COMPACTIFICATIONS OF THE MODULI SPACE OF POINTS IN PROJECTIVE SPACE

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ABSTRACT. We introduce and study smooth compactifications of the moduli space of n labeled points with weights in projective space, which have normal crossings boundary and are defined as GIT quotients of the weighted Fulton MacPherson compactification. We show the more general result that the GIT quotient of a wonderful compactification is also a wonderful compactification under certain hypotheses. We also study a weighted version of the configuration spaces parametrizing n points in affine space up to translation and homothety. For dimension one, the above compactifications are isomorphic to Hassett's moduli space of rational weighted stable curves.

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1. INTRODUCTION

For any smooth variety X, Fulton and MacPherson constructed a smooth compactification $X[n]$ of the configuration space of n distinct labeled points in X, such that all points remain distinct in the degenerate configurations [FM94]. This compactification was used by Hu and Keel to construct $\overline{M}_{0,n}$ as a GIT quotient of $\mathbb{P}^1[n]$ by SL_2 [HK00]. Recently, the second author extended $X[n]$ by including weight data which allow points to collide depending on the accumulation of their weights [Rou14]. These generalizations can be used to recover the moduli spaces of weighted stable rational curves defined by Hassett [Has03].

The purpose of this article is to introduce weighted compactifications $\overline{P}^{\mathscr{A}}_{d,n}$ of the moduli space of n points in $\mathbb{P}^d$. These spaces can be understood as the GIT quotients of the 'weighted Fulton MacPherson' compactification $\mathbb{P}^d_{\mathscr{A}}[n]$ (Section 3.2) of n points in $\mathbb{P}^d$ with respect to SL_{d+1} and as an iterative blow up construction similar to the one used by Kapranov for the moduli space of stable rational curves [Kap93]. The intuitive geometric picture is similar to the one for $\overline{M}_{0,n}$: we start with an equivalence class of n points in $\mathbb{P}^d$ defined up to the usual SL_{d+1} action and an ordered set of weights $\mathscr{A}$, i.e. an ordered set of numbers between 0 and 1, and we attach each number to the corresponding labeled point. If a subset of points in $\mathbb{P}^d$ with weight sum larger than one collides, then we blow up the locus where it collides and attach a new $\mathbb{P}^d$, which we glue along the exceptional divisor. The points then 'move' to the new $\mathbb{P}^d$ and are not simultaneously coincident anymore. Also, the points in this new component are defined up to the natural SL_{d+1} action that fixes the exceptional divisor pointwise. We continue inductively until all colliding points with weight larger than one are separated. The resulting degenerations are called *weighted stable trees* with respect to the set of weights $\mathscr{A}$ (Section 2.6). Let $P^{\mathscr{A}}_{d,n}$ be the open locus in $\overline{P}^{\mathscr{A}}_{d,n}$ parametrizing n labeled points in $\mathbb{P}^d$ up to an action of SL_{d+1} with total weight less than or equal to 1 (Definition 4.2). The following theorem is proven in Section 4.1.

Theorem 1.1. *For any set of admissible weights $\mathscr{A}$:*

(1) The compactification $\overline{P}^{\mathscr{A}}_{d,n}$ of $P^{\mathscr{A}}_{d,n}$ is a smooth variety, whose boundary is a union of smooth divisors that intersect with normal crossings.

(2) *There exists a smooth variety $\overline{U}^{\mathscr{A}}_{d,n}$ and a flat proper morphism*

$$\hat{\phi}_{\mathscr{A}} : \overline{U}^{\mathscr{A}}_{d,n} \to \overline{P}^{\mathscr{A}}_{d,n}$$

equipped with n sections $\hat{\sigma}_i : \overline{P}^{\mathscr{A}}_{d,n} \to \overline{U}^{\mathscr{A}}_{d,n}$ such that
- *the images of $\hat{\sigma}_i$ lie in the relative smooth locus of $\hat{\phi}_{\mathscr{A}}$ and*
- *the geometric fibers of $\hat{\phi}_{\mathscr{A}}$ are precisely the weighted stable trees.*

While our focus is the study of $\overline{P}^{\mathscr{A}}_{d,n}$, the theoretical framework developed in this paper allows for many more compactifications of moduli problems arising in different contexts. In particular, in Section 3.3, we study GIT quotients of so called 'wonderful compactifications' (see [Li09]). Wonderful compactifications are always smooth with normal crossings boundary and can be described as a sequence of smooth blowups. Quite a few compactifications in the literature can be obtained as wonderful compactifications: among these are the Fulton-MacPherson compactification, Keel's construction of $\overline{M}_{0,n}$, Kuperberg-Thurston's compactifications, Ulyanov's polydiagonal compactification and Hu's compactification of open varieties (see Section 4 of [ibid.]). In Lemma 3.9, we show that the GIT quotients of wonderful compactifications are also wonderful compactifications under certain conditions. As a result, we can study degenerations of equivalence classes of points in an arbitrary smooth variety X with a given group action (see Section 2.3 for a description). We use stability conditions which are numerical generalizations of the ones used to define $\overline{M}_{0,\omega}$. Our requirement that the total weight of colliding points is less than or equal to one and away from the double locus resembles the one asking for at worst log canonical pairs. We also require a minimum weight on each component which is similar to the ampleness condition and is used to prevent additional blow ups.

As far as applications of $\overline{P}^{\mathscr{A}}_{d,n}$ are concerned, we remark that the spaces $\overline{P}^{\mathscr{A}}_{2,9-k}$ can be viewed as smooth weighted compactifications of the moduli of marked Del Pezzo surfaces for $1 \le k \le 4$. Indeed, a generic smooth Del Pezzo surface is the blow up of $\mathbb{P}^2$ along $9-k$ distinct points with the labels of the points inducing markings on it. The comparison of $\overline{P}^{\mathscr{A}}_{2,9-k}$ with other compactifications of the moduli space of marked Del Pezzo surfaces [HKT09] is the subject of work in progress. We discuss the relationship of our work with the moduli space of hyperplane arrangements constructed by Hacking-Keel-Tevelev [HKT06] and Alexeev [Ale08, Ale13] in Section 1.2. Futhermore, by examining the boundary of $\overline{P}^{\mathscr{A}}_{d,n}$ we run into the space of so called *weighted stable rooted trees* $T^{\mathscr{A}}_{d,n}$ (Section 2.5), recently introduced by the second author [Rou15] for the purpose of calculating the Chow ring of the Hassett's spaces in genus 0. $T^{\mathscr{A}}_{d,n}$ is a weighted generalization of the Chen-Gibney-Krashen compactification, $T_{d,n}$, of the parameter space of n distinct labeled points in $\mathbb{A}^d$ up to translation and homothety ([CGK09]). The configuration spaces $T_{d,n}$ are smooth projective varieties, their boundary is a union of smooth divisors that meet with normal crossings and they can be understood as non-reductive Chow quotients [GG15]. The closed points of $T_{d,n}$ parametrize a direct generalization of stable pointed rational curves known as stable pointed rooted trees. We further explore the geometry of $T^{\mathscr{A}}_{d,n}$ and generalize some of the main results in [CGK09]. Let $\mathscr{A} = \{a_1, a_2, \dots, a_n\}$ be an admissible set of weights (see Section 2.1) and consider the admissible weight sets $\mathscr{A}(I) := \{a_i \mid i \in I\}$ and $\mathscr{A}_+(I^c) := \{a_i \mid i \notin I\} \cup \{a_{n+1} = 1\}$.

Theorem 1.2. *Let $\mathscr{A} = \{a_1, a_2, \dots, a_n\}$ be an admissible set of weights (see Section 2.1). Then:*

(1) *$T^{\mathscr{A}}_{d,n}$ is a smooth variety, whose boundary is a union of smooth divisors that intersect with normal crossings.*

(2) *Each divisor D_I in the boundary of $T^{\mathscr{A}}_{d,n}$ factors as:*

$$T^{\mathscr{A}(I)}_{d,|I|} \times T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1}$$

Moreover, the spaces $T^{\mathscr{A}}_{d,n}$ appear in the boundary of $\overline{P}^{\mathscr{A}}_{d,n}$. In particular, every divisor $E_I \subset \overline{P}^{\mathscr{A}}_{d,n}$ factors as

$$E_I \cong T^{\mathscr{A}(I)}_{d,|I|} \times \overline{P}^{\mathscr{A}+(I^c)}_{n-|I|+1}$$

The previous theorem is proven in Section 4.2. By varying the weights, we show the existence of reduction morphisms among our configuration spaces(see Section 5). In particular, we obtain toric ones (for the proof see Section 5.2).

Theorem 1.3. *There exist birational contractions, obtained by decreasing the weights of the marked points, of the form:*

$$T_{d,n} \to T^{LM}_{d,n} \to \mathbb{P}^{nd-d-1} \qquad\qquad \overline{P}_{d,n} \to \overline{P}^{LM}_{d,n} \to (\mathbb{P}^{n-d-2})^d$$

where $T^{LM}_{d,n}$ and $\overline{P}^{LM}_{d,n}$ are toric varieties. Moreover, these toric varieties are 'maximal', in the sense that any toric variety $T^{\mathscr{A}}_{d,n}$ (resp. $\overline{P}^{\mathscr{A}}_{d,n}$) obtained by a choice of admissible weights $\mathscr{A}$, factors through $T^{LM}_{d,n} \to \mathbb{P}^{nd-d-1}$ (resp. $\overline{P}^{LM}_{d,n} \to (\mathbb{P}^{n-d-2})^d$).

Our proofs are constructive. We show that relevant GIT quotients parametrizing configurations with multiple points are products of projective spaces. Then, we characterize the loci parametrizing non-stable configurations with respect to a given set of weights. These loci are unions of smooth varieties intersecting cleanly (Definition 3.1) and the wall crossing associated to changing the weights are smooth blow ups. For example, for the configuration spaces associated to points in $\mathbb{P}^d$ we find (see Corollary 4.21 for the analogous result on $T_{d,n}$):

Corollary 1.4. *For $n \geq d+3$, the morphism $\overline{P}_{d,n} \to \left(\mathbb{P}^{n-d-2}\right)^d$ can be understood as completing the following $(n-d-2)$ steps successively:*

(1) *blow up $(n-d)$ disjoint points parametrizing configurations with a $(n-(d+1))$ multiple point.*
(2) *blow up the strict transforms of the $\left(\mathbb{P}^1\right)^d$s spanned by the previous $(n-d)$ disjoint points; they generically parametrize configurations with a $(n-(d+2))$ multiple point.*

$$\vdots$$

(n-d-2) *blow up the strict transforms of $\left(\mathbb{P}^{n-d-3}\right)^d$'s spanned by the $\left(\mathbb{P}^{n-d-4}\right)^d$'s of step $(n-d-3)$; they generically parametrize configurations with a double point.*

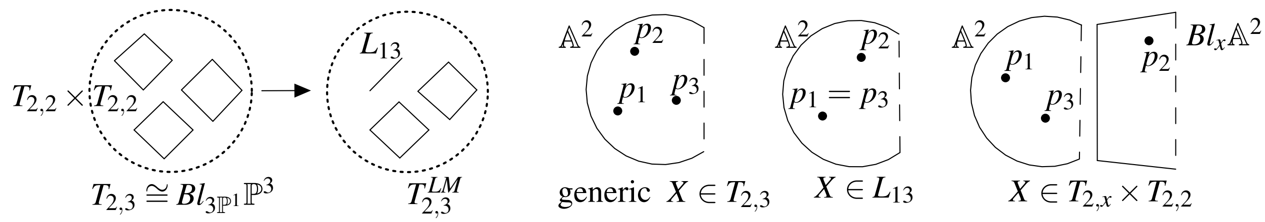
Finally, we recall that for $T_{1,n} \cong \overline{M}_{0,n+1}$ the product of forgerful morphism is injective (see [GG10, Thm 1.3]). We generalize this result for all $T_{d,n}$ (see Example 5.4 and afterwards):

Theorem 1.5. *For any $3 \leq k \leq n$ the product of forgetful morphisms $\pi_I : T_{d,n} \to T_{d,|I|}$ over all subsets $I \subset \{1,\dots,n\}$ of cardinallity $|I| = k$ is injective*

$$\pi_k : T_{d,n} \to \prod_{|I|=k} T_{d,I}$$

In contrast, if $k = 2$ the morphism π_2 has positive dimensional fibers.

1.1. **Explicit examples.** Next, we present a few examples to illustrate our results. Two points in $\mathbb{A}^2$ have one degree of freedom up to translation and homothety, because we can always translate one of them to the origin, and we can scale the second point along the line generated by the two points. Therefore, $T_{2,2} \cong \mathbb{P}^1$. To describe $T_{2,3}$, we notice that the open loci parametrizing configurations of three distinct points in $\mathbb{A}^2$ up to translation and homothety is $\mathbb{P}^3 \setminus \{L_{12}, L_{13}, L_{23}\}$ where L_{ij} are disjoint lines. Each line L_{ij} parametrizes a configuration with the double point $p_i = p_j$.

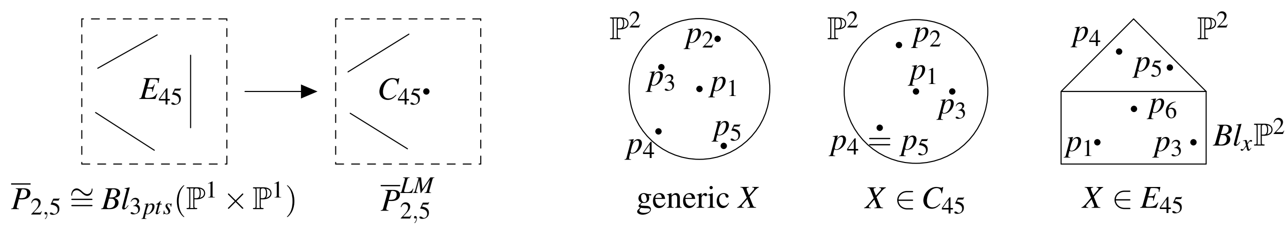


(A) Space parametrizing three points in $\mathbb{A}^2$ up to translation and homothety.

(B) Parametrized stable rooted trees by the interior and the boundary.

FIGURE 1. (A) depicts the compactifications $T^{LM}_{2,3}$ and $T_{2,3}$, while (B) depicts the objects they parametrize .

For weights equal to 1 those double points are not allowed. And, $T_{2,3}$ is the blow up of $\mathbb{P}^3$ along these three lines L_{ij}. The boundary component can be interpreted as $T_{2,2} \times T_{2,2}$; it parametrizes stable rooted trees that decompose as the union of two components. On the other hand, we can choose weights $\mathscr{A}_{LM}$ allowing $p_1 = p_3$ while forbidding the other double points. The respective model $T^{LM}_{2,3}$ is the blow up of $\mathbb{P}^3$ along the lines L_{12} and L_{23}.
The space $\overline{P}_{2,5}$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points and the boundary divisor is the union of three disjoint $T_{2,2}$'s. Indeed, we fix the points p_1, p_2 and p_3 in general position and away from the other ones. The open locus parametrizing five distinct points in $\mathbb{P}^2$ up to an action of $Aut(\mathbb{P}^2)$ is $\mathbb{P}^1 \times \mathbb{P}^1$ minus three zero-dimensional loci. These loci parametrize configurations with either $p_4 = p_5$, $p_5 = p_3$ or $p_4 = p_3$. Blowing up these loci creates three boundary divisors E_{ik}.



(A) Space parametrizing five points in $\mathbb{P}^2$ up to the SL_3 action.

(B) Stable trees parametrized by the interior and the boundary.

FIGURE 2. (A) depicts the compactifications $P^{LM}_{2,3}$ and $P_{2,3}$, while (B) depicts the objects they parametrize

Each of them parametrizes a configuration with two distinct points p_i, p_k lying in a new $\mathbb{P}^2$ together with a distinguished hyperplane H glued along the original $\mathbb{P}^2$ blow up at a point, where the rest of the points lie. The two points p_i and p_k are thought to be in $\mathbb{A}^2 = \mathbb{P}^2 \setminus H$ because the points never touch H. Then, the exceptional divisors are in fact isomorphic to $T_{2,2}$.

1.2. **Comparing $\overline{P}_{d,n}$ with the moduli space of hyperplane arrangements.** The moduli space of n generic points is related to the one of generic hyperplane arrangements because parametrizing n distinct hyperplanes in $\mathbb{P}^d$ amounts to parametrizing n distinct points in the dual projective space $\hat{\mathbb{P}}^d$. The perspective of hyperplanes is a fruitful one. A compact moduli space $\overline{M}_{\vec{w}}(\hat{\mathbb{P}}^d, n)$ was constructed for hyperplanes of weight one by Hacking-Keel-Tevelev [HKT06], and for arbitrary weights by Alexeev [Ale08, Ale13]. The space $\overline{M}(\hat{\mathbb{P}}^d, n)$ is quite intricate– it can be arbitrarily singular, and can contain many irreducible components. However, it has a main component $\overline{M}^m(\hat{\mathbb{P}}^d, n)$ parametrizing n hyperplanes in the projective space and their log canonical degenerations. This main component is isomorphic up to normalization to the Chow quotient $(\mathbb{P}^d)^n /\!/_{Ch} SL_{d+1}$ defined by Kapranov [Kap93]. Therefore, there exists a morphism from the main component $\overline{M}^m(\hat{\mathbb{P}}^d, n)$ to any GIT quotient of n points in $\mathbb{P}^d$ (see [Ale13, Sec 5.5]). In particular, by Lemma 4.1 there is a blow up

$$\Phi : \overline{M}^m(\hat{\mathbb{P}}^d, n) \to \left(\mathbb{P}^{n-d-2}\right)^d$$

The center of this blow up is supported at the union of loci parametrizing configuration of hyperplanes with non-log canonical singularities. We can classify these loci as follows:

Type I: They parametrize overlapping hyperplanes. In the dual projective space, they correspond to the loci parametrizing colliding points.

