Perverse sheaves and graphs on surfaces

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1 Introduction

The aim of this note is to propose a combinatorial description of the categories $\mathrm{Perv}(S, N)$ of perverse sheaves on a Riemann surface S with possible singularities at a finite set of points N. Here we consider the sheaves with values in the category of vector spaces over a fixed base field $\mathbf{k}$. We allow (in fact, require) the surface S to have a boundary. Categories of the type $\mathrm{Perv}(S, N)$ can be used as inductive building blocks in the study of any category of perverse sheaves, see [GMV]. Therefore explicit combinatorial descriptions of them are desirable.

We proceed in a manner similar to [KS1]. To define a combinatorial data we need to fix a *Lagrangian skeleton* of S. In our case it will be a *spanning graph* $K \subset S$ with the set of vertices $\mathrm{Vert}(K) = N$ (for the precise meaning of the word "spanning" see Section 3B). (In the case of a hyperplane arrangement in $\mathbb{C}^n$ discussed in [KS1] the Lagrangian skeleton was $\mathbb{R}^n \subset \mathbb{C}^n$.) We denote by $\mathrm{Ed}(K)$ the set of edges of K. We suppose for simplicity in this Introduction that K has no loops. As any graph embedded into an oriented surface, K is naturally a *ribbon graph*, i.e., it is equipped with a cyclic order on the set of edges incident to any vertex.

To any ribbon graph K we associate a category $\mathcal{A}_K$ whose objects are collections $\{E_x, E_e \in \mathrm{Vect}(\mathbf{k}),\ x \in \mathrm{Vert}(K), e \in \mathrm{Ed}(K)\}$ together with linear maps

$$E_x \underset{\delta}{\overset{\gamma}{\rightleftarrows}} E_e.$$

given for each couple (x, e) with x being a vertex of an edge e. Here $\mathrm{Vect}(\mathbf{k})$ denotes the category of finite dimensional $\mathbf{k}$-vector spaces. These maps must satisfy the relations which use the ribbon structure on K and are listed in Section 3C below,.

If $K \subset S$ is a spanning graph as above, then our main result (see Theorem 3.6) establishes an equivalence of categories

$$Q_K : \ \mathrm{Perv}(S, N) \xrightarrow{\sim} \mathcal{A}_K$$

For $\mathcal{F} \in \mathrm{Perv}(S, N)$ the vector spaces $Q_K(\mathcal{F})_x, Q_K(\mathcal{F})_v$ are the stalks of the constructible complex $\mathcal{R}_K(\mathcal{F}) = Ri^!_K(\mathcal{F})[1]$ on K which, as we prove, is identified with a constructible sheaf in degree 0.

A crucial particular case is $S =$ the unit disc $D \subset \mathbb{C}$, $N = \{0\}$. Take for the skeleton a corolla K_n with center at 0 and n branches. Thus, the same category $\mathrm{Perv}(D, 0)$ has infinitely many incarnations, being equivalent to $\mathcal{A}_n := \mathcal{A}_{K_n}, \ n \geq 1$.

The corresponding equivalence Q_n is described in Section 2, see Theorem 2.1. The special cases of this equivalence are:

(i) $Q_1 : \mathrm{Perv}(D, 0) \xrightarrow{\sim} \mathcal{A}_1$: this is a classical theorem, in the form given in [GGM].

(ii) $Q_2 : \mathrm{Perv}(D, 0) \xrightarrow{\sim} \mathcal{A}_2$ is a particular case of the main result of [KS1], see *op. cit.*, §9A. In *loc. cit.* we have also described the resulting equivalence

$$\mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2$$

explicitely. In a way, objects of $\mathcal{A}_2$ are "square roots" of objects of $\mathcal{A}_1$, in the same manner as the Dirac operator is a square root of the Schrödinger operator. That is why we call $\mathcal{A}_n$ a "$1/n$-spin (parafermionic) incarnation" of $\mathrm{Perv}(D, 0)$.

Finally, in Section 3D we give (as an easy corollary of the previous discussion) a combinatorial description of the category $\mathrm{PolPerv}(S, N)$ of *polarized* perverse sheaves, cf. [S]. These objects arise "in nature" as decategorified perverse Schobers, cf. [KS2], the polarization being induced by the Euler form $(X, Y) \mapsto \chi(R\,\mathrm{Hom}(X, Y))$.

The idea of localization on a Lagrangian skeleton was proposed by M. Kontsevich in the context of Fukaya categories. The fact that it is also applicable to the problem of classifying perverse sheaves (the constructions of [GGM] and [KS1] can be seen, in retrospect, as manifestations of this idea) is a remarkable phenomenon. It indicates a deep connection between Fukaya categories and perversity. A similar approach will be used in [DKSS] to construct the Fukaya category of a surface with coefficients in a perverse Schober.

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2 The "fractional spin" description of perverse sheaves on the disk

A. Statement of the result. Let X be a complex manifold. By a *perverse sheaf* on X we mean a $\mathbb{C}$-constructible complex $\mathcal{F}$ of sheaves of $\mathbf{k}$-vector spaces on X, which satisfies the middle perversity condition, normalized so that a local system in degree 0 is perverse. Thus, if $\mathbf{k} = \mathbb{C}$ and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $\underline{R\operatorname{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is a perverse sheaf.

Let $D = \{|z| < 1\} \subset \mathbb{C}$ be the unit disk. Let $\mathbf{k}$ be a field. We denote by $\operatorname{Perv}(D, 0)$ the abelian category of perverse shaves of $\mathbf{k}$-vector spaces on D which are smooth (i.e., reduce to a local system in degree 0) outside $0 \in D$.

Let $n \geq 1$ be an integer. Let $\mathcal{A}_n$ be the category of diagrams of finite-dimensional $\mathbf{k}$-vector spaces (quivers) Q , consisting of spaces $E_0, E_1, \cdots, E_n$ and linear maps

$$E_0 \underset{\delta_i}{\overset{\gamma_i}{\rightleftarrows}} E_i \,, \quad i = 1, \cdots, n,$$

satisfying the conditions (for $n \geq 2$):

(C1) $\gamma_i \delta_i = \mathrm{Id}_{E_i}$.

