatorname{Adiag}(a_1, \ldots, a_m)_{ij} := \begin{cases} a_k & \text{if } i = k, j = m + 1 - k \\ 0 & \text{otherwise.} \end{cases}$$

2. Preliminaries

2.1. **Very weak stability conditions.** Let us recall the general arguments of very weak stability conditions, which are the variants of weak stability of [Tod2] introduced in [BMS, Definition B.1]. We closely follow the notions as in [PT].

Let $\mathcal{D}$ be a triangulated category, and $K(\mathcal{D})$ its Grothendieck group. We fix a finitely generated free abelian group Γ and a group homomorphism

$$\operatorname{cl} \colon K(\mathcal{D}) \to \Gamma.$$

Definition 2.1. A *very weak stability condition* on $\mathcal{D}$ is a pair $(Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z \colon \Gamma \to \mathbb{C}$ is a group homomorphism satisfying the following conditions:

(i) For any $E \in \mathcal{A}$, we have $Z(E) \in \mathbb{H} \cup \mathbb{R}_{\leqslant 0}$. Here and elsewhere we write $Z(E) = Z(\operatorname{cl}(E))$, and $\mathbb{H}$ is the upper half plane.

(ii) The associated slope function $\mu \colon \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mu(E) = \begin{cases} +\infty & \text{if } \operatorname{Im} Z(E) = 0 \\ -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)} & \text{otherwise,} \end{cases}$$

and it satisfies the Harder-Narasimhan property.

We say that $E \in \mathcal{A}$ is μ-(semi)stable if for any non-zero subobject $F \subset E$ in $\mathcal{A}$, we have the inequality: $\mu(F) < (\leqslant) \mu(E/F)$.

The Harder-Narasimhan filtration of an object $E \in \mathcal{A}$ is a chain of subobjects $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ in $\mathcal{A}$ such that each $F_i = E_i/E_{i-1}$ is μ-semistable with $\mu(F_i) > \mu(F_{i+1})$. If such Harder-Narasimhan filtrations exists for all objects in $\mathcal{A}$, we say that μ satisfies the Harder-Narasimhan property.

For a given a very weak stability condition $(Z, \mathcal{A})$, we define its slicing on $\mathcal{D}$ (see [Bri1, Definition 3.3])

$$\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}, \ \mathcal{P}(\phi) \subset \mathcal{D}$$

as in the case of Bridgeland stability conditions (see [Bri1, Proposition 5.3]). Namely, for $0 < \phi \leqslant 1$, the category $\mathcal{P}(\phi)$ is defined to be

$$\mathcal{P}(\phi) = \{E \in \mathcal{A} : E \text{ is } \mu\text{-semistable with } \mu(E) = -1/\tan(\pi\phi)\} \cup \{0\}.$$

Here we set $-1/\tan \pi = \infty$. The other subcategories are defined by setting

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$

For an interval $I \subset \mathbb{R}$, we define $\mathcal{P}(I)$ to be the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{P}(\varphi)$ for each $\varphi \in I$. For $0 \leqslant s \leqslant 1$, the pair $(\mathcal{P}((s,1]), \mathcal{P}((0,s]))$ of subcategories of $\mathcal{A} = \mathcal{P}((0,1])$ is a torsion pair, and the corresponding tilt is $\mathcal{P}((s, s+1])$.

Note that the category $\mathcal{P}(1)$ contains the following category

$$\mathcal{C} := \{E \in \mathcal{A} : Z(E) = 0\}.$$

It is easy to check that $\mathcal{C}$ is closed under subobjects and quotients in $\mathcal{A}$. In particular, $\mathcal{C}$ is an abelian subcategory of $\mathcal{A}$. Moreover, the pair $(Z, \mathcal{A})$ gives a *Bridgeland stability condition* on $\mathcal{D}$ if $\mathcal{C} = \{0\}$.

2.2. Bridgeland stability on varieties.

Let X be a smooth projective variety and let $D^b(X)$ be the bounded derived category of coherent sheaves on X. A stability condition σ on $D^b(X)$ is *numerical* if the central charge function $Z : K(X) \to \mathbb{C}$ factors through the Chern character map $\mathrm{ch} : K(X) \to H^{2*}_{\mathrm{alg}}(X, \mathbb{Q})$.

Usually, a stability condition σ on $D^b(X)$ is called *geometric* if all the skyscraper sheaves $\mathcal{O}_x$ of $x \in X$ are σ-stable of the same phase. The following result gives some properties of geometric stability conditions on varieties.

Proposition 2.2. *Let X be a smooth projective variety of dimension n. Let $\sigma = (Z, \mathcal{A})$ be a geometric stability condition on $D^b(X)$ with all skyscraper sheaves $\mathcal{O}_x$ of $x \in X$ are σ-stable with phase one. If $E \in \mathcal{A}$ then $H^i_{\mathrm{Coh}(X)}(E) = 0$ for $i \notin \{-n+1, -n+2, \ldots, 0\}$.*

Proof. The following proof is adapted from [Bri2, Lemma 10.1]. Let $\mathcal{P}$ be the corresponding slicing of σ. Since $\mathcal{A} = \mathcal{P}((0,1])$ and $\mathrm{Coh}_0(X) \subset \mathcal{P}(1)$, from the Harder-Narasimhan property, we only need to consider $E \in \mathcal{A}$ such that $\mathrm{Hom}_X(\mathrm{Coh}_0(X), E) = 0$. For any skyscraper sheaf $\mathcal{O}_x$ of $x \in X$ we have $\mathcal{O}_x[i] \in \mathcal{P}(1+i)$ and $E[i] \in \mathcal{P}((i, 1+i])$. Therefore, for all $i < 0$, $\mathrm{Hom}_X(E, \mathcal{O}_x[i]) = 0$, and $\mathrm{Hom}_X(\mathcal{O}_x, E[1+i]) \cong \mathrm{Hom}_X(E, \mathcal{O}_x[n-1-i])^* = 0$. So by [BM, Proposition 5.4], E is quasi-isomorphic to a complex of locally free sheaves of length n. This completes the proof as required. $\square$

When X is a smooth projective curve, the central charge function $Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$ together with the heart $\mathrm{Coh}(X)$ of the standard t-structure defines a geometric stability condition on $D^b(X)$. However, for a smooth projective variety X with $\dim X \geqslant 2$, there is no numerical stability condition on $D^b(X)$ with $\mathrm{Coh}(X)$ as the heart of a stability condition (see [Tod1, Lemma 2.7] for a proof). Motivated by the constructions for smooth projective surfaces (see [Bri2, AB]) together with some observations in Mathematical Physics, for any dimensional smooth projective variety X, it is expected that the function defined by

$$Z_{B+i\omega}(-) = -\int_X e^{-B-i\omega} \mathrm{ch}(-)$$

is a central charge function of some geometric stability condition on $D^b(X)$ (see [BMT, Conjecture 2.1.2]). Here $B + i\omega \in \mathrm{NS}_{\mathbb{C}}(X)$ is a complexified ample class on X, that is by definition $B, \omega \in \mathrm{NS}_{\mathbb{R}}(X)$ with ω an ample class. By using the notion of very weak stability, let us conjecturally construct a heart for this central charge function.