Type II: They parametrize hyperplanes concurrent along a linear subspace. In the dual projective space, they correspond to the loci parametrizing points satisfying a co-linearity condition.

We may consider $\overline{P}_{d,n}$ as an approximation of $\overline{M}^m(\hat{\mathbb{P}}^d, n)$, because our construction (see Theorem 1.1) is equivalent to a sequence of smooth blow ups of $\left(\mathbb{P}^{n-d-2}\right)^d$ along the centers of Type I, while the morphism Φ is a sequence of non-necessarily smooth blow ups along centers of Type I and II:

$$\begin{tikzcd}
& \overline{M}^m(\hat{\mathbb{P}}^d, n) \arrow[dl, "\Phi"'] \arrow[d, dotted, leftrightarrow] & \Longleftarrow \text{blowing up loci of Type I and II} \\
\left(\mathbb{P}^{n-d-2}\right)^d & \overline{P}_{d,n} \arrow[l] & \Longleftarrow \text{blowing up loci of Type I}
\end{tikzcd}$$

We warn that in general, the behavior of the generic fiber of Φ at the centers of Type II is not known and there is no explicit description of the morphism Φ. Moreover, we do not interpret $\overline{P}_{d,n}$ as a configuration space of hyperplanes, because our universal family $\overline{U}_{d,n} \to \overline{P}_{d,n}$ is constructed from the one associated to the Fulton-MacPherson spaces. The objects in this family are surfaces with n distintic points on them (see Figure 2).

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CONTENTS

1.4. **Conventions.** Throughout this paper, the term variety will be understood as a reduced and irreducible scheme defined over an algebraically closed field of characteristic 0. Also we will often denote the set of integers $\{1,2,\dots,n\}$ by the capital letter N.

2. Description of the Parametrized Objects

Here, we describe the three types of parametrized objects that appear in our work. First, the weighted stable degenerations of n labelled points in an arbitrary nonsingular variety X. Second, the weighted stable rooted trees which are degenerations of n labelled points in $\mathbb{A}^d$ defined up to translation and homothety. Third, the weighted stable trees which are degenerations of n points in $\mathbb{P}^d$ defined up an action of SL_{d+1}. Next, we discuss the weight domain for each of these cases.

2.1. **Weight Domains.** Let X be a smooth variety with $\dim X = d \geq 1$ and let $n \geq 2$. The domain of admissible weights for the *weighted compactifications* of the configuration space of n labeled points in X (section 3.2) is given by

$$\mathscr{D}^{FM}_{d,n} := \{(a_1,\dots,a_n) \in \mathbb{Q}^n \ : \ 0 < a_i \leq 1\}$$

Now let $d \geq 1$ and $n \geq 2$. The domain of admissible weights for the space of *weighted stable rooted trees* (Section 4.2) is given by

$$\mathscr{D}^{T}_{d,n} := \{(a_1,\dots,a_n) \in \mathbb{Q}^n \ : \ 0 < a_i \leq 1 \ \text{ and } 1 < a_1+\ldots+a_n\}$$

Finally, let us fix d and n such that $d \geq 1$ and $n \geq d+2$. Given arbitrarily small numbers $\hat{\varepsilon} \ll \varepsilon$, we consider the set of weights

$$w_1 = \ldots = w_d = 1-\hat{\varepsilon}, \qquad w_{d+1} = 1-(n-(d+1))\varepsilon + d\hat{\varepsilon}, \qquad w_{d+2} = \ldots = w_n = \varepsilon.$$

Then the domain of admissible weights for the space of *weighted stable trees* (Section 4.1) is

$$\mathscr{D}^{P}_{d,n} = \{(a_1,\dots a_n) \in \mathbb{Q}^n \ : w_i \leq a_i \leq 1\}$$

These last constraints are motivated by a technical requirement in Lemma 4.1. In the sequel, we will often refer to the number a_i as a *weight* (of some labeled point p_i in a configuration). Given $I \subset N := \{1,\dots,n\}$ and $\mathscr{A} = \{a_1,\dots,a_n\}$, we define

$$\mathscr{A}(I) := \{a_i \mid i \in I\} \ \text{ and } \ \mathscr{A}_+(I^c) := \{a_i \mid i \notin I\} \cup \{a_{n+1} = 1\}.$$

2.2. **Weighted stable degenerations.** (see also the descriptions in [FM94] and [Pan95] for the case where all weights are equal to 1) Let X be a nonsingular variety of dimension d. Let $(x_1, x_2, \dots, x_n)$ be an ordered n tuple of labeled points $x_i \in X$ and consider an ordered set $\mathscr{A} \in \mathscr{D}^{FM}_{d,n}$. We say that x_i has weight a_i. A subset $S \subset \{1, \dots, n\}$ is said to be *coincident* at $x \in X$ if

- S contains at least two indices, that is, $|S| \geq 2$;
- the total weight of the points labeled S is larger than one, that is, $\sum_{i \in S} a_i > 1$ and
- for all $i \in S$, $x_i = x$.

A *screen* of a coincident set S at x consists of the data $(t_i)_{i \in S}$ such that (see Example 2.1):

(1) $t_i \in T_x$, the tangent space of X at x (i.e. the tangent direction at which x_i approaches x);
(2) there exist $i, j \in S$ such that $t_i \neq t_j$

Two data sets $(t_i)_{i \in S}$ and $(t'_i)_{i \in S}$ are equivalent if there exist $c \in \mathbb{G}_m$ and $v \in T_x$ such that

$$c \cdot t_i + v = t'_i$$

for all $i \in S$. In other words, if we identify T_x with the affine space $\mathbb{A}^d$, then $(t_i)_{i \in S}$ defines an equivalence class of points in $\mathbb{A}^d$ up to translation and homothety. Now, consider the n-tuple $(x_1, x_2, \dots x_n)$ together with the collection of all coincident sets S. We construct the (weighted) n-pointed $\mathscr{A}$ stable degeneration of X as follows. Let z be a coordinate that occurs multiple times in $(x_1, \dots x_n)$ and generates a coincident set at z. We blow up X at z and attach the projective completion $\mathbb{P}(T_z \oplus 1) \cong \mathbb{P}^d$ along the exceptional divisor $\mathbb{P}(T_z) \cong \mathbb{P}^{d-1}$, which is identified with the infinity section. Note that the complement $\mathbb{P}(T_z \oplus 1) \setminus \mathbb{P}(T_z)$ is isomorphic to the affine space $T_z \cong \mathbb{A}^d$. Let S_z be the maximal coincident subset at z. The screen corresponding to S_z associates points of T_z to the indices that lie in S_z. By condition (2) for screens, we see that some separation of those points occurs inside the new component $\mathbb{P}(T_z \oplus 1)$ and these points are defined up to translation and homothety. We continue this process by blowing up points in the new spaces T_z specified by the subsequent screens until all screens have been used, for all such coordinates z. The resulting variety is equipped with n points s_i lying in the smooth locus. By this description we see that if $S \subset \{1, \dots, n\}$ and $\sum_{i \in S} a_i > 1$ then some separation of the points $(s_i)_{i \in S}$ necessarily occurs. This means that if the sections $(s_i)_{i \in S}$ all coincide for some S, then $\sum_{i \in S} a_i \leq 1$.

Example 2.1. (see Figure 3) We describe the weighted stable degeneration of a nonsingular variety X associated to the coincident sets:

$$\{1,2,3,4,5,6,7\}, \{1,2,3,4,5\}, \{6,7\}.$$

The distinguished component is a blowup of the original variety X at a point. The two end components are isomorphic to $\mathbb{P}^r$, where $r = \dim(X)$; on each of the end components we have two distinct loci of (possibly coincident) smooth markings. They satisfy $a_1 + a_2 \leq 1$, $a_3 + a_4 + a_5 \leq 1$, $a_1 + a_2 + a_3 + a_4 + a_5 > 1$ and $a_6 + a_7 > 1$. □

To any $\mathscr{A}$-stable degeneration we associate a tree, its dual graph, whose vertices are in one to one correspondence with its components and whose vertices are in one to one correspondence with the intersections of its components. In general, we have the following types of components:

(1) A distinguished component which is a blowup of X at a finite set of points.
(2) *End* components are the irreducible components whose vertex has valence equal to 1 and are different from distinguished component. Any end component comes with at least three

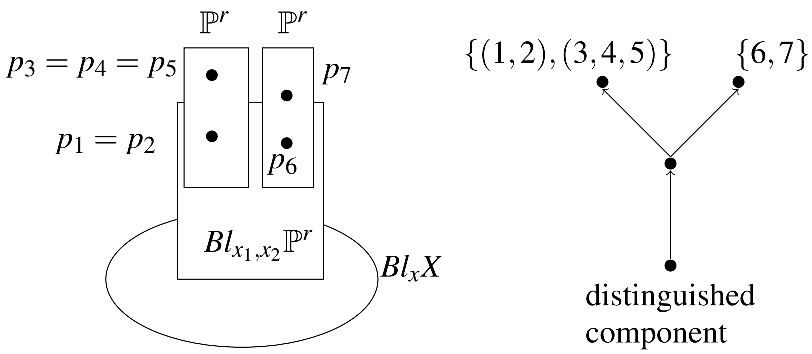


FIGURE 3. *The weighted stable degeneration described in Example 2.1 and its associated dual graph.*

distinct markings: at least two coming from distinct smooth points and exactly one from an intersection with another component, which is a divisor of that end component.

(3) *Ruled* components are the irreducible components whose vertex has valence 2 ; they isomorphic to $\mathbb{P}^d$ blown up at a point. Any ruled component also come with three distinct markings: at least one from a smooth point and exactly two from intersections with other components (which are divisors of the ruled component).

(4) Any other component different to the above ones is isomorphic to $\mathbb{P}^d$ blown up at at least two distinct points. It also comes with at least three distinct markings which can be either from a smooth point or from intersections with other components.

In other words all components, except for the distinguished one, come with at least three distinct markings, so there are no nontrivial automorphisms of an n-pointed $\mathscr{A}$ stable degeneration preserving the distinguished component pointwise. This justifies the term *stability*.

2.3. **Weighted stable degenerations of X with respect to a group action.** If we have an algebraic group G acting on our variety X, then the construction of Section 2.2 can be generalized to configurations of n labelled points in X defined up to the action of G. Indeed, we have

An equivalence class of n weighted points in X. Let $\overline{(x_1,x_2,\dots,x_n)} := \{(gx_1,gx_2\dots gx_n)|\, g\in G\}$ be the G-orbit of the n tuple of labeled points $x_i \in X$. For any $x \in X$, we write $\bar{x}$ in place of the orbit $G\cdot x$ for convenience. We say that $\bar{x_i}$ has weight a_i, and we say that a subset $S \subset N$ is *coincident* at $\bar{x}$ if $|S| \geq 2$, $\sum_{i\in S} a_i > 1$ and for all $i \in S$, $\bar{x_i} = \bar{x}$.

An equivalence class of screens can be associated to a coincident set S at the orbit $\bar{x}$ as follows. Indeed, let $(t_i)_{i\in S}$ be the screen data of a coincident set S at $x \in \bar{x}$. Any element $g \in G$ induces a map from the tangent space T_x to $T_{g\cdot x}$. Then, we define the screen data $(g\cdot t_i)_{i\in S}$ in $T_{g\cdot x}$ to be the image of t_i via $T_x \to T_{g\cdot x}$, and we consider them equivalent to $(t_i)_{i\in S}$ in T_x. This equivalence relation respects the equivalence under translation and homothety $(\sim)$ among the screen data. That is

$$(t_i)_{i\in S} \sim (t'_i)_{i\in S} \iff (g\cdot t_i)_{i\in S} \sim (g\cdot t'_i)_{i\in S}$$

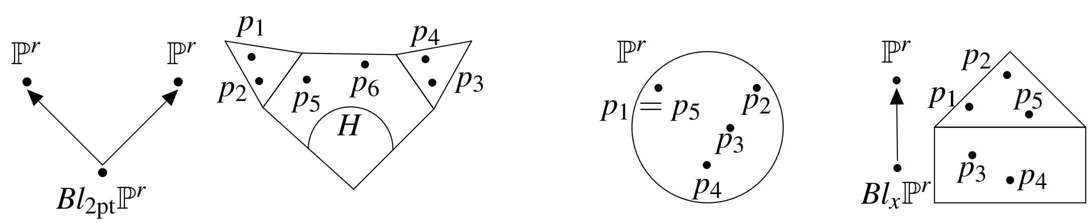
An equivalence class of n-pointed $\mathscr{A}$ stable degenerations of X with respect to G. Let $(x_1,x_2,\dots x_n)$ be a n-tuple which is a representative of the equivalence class $\overline{(x_1,x_2,\dots,x_n)}$ together with the collection of all coincident sets S. We construct the associated weighted stable degeneration of X as in Section 2.2. The action of the group allows us to identify all weighted stable degenerations constructed at the n tuples in the orbit. In particular, two degenerations are equivalent if and only if

(1) The markings on the distinguished component of each of the degenerations are equivalent under the G action.
(2) Their non-distinguished components are equivalent in the following sense: each $\mathbb{P}(T_z \oplus 1)$ arising in the construction of 2.2 from a multiple coordinate z in $(x_1, \ldots x_n)$ is identified with $\mathbb{P}(T_{g\cdot z} \oplus 1)$ via $T_z \to T_{g\cdot z}$. Moreover, $\mathbb{P}(T_z)$ is identified with $\mathbb{P}(T_{g\cdot z})$ and the corresponding markings, which lie in $T_z = \mathbb{P}(T_z \oplus 1) \setminus \mathbb{P}(T_z)$ by construction, are identified with the markings in $T_{g\cdot z} = \mathbb{P}(T_{g\cdot z} \oplus 1) \setminus \mathbb{P}(T_{g\cdot z})$.

Next, we describe how all geometric objects parametrized by $X_{\mathscr{A}}[n]$, $T^{\mathscr{A}}_{d,n}$ and $\overline{P}^{\mathscr{A}}_{d,n}$ are obtained by the above procedure. They will be weighted pointed stable degenerations of X with respect to G for suitably chosen input data $X, \mathscr{A}, G$ and an action $G \times X \to X$.

2.4. **Weighted stable degenerations (revisited).** (see Section 2.2) Let X be a nonsingular variety over an algebraically closed field k and $\mathscr{A} \in \mathscr{D}^{FM}_{d,n}$. The n-pointed $\mathscr{A}$-stable degenerations of X parametrized by the geometric points of $X_{\mathscr{A}}[n]$ are obtained by the procedure in 2.3 for input data $X, \mathscr{A}$ and trivial group G.

2.5. **Weighted stable rooted trees.** Let $\mathscr{A}$ be a set of weights in $\mathscr{D}^T_{d,n}$. The geometric points of $T^{\mathscr{A}}_{d,n}$ are obtained by the procedure in Section 2.3 for input data $\mathbb{A}^d, \mathscr{A}$ and G the group that acts by translation and homothety.



(A) weighted stable rooted tree

(B) weighted stable trees

FIGURE 4. Examples of parametrized objects and their dual graphs

It is convenient to think of points in $\mathbb{A}^d$ as points in $\mathbb{P}^d$ that lie away from a fixed hyperplane $H \subset \mathbb{P}^d$ which is called the *root*. Let $G \cong \mathbb{G}_m \rtimes \mathbb{G}_a^d \subset Aut(\mathbb{P}^d)$ be the group which fixes the root pointwise. Under this interpretation, $X = \mathbb{P}^d \setminus H$ and the equivalence class of n points is determined by the restriction of the action of G to $X = \mathbb{P}^d \setminus H$. Informally, we say that we have n points in $\mathbb{P}^d$ away from a given hyperplane H which is fixed pointwise by G.

The dual graph of the resulting variety is a rooted tree. The distinguished vertex corresponds to the variety that contains the root H.

Lemma 2.2. *For any weight $\mathscr{A}$ in $\mathscr{D}^T_{d,n}$, a stable rooted tree has only trivial automorphisms.*

Proof. The conditions on the weights imply every component in a stable rooted tree have at least two markings. Any subgroup of $\mathbb{G}_m \rtimes \mathbb{G}_a^d$ fixing them is necessarily trivial. $\square$

2.6. **Weighted stable trees.** Let $\mathscr{A}$ be a set of weights in $\mathscr{D}^P_{d,n}$. The geometric points of $\overline{P}^{\mathscr{A}}_{d,n}$ are obtained by the procedure in Section 2.3 for input data $\mathbb{P}^d, \mathscr{A}$, $G \cong SL_{d+1}$ together with its natural action in $\mathbb{P}^d$ and the condition that every representative of a configuration $\overline{(p_1, p_2, \ldots, p_n)}$ satisfy the following conditions:

(1) $p_1,\dots,p_d,p_{d+1}$ are in general position;
(2) none of the $p_i, i\in\{d+2,\dots n\}$ can lie in the linear subspace spanned by $p_1,\dots,p_d$;
(3) we cannot have $p_{d+1}=\dots=p_n$ and
(4) the points $p_i, i=d+2,\dots,n$ cannot all lie on the hyperplane spanned by $\{p_1,\dots\hat{p}_k\dots p_{d+1}\}$ simultaneously.

The geometric meaning of these last conditions will become apparent in Lemma 4.1. The resulting variety has a dual graph that is a tree whose distinguished vertex corresponds to the original $\mathbb{P}^d$, where $p_1,\dots,p_d,p_{d+1}$ lie.