(C2) The operator $T_i := \gamma_{i+1}\delta_i : E_i \to E_{i+1}$ (where $i+1$ is considered modulo n), is an isomorphism for each $i = 1, \cdots, n$.

(C3) For $i \neq j, j+1 \mod n$, we have $\gamma_i \delta_j = 0$.

For $n = 1$ we impose the standard relation:

(C) The operator $T = \mathrm{Id}_{E_1} - \gamma_1\delta_1 : E_1 \to E_1$ is an isomorphism.

Theorem 2.1. *For each $n \geq 1$, the category* $\operatorname{Perv}(D, 0)$ *is equivalent to* $\mathcal{A}_n$.

For $n = 1$ this is the standard (Φ, Ψ) description of perverse sheaves on the disk [GGM], [Be]. For $n = 2$ this is a particular case of the description in [KS1] (§9 there).

B. Method of the proof. For the proof we consider a star shaped graph $K = K_n \subset D$ obtained by drawing n radii $R_1, ..., R_n$ from 0 in the counter-clockwise order, see Fig. 1. Then $D - K$ is the union of n open sectors $U_1, \cdots, U_n$ numbered so that U_ν is bordered by R_ν and $R_{\nu+1}$.

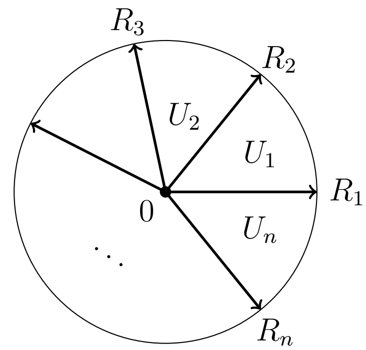

Figure 1: The graph $K = K_n$.

Proposition 2.2. *For any $\mathcal{F} \in \mathrm{Perv}(D, 0)$ we have $\underline{\mathbb{H}}^i_K(\mathcal{F}) = 0$ for $i \neq 1$. Therefore the functor*

$$\mathcal{R} : \mathrm{Perv}(D, 0) \longrightarrow \mathrm{Sh}_K, \quad \mathcal{F} \mapsto \mathcal{R}(\mathcal{F}) = \mathcal{R}_K(\mathcal{F}) := \underline{\mathbb{H}}^1_K(\mathcal{F})$$

is an exact functor of abelian categories.

Proof: Near a point $x \in K$ other than 0, the graph K is a real codimension 1 submanifold in D, and $\mathcal{F}$ is a local system in degree 0, so the statement is obvious ("local Poincaré duality"). So we really need only to prove that the space $\underline{\mathbb{H}}^j_{K_n}(\mathcal{F})_0 = \mathbb{H}^j_{K_n}(D, \mathcal{F})$ vanishes for $j \neq 1$. The case $n = 0$ is known, $\underline{\mathbb{H}}^1_{K_1}(D, \mathcal{F})$ being identified with $\Phi(\mathcal{F})$, see [GGM]. The general case is proved by induction on n. We consider an embedding of K_n into K_{n+1} so that the new radius R_{n+1} subdivides the sector U_n into two. This leads to a morphism between the long exact sequences relating hypercohomology with and without support in K_n and K_{n+1}:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^j(D - K_n, \mathcal{F}) \arrow[r] \arrow[d, "\alpha_j"'] & \mathbb{H}^{j+1}_{K_n}(D, \mathcal{F}) \arrow[r] \arrow[d, "\beta_j"] & \mathbb{H}^{j+1}(D, \mathcal{F}) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & \mathbb{H}^j(D - K_{n+1}, \mathcal{F}) \arrow[r] & \mathbb{H}^{j+1}_{K_{n+1}}(D, \mathcal{F}) \arrow[r] & \mathbb{H}^{j+1}(D, \mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

For $j \neq 0$ the map α_j is an isomorphism because its source and target are 0. Indeed, $D - K_n$ as well as $D - K_{n+1}$ is the union of contractible sectors, and $\mathcal{F}$ is a local system in degree 0 outside 0, so the higher cohomology of each sector with coefficients in $\mathcal{F}$ vanishes. This means that

$$\mathbb{H}^{j+1}_{K_n+1}(D, \mathcal{F}) \;=\; H^{j+1}_{K_n}(D, \mathcal{F}) \;=\; 0, \quad j \neq 0$$

and so by induction all these spaces are equal to 0. $\square$

Remark 2.3. An alternative proof of the vanishing of $\underline{\mathbb{H}}^{\neq 1}_{K_n}(\mathcal{F})_0$ can be obtained by noticing that $\underline{R\Gamma}_{K_n}(\mathcal{F})_0[1]$ can be identified with $\Phi_{z^n}(\mathcal{F})$, the space of vanishing cycles with respect to the function z^n. It is known that forming the sheaf of vanishing cycles with respect to any holomorphic function preserves perversity.

The graph $K = K_n$ is a regular cellular space with cells being $\{0\}$ and the open rays $R_1, \cdots, R_n$. For $\mathcal{F} \in \mathrm{Perv}(D, 0)$ the sheaf $\mathcal{R}(\mathcal{F})$ is a cellular sheaf on K and as such is completely determined by the linear algebra data of:

(1) Stalks at the (generic point of the) cells, which we denote:

$$E_0 = E_0(\mathcal{F}_n) := \mathcal{R}(\mathcal{F})_0 = \text{ stalk at } 0;$$
$$E_i = E_i(\mathcal{F}) = \mathcal{R}(\mathcal{F})_{R_i} = \text{ stalk at } R_i, \quad i = 1, \cdots, n.$$

(2) Generalization maps corresponding to inclusions of closures of the cells, which we denote

$$\gamma_i : E_0 \longrightarrow E_i, \quad i = 1, \cdots, n.$$

This gives "one half" of the quiver we want to associate to $\mathcal{F}$.

C. Cousin complex. In order to get the second half of the maps (the δ_i), we introduce, by analogy with [KS1], a canonical "Cousin-type" resolution of any $\mathcal{F} \in \mathrm{Perv}(D, 0)$.