Let $n = \dim X$. For $0 \leqslant k \leqslant n$, we define the k-truncated Chern character by

$$\mathrm{ch}_{\leqslant k}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \ldots, \mathrm{ch}_k(E), 0, \ldots, 0),$$

and the function $Z_{B+i\omega}^{(k)} : K(X) \to \mathbb{C}$ by

$$Z_{B+i\omega}^{(k)}(E) = -i^{n-k} \int_X e^{-B-i\omega} \operatorname{ch}_{\leqslant k}(E).$$

The usual slope stability on sheaves gives the very weak stability condition $(Z_{B+i\omega}^{(1)}, \operatorname{Coh}(X))$. Moreover, we formulate the following:

Conjecture 2.3 ([Piy2]). *For each $1 \leqslant k < n$, the pair $\sigma_k = (Z_{B+i\omega}^{(k)}, \mathcal{A}_{B+i\omega}^{(k)})$ gives a very weak stability condition on $D^b(X)$, where the hearts $\mathcal{A}_{B+i\omega}^{(k)}$, $1 \leqslant k \leqslant n$ are defined by*

$$\left. \begin{array}{l} \mathcal{A}_{B+i\omega}^{(1)} = \operatorname{Coh}(X) \\ \mathcal{A}_{B+i\omega}^{(k+1)} = \mathcal{P}_{\sigma_k}((1/2, 3/2]) \end{array} \right\}.$$

Moreover, the pair $\sigma_n = (Z_{B+i\omega}, \mathcal{A}_{B+i\omega}^{(n)})$ is a Bridgeland stability condition on $D^b(X)$.

Remark 2.4. As mentioned before, $(Z_{\omega,B}^{(1)}, \mathcal{A}_{B+i\omega}^{(1)} = \operatorname{Coh}(X))$ is a very weak stability condition for any variety and a Bridgeland stability condition for curves. By [Bri2, AB], $(Z_{\omega,B}^{(2)}, \mathcal{A}_{B+i\omega}^{(2)})$ is a Bridgeland stability condition for surfaces. In [BMT], the authors proved that the pair $(Z_{B+i\omega}^{(2)}, \mathcal{A}_{B+i\omega}^{(2)})$ is again a very weak stability condition for 3-folds. Here the stability was called tilt slope stability. The usual Bogomolov-Gieseker inequality for $Z_{B+i\omega}^{(1)}$ stable sheaves plays a crucial role in these proofs. Clearly the same arguments work for any higher dimensional varieties. Therefore, we can always construct the category $\mathcal{A}_{B+i\omega}^{(3)}$ when $\dim X \geqslant 2$. In [BMT], the authors conjectured that this category is a heart of a Bridgeland stability condition with the central charge $Z_{B+i\omega}$. Moreover, they reduced it to prove Bogomolov-Gieseker type inequalities for $Z_{B+i\omega}^{(2)}$ stable objects $E \in \mathcal{A}_{B+i\omega}^{(2)}$ with $\operatorname{Re} Z_{B+i\omega}^{(2)}(E) = 0$:

(1) $$\operatorname{ch}_3^B(E) \leqslant \frac{\omega^2}{18} \operatorname{ch}_1^B(E).$$

So far this conjectural inequality is known to hold for some Fano 3-folds [BMT, Mac, Sch, Li], abelian 3-folds [MP1, MP2, Piy1, BMS] and the étale quotients of abelian 3-folds [BMS].

2.3. **Fourier-Mukai theory.** Let us quickly recall some of the important notions in Fourier-Mukai theory. Further details can be found in [Huy2].

Let X, Y be smooth projective varieties and let p_i, $i = 1, 2$ be the projection maps from $X \times Y$ to X and Y, respectively. The *Fourier-Mukai functor* $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ with kernel $\mathcal{E} \in D^b(X \times Y)$ is defined by

$$\Phi_{\mathcal{E}}^{X \to Y}(-) = \mathbf{R}p_{2*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_1^*(-)).$$

The map $\Sigma : Y \times X \to X \times Y$ is defined by $\Sigma(y, x) = (x, y)$ for any $x \in X$ and $y \in Y$. Let $\mathcal{E}_L = \Sigma^*(\mathcal{E}^\vee \overset{\mathbf{L}}{\otimes} p_2^* \omega_Y)[\dim Y]$, and $\mathcal{E}_R = \Sigma^*(\mathcal{E}^\vee \overset{\mathbf{L}}{\otimes} p_1^* \omega_X)[\dim X]$. Then we have the adjunctions (see [Huy2, Proposition 5.9]): $\Phi_{\mathcal{E}_L}^{Y \to X} \dashv \Phi_{\mathcal{E}}^{X \to Y} \dashv \Phi_{\mathcal{E}_R}^{Y \to X}$.

When $\Phi_{\mathcal{E}}^{X \to Y}$ is an equivalence of the derived categories, usually it is called a *Fourier-Mukai transform*. On the other hand by Orlov's Representability Theorem (see [Huy2, Theorem

5.14]), any equivalence between $D^b(X)$ and $D^b(Y)$ is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X \to Y}$ for some $\mathcal{E} \in D^b(X \times Y)$.

Any Fourier-Mukai functor $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ induces a linear map $\Phi_{\mathcal{E}}^H : H_{alg}^{2*}(X, \mathbb{Q}) \to H_{alg}^{2*}(Y, \mathbb{Q})$, usually called the cohomological Fourier-Mukai functor, and it is an isomorphism when $\Phi_{\mathcal{E}}^{X \to Y}$ is a Fourier-Mukai transform. The induced transform fits into the following commutative diagram, due to the Grothendieck-Riemann-Roch theorem.

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\;\;\Phi_{\mathcal{E}}^{X \to Y}\;\;} & D^b(Y) \\
{\scriptstyle [-]}\downarrow & & \downarrow{\scriptstyle [-]} \\
K(X) & \xrightarrow{\;\;\Phi_{\mathcal{E}}^K\;\;} & K(Y) \\
{\scriptstyle \nu_X(-)}\downarrow & & \downarrow{\scriptstyle \nu_Y(-)} \\
H_{alg}^{2*}(X, \mathbb{Q}) & \xrightarrow{\;\;\Phi_{\mathcal{E}}^H\;\;} & H_{alg}^{2*}(Y, \mathbb{Q})
\end{array}
$$

Here $\nu_Z(-) = \mathrm{ch}(-)\sqrt{\mathrm{td}_Z}$ is the Mukai vector map, where $\mathrm{ch} : K(Z) \to H_{alg}^{2*}(Z, \mathbb{Q})$ is the Chern character map and td_Z is the Todd class of Z.