Lemma 2.3. *For any weight $\mathscr{A}$ in $\mathscr{D}^P_{d,n}$, a weighted stable tree has only trivial automorphisms.*

Proof. By the conditions on the weights, the distinguished component has at least $d+2$ markings and the other components have at least two. The result follows on the components associated to the non-distinguished vertices by Lemma 2.2, and on the distinguished component from the fact that a configuration of $d+2$ labelled points in general position in $\mathbb{P}^d$ has trivial automorphisms. $\square$

3. Wonderful Compactifications and GIT

To construct our compactifications we make use of two technical tools: the theory of wonderful compactifications ([Li09]; Section 3.1) and relative GIT ([Hu96]; Section 3.3). Wonderful compactifications will allow us to describe the behavior of iterated blow ups along smooth loci. In section 3.9 we show that wonderful compactifications descend to wonderful compactifications after taking GIT quotients under certain hypotheses. We also need a master space whose role is similar to the role played by the Hilbert scheme in the construction of the moduli of curves; this space is, in our case, a weighted generalization of the Fulton-MacPherson compactification ([Rou14]; Section 3.2).

3.1. **Wonderful Compactifications.** In this section we give a brief account of the theory of wonderful compactifications. For details and full proofs see [Li09].

Definition 3.1. [Li09, Sec 2.1] An **arrangement** of subvarieties of a nonsingular variety Y is a finite set $\mathscr{S}=\{S_i\}$ of nonsingular closed subvarieties $S_i\subset Y$ such that

(1) any two varieties S_i and S_j **intersect cleanly**, i.e. their scheme-theoretic intersection is nonsingular and $T_y=T_{S_i,y}\cap T_{S_j,y}$ for all $y\in S_i\cap S_j$.
(2) $S_i\cap S_j$ is either equal to some S_k or empty.

We say that S_i and S_j intersect **transversally** if, for every point y in Y, $T_y=T_{S_i,y}+T_{S_j,y}$; if $y\notin S_i$, then we adopt the convention that $T_{S_i,y}:=T_y$.
A subset $\mathscr{G}\subset\mathscr{S}$ is called a **building set** of $\mathscr{S}$ if for all $S\in\mathscr{S}$ the minimal elements of $\mathscr{G}$ containing S intersect transversally and their intersection is equal to S (this condition is trivially satisfied if $S\in\mathscr{G}$). These minimal elements are called the $\mathscr{G}$-factors of S.
A finite set $\mathscr{G}$ of nonsingular subvarieties of Y is called a building set if the set of all possible intersections of collections of subvarieties of $\mathscr{G}$ forms an arrangement $\mathscr{S}$ and $\mathscr{G}$ is a building set of $\mathscr{S}$. In this situation, $\mathscr{S}$ is called the **arrangement induced** by $\mathscr{G}$.

Definition 3.2. [Li09, Def 1.1] Let $\mathscr{G}$ be a building set. The wonderful compactification $Y_{\mathscr{G}}$ of $\mathscr{G}$ is the closure of the image of the natural embedding

$$Y^0:=Y\setminus\bigcup_{S_k\in\mathscr{G}}S_k\hookrightarrow\prod_{S_k\in\mathscr{G}}Bl_{S_k}Y$$

Sometimes, we also denote it as $Bl_{\mathscr{G}}Y$ by reasons that will be made apparent in Theorem 3.4 (2).

Definition 3.3. The **dominant transform** of a variety under a blowup is either the strict transform, in case the variety is not contained in the center or the blowup, or the inverse image of the variety via the blowup map if the variety is contained in the center.

The following theorem will be central for our construction:

Theorem 3.4. *(*[Li09, Thm 1.3, Proposition 2.13]*) Let Y be a nonsingular variety and $\mathscr{G} = \{S_1, S_2, \ldots, S_N\}$ be a building set of subvarieties of Y. Then,*

(1) the wonderful compactification $Y_{\mathscr{G}}$ is a nonsingular variety. Moreover, for each $S_i \in \mathscr{G}$ there is a nonsingular divisor $D_{S_i} \subset Y_{\mathscr{G}}$, such that
  (a) The union of the divisors is $Y_{\mathscr{G}} \setminus Y^o$.
  (b) Any set of divisors, with not empty intersection, intersects transversally.
  (c) The divisors D_{S_i} are the iterated dominant transforms of the S_i under $Y_{\mathscr{G}} \to Y$

(2) if we arrange the elements of $\mathscr{G}$ in an ascending dimension order or in such an order that the first i terms $S_1, S_2, \ldots, S_i$ form a building set for all $1 \leq i \leq N$, then

$$Y_{\mathscr{G}} = Bl_{\widetilde{S}_N} \cdots Bl_{\widetilde{S}_2} Bl_{S_1} Y$$

where $Y_{\mathscr{G}}$ is the wonderful compactification of Y with respect to $\mathscr{G}$ and the $\sim$ sign on top of each S_i stands for the iterated dominant transform of the latter in the corresponding blowup.

(3) let I_i be the ideal sheaf of S_i. Then, the wonderful compactification $Y_{\mathscr{G}}$ is equal to the blowup of Y along the ideal sheaf $I_1 I_2 \ldots I_N$,

$$Y_{\mathscr{G}} \cong Bl_{I_N} \ldots Bl_{I_2} Bl_{I_1} Y \cong Bl_{I_1 I_2 \ldots I_N} Y$$

3.2. **Weighted Compactifications of Configuration Spaces.** We recall some results from [Rou14]. Let X be a nonsingular variety over an algebraically closed field k and $\mathscr{A} := \{a_1, a_2, \ldots, a_n\}$ be a set of rational numbers such that $0 < a_i \leq 1, i = 1, 2, \ldots, n$. Also, let

$$\mathscr{K}_{\mathscr{A}} := \{\Delta_I \subset X^n | I \subset N \text{ and } \sum_{i \in I} a_i > 1\}, \text{ where } \Delta_I := \{(x_1, \ldots, x_n) \in X^n | x_{i_1} = \ldots = x_{i_k},\ i_s \in I\}$$

and list its elements in ascending dimension order. The above set is shown in [Rou14] to be a building set which satisfies the hypothesis of Theorem 3.4(2). The work [ibid] is concerned with the study of a natural compactification of the configuration space $X^n \setminus \bigcup_{\Delta_I \in \mathscr{K}_{\mathscr{A}}} \Delta_I$, i.e. the parameter space of n labeled points in X carrying weights a_i subject to the following condition:

- for any set of labels $I \subset N$ of coincident points we have $\sum_{i \in I} a_i \leq 1$.

Definition 3.5. The weighted compactification $X_{\mathscr{A}}[n]$ of $X^n \setminus \bigcup_{\Delta_I \in \mathscr{K}_{\mathscr{A}}} \Delta_I$ is the wonderful compactification of $\mathscr{K}_{\mathscr{A}}$.

We have the following result ([Rou14, Theorems 2 and 3]):

Theorem 3.6. *For any set of admissible weights $\mathscr{A}$,*

(1) $X_{\mathscr{A}}[n]$ is a nonsingular variety. The boundary $X_{\mathscr{A}}[n] \setminus (X^n \setminus \bigcup_{\Delta_I \in \mathscr{K}_{\mathscr{A}}} \Delta_I)$ is the union of $|\mathscr{K}_{\mathscr{A}}|$ divisors D_I, where $I \subset N, |I| \geq 2$ and $\sum_{i \in I} a_i > 1$.

(2) *Any set of boundary divisors intersects transversally. An intersection of divisors $D_{I_1} \cap D_{I_2} \cap \dots D_{I_k}$ is nonempty precisely when the sets are nested in the sense that any pair $\{I_i, I_j\}$ either has empty intersection or one set is contained in the other.*
(3) *$X_{\mathscr{A}}[n]$ is the iterated blowup of X^n along the dominant transforms of the elements of $\mathscr{K}_{\mathscr{A}}$. Moreover, each divisor D_I is the iterated dominant transform of Δ_I in $X_{\mathscr{A}}[n]$.*
(4) *There exists a 'universal' family $\phi_{\mathscr{A}} : X_{\mathscr{A}}[n]^+ \to X_{\mathscr{A}}[n]$ equipped with n sections $\sigma_i : X_{\mathscr{A}}[n] \to X_{\mathscr{A}}[n]^+, i = 1, \dots n$ whose images lie in the relative smooth locus of $\phi_{\mathscr{A}}$. It is a flat morphism between nonsingular varieties, whose fibers are the n pointed $\mathscr{A}$ stable degenerations of X described in Section 2.2.*

Now, with notation as above, consider $X = \mathbb{P}^d$ with the natural SL_{d+1} action. We will use the following lemma in our subsequent construction of $\overline{P}^{\mathscr{A}}_{d,n}$:

Lemma 3.7. *(1) There is a lift of the SL_{d+1} action on $\mathbb{P}^d$ to $\mathbb{P}^d_{\mathscr{A}}[n]$ and $\mathbb{P}^d_{\mathscr{A}}[n]^+$. Moreover, the morphisms $\pi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n] \to (\mathbb{P}^d)^n$ (cf Theorem 3.6(3)) and $\phi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n]^+ \to \mathbb{P}^d_{\mathscr{A}}[n]$ (cf Theorem 3.6(4)), as well as the sections $\sigma_i : \mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}}[n]^+$, $i = 1, \dots, n$ are SL_{d+1}-equivariant.*
(2) $\phi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n]^+ \to \mathbb{P}^d_{\mathscr{A}}[n]$ is a projective morphism.

Proof. (1) $\mathbb{P}^d_{\mathscr{A}}[n]$ is obtained by successively blowing up strict transforms of diagonals, which are SL_{d+1}-invariant. Therefore the SL_{d+1} action on $\mathbb{P}^d$ lifts to $\mathbb{P}^d_{\mathscr{A}}[n]$ so that $\pi_{\mathscr{A}}$ becomes SL_{d+1} equivariant. Also, by the construction in [Rou14, Section 3], $\mathbb{P}^d_{\mathscr{A}}[n]^+$ is the iterated blowup of $\mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d$ along strict transforms of diagonals in $(\mathbb{P}^d)^n \times \mathbb{P}^d$, hence a blowup along SL_{d+1}-equivariant centers. Therefore the morphism $\mathbb{P}^d_{\mathscr{A}}[n]^+ \to \mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d$ is SL_{d+1}-equivariant. Since the projection $\mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d \to \mathbb{P}^d_{\mathscr{A}}[n]$ is also SL_{d+1}-equivariant, we deduce that $\phi_{\mathscr{A}}$ is equivariant. Finally, the sections σ_i are obtained by blowing up centers inside $\mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d$ that are isomorphic to the divisor D_N corresponding to the small diagonal Δ_N [ibid.]; indeed those centers are the graphs of the morphisms $D_N \hookrightarrow \mathbb{P}^d_{\mathscr{A}}[n] \xrightarrow{p_i} \mathbb{P}^d$, where p_i is the projection to the i-th factor. Since Δ_N is invariant under the SL_{d+1} action, so are the above centers, hence so are σ_i.
(2) As noted above, $\phi_{\mathscr{A}}$ is the composition of a sequence of blowups $\mathbb{P}^d_{\mathscr{A}}[n]^+ \to \mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d$ with the projection $\mathbb{P}^d_{\mathscr{A}}[n] \times \mathbb{P}^d \to \mathbb{P}^d_{\mathscr{A}}[n]$, i.e. a composition of projective morphisms, hence projective. □

3.3. **Relative GIT and blowing up.** In order to construct our compactification $\overline{P}^{\mathscr{A}}_{d,n}$, we will descend the blowup construction of the weighted Fulton MacPherson space $\mathbb{P}^d_{\mathscr{A}}[n]$ to appropriately defined GIT quotients. To this end, we will make use of the machinery of relative GIT developed by Hu [Hu96]:

Lemma 3.8. [Hu96, Thm 3.11, Thm 3.13] *Let $\pi : Y \to Z$ be a G-equivariant projective morphism between two (possibly singular) quasi-projective varieties. Given any linearized ample line bundle L on Z such that the GIT stable loci of Z is equal to its strictly semistable point that is*

$$Z^{ss}(L) = Z^s(L)$$

and choose a relatively ample linearized line bundle M on Y . Then

(1) *there exists n_0 such that when $n \geq n_0$, we have*

$$Y^{ss}(\pi^* L^n \otimes M) = Y^s(\pi^* L^n \otimes M) = \pi^{-1}(Z^s(L))$$

(2) *Given $\tilde{L} := \pi^* L^n \otimes M$, then there is a projective morphism*

$$\hat{\pi} : Y /\!/_{\tilde{L}} G \to Z /\!/_L G$$

(3) *for any $z \in Z^s(L)$ with stabilizer G_z, we have*

$$\hat{\pi}^{-1}([G \cdot z]) \cong \pi^{-1}(z)/G_z$$

(4) *if π is a fibration and G acts freely on $Z^s(L)$, then $\hat{\pi}$ is also a fibration with the same fibers as π.*

The following result shows that blowing up along building sets is compatible with taking GIT quotients under certain hypotheses.

Lemma 3.9. *Let Y be a nonsingular projective variety such that*

(1) *Y admits an action by a reductive algebraic group G, and*
(2) *there is a G-linearized ample line bundle L on Y such that $Y^s(L) = Y^{ss}(L)$ and G acts with trivial stabilizers on $Y^s(L)$.*

Let $\mathscr{G}$ be a building set that consists of G-invariant subvarieties of Y listed in ascending dimension order. Then

(1) *Let $\mathscr{G}^s := \{S_k \cap Y^s \mid S_k \in \mathscr{G}$ and $S_k \cap Y^s \neq \emptyset\}$ and let $\hat{\mathscr{G}}$ be the set which consists of the images of its elements under $Y^s \to Y^s/\!/G$. Then $\mathscr{G}^s$ and $\hat{\mathscr{G}}$ are building sets.*
(2) *Let $p : Bl_{\mathscr{G}^s} Y^s \to Y^s$ be the iterated blowup of Y^s along the dominant transforms of the elements of $\mathscr{G}^s$ in ascending dimension order. Also, let $\widetilde{L_d} := p^*(L^d) \otimes \mathscr{O}(-E)$ on $Bl_{\mathscr{G}^s} Y^s$, where E is the total boundary divisor i.e. the union of the exceptional divisors of this blow up. Then, for sufficiently large d, $\widetilde{L_d}$ admits a linearization such that $(Bl_{\mathscr{G}^s} Y^s)^{ss}(\widetilde{L_d}) = (Bl_{\mathscr{G}^s} Y^s)^s(\widetilde{L_d}) = p^{-1}(Y^s(L))$ and p descends to a morphism*

$$\hat{p} : (Bl_{\mathscr{G}^s} Y^s)^s /\!/ G \to Y^s /\!/ G$$

which is the iterated blowup of $Y^s/\!/G$ along the dominant transforms of the elements of $\hat{\mathscr{G}}$ in ascending dimension order.

Proof. We modify the arguments in [Hu03, Section 7]. For (1), note that by Definition 3.1 all defining properties of a building set and its induced arrangement are Zariski local, so the restriction $\mathscr{G}^s$ is readily seen to be a building set. We now show that $\hat{\mathscr{G}}$ is a building set. Let $\{T_i\}$, $\{T_i^s\}$ and $\{\hat{T}_i\}$ be the sets that consist of all possible intersections of the varieties in $\mathscr{G}$, $\mathscr{G}^s$ and $\hat{\mathscr{G}}$ respectively.
Claim: The set $\{\hat{T}_i\}$ is an arrangement of subvarieties of $\hat{\mathscr{G}}$.

Proof: We only need to show $\{\hat{T}_i\}$ intersect cleanly. It suffices to check this locally. Let x be a point in Y^s; then, since G acts with trivial stabilizers on Y^s, by Luna's étale slice Theorem, there exists a locally closed smooth subvariety W_x of Y^s containing x and an open G-invariant subvariety $U_x \subset Y^s$ containing x such that the morphism

$$G \times W_x \to U_x$$

is strongly étale and $G \cdot W_x = U_x$. Therefore, by pulling back via $U_x \cap T_i \to U_x$, we obtain étale morphisms

$$G \times (W_x \cap T_i) \to U_x \cap T_i \tag{1}$$

By the hypothesis, any T_i and T_j intersect cleanly so their restrictions to U_x must also intersect cleanly. Since the morphism 1 induces an isomorphism on tangent spaces the intersection $(W_x \cap T_i)$

with $(W_x \cap T_j)$ must also be clean. Now, the morphism $G \times W_x \to U_x$ induces an étale surjective morphism $W_x \to U_x//G$, hence also étale surjective morphisms

$$(W_x \cap T_i) \to (U_x \cap T_i)//G = \hat{T}_i \cap (U_x//G) \text{ and } (W_x \cap T_j) \to (U_x \cap T_j)//G = \hat{T}_j \cap (U_x//G)$$

which take $((W_x \cap T_i) \cap (W_x \cap T_j))$ to $(U_x \cap T_i \cap T_j)//G = \hat{T}_i \cap \hat{T}_j \cap (U_x//G)$. Consequently, $\hat{T}_i \cap \hat{T}_j$ is a clean intersection. **End.**

Now let $\hat{T}$ be an arbitrary element in the induced arrangement of $\hat{\mathscr{G}}$. It remains to show that the minimal elements of $\hat{\mathscr{G}}$ that contain $\hat{T}$ intersect transversally. Assume that these minimal elements are $\hat{S}_1, \hat{S}_2, \dots \hat{S}_m$. Then we see that S_i are the minimal elements of $\mathscr{G}$ that contain T. Indeed, if $\hat{S}_i \supset \hat{S}' \supset \hat{T}$ for some $S' \in \mathscr{G}$, then $S_i \cap Y^s \supset S' \cap Y^s$, therefore $S_i = \overline{S_i \cap Y^s} \supset \overline{S' \cap Y^s} = S'$. Now, since, by definition, S_i intersect transversally, we may repeat the argument of the previous paragraph (with S_i in place of the T_i above) to actually deduce that $\hat{S}_i$ intersect tranversally as well. Hence we have verified part (1) of the lemma.