Denote by

$$i : K \hookrightarrow D, \quad j : D - K \hookrightarrow D$$

the embeddings of the closed subset K and of its complement $D - K = \bigsqcup_{\nu=1}^n U_\nu$. For any complex of sheaves $\mathcal{F}$ on D (perverse or not) we have a canonical distinguished triangle in $D^b \,\mathrm{Sh}_D$:

$$i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F} \xrightarrow{\delta} i_* i^! \mathcal{F}[1] \tag{2.4}$$

(here and elsewhere j_* means the full derived direct image). Recalling that $i^!$ has the meaning of cohomology with support, and denoting $j_\nu : U_\nu \hookrightarrow D$ the embeddings of the connected components of $D - K$, we conclude, from Proposition 2.2:

Corollary 2.5. *Let $\mathcal{F} \in \mathrm{Perv}(D,0)$. Then $\mathcal{F}$ is quasi-isomorphic (i.e., can be thought of as represented by) the following 2-term complex of sheaves on D:*

$$\mathcal{E}^\bullet(\mathcal{F}) \;=\; \left\{ \bigoplus_{\nu=1}^{n} j_{\nu *}(\mathcal{F}|_{U_\nu}) \xrightarrow{\underline{\delta}} \mathcal{R}(\mathcal{F}) \right\}.$$

Here the grading of the complex is in degrees $0,1$ *and the map* $\underline{\delta}$ *is induced by the boundary morphism* δ *from* (2.4). □

We further identify

$$\mathcal{F}|_{U_\nu} \simeq \underline{E}_{\nu U_\nu} = \text{ constant sheaf on } U_\nu \text{ with stalk } E_\nu$$

as follows. As $\mathcal{F}$ is locally constant (hence constant) on U_ν, it is enough to specify an isomorphism

$$H^0(U_\nu, \mathcal{F}) \xrightarrow{\alpha_\nu} E_\nu = \underline{\mathbb{H}}^1_K(\mathcal{F})_x,$$

where x is any point on R_ν. Taking a small disk V around x, we have $E_\nu = \mathbb{H}^1_{V\cap K}(V,\mathcal{F})$. This hypercohomology with support is identified, via the coboundary map of the standard long exact sequence relating cohomology with and without support), with the cokernel of the map

$$H^0(V,\mathcal{F}) \longrightarrow H^0(V\cap U_\nu, \mathcal{F}) \oplus H^0(V\cap U_{\nu-1},\mathcal{F}) = H^0(V-K,\mathcal{F})$$

Since $\mathcal{F}$ is constant on V, the projection of $H^0(V\cap U_\nu,\mathcal{F}) = H^0(U_\nu,\mathcal{F})$ to the cokernel, i.e., to E_ν, is an isomorphism. We define α_ν to be this projection.

We now assume $n \geq 2$. Then the closure of U_ν is a proper closed sector $\overline{U}_\nu$, and so we can rewrite the complex $\mathcal{E}^\bullet(\mathcal{F})$ as

$$\mathcal{E}^\bullet(\mathcal{F}) \;=\; \left\{ \bigoplus_{\nu=1}^{n} \underline{E_\nu}_{\overline{U}_\nu} \xrightarrow{\underline{\delta}} \mathcal{R}(\mathcal{F}) \right\}.$$

Indeed,

$$j_{\nu *}(\mathcal{F}|_{U_\nu} = j_{\nu *}(\underline{E}_{\nu U_\nu}) \;=\; \underline{E_\nu}_{\overline{U}_\nu}.$$

D. Analysis of the morphism $\underline{\delta}$. We now analyze the maps of stalks over various points induced by $\underline{\delta}$. Since the target of $\underline{\delta}$ is supported on K, it is enough to consider two cases:

Stalks over 0. We get maps of vector spaces

$$\delta_\nu : \left(\underline{E}_{\nu \overline{U}_\nu}\right)_0 = E_\nu \longrightarrow \mathcal{R}(\mathcal{F})_0 = E_0, \quad \nu = 1, \cdots, n.$$

These maps, together with the generalization maps γ_ν, form the quiver

$$Q = Q(\mathcal{F}) \;=\; \Big\{ \, E_0 \underset{\delta_i}{\overset{\gamma_i}{\rightleftarrows}} E_i \,, \quad i = 1, \cdots, n \Big\}$$

which we associate to $\mathcal{F}$.

Stalks over a generic point $x \in R_\nu$. As x lies in two closed subsets $\underline{U}_\nu$ and $\underline{U}_{\nu-1}$, we have two maps

$$\delta_{U_\nu, R_\nu} : \left(\underline{E}_{\nu \overline{U}_\nu}\right)_x = E_\nu \longrightarrow \mathcal{R}(\mathcal{F})_x = E_\nu,$$
$$\delta_{U_{\nu-1}, R_\nu} : \left(\underline{E}_{\nu-1 \overline{U}_{\nu-1}}\right)_x = E_{\nu-1} \longrightarrow \mathcal{R}(\mathcal{F})_x = E_\nu.$$

Proposition 2.6. *We have the following relations:*

$$\delta_{U_\nu, R_\nu} = \mathrm{Id}_{E_\nu}, \quad \delta_{U_{\nu-1}, R_\nu} = -T_{\nu-1},$$

where

$$T_{\nu-1} : E_{\nu-1} = h^0(U_{\nu-1}, \mathcal{F}) \longrightarrow E_\nu = H^0(U_\nu, \mathcal{F})$$

is the counterclockwise monodromy map for the local system $\mathcal{F}|_{D-\{0\}}$.

Proof: We start with the first relation. Recal that the identification $\alpha_\nu : H^0(U_\nu, \mathcal{F}) \to E_\nu$ was defined in terms of representation of E_ν as a quotient, i.e., in terms of the coboundary map in the LES relating hypercohomology with and without support. So the differential $\underline{\delta}$ in $\mathcal{E}^\bullet(\mathcal{F})$, applied to a section $s \in H^0(U_\nu, \mathcal{F})$, gives precisely $\alpha_\nu(s)$, so after the identification by α_ν, the map on stalks over $x \in R_\nu$, becomes the identity.