Let $\nu \in H_{alg}^{2*}(X, \mathbb{Q})$ be a Mukai vector. Then $\nu = \sum_{i=0}^{\dim X} \nu_i$ for $\nu_i \in H_{alg}^{2i}(X, \mathbb{Q})$ and the Mukai dual of ν is defined by $\nu^* = \sum_{i=0}^{\dim X}(-1)^i \nu_i$. A symmetric bilinear form $\langle -, - \rangle_X$ called *Mukai pairing* is defined by the formula

$$
\langle \nu, w \rangle_X = -\int_X \nu^* \cdot w \cdot e^{c_1(X)/2}.
$$

Note that for an abelian variety X, $\mathrm{td}_X = 1$ and $c_1(X) = 0$. Hence the Mukai vector $\nu(E)$ of $E \in D^b(X)$ is the same as its Chern character $\mathrm{ch}(E)$.

Due to Mukai and Căldăraru-Willerton, for any $u \in H_{alg}^{2*}(Y, \mathbb{Q})$ and $\nu \in H_{alg}^{2*}(X, \mathbb{Q})$ we have

$$
(2) \qquad\qquad \left\langle \Phi_{\mathcal{E}_L}^H(u), \nu \right\rangle_X = \left\langle u, \Phi_{\mathcal{E}}^H(\nu) \right\rangle_Y
$$

(see [Huy2, Proposition 5.44], [CW]).

2.4. **Abelian varieties.** Over any field, an *abelian variety* X is a complete group variety, that is X is an algebraic variety equipped with the maps $m : X \times X \to X$, $(x, y) \mapsto x + y$ (the group law), and $(-1) : X \to X$, $x \mapsto -x$ (the inverse map), together with the identity element $e \in X$. For $a \in X$, the morphism $t_a : X \to X$ is defined by $t_a = m(-, a) : x \mapsto x + a$. Over the field of complex numbers, an abelian variety is a complex torus with the structure of a projective algebraic variety.

Let $\mathrm{Pic}^0(X)$ be the subgroup of the abelian group $\mathrm{Pic}(X)$ consisting of elements represented by line bundles which are algebraically equivalent to zero, and the corresponding quotient $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the Néron-Severi group $\mathrm{NS}(X)$. The group $\mathrm{Pic}^0(X)$ is naturally isomorphic to an abelian variety called the *dual abelian variety* of X, denoted by $\widehat{X}$.

The *Poincaré line bundle* $\mathcal{P}$ on the product $X \times \widehat{X}$ is the uniquely determined line bundle satisfying (i) $\mathcal{P}_{X \times \{\widehat{x}\}} \in \mathrm{Pic}(X)$ is represented by $\widehat{x} \in \widehat{X}$, and (ii) $\mathcal{P}_{\{e\} \times \widehat{X}} \cong \mathcal{O}_{\widehat{X}}$. In [Muk2], Mukai proved that the Fourier-Mukai functor $\Phi_{\mathcal{P}}^{X \to \widehat{X}} : D^b(X) \to D^b(\widehat{X})$ is an equivalence of the derived categories, that is a Fourier-Mukai transform.

A vector bundle E on an abelian variety X is called *homogeneous* if we have $t_x^* E \cong E$ for all $x \in X$. A vector bundle E on X is homogeneous if and only if E can be filtered by line bundles from $\mathrm{Pic}^0(X)$ (see [Muk1]). We call a vector bundle E is *semihomogeneous* if for every $x \in X$ there exists a flat line bundle $\mathcal{P}_{X \times \{\hat{x}\}}$ on X such that $t_x^* E \cong E \otimes \mathcal{P}_{X \times \{\hat{x}\}}$. A vector bundle E is called *simple* if we have $\mathrm{End}_X(E) \cong \mathbb{C}$. A rank r simple semihomogeneous bundle E has the Chern character

$$(3) \qquad \mathrm{ch}(E) = r \, e^{c_1(E)/r}.$$

Furthermore, due to Mukai, for any $D_X \in \mathrm{NS}_{\mathbb{Q}}(X)$, there exists simple semihomogeneous bundles E on X with $\mathrm{ch}(E) = r \, e^{D_X}$ for some $r \in \mathbb{Z}_{>0}$. See [Orl] for further details.

The image of an ample line bundle L on X under the Fourier-Mukai transform $\Phi_{\mathcal{P}}^{X \to \hat{X}}$ is

$$\Phi_{\mathcal{P}}^{X \to \hat{X}}(L) \cong \widehat{L}$$

for some rank $\chi(L) = c_1(L)^g/g!$ semihomogeneous bundle $\widehat{L}$. Moreover, $-c_1(\widehat{L})$ is an ample divisor class on $\widehat{X}$ (see [BL]). Therefore, we have the following:

Lemma 2.5. *Let $\ell_X \in \mathrm{NS}_{\mathbb{Q}}(X)$ be an ample class on X. Under the induced cohomological transform $\Phi_{\mathcal{P}}^{\mathrm{H}}$ of $\Phi_{\mathcal{P}}^{X \to \hat{X}}$ we have*

$$\Phi_{\mathcal{P}}^{\mathrm{H}}(e^{\ell_X}) = (\ell_X^g/g!) \, e^{-\ell_{\hat{X}}}$$

for some ample class $\ell_{\hat{X}} \in \mathrm{NS}_{\mathbb{Q}}(\widehat{X})$, satisfying

$$(\ell_X^g/g!)(\ell_{\hat{X}}^g/g!) = 1.$$

Moreover, for each $0 \leqslant i \leqslant g$,

$$\Phi_{\mathcal{P}}^{\mathrm{H}} \left(\frac{\ell_X^i}{i!} \right) = \frac{(-1)^{g-i} \ell_X^g}{g!(g-i)!} \, \ell_{\hat{X}}^{g-i}.$$

3. Cohomological Fourier-Mukai Transforms

Let Y be a g-dimensional abelian variety and let $D_Y \in \mathrm{NS}_{\mathbb{Q}}(Y)$. Let X be the fine moduli space of rank r simple semihomogeneous bundles E on Y with $c_1(E)/r = D_Y$. Due to Mukai X is a g-dimensional abelian variety. Let $\mathcal{E}$ be the associated universal bundle on $X \times Y$; so by (3) we have

$$\mathrm{ch}(\mathcal{E}_{\{x\} \times Y}) = r \, e^{D_Y}.$$

Let $\Phi_{\mathcal{E}}^{X \to Y} : D^b(X) \to D^b(Y)$ be the corresponding Fourier-Mukai transform from $D^b(X)$ to $D^b(Y)$ with kernel $\mathcal{E}$. See [Orl] for further details. Then its quasi inverse is given by $\Phi_{\Sigma^* \mathcal{E}^\vee}^{Y \to X}[g]$. Again, by (3) we have

$$\mathrm{ch}(\mathcal{E}_{X \times \{y\}}) = r \, e^{D_X}$$

for some $D_X \in \mathrm{NS}_{\mathbb{Q}}(X)$.