Next, we show part (2); let $\mathscr{G}^s_k$ be the subset which consists of the first k elements of $\mathscr{G}^s$. Also, let T_{k+1} be the unique element of $\mathscr{G}^s_{k+1}$ such that $\mathscr{G}^s_{k+1} = \{T_{k+1}\} \cup \mathscr{G}^s_k$. We will show by induction that, for any k such that $0 \le k \le |\mathscr{G}^s|$, the following statement is true:

i Let $p_k : Bl_{\mathscr{G}^s_k} Y^s \to Y^s$ be the natural blowup morphism and set $L_{k,d} := p_k^*(L^d) \otimes \mathscr{O}(-\sum_{i=1}^{k} E_i)$, where E_i are the exceptional divisors of $Bl_{\mathscr{G}^s_k} Y^s$. Then, for sufficiently large d, $L_{k,d}$ admits a linearization such that $(Bl_{\mathscr{G}^s_k} Y^s)^s(L_{k,d}) = (Bl_{\mathscr{G}^s_k} Y^s)^{ss}(L_{k,d}) = p_k^{-1}(Y^s(L))$ and $Bl_{\mathscr{G}^s_k} Y^s //_{L_{k,d}} G = (Bl_{\mathscr{G}^s_k} Y^s)^s//G$ is the iterated blowup of $Y^s//G$ along the iterated dominant transforms of $\hat{T}_1, \dots \hat{T}_k$.

ii There is a commutative diagram

$$\begin{tikzcd}
(Bl_{\mathscr{G}^s_k} Y^s)^s \arrow[r, "q_k"] \arrow[d, "p_k"'] & (Bl_{\mathscr{G}^s_k} Y^s)^s//G \arrow[d, "\hat{p}_k"] \\
Y^s \arrow[r, "q_0"'] & Y^s//G
\end{tikzcd}$$

such that for every l with $|\mathscr{G}| \ge l > k$ the iterated strict transform $(\hat{T}_l)^{(k)}$ of $\hat{T}_l$ in $(Bl_{\mathscr{G}^s_k} Y^s)^s//G$ is the image of the iterated strict transform $(T^s_l)^{(k)}$ of T^s_l in $(Bl_{\mathscr{G}^s_k} Y^s)^s$ via q_k.

For $k = 0$ there is nothing to prove. Assume the above statement is true for some $k \ge 0$. Then $p_k^*(L^d) \otimes \mathscr{O}(-\sum_{i=1}^{k} E_i)$ is well known to be relatively ample for large d. Therefore, by lemma 3.8(1)- after twisting by a large enough power of $p_k^*(L)$ if necessary- we deduce that $(Bl_{\mathscr{G}^s_k} Y^s)^s(L_{k,d}) = (Bl_{\mathscr{G}^s_k} Y^s)^{ss}(L_{k,d}) = p_k^{-1}(Y^s(L))$. Now, by [Kir85, Lemma 3.11], it holds that

$$(Bl_{\mathscr{G}^s_{k+1}} Y^s)^s//G = \left(Bl_{(T^s_{k+1})^{(k)}}(Bl_{\mathscr{G}^s_k} Y^s)\right)^s //G$$

is the blowup of $(Bl_{\mathscr{G}^s_k} Y^s)^s//G$ along the image of $(T^s_{k+1})^{(k)}$ via q_k, which, by the inductive hypothesis, is equal to the iterated strict transform of $\hat{T}_{k+1}$ along $\hat{p}_k$. Therefore, by part (i) of the inductive hypothesis for k, we establish part (i) for $k+1$. We also obtain a commutative diagram

$$\begin{tikzcd}
(Bl_{\mathscr{G}^s_{k+1}} Y^s)^s \arrow[r, "q_{k+1}"] \arrow[d, "p_{k+1,k}"'] & (Bl_{\mathscr{G}^s_{k+1}} Y^s)^s /\!/ G \arrow[d, "\hat{p}_{k+1,k}"] \\
(Bl_{\mathscr{G}^s_k} Y^s)^s \arrow[r, "q_k"'] & (Bl_{\mathscr{G}^s_k} Y^s)^s /\!/ G
\end{tikzcd}$$

It remains to show that for $l > k+1$ the strict transform of $(\hat{T}_l)^{(k)}$ along $\hat{p}_{k+1,k}$ is equal to the image of $(T_l^s)^{(k+1)}$ via q_{k+1}. But the strict transform of $(\hat{T}_l)^{(k)}$ along $\hat{p}_{k+1,k}$ is, by definition, equal to the blowup of $(\hat{T}_l)^{(k)}$ along its intersection with the center $(\hat{T}_l)^{(k)} \cap (\hat{T}_{k+1})^{(k)}$, which is nonsingular; indeed, by [Li09, Definition 2.8] and the remark following it, for any k, the set $\{(\hat{T}_l)^{(k)} | \hat{T}_l \in \hat{\mathscr{G}}\}$- given an ascending dimension order - is a building set and the element $(\hat{T}_{k+1})^{(k)}$ is minimal. Therefore, by [Li09, Lemma 2.6 (i)] $(\hat{T}_l)^{(k)} \cap (\hat{T}_{k+1})^{(k)}$ is nonsingular and also equivariant by the hypothesis. By part (ii) of the inductive hypothesis, the blowup of $(\hat{T}_l)^{(k)}$ along $(\hat{T}_l)^{(k)} \cap (\hat{T}_{k+1})^{(k)}$ is equal to the blowup of $q_k((T_l^s)^{(k)})$ along $q_k((T_l^s)^{(k)} \cap (T_{k+1}^s)^{(k)})$, which in turn, by [Kir85, Lemma 3.11], is equal to $(Bl_{(T_l^s)^{(k)} \cap (T_{k+1}^s)^{(k)}} (T_l^s)^{(k)})^s /\!/ G = ((T_l^s)^{(k+1)})^s /\!/ G$, so we are done. $\square$

The following result shows that, under the hypotheses of Lemma 3.9, a wonderful compactification descends to a wonderful compactification in the GIT.

Corollary 3.10. *We suppose that the hypotheses of Lemma 3.9 are satisfied. Let $\mathscr{G}$ be a building set of subvarieties of Y, let $\mathscr{G}^s$ be its restriction to the stable locus, and let $\widetilde{L_d}$, L be the line bundles of Lemma 3.9 (2). Then, the following hold:*

(1) The variety $Bl_{\mathscr{G}^s} Y^s /\!/_{\widetilde{L^d}} G$ is the wonderful compactification of the arrangement $\hat{\mathscr{G}}$ of subvarieties of $Y /\!/_L G$.

(2) Let $\bar{p} : Bl_{\mathscr{G}} Y \to Y$ be the iterated blowup of Y along the dominant transforms of the elements of $\mathscr{G}$ in ascending dimension order. Also, let $\bar{L}_d := \bar{p}^*(L^d) \otimes \mathscr{O}(-\overline{E})$ on $Bl_{\mathscr{G}^s} Y^s$, where $\overline{E}$ is the total boundary divisor, i.e. the union of the exceptional divisors of this blow up. Then $Bl_{\mathscr{G}^s} Y^s /\!/_{\widetilde{L^d}} G \cong Bl_{\mathscr{G}} Y /\!/_{\bar{L}_d} G$.

Proof. (1) follows immediately by Theorem 3.4(2) and Lemma 3.9 (2). For (2) note that blowing up is compatible with restricting to an open set. Therefore, we have the commuting diagram

$$\begin{tikzcd}
Bl_{\mathscr{G}^s} Y^s \arrow[r] \arrow[d] & Bl_{\mathscr{G}} Y \arrow[d, "\bar{p}"] \\
Y^s \arrow[r] & Y
\end{tikzcd}$$

where $(\bar{p})^{-1}(Y^s) = Bl_{\mathscr{G}^s} Y^s$. By Lemma 3.8, we have $(\bar{p})^{-1}(Y^s) = (Bl_{\mathscr{G}} Y)^s$. Consequently, $Bl_{\mathscr{G}^s} Y^s = (Bl_{\mathscr{G}} Y)^s$ and

$$Bl_{\mathscr{G}^s} Y^s /\!/_{\widetilde{L_d}} G \cong Bl_{\mathscr{G}} Y /\!/_{\bar{L}_d} G$$

$\square$

4. THE COMPACTIFICATIONS $\overline{P}_{d,n}^{\mathscr{A}}$ AND $T_{d,n}^{\mathscr{A}}$ (THEOREMS 1.1 AND 1.2).

In this section, we construct our configuration spaces, their respective universal families and we describe their boundary.

4.1. **Points in projective space.** Our first step is to give a model of $\overline{P}^{\mathscr{A}}_{d,n}$ isomorphic to a product of projective spaces.

Lemma 4.1. *There exists an ample SL_{d+1}-linearized line bundle $L_{d,n}$ such that the GIT quotient $(\mathbb{P}^d)^n/\!/_{L_{d,n}}SL_{d+1}$ is isomorphic to $\left(\mathbb{P}^{n-d-2}\right)^d$. Futhermore,*

1. *there is no strictly semistable locus in $(\mathbb{P}^d)^n$ with respect to $L_{d,n}$ and*
2. *we can choose coordinates so that the point*

$$\prod_{k=0}^{d-1}[b^k_{d+2}:\ldots:b^k_n]\in\left(\mathbb{P}^{n-d-2}\right)^d$$

parametrizes the equivalence class of n points induced by:

$$p_1=[1:\ldots:0],\quad p_2=[0:1:\ldots 0],\quad p_{d+1}=[0:\ldots:1]\text{ and }p_i=[b^0_i:\ldots:b^{d-1}_i:1]$$

with $d+2\le i\le n$.

Proof. We define $L_{d,n}$ to be the fractional line bundle $\mathscr{O}(w_1,w_2,\ldots,w_n)$ on $(\mathbb{P}^d)^n$ such that

$$w_1=\ldots=w_d=1-\hat{\varepsilon},w_{d+1}=1-(n-(d+1))\varepsilon+d\hat{\varepsilon}\text{ and }w_{d+2}=\ldots=w_n=\varepsilon$$

with $1\gg\varepsilon\gg\hat{\varepsilon}>0$ with its canonical SL_{d+1} linearization ([Dol03, Chapter 11]) and we take the GIT quotient of n points in $\mathbb{P}^d$ with respect to that linearization. A configuration of points is GIT semistable (resp. stable) if and only if $\sum_{p_i\in W}w_i\le(\dim(W)+1)$ (resp. $\sum_{p_i\in W}w_i<(\dim(W)+1)$) for any proper subspace $W\subset\mathbb{P}^d$ (see [Dol03, Thm 11.2]). Observe that the total GIT weights of any subset of points is never equal to an integer, so the above inequality is always strict. Therefore, there is no strictly semistable locus. The above inequality is equivalent to the following conditions:

1. $p_1,\ldots,p_d,p_{d+1}$ must be in general position
2. none of the $p_i,i\in\{d+2,\ldots n\}$ can lie in the linear subspace spanned by $p_1,\ldots,p_d$
3. we cannot have $p_{d+1}=\ldots=p_n$ and
4. the points $p_i,i=d+2,\ldots,n$ cannot all lie on the hyperplane spanned by $\{p_1,\ldots\hat{p}_k\ldots p_{d+1}\}$ simultaneously.

Then, we can fix the configuration of points $\{p_1,\ldots,p_n\}$ to be as in the statement. Consequently, the automorphism group of the resulting configuration is isomorphic to $\mathbb{G}^d_m$. By our conditions on the weights, the parameter space of each point p_i with $(d+2)\le i\le n$ is contained in $\mathbb{A}^d$, because p_i cannot lie in the hyperplane $(x_{n+1}=0)$ determined by the points $\{p_1,\ldots,p_d\}$. The only other restriction on the points $p_i,i=d+2,\ldots,n$ is that they cannot all lie on the hyperplane spanned by $\{p_1,\ldots\hat{p}_k\ldots p_{d+1}\}$ at the same time. This means that configurations with points with

$$b^k_{d+2}=\ldots=b^k_n=0$$

are forbidden as well. We denote the loci parametrizing these last configurations of points as $\mathbb{A}^{(d-1)(n-(d+1))}_k$ with $0\le k\le d-1$. Then, we obtain

$$\left(\prod_{i=d+2}^{n}\mathbb{A}^d\setminus\bigcup_{k=0}^{d-1}\mathbb{A}^{(d-1)(n-(d+1))}_k\right)/\!/\mathbb{G}^d_m=\left((\mathbb{A}^{n-(d+1)})^d\setminus\bigcup_{k=0}^{d-1}\mathbb{A}^{(d-1)(n-(d+1))}_k\right)/\!/\mathbb{G}^d_m$$

$$=\left(\mathbb{A}^{n-(d+1)}\setminus 0\right)^d/\!/\mathbb{G}^d_m=\left(\mathbb{P}^{n-(d+2)}\right)^d.$$

□

We now define the open locus in $(\mathbb{P}^d)^n//_{L_{d,n}}SL_{d+1}$ that we want to compactify. Let $\mathscr{A} \in \mathscr{D}^P_{d,n}$ be an admissible weight set for the weighted compactification $\mathbb{P}^d_{\mathscr{A}}[n]$. Also, let $\mathscr{K}_{\mathscr{A}} = \{\Delta_I \subset (\mathbb{P}^d)^n | I \subset N \text{ and } \sum_{i\in I} a_i > 1\}$ be the set associated to the construction of $\mathbb{P}^d_{\mathscr{A}}[n]$ (see Section 3.2) and consider the line bundle $L_{d,n}$ of Lemma 4.1. Following the notation of Lemma 3.9, let $\hat{\Delta}_I$ be the descent of $\Delta_I \in \mathscr{K}_{\mathscr{A}}$, such that $\Delta_I \cap \big((\mathbb{P}^d)^n\big)^s \neq \emptyset$, to $(\mathbb{P}^d)^n//_{L_{d,n}}SL_{d+1}$ and let $\hat{\mathscr{K}}_{\mathscr{A}}$ be the set of all such $\hat{\Delta}_I$.

Definition 4.2. The *weighted moduli space* of n labeled points in $(\mathbb{P})^d$ with respect to $\mathscr{A}$ and $L_{d,n}$ is

$$P^{\mathscr{A}}_{d,n} := \Big((\mathbb{P}^d)^n//_{L_{d,n}}SL_{d+1}\Big) \setminus \bigcup_{\hat{\Delta}_I \in \hat{\mathscr{K}}_{\mathscr{A}}} \hat{\Delta}_I$$

Recall that, by Theorem 3.6, there is a sequence of blowups $\pi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n] \longrightarrow (\mathbb{P}^d)^n$ along the dominant transforms of the elements of a building set $\mathscr{K}_{\mathscr{A}}$. Let E be the total boundary divisor, i.e the union of the exceptional divisors of this blowup. Now let $L_{d,n}$ be the line bundle of lemma 4.1; then, the strictly semistable locus of $(\mathbb{P}^d)^n$ with respect to $L_{d,n}$ is empty, by [ibid.], so the hypotheses of Lemma 3.9(2) are satisfied. Hence, there is a canonically linearized line bundle $\tilde{L}_{\mathscr{A}} := \pi^*_{\mathscr{A}}(L^{\otimes e}_{d,n}) \otimes \mathscr{O}(-E)$, such that, for e sufficiently large, $(\mathbb{P}^d_{\mathscr{A}}[n])^{ss}(\tilde{L}_{\mathscr{A}}) = (\mathbb{P}^d_{\mathscr{A}}[n])^{s}(\tilde{L}_{\mathscr{A}}) = \pi^{-1}_{\mathscr{A}}[\big((\mathbb{P}^d)^n\big)^s]$, where the stable locus $\big((\mathbb{P}^d)^n\big)^s$ is induced by $L_{d,n}$.

Definition 4.3. Let $\mathscr{A} \in \mathscr{D}^P_{d,n}$ and let $\tilde{L}_{\mathscr{A}}$ be the line bundle defined above. The *weighted compactification* of $P^{\mathscr{A}}_{d,n}$ is

$$\overline{P}^{\mathscr{A}}_{d,n} := (\mathbb{P}^d_{\mathscr{A}}[n])//_{\tilde{L}_{\mathscr{A}}}SL_{d+1}$$

Remark 4.4. We will see below (Lemma 4.8 and Corollary 4.9) that $\hat{\mathscr{K}}_{\mathscr{A}}$ is a building set and $\overline{P}^{\mathscr{A}}_{d,n}$ is the wonderful compactification of $\hat{\mathscr{K}}_{\mathscr{A}}$.