We now prove the second relation. Representing $e \in E_{\nu-1}$ by a section $s \in H^0(U_{\nu-1}, \mathcal{F})$, we see that $\delta_{U_{\nu-1}, R_\nu}(e)$ is represented by the image of

$$(0, s) \;\in\; H^0(U_\nu, \mathcal{F}) \oplus H^0(U_{\nu-1}, \mathcal{F})$$

in the quotient

$$\big(H^0(U_\nu, \mathcal{F}) \oplus H^0(U_{\nu-1}, \mathcal{F})\big)\big/H^0(V, \mathcal{F}).$$

But the identification of this quotient with E_ν is via the projection to the first, not second, summand, i.e., to $H^0(V \cap U_\nu, \mathcal{F}) = H^0(U_\nu, \mathcal{F})$. The element $(t, 0)$ projecting to the same element of the quotient as $(0, s)$, has $t = -T_{\nu-1}(s) =$ *minus* the analytic continuation of s to U_ν. $\square$

Proposition 2.7. *The maps γ_ν, δ_ν in the diagram $Q(\mathcal{F})$ satisfy the conditions (C1)-(C3), i.e., $Q(\mathcal{F})$ is an object of the category $\mathcal{A}_n$.*

Proof:/ We spell out the conditions that the differential $\underline{\delta}$ in the Cousin complex $\mathcal{E}^\bullet(\mathcal{F})$ is a morphism of sheaves. More precisely, both terms of the complex are cellular sheaves on D with respect to the regular cell decomposition given by $0, R_1, \cdots, R_n, U_1, \cdots, U_n$. So the maps of the stalks induced by $\underline{\delta}$, must commute with the generalization maps.

Consider the generalization maps from 0 to R_ν. In thje following diagram the top row is the stalk of the complex $\mathcal{E}^\bullet(\mathcal{F})$ over 0, the bottom row is the stalk over R_ν, and the vertical arrows are the generalization maps:

$$\begin{tikzcd} \bigoplus_{\mu=1}^n E_\mu \arrow[r, "\sum \delta_\mu"] \arrow[d, "p_{\nu,\nu-1}"'] & E_0 \arrow[d, "\gamma_\nu"] \\ E_\nu \oplus E_{\nu-1} \arrow[r, "\mathrm{Id}-T_{\nu-1}"'] & E_\nu, \end{tikzcd}$$

the lower horizontal arrow having been described in Proposition 2.6. We now spell out the condition of commutativity on each summand E_μ inside $\bigoplus_{\mu=1}^n E_\mu$.

<u>Commutativity on E_ν</u>: this means $\gamma_\nu \delta_\nu = \mathrm{Id}$.

<u>Commutativity on $E_{\nu-1}$</u>: this means $\gamma_\nu \delta_{\nu-1} = -T_{\nu-1}$, in particular, this composition is an isomorphism.

<u>Commutativity on E_μ for $\mu \neq \nu, \nu-1$</u>: This means that $\gamma_\nu \delta_\mu = 0$, since the projection $p_{\nu,\nu-1}$ annihilates E_μ. The proposition is proved. $\square$

E. $Q(\mathcal{F})$ **and duality.** Recall that the category $\mathrm{Perv}(D, 0)$ has a perfect duality

$$\mathcal{F} \mapsto \mathcal{F}^\star = \mathbb{D}(\mathcal{F})[2], \tag{2.8}$$

i.e., the shifted Verdier duality normalized so that for $\mathcal{F}$ being a local system (in our case, constant sheaf) in degree 0, we have that $\mathcal{F}^\star$ is the dual local system in degree 0. We will use the notation (2.8) also for more general complexes of sheaves on D.

On the other hand, the category $\mathcal{A}_n$ also has a perfect duality

$$Q = \left\{ E_0 \underset{\delta_\nu}{\overset{\gamma_\nu}{\rightleftarrows}} E_\nu \right\}_{\nu=1}^n \mapsto Q^* = \left\{ E_0^* \underset{\gamma_\nu^*}{\overset{\delta_\nu^*}{\rightleftarrows}} E_\nu^* \right\}_{\nu=1}^n.$$

Proposition 2.9. *The functor $Q : \mathrm{Perv}(D, 0) \to \mathcal{A}_n$ commutes with duality, i.e., we have canonical identifications $Q(\mathcal{F}^\bigstar) \simeq Q(\mathcal{F})^*$.*

Proof: We modify the argument of [KS1], Prop. 4.6. That is, we think of K as consisting of n "equidistant" rays R_ν, joining 0 with

$$\zeta^{\nu-1}, \;\; \zeta = e^{2\pi i/n}, \;\; \nu = 1, 2, \cdots, n.$$

We consider another star-shaped graph K' formed by the radii $R'_1, \cdots, R'_n$ so that R'_ν is in the middle of the sector U_ν. Thus, the rotation by $e^{i\pi/n}$ identifies R_ν with R'_ν and K' with K.

We can use K' instead of K to define $\mathcal{R}(\mathcal{F})$ and $\mathcal{R}(\mathcal{F}^\bigstar)$.We will denote the corresponding sheaves $\mathcal{R}_{K'}(\mathcal{F}) = \underline{\mathbb{H}}^1_{K'}(\mathcal{F})$, and similarly for $\mathcal{R}_{K'}(\mathcal{F}^\bigstar)$. The sheaves defined using K, will be denoted by $\mathcal{R}_K(\mathcal{F})$ and so on.

Since Verdier duality interchanges $i^!$ and i^* (for $i : K' \to D$ being the embedding), we have

$$\mathcal{R}_{K'}(\mathcal{F}^\bigstar)^\bigstar \;\simeq\; \mathcal{F}|_{K'}$$

(usual restriction). To calculate this restriction, we use the Cousin resolution of $\mathcal{F}$ defined by using K and the U_ν:

$$\mathcal{F} \;\simeq\; \mathcal{E}^\bullet \;=\; \left\{ \bigoplus_{\nu=1}^{n} \underline{E}_{\nu \overline{U}_\nu} \xrightarrow{\delta} \mathcal{R}_K(\mathcal{F}) \right\}.$$

So we restrict $\mathcal{E}^\bullet$ to K'. SInce $K' \cap K = \{0\}$ and $\mathcal{R}_K(\mathcal{F})$ is supported on K, the restriction $\mathcal{R}_K(\mathcal{F})|_{K'} = \underline{E}_{00}$ is the skyscraper sheaf at 0 with stalk E_0. So

$$\mathcal{F}|_{K'} \;\sim\; \left\{ \bigoplus_{\nu=1}^{n} (\underline{E}_\nu)_{\overline{R}'_\nu} \xrightarrow{\delta'|_{K'} = \sum \delta_\nu} (\underline{E}_0)_0 \right\}.$$

