ON FUJITA INVARIANTS OF SUBVARIATIES OF A UNIRULED VARIETY

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Abstract. We show that if X is a smooth uniruled projective variety and L a big and semiample $\mathbb{Q}$-divisor on X, then there exists a proper closed subset $W \subset X$ such that every subvariety Y satisfying $a(Y, L) > a(X, L)$ is contained in W.

1. Introduction

If X is a smooth projective variety and L is a big $\mathbb{Q}$-divisor on X, then the *Fujita invariant*, or *a-constant* is defined as follows

$$a(X, L) = \inf\{t > 0 \mid K_X + tL \text{ is big}\}.$$

Note that $a(X, L) \in \mathbb{R}_{\geq 0}$ is well defined since $K_X + tL$ is big for all $t > 0$ sufficiently large, and that $a(X, L) > 0$ if and only if K_X is not pseudo-effective. It is easy to see that the a-constant is a birational invariant in the sense that if $\nu : X' \to X$ is a birational morphism of smooth varieties and $L' = \nu^*L$, then $a(X, L) = a(X', L')$. Therefore we may also define the a-constant for a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on an arbitrary normal projective variety X by letting

$$a(X, L) := a(X', L')$$

where $\nu : X' \to X$ is a resolution of singularities and $L' = \nu^*L$. Note that if X is smooth, then the a-constant is the usual pseudo-effective threshold, however if X is singular, these numbers may be different.

In [8], motivated by a conjecture of Batyrev and Manin that relates arithmetic properties of varieties with ample anticanonical class to geometric invariants, a-constants were intensively studied by Lehmann, Tanimoto and Tschinkel. They show that ([8, Theorem 1.1]), if X is a smooth uniruled projective variety and L an ample $\mathbb{Q}$-divisor on X, then there exists a countable union of proper closed subsets $W \subset X$ such that every subvariety Y satisfying $a(Y, L) > a(X, L)$ is contained in W. For the purpose of applications, it is expected that one may choose W to be a proper closed subset of X. The purpose of this note is to prove that this is indeed the case:

Theorem 1.1. *Let X be a smooth uniruled projective variety and L a big and semiample $\mathbb{Q}$-divisor on X. Then there exists a proper closed subset $W \subset X$ such that every subvariety Y satisfying $a(Y, L) > a(X, L)$ is contained in W.*

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Note that this result is proven in [8, Theorem 1.2] assuming that a weak version of the BAB conjecture holds in dimension $n-1=\dim X-1$. We expect that Theorem 1.1 holds also if we just assume that L is nef and big (rather than big and semiample).

Our idea is to replace the WBAB conjecture assumed in [8, Theorem 1.2] by constructing non-klt centers (see Proposition 2.8) and applying finiteness of a-constants (see Corollary 2.15). This is an application of a recent result of Di Cerbo [3] based on a boundedness result proved by Birkar [2].

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2. Preliminaries

In this paper we work over the field of complex numbers $\mathbb{C}$.

2.1. **Facts on a-constants.** In this subsection, for the convenience of the reader, we collect several facts about a-constants that were proven in [8].

Proposition 2.1 ([8, Proposition 4.1])**.** *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor. Let $\mathcal{U}\to W$ be a family of subvarieties of X such that $\mathcal{U}\to X$ is dominant. Then a general member Y of the family $\mathcal{U}$ satisfies $a(Y,L)\le a(X,L)$.*

Theorem 2.2 ([8, Theorem 4.2])**.** *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor. Let $\pi:\mathcal{U}\to W$ be a family of subvarieties of X. There exists a proper closed subset $V\subset X$ such that if a member Y of the family $\mathcal{U}$ satisfies $a(Y,L)>a(X,L)$ then $Y\subset V$.*

Proposition 2.3 ([8, Proposition 4.6])**.** *Let X be a smooth uniruled projective variety and L a big and nef $\mathbb{Q}$-divisor. Then either*

(1) *X is covered by proper subvarieties Y satisfying $a(Y,L)=a(X,L)$, or*
(2) *X is birational to a $\mathbb{Q}$-factorial terminal Fano variety X' of Picard number 1.*

Lemma 2.4 ([8, Lemma 4.7])**.** *Let X be a smooth projective variety and L a big and nef $\mathbb{Q}$-divisor on X. Fix a constant C. Then the subset of* $\mathrm{Chow}(X)$ *parametrizing subvarieties of X that are not contained in $\mathbf{B}_+(L)$ and are of L-degree at most C is bounded.*

2.2. **Non-klt centers.** We follow the standard notation and conventions of the minimal model program, see eg. [5].