Remark 4.5. When $\mathscr{A}$ is such that $a_1 = \ldots = a_d = 1-\hat{\varepsilon}$, $a_{d+1} = 1-(n-(d+1))\varepsilon + d\hat{\varepsilon}$ and $a_{d+2} = \ldots = a_n = \varepsilon$, then $\mathscr{A}$ coincides with the set of GIT weights corresponding to $L_{d,n}$. Then $\overline{P}^{\mathscr{A}}_{d,n}$ is the GIT quotient of Lemma 4.1. Indeed,

$$\overline{P}^{\mathscr{A}}_{d,n} = (\mathbb{P}^d_{\mathscr{A}}[n])^s(\tilde{L}_{\mathscr{A}})//SL_{d+1} = \pi^{-1}_{\mathscr{A}}[\Big((\mathbb{P}^d)^n\Big)^s]//SL_{d+1} = \Big((\mathbb{P}^d)^n\Big)^s//SL_{d+1}.$$

The last equality follows because $\big((\mathbb{P}^d)^n\big)^s$ is contained in the locus where the points with labels $\{1,\ldots,d\}$ are distinct, that is, in the open locus $(\mathbb{P}^d)^n \setminus \bigcup_{\Delta_I \in \mathscr{K}_{\mathscr{A}}} \Delta_I$, hence $\pi^{-1}_{\mathscr{A}}[\big((\mathbb{P}^d)^n\big)^s]$ is isomorphic to $\big((\mathbb{P}^d)^n\big)^s$. The above equalities still hold if we increase $a_1,\ldots,a_d$ to any number between $1-\hat{\varepsilon}$ and 1, because $\mathscr{K}_{\mathscr{A}}$ remains invariant. $\overline{P}^{\mathscr{A}}_{d,n}$ changes only if we increase the weights $a_{d+1},\ldots,a_n$.

We now describe the loci parametrizing coincident points in the GIT of Lemma 4.1.

Lemma 4.6. *The loci of coincident points in $(\mathbb{P}^{n-d-2})^d$ labeled by $I \subsetneq \{d+1,\ldots n\}$ such that $2 \leq |I| \leq (n-d-1)$ are given by*

$$H_I = \begin{cases} \bigcap_{i,j\in I}(b^0_i - b^0_j = \ldots = b^{d-1}_i - b^{d-1}_j = 0) & |\ \mathit{if}\ d+1 \notin I \\ \bigcap_{i\in I\setminus d+1}(b^0_i = \ldots = b^{d-1}_i = 0) & |\ \mathit{if}\ d+1 \in I \end{cases}$$

They parametrize configurations with either the multiple point $\{p_i \mid i \in I\}$ with $i \in I$ or a degeneration of it. Furthermore, $H_I \cong (\mathbb{P}^{(n-|I|)-d-1})^d$.

Next, we define the set of subvarieties of $(\mathbb{P}^{n-d-2})^d$ parametrizing coincident points. Let $\mathscr{A} \in \mathscr{D}^P_{d,n}$, set $D := \{d+1, \dots n\}$ and consider the subset $\mathscr{A}(D)$ of $\mathscr{A}$ that consists of the elements of $\mathscr{A}$ whose labels lie in D (see Section 2.1). Let H_I be the subvarieties of $(\mathbb{P}^{n-d-2})^d$ defined in 4.6.

Definition 4.7. The set of subvarieties of $(\mathbb{P}^{n-d-2})^d$ parametrizing coincident points is

$$\mathscr{G}_{\mathscr{A}(D)} := \{H_I \subset (\mathbb{P}^{n-d-2})^d \mid \sum_{i \in I} a_i > 1\}$$

Now let $\mathscr{K}_{\mathscr{A}} = \{\Delta_I \subset (\mathbb{P}^d)^n | I \subset N \text{ and } \sum_{i \in I} a_i > 1\}$ be the set associated to the construction of $\mathbb{P}^d_{\mathscr{A}}[n]$ (see Section 3.2) and let $L_{d,n}$ be the line bundle used in Lemma 4.1. The set $\mathscr{K}_{\mathscr{A}}$ is given an ascending dimension partial order. In addition, consider its descent $\hat{\mathscr{K}}_{\mathscr{A}}$ (see Lemma 3.9) to $(\mathbb{P}^d)^n /\!/_{L_{d,n}} SL_{d+1}$, which is given also an ascending dimension partial order compatible with the order of $\mathscr{K}_{\mathscr{A}}$.

Lemma 4.8. *Let $\mathscr{K}_{\mathscr{A}}$ and $\hat{\mathscr{K}}_{\mathscr{A}}$ as above. Then*

- *$\hat{\mathscr{K}}_{\mathscr{A}}$ is a building set and*
- *$\hat{\mathscr{K}}_{\mathscr{A}} = \mathscr{G}_{\mathscr{A}(D)}$.*

Proof. By Lemma 3.9(1), $\hat{\mathscr{K}}_{\mathscr{A}}$ is a building set. Then, we only need to prove that the equality holds. The equivalence class $[SL_{d+1} \cdot (p_1, \dots p_n)]$ is in $\mathscr{G}_{\mathscr{A}(D)}$ if and only if we can select a representative $(p_1, \dots p_n)$ of the orbit such that it is contained in the stable locus and it has at least a subset of overlapping points $\{p_{i_1} = \dots = p_{i_s} \mid i_s \in I\}$ such that $\sum_{i_k \in I} a_i > 1$. These two conditions are the ones defining $\mathscr{K}^s_{\mathscr{A}} = \mathscr{K}_{\mathscr{A}} \cap ((\mathbb{P}^d)^n)^s$. ☐

Corollary 4.9. $(\mathbb{P}^d_{\mathscr{A}}[n]) /\!/_{\tilde{L}_{\mathscr{A}}} SL_{d+1}$ *is the wonderful compactification of the arrangement $\hat{\mathscr{K}}_{\mathscr{A}}$.*

Proof. By Section 3.2, the weighted Fulton-Macpherson spaces $\mathbb{P}^d_{\mathscr{A}}[n]$ are wonderful compactifications with respect to the building set $\mathscr{K}_{\mathscr{A}}$; and the blow up $\pi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n] \longrightarrow (\mathbb{P}^d)^n$ is SL_{d+1} equivariant by Lemma 3.7. Moreover, the conditions required in Lemma 3.9 and Corollary 3.10 are satisfied by $(\mathbb{P}^d)^n$ if we take the line bundle $L_{d,n}$ used in Lemma 4.1. Indeed, there is no strictly semistable locus induced by $L_{d,n}$ (Lemma 4.1); and from the smoothness of the quotient in Lemma 4.1 we conclude the stabilizers are trivial. Therefore, by Lemma 3.8 and Corollary 3.10, $(\mathbb{P}^d_{\mathscr{A}}[n])^s \cong (\mathbb{P}^d_{\mathscr{A}}[n])^{ss}$ and the variety $(\mathbb{P}^d_{\mathscr{A}}[n]) /\!/_{\tilde{L}_{\mathscr{A}}} SL_{d+1}$ is the wonderful compactification of the building set $\hat{\mathscr{K}}_{\mathscr{A}}$. ☐

Proof of Theorem 1.1:
Part 1: Follows immediately from Corollary 4.9 and Theorem 3.4 (1).
Part 2: First we define $\hat{\phi}_{\mathscr{A}}$ and its sections $\hat{\sigma}_i$ by descending the analogous morphisms that appear in the weighted Fulton MacPherson construction. Let $\phi_{\mathscr{A}} : \mathbb{P}^d_{\mathscr{A}}[n]^+ \to \mathbb{P}^d_{\mathscr{A}}[n]$ be the universal family of the weighted Fulton MacPherson compactification $\mathbb{P}^d_{\mathscr{A}}[n]$ (Theorem 3.6(4)). We have already seen that the action of SL_{d+1} on $\mathbb{P}^d$ lifts to $\mathbb{P}^d_{\mathscr{A}}[n]$ and $\mathbb{P}^d_{\mathscr{A}}[n]^+$ so that $\phi_{\mathscr{A}}$ becomes equivariant (Lemma 3.7). Now let $\tilde{L}_{\mathscr{A}}$ be the line bundle of definition 4.3. Recall by Lemma 3.7(2) that $\phi_{\mathscr{A}}$ is projective. Now let us choose an arbitrary relatively ample linearized line bundle $M^+_{\mathscr{A}}$

for $\phi_{\mathscr{A}}$. Set $L^+_{\mathscr{A}} = M^+_{\mathscr{A}} \otimes \tilde{L}^{\otimes m}_{\mathscr{A}}$ for sufficiently large m and $\overline{U}^{\mathscr{A}}_{d,n} := \mathbb{P}^d_{\mathscr{A}}[n]^+ /\!/_{L^+_{\mathscr{A}}} SL_{d+1}$. By 3.8(1), the above GIT quotient is independent of the choice of $M^+_{\mathscr{A}}$ for large m. Then, by Lemma 3.8 (2), we may descend $\phi_{\mathscr{A}}$ to obtain a morphism

$$\hat{\phi}_{\mathscr{A}} : \overline{U}^{\mathscr{A}}_{d,n} \to \overline{P}^{\mathscr{A}}_{d,n}$$

Moreover, $\phi_{\mathscr{A}}$ is equipped with n sections $\sigma_i : \mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}}[n]^+$([ibid.]), which are SL_{d+1} equivariant by Lemma 3.7. By Lemma 4.1(1), we have $\mathbb{P}^d_{\mathscr{A}}[n]^{ss}(\tilde{L}_{\mathscr{A}}) = \mathbb{P}^d_{\mathscr{A}}[n]^s(\tilde{L}_{\mathscr{A}})$. Consequently, by Lemma 3.8(1), we deduce that $\phi^{-1}_{\mathscr{A}}(\mathbb{P}^d_{\mathscr{A}}[n]^s(\tilde{L}_{\mathscr{A}})) = (\mathbb{P}^d_{\mathscr{A}}[n]^+)^{ss}(L^+_{\mathscr{A}}) = (\mathbb{P}^d_{\mathscr{A}}[n]^+)^s(L^+_{\mathscr{A}})$. Therefore, the restriction σ^s_i of the section σ_i to $\mathbb{P}^d_{\mathscr{A}}[n]^s(\tilde{L}_{\mathscr{A}})$ maps to the subvariety $(\mathbb{P}^d_{\mathscr{A}}[n]^+)^s(L^+_{\mathscr{A}})$ of $\mathbb{P}^d_{\mathscr{A}}[n]^+$. Let $\phi^s_{\mathscr{A}}$ be the restriction of $\phi_{\mathscr{A}}$ to the stable locus. We may now descend σ^s_i to the associated geometric quotients to obtain sections $\hat{\sigma}_i : \overline{P}^{\mathscr{A}}_{d,n} \to \overline{U}^{\mathscr{A}}_{d,n}$, $i = 1, \ldots, n$, that fit in the following commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^d_{\mathscr{A}}[n]^+)^s \arrow[r] \arrow[d, shift left, "\phi^s_{\mathscr{A}}"] & (\mathbb{P}^d_{\mathscr{A}}[n]^+)^s /\!/ SL_{d+1} \arrow[d, shift left, "\hat{\phi}_{\mathscr{A}}"] \\
\mathbb{P}^d_{\mathscr{A}}[n]^s \arrow[r] \arrow[u, shift left, bend left, "\sigma^s_i"] & (\mathbb{P}^d_{\mathscr{A}}[n])^s /\!/ SL_{d+1} \arrow[u, shift left, bend left, "\hat{\sigma}_i"]
\end{tikzcd}$$

By Theorem 3.6(4), σ_i lie in the relative smooth locus of $\phi_{\mathscr{A}}$. Now, SL_{d+1} acts with trivial stabilizers on $\mathbb{P}^d_{\mathscr{A}}[n]^s$, so, by Lemma 3.8(3) the fiber of $\hat{\phi}_{\mathscr{A}}$ over a geometric point (orbit) in the stable locus is isomorphic to the fiber of $\phi^s_{\mathscr{A}}$ over any point in that orbit. Therefore, we see that the relative smooth locus of $\phi^s_{\mathscr{A}}$ maps to the relative smooth locus $\hat{\phi}_{\mathscr{A}}$ via the quotient morphism. Hence, by the commutativity of the above diagram we establish part (1) of the lemma. Additionally, since $\mathbb{P}^d_{\mathscr{A}}[n]$ and $\mathbb{P}^d_{\mathscr{A}}[n]^+$ are projective, the GIT quotients $\overline{P}^{\mathscr{A}}_{d,n}$ and $\overline{U}^{\mathscr{A}}_{d,n}$ are projective, so $\hat{\phi}_{\mathscr{A}}$ is proper. By Lemma 3.8 (3) again and the fact that fibers of $\phi^s_{\mathscr{A}}$ are equidimensional, it follows that the fibers of $\hat{\phi}_{\mathscr{A}}$ are also equidimensional. Moreover, since $\phi^s_{\mathscr{A}}$ is equivariant and SL_{d+1} acts with trivial stabilizers on $\mathbb{P}^d_{\mathscr{A}}[n]^s$, we see that SL_{d+1} acts with trivial stabilizers on the smooth variety $(\mathbb{P}^d_{\mathscr{A}}[n]^+)^s$ as well. Therefore $(\mathbb{P}^d_{\mathscr{A}}[n]^+)^s /\!/ SL_{d+1}$ is smooth (for example, by Luna's étale slice theorem) and, since $\overline{P}^{\mathscr{A}}_{d,n}$ is also smooth (Theorem 1.1), we deduce that $\hat{\phi}_{\mathscr{A}}$ is flat. It remains to verify part (2); once again, Lemma 3.8 (3) guarantees that if x is a geometric point in $\mathbb{P}^d_{\mathscr{A}}[n]^s$, then $\hat{\phi}^{-1}_{\mathscr{A}}([SL_{d+1} \cdot x]) \cong \phi^{-1}_{\mathscr{A}}(x)$. Then (2) is equivalent to saying that the weighted stable n-pointed degeneration $(\phi^{-1}_{\mathscr{A}}(x), \sigma^s_i(x))$ is isomorphic to $(\hat{\phi}^{-1}_{\mathscr{A}}([SL_{d+1} \cdot x]), \hat{\sigma}^s_i([SL_{d+1} \cdot x]))$. Then (2) follows immediately from the descriptions in Sections 2.6 and 3.2. □

Corollary 4.10. *For every weight set $\mathscr{A} \in \mathscr{D}^P_{d,n}$:*

1. *(1) $\overline{P}^{\mathscr{A}}_{d,n}$ is the iterated blowup of $(\mathbb{P}^{n-d-2})^d$ along the elements of $\mathscr{G}_{\mathscr{A}(D)}$ in ascending dimension order.*
2. *(2) The boundary $\overline{P}^{\mathscr{A}}_{d,n} \setminus P^{\mathscr{A}}_{d,n}$ is the union of $|\mathscr{G}_{\mathscr{A}(D)}|$ divisors E_I.*
3. *(3) Each of the divisors E_I is the iterated dominant transform (cf definition 4.25) of H_I in the sequence of blowups $\overline{P}^{\mathscr{A}}_{d,n} \to (\mathbb{P}^{n-d-2})^d$.*
4. *(4) Any set of boundary divisors intersects transversely.*

Proof. By Theorem 3.4(2) the first statement is equivalent to the claim that $\overline{P}^{\mathscr{A}}_{d,n}$ is the wonderful compactification with respect to the elements of $\mathscr{G}_{\mathscr{A}(D)}$. This last claim is proven in the first part of the proof of Theorem 1.1. The other statements follow from Theorem 3.4 (1). □

4.2. **Points in affine space.** We first recall the definition of weighted $T_{d,n}$ and then we give a birational model of $T_{d,n}$ isomorphic to a projective space.
Let $X_{\mathscr{A}}[n]$ be the weighted compactification of the weighted configuration space of n points in a smooth variety X (Section 3.2), where $\mathscr{A} \in \mathscr{D}^T_{d,n}$, that is $\sum_{i=0}^{n} a_i > 1$. The above inequality implies that $\Delta_N \in \mathscr{K}_{\mathscr{A}}$. Now let $D_N \subset X_{\mathscr{A}}[n]$ be the divisor corresponding to the locus where all n points in the various weighted stable degenerations of X lie over an arbitrary point in X. This is precisely the iterated dominant transform of the small diagonal Δ_N under the sequence of blowups $X_{\mathscr{A}}[n] \to X^n$ (Theorem 3.6(3)). We have a morphism (induced by the blowdown)

$$q_{\mathscr{A}} : D_N \to \Delta_N \cong X.$$

Definition 4.11. [Rou15] Let X be a smooth variety of dimension d and consider a geometric point $x \in X$. We define $T^{\mathscr{A}}_{d,n} := {q_{\mathscr{A}}}^{-1}(x)$.

$$\begin{tikzcd} T^{\mathscr{A}}_{d,n} \arrow[r] \arrow[d] & D_N \arrow[r] \arrow[d, "q_{\mathscr{A}}"] & X_{\mathscr{A}}[n] \arrow[d] \\ x \arrow[r] & \Delta_N \cong X \arrow[r] & X^n \end{tikzcd}$$

Remark 4.12. It is shown in [ibid.] that the above definition is independent of x and X, as long as d, n and $\mathscr{A}$ are fixed.

Remark 4.13. The universal family $(T^{\mathscr{A}}_{d,n})^+ \to T^{\mathscr{A}}_{d,n}$ is constructed in [Rou15]. Its geometric fibers are precisely the objects described in Section 2.5.

Recall that given a configuration of n points in affine space defined up to translation and homothety, it is convenient to think of them as points in $\mathbb{P}^d$ that lie away from a fixed hyperplane $H \subset \mathbb{P}^d$ called the *root* and defined up to the action of the subgroup $G \subset SL_{d+1}$ that fixed the root pointwise.

