$\mathcal{W}$-ALGEBRA CONSTRAINTS AND TOPOLOGICAL RECURSION FOR A_N-SINGULARITY

(WITH AN APPENDIX BY DANILO LEWANSKI)

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Abstract. We derive a Bouchard–Eynard type topological recursion for the total descendant potential of A_N-singularity. Our argument relies on a certain twisted representation of a Heisenberg Vertex Operator Algebra (VOA) constructed via the periods of A_N-singularity. In particular, our approach allows us to prove that the topological recursion for the total descendant potential is equivalent to a certain generating set of $\mathcal{W}$-algebra constraints.

1. Introduction

Motivated by his work in Gromov–Witten theory, Givental has introduced the notion of a total descendant and a total ancestor potential (see [7]). The definition makes sense for every conformal semi-simple Frobenius manifold. The main input is the so-called R-matrix and several copies of the Witten–Kontsevich τ-function normalised in an appropriate way (see [7, 8]). On the other hand, it was proved by [5] and [11] that the total ancestor potential can be reconstructed only in terms of the R-matrix by using the *local* Eynard–Orantin recursion. The main problem addressed in this paper is to find a topological recursion for the total descendant potential. The first step in solving this problem was suggested by Bouchard and Eynard in [2]. Their construction was successfully applied to obtain a recursion for the total descendant potential of A_N-singularity in [6] (see Section 7). In general however, the method of Bouchard and Eynard is not directly applicable, because the spectral curve is an infinite sheet covering, i.e., not a Riemann surface. In this paper we would like to suggest an approach based on the VOA construction of [3]. We will focus on the case of A_N-singularity and hence we will recover Theorem 7.3 in [6]. Furthermore, our approach allows us to compare the topological recursion and the $\mathcal{W}$-constraints for the total descendant potential of A_N-singularity (see [3]). More precisely, we prove that the so called *dilaton shift* identifies the differential operators of the topological recursion with states in the $\mathcal{W}$-algebra corresponding to the elementary symmetric polynomials. Constructing explicitly elements of the $\mathcal{W}$-algebra is in general very difficult problem. It would be interesting to find out other examples in which the topological recursion can be used to construct generators of a $\mathcal{W}$-algebra.

1.1. **Results.** Our main result will be stated entirely in terms of the root system of type A_N. The formulation in terms of vertex algebras requires a little bit more notation, so it will be given later on in Section 3. Let us fix the notation and recall the necessary background. Let $\mathfrak{h} \subset \mathbb{C}^{N+1}$ be the hyper-plane $\chi_1 + \cdots + \chi_{N+1} = 0$, where χ_i are the standard coordinate

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functions on $\mathbb{C}^{N+1}$. Recall that the root system of type A_N can be realised as

$$\Delta = \{\chi_i - \chi_j \mid 1 \le i \ne j \le N\} \quad \subset \quad \mathfrak{h}^*.$$

The corresponding Weyl group is the symmetric group on $N+1$ elements, while its action on $\mathfrak{h}^*$ is induced from the standard action on $(\mathbb{C}^{N+1})^*$ given by permuting $\chi_1, \dots, \chi_{N+1}$. Furthermore, the unique W-invariant bilinear form $(\ |\ )$ for which $(\alpha|\alpha) = 2$ for all $\alpha \in \Delta$ is induced by the following bilinear form on $(\mathbb{C}^{N+1})^*$:

$$(\chi_i|\chi_j) = -\frac{1}{h} + \delta_{ij},$$

where $h := N+1$.

We define a set of differential operators on the infinitely many variables

$$\mathbf{t} = \{t_{k,a}\}, \quad 1 \le a \le N, \quad k \ge 0.$$

Sometimes it is convenient to rescale the above variables and to work with

$$x_{k,a} = \frac{t_{k,a}}{(-a+h)(-a+2h)\cdots(-a+kh)}, \quad 1 \le a \le N, \quad k \ge 0.$$

First, we define a set of linear differential operators

$$\Phi_a(\lambda) := \sum_{m=0}^{\infty} \Big(\lambda^m x_{m,a} \hbar^{-1/2} + \lambda^{-m-1}(a+mh)\hbar^{1/2}\partial/\partial x_{m,h-a} \Big), \quad 1 \le a \le h,$$

where $\hbar$ is a formal parameter. Next we introduce the so called *propagators*

$$P_{ij}(\lambda) := \frac{\eta^{i+j}}{(\eta^i - \eta^j)^2}\, \lambda^{-2}, \quad 1 \le i \ne j \le h,$$

where $\eta = e^{2\pi\sqrt{-1}/h}$. Finally, the differential operators that we need are

$$X_j(\lambda) = \sum_{a=1}^{N} \eta^{-ja} \Phi_a(\lambda) \lambda^{-a/h}, \quad 1 \le j \le h$$

and

$$X_{j_1,\dots,j_r}(\lambda) = \sum_{i_1,\dots,i_{r'}} \Big(\prod_{s=1}^{r'} P_{i_s}(\lambda) \Big) : \prod_{j \in J \setminus I} X_j(\lambda) :, \tag{1}$$

where the sum is over all disjoint pairs $i_s = (i_s^{(1)}, i_s^{(2)})$, $1 \le s \le r'$, s.t.,

$$1 \le i_s^{(1)} < i_s^{(2)} \le h, \quad i_1^{(1)} < \cdots < i_{r'}^{(1)},$$

we have used the notation

$$I = \bigcup_{s=1}^{r'} \{i_s^{(1)}, i_s^{(2)}\}, \quad J = \{j_1, \dots, j_r\}, \quad P_{i_s}(\lambda) = P_{i_s^{(1)}, i_s^{(2)}}(\lambda),$$

and $::$ is the normal ordering in which all differentiation operations are applied before the multiplication ones.

The total descendant potential is a formal series of the type

$$\mathcal{D}(\hbar; \mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}^{(g)}(\mathbf{t}) \Big),$$

where $\mathcal{F}^{(g)}$ are formal power series in $\mathbf{t}$. We refer to [8] for the precise definition. Let us define $\Omega^{(g)}_{j_1,\dots,j_r}$ by the following identity

$$X_{j_1,\dots,j_r}\,\mathcal{D}(\hbar;\mathbf{t}) = \Big(\sum_{g=0}^{\infty} \hbar^{g-r/2}\Omega^{(g)}_{j_1,\dots,j_r}(\lambda;\mathbf{t})\Big)\mathcal{D}(\hbar;\mathbf{t}),$$

where $1\leq j_1<\cdots<j_r\leq h$.

Theorem 1.1. *The following identity holds:*

$$(-a+(m+1)h)\frac{\partial\mathcal{F}^{(g)}}{\partial x_{m,a}} = -\operatorname{Res}_{\lambda=0}\frac{1}{h}\sum_{i=1}^{h}\sum_{j_1,\dots,j_r}\frac{\eta^{-ia}\lambda^{m+1-\frac{1}{h}(a+r)}}{\prod_{s=1}^{r}(\eta^i-\eta^{j_s})}\,\Omega^{(g)}_{i,j_1,\dots,j_r}(\lambda;\mathbf{t})d\lambda,$$

where the 2nd sum is over all non-empty subsets $\{j_1,\dots,j_r\}$ *of* $\{1,\dots,i-1,i+1,\dots,h\}$.

It is not hard to see that if we give an appropriate weight to each variable $x_{k,i}$, so that the functions $\mathcal{F}^{(g)}$ are homogeneous, then the identity in Theorem 1.1 will give us a recursion that uniquely determines $\mathcal{F}^{(g)}$ for all $g\geq 0$.