Definition 2.5. Let (X,Δ) be a pair so that X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor, and $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We say that a subvariety $V\subset X$ is a *non-klt center* of (X,Δ) if it is the image of a divisor of discrepancy at most -1. We will denote by $\mathrm{Nklt}(X,\Delta)$ the union of all non-klt centers of (X,Δ). A *non-klt place* is a valuation corresponding to a divisor of discrepancy at most -1. A non-klt center is *pure* if $K_X+\Delta$ is log canonical at the generic point of V. If moreover there is a unique non-klt place lying over the generic point of V, we will say that V is an *exceptional* non-klt center.

The following is a weak form of Kawamata's subadjunction theorem.

Theorem 2.6 (Subadjunction, see [4, Proposition 5.1])**.** *Let $V \subset X$ be a non-klt center of a pair (X, Δ) which is lc at a general point of V. Let $\nu : V^\nu \to V$ be the normalization. Then there is an effective $\mathbb{Q}$-divisor Δ_{V^ν} on V^ν such that*

$$\nu^*(K_X + \Delta)|_{V_\nu} \sim_{\mathbb{Q}} K_{V^\nu} + \Delta_{V^\nu}.$$

We have the following connectedness lemma of Kollár and Shokurov for the non-klt locus (cf. Shokurov [9], Kollár [6, 17.4]).

Theorem 2.7 (Connectedness Lemma)**.** *Let $f : X \to Z$ be a proper morphism of normal varieties with connected fibers and D a $\mathbb{Q}$-divisor such that $-(K_X + D)$ is $\mathbb{Q}$-Cartier, f-nef, and f-big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f-exceptional (i.e. all of its components have image of codimension at least* 2*), then* $\mathrm{Nklt}(X, D) \cap f^{-1}(z)$ *is connected for any $z \in Z$.*

We can use the following proposition to construct non-klt centers.

Proposition 2.8 (cf. [7, Lemma 3.2])**.** *Let X be a $\mathbb{Q}$-factorial terminal Fano variety of dimension n. Assume $(-K_X)^n > (wn)^n$ for some positive rational number w. Then for every point $p \in X$ there is an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ such that the unique minimal non-klt center $V_p \subset$* $\mathrm{Nklt}(X, \Delta_p)$ *containing p is exceptional.*

Proof. Fix a point p. Fix a positive rational number w' such that $(-K_X)^n > (w'n)^n > (wn)^n$. By [5, 6.7.1 Theorem], there is an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$ such that (X, Δ'_p) is not lc at p. Take $0 < t \le 1$ the unique rational number such that $(X, t\Delta'_p)$ is log canonical but not klt at p. By [1, Proposition 3.2, Lemma 3.4], we can find an effective $\mathbb{Q}$-divisor $M_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$ and some rational number $a > 0$ such that for any rational number $0 < \epsilon \ll 1$, the pair $(X, (1-\epsilon)t\Delta'_p + \epsilon a M_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. Note that

$$(1-\epsilon)t\Delta'_p + \epsilon a M_p \sim_{\mathbb{Q}} -\frac{(1-\epsilon)t + \epsilon a}{w'}K_X$$

and $\frac{(1-\epsilon)t+\epsilon a}{w'} < \frac{1}{w}$ for $0 < \epsilon \ll 1$. Since $-K_X$ is ample, by adding a $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to a multiple of $-K_X$ to Δ'_p, we conclude that there exists an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ and (X, Δ_p) has a unique minimal non-klt center V_p passing through p which is exceptional. □

Lemma 2.9. *Keep the notation in Proposition 2.8. If $w > 2$, then* $\dim V_p >$ 0 *for a general point p.*