Lemma 4.14. *Let $\mathscr{B} := \left(\frac{1}{n}+\varepsilon, \dots, \frac{1}{n}+\varepsilon\right) \in \mathscr{D}^T_{d,n}$. Then $T^{\mathscr{B}}_{d,n} \cong \mathbb{P}^{dn-d-1}$ and there is a choice of coordinates so that the point*

$$[x_{11} : x_{12} : \ldots : x_{1d} : \ldots : x_{21} : x_{22} : \ldots x_{2d} : \ldots : x_{(n-1)1} : x_{(n-1)2} : \ldots : x_{(n-1)d}] \in T^{\mathscr{B}}_{d,n}$$

parametrizes the equivalence class associated to the collection of n points:

$$p_1 := [1 : x_{11} : \ldots : x_{1d}], \quad \ldots \quad p_{n-1} := [1 : x_{(n-1)1} ; x_{(n-1)2} : \ldots : x_{(n-1)d}], \quad p_n := [1 : 0 : \ldots : 0]$$

Proof. By Definition 4.11, $T^{\mathscr{B}}_{d,n}$ is the fiber $q^{-1}_{\mathscr{B}}(x)$ of a point in the divisor D_N over X where D_N is the iterated dominant of the small diagonal Δ_N along the sequence of blowups $X_{\mathscr{B}}[n] \to X^n$. For our choice of weights there is only one blow up involved, hence the dominant transform of Δ_N in $X_{\mathscr{B}}[n]$ is the projective bundle $\mathbb{P}(N_{\Delta_N/X^n})$. Therefore, its fiber over $\Delta_N = X$ is isomorphic to the projective space $\mathbb{P}^{dn-d-1}$. To obtain the coordinates, we describe an alternative and instructive construction. We consider $\mathbb{P}^d$ with homogeneous coordinates $[x_0; \ldots ; x_d]$ and take the root H to be $(x_0 = 0)$. We can choose the location of one of the points, say p_n, to be $[1 : 0 \cdots : 0] \in (\mathbb{P}^d \setminus H) = \mathbb{A}^d$. The location of the other $(n-1)$ points can be anywhere in $\mathbb{P}^d \setminus H$, but they cannot all overlap with p_n simultaneously. The automorphism group of $\mathbb{P}^d$ that fixes the hyperplane H pointwise and the point p_n is $\mathbb{G}_m$. Then, we conclude that the our parameter space is

$$\left((\mathbb{A}^d)^{n-1} \setminus (0,0,\ldots 0)\right) /\!/ \mathbb{G}_m \cong \mathbb{P}^{d(n-1)-1}.$$

with the coordinates described in the statement. □

Next, we describe the loci in the above $T^{\mathscr{B}}_{d,n}$ that parametrize configurations with overlapping points.

Lemma 4.15. *Let $\mathscr{B}$ be as in lemma 4.14. The loci of coincident points in $T^{\mathscr{B}}_{d,n}$ are given by*

$$\delta_I = \begin{cases} \bigcap_{i,j\in I}(x_{i1}-x_{j1} = \ldots = x_{id}-x_{jd} = 0) & \mid \text{if } n \notin I \\ \bigcap_{i\in I\setminus n}(x_{i1} = \ldots = x_{1d} = 0) & \mid \text{if } n \in I \end{cases}$$

with $2 \leq |I| \leq (n-1)$. The locus δ_I is isomorphic to $\mathbb{P}^{d(n-|I|)-1}$, and it parametrizes configurations where points whose labels lie in I coincide.

Proof. It follows from Lemma 4.14. □

Definition 4.16. Given the set of weights $\mathscr{A} = \{a_1,\ldots,a_n\} \in \mathscr{D}^T_{d,n}$, we define the partially ordered set

$$\mathscr{H}_{\mathscr{A}} = \left\{\delta_I \subset \mathbb{P}^{d(n-1)-1} \mid \textstyle\sum_{i\in I} a_i > 1\right\}$$

with partial order $(<)$ given by $\delta_I < \delta_J$, if and only if $|J| < |I|$.

Lemma 4.17. *The ordered set $(\mathscr{H}_{\mathscr{A}}, <)$ is a building set.*

Proof. Consider an arbitrary nonempty intersection $S := \delta_{I_1} \cap \cdots \cap \delta_{I_k}$ of varieties that belong to the set $\mathscr{H}_{\mathscr{A}}$. We need to establish that the minimal elements of $\mathscr{H}_{\mathscr{A}}$ containing S intersect transversally and their intersection is S. To see this, observe that the above intersection can be written uniquely as an intersection of the form $\delta_{I'_1} \cap \cdots \cap \delta_{I'_m}$, where $m \leq k$, the I'_i are pairwise disjoint and each of the I'_i is a union of I_j's. Each of the $\delta_{I'_i}$'s belongs to $\mathscr{H}_{\mathscr{A}}$, because I'_i cannot be the set $\{1,\ldots,n\}$ (otherwise S would be empty), and since I'_i contains some I_j, we have

$$\sum_{k\in I'_i} a_k \geq \sum_{k\in I_j} a_k > 1.$$

By construction, the $\delta_{I'}$'s are the minimal elements of $\mathscr{H}_{\mathscr{A}}$ that contain S. Finally, since the indices

I'_i are disjoint, we see that the varieties $\delta_{I'_1},\ldots,\delta_{I'_m}$ intersect transversally. □

Definition 4.18. Let $\mathscr{A} \in \mathscr{D}^T_{d,n}$. The weighted configuration space of n points in $\mathbb{A}^d$ up to translation and homothety with respect to $\mathscr{A}$ is

$$(T^{\mathscr{A}}_{d,n})^o := \mathbb{P}^{d(n-1)-1} \setminus \bigcup_{\Delta_I \in \mathscr{H}_{\mathscr{A}}} \delta_I$$

Lemma 4.19. *$T^{\mathscr{A}}_{d,n}$ is isomorphic to the wonderful compactification of the arrangement $\mathscr{H}_{\mathscr{A}}$.*

Proof. Let X as in Definition 4.11. For any $Y \subset X^n$ (resp. $Y \subset \mathbb{P}^{d(n-1)-1}$), we denote by $Y^{(i)}$ the iterated dominant transform of Y in the i-th step of the sequence of blowups $X_{\mathscr{A}}[n] \to X^n$ (resp. $Bl_{\mathscr{H}_{\mathscr{A}}}\mathbb{P}^{d(n-1)-1} \to \mathbb{P}^{d(n-1)-1}$). The i-th dominant transform $\Delta_N^{(i)}$ is the blowup of the $(i-1)$-th dominant transform $\Delta_N^{(i-1)}$ along the intersection $\Delta_N^{(i-1)} \cap \Delta_I^{(i-1)}$, where $\Delta_I^{(i-1)}$ is the center of the i-th blowup. We show that the fiber of the i-th iterated dominant transform of D_N is the $(i-1)$-th blowup of $\mathbb{P}^{d(n-1)-1}$. The proof of the lemma follows from this if we set i equal to the total number of blowups in $X_{\mathscr{A}}[n] \to X^n$. To this end, we show by induction that, for *every* $\Delta_I \subset X^n$ and $i \geq 1$:

(1) the fiber $\Delta_N^{(i)} \times_X x$ is isomorphic to the $(i-1)$-th iterated blowup of $\mathbb{P}^{d(n-1)-1}$ along $\mathscr{H}_{\mathscr{A}}$;
(2) the fiber $(\Delta_N^{(i)} \cap \Delta_I^{(i)}) \times_X x$ is isomorphic to $\delta_I^{(i-1)}$.

To prove the above claim for $i = 1$, observe that $\Delta_N^{(1)}$ is the exceptional divisor of the first blowup, hence isomorphic to $\mathbb{P}(N_{\Delta_N/X^n}) = \mathbb{P}(T_{X^n}/T_X)$ and $\Delta_I^{(1)} \cap \Delta_N^{(1)} = \mathbb{P}(N_{\Delta_N/\Delta_I}) = \mathbb{P}(T_{\Delta_I}/T_X)$. Also, observe that the δ_I in lemma 4.15 are obtained by descending the diagonals of $(\mathbb{A}^d)^{n-1}$ labeled by I (i.e. the loci in $(\mathbb{A}^d)^{n-1}$ where all factors of $(\mathbb{A}^d)^{n-1}$ labeled by I are equal) to the quotient $\big((\mathbb{A}^d)^{n-1} \setminus (0,0,\dots 0)\big) /\!/ \mathbb{G}_m \cong \mathbb{P}^{d(n-1)-1}$. Therefore the embedding $\mathbb{P}(T_{\Delta_I}/T_X) \hookrightarrow \mathbb{P}(T_{X^n}/T_X)$ over X pulls back to $\delta_I \hookrightarrow \mathbb{P}^{d(n-1)-1}$ via $x \to X$.
Now let $\Delta_J^{(i)}$ be the center of the $i+1$-th blowup of X^n and assume the claim is true for some $i \geq 1$. We will show the claim is true for $i+1$. By [Li09, Proposition 2.8], the set that consists of the i-th iterated dominant transforms of the elements of a building set (which is given an ascending dimension order) is also a building set, so the set $\{\Delta_I^{(i)} | \Delta_I \in \mathscr{K}_{\mathscr{A}}\}$ is a building set, where $\mathscr{K}_{\mathscr{A}}$ is the building set described in section 3.2. Therefore, by [Li09, Lemma 2.6], the center $\Delta_J^{(i)}$ intersects transversally with the divisor $\Delta_N^{(i)}$, so the embedding $(\Delta_N^{(i)} \cap \Delta_J^{(i)}) \hookrightarrow \Delta_N^{(i)}$ is regular; let $\mathscr{I}$ be the ideal sheaf corresponding to this embedding. Therefore, $\mathscr{I}^n/\mathscr{I}^{n+1}$ is locally free, hence flat over $\mathscr{O}_{\Delta_N^{(i)}}/\mathscr{I}$. Then, by the exact sequence

$$0 \to \frac{\mathscr{I}^n}{\mathscr{I}^{n+1}} \to \frac{\mathscr{O}_{\Delta_N^{(i)}}}{\mathscr{I}^{n+1}} \to \frac{\mathscr{O}_{\Delta_N^{(i)}}}{\mathscr{I}^n} \to 0$$

we deduce by induction that $\mathscr{O}_{\Delta_N^{(i)}}/\mathscr{I}^n$ is also flat over $\mathscr{O}_{\Delta_N^{(i)}}/\mathscr{I}$ for all n. But, by part (2) of the claim, $\mathscr{O}_{\Delta_N^{(i)}}/\mathscr{I}$ is flat over $\mathscr{O}_X$, so $\mathscr{O}_{\Delta_N^{(i)}}/\mathscr{I}^n$ is also flat over $\mathscr{O}_X$ for all n. Also , by (1) and (2) for i, we deduce that $\Delta_N^{(i)}$ and $\Delta_N^{(i)} \cap \Delta_J^{(i)}$ are equidimensional over X, hence flat over X. Now, by [I$^+$82], we deduce claim (1) for $i+1$.
To see (2) also holds, let $(\Delta_N^{(i)} \cap \Delta_I^{(i)})\tilde{}$ be the strict transform of $\Delta_N^{(i)} \cap \Delta_I^{(i)}$ in the $i+1$-th blowup of X^n. Then observe that $(\Delta_N^{(i)} \cap \Delta_I^{(i)})\tilde{} = \Delta_N^{(i+1)} \cap \Delta_I^{(i+1)}$: indeed, as noted above, $\Delta_N^{(i)}$ intersects transversally with the center $\Delta_J^{(i)}$. Also, since $\{\Delta_I^{(i)} | \Delta_I \in \mathscr{K}_{\mathscr{A}}\}$ is a building set, by [Li09, Lemma 2.6] again, we see that any $\Delta_I^{(i)}$ either intersects transversally or contains the center $\Delta_J^{(i)}$. Then, by a repeated application of lemma 6.1 we may deduce that

- the intersection of $\Delta_N^{(i)} \cap \Delta_I^{(i)}$ with the center $\Delta_J^{(i)}$ is transversal (possibly empty) and consequently (by part (b) and (c) of the same lemma)
- $(\Delta_N^{(i)} \cap \Delta_I^{(i)})\tilde{} = \Delta_N^{(i+1)} \cap \Delta_I^{(i+1)}$.

Therefore, the embedding $(\Delta_N^{(i)} \cap \Delta_I^{(i)} \cap \Delta_J^{(i)}) \hookrightarrow (\Delta_N^{(i)} \cap \Delta_J^{(i)})$ is regular and now we may repeat the argument of the previous paragraph to deduce that

$$(\Delta_N^{(i+1)} \cap \Delta_I^{(i+1)}) \times_X x = (\Delta_N^{(i)} \cap \Delta_I^{(i)})\tilde{} \times_X x = ((\Delta_N^{(i)} \cap \Delta_I^{(i)}) \times_X x)\tilde{} = (\delta_I^{(i-1)})\tilde{} = \delta_I^{(i)}$$

, where the third equality holds because of the inductive hypothesis for i. This concludes part (2) of the claim for $i+1$.

□

Proof of Theorem 1.2(1): Immediate by Lemma 4.19 and Theorem 3.4.

Corollary 4.20. *For every weight set $\mathscr{A} \in \mathscr{D}_{d,n}^T$,*

(1) $T_{d,n}^{\mathscr{A}}$ is the iterated blowup of $\mathbb{P}^{d(n-1)-1}$ along the dominant transforms of the varieties that belong to $\mathscr{H}_{\mathscr{A}}$ in ascending dimension order.
(2) The boundary $T_{d,n}^{\mathscr{A}} \setminus (T_{d,n}^{\mathscr{A}})^o$ is the union of $|\mathscr{H}_{\mathscr{A}}|$ divisors D_I.
(3) Each of the divisors D_I is the iterated dominant transform (cf definition 4.25) of δ_I in the sequence of blowups $T_{d,n}^{\mathscr{A}} \to \mathbb{P}^{d(n-1)-1}$.
(4) Any set of boundary divisors intersects transversally.

Proof. The proof is a direct consequence of Lemma 4.17 and Theorem 3.4. □

Corollary 4.21. *For $dn \geq d+3$, the morphism $T_{d,n} \to \mathbb{P}^{dn-d-1}$ can be understood as completing the following steps successively*

(1) blow up n disjoint loci isomorphic to $\mathbb{P}^{d-1}$ parametrizing configurations with a $(n-1)$-multiple point.
(2) blow up the strict transforms of the $(2d-1)$-dimensional planes spanned by $d+1$ distinct loci isomorphic to $\mathbb{P}^{d-1}$; they generically parametrize configurations with a $(n-2)$-multiple point.

$\vdots$

(n-2) blow up the strict transforms of $(d(n-2)-1)$ planes spanned by the $\mathbb{P}^{d(n-3)-1}$'s of step $(n-3)$; they generically parametrize configurations with a double point.

Proof. This is the loci $\mathscr{H}$ associated to all weights equal to one (see Definition 4.16) and described in Lemma 4.15. □

4.3. **Structure of the boundary.** From the theory of wonderful compactifications, we obtain a criterion for deciding whether a set of divisors intersect or not.

Corollary 4.22. *A set of divisors $D_{I_1}, \ldots D_{I_r}$ in $T_{d,n}^{\mathscr{A}}$ (resp. $\overline{P}_{d,n}^{\mathscr{A}}$) has nonempty intersection if and only if the indices I_k are either pairwise contained in each other or disjoint. Equivalently, the divisors D_{I_k} have nonempty pairwise intersection.*

Proof. We only show the statement for $T_{d,n}^{\mathscr{A}}$, since the proof of the statement for $\overline{P}_{d,n}^{\mathscr{A}}$ is identical. We first show the condition about the divisors. If the divisors intersect, then every pair intersects as well. Next, suppose that $D_{I_k} \cap D_{I_j} \neq \emptyset$ for every I_j and I_k in our set $\mathscr{I} = \{I_1, \ldots I_r\}$. Then either $I_k \cap I_j = \emptyset$ or one set of indexes is contained in the other one. This implies $I_k \cap I_j = \emptyset$ for every I_k and I_j in the minimal set of $\mathscr{I}$ (N.B. the minimal elements with respect to the containment order). Clearly, we can find a configuration of points with multiple points defined by disjoint I_k and I_j. Let S be the loci parametrizing these configuration of points. By [Li09, Def 2.3] and [Li09, Thm 1.2.ii], a set of divisors $D_{I_1} \cap D_{I_2} \cap \ldots D_{I_k}$ has not empty intersection if and only if there exist a $S_{\mathscr{I}}$

in the arrangement generated by $\mathscr{H}_{\mathscr{A}}$ such that *all* the $\mathscr{H}_{\mathscr{A}}$-factors of $S_{\mathscr{I}}$ are the minimal elements of $\mathscr{I}$ (see Definition 3.1). We can take S to be our $S_{\mathscr{I}}$. The first condition follows from the fact that if two divisors D_I and D_K intersect, then either $I \subset K$ or $I \cap K \neq \emptyset$. $\square$

Next, we give a proof of Theorem 1.2 (2). We only prove the result about $T_{d,n}^{\mathscr{A}}$; by our proof it will be made apparent that the proof for $\overline{P}_{d,n}^{\mathscr{A}}$ follows in the exact same way from its construction as the iterated blowup of $(\mathbb{P}^{n-d-2})^d$ (Corollary 4.10). Recall that the divisor D_I is the dominant transform of the locus δ_I which generically parametrizes configurations with a multiple point $p_I := \cap_{i\in I} p_i$. Let $D_I \subset T_{d,n}^{\mathscr{A}}$ be the divisor corresponding to the set I and consider the ordered sets $\mathscr{A}(I) := \{a_i \mid i \in I\}$ and $\mathscr{A}_+(I^c) := \{a_i \mid i \notin I\} \cup \{a_{n+1} = 1\}$.

First, consider the set $\mathscr{H}_{\mathscr{A}}$ and the varieties $\delta_I \cong \mathbb{P}^{d(n-|I|)-1}$ as in Definition 4.16. We now give a different order $(\prec)$ on the above set by reshuffling its elements.

Definition 4.23. The degenerations of a multiple point $p_I = \cap_{i\in I} p_i$ induce the following partition of the set $\mathscr{H}_{\mathscr{A}}$.