On the other hand, the shifted Verdier dual to $\mathcal{R}_{K'}(\mathcal{F}^\bigstar)$, as a sheaf on K' is identified by, e.g., [KS1], Prop. 1.11 with the complex of sheaves

$$\left\{ \bigoplus_{\substack{C \subset K' \\ \mathrm{codim}(C) = 0}} \underline{E_C(\mathcal{F}^\bigstar)^*}_{\overline{C}} \otimes \mathrm{or}(C) \xrightarrow{\sum (\gamma^{\mathcal{F}^\bigstar}_{C'C})^*} \bigoplus_{\substack{C \subset K' \\ \mathrm{codim}(C) = 1}} \underline{E_C(\mathcal{F}^\bigstar)^*}_{\overline{C}} \otimes \mathrm{or}(C) \right\}.$$

Here C runs over all cells of the cell complex K', and $E_C(\mathcal{F}^\bigstar)$ is the stalk of the cellular sheaf $\mathcal{R}_{K'}(\mathcal{F}^\bigstar)$ at the cell C. Explicitly, C is either 0 or one of the R'_ν, so

$$\mathcal{R}_{K'}(\mathcal{F}^\bigstar)^\bigstar \;=\; \left\{ \bigoplus_{\nu=1}^{n} \underline{E_\nu(\mathcal{F}^\bigstar)^*}_{\overline{R}_\nu} \xrightarrow{\sum (\gamma^{\mathcal{F}^\bigstar}_{C'C})^*} \underline{E_0(\mathcal{F}^\bigstar)^*}_0 \right\}.$$

By the above, this complex is quasi-isomorphic to

$$\mathcal{F}|_{K'} \;=\; \left\{ \bigoplus_{\nu=1}^{n} \underline{E_\nu(\mathcal{F})}_{\overline{R_\nu}} \xrightarrow{\sum \delta_\nu^{\mathcal{F}}} \underline{E_0(\mathcal{F})}_0 \right\}.$$

So we conclude that

$$E_\nu(\mathcal{F}^\bigstar) \;=\; E_\nu(\mathcal{F})^*, \quad \gamma_\nu^{\mathcal{F}^\bigstar} = (\delta_\nu^{\mathcal{F}})^*.$$

This proves the proposition.

Proof of Theorem 2.1. We already have the functor

$$Q : \mathrm{Perv}(D, 0) \longrightarrow \mathcal{A}_n, \quad \mathcal{F} \mapsto Q(\mathcal{F}).$$

Let us define a functor $\mathcal{E} : \mathcal{A}_n \to D^b \, \mathrm{Sh}_D$. Suppose we are given

$$Q = \left\{ E_0 \underset{\delta_\nu}{\overset{\gamma_\nu}{\rightleftarrows}} E_\nu \right\}_{\nu=1}^{n} \in \mathcal{A}_n.$$

We associate to it the Cousin complex

$$\mathcal{E}^\bullet(Q) \;=\; \left\{ \bigoplus_{\nu=1}^{n} \underline{E_\nu}_{\overline{U}_\nu} \xrightarrow{\delta} \mathcal{R}(Q) \right\}.$$

Here $\mathcal{R}(Q)$ is the cellular sheaf on K with stalk E_0 at 0, stalk E_ν at R_ν and the generalization map from 0 to R_ν given by γ_ν. The map $ul\delta$ is defined on the stalks as follows:

Over 0: the map

$$(\underline{E_\nu}_{\overline{U}_\nu})_0 = E_\nu \longrightarrow \mathcal{R}(Q)_0 = E_0$$

is given by $\delta_\nu : E_\nu \to E_0$.

Over R_ν: The map

$$\left(\bigoplus_{\mu=1}^{n} \underline{E_\mu}_{\overline{U}_\mu} \right)_{R_\nu} \;=\; E_\nu \oplus E_{\nu-1} \longrightarrow \; \mathcal{R}(Q)_{R_|nu} = E_\nu$$

is given by

$$\mathrm{Id} - T_{\nu-1} : E_\nu \oplus E_{\nu-1} \longrightarrow E_\nu.$$

Reading the proof of Proposition 2.6 backwards, we see that the conditions (C1)-(C3) mean that in this way we get a morphism $\underline{\delta}$ of cellular sheaves on D, so $\mathcal{E}^\bullet(Q)$ is an object of $D^b \, \mathrm{Sh}_D$.

Further, similarly to Proposition 2.9, we see that $\mathcal{E}^\bullet(Q^*) \simeq \mathcal{E}(Q)^\bigstar$.

Proposition 2.10. $\mathcal{E}^\bullet(Q)$ *is constructible with respect to the stratification* $(\{0\}, D - \{0\})$ *and is perverse.*

Proof: Constructibility. It is sufficient to prove the following:

(a) The sheaf $\underline{H}^0(\mathcal{E}^\bullet(Q))|_{D-\{0\}}$ is locally constant.

(b) The sheaf $\underline{H}^1(\mathcal{E}^\bullet(Q))|_{D-\{0\}}$ is equal to 0.

To see (a), we look at the map of stalks over R_ν:

$$\mathrm{Id} - T_{\nu-1} : E_\nu \oplus E_{\nu-1} \longrightarrow E_\nu$$

given by the differential $\underline{\delta}$ in $\mathcal{E}^\bullet(Q)$. So, by definition, $\underline{H}^0(\mathcal{E}^\bullet(Q))_{R_\nu}$ and $\underline{H}^1(\mathcal{E}^\bullet(Q))_{R_\nu}$ are the kernel and cockerel of this map.

Now, since $T_{\nu-1}$ is an isomorphism, $\mathrm{Ker}(\mathrm{Id} - T_{\nu-1})$ projects to both E_ν and $E_{\nu-1}$ isomorphically. This means that $\underline{H}^0(\mathcal{E}^\bullet(Q))$ is locally constant over R_ν: the stalk at R_ν projects ("generalizes") to the stalks at U_ν and $U_{|nu-1}$ in an isomorphic way.