Proof. Assume to the contrary that there exist $p_1 \in X$ such that $V_{p_1} = \{p_1\}$ and $p_2 \in X \backslash \mathrm{Supp}(\Delta_{p_1})$ such that $V_{p_2} = \{p_2\}$. Then p_1 and p_2 are contained in $\mathrm{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$ and p_2 is isolated by construction. On the other hand,

$$-(K_X + \Delta_{p_1} + \Delta_{p_2}) \sim_{\mathbb{Q}} \Big(1 - \frac{2}{w}\Big)(-K_X)$$

is ample. By the connectedness lemma, $\mathrm{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$ is connected, which is a contradiction. □

2.3. **Finiteness of a-constants.** We recall the main result of [3] in this subsection.

Definition 2.10. Let X be a normal projective variety and H a big $\mathbb{Q}$-divisor. We define the *pseudo-effective threshold* to be

$$\tau(X, H) := \inf\{t \geq 0 \mid K_X + tH \text{ is big}\}.$$

Note that if X is smooth, a-constant and pseudo-effective threshold just coincide.

Definition 2.11 (cf. [3, Definition 3.1])**.** Fix a positive integer n and two positive real numbers ϵ and δ. We define $\mathcal{D}_n(\epsilon, \delta)$ to be the set of lc pairs (X, Δ) such that:

(1) X is a normal projective variety of dimension n,
(2) Δ is a big $\mathbb{Q}$-divisor with coefficients $\geq \delta$, and
(3) $(X, t\Delta)$ is ϵ-lc and $K_X + t\Delta$ is pseudo-effective for some $0 \leq t \leq 1$.

Definition 2.12 (cf. [3, Definition 3.2])**.** Fix a positive integer n and two positive real numbers ϵ and δ. We define the set

$$\mathcal{T}_n(\epsilon, \delta) := \{\tau(X, \Delta) \mid (X, \Delta) \in \mathcal{D}_n(\epsilon, \delta)\}.$$

Theorem 2.13 ([3, Corollary 3.6])**.** *Fix a positive integer n and three positive real numbers ϵ, δ and η. Then the set $\mathcal{T}_n(\epsilon, \delta) \cap [\eta, 1]$ is a finite set.*

To apply this theorem in our situation, we have the following corollary.

Definition 2.14. Fix a positive integer n. We define $\mathcal{P}_n$ to be the set of pairs (Y, L) such that:

(1) Y is a normal projective variety of dimension n,
(2) L is a base point free big Cartier divisor.

Corollary 2.15. *Fix a positive integer n and a positive real number η. Then the set*

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \infty)$$

is a finite set.

Proof. We may assume that $\eta \leq \frac{1}{4(n+1)}$.

Firstly, we show that the set

$$\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap \left[\eta, \frac{1}{2}\right]$$

is a finite set. Take $(Y, L) \in \mathcal{P}_n$ and assume that $a(Y, L) \in [\eta, \frac{1}{2}]$. Note that $a(Y, \frac{1}{2}L) = 2a(Y, L) \in [2\eta, 1]$. By taking a resolution, we may assume that Y is smooth. In this case $a(Y, \frac{1}{2}L) = \tau(Y, \frac{1}{2}L)$. Replacing L by a general element in $|L|$, we may assume that L is irreducible and smooth. Moreover, $(Y, \frac{1}{2}L)$ is $\frac{1}{2}$-lc and $K_Y + \frac{1}{2}L$ is pseudo-effective, that is, $(Y, \frac{1}{2}L) \in \mathcal{D}_n(\frac{1}{2}, \frac{1}{2})$. This implies that the set

$$\left\{a\left(Y, \frac{1}{2}L\right) \;\middle|\; (Y, L) \in \mathcal{P}_n\right\} \cap [2\eta, 1]$$

is finite by Theorem 2.13, and so is $\{a(Y,L) \mid (Y,L) \in \mathcal{P}_n\} \cap [\eta, \frac{1}{2}]$.