A *polarization* on X is by definition the first Chern class $c_1(L)$ of an ample line bundle L on X. However, it is usual to say the line bundle L itself a polarization.

Let $a \in X$ and $b \in Y$. Consider the Fourier-Mukai functor Γ from $D^b(X)$ to $D^b(\widehat{Y})$ defined by

$$\Gamma = \Phi_{\mathcal{P}}^{Y \to \hat{Y}} \circ \mathcal{E}_{\{a\} \times Y}^* \circ \Phi_{\mathcal{E}}^{X \to Y} \circ \mathcal{E}_{X \times \{b\}}^* [g],$$

where $\mathcal{E}^*_{\{a\}\times Y}$ denotes the functor $\mathcal{E}_{\{a\}\times Y} \otimes^* (-)$ and similar for $\mathcal{E}^*_{X\times\{b\}}$. Let $\widehat{\Gamma} : D^b(\widehat{Y}) \to D^b(X)$ be the Fourier-Mukai functor defined by

$$\widehat{\Gamma} = \mathcal{E}_{X\times\{b\}} \circ \Phi^{Y\to X}_{\Sigma^*\mathcal{E}^\vee} \circ \mathcal{E}_{\{a\}\times Y} \circ \Phi^{\widehat{Y}\to Y}_{\Sigma^*\mathcal{P}^\vee}[g].$$

Then $\widehat{\Gamma}$ and Γ are adjoint functors to each other. By direct computation, $\Gamma(\mathcal{O}_x) = \mathcal{O}_{Z_x}$ for some 0-subscheme $Z_x \subset \widehat{Y}$, and $\Gamma(\mathcal{O}_{\widehat{y}}) = \mathcal{O}_{Z_{\widehat{y}}}$ for some 0-subscheme $Z_{\widehat{y}} \subset X$; where the lengths of Z_x and $Z_{\widehat{y}}$ are r^3 and r respectively. Therefore, the Fourier-Mukai kernel $\mathcal{F}$ of Γ is $\mathcal{F} \in \mathrm{Coh}_g(X \times \widehat{Y})$, with $\mathcal{F}^\vee \cong \mathcal{E}xt^g(\mathcal{F}, O_{X\times\widehat{Y}})[-g]$. So $\Gamma(\mathrm{Coh}_i(X)) \subset \mathrm{Coh}_i(\widehat{Y})$ and $\widehat{\Gamma}(\mathrm{Coh}_i(\widehat{Y})) \subset \mathrm{Coh}_i(X)$ for all i. Also by direct computation, $\Gamma(\mathcal{O}_X)$ and $\widehat{\Gamma}(\mathcal{O}_{\widehat{Y}})$ are homogeneous bundles of rank r and r^3 respectively. Let $\ell_X \in \mathrm{NS}_\mathbb{Q}(X)$ be an ample class. Under the induced cohomological map Γ^H we have

$$\Gamma^H(e^{\ell_X}) = r\, e^{\ell_{\widehat{Y}}},$$

for some class $\ell_{\widehat{Y}} \in \mathrm{NS}_\mathbb{Q}(\widehat{Y})$ satisfying $r^2 \ell_X^g = \ell_{\widehat{Y}}^g$. So we have

$$\widehat{\Gamma}^H(e^{\ell_{\widehat{Y}}}) = r^3\, e^{\ell_X}.$$

Moreover, for each $0 \leqslant i \leqslant g$,

$$\Gamma^H(\ell_X^i) = r\,\ell_{\widehat{Y}}^i, \quad \widehat{\Gamma}^H(\ell_{\widehat{Y}}^i) = r^3\,\ell_X^i.$$

By the Nakai–Moishezon criterion, one can show $\ell_{\widehat{Y}}$ is an ample class on $\widehat{Y}$. By Theorem 2.5, under the induced cohomological map of $\Phi^{\widehat{Y}\to Y}_{\Sigma^*\mathcal{P}^\vee}$ we have

$$\Phi^H_{\Sigma^*\mathcal{P}^\vee}(e^{\ell_{\widehat{Y}}}) = (\ell_{\widehat{Y}}^g/g!)\, e^{-\ell_Y},$$

for some ample class $\ell_Y \in \mathrm{NS}_\mathbb{Q}(Y)$.

Let $\Xi : D^b(X) \to D^b(Y)$ be the Fourier-Mukai functor defined by

$$\Xi = \mathcal{E}^*_{\{a\}\times Y} \circ \Phi^{X\to Y}_{\mathcal{E}} \circ \mathcal{E}^*_{X\times\{b\}} = \Phi^{\widehat{Y}\to Y}_{\Sigma^*\mathcal{P}^\vee} \circ \Gamma.$$

The image of e^{ℓ_X} under its induced cohomological transform Ξ^H is $(r^3\ell_X^g/g!)\, e^{-\ell_Y}$. Therefore, we deduce the following.

Theorem 3.1 ([Piy2]). *If $\ell_X \in \mathrm{NS}_\mathbb{Q}(X)$ is an ample class then*

$$e^{-D_Y}\, \Phi^H_{\mathcal{E}}\, e^{-D_X}(e^{\ell_X}) = (r\,\ell_X^g/g!)\, e^{-\ell_Y},$$

for some ample class $\ell_Y \in \mathrm{NS}_\mathbb{Q}(Y)$, satisfying $(\ell_X^g/g!)(\ell_Y^g/g!) = 1/r^2$. Moreover, for each $0 \leqslant i \leqslant g$,

$$e^{-D_Y}\, \Phi^H_{\mathcal{E}}\, e^{-D_X}\left(\frac{\ell_X^i}{i!}\right) = \frac{(-1)^{g-i}r\,\ell_X^g}{g!(g-i)!}\, \ell_Y^{g-i}.$$

Theorem 3.2 ([Piy2]). *If the ample line bundle L defines a polarization on X, then the line bundle $\det(\Xi(L))^{-1}$ is ample and so it defines a polarization on Y.*

Let us introduce the following notation:

Notation 3.3. Let $B, \ell_X \in \mathrm{NS}_\mathbb{Q}(X)$. For $E \in D^b(X)$, the entries $v_i^{B,\ell_X}(E)$, $i = 0, \ldots, g$ are defined by

$$v_i^{B,\ell_X}(E) = i!\,\ell_X^{g-i} \cdot \mathrm{ch}_i^B(E).$$

Here $\mathrm{ch}_i^B(E)$ is the i-th component of the B-twisted Chern character $\mathrm{ch}^B(E)$ defined by $\mathrm{ch}^B(E) = e^{-B}\,\mathrm{ch}(E)$. The vector $v^{B,\ell_X}(E)$ is defined by

$$v^{B,\ell_X}(E) = \left(v_0^{B,\ell_X}(E),\ldots,v_g^{B,\ell_X}(E)\right).$$

Theorem 3.4 ([Piy2]). *If we consider* $v^{-D_X,\ell_X}, v^{D_Y,\ell_Y}$ *as column vectors, then*

$$v^{D_Y,\ell_Y}\left(\Phi_{\mathcal{E}}^{X\to Y}(E)\right) = \frac{g!}{r\,\ell_X^g}\,\mathrm{Adiag}\left(1,-1,\ldots,(-1)^{g-1},(-1)^g\right)\,v^{-D_X,\ell_X}(E).$$

4. Stability Conditions Under Fourier-Mukai Transforms

This section generalizes some of the similar results in [MP2, Piy1]. Recall that a Bridgeland stability condition σ on a triangulated category $\mathcal{D}$ consists of a stability function Z together with a slicing $\mathcal{P}$ of $\mathcal{D}$ satisfying certain axioms. Equivalently, one can define σ by giving a bounded t-structure on $\mathcal{D}$ together with a stability function Z on the corresponding heart $\mathcal{A}$ satisfying the Harder-Narasimhan property. Then σ is usually written as the pair $(Z,\mathcal{P})$ or $(Z,\mathcal{A})$.

Let $\Upsilon : \mathcal{D} \to \mathcal{D}'$ be an equivalence of triangulated categories, and let $W : K(\mathcal{D}) \to \mathbb{C}$ be a group homomorphism. Then

$$(\Upsilon \cdot W)\left([E]\right) = W\left([\Upsilon^{-1}(E)]\right)$$

defines an induced group morphism $\Upsilon \cdot W$ in $\mathrm{Hom}(K(\mathcal{D}'),\mathbb{C})$ by the equivalence Υ. Moreover, this can be extended to a natural induced stability condition on $\mathcal{D}'$ by defining $\Upsilon \cdot (Z,\mathcal{A}) = (\Upsilon \cdot Z, \Upsilon(\mathcal{A}))$.

Let X, Y be two derived equivalent g-dimensional abelian varieties as in Section 3, which is given by the Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X\to Y} : D^b(X) \to D^b(Y)$. Let $\ell_X \in \mathrm{NS}_{\mathbb{Q}}(X)$ be an ample class on X and let $\ell_Y \in \mathrm{NS}_{\mathbb{Q}}(Y)$ be the induced ample class on Y as in Theorem 3.1.

Let $u = \lambda e^{i\alpha}$ be a complex number. Consider the function $Z_{-D_X+u\ell_X}^X : K(X) \to \mathbb{C}$ defined by

$$Z_{-D_X+u\ell_X}^X(E) = -\int_X e^{-(-D_X+u\ell_X)}\,\mathrm{ch}(E).$$

For $E \in D^b(Y)$ we have

$$
\begin{aligned}
\left(\Phi_{\mathcal{E}}^{X\to Y} \cdot Z_{-D_X+u\ell_X}^X\right)(E) &= \left\langle e^{-D_X+u\ell_X},\,\mathrm{ch}\left((\Phi_{\mathcal{E}}^{X\to Y})^{-1}(E)\right)\right\rangle_X \\
&= \left\langle e^{-D_X+u\ell_X},\,(\Phi_{\mathcal{E}}^H)^{-1}(\mathrm{ch}(E))\right\rangle_X \\
&= \left\langle \Phi_{\mathcal{E}}^H\left(e^{-D_X+u\ell_X}\right),\,\mathrm{ch}(E)\right\rangle_Y \\
&= \left\langle e^{D_Y}\left(e^{-D_Y}\Phi_{\mathcal{E}}^H e^{-D_X}\right)(e^{u\ell_X}),\,\mathrm{ch}(E)\right\rangle_Y \\
&= (r\,\ell_X^g u^g/g!)\,\left\langle e^{D_Y-\ell_Y/u},\,\mathrm{ch}(E)\right\rangle_Y,
\end{aligned}
$$

since by Theorem 3.1, $e^{-D_Y}\Phi_{\mathcal{E}}^H e^{-D_X}(e^{u\ell_X}) = (r\,\ell_X^g u^g/g!)\,e^{-\ell_Y/u}$. So we have the following relation:

Lemma 4.1. *We have* $\Phi_{\mathcal{E}}^{X\to Y} \cdot Z_{-D_X+u\ell_X}^X = \zeta\,Z_{D_Y-\ell_Y/u}^Y$, *for* $\zeta = r\,\ell_X^g u^g/g!$.

Assume there exist a stability condition for any complexified ample class $-D_X + u\ell_X$ with a heart $\mathcal{A}^X_{-D_X+u\ell_X}$ and a slicing $\mathcal{P}^X_{-D_X+u\ell_X}$ associated to the central charge function $Z^X_{-D_X+u\ell_X}$. From Lemma 4.1 for any $\phi \in \mathbb{R}$, $\zeta Z^Y_{D_Y-\ell_Y/u}\left(\Phi^{X\to Y}_{\mathcal{E}}\left(\mathcal{P}^X_{-D_X+u\ell_X}(\phi)\right)\right) \subset \mathbb{R}_{>0}e^{i\pi\phi}$; that is

$$Z^Y_{D_Y-\ell_Y/u}\left(\Phi^{X\to Y}_{\mathcal{E}}\left(\mathcal{P}^X_{-D_X+u\ell_X}(\phi)\right)\right) \subset \mathbb{R}_{>0}e^{i(\pi\phi-\arg(\zeta))}.$$

So we would expect

$$\Phi^{X\to Y}_{\mathcal{E}}\left(\mathcal{P}^X_{-D_X+u\ell_X}(\phi)\right) = \mathcal{P}^Y_{D_Y-\ell_Y/u}\left(\phi - \frac{\arg(\zeta)}{\pi}\right),$$

and so

$$\Phi^{X\to Y}_{\mathcal{E}}\left(\mathcal{P}^X_{-D_X+u\ell_X}((0,\,1])\right) = \mathcal{P}^Y_{D_Y-\ell_Y/u}\left(\left(-\frac{\arg(\zeta)}{\pi},\, -\frac{\arg(\zeta)}{\pi}+1\right]\right).$$