(1) The set $\mathscr{H}_1 := \{\delta_J \in \mathscr{H}_{\mathscr{A}} \,|\, J \supsetneq I\}$ parametrizes multiple points obtained as a degeneration of p_I.
(2) The set $\mathscr{H}_2 := \{\delta_J \in \mathscr{H}_{\mathscr{A}} \,|\, J \cap I = \emptyset\}$ parametrizes multiple points than can coexist with p_I in a given configuration.
(3) The set $\mathscr{H}_3 := \{\delta_J \in \mathscr{H}_{\mathscr{A}} \,|\, J \text{ overlaps with } I\}$ parametrizes multiple points that cannot coexist with p_I in any configuration.
(4) The set $\mathscr{H}_4 := \{\delta_J \in \mathscr{H}_{\mathscr{A}} \,|\, J \subseteq I\}$ parametrizes p_I and multiple points that degenerate to p_I.

The partial order $(\prec)$ on the set $\mathscr{H}_{\mathscr{A}}$ is determined by the following rules:

- let $i \in \{1,2,3,4\}$. Then the $\delta_J \in \mathscr{H}_i$ are given an ascending dimension order.
- $\delta_{J_i} \prec \delta_{J_j}$ for any $\delta_{J_i} \in \mathscr{H}_i$ and $\delta_{J_j} \in \mathscr{H}_j$ such that $i < j$.

Also, for any $i \in \{1,2,3,4\}$ we denote by $\mathbf{P}^{[i]}$ the sequence of blowups of $\mathbb{P}^{d(n-1)-1}$ along the iterated dominant transforms of the varieties that belong to $\mathscr{H}_1 \cup \cdots \cup \mathscr{H}_i$ with order $(\prec)$.

Lemma 4.24. *The ordered set $(\mathscr{H}_{\mathscr{A}}, \prec)$ satisfies the condition of part (2) of Theorem 3.4. Therefore, $T_{d,n}^{\mathscr{A}}$ is isomorphic to the iterated blowup of $\mathbb{P}^{d(n-1)-1}$ along the varieties that belong to $\mathscr{H}_{\mathscr{A}}$ with the order $(\prec)$ defined above.*

Proof. Almost identical to the proof of Theorem 1.2 (1) $\square$

To prove Theorem 1.2 (2) suffices to consider the iterated dominant transform of δ_I along $\mathscr{H}_{\mathscr{A}}$ in the order $(\prec)$ described above. First, we look at the iterated dominant transform of δ_I in the sequence of blowups of $\mathbb{P}^{d(n-1)-1}$ along the varieties that belong to $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$. We will show the following:

Lemma 4.25. *$T_{d,n-|I|+1}^{\mathscr{A}_+(I^c)}$ is isomorphic to the sequence of blowups of $\delta_I \cong \mathbb{P}^{d(n-|I|)-1}$ along the iterated strict transforms of the varieties in $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$ in the order $(\prec)$ described above.*

Proof. See Appendix 6. $\square$

Let $\widetilde{\delta}_I$ be the strict transform of δ_I in the sequence of blowups of $\mathbf{P}^{[3]} \to \mathbb{P}^{d(n-1)-1}$ along the centers corresponding to $\mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3$. Next, we look at the iterated dominant transform of $\widetilde{\delta}_I$

in the sequence of blowups $\mathbf{P}^{[4]} \to \mathbf{P}^{[3]}$ along the centers corresponding to $\mathscr{H}_4$. We will need the following lemma:

Lemma 4.26. *With notation as above*

(1) Let $\widetilde{\delta_I}$ be the strict transform of δ_I in $\mathbf{P}^{[3]}$. The normal bundle of $\widetilde{\delta_I}$ in $\mathbf{P}^{[3]}$ is isomorphic to the direct sum $\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}(p-1)$, where p is the cardinality of $\mathscr{H}_1$.

(2) Let $\delta_{I'} \in \mathscr{H}_4$ so that $\delta_{I'} \supset \delta_I$ and let $\widetilde{\delta_{I'}}$ be its strict transform in $\mathbf{P}^{[3]}$. Then the normal bundle of $\widetilde{\delta_I}$ in $\widetilde{\delta_{I'}}$ is isomorphic to $\bigoplus_{i=1}^{d(|I|-|I'|)} \mathscr{O}(p-1)$, where p is the cardinality of $\mathscr{H}_1$.

Proof. See Appendix 6. $\square$

Now, we are ready to finish the proof of our main theorem.

Proof. (Theorem 1.2(2)) With notation as above, let $\widetilde{\delta_I}$ be the sequence of blowups corresponding to elements of $\mathscr{H}_3$. We need to keep track of the iterated dominant transform of $\widetilde{\delta_I}$ in the sequence of blowups $\mathbf{P}^{[4]} \to \mathbf{P}^{[3]}$ defined above. By definition, the first blowup in this sequence has center $\widetilde{\delta_I}$. As a result, the dominant transform of $\widetilde{\delta_I}$ in this first blowup is the exceptional divisor. The latter is equal to $\mathbb{P}(N_{\widetilde{\delta_I}/\mathbf{P}^{[3]}})$, which in turn is equal to (cf Lemma 4.26(1))

$$\mathbb{P}\left(\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\widetilde{\delta_I}}(p-1)\right) \cong \mathbb{P}\left(\Big(\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\widetilde{\delta_I}}(p-1)\Big) \otimes \mathscr{O}_{\widetilde{\delta_I}}(1-p)\Big)\right) = \mathbb{P}\left(\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\widetilde{\delta_I}}\right)$$
$$= \mathbb{P}^{d(|I|-1)-1} \times \widetilde{\delta_I} = \mathbb{P}^{d(|I|-1)-1} \times T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1}$$

where the last equality follows from Lemma 4.25. Moreover the dominant transform of $\widetilde{\delta_{I'}}$ in the blowup of $\mathbf{P}^{[3]}$ along $\widetilde{\delta_I}$ intersects the above exceptional divisor in $\mathbb{P}(N_{\widetilde{\delta_I}/\widetilde{\delta_{I'}}})$. As above, using Lemmas 4.25 and 4.26(2) we have

$$\mathbb{P}(N_{\widetilde{\delta_I}/\widetilde{\delta_{I'}}}) \cong \mathbb{P}^{d(|I|-|I'|)-1} \times T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1}$$

From the above we deduce that the further stepwise dominant transforms of $\widetilde{\delta_I}$ in the sequence $\mathbf{P}^{[4]} \to \mathbf{P}^{[3]}$ are iterated blowups of $T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1} \times \mathbb{P}^{d(|I|-1)-1}$ along $T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1} \times \mathbb{P}^{d(|I|-|I'|)-1}$ in ascending dimension order. Equivalently, since the formation of blowup commutes with smooth base change, the further stepwise dominant transforms of $\widetilde{\delta_I}$ are isomorphic to the product

$$Bl_{\mathscr{H}_{\mathscr{A}(I)}}\mathbb{P}^{d(|I|-1)-1} \times T^{\mathscr{A}_+(I^c)}_{d,n-|I|+1}$$

where $Bl_{\mathscr{H}_{\mathscr{A}(I)}}\mathbb{P}^{d(|I|-1)-1}$ is the iterated blowup of $\mathbb{P}^{d(|I|-1)-1}$ along the set

$$\mathscr{H}_{\mathscr{A}(I)} := \{\mathbb{P}^{d(|I|-|I'|)-1} \subset \mathbb{P}^{d(|I|-1)-1} \mid I' \subsetneq I \text{ and } \sum_{i \in I'} a_i > 1\}$$

in ascending dimension order. This is precisely the set of *loci of coincident points* of Definition 4.15 for input data $d, \mathscr{A}(I)$ and I, where $\mathscr{A}(I)$ is defined in Section 2.1. This concludes the proof of our theorem. $\square$

5. Reduction, Forgetful Morphisms and Toric models (Theorems 1.3 and 1.5)

5.1. **Reduction and forgetful morphisms.** By modifying the weights appropriately, we induce morphisms among our compactifications.

Proposition 5.1. *Let $\mathscr{A} := \{a_1, a_2, \ldots, a_n\}$ and $\mathscr{B} := \{b_1, b_2, \ldots, b_n\}$ be two weight sets in $\mathscr{D}^T_{d,n}$ (resp. $\mathscr{D}^P_{d,n}$) as in Section 2.1 such that $b_i \leq a_i$ for all $i = 1, 2, \ldots n$ (resp. for all $i = d+2, \ldots n$). There exists a natural reduction morphism*

$$\rho_{\mathscr{B},\mathscr{A}} : T^{\mathscr{A}}_{d,n} \to T^{\mathscr{B}}_{d,n} \qquad\qquad (\text{resp. } \hat{\rho}_{\mathscr{B},\mathscr{A}} : \overline{P}^{\mathscr{A}}_{d,n} \to \overline{P}^{\mathscr{B}}_{d,n})$$

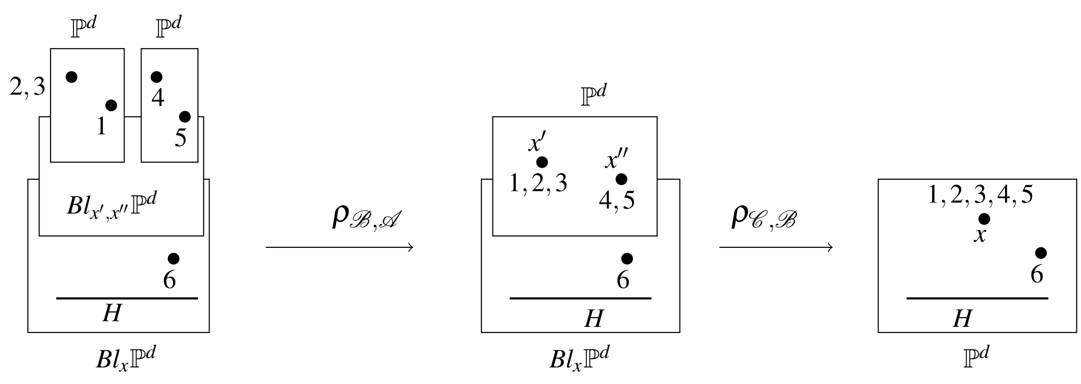
which is a blowdown in case one of the above inequalities is strict. At the level of k-points, the morphism $\rho_{\mathscr{B},\mathscr{A}}$ (resp. $\hat{\rho}_{\mathscr{B},\mathscr{A}}$) reassigns the weights of the sections of an $\mathscr{A}$-stable rooted tree (resp. $\mathscr{A}$-stable tree) and then successively collapses all components that are unstable with respect to $\mathscr{B}$.

Proof. Our argument follows closely the argument in the proof of [Rou14, Theorem 5]. Let $\mathscr{H}_{\mathscr{A}}$ and $\mathscr{H}_{\mathscr{B}}$ with notation as in Lemma 4.16. with partial order $(<)$ given by $\delta_I < \delta_J$, if and only if $|J| < |I|$. By Theorem 4.20, we know that $\mathscr{H}_{\mathscr{A}}$ and $\mathscr{H}_{\mathscr{B}}$ are building sets and, by the hypothesis, $\mathscr{H}_{\mathscr{B}} \subset \mathscr{H}_{\mathscr{A}}$. For ease of notation, denote the ideal sheaves of the $\delta_I \in \mathscr{H}_{\mathscr{A}}$ by $\mathscr{I}_1, \mathscr{I}_2, \ldots \mathscr{I}_k$ in order preserving bijection with the δ_I in $\mathscr{H}_{\mathscr{A}}$. Similarly, let $\{\mathscr{I}_{i_1}, \mathscr{I}_{i_2}, \ldots \mathscr{I}_{i_l}\} \subset \{\mathscr{I}_1, \mathscr{I}_2, \ldots \mathscr{I}_k\}$ be the set of ideal sheaves of the $\delta_I \in \mathscr{H}_{\mathscr{B}}$, listed again in order preserving bijection with the δ_I in $\mathscr{H}_{\mathscr{B}}$. By Theorem 3.4(3), we have

$$T^{\mathscr{A}}_{d,n} \cong Bl_{\mathscr{I}_k} \ldots Bl_{\mathscr{I}_2} Bl_{\mathscr{I}_1} \mathbb{P}^{d(n-1)-1} \cong Bl_{\mathscr{I}_k} \ldots \left(Bl_{\mathscr{I}_{i_l}} \ldots Bl_{\mathscr{I}_{i_2}} Bl_{\mathscr{I}_{i_1}} \mathbb{P}^{d(n-1)-1} \right)$$

where the ideal sheaves outside the parenthesis belong to $\{\mathscr{I}_1, \mathscr{I}_2, \ldots \mathscr{I}_k\} \setminus \{\mathscr{I}_{i_1}, \mathscr{I}_{i_2}, \ldots \mathscr{I}_{i_l}\}$.. Therefore, $T^{\mathscr{A}}_{d,n} \cong Bl_{\mathscr{I}_k} \ldots T^{\mathscr{B}}_{d,n}$ from which we obtain the morphism $\rho_{\mathscr{B},\mathscr{A}} : T^{\mathscr{A}}_{d,n} \to T^{\mathscr{B}}_{d,n}$. The proof for $\hat{\rho}_{\mathscr{B},\mathscr{A}}$ is entirely analogous so we omit it. □

FIGURE 5. *Example of reduction morphisms for stable rooted trees with $n = 6$.*



$$\mathscr{A} = \left\{1, \frac{1}{3}+\varepsilon, \frac{1}{3}+\varepsilon, 1, 1, 1\right\}, \qquad \mathscr{B} = \left\{\frac{1}{5}+\varepsilon, \ldots, \frac{1}{5}+\varepsilon, 1\right\}, \qquad \mathscr{C} = \left\{\frac{1}{6}+\varepsilon, \ldots, \frac{1}{6}+\varepsilon, 1\right\}$$

The above morphisms behave favourably under weight reduction, as the following proposition suggests. We omit its proof, since it is identical to the proof of [Rou14, Proposition 5].

Proposition 5.2. *Let $\mathscr{A} := \{a_1, a_2, \dots, a_n\}$, $\mathscr{B} := \{b_1, b_2, \dots, b_n\}$ and $\mathscr{C} := \{c_1, c_2, \dots, c_n\}$ be weight sets in $\mathscr{D}^T_{d,n}$ (resp. $\mathscr{D}^P_{d,n}$) such that $c_i \leq b_i \leq a_i$ for all $i = 1, 2, \dots n$. Then:*

$$\rho_{\mathscr{C},\mathscr{A}} = \rho_{\mathscr{C},\mathscr{B}} \circ \rho_{\mathscr{B},\mathscr{A}}, \qquad\qquad (resp.\ \hat{\rho}_{\mathscr{C},\mathscr{A}} = \hat{\rho}_{\mathscr{C},\mathscr{B}} \circ \hat{\rho}_{\mathscr{B},\mathscr{A}})$$

We can also pass from weighted (pointed) trees with fixed data to others, by forgetting subsets of points and stabilizing.

Proposition 5.3. *Let R be a subset of $N = \{1, 2, \dots, n\}$ and $\mathscr{A}$ be a weight set in $\mathscr{D}^T_{d,n}$ (resp. $\mathscr{D}^P_{d,r}$, where $r = |R|$). Let $\mathscr{A}(R)$ be the subset of $\mathscr{A} := \{a_1, a_2, \dots, a_n\}$ described in 2.1 (resp. with the additional assumption that $R \supseteq \{1, \dots, d+1\}$) . Then, there exists a natural forgetful morphism*

$$\phi_{\mathscr{A},\mathscr{A}(R)} : T^{\mathscr{A}}_{d,n} \to T^{\mathscr{A}(R)}_{d,r} \qquad\qquad (resp.\ \hat{\phi}_{\mathscr{A},\mathscr{A}(R)} : \overline{P}^{\mathscr{A}}_{d,n} \to \overline{P}^{\mathscr{A}(R)}_{d,r})$$

At the level of k-points, the above morphism successively collapses all components of an $\mathscr{A}$-stable pointed rooted tree (resp. $\mathscr{A}(R)$-stable pointed tree) that are unstable with respect to $\mathscr{A}(R)$.

Proof. We start with the morphism $\phi_{\mathscr{A},\mathscr{A}(R)}$. By [Rou14, Theorem 6] and its proof, there exists a morphism

$$\mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r}$$

Let $D_N \subset \mathbb{P}^d_{\mathscr{A}}[n]$ and $D_R \subset \mathbb{P}^d_{\mathscr{A}(R)}[r]$ be the divisors corresponding to the small diagonals $\Delta_N \subset (\mathbb{P}^d)^N$ and $\Delta_R \subset (\mathbb{P}^d)^R$ respectively.

Claim: The restriction of $\mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r}$ to D_N surjects onto (a subvariety isomorphic to) D_R.
Proof of Claim: By the proof of [Rou14, Theorem 6], $\mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \to (\mathbb{P}^d)^n$ is obtained as a sequence of blowups of $(\mathbb{P}^d)^n$ along the iterated dominant transforms of the set

$$\mathscr{K}_{\mathscr{A}(R)} = \{\Delta_I \subset (\mathbb{P}^d)^n | I \subset R \text{and} \sum_{i_k \in I} a_{i_k} > 1\}$$

in ascending dimension order.