Then we show that the set

$$\{a(Y,L) \mid (Y,L) \in \mathcal{P}_n\} \cap \left[\frac{1}{2}, \infty\right)$$

is a finite set. Take $(Y,L) \in \mathcal{P}_n$ and assume that $a(Y,L) \geq \frac{1}{2}$. By taking a resolution, we may assume that Y is smooth. By [8, Proposition 2.10], $a(Y,L) \leq n+1$. Now we consider $(Y, 2(n+1)L) \in \mathcal{P}_n$. Note that $a(Y, 2(n+1)L) = \frac{1}{2(n+1)} a(Y,L)$, hence $a(Y, 2(n+1)L) \in [\frac{1}{4(n+1)}, \frac{1}{2}]$. By the first step, $a(Y, 2(n+1)L)$ belongs to a finite set. Hence $a(Y,L)$ belongs to a finite set. □

3. Proof of Theorem 1.1

We prove the following proposition suggested by B. Lehmann.

Proposition 3.1. *Fix a positive real number t. Let X be a smooth projective variety and L a big and semiample $\mathbb{Q}$-divisor. Then there is a bounded family $\mathcal{U}$ of subvarieties of X such that any subvariety Y not contained in $\mathbf{B}_+(L)$, with $a(Y,L) > t$ is dominated by some members Z of $\mathcal{U}$, such that $a(Z,L) = a(Y,L)$.*

Proof. Note that for a subvariety Y not contained in $\mathbf{B}_+(L)$, $L|_Y$ is nef and big, and so $a(Y,L)$ is well defined. Therefore we will only consider subvarieties not contained in $\mathbf{B}_+(L)$.

Replacing L by some multiple, we may assume that L is a base point free Cartier divisor.

We construct $\mathcal{U}$ inductively by increasing induction on the dimension of Y.

For a subvariety Y with $a(Y,L) > t$ and $\dim Y = 1$, it is easy to see that Y is a rational curve with

$$\deg_Y(L) = Y \cdot L = \frac{2}{a(Y,L)} < \frac{2}{t}.$$

By Lemma 2.4, such Y form a bounded family $\mathcal{U}_1$.

Suppose that we have constructed a bounded family $\mathcal{U}_i$ of subvarieties such that every subvariety Y with $a(Y,L) > t$ and $\dim Y \leq i$ is dominated by some members Z of $\mathcal{U}$ such that $a(Z,L) = a(Y,L)$. We construct $\mathcal{U}_{i+1}$ as follows. Suppose that Y is an $(i+1)$-dimensional subvariety satisfying $a(Y,L) > t$. By taking a resolution, we may assume that Y is smooth. Proposition 2.3 shows that either

(1) Y is covered by proper subvarieties Z with $a(Z,L) = a(Y,L)$, or
(2) Y is birational to a $\mathbb{Q}$-factorial terminal Fano variety Y' of Picard number 1.

In Case (1), by induction, Z is dominated by some members Z' of $\mathcal{U}_i$ such that $a(Z',L) = a(Z,L)$, and so is Y.

In Case (2), by taking a resolution, we may assume $\phi : Y \dashrightarrow Y'$ is a morphism. By the proof of [8, Proposition 4.6], $K_{Y'} + a(Y,L)\phi_*(L|_Y) \equiv 0$.

We define constant $c_0 < 1$ and $w > 2$ as follows: since L is base point free, we know that the set

$$\{a(Z,L) \mid Z \text{ is a subvariety of } X\} \cap (t, \infty]$$

is finite by Corollary 2.15. Hence we may take a rational number $c_0 < 1$ such that the set

$$\{a(Z,L) \mid Z \text{ is a subvariety of } X\} \cap [c_0 a(Z',L), a(Z',L))$$

is empty for any subvariety Z' with $a(Z',L) > t$. Take $w = \frac{1}{1-c_0}$. We may assume $w > 2$ by decreasing c_0.

If $(-K_{Y'})^{i+1} \leq (w(i+1))^{i+1}$, then

$$(L|_Y)^{i+1} \leq (\phi_*(L|_Y))^{i+1} \leq \frac{(w(i+1))^{i+1}}{a(Y,L)^{i+1}} < \frac{(w(i+1))^{i+1}}{a(X,L)^{i+1}}.$$

Then by Lemma 2.4, such Y form a bounded family $\mathcal{U}'_{i+1}$.