For $0 \leqslant \alpha < 1$, $\mathcal{P}^Y_{D_Y-\ell_Y/u}((\alpha,\alpha+1]) = \left\langle \mathcal{P}^Y_{D_Y-\ell_Y/u}((0,\alpha])\,[1],\, \mathcal{P}^Y_{D_Y-\ell_Y/u}((\alpha,1])\right\rangle$ is a tilt of $\mathcal{A}^Y_{D_Y-\ell_Y/u} = \mathcal{P}^Y_{D_Y-\ell_Y/u}((0,1])$ associated to a torsion theory coming from $Z^Y_{D_Y-\ell_Y/u}$ stability. Therefore, one would expect $\Phi^{X\to Y}_{\mathcal{E}}\left(\mathcal{A}^X_{-D_X+u\ell_X}\right)$ is a tilt of $\mathcal{A}^Y_{D_Y-\ell_Y/u}$ associated to a torsion theory coming from $Z^Y_{D_Y-\ell_Y/u}$ stability, up to shift.

Moreover, for the Fourier-Mukai transform $\Phi^{X\to Y}_{\mathcal{E}}$ when ζ is real, that is, $u^g \in \mathbb{R}$, we would expect the following equivalence:

Conjecture 4.2 ([Piy2]). *The Fourier-Mukai transform $\Phi^{X\to Y}_{\mathcal{E}} : D^b(X) \to D^b(Y)$ gives the equivalence of stability condition hearts conjecturally constructed in Conjecture 2.3:*

$$\Phi^{X\to Y}_{\mathcal{E}}[k]\left(\mathcal{A}^X_{\Omega_k}\right) = \mathcal{A}^Y_{\Omega'_k}.$$

Here $\Omega_k = -D_X + \lambda e^{ik\pi/g}\ell_X$ and $\Omega'_k = D_Y - e^{-ik\pi/g}\ell_Y/\lambda$ are complexified ample classes on X and Y respectively, for $k \in \{1,2,\dots,(g-1)\}$, and some $\lambda \in \mathbb{R}_{>0}$.

5. Equivalences of Stability Condition Hearts

The main aim of this section is to show that Conjecture 4.2 is true for abelian surfaces and abelian 3-folds. In order to achieve this we study various stabilities of sheaves and complexes of them under the Fourier-Mukai transforms. This generalizes author's joint work [MP1, MP2] with A. Maciocia.

5.1. **Abelian surfaces.** Let X, Y be derived equivalent abelian surfaces and let ℓ_X, ℓ_Y be ample classes on them respectively as in Theorem 3.1. Let Ψ be the Fourier-Mukai transform $\Phi^{X\to Y}_{\mathcal{E}}$ from X to Y with kernel $\mathcal{E}$, and let $\widehat{\Psi} = \Phi^{Y\to X}_{\Sigma^*\mathcal{E}^\vee}$. We have

$$(4) \qquad\qquad \Psi(\mathrm{Coh}(X)) \subset \langle \mathrm{Coh}(Y), \mathrm{Coh}(Y)[-1], \mathrm{Coh}(Y)[-2]\rangle,$$

and similar relation for $\widehat{\Psi}$. Since $\widehat{\Psi} \circ \Psi \cong [-2]$ and $\Psi \circ \widehat{\Psi} \cong [-2]$, we have the following convergence of the spectral sequences.

$$(5) \qquad\qquad \left.\begin{array}{l} E^{p,q}_2 = \widehat{\Psi}^p\Psi^q(E) \Longrightarrow H^{p+q-2}_{\mathrm{Coh}(X)}(E) \\[2mm] E^{p,q}_2 = \Psi^p\widehat{\Psi}^q(E) \Longrightarrow H^{p+q-2}_{\mathrm{Coh}(Y)}(E) \end{array}\right\}.$$

Here and elsewhere we write $\widehat{\Psi}^p(E) = H^p_{\mathrm{Coh}(X)}(\widehat{\Psi}(E))$ and $\Psi^q(E) = H^q_{\mathrm{Coh}(Y)}(\Psi(E))$. Immediately from the convergence of this spectral sequence for $E \in \mathrm{Coh}(X)$, we deduce that

- $\widehat{\Psi}\Psi^0(\mathsf{E}) \in \mathrm{Coh}(\mathsf{X})[-2]$, and $\widehat{\Psi}\Psi^2(\mathsf{E}) \in \mathrm{Coh}(\mathsf{X})$;
- there is an injection $\widehat{\Psi}^0\Psi^1(\mathsf{E}) \hookrightarrow \widehat{\Psi}^2\Psi^0(\mathsf{E})$, and a surjection $\widehat{\Psi}^0\Psi^2(\mathsf{E}) \twoheadrightarrow \widehat{\Psi}^2\Psi^1(\mathsf{E})$.

Let us recall the notation in Conjecture 4.2 for our derived equivalent abelian surfaces. Consider the complexified ample classes $\Omega = -D_X + i\lambda\ell_X$, $\Omega' = D_Y + i\ell_Y/\lambda$ on X, Y respectively. The function defined by $Z_\Omega^{(1)} = -i\int_X e^{-\Omega}(\mathrm{ch}_0, \mathrm{ch}_1, 0)$ together with the standard heart $\mathrm{Coh}(X)$ defines a very weak stability condition σ_1 on $D^b(X)$. Define the subcategories

$$\mathcal{F}^X = \mathcal{P}_{\sigma_1}^X((0,\ 1/2]), \quad \mathcal{T}^X = \mathcal{P}_{\sigma_1}^X((1/2,\ 1])$$

of $\mathrm{Coh}(X)$ in terms of the associated slicing $\mathcal{P}_{\sigma_1}^X$. Then the Bridgeland stability condition heart in Conjecture 4.2 is

$$\mathcal{A}^X = \langle \mathcal{F}^X[1], \mathcal{T}^X \rangle = \mathcal{P}_{\sigma_1}^X((1/2,\ 3/2]).$$

We consider similar subcategories associated to Ω' on Y.

Let us sketch the proof of the equivalence in Conjecture 4.2. This is essentially a generalization of Yoshioka's in [Yos], and closely follow the one in the author's PhD thesis [Piy1, Section 6].

We need the following results about cohomology sheaves of the images under the Fourier-Mukai transforms.