1.2. **Genus-0.** Since the propagators do not contribute to genus 0, the genus-0 reduction of the identity in Theorem 1.1 takes a very simple form. Put

$$p_{m,a} = (-a+(m+1)h)\frac{\partial\mathcal{F}^{(0)}}{\partial x_{m,a}},\quad 1\leq a\leq N,\quad m\geq 0,$$

$$\Phi^{(0)}_a(\lambda,\mathbf{t}) := \sum_{m=0}^{\infty}(x_{m,a}\lambda^m + p_{m,h-a}\lambda^{-m-1}),$$

and define the following numbers

$$C(a_1,\dots,a_r) = \sum_{1\leq j_1<\cdots<j_r\leq h-1}\frac{\eta^{-j_1a_1}}{1-\eta^{j_1}}\cdots\frac{\eta^{-j_ra_r}}{1-\eta^{j_r}},\quad 1\leq a_1,\dots,a_r\leq N.$$

Corollary 1.2. *The following identity holds*

$$p_{m,a} = -\operatorname{Res}_{\lambda=0}\sum_{a_1,\dots,a_r=1}^{h-1} C(a_1,\dots,a_r)\,\Phi^{(0)}_{a_0}(\lambda,\mathbf{t})\Phi^{(0)}_{a_1}(\lambda,\mathbf{t})\cdots\Phi^{(0)}_{a_r}(\lambda,\mathbf{t})\lambda^{m+n+1}\,d\lambda,$$

where the numbers $n\in\mathbb{Z}$ *and* a_0, $0\leq a_0\leq h-1$ *are defined by*

$$-(a+r+a_1+\cdots+a_r) = nh+a_0$$

and if $a_0=0$ *then we set* $\Phi^{(0)}_{a_0}=0$.

If we set $x_{0,a}:=t_a$ and $x_{m,a}=0$ for $m>0$, then the identity in Corollary 1.2 allows us to compute the primary potential of the Frobenius structure.

1.3. **$\mathcal{W}$-constraints.** Recall that the vector space $\mathcal{F} := \mathrm{Sym}(\mathfrak{h}[\zeta^{-1}]\zeta^{-1})$ has the structure of a highest weight $\widehat{\mathfrak{h}}$-module, where $\widehat{\mathfrak{h}} := \mathfrak{h}[\zeta,\zeta^{-1}]\oplus\mathbb{C}$ is the Heisenberg Lie algebra with Lie bracket defined via the invariant bi-linear form $(\ |\ )$ (see Section 3). Following the construction in [3] we define a state-field correspondence $v\mapsto X(v)$, which to every $v\in\mathcal{F}$ associates a *twisted field* $X(v)$. The latter is a differential operator on a set of formal variables $q_{k,i}$, $1\leq i\leq N$, $k\geq 0$ whose coefficients are Laurent polynomials in $\lambda^{1/h}$. Let us point out that under the *dilaton shift*

$$t_{k,i} = q_{k,i}+\delta_{k,0}\delta_{i,N},\quad 1\leq i\leq N,\quad k\geq 0, \tag{2}$$

the differential operators

$$X_{j_1,\dots,j_r}(\lambda) = X(\chi_{j_1}\cdots\chi_{j_r},\lambda),$$

where $X(v,\lambda)$ denotes the value of $X(v)$ at the point λ and we identify $\mathfrak{h}\subset\mathcal{F}$ via $a\mapsto a\,\zeta^{-1}$.

Let $e_r\in\mathrm{Sym}(\mathfrak{h})$, $2\leq r\leq h$, be the degree-r elementary symmetric polynomials in $\chi_1,\dots,\chi_h$. Note that from the topological recursion in Theorem 1.1 we get a set of differential operators that annihilates the total descendant potential $\mathcal{D}(\hbar;\mathbf{t})$.

Theorem 1.3. *Under the dilaton shift* (2) *the set of differential constraints corresponding to the topological recursion turns into*

$$\mathrm{Res}_{\lambda=0}\ \lambda^m X(e_{h+1-a},\lambda)\,\mathcal{D}(\hbar;\mathbf{q}) = 0,\quad 1\leq a\leq N,\quad m\geq 0.$$

The proof of Theorem 1.3 will be reduced to a combinatorial identity, whose proof will be given in the Appendix. It is easy to check that all e_r, $2\leq r\leq h$, are in the kernel of the screening operators $e^{\beta}_{(0)}$, $\beta\in\Delta$. Therefore the main result in [3] and Theorem 1.3 give an alternative proof of Theorem 1.1. Let us point out that in general the invariant polynomials are not in the $\mathcal{W}$-algebra, so at least to the author, it is a little bit surprising that the elementary symmetric polynomials have this property.

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2. Conformal Frobenius structure

Let us recall the construction of a Frobenius structure on the space of miniversal unfolding of A_N-singularity (see [4, 9, 13]). Let

$$F(s,x) = \frac{x^{N+1}}{N+1} + s_1x^{N-1}+\cdots+s_N$$

be a miniversal unfolding of singularity of type A_N. The deformation parameters are allowed to take arbitrary complex values, i.e.,

$$s=(s_1,\dots,s_N)\quad\in\quad B:=\mathbb{C}^N.$$

The space B is equipped with a semi-simple Frobenius structure as follows. Using the so called *Kodaira–Spencer* isomorphism

$$T_sB \cong \mathbb{C}[x]/(\partial_x F(s,x)), \quad \partial/\partial s_i \mapsto \partial_{s_i} F \pmod{\partial_x F} \tag{3}$$

we can equip each tangent space T_sB with a multiplication $\bullet_s$ and with a residue pairing

$$(\partial/\partial s_i, \partial/\partial s_j) = \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{\partial_{s_i} F \partial_{s_j} F}{\partial_x F}\, dx, \tag{4}$$

where the contour of integration C is a big loop enclosing the critical points of F. The main property of the above pairing and multiplication is that the family of connections

$$\nabla = \nabla^{\mathrm{LC}} - z^{-1} \sum_{i=1}^{N} (\partial_{s_i} \bullet_s) ds_i \tag{5}$$

is flat. Here z is a formal parameter, ∇^{LC} is the Levi–Civita connection of the residue pairing, and $\partial_{s_i}\bullet_s$ denotes the linear operator in T_sB of multiplication by the tangent vector $\partial/\partial s_i$.

The flatness of ∇ implies the flatness of ∇^{LC}. We construct a trivialisation of the tangent and the cotangent bundle as follows. Let us denote by $H = \mathbb{C}[x]/x^N$ the local algebra of $F(0,x)$. Then we have the following identifications

$$T^*B \cong TB \cong B \times T_0B \cong B \times H,$$

where the first isomorphism is given by the residue pairing, the second one uses the parallel transport with ∇^{LC}, and the last one is the Kodaira–Spencer isomorphism. Let us choose a flat coordinate system $t = (t_1, \ldots, t_N)$, s.t., the point $t = 0$ corresponds to $s = 0$, and the vector fields $\partial/\partial t_i$ correspond to the basis $\phi_i(x) = x^{N-i}$ $(1 \leq i \leq N)$ of H.

The connection (5) can be extended also in the z-direction

$$\nabla_{\partial/\partial z} = \frac{\partial}{\partial z} - \theta z^{-1} + (E\bullet) z^{-2}, \tag{6}$$

where θ is the so called Hodge grading operator and E is the Euler vector field. Recall that via the Kodaira-Spencer isomorphism (3) E corresponds to F. In flat coordinates we have

$$E = \sum_{i=1}^{N} (1 - d_i) t_i \partial/\partial t_i,$$

where $d_i = \deg(\phi_i) = (N-i)/(N+1)$ $(1 \leq i \leq N)$ is the so-called degree spectrum. The maximal degree $D = d_1 = (N-1)/(N+1)$ is called the *conformal dimension* of the Frobenius manifold. The operator $\theta = \frac{D}{2} - \deg$, i.e.,

$$\theta : H \to H, \quad \theta(\phi_i) = (D/2 - d_i)\phi_i = \Big(-\frac{1}{2} + \frac{i}{N+1} \Big) \phi_i.$$

The connection operators (5) and (6) give rise to a flat connection on the trivial bundle $(B \times \mathbb{C}^*) \times H \to B \times \mathbb{C}^*$, which is also known as the *Dubrovin's connection.*

2.1. **The periods of A_N-singularity.** Put $X = B\times\mathbb{C}$ and let

$$\varphi: X\to B\times\mathbb{C},\quad \varphi(t,x) = (t, F(t,x)).$$

The non-singular fibers $X_{t,\lambda} := \varphi^{-1}(t,\lambda)$ form a smooth fibration called the *Milnor fibration.* Let us denote by $(B\times\mathbb{C})'$ the complement of the *discriminant* locus of the map φ, i.e., $(B\times\mathbb{C})'$ is the set of all (t,λ), s.t., the fiber $X_{t,\lambda}$ consists of $N+1$ pairwise distinct points.