To see (b), we notice that $\mathrm{Id}-T_{\nu-1}$ is clearly surjective and so $\underline{H}^1(\mathcal{E}^\bullet(Q))_{R_\nu} = 0$. Since $\mathcal{E}^1(Q) = \mathcal{R}(Q)$ is supported on K, this means that $\underline{H}^1(\mathcal{E}^\bullet(Q))$ is supported at 0, and its restriction to $D - \{0\}$ vanishes. So $\underline{H}^i(\mathcal{E}^\bullet(Q))$ are $\mathbb{C}$-constructible as claimed.

Perversity. By the above, $\mathcal{E}^\bullet(Q)$ is *semi-perverse*, i.e., lies in the non-positive part of the perverse t-structure, that is, $\underline{H}^i(\mathcal{E}^\bullet(Q))$ is supported on complex codimension $\geq i$. Further, $\mathcal{E}^\bullet(Q)^\star$ also satisfies the same semi-perversity since it is identified with $\mathcal{E}^\bullet(Q^*)$. This means that $\mathcal{E}^\bullet(Q)$ is fully perverse. Proposition 2.10 is proved.

It remains to show that the functors

$$\mathrm{Perv}(D, 0) \underset{\mathcal{E}}{\overset{Q}{\rightleftarrows}} \mathcal{A}_n$$

are quasi-inverse to each other. This is done in a way completely parallel to [KS1], Prop. 6.2 and Lemma 6.3 ("orthogonality relations"). Theorem 2.1 is proved. $\square$

See Appendix for some further study of the categories $\mathcal{A}_n$.

3 The graph description of perverse sheaves on an oriented surface

A. Generalities. The purity property. Let S be a compact topological surface, possibly with boundary ∂S; we denote $S^\circ = S - \partial S$ the interior. Let $N \subset S^\circ$ be a finite subset. We then have the category $\mathrm{Perv}(S, N)$ formed by perverse sheaves of $\mathbf{k}$-vector spaces on S, smooth outside N.

By a *graph* we mean a topological space obtained from a finite 1-dimensional CW-complex by removing finitely many points. Thus we do allow edges not terminating in a vertex on some side (“legs”), as well as 1-valent and 2-valent vertices as well as loops. For a vertex x of a graph K we denote by $H(x)$ the set of half-edges incident to x. We can, if we wish, consider any point $x \in K$ as a vertex: if it lies on an edge, we consider it as a 2-valent vertex, so $H(x)$ is this case is the set of the two orientations of the edge containing x. Further, for a graph K we denote by $\mathrm{Vert}(K)$ and $\mathrm{Ed}(K)$ the sets of vertices and edges of K.

We denote by $\mathcal{C}_K$ the *cell category* of K defined as follows. The set $\mathrm{Ob}(\mathcal{C}_K)$ is $\mathrm{Vert}(K) \sqcup \mathrm{Ed}(K)$ (“cells"). Non-identity morphisms can exist only between a vertex x and an edge e, and

$$\mathrm{Hom}(x, e) \;=\; \big\{\text{half-edges } h \in H(x) \text{ contained in } e\big\}.$$

So $|\mathrm{Hom}(x, e)|$ can be $0, 1$ or 2 (the last posibility happens when e is a loop beginning and ending at x). If K has no loops, then $\mathcal{C}_K$ is a poset. We denote by $\mathrm{Rep}(\mathcal{C}_K) = \mathrm{Fun}(\mathcal{C}_K, \mathrm{Vect}_{\mathbf{k}})$ the category of representations of $\mathcal{C}_K$ over $\mathbf{k}$.

Proposition 3.1. *The category of cellular sheaves on K is equivalent to* $\mathrm{Rep}(\mathcal{C}_K)$.

Proof: This is a particular case of general statement [T] which describes constructible sheaves on any stratified space in terms of representations of the category of exit paths. □

Let now $K \subset S$ be any embedded graph (possibly passing through some points of N). We allow 1-valent vertices of K to be situated inside S, as well as on ∂S.

Proposition 3.2. *For $\mathcal{F} \in \mathrm{Perv}(S, N)$ we have $\mathbb{H}^i_K(\mathcal{F}) = 0$ for $i \neq 1$.*

Proof: This follows from Proposition 2.2. Indeed, the statement is local on S, and the graph K is modeled, near each of its points, by a star shaped graph in a disk. $\square$

We denote

$$\mathcal{R}(\mathcal{F}) \;=\; \mathcal{R}_K(\mathcal{F}) \;:=\; \underline{\mathbb{H}}^1_K(\mathcal{F}),$$

this is a cellular sheaf on K.

B. Spanning ribbon graphs. From now on we assume that S is oriented. Graphs embedded into S have, therefore, a canonical *ribbon structure*, i.e., a choice of a cyclic ordering on each set $H(x)$. See, e.g., [DK] for more background on this classical concept.

For a ribbon graph K we have a germ of an oriented surface with boundary $\mathrm{Surf}(K)$ obtained by thickening each edge to a ribbon and gluing the ribbons at vertices according to the cyclic order. In the case of a 1-valent vertex x we take the ribbon to contain x, so that x will be inside $\mathrm{Surf}(K)$.

By a *spanning graph* for S we mean a graph K, embedded into S° as a closed subset, such that the closure $\overline{K} \subset S$ is a graph embedded into S, and the embedding $K \hookrightarrow S^\circ$ is a homotopy equivalence. Thus we allow for legs of K to touch the boundary of S.

C. The category associated to a ribbon graph.

Definition 3.3. Let K be a graph, and $\mathcal{C}_K$ be its cell category. By a *double representation* of $\mathcal{C}_K$ we mean a datum Q of:

(1) For each $x \in \mathrm{Vert}(K)$, a vector space E_x.

(2) For each $e \in \mathrm{Ed}(K)$, a vector space E_e.

(2) For each half-edge h incident to a vertex x and an edge e, linear maps

$$E_x \underset{\delta_h}{\overset{\gamma_h}{\rightleftarrows}} E_e.$$

Let $\mathrm{Rep}^{(2)}\,\mathcal{C}_K$ be the categry of double representations of $\mathcal{C}_K$.