Proposition 5.1 ([Piy2]). *Let* $\mathsf{E} \in \mathrm{Coh}(X)$. *Then we have the following:*

1. *(i) If* $\mathsf{E} \in \mathcal{T}^X$ *then* $\Psi^2(\mathsf{E}) = 0$, *and (ii) if* $\mathsf{E} \in \mathcal{F}^X$ *then* $\Psi^0(\mathsf{E}) = 0$.
2. *(i)* $\Psi^2(\mathsf{E}) \in \mathcal{T}^Y$, *and (ii)* $\Psi^0(\mathsf{E}) \in \mathcal{F}^Y$.
3. *(i) if* $\mathsf{E} \in \mathcal{T}^X$ *then* $\Psi^1(\mathsf{E}) \in \mathcal{T}^Y$, *and (ii) if* $\mathsf{E} \in \mathcal{F}^X$ *then* $\Psi^1(\mathsf{E}) \in \mathcal{F}^Y$.

In other words, the results of the above proposition say

$$\left. \begin{array}{l} \Psi(\mathcal{T}^X) \subset \langle \mathcal{F}^Y, \mathcal{T}^Y[-1] \rangle \\ \Psi(\mathcal{F}^X) \subset \langle \mathcal{F}^Y[-1], \mathcal{T}^Y[-2] \rangle \end{array} \right\}.$$

Similar results hold for $\widehat{\Psi}$. Since $\mathcal{A}^X = \langle \mathcal{F}^X[1], \mathcal{T}^X \rangle$ and $\mathcal{A}^Y = \langle \mathcal{F}^Y[1], \mathcal{T}^Y \rangle$, we have $\Psi[1](\mathcal{A}^X) = \mathcal{A}^Y$ as required.

5.2. **Abelian 3-folds.** Let X, Y be derived equivalent abelian 3-folds and let ℓ_X, ℓ_Y be ample classes on them respectively as in Theorem 3.1. Let Ψ be the Fourier-Mukai transform $\Phi_{\mathcal{E}}^{X \to Y}$ from X to Y with kernel $\mathcal{E}$, and let $\widehat{\Psi} = \Phi_{\Sigma^*\mathcal{E}^\vee}^{Y \to X}$. Then $\widehat{\Psi} \circ \Psi \cong [-3]$ and $\Psi \circ \widehat{\Psi} \cong [-3]$.

Let us recall the notation in Conjecture 4.2 for our derived equivalent abelian 3-folds. Consider the complexified ample classes

$$\left. \begin{array}{l} \Omega = (-D_X + \lambda\ell_X/2) + i\sqrt{3}\lambda\ell_X/2 \\ \Omega' = (D_Y - \ell_Y/(2\lambda)) + i\sqrt{3}\ell_Y/(2\lambda) \end{array} \right\}$$

on X, Y respectively. The function defined by $Z_\Omega^{(1)} = \int_X e^{-\Omega}(\mathrm{ch}_0, \mathrm{ch}_1, 0, 0)$ together with the standard heart $\mathrm{Coh}(X)$ defines a very weak stability condition σ_1 on $D^b(X)$. Define the subcategories

$$\mathcal{F}_1^X = \mathcal{P}_{\sigma_1}^X((0,\ 1/2]), \quad \mathcal{T}_1^X = \mathcal{P}_{\sigma_1}^X((1/2,\ 1])$$

of $\mathrm{Coh}(X)$ in terms of the associated slicing $\mathcal{P}_{\sigma_1}^X$; and the corresponding tilt be

$$\mathcal{B}^X = \langle \mathcal{F}_1^X[1], \mathcal{T}_1^X \rangle = \mathcal{P}_{\sigma_1}^X((1/2,\ 3/2]).$$

Then as mentioned in Remark 2.4, due to [BMT] the function defined by

$$Z_\Omega^{(2)} = -i \int_X e^{-\Omega}(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, 0)$$

together with the heart $\mathcal{B}^X$ defines a very weak stability condition σ_2 on $D^b(X)$. Define the subcategories

$$\mathcal{F}_2^X = \mathcal{P}_{\sigma_2}^X((0,\,1/2]), \quad \mathcal{T}_2^X = \mathcal{P}_{\sigma_2}^X((1/2,\,1])$$

of $\mathcal{B}^X$ in terms of the associated slicing $\mathcal{P}_{\sigma_2}^X$. Then the expected Bridgeland stability condition heart in Conjecture 2.3 is

$$\mathcal{A}^X = \langle \mathcal{F}_2^X[1], \mathcal{T}_2^X \rangle = \mathcal{P}_{\sigma_2}^X((1/2,\,3/2]).$$

We consider similar subcategories on Y associated to Ω'.

Under the Fourier-Mukai transform Ψ, for any $E \in \mathrm{Coh}(X)$, $H_{\mathrm{Coh}(Y)}^i(\Psi(E)) = 0$ when $i \notin \{0, 1, 2, 3\}$.
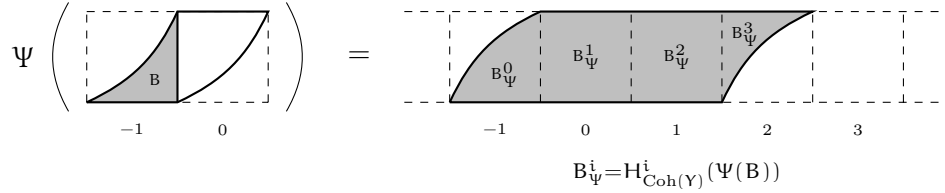
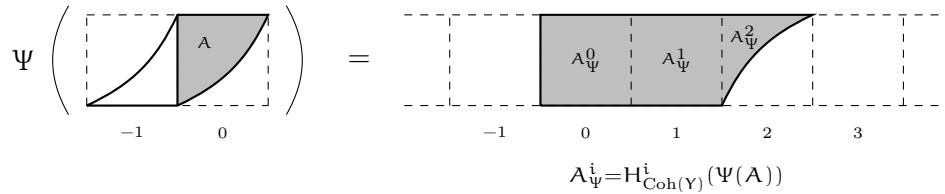We can visualize $\mathcal{B}^X$ and $\mathcal{B}^Y$ as follows:
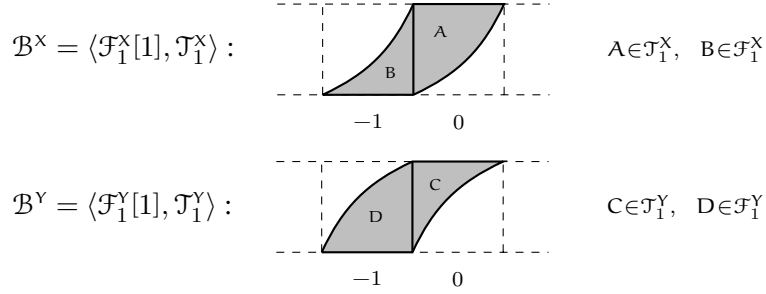
Following similar arguments in [MP1, MP2, Piy1], one can prove

$$\left. \begin{array}{ll} H_{\mathrm{Coh}(Y)}^0(\Psi(\mathcal{F}_1^X)) \subset \mathcal{F}_1^Y, & H_{\mathrm{Coh}(Y)}^3(\Psi(\mathcal{F}_1^X)) \subset \mathcal{T}_1^Y \\ H_{\mathrm{Coh}(Y)}^3(\Psi(\mathcal{T}_1^X)) = \{0\}, & H_{\mathrm{Coh}(Y)}^2(\Psi(\mathcal{T}_1^X)) \subset \mathcal{T}_1^Y \end{array} \right\}.$$

That is $\Psi(\mathcal{F}_1^X)[1] \subset \langle \mathcal{B}^Y, \mathcal{B}^Y[-1], \mathcal{B}^Y[-2] \rangle$,

and $\Psi(\mathcal{T}_1^X) \subset \langle \mathcal{B}^Y, \mathcal{B}^Y[-1], \mathcal{B}^Y[-2] \rangle$.