The periods of A_N-singularity are defined by

$$I^{(n)}_a(t,\lambda) = -d_t\partial_\lambda^n \int_{a_{t,\lambda}} d^{-1}\omega \quad \in T_t^*B\cong H,$$

where n is an arbitrary integer and $a\in\mathfrak{h} := \widetilde{H}_0(X_{0,1})\otimes\mathbb{C}$ is a reduced homology cycle. The notation on the RHS is as follows: we denote by $\omega = dx$ and $d^{-1}\omega = x$ (this is a 0-form), the integration cycle $a_{t,\lambda}$ is obtained from a after choosing a reference path in $(B\times\mathbb{C})'$ from $(0,1)$ to (t,λ) and using the parallel transport with respect to the corresponding Gauss–Manin connection. Finally, d_t is the De Rham differential on B.

The period integrals are solutions to a connection $\nabla^{(n)}$, which is a Laplace transform of the Dubrovin's connection

$$\begin{aligned}\nabla^{(n)}_{\partial_{t_i}} &= \partial_{t_i} + \frac{\phi_i\bullet}{\lambda - E\bullet}\Big(\theta - \frac{1}{2} - n\Big),\quad 1\le i\le N,\\ \nabla^{(n)}_{\partial_\lambda} &= \partial_\lambda - \frac{1}{\lambda - E\bullet}\Big(\theta - \frac{1}{2} - n\Big).\end{aligned}$$

The above system of equations can be solved in a neighbourhood of $\lambda=\infty$ in the following way. Let us choose a solution to Dubrovin's connection in the form $\Phi(t,z) = S(t,z)z^\theta$, where $S(t,z) = 1 + S_1(t)z^{-1}+\cdots$ is an operator series whose coefficients $S_k(t)\in\mathrm{End}(H)$. Such a solution is unique and it satisfies the *symplectic condition* $S(t,z)S(t,-z)^T = 1$, where ^{T} is transposition with respect to the residue pairing. The function

$$Y^{(n)}(t,\lambda) = S(t,-\partial_\lambda^{-1})\frac{\lambda^{\theta-n-1/2}}{\Gamma(\theta-n+1/2)}$$

is a fundamental solution to $\nabla^{(n)}$. Moreover, the reference point $(0,1)$ is within the range of convergence (because $S(0,z)=1$), therefore we can define an isomorphism $\mathfrak{h}\cong H$, s.t.,

$$I^{(n)}_a(t,\lambda) = Y^{(n)}(t,\lambda)a$$

for all (t,λ) sufficiently close to $(0,1)$.

2.2. **Monodromy representation.** Let us denote by $\Delta\subset H$ the set of vanishing cycles.

Lemma 2.1. *The set of vanishing cycles* $\Delta = \{\chi_i - \chi_j \mid 1\le i\ne j\le N+1\}$*, where*

$$\chi_i = \sum_{a=1}^N \eta^{-ia}(N+1)^{-a/(N+1)}\Gamma\Big(1-\frac{a}{N+1}\Big)\,\phi_{N+1-a},$$

where $\eta = e^{2\pi\sqrt{-1}/(N+1)}$.

Proof. The fiber $X_{t,\lambda}$ consists of the zeroes $x_i(t,\lambda)$ $(1 \le i \le N+1)$ of the equation $F(s,x) = \lambda$. The vanishing cycles have the form $\alpha = [x_i(0,1)] - [x_j(0,1)]$, where $x_i(0,1) = (N+1)^{1/(N+1)}\eta^i$. By definition

$$I^{(0)}_\alpha(t,\lambda) = -d_t \int_{\alpha_{t,\lambda}} x = -d_t(x_i(t,\lambda) - x_j(t,\lambda)).$$

Furthermore,

$$-d_t\, x_i(t,\lambda) = \sum_{a=1}^N \frac{x_i(t,\lambda)^{N-a}}{\partial_x F(t,x_i)}\, ds_a.$$

On the other hand, note that the residue pairing has the form

$$(\partial/\partial t_a, \partial/\partial t_b) = (x^{N-a}, x^{N-b}) = \delta_{a+b,N+1}.$$

Therefore, at $t = 0$ we have

$$I^{(0)}_\alpha(0,\lambda) = \sum_{a=1}^N (x_i(0,\lambda)^{-a} - x_j(0,\lambda)^{-a}) ds_a$$

and since at $t = 0$: $ds_a = dt_a = x^{a-1} = \phi_{N+1-a}$, we get

$$\sum_{a=1}^N (N+1)^{-a/(N+1)}(\eta^{-ia} - \eta^{-ja})\lambda^{-a/(N+1)}\phi_{N+1-a} = Y^{(0)}(0,\lambda)(\chi_i - \chi_j). \quad \square$$

Using the above Lemma we can verify Saito's formula for the intersection pairing (see [12]), i.e., the bi-linear form

$$(a|b) := (I^{(0)}_a(t,\lambda), (\lambda - E\bullet) I^{(0)}_b(t,\lambda))$$

coincides with the intersection pairing in $\widetilde{H}_0(X_{0,1})$. The Picard–Lefschetz formula for the monodromy of the Gauss–Manin connection (see [1]) takes the form

$$w_\alpha(y) = y - (\alpha|y)\alpha, \quad y \in H,$$

where w_α is the image of the monodromy representation of $\nabla^{(n)}$

$$\pi_1(B \times \mathbb{C})' \to \mathrm{GL}(H)$$

of a simple loop around the discriminant corresponding to a path along which the cycle vanishes. In particular, we get that the monodromy group is the symmetric group S_{N+1} acting by permutation on the set $(\chi_1, \dots, \chi_{N+1})$, while w_α for $\alpha = \chi_i - \chi_j$ is just the transposition swapping χ_i and χ_j.

3. Heisenberg Vertex Operator Algebra

Let us denote by $\widehat{\mathfrak{h}}$ the Heisenberg Lie algebra $H[\zeta,\zeta^{-1}] \oplus \mathbb{C}$ with bracket

$$[f(\zeta), g(\zeta)] = \mathrm{Res}_{\zeta=0}(f'(\zeta)|g(\zeta))d\zeta.$$

It is convenient to denote $a_{(n)} = a\zeta^n$ for $a \in H$ and $n \in \mathbb{Z}$. Then the above formula is equivalent to

$$[a_{(m)}, b_{(n)}] = m(a|b)\delta_{m+n,0}, \quad m,n \in \mathbb{Z}.$$

The vector space $\mathcal{F} = \mathrm{Sym}(H[\zeta^{-1}]\zeta^{-1})$ has a natural structure of a highest-weight $\widehat{\mathfrak{h}}$-module, s.t., $a_{(n)}1 = 0$ for all $a \in H$ and $n \ge 0$.