Let now K be a ribbon graph. Denote by $\mathcal{A}_K$ the full subcategory in $\mathrm{Rep}^{(2)}\,\mathcal{C}_K$ formed by double representations $Q = (E_x, E_e, \gamma_h, \delta_h)$ such that for each vertex $x \in K$ the following conditions are satisfied (depending on the valency of x):

- If x is 1-valent, then we require:

  (C_x) $\mathrm{Id}_{E_e} - \gamma_h \delta_h : E \to E$ is an isomorphism.

- If the valency of x is ≥ 2, then we require:

  $(C1_x)$ For each half-edge h incident to x, we have $\gamma_h \delta_h = \mathrm{Id}_{E_e}$.

  $(C2_x)$ Let h, h' be any two half-edges incident to x such that h' *immediately follows* h in the cyclic order on $H(x)$. Let e, e' be the edges containing h, h'. Then $\gamma_{h'} \delta_h : E_e \to E_{e'}$ is an isomorphism.

  $(C3_x)$ If h, h' are two half edges incident to x such that $h \neq h'$ and h' does not immediately follow h, then $\gamma_{H'} \delta_h = 0$.

Example 3.4. If $K = K_n$ is a "ribbon corolla" with one vertex and n legs, then $\mathcal{A}_K = \mathcal{A}_n$ is the category from §2.

D. Description of $\mathrm{Perv}(S, N)$ **in terms of spanning graphs.** Let S be an oriented surface, $K \subset S$ be a spanning graph and $N = \mathrm{Vert}(K)$. For $\mathcal{F} \in \mathrm{Perv}(S, N)$ we have the sheaf $\mathcal{R}_K(\mathcal{F})$ on K, cellular with respect to the cell structure given by the vertices and edges. Therefore by Proposition 3.1 it gives the representation of $\mathcal{C}_K$ which, explicitly, consisting of:

- The stalks E_x, E_e at the vertices and edges of K. We write $E_x(\mathcal{F}), E_e(\mathcal{F})$ if needed.

- The generalization maps $\gamma_h : E_x \to E_e$ for any incidence, i.e. half-edge h containing x and contained in e.

Proposition 3.5. *For the Verdier dual perverse sheaf $\mathcal{F}^\bigstar$ we have canonical identifications*

$$E_x(\mathcal{F}^\bigstar) \simeq E_x(\mathcal{F})^*, \quad E_e(\mathcal{F}^\bigstar) \simeq E_e(\mathcal{F})^*.$$

Proof: Follows from the local statement for a star shaped graph in a disk, Prop. 2.9. $\square$

So we define

$$\delta_h = (\gamma_h^{\mathcal{F}^\bigstar})^* : E_e \to E_x.$$

Theorem 3.6. *(a) The data $Q = Q(\mathcal{F}) = (E_x, E_e, \gamma_h, \delta_h)$ form an object of the category $\mathcal{A}_K$.*

(b) If K is a spanning graph for S, and $N = \mathrm{Vert}(K)$, then the functor

$$Q_K : \mathrm{Perv}(S, N) \longrightarrow \mathcal{A}_K, \quad \mathcal{F} \mapsto Q(\mathcal{F})$$

is an equivalence of categories.

Proof: (a) The relations $(C1_x)$-$(C3_x)$ resp. (C_x) defining $\mathcal{A}_K$, are of local nature, so they follow from the local statement (Proposition 2.6) about a star shaped graph in a disk.

(b) This is obtained by gluing the local results (Theorem 2.1). More precisely, perverse sheaves smooth outside N, form a stack $\mathfrak{P}$ of categories on S. We can assume that $S = \mathrm{Surf}(K)$, so $\mathfrak{P}$ can be seen as a stack on K, and $\mathrm{Perv}(S, N) = \Gamma(K, \mathfrak{P})$ is the category of global sections of this stack. Similarly, $\mathcal{A}_K$ also appears as $\Gamma(K, \mathfrak{A})$, where $\mathfrak{A}$ is the stack of categories on K given by $K' \mapsto \mathcal{A}_{K'}$ (here K' runs over open subgraphs of K). Our functor Q comes from a morphism of stacks $\mathfrak{Q} : \mathfrak{P} \to \mathfrak{A}$, so it is enough to show that $\mathfrak{Q}$ is an equivalence of stacks. This can be verified locally, at the level fo stalks at arbitrary points $x \in K$, where the statement reduces to Theorem 2.1. $\square$

Cf. [KS1], §9B for a similar argument.

D. Polarized sheaves. Let us call a *polarized* space an object $E \in \mathrm{Vect}(\mathbf{k})$ equipped with a nondegenerate $\mathbf{k}$-bilinear form

$$\langle \ , \ \rangle : \ E \times E \longrightarrow \mathbf{k},$$

not necessarily symmetric. A linear map $f : \ E \to E'$ between polarized spaces has two adjoints: the left and the right one, ${}^\top f, f^\top : \ E' \to E$, defined by

$$\langle {}^\top f(x), y\rangle = \langle x, f(y)\rangle; \ \langle y, f^\top(x)\rangle = \langle f(y), x\rangle.$$

Polarized spaces give rise to several interesting geometric structures motivated by category theory, see [Bon].

Let us call a *polarized perverse sheaf* over S an object $\mathcal{F} \in \mathrm{Perv}(S, N)$ equipped with an isomorphism with its Verdier dual $B : \mathcal{F} \to \mathcal{F}^\star$. This concept can be compared with that of [S]; however we do not require any symmetry of B. Polarized perverse sheaves on S with singularities in N form a category $\mathrm{PolPerv}(S, N)$ whose morphisms are morphisms of perverse sheaves commuting with the isomorphisms B.

Definition 3.7. Given a ribbon graph K, we define the category ${}^p\mathcal{A}_K$ (resp. $\mathcal{A}^p_K$) whose objecs are collections $Q = (E_x, E_e, \gamma_h, \delta_h)$ as in $\mathcal{A}_K$, together with an additional data of polarizations on all spaces E_e, E_x, subject to an additional condition: for each h, we have $\delta_h = {}^\top\gamma_h$ (resp. $\delta_h = \gamma_h^\top$).