Now we assume that $(-K_{Y'})^{i+1} > (w(i+1))^{i+1}$. By Proposition 2.8, for a general point $p \in Y'$, there exists an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w}K_{Y'}$ such that $V'_p \subset \mathrm{Nklt}(Y', \Delta'_p)$ is the minimal exceptional non-klt center containing p. Note that by Lemma 2.9 and $w > 2$, $\dim V'_p > 0$. Let $\nu : \tilde{V}^\nu_p \to V'_p$ be the normalization. For any $\mathbb{Q}$-Cartier divisor G on V'_p, we denote $G|_{\tilde{V}^\nu_p} = \nu^* G$. By Theorem 2.6, there is an effective $\mathbb{Q}$-divisor $\Delta_{\tilde{V}^\nu_p}$ such that

$$(K_{Y'} + \Delta'_p)|_{\tilde{V}^\nu_p} \sim_{\mathbb{Q}} K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p}.$$

Note that since $K_{Y'} + a(Y,L)\phi_* L \equiv 0$, we have

$$K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p} + \Big(1 - \frac{1}{w}\Big) a(Y,L)\phi_* L|_{\tilde{V}^\nu_p} \sim_{\mathbb{Q}} 0.$$

Let V_p be the strict transform of V'_p on Y. Let $\tilde{V}_p$ be a common resolution of $\tilde{V}^\nu_p$ and V_p, $f : \tilde{V}_p \to V_p$, $g : \tilde{V}_p \to \tilde{V}^\nu_p$. Then

$$\begin{aligned} & K_{\tilde{V}_p} + \Big(1 - \frac{1}{w}\Big) a(Y,L) f^*(L|_{V_p}) \\ = \ & g^*\Big(K_{\tilde{V}^\nu_p} + \Delta_{\tilde{V}^\nu_p} + \Big(1 - \frac{1}{w}\Big) a(Y,L)\phi_* L|_{\tilde{V}^\nu_p}\Big) - g_*^{-1}\Delta_{\tilde{V}^\nu_p} + E \\ \sim_{\mathbb{Q}} & -g_*^{-1}\Delta_{\tilde{V}^\nu_p} + E, \end{aligned}$$

where E is a g-exceptional $\mathbb{Q}$-divisor. Note that the $\mathbb{Q}$-divisor $-g_*^{-1}\Delta_{\tilde{V}^\nu_p} + E$ is not big. Hence $K_{\tilde{V}_p} + (1 - \frac{1}{w}) a(Y,L) f^*(L|_{V_p})$ is not big and therefore

$$a(V_p, L) \geq \Big(1 - \frac{1}{w}\Big) a(Y,L) = c_0 a(Y,L).$$

By the definition of c_0, this implies that $a(V_p, L) \geq a(Y,L)$. Since p is a general point, Y is dominated by such V_p. By induction, V_p is dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) = a(V_p,L) \geq a(Y,L)$. Hence Y is dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) \geq a(Y,L)$. By Proposition 2.1, by taking general members, Y is dominated by some members Z of $\mathcal{U}_i$ such that $a(Z,L) = a(Y,L)$.

Hence we may take $\mathcal{U}_{i+1} = \mathcal{U}_i \cup \mathcal{U}'_{i+1}$, and the proof is completed. □

Proof of Theorem 1.1. Take $t = a(X,L)$ in Proposition 3.1, there is a bounded family $\mathcal{U}$ of subvarieties of X such that any subvariety Y not contained in $\mathbf{B}_+(L)$, with $a(Y,L) > a(X,L)$ is dominated by some members Z of $\mathcal{U}$, such that $a(Z,L) = a(Y,L) > a(X,L)$. By Theorem 2.2, there exists a proper closed subset $W \subset X$ such that any member Z of the family $\mathcal{U}$ satisfying $a(Z, L) > a(X, L)$ is contained in W. Hence any subvariety Y with $a(Y, L) > a(X, L)$ is contained in W. □

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