3.1. **The tame Fock space.** Given a commutative ring R, let us denote by $\widehat{\mathbb{V}}_R$ the space of formal series of the form

$$\sum_{g\in\frac{1}{2}\mathbb{Z}}\ \sum_{K=((k_1,i_1),\dots,(k_s,i_s))} c^{(g)}_{K,I}\hbar^{g-1}t_{k_1,i_1}\cdots t_{k_s,i_s},\quad c^{(g)}_{K,I}\in R,$$

where the 2nd sum is over all lexicographically increasing sequences K of pairs (k,i), $k\geq 0, 1\leq i\leq N$, i.e., either $k_p<k_{p+1}$ or $k_p=k_{p+1}$ and $i_p\leq i_{p+1}$ If $R=\mathbb{C}$, then we simply put $\widehat{\mathbb{V}}:=\widehat{\mathbb{V}}_{\mathbb{C}}$. Let us denote by $\mathbb{V}_{\rm tame}\subset\mathbb{V}$ the subspace of formal series satisfying the *tameness condition*: if $c^{(g)}_{K,I}\neq 0$, then

$$k_1+\cdots+k_s\leq 3g-3+s.$$

Let us denote by $\mathcal{O}$ the algebra of holomorphic functions on the monodromy covering space of $(B\times\mathbb{C})'$. A *twisted field* on $(B\times\mathbb{C})$ is a $\mathbb{C}$-linear map $\mathbb{V}_{\rm tame}\to\widehat{\mathbb{V}}_{\mathcal{O}}$. The space of all twisted fields will be denoted by $\operatorname{Hom}_{\mathbb{C}}(\mathbb{V}_{\rm tame},\widehat{\mathbb{V}}_{\mathcal{O}})$.

3.2. **Twisted representation.** Following Givental [8], we introduce the symplectic vector space $\mathcal{H}=H(\!(z^{-1})\!)$ with the symplectic form

$$\Omega(f(z),g(z))=\operatorname{Res}_{z=0}(f(-z),g(z))dz.$$

Recall, also the following quantisation rules

$$(\phi_i z^k)\widehat{\ }=-\hbar^{1/2}\partial_{t_{k,i}},\quad (\phi^i(-z)^{-k-1})\widehat{\ }=\hbar^{-1/2}\,t_{k,i},$$

where $\phi^i:=\phi_{N+1-i}$ is the dual to ϕ_i with respect to the residue pairing. These rules extend by linearity to define a representation of the Poisson Lie algebra of linear and constant functions on $\mathcal{H}$. We define a *State-Field* correspondence

$$X:\mathcal{F}\to\operatorname{Hom}_{\mathbb{C}}(\mathbb{V}_{\rm tame},\widehat{\mathbb{V}}_{\mathcal{O}})$$

as follows

$$X(a\zeta^{-1}):=\phi_a(t,\lambda):=(\phi_a(t,\lambda;z))\widehat{\ },\quad a\in H,\quad n\in\mathbb{Z}_{\geq 0},$$

where

$$\phi_a(t,\lambda;z)=\sum_{n\in\mathbb{Z}} I^{(n+1)}_a(t,\lambda)\,(-z)^n.$$

For the remaining states the definition is such that

$$X_t(a_{(-n-1)}v,\lambda)=\operatorname{Res}_{\lambda'=\lambda}\Big(X_t(a,\lambda')X_t(v,\lambda)\frac{d\lambda'}{(\lambda'-\lambda)^{n+1}}\Big),$$

where we denoted by $X_t(v,\lambda)$ the value of the field $X(v)$ at a point $(t,\lambda)\in(B\times\mathbb{C})'$.

More explicitly, if $v=\alpha^1_{-k_1-1}\cdots\alpha^r_{-k_r-1}1\in\mathcal{F}$, then the field $X(v)$ can be computed explicitly in terms of the generating fields $\phi_{\alpha^i}(t,\lambda)$ and the so called *propagators*

$$P^{(k)}_{\alpha,\beta}(t,\lambda)\in\mathcal{O},\quad \alpha,\beta\in H,\quad k\in\mathbb{Z}_{\geq 0}$$

defined by the Laurent series expansion

$$\Omega(\phi^+_\alpha(t,\lambda_1;z),\phi_\beta(t,\lambda_2;z))=\frac{(\alpha|\beta)}{(\lambda_1-\lambda_2)^2}+\sum_{k=0}^\infty P^{(k)}_{\alpha,\beta}(t,\lambda_2)(\lambda_1-\lambda_2)^k.$$

The formula for the field $X(v)$ is reminiscent of the Whick formula

$$X_t(v,\lambda) = \sum_J \Big(\prod_{(i,j)\in J} \partial_\lambda^{(k_j)} P^{(k_i)}_{\alpha^i,\alpha^j}(t,\lambda)\Big) :\Big(\prod_{l\in J'} \partial_\lambda^{(k_l)} X_{s,\lambda}(\alpha^l)\Big):, \tag{7}$$

where $\partial_\lambda^{(k)} := \frac{\partial_\lambda^k}{k!}$ and the sum is over all collections J of disjoint ordered pairs (i_1,j_1), …, $(i_s,j_s) \subset \{1,\dots,r\}$ such that $i_1 < \cdots < i_s$ and $i_l < j_l$ for all l, and $J' = \{1,\dots,r\} \setminus \{i_1,\dots,i_s,j_1,\dots,j_s\}$.

It is proved in [10] that the analytic continuation of the propagators is compatible with the monodromy action on α and β. Moreover, we have the following explicit formulas

$$\Omega(\phi^+_\alpha(t,\lambda_1;z), \mathbf{f}_\beta(t,\lambda_2;z)) = \frac{1}{\lambda_1-\lambda_2}\,(I^{(0)}_\alpha(t,\lambda_1), (\lambda_2 - E\bullet)I^{(0)}_\beta(t,\lambda_2))$$

and

$$P^{(0)}_{\alpha,\beta}(t,\lambda) = \frac{1}{2}((\lambda - E\bullet)I^{(1)}_\alpha(t,\lambda), I^{(1)}_\beta(t,\lambda)),$$

where

$$\mathbf{f}_\beta(t,\lambda;z) = \sum_{n\in\mathbb{Z}} I^{(n)}_\beta(t,\lambda)\,(-z)^n.$$

The monodromy representation extends naturally to $\mathcal{F}$. It follows from formula (7) that the analytic continuation of $X_t(v,\lambda)$ in (t,λ) is compatible (or equivalent) to the monodromy action on v.

3.3. **Global recursion.** If $(t,\lambda) \in (B\times\mathbb{C})'$ and $c^1,\dots,c^r \in H$, then we define

$$\Omega^{(g)}_{c^1\cdots c^r}(t,\lambda;\mathbf{t}) \;\in\; \mathbb{C}[\![t_0,t_1,t_2,\dots]\!]$$

by the following equation

$$X_t(c^1_{(-1)}\dots c^r_{(-1)}1,\lambda)\,\mathcal{A}_t(\hbar;\mathbf{t}) = \sum_{g=0}^\infty \hbar^{g-\frac{r}{2}}\,\Omega^{(g)}_{c^1\cdots c^r}(t,\lambda;\mathbf{t})\mathcal{A}_t(\hbar;\mathbf{t}).$$

Recall also, that the total ancestor potential has the form

$$\mathcal{A}_t(\hbar;\mathbf{t}) = \exp\Big(\sum_{n,g=0}^\infty \frac{\hbar^{g-1}}{n!}\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle_{g,n}(t)\Big)$$

where $\mathbf{t}(\psi) = \sum_{k=0}^\infty\sum_{a=1}^N t_{k,a}\phi_a\psi^k$ and the correlator has the form

$$\langle \phi_{a_1}\psi^{k_1},\dots,\phi_{a_n}\psi^{k_n}\rangle_{g,n}(t) = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^t_{g,n}(\phi_{a_1},\dots,\phi_{a_n})\psi_1^{k_1}\cdots\psi_n^{k_n},$$

where $\Lambda^t_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$ is a certain Cohomological Field Theory defined through the Frobenius structure. According to the main result in [11], the total ancestor potential is uniquely determined by the following recursion:

$$\sum_{n=0}^\infty \frac{1}{n!}\langle \phi_a\,\psi^m,\mathbf{t},\dots,\mathbf{t}\rangle_{g,n+1}(t) = \frac{1}{4}\sum_{i=1}^N \mathrm{Res}_{\lambda=u_i}\,\frac{(I^{(-m-1)}_{\beta_i}(t,\lambda),\phi_a)}{(I^{(-1)}_{\beta_i}(t,\lambda),1)}\,\Omega^{(g)}_{\beta_i,\beta_i}(t,\lambda;\mathbf{t})\,d\lambda, \tag{8}$$

where u_i $(1\le i\le N)$ are the critical values of $F(t,x)$ and β_i is a cycle vanishing over $\lambda = u_i$.