Hence,

$$\Psi(\mathcal{B}^X) \subset \langle \mathcal{B}^Y, \mathcal{B}^Y[-1], \mathcal{B}^Y[-2] \rangle.$$

Similarly, one can prove
$$\Psi[1](\mathcal{B}^\mathsf{Y}) \subset \langle \mathcal{B}^\mathsf{X}, \mathcal{B}^\mathsf{X}[-1], \mathcal{B}^\mathsf{X}[-2] \rangle.$$

Therefore, under the Fourier-Mukai transforms Ψ and $\widehat{\Psi}[1]$, the images of the complexes in the subcategories $\mathcal{B}^\mathsf{X}$ and $\mathcal{B}^\mathsf{Y}$ behave somewhat like the images of sheaves on abelian surfaces under the Fourier-Mukai transform (see (4)). Also as similar to [MP1, MP2, Piy1], one can prove

$$\left. \begin{array}{c} \Psi(\mathcal{T}_2^\mathsf{X}) \subset \langle \mathcal{F}_2^\mathsf{Y}, \mathcal{T}_2^\mathsf{Y}[-1] \rangle \\[4pt] \Psi(\mathcal{F}_2^\mathsf{X}) \subset \langle \mathcal{F}_2^\mathsf{Y}[-1], \mathcal{T}_2^\mathsf{Y}[-2] \rangle \end{array} \right\},$$

and similar relations for $\widehat{\Psi}[1]$. Since $\mathcal{A}^\mathsf{X} = \langle \mathcal{F}_2^\mathsf{X}[1], \mathcal{T}_2^\mathsf{X} \rangle$ and $\mathcal{A}^\mathsf{Y} = \langle \mathcal{F}_2^\mathsf{Y}[1], \mathcal{T}_2^\mathsf{Y} \rangle$ we have $\Psi[1](\mathcal{A}^\mathsf{X}) = \mathcal{A}^\mathsf{Y}$ as required.

Consequently, by similar arguments as in [MP1, MP2, Piy1] for principally polarized abelian 3-folds with Picard rank one, one can prove the strong Bogomolov-Gieseker type inequality introduced in [BMT] holds for both X and Y.

Theorem 5.2 ([Piy2])**.** *Conjecture 4.2 is true for abelian 3-folds. Moreover, the strong Bogomolov-Gieseker type inequality introduced in* [BMT] *holds for abelian 3-folds.*

Acknowledgements

References

[AB]   D. Arcara and A. Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces. With an appendix by M. Lieblich*, J. Eur. Math. Soc. 15 (2013), 1–38.

[BMS]   A. Bayer, E. Macrì, and P. Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds*, preprint, arXiv:1410.1585.

[BMT]   A. Bayer, E. Macrì, and Y. Toda, *Bridgeland stability conditions on 3-folds I: Bogomolov-Gieseker type inequalities*, J. Algebraic Geom. 23 (2014), 117–163.

[BL]   C. Birkenhake, H. Lange, *The dual polarization of an abelian variety*, Arch. Math. (Basel) 73 (1999), no. 5, 380−389.

[Bri1] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math 166 (2007), 317–345.

[Bri2] _____, *Stability conditions on K3 surfaces*, Duke Math. J. 141 (2008), 241–291.

[BM]   T. Bridgeland and A. Maciocia, *Fourier-Mukai transforms for K3 and elliptic fibrations*, J. Algebraic Geom. 11 (2002), no. 4, 629−657.

[CW]   A. Căldăraru and S. Willerton, *The Mukai pairing, I: A categorical approach*, New York J. Math. 16 (2010), 61−98.

[Huy1] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.

[Huy2] _____, *Derived and abelian equivalence of K3 surfaces*, J. Algebraic Geom. 17 (2008), no. 2, 375–400.

[Li]   C. Li, *Stability conditions on Fano threefolds of Picard number one*, preprint, arXiv:1510.04089.

[Muk1] S. Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. 18 (1978), no. 2, 239−272.

[Muk2] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153−175.

[Mac]   E. Macrì, *A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space*, Algebra Number Theory 8 (2014), 173–190.

[MP1]   A. Maciocia and D. Piyaratne, *Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds*, Algebr. Geom. 2 (2015), no. 3, 270 – 297.

[MP2]   _____, *Fourier-Mukai Transforms and Bridgeland Stability Conditions on Abelian Threefolds II*, preprint, arXiv:1310.0299v2.

[Orl]   D. Orlov, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 3, 131 – 158; translation in Izv. Math. 66 (2002), no. 3, 569 – 594.

[Piy1] D. Piyaratne, *Fourier-Mukai Transforms and Stability Conditions on Abelian Threefolds*, PhD thesis, University of Edinburgh (2014), `http://hdl.handle.net/1842/9635`.

[Piy2] _____, *TBA*, in preparation.

[PT]   D. Piyaratne and Y. Toda, *Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants*, preprint, arXiv:1504.01177.

[Sch]   B. Schmidt, *A generalized Bogomolov-Gieseker inequality for the smooth quadric threefold*, Bull. Lond. Math. Soc. 46 (2014), 915–923.

[Tod1] Y. Toda, *Limit stable objects on Calabi-Yau 3-folds*, Duke Math. J. 149 (2009), no. 1, 157 – 208.

[Tod2] _____, *Curve counting theories via stable objects I: DT/PT correspondence*, J. Amer. Math. Soc. 23 (2010), 1119–1157.

[Yos]   K. Yoshioka, *Stability and the Fourier-Mukai transform II*, Compos. Math. 145 (2009), no. 1, 112 – 142.

KAVLI INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE (WPI), THE UNIVERSITY OF TOKYO INSTITUTES FOR ADVANCED STUDY, THE UNIVERSITY OF TOKYO, KASHIWA, CHIBA 277-8583, JAPAN.

E-mail address: `dulip.piyaratne@ipmu.jp`