Since by definition the equivalence Q_K from Theorem 3.6 commutes with the duality, we obtain:

Corollary 3.8. *If K is a spanning graph for S and $N = \mathrm{Vert}(K)$, then we have two equivalences*

$$ {}^pQ_K: \ \mathrm{PolPerv}(S,N) \xrightarrow{\sim} {}^p\mathcal{A}_K; \quad Q^p_K: \ \mathrm{PolPerv}(S,N) \xrightarrow{\sim} \mathcal{A}^p_K. \quad \square $$

A Appendix. Coboundary actions and helices

Since the same marked surface (S, N) has many spanning graphs $K \supset N$, the corresponding categories $\mathcal{A}_K$ are all equivalent to each other, being identified with $\mathrm{Perv}(S, N)$. One can continue the analysis of this paper by constructing a system of explicit identifications $\mathcal{A}_K \to \mathcal{A}_{K'}$ for pairs of spanning graphs $K, K' \supset N$ connected by "elementary moves", in the spirit of [DK]. To keep the paper short, we do not do it here, but discuss a local aspect of this issue: the action of $\mathbb{Z}/n$ on $\mathcal{A}_n$.

Let G be a discrete group which we consider as a category with one object pt. Recall (see, e.g., [De], [GK]) that a *category with G-action*, or a *categorical representation* of G is a lax 2-functor $F : G \to \mathcal{C}at$ from G to the 2-category of categories.

Explicitly, it consists of the following data (plus the data involving the unit of G, see [GK]):

(0) A category $\mathcal{C} = F(\mathrm{pt})$.

(1) For each $g \in G$, a functor $g_* = F(g) : \mathcal{C} \to \mathcal{C}$.

(2) For each $h, g \in G$, an isomorphism of functors $\alpha_{h,g} = F(h, g) : h_*g_* \Rightarrow (hg)_*$.

(3) It is required that for any three elements $h, g, f \in G$ the square

$$\begin{array}{ccc} h_*g_*f_* & \xrightarrow{\alpha_{h,g}} & (hg)_*f_* \\ \alpha_{g,f}\downarrow & & \downarrow \alpha_{hg,f} \\ h_*(gf)_* & \xrightarrow{\alpha_{h,gf}} & (hgf)_* \end{array}$$

is commutative, i.e.

$$\alpha_{hg,f}\alpha_{h,g} = \alpha_{h,gf}\alpha_{g,f}.$$

Example A.1. If $F(g) = \mathrm{Id}_{\mathcal{C}}$ for all $g \in G$, then F is the same as a 2-cocycle $\alpha \in Z^2(G; Z(\mathcal{C}))$ where $Z(\mathcal{C})$ is the *center* of $\mathcal{C}$:= the group of automorphisms of the identity functor $\mathrm{Id}_{\mathcal{C}}$, the action of G on $Z(\mathcal{C})$ being trivial.

We say that the action is *strict* if $(gf)_* = g_* f_*$, and $\alpha_{g,f} = \mathrm{Id}_{g_* f_*}$ for all composable g, f. In other words, a strict action is simply a group homomorphism $F : G \to \mathrm{Aut}(\mathcal{C})$.

All actions of G on a given category $\mathcal{C}$ form themselves a category, denoted $\mathcal{A}ct(G, \mathcal{C})$. It has a distinguished object I, the trivial action, with all $g_* = \mathrm{Id}_{\mathcal{C}}$ and all $\alpha_{h,g} = \mathrm{Id}$. Given an action F as above, *a coboundary structure* on F is an isomorphism $\beta : I \to F$ in $\mathcal{A}ct(G, \mathcal{C})$. Explicitly, it consists of:

- A collection of natural transformations $\beta_g : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} g_*$, given for all $g \in G$ such that
- For each $g, f \in G$ the square

$$\begin{array}{ccc} \mathrm{Id}_{\mathcal{C}} & \xrightarrow{\beta_f} & f_* \\ \beta_{gf} \downarrow & & \downarrow \beta_g \\ (gf)_* & \xrightarrow{\alpha_{g,f}^{-1}} & g_* f_* \end{array}$$

is commutative, in other words, $\alpha_{g,f} = \beta_{gf}\beta_f^{-1}\beta_g^{-1}$.

Examples A.2. (a) In the situation of Example A.1 a coboundary structure on F is the same as a 1-cochain

$$\beta \in C^1(G; Z(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Set}}(G, Z(\mathcal{C})), \quad d\beta = \alpha.$$

(b) If our action is strict, then a coboundary structure on it is a collection of natural transformations $\{\beta_g\}$ as above, such that $\beta_{gf} = \beta_g \beta_f$.

Returning now to the situation of §2, we have a strict action of $\mathbb{Z}/n$ on $\mathcal{A}_n$ such that $k \in \mathbb{Z}/n$ acts by rotation by $2\pi k/n$. More precisely, for

$$x = (E_0, E_1, \ldots, E_n;\ \gamma_i, \delta_i) \in \mathrm{Ob}(\mathcal{A}_n)$$

we define

$$k_* x = (E_0, E_{1+k}, \ldots, E_{n+k}; \gamma_{i+k}, \delta_{i+k}), \quad k \in \mathbb{Z}/n$$

where the indices (except for E_0) are understood modulo n.

Proposition A.3. *The strict action of $\mathbb{Z}$ on $\mathcal{A}_n$ induced by the composition*

$$\mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow Aut(\mathcal{A}_n)$$

is coboundary.

Proof: For $x \in \mathcal{A}_n$ as above we have an arrow $\beta_1(x): \ x \to 1_* x$ in $\mathcal{A}_n$, induced be the fractional monodromies

$$T_i = \gamma_{i+1}\delta_i : E_i \longrightarrow E_{i+1}, \ i \in \mathbb{Z}/n,$$

which give rise to natural isomorphisms $\beta_1: \ \mathrm{Id}_{\mathcal{A}_n} \to 1_*$ (here $1 \in \mathbb{Z}/n$ is the generator). More generally, putting $\beta_k := (\beta_1)^k, \ k \in \mathbb{Z}$, we get a coboundary structure on the composed action. □

Note that in particular the global monodromy $T = \beta_n$ is a natural transformation $\mathrm{Id}_{\mathcal{A}_n} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}_n}$, so it is an element of $Z(\mathcal{A}_n)$. For an object $x \in \mathcal{A}_n$ the sequence

$$\cdots (-1)_* x, x, 1_* x, \cdots, n_* x, \cdots$$

can be seen as a decategorified analog of a *helix*, see [BP], with the monodromy T playing the role of the Serre functor.

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