Let C be a loop that encloses all critical values. Motivated by the work of Bouchard and Eynard [2], we would like to compare the RHS of (8) with the following integral

$$-\frac{1}{2\pi\sqrt{-1}}\oint_C \sum_{i=1}^{N+1}\sum_J \frac{(I^{(-m-1)}_{\chi_i}(t,\lambda),\phi_a)}{\prod_{j\in J}(I^{(-1)}_{\chi_i-\chi_j}(t,\lambda),1)}\Omega^{(g)}_{\chi_i,\chi_{j_1},\dots,\chi_{j_r}}(t,\lambda;\mathbf{t})d\lambda, \tag{9}$$

where the 2nd sum is over all non-empty subsets $J\subset\{1,\dots,N+1\}\setminus\{i\}$ and $j_1,\dots,j_r$ are the elements of J.

Theorem 3.1. *The RHS of the local recursion* (8) *coincides with the integral* (9).

Proof. The integral (9) can be evaluated with the residue theorem. It is a sum of the residues at the critical values. Let us verify that the residue at $\lambda=u_1$ coincides with the corresponding residue in the local recursion (8). Similar argument applies to the remaining critical values. We may assume that $\beta=\chi_1-\chi_2$ is the cycle vanishing at $\lambda=u_1$.

The terms with $r=1$ contribute to the residue only if the set $J\cup\{i\}$ contains 1 or 2, otherwise $(\chi_i|\beta)=(\chi_j|\beta)=0$ and the entire expression is analytic at $\lambda=u_1$. The 2 terms for which $i=1$, $J=\{2\}$ and $i=2$, $J=\{1\}$ add up to

$$-\operatorname{Res}_{\lambda=u_1}\frac{(I^{(-m-1)}_{\chi_1-\chi_2}(t,\lambda),\phi_a)}{I^{(-1)}_{\chi_1-\chi_2}(t,\lambda),1)}\,\Omega^{(g)}_{\chi_1\chi_2}(t,\lambda;\mathbf{t})d\lambda.$$

However, $-\Omega^{(g)}_{\chi_1\chi_2}=\frac14(\Omega^{(g)}_{\beta,\beta}-\Omega^{(g)}_{\chi_1+\chi_2,\chi_1+\chi_2})$ and since $(\chi_1+\chi_2|\beta)=0$, the form $\Omega^{(g)}_{\chi_1+\chi_2,\chi_1+\chi_2}$ is analytic at $\lambda=u_1$, so it does not contribute to the residue. The above residue coincides with the residue contribution at $\lambda=u_1$ of (8).

We claim that the terms for which the set $J\cup\{i\}$ contains precisely one of the elements 1 or 2 cancel with the terms for which $J\cup\{i\}$ contains both 1 and 2. To avoid cumbersome notation put $\chi_i:=\chi_i\zeta^{-1}\in\mathcal{F}$ and

$$X_I(t,\lambda):=X_t(\chi_{i_1}\cdots\chi_{i_s}1,\lambda)$$

where $I=\{i_1,\dots,i_s\}\subset\{1,2,\dots,N+1\}$. Let us compute

$$-\operatorname{Res}_{\lambda=u_1}\sum_{s=1}^{2}\sum_{i=1}^{N+1}\sum_{J:(J\cup\{i\})\cap\{1,2\}=\{s\}}\frac{(I^{(-m-1)}_{\chi_i}(t,\lambda),\phi_a)}{\prod_{j\in J}(I^{(-1)}_{\chi_i-\chi_j}(t,\lambda),1)}X_{J\cup\{i\}}(t,\lambda)\mathcal{A}_t(\hbar;\mathbf{t}). \tag{10}$$

We may replace $X_{J\cup\{i\}}(t,\lambda)\mathcal{A}_t$ by

$$X_{(J\cup\{i\})\setminus\{s\}}(t,\lambda)(\phi^+_{\chi_s}(t,\lambda;z))\widehat{\ }\mathcal{A}_t \tag{11}$$

because the remaning terms do not contribute to the residue. Note that

$$(\phi^+_{\chi_s}(t,\lambda;z))\widehat{\ }\mathcal{A}_t=-\hbar^{1/2}\sum_{g,n=0}^{\infty}\frac{\hbar^{g-1}}{n!}\langle\phi^+_{\chi_s}(t,\lambda;\psi),\mathbf{t},\dots,\mathbf{t}\rangle_{g,n+1}.$$

Recalling the local recursion (8) the expression (11) is transformed into

$$\frac14\hbar^{1/2}\sum_{k=1}^{N}\operatorname{Res}_{\lambda'=u_k}\frac{\Omega(\phi^+_{\chi_s}(t,\lambda;z),\mathbf{f}_{\beta_k}(t,\lambda';z))}{(I^{(-1)}_{\beta_k}(t,\lambda'),1)}X_{(J\cup\{i\})\setminus\{s\}}(t,\lambda)X_t(\beta_k^2,\lambda')\mathcal{A}_t(\hbar;\mathbf{t}).$$

On the other hand

$$\Omega(\phi^+_{\chi_s}(t,\lambda;z),\mathbf{f}_{\beta_k}(t,\lambda';z))=\frac{1}{\lambda-\lambda'}(I^{(0)}_{\chi_s}(t,\lambda),(\lambda'-E\bullet)I^{(0)}_{\beta_k}(t,\lambda'))=\frac{(\chi_s|\beta_k)}{\lambda-\lambda'}+\cdots,$$

where the dots stand for a term analytic at $\lambda' = \lambda$. The sum (10) turns into

$$-\frac{1}{4}\hbar^{1/2}\sum_{k=1}^{N}\operatorname{Res}_{\lambda=u_1}\operatorname{Res}_{\lambda'=u_k}\sum_{s=1}^{2}\sum_{i=1}^{N+1}\sum_{J:(J\cup\{i\})\cap\{1,2\}=\{s\}}\frac{1}{\lambda-\lambda'}(I^{(0)}_{\chi_s}(t,\lambda),(\lambda'-E\bullet)I^{(0)}_{\beta_k}(t,\lambda'))$$
$$\frac{(I^{(-m-1)}_{\chi_i}(t,\lambda),\phi_a)}{(I^{(-1)}_{\beta_k}(t,\lambda'),1)\prod_{j\in J}(I^{(-1)}_{\chi_i-\chi_j}(t,\lambda),1)}X_{(J\cup\{i\})\setminus\{s\}}(t,\lambda)X_t(\beta_k^2,\lambda')\mathcal{A}_t(\hbar;\mathbf{t})d\lambda' d\lambda.$$

Note that if we compute first the residue with respect to $\lambda = u_1$ we would get 0. Furthermore, the two residue operations commute unless $k = 1$. If $k = 1$, then

$$\operatorname{Res}_{\lambda=u_1}\operatorname{Res}_{\lambda'=u_1} = \operatorname{Res}_{\lambda'=u_1}\operatorname{Res}_{\lambda=\lambda'}.$$

Recalling the definition of the State-Field correspondence we get

$$-\frac{1}{4}\hbar^{1/2}\operatorname{Res}_{\lambda=u_1}\sum_{s=1}^{2}\sum_{i=1}^{N+1}\sum_{J:(J\cup\{i\})\cap\{1,2\}=\{s\}}(\chi_s|\beta_1)\times$$
$$\frac{(I^{(-m-1)}_{\chi_i}(t,\lambda),\phi_a)}{(I^{(-1)}_{\beta_1}(t,\lambda),1)\prod_{j\in J}(I^{(-1)}_{\chi_i-\chi_j}(t,\lambda),1)}X_t(\chi_{j_1}\dots\chi_{j_r}\beta_1^2,\lambda)\mathcal{A}_t(\hbar;\mathbf{t})d\lambda,$$

where $j_1,\dots,j_r$ are the elements of the set $J' := (J\cup\{i\})\setminus\{s\}$. Just like before we can replace $-\frac{1}{4}\beta_1^2$ with $\chi_1\chi_2$. Rearranging the sum so that the summation over J' is first we get

$$\hbar^{1/2}\operatorname{Res}_{\lambda=u_1}\sum_{J'=\{j_1,\dots,j_r\}}\Bigg(\sum_{s=1}^{2}\sum_{j'\in J'\cup\{s\}}(\chi_s|\beta)\times$$
$$\frac{(I^{(-m-1)}_{\chi_{j'}}(t,\lambda),\phi_a)}{(I^{(-1)}_{\chi_1-\chi_2}(t,\lambda),1)\prod_{j\in J'\cup\{s\}\setminus\{j'\}}(I^{(-1)}_{\chi_{j'}-\chi_j}(t,\lambda),1)}\Bigg)X_{J'\cup\{1,2\}}(t,\lambda)\mathcal{A}_t(\hbar;\mathbf{t}),$$

where the outer sum is over all subsets J' that do not contain 1 and 2. Note that the sum over s and j' in the brackets yields

$$\frac{(I^{(-m-1)}_{\chi_1}(t,\lambda),\phi_a)}{(I^{(-1)}_{\chi_1-\chi_2}(t,\lambda),1)\prod_{j\in J'}(I^{(-1)}_{\chi_1-\chi_j}(t,\lambda),1)}-\frac{(I^{(-m-1)}_{\chi_2}(t,\lambda),\phi_a)}{(I^{(-1)}_{\chi_1-\chi_2}(t,\lambda),1)\prod_{j\in J'}(I^{(-1)}_{\chi_2-\chi_j}(t,\lambda),1)}+$$

$$\sum_{j'\in J'}\left(\frac{1}{(I^{(-1)}_{\chi_{j'}-\chi_1}(t,\lambda),1)}-\frac{1}{(I^{(-1)}_{\chi_{j'}-\chi_2}(t,\lambda),1)}\right)\frac{(I^{(-m-1)}_{\chi_{j'}}(t,\lambda),\phi_a)}{(I^{(-1)}_{\chi_1-\chi_2}(t,\lambda),1)\prod_{j\in J'\setminus\{j'\}}(I^{(-1)}_{\chi_{j'}-\chi_j}(t,\lambda),1)}.$$

The above sum is precisely

$$\sum_{j'\in J'\cup\{1,2\}}\frac{(I^{(-m-1)}_{\chi_{j'}}(t,\lambda),\phi_a)}{\prod_{j\in J'\cup\{1,2\}\setminus\{j'\}}(I^{(-1)}_{\chi_{j'}-\chi_j}(t,\lambda),1)}.$$

Note that the sum over J and i of the terms in (9) for which $J\cup\{i\}$ contains precisely one of the elements 1 or 2 coincides with (10). While the above argument shows that the sum (10) cancels with the sum over J and i of the terms in (9) for which $J\cup\{i\}$ contains both 1 and 2. Therefore our claim follows and the proof of the Theorem is completed. □

Theorem 1.1 is an immediate corollary of the above theorem, because in the case of A_N-singularity the restriction of the total ancestor potential $\mathcal{A}_t(\hbar;\mathbf{t})$ to $t=0$ coincides with $\mathcal{D}(\hbar;\mathbf{t})$.

3.4. **Example.** Let us use Corollary 1.2 to compute the primary genus-0 potential of the A_3-singularity. Put $p_a := p_{0,a}$, $x_{m,a}=0$ for $m>0$, and $t_a := x_{0,a} = t_{0,a}$. Note that

$$\Phi_a^{(0)} = t_a + p_{4-a}\lambda^{-1}.$$

The identities in Corollary 1.2 yield

$$\begin{aligned} p_3 &= t_1t_3 + \frac{1}{2}t_2^2, \\ p_2 &= 2t_2t_3 - (C(1,1)+C(1,2)+C(2,1))t_1^2t_2, \\ p_1 &= -t_1p_3 + \frac{3}{2}\,t_3^2 - C(1,1)t_1^2t_3 - (C(1,2)+C(2,1)+C(2,2))t_1t_2^2 \\ &\quad -(C(1,3)+C(3,1))t_1^2t_3 - C(1,1,1)t_1^4. \end{aligned}$$

A straightforward computation gives

$$\begin{aligned} &C(1,1)=C(2,2)=0, \\ &C(1,2)=1/2, \quad C(2,1)=1/2, \quad C(1,3)=(\eta-1)/2, \quad C(3,1)=(-\eta-1)/2, \\ &C(1,1,1)=-1/4. \end{aligned}$$

Therefore

$$p_3 = t_1t_3+\frac{1}{2}t_2^2, \quad p_2 = 2t_2t_3 - t_1^2t_2, \quad p_1 = -\frac{3}{2}t_1t_2^2 + \frac{3}{2}\,t_3^2 + \frac{1}{4}t_1^4,$$

i.e.,

$$\frac{\partial F}{\partial t_3} = t_1t_3 + \frac{1}{2}t_2^2, \quad \frac{\partial F}{\partial t_2} = t_2t_3 - \frac{1}{2}t_1^2t_2, \quad \frac{\partial F}{\partial t_1} = -\frac{1}{2}t_1t_2^2 + \frac{1}{2}\,t_3^2 + \frac{1}{12}t_1^4,$$

where F is the restriction of $\mathcal{F}^{(0)}$ to $t_{0,a}=t_a$, $t_{m,a}=0$ for $m>0$. Now it is easy to find that

$$F(t_1,t_2,t_3) = \frac{1}{2}(t_1t_3^2 + t_2^2t_3) - \frac{1}{4}t_1^2t_2^2 + \frac{1}{60}t_1^5.$$

4. The topological recursion and $\mathcal{W}$-constraints

The goal in this section is to prove Theorem 1.3. Note that the differential operators corresponding to the topological recursion have the form

$$\sum_{r=0}^{h-1} \operatorname{Res}_{\lambda=0} \sum_{i=1}^{h} \sum_{\substack{1\le j_1<\cdots<j_r\le h \\ j_s\neq i}} \frac{(I^{(-m-1)}_{\chi_i}(0,\lambda),\phi_a)}{\prod_{s=1}^r (I^{(-1)}_{\chi_i-\chi_{j_s}}(0,\lambda),1)}\,\hbar^{(r-1)/2}\,X_0(\chi_i\chi_{j_1}\cdots\chi_{j_r},\lambda). \tag{12}$$

Note that by definition if $r=0$, then the product over s is 1 and the corresponding contribution to the sum is $\partial_{t_{m,a}}$. To avoid cumbersome notation we set $X(v,\lambda):=X_0(v,\lambda)$. It is convenient to rewrite the above differential operator in terms of the cycles

$$\gamma_a := h^{-a/h}\Gamma(1-a/h)\,\phi_{h-a}, \quad 1\le a\le N.$$

Note that $\chi_i = \sum_{a=1}^N \eta^{-ia}\gamma_a$ and that

$$I^{(0)}_{\gamma_a}(0,\lambda) = (h\lambda)^{-a/h}\,\phi_{h-a}.$$

We get

$$(I^{(-1)}_{\chi_i-\chi_j}(0,\lambda),1) = (\eta^i-\eta^j)\, I^{(-1)}_{\gamma_N}(0,\lambda),1) = (\eta^i-\eta^j)\,(h\lambda)^{1/h} \tag{13}$$

and

$$(I^{(-m-1)}_{\gamma_a}(0,\lambda),\phi_a) = \frac{(h\lambda)^{m+1-a/h}}{(-a+h)\cdots(-a+(m+1)h)}. \tag{14}$$

The differential operator (12) takes the form

$$\sum_{r=0}^{h-1} \mathrm{Res}_{\lambda=0}\ d\lambda \frac{(I^{(-m-1)}_{\gamma_a}(0,\lambda),\phi_a)}{(I^{(-1)}_{\gamma_N}(0,\lambda),1)^r}\,\hbar^{(r-1)/2}\times$$

$$\sum_{a_0,\dots,a_r=1}^{N}\sum_{i=1}^{h}\sum_{\substack{1\le j_1<\cdots<j_r\le h\\ j_s\neq i}} \eta^{-i(r+a+a_0+a_1+\cdots+a_r)}\Big(\prod_{s=1}^{r}\frac{\eta^{-(j_s-i)a_s}}{1-\eta^{j_s-i}}\Big)X(\gamma_{a_0}\gamma_{a_1}\cdots\gamma_{a_r},\lambda).$$

Shifting the summation indexes $j_s\mapsto j_s+i$ and summing over i we get

$$\sum_{r=0}^{h-1}\mathrm{Res}_{\lambda=0}\ hd\lambda\frac{(I^{(-m-1)}_{\gamma_a}(0,\lambda),\phi_a)}{(I^{(-1)}_{\gamma_N}(0,\lambda),1)^r}\,\hbar^{(r-1)/2}\sum_{a_1,\dots,a_r=1}^{N} C(a_1,\dots,a_r)X(\gamma_{a_0}\gamma_{a_1}\cdots\gamma_{a_r},\lambda), \tag{15}$$

where a_0 is such that $0\le a_0\le h-1$, $r+a+a_0+\cdots+a_r\equiv 0 \pmod h$, we assume that $\gamma_{a_0}=0$ if $a_0=0$, and

$$C(a_1,\dots,a_r) := \sum_{1\le j_1<\cdots<j_r\le h-1}\prod_{s=1}^{r}\frac{\eta^{-j_s a_s}}{1-\eta^{j_s}},$$

where for $r=0$ the RHS is by definition 1. Since the differential operator $X(\gamma_{a_0}\cdots\gamma_{a_r},\lambda)$ is invariant under the permutations of $(a_0,\dots,a_r)$ we can arrange the 2nd sum in (15) to be over all increasing sequences $a_0\le a_1\le\cdots\le a_r$, i.e.,

$$\sideset{}{'}\sum_{1\le a_0\le\cdots\le a_r\le N} C[a_0,\dots,a_r]X(\gamma_{a_0}\gamma_{a_1}\cdots\gamma_{a_r},\lambda), \tag{16}$$

where $'$ means that we allow only sequences $(a_0,\dots,a_r)$ that satisfy the condition

$$r+a+a_0+\cdots+a_r\equiv 0 \pmod h$$

and the numbers $C[a_0,\dots,a_r]$ are defined as follows. If $r=0$, then we put $C[a_0]:=1$. Otherwise,

$$C[a_0,\dots,a_r] := \sum_{i=0}^{r}\frac{1}{m_i}\,\mathrm{SymC}(a_0,\dots,\widehat{a_i},\dots,a_r), \tag{17}$$

where m_i denotes the multiplicity of a_i in the sequence $(a_0,\dots,a_r)$ and SymC is the symmetrisation of C

$$\mathrm{SymC}(b_1,\dots,b_r) = \frac{1}{|\operatorname{Aut}(b_1,\dots,b_r)|}\sum_{\sigma\in S_r} C(a_{\sigma(1)},\dots,a_{\sigma(r)}).$$

Let us fix a summand in the sum (16). The corresponding sequence has the form

$$(a_0,a_1,\dots,a_r) = (b_1,\dots,b_{r'},N,\dots,N),\quad b_i<N,\quad 1\le i\le r'.$$

Put $m=r+1-r'$. Since the dilaton shift is equivalent to shifting

$$\gamma_a \mapsto \gamma_a + (I^{(-1)}_{\gamma_N}(0,\lambda),1)\,\hbar^{-1/2}\delta_{a,N}.$$

our summand is transformed into

$$C[b_1,\dots,b_{r'},\underbrace{N,\dots,N}_{m}]\sum_{m'=0}^{m}\binom{m}{m'}X(b_1\cdots b_{r'}\underbrace{N\cdots N}_{m'},\lambda)\,(I^{(-1)}_{\gamma_N}(0,\lambda),1)^{m-m'}\hbar^{-(m-m')/2}.$$

The key step now is the following identity.

Lemma 4.1. *The following identity holds*

$$C[b_1,\dots,b_r,\underbrace{N,\dots,N}_{m}] = (-1)^m\binom{[\sum_{i=1}^r b_i]_h}{m}C[b_1,\dots,b_r],$$

where $[b]_h$ *denotes the remainder of* b *modulo* h.

The proof will be given in the appendix. Using this Lemma we get

$$C[b_1,\dots,b_{r'},\underbrace{N,\dots,N}_{m}]\binom{m}{m'} = C[b_1,\dots,b_{r'},\underbrace{N,\dots,N}_{m'}](-1)^{m-m'}\binom{[\sum_{i=1}^{r'} b_i]_h-m'}{m-m'}.$$

Note that in particular, the multiplicity m of N in the sequence $(a_0,\dots,a_r)$ does not exceed $[\sum_{i=1}^{r'} b_i]_h$. The sum (16) can be written as follows:

$$\sum_{r'=0}^{r+1}\ \sideset{}{'}\sum_{1\leq b_1\leq\cdots\leq b_{r'}<N}\ \sum_{m'=0}^{r+1-r'} C[b_1,\dots,b_{r'},\underbrace{N,\dots,N}_{m'}]X(b_1\cdots b_{r'}\underbrace{N\cdots N}_{m'},\lambda)\times$$
$$\times(-1)^{r+1-r'-m'}\binom{[\sum_{i=1}^{r'} b_i]_h-m'}{r+1-r'-m'}(I^{(-1)}_{\gamma_N}(0,\lambda),1)^{r+1-r'-m'}\hbar^{-(r+1-r'-m')/2},$$

where the $'$ in the summation over $(b_1,\dots,b_{r'})$ means that

$$r'-1+b_1+\cdots+b_{r'}+a\equiv 0 \pmod h. \tag{18}$$

Substituting the above expression in (15), changing the summation index r via $s=r+1-r'-m'$, and changing the order of the summation we get

$$\sum_{r'=0}^{h}\ \sideset{}{'}\sum_{1\leq b_1\leq\cdots\leq b_{r'}<N}\ \sum_{m'=0}^{[\sum_{i=1}^{r'}b_i]_h}\mathrm{Res}_{\lambda=0}\ hd\lambda\frac{(I^{(-m-1)}_{\gamma_a}(0,\lambda),\phi_a)}{(I^{(-1)}_{\gamma_N}(0,\lambda),1)^{r'+m'-1}}\,\hbar^{-1+(r'+m')/2}\times$$
$$\times C[b_1,\dots,b_{r'},\underbrace{N,\dots,N}_{m'}]X(b_1\cdots b_{r'}\underbrace{N\cdots N}_{m'},\lambda)\times$$
$$\times\sum_{s=0}^{[\sum_{i=1}^{r'}b_i]_h-m'}(-1)^s\binom{[\sum_{i=1}^{r'} b_i]_h-m'}{s}.$$

Note that the sum over s on the 3rd line of the above formula is 0 unless $m'=[\sum b_i]_h$. Note also that $r'+m'\leq h$, otherwise the 2nd line of the formula vanishes. Recalling (18) we get that $r'+m'=h+1-a$. Using formulas (13) and (14) we get

$$hd\lambda\frac{(I^{(-m-1)}_{\gamma_a}(0,\lambda),\phi_a)}{(I^{(-1)}_{\gamma_N}(0,\lambda),1)^{r'+m'-1}}\,\hbar^{-1+(r'+m')/2} = \mathrm{const}\ \hbar^{(h-a-1)/2}\ d\lambda\,\lambda^m,$$

where the value of the constant is not important. We get that up to a constant the dilaton shift transforms the differential operator (12) into

$$\mathrm{Res}_{\lambda=0}\, d\lambda\lambda^m \sideset{}{'}\sum_{1\le b_1\le\cdots\le b_{h+1-a}\le N} C[b_1,\dots,b_{h+1-a}]X(\gamma_{b_1}\cdots\gamma_{b_{h+1-a}},\lambda),$$

where the $'$ indicates that the sum is over $(b_1,\dots,b_{h+1-a})$, s.t., $\sum b_i \equiv 0 (\mathrm{mod}\ h)$. Put $r = h+1-a$. We claim that

$$\sideset{}{'}\sum_{1\le b_1\le\cdots\le b_r\le N} C[b_1,\dots,b_r]\gamma_{b_1}\cdots\gamma_r$$

coincides with the elementary symmetric polynomial in $\chi_1,\dots,\chi_h$ of degree r. Similarly to what we did in the beginning of this Section we can rewrite the above sum as

$$\sum_{1\le i_1<\cdots<i_r\le h}\Big(\sum_{s=1}^{r}\frac{\eta^{i_s(r-1)}}{\prod_{\substack{t=1\\t\ne s}}^{r}(\eta^{i_s}-\eta^{i_t})}\Big)\chi_{i_1}\cdots\chi_{i_r}.$$

The coefficient in front of $\chi_{i_1}\cdots\chi_{i_r}$ is 1, because if we introduce the Vandermonde matrix $A_{s,t} := \eta^{(s-1)i_t}$, $1\le s,t\le r$, then the sum in the brackets can be interpreted as the quotient of the expansion of $\det(A)$ with respect to the last row and

$$\det(A) = \prod_{1\le s<t\le r}(\eta^{i_t}-\eta^{i_s}).$$

Appendix A. Proof of Lemma 4.1

by D. Lewanski

Recall the definition of the numbers

$$C[a_1,\dots,a_r],\quad 1\le r\le h,\quad 1\le a_i\le N$$

given by formula (17). It is convenient to extend the above definition by setting $C[a_1,\dots,a_r] = 0$ for $r > h$.

Lemma A.1. *Let $r\ge 1$ and $1\le a_1\le\cdots\le a_r\le N-1$ be an arbitrary sequence. The following identity holds:*

$$\sum_{m=0}^{\infty}\mathrm{SymC}[a_1,\dots,a_r,\underbrace{N,\dots,N}_{m}](1-Y)^m =$$

$$=\frac{1-Y^h}{h(1-Y)}\sum_{k_1,\dots,k_r=0}^{\infty}\sum_{\substack{I\subset\{1,\dots N\}\\|I|=r}}\Big(\prod_{j=1}^{r}\eta^{-i_j(a_j-k_j)}\Big)Y^{\sum_{i=1}^{r}k_i}, \tag{19}$$

where the 2nd sum on the RHS is over all sequences $I = (i_1,\dots,i_r)$ of pairwise different numbers.

Proof. Let us use the notation $I \subset \{1, 2, \ldots, N\}$ to denote that I is a sequence $(i_1, \ldots, i_r)$ of pairwise distinct numbers, while $I \subset (1, 2, \ldots, N)$ is a subsequence, i.e., a sequence of increasing numbers $i_1 < \cdots < i_r$. Recalling the definition of SymC we get

$$\sum_{m=0}^{\infty} \mathrm{SymC}[a_1, \ldots, a_r, \underbrace{N, \ldots, N}_{\mathrm{m}}](1-Y)^m =$$
$$= \sum_{m=0}^{\infty} \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=r}} \prod_{s=1}^{r} \frac{\eta^{-i_s a_s}}{1-\eta^{i_s}} \sum_{\substack{J \subset (1,\ldots,N) \setminus I \\ |I|=m}} \prod_{t=1}^{m} \frac{\eta^{j_s}}{1-\eta^{j_s}} (1-Y)^m$$
$$= \sum_{m=0}^{\infty} \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=r}} \prod_{s=1}^{r} \frac{-\zeta^{i_s(a_s+1)}}{1-\zeta^{i_s}} \sum_{\substack{J \subset (1,\ldots,N) \setminus I \\ |J|=m}} \prod_{t=1}^{m} \frac{1}{1-\zeta^{j_t}} (Y-1)^m$$

where $\zeta = \eta^{-1}$. Observe that for the function

$$f_I(x) := \prod_{i \in \{1,\ldots,N\} \setminus I} (x - \zeta^i) = \frac{x^h - 1}{x - 1} \prod_{i \in I} \frac{1}{(x - \zeta^i)}$$

we have

$$\frac{1}{m!} \frac{\partial_Y^m f_I(Y)}{f_I(Y)} \bigg|_{Y=1} = \sum_{\substack{J \subset (1,\ldots,N) \setminus I \\ |J|=m}} \prod_{t=1}^{m} \frac{1}{1-\zeta^{j_t}}$$

contracting the Taylor expansion the initial term is:

$$\sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=r}} \prod_{s=1}^{r} \frac{-\zeta^{i_s(a_s+1)}}{1-\zeta^{i_s}} \frac{f_I(Y)}{f_I(1)} = \frac{Y^h - 1}{h(Y-1)} \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=r}} \prod_{s=1}^{r} \frac{-\zeta^{i_s(a_s+1)}}{Y - \zeta^{i_s}}$$

Substituting back $\eta = \zeta^{-1}$ and expanding in geometric power series in the variables $Y\eta^{i_s}$ proves the lemma.

□

The statement in Lemma 4.1 is equivalent to the following identity.

Lemma A.2. *We have*

$$\sum_{m=0}^{\infty} C[a_1, \ldots, a_r, \underbrace{N, \ldots, N}_{m}](1-Y)^m = Y^{[\sum_{i=1}^r a_i]_h} C[a_1, \ldots, a_r],$$

where $[a]_h$ *denotes the remainder of* a *modulo* h.

Proof. By definition

$$\sum_{m=0}^{\infty} C[a_1,\dots,a_r,\underbrace{N,\dots,N}_{\text{m}}](1-Y)^m =$$

$$(1-Y)\sum_{m=0}^{\infty} \mathrm{SymC}[a_1,\dots,a_r,\underbrace{N,\dots,N}_{\text{m}}](1-Y)^m +$$

$$\sum_{i=1}^{r} \mathrm{SymC}[a_1,\dots,\hat{a}_i,\dots,a_r,\underbrace{N,\dots,N}_{\text{m}}](1-Y)^m.$$

Let us substitute Equation (19) in the right hand side: the factor $(1-Y)^{-1}$ cancels out in the first summand, while in the i-th summand can be expanded as $\sum_{k_i=0}^{\infty} \eta^{-0(a_i-k_i)} Y^{k_i}$. Thus the first summand collects all the subsets of $\{0,\dots,N\}$ of cardinality r not containing zero while the second summand collects all the subsets containing zero with the same cardinality r. Hence we get

$$\frac{(1-Y^h)}{h} \sum_{k_1,\dots,k_r=0}^{\infty} \sum_{\substack{I\subset\{0,\dots,N\} \\ |I|=r}} \prod_{j=1}^{r} \eta^{-i_j(a_j-k_j)} Y^{\sum k_i}$$

Now the set $\{0,1,\dots N\}$ is symmetric with respect to the shift $i_j \mapsto i_j+1$ simultaneously for all j. This implies $\eta^{-\sum(a_j-k_j)} = 1$, hence $\sum k_i = [\sum a_i]_h + hl$, for $l \in \mathbb{Z}_{\geq 0}$. The initial term can now be expanded in powers of Y as

$$Y^{[\sum a_i]_h}(1-Y^h)\frac{1}{h}\sum_{l=0} c_l (Y^h)^l$$

Since the expression is polynomial in Y, we should have $c_l = c_{l+1} = c$ for all indexes $l \geq 0$. We showed:

$$\sum_{m=0}^{\infty} C[a_1,\dots,a_r,\underbrace{N,\dots,N}_{\text{m}}](1-Y)^m = Y^{[\sum a_i]_h}\frac{c}{h}$$

Now evaluating at $Y=1$ gives $c/h = C[a_1,\dots,a_r]$ as desired.

□

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