Moreover, $\mathbb{P}^d_{\mathscr{A}}[n]$ is obtained from $\mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r}$ by blowing up ideal sheaves corresponding to $\mathscr{K}_{\mathscr{A}} \setminus \mathscr{K}_{\mathscr{A}(R)}$. By [Li09], D_N is the iterated dominant transform of Δ_N along the sequence of blowups $\mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \to (\mathbb{P}^d)^n$. It therefore suffices to observe that the iterated dominant transform of Δ_N along the sequence of blowups $\mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \to (\mathbb{P}^d)^n$ is isomorphic to the divisor $D_R \subset \mathbb{P}^d_{\mathscr{A}(R)}[r]$. Indeed, consider the embedding $j : (\mathbb{P}^d)^r \to (\mathbb{P}^d)^r \times (\mathbb{P}^d)^{n-r}$, which is obtained as the graph of the morphism $(\mathbb{P}^d)^r \xrightarrow{p_i} \mathbb{P}^d \xrightarrow{diag} (\mathbb{P}^d)^{n-r}$ where p_i is the projection to the i-th factor. Then $\Delta_N \subset (\mathbb{P}^d)^n$ is the image of $\Delta_R \subset (\mathbb{P}^d)^r$ via j. Therefore, since the iterated dominant transform of Δ_R along $\mathbb{P}^d_{\mathscr{A}(R)}[r] \to (\mathbb{P}^d)^r$ is D_R (theorem 3.4), we conclude that the iterated dominant transform of Δ_N is the graph of $D_R \to (\mathbb{P}^d)^{n-r}$. **End.**

$$\begin{tikzcd}
T^{\mathscr{A}}_{d,n} \arrow[rr] \arrow[dr] \arrow[dd] & & D_N \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{P}^d_{\mathscr{A}}[n] \arrow[dr] \arrow[dd] & \\
& T^{\mathscr{A}(R)}_{d,r} \arrow[rr] \arrow[dl] & & D_R \arrow[rr] \arrow[dl] & & \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \arrow[dl] \\
x \arrow[rr] & & \mathbb{P}^d \arrow[rr, "\text{”}diag\text{”}"] & & (\mathbb{P}^d)^n = (\mathbb{P}^d)^r \times (\mathbb{P}^d)^{n-r} &
\end{tikzcd}$$

In view of the claim, we have a morphism of D_N to D_R over $\mathbb{P}^d$, which pulls back to a morphism $T^{\mathscr{A}}_{d,n} \to T^{\mathscr{A}(R)}_{d,r}$ between their fibers over $x \in \mathbb{P}^d$.

Next we prove the existence of the map $\hat{\phi}_{\mathscr{A},\mathscr{A}(R)}$. By [Rou14, Theorem 6], there exists a natural forgetful morphism $\psi_R : \mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}(R)}[r]$. This morphism is SL_{d+1} invariant because it is constructed by blowing up and then projecting invariant loci with respect to our action. To verify our statement, we check that ψ_R takes $(\mathbb{P}^d_{\mathscr{A}}[n])^s$ to $(\mathbb{P}^d_{\mathscr{A}(R)}[r])^s$. From the proof in [ibid.], ψ_R is the composition of a blowdown $\mathbb{P}^d_{\mathscr{A}}[n] \to \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r}$ and the projection of $\mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r}$ to the first factor. Therefore we have a commutative diagram

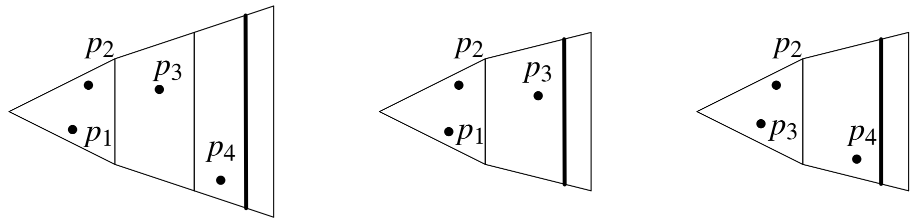
$$\begin{tikzcd}
\mathbb{P}^d_{\mathscr{A}}[n] \arrow[r] \arrow[dr, "\pi_{\mathscr{A}}"'] & \mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \arrow[r] \arrow[d, "\pi_{\mathscr{A}(R)} \times id"'] & \mathbb{P}^d_{\mathscr{A}(R)}[r] \arrow[d, "\pi_{\mathscr{A}(R)}"] \\
& (\mathbb{P}^d)^r \times (\mathbb{P}^d)^{n-r} \arrow[r, "p_R"] & (\mathbb{P}^d)^r
\end{tikzcd}$$

where the morphism p_R is the projection from $(\mathbb{P}^d)^n = (\mathbb{P}^d)^r \times (\mathbb{P}^d)^{n-r}$ to $(\mathbb{P}^d)^r$. Now, recall that $(\mathbb{P}^d_{\mathscr{A}}[n])^s$ is equal to the preimage of $((\mathbb{P}^d)^n)^s$ under $\pi_{\mathscr{A}}$ (see discussion before Definition 4.3). It would therefore suffice to show that the preimage of $((\mathbb{P}^d)^n)^s$ under $\pi_{\mathscr{A}(R)} \times id$ maps to $(\mathbb{P}^d_{\mathscr{A}(R)}[r])^s$ via the projection $\mathbb{P}^d_{\mathscr{A}(R)}[r] \times (\mathbb{P}^d)^{n-r} \to \mathbb{P}^d_{\mathscr{A}(R)}[r]$. But $(\mathbb{P}^d_{\mathscr{A}(R)}[r])^s$ is in turn equal to the preimage of $((\mathbb{P}^d)^r)^s$ under $\pi_{\mathscr{A}(R)}$. Consequently, it is enough to show that the projection p_R takes $((\mathbb{P}^d)^n)^s$ to $((\mathbb{P}^d)^r)^s$, which can be seen directly using [Dol03, Thm 11.2].

Since ψ_R-at the level of k-points- successively collapses all components of an $\mathscr{A}$ stable degeneration that are unstable with respect to $\mathscr{A}(R)$, we deduce, by (3) in Lemma 3.8, that the morphism $\overline{P}^{\mathscr{A}}_{d,n} \to \overline{P}^{\mathscr{A}(R)}_{d,n}$ has the desired moduli interpretation at geometric points. □

Next, we denote as π_I the forgetful map $\pi_I : T_{d,n} \to T_{d,|I|}$ obtained by forgetting the points $\{p_i \mid i \in I^c\}$ and stabilizing afterwards. First, we illustrate a particular case which leads us to Theorem 1.5.

Example 5.4. Consider the three dimensional loci $T_{2,2} \times (T_{2,2} \times T_{2,2}) \subset T_{2,4}$ that parametrizes a stable tree $X = X_1 \cup X_2 \cup X_3$ as on the adjacent figure.



$X = X_1 \cup X_2 \cup X_3 \cup X_4$ $\qquad$ $\pi_{123}(X)$ $\qquad$ $\pi_{234}(X)$

The points p_1 and p_2 are supported in the first surface $X_1 \cong \mathbb{P}^2$. The point p_3 is at the second surface $X_2 \cong Bl_x(\mathbb{P}^d)$, and the point p_4 are at the rooted component $X_3 \cong Bl_y(\mathbb{P}^d)$. The morphism π_I with $I = \{1,2,3\}$ contracts the last component X_3 to a line. We obtain a configuration of points parametrized by $T_{2,2} \times T_{2,2}$ with the point p_1 and p_2 supported in the first surface and p_3 supported in the last one.

We can recover the position of p_1, p_2 and p_3 in X from $\pi_I(X)$ but we lost the information of p_4. Similarly, the morphism π_J with $J = \{2,3,4\}$ contracts X_1 and we lost the information of the points p_1 and p_2, but the position of the points p_3 and p_4 can be recovered from $\pi_J(X)$. By using all possible subsets $|I| = 3$ we can recover the initial configuration of points in X uniquely.

The following argument is essential the one used in above example.

Proof. (of Theorem 1.5) We first show the statement for the open loci $T^0_{d,n}$ that parametrizes n distinct points. Let X be one of those stable rooted trees, and let $p_1, \ldots p_n$ be its marked points. For the sake of clarity and only in this paragraph we write the argument for $k = 3$, the one for general k follows verbatim. Select a set I with three indices say $I = \{1,2,j\}$. Recall that $\pi_I : T_{d,n} \to T_{d,|I|}$, the support of both X and $\pi_I(X)$ is $\mathbb{P}^d$, and without loss of generality, we can fix the same position for p_1 and p_2 in both X and $\pi_I(X)$. The key observation is that fixing p_1 and p_2 fixes the location of p_j in both X and $\pi_I(X)$ completely. The situation is identical to the one for $M_{0,n}$ where fixing three points in a $\mathbb{P}^1$ assigns unique coordinates to the other $(n-3)$ points in that projective line. Then, we can uniquely recover the coordinates of p_j in X from $\pi_I(X)$. Since we are considering *all* subsets I with $|I| = 3$, we recover uniquely all points in the stable tree X from their images $\pi_I(X)$.

Next, we consider a stable rooted tree $X = \cup_v X_v$ parametrized by the boundary. Let k be a fixed integer with $3 \leq k \leq n$ and let $I(v)$ be the set of indices of the marked points contained in the component X_v. For instance, in Example 5.4, we have $I(1) = \{1,2\}$, $I(2) = \{3\}$ and $I(3) = \{4\}$. Suppose that X_v is a component such that $|I(v)| \leq k$. Then, there is a set of indices K with $|K| = k$ such that π_K leaves the positions of the points in X_v unchanged because we can choose it to be $I(v) \subset K$. This means that from $X \to \pi_K(X)$, we recover all the points $p_i \in X_v$. For instance in Example 5.4, the set $K = \{1,2,3\}$ allows us to recover the points in the component X_1. Next, suppose that $X_{\tilde{v}}$ is a component such that $3 \leq k < |I(\tilde{v})|$. If we choose a $J \subset I(\tilde{v})$ then it holds that $X \to \pi_J(X) \cong \mathbb{P}^d$. We can uniquely determine the points in $X_{\tilde{v}}$ by using *all* the indices J such that $J \subset I(\tilde{v})$ and $|J| = k$. The argument is the same as the one used in the previous paragraph: Fixing two points, say p_{i_1} and p_{i_2} in both $X_{\tilde{v}}$ and $\pi_J(X)$, will completely determine the positions of all p_i with $i \in J$. Therefore, the position of the points in any component of X can be recovered by considering all such subsets J and our statement follows.

Finally, we treat $k = 2$. The problem is that we cannot distinguish configurations where all points are collinear. Let $I \subset \{1,\ldots,n\}$ be a subset of two elements and let $l(I)$ be the line in $\mathbb{P}^d$ generated by two points p_i with $i \in I$. Notice that the image of forgetful morphism $\pi_I : T^o_{d,n} \to T_{d,|I|} \cong \mathbb{P}^{d-1}$ is defined by intersecting the line $l(I)$ with the root H. Indeed, without loss of generality we may take

$I = \{1,2\}$. After making an appropriate translation, we may assume $p_1 = [1:0:\cdots:0]$. After this choice, our automorphism group is $\mathbb{G}_m$. This $\mathbb{G}_m$-action fixes both the root and the point p_1. It acts on the line $l(I)$ by translating p_2 along it. Therefore, all pairs of distinct points p_1, p_2 supported on $l(I)$ define the same G-orbit; and we can take $l(I) \cap H$ to be the image of X in $\mathbb{P}^{d-1}$. The other points are not relevant because we assign weight zero to them. The above argument implies that the product of forgetful morphisms π_2 is generated by the intersection of the root H with the lines $l(I)$ such that $I \subset \{1\ldots,n\}$ and $|I| = 2$. If the n points are collinear, there is only one line $l(I)$ generated by all points. This line intersects the root at the same point regardless of the points' positions inside $l(I)$. The loci parametrizing configurations of n collinear distinct points is positive dimensional in $T^0_{d,n}$ for any $n \geq 3$ and it will be contracted by π_2. □

5.2. **Toric Compactificiations.** There is a toric compactification of $M_{0,n}$ known as the Losev-Manin space [LM00]. This toric model can be identified with Hassett's moduli space of weighted stable curves for the set of weights $\mathscr{A}_{LM} = (1,1,\varepsilon,\ldots,\varepsilon)$, and its associated polytope is known as the Permutahedron (see [Has03, Sec 6]). Similarly, we can also find toric models for our configuration spaces by choosing appropriate weights. These models are maximal in the sense that any other toric model obtained from a set of weights will be necessarily a blow down of them. To describe the toric model of $T_{d,n}$, we denote the rays of the fan associated to $\mathbb{P}^{d(n-1)-1}$ as

$$\{\vec{e}^1_1,\ldots,\vec{e}^d_1,\vec{e}^1_2\ldots,\vec{e}^d_2,\ldots\vec{e}^1_{n-1},\ldots,\vec{e}^d_{n-1}\} \qquad \text{with} \qquad \vec{e}^k_i \in \mathbb{Z}^{d(n-1)}/\sum_{i,k}\vec{e}^k_i = 0$$

where $\vec{e}^k_i$ has its unique non-zero entry at the index $d(i-1)+k-1$. For example, for $\mathbb{P}^3$ we have

$$\vec{e}^1_1 = (1,0,0,0), \qquad \vec{e}^2_1 = (0,1,0,0), \qquad \vec{e}^1_2 = (0,0,1,0), \qquad ,\vec{e}^2_2 = (0,0,0,1)$$

Corollary 5.5. *The compactification $T^{LM}_{d,n}$ associated to the set of weights $\{1,\varepsilon,\ldots\varepsilon\} \in \mathscr{D}^T_{d,n}$ is a toric variety whose fan has rays of the form $\vec{e}^1_1,\ldots,\vec{e}^d_{n-1}$ and $\sum_{i\in I}\left(\vec{e}^1_i+\ldots+\vec{e}^d_i\right)$ where $1 \leq |I| \leq n-2$ and $I \subsetneq \{1,\ldots,n-1\}$.*

Proof. The building set associated to this set of weights is the largest one contained in the toric boundary of $\mathbb{P}^{d(n-1)-1}$. This claim follows at once by restricting the building set of Corollary 4.21, i.e. the set where all weights are equal to 1, to the toric boundary of $\mathbb{P}^{d(n-1)-1}$. In the notation of Lemma 4.15, the center of the blow ups have the form:

$$[0:\ldots 0, a_{k_1 1}:\ldots:a_{k_1 d}:\ldots:a_{k_s 1}:\ldots:a_{k_s(n-1)}:0\ldots 0] \qquad k_r \in \{1,\ldots,n-1\}\setminus I$$

for some $I \subset \{1,\ldots,n-1\}$ with $1 \leq |I| \leq n-2$. Each center is the intersection of divisors associated to the rays $\vec{e}^1_{i_k},\ldots\vec{e}^d_{i_k}$ where $i_k \in I$. The wonderful compactification involves blowing up these loci, each of which generates a divisor associated to the ray $\sum_{i\in I}\left(\vec{e}^1_i+\ldots+\vec{e}^d_i\right)$, because the blow up is smooth. □

Example 5.6. $T^{LM}_{2,3}$ is the blow up of $\mathbb{P}^3$ along two disjoint lines. $T^{LM}_{2,4}$ is the sequence of successive blow ups of $\mathbb{P}^5$ along three disjoint lines followed by the blowups of the strict transforms of three $\mathbb{P}^3$'s.

Next, we describe the toric model of $\overline{P}_{d,n}$. We denote the rays of the fan of $\left(\mathbb{P}^{n-d-2}\right)^d$ as $\{e^i_{d+2},\ldots e^i_n\}$ where $e^i_k \in \mathbb{Z}^{d(n-1)-1}$ with $1 \leq i \leq d$.

Corollary 5.7. *The compactification $\overline{P}^{LM}_{d,n}$ associated to the weights $a_1 = \ldots = a_{d+1} = 1, a_{d+2} = \ldots = a_n = \varepsilon$ is a toric variety whose fan has rays of the form $e^i_{d+2}, \ldots, e^i_n$ and $\sum_{i \in I} \left(e^1_i + \ldots + e^d_i\right)$ where $1 \leq |I| \leq n-d-2$ and $I \subset \{d+2, \ldots, n\}$.*

Proof. The proof is the same as the proof of Corollary 5.5, only now we use Corollary 1.4. By Lemma 4.8, $\overline{P}^{LM}_{d,n}$ is the wonderful compactification of the building set associated to the weights of the statement. Therefore, by Theorem 3.4 and Corollary 4.9, $\overline{P}^{\mathscr{A}}_{d,n}$ is a sequence of blowups of $\left(\mathbb{P}^{n-d-2}\right)^d$, whose centers are the dominant transforms of the elements of that building set. In the notation of Lemmas 4.1 and 4.6, these centers have the form:

$$\prod_{k=0}^{d-1} [0 \ldots 0 : b^k_{r_1} : \ldots : b^k_{r_s} : 0 \ldots 0] \qquad\qquad r_s \in \{d+2, \ldots n\} \setminus I$$

where $1 \leq |I| \leq n-d-2$. Each center is the intersection of the divisors associated to the rays $e^1_{i_1}, \ldots e^1_{i_s}$ with $i_k \in I$. Then, the rays obtained by blowing up these loci are the ones in the statement. $\square$

Example 5.8. $\overline{P}^{LM}_{2,5}$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two points. $\overline{P}^{LM}_{2,6}$ is the sequence of successive blow ups of $\mathbb{P}^2 \times \mathbb{P}^2$ at three points followed by the strict transforms of three $\mathbb{P}^1 \times \mathbb{P}^1$'s.

Proof of Theorem 1.3: Immediate from Proposition 5.1 and Corollaries 5.5, 5.7. $\square$

6. Appendix

We complete the details required for the proof of Theorem 1.2 (2). First, we state the following basic lemma, which we will use in the sequel.

Lemma 6.1. *Let Z be a smooth subvariety of a smooth variety Y and let $\pi : Bl_Z Y \to Y$ be the blowup, with exceptional divisor $E = \pi^{-1}(Z)$.*

(1) Let V be a smooth subvariety of Y, not contained in Z, and let $\widetilde{V} \subset Bl_Z Y$ be its strict transform. Then,

(a) if V meets Z transversally (or is disjoint from Z), then $\widetilde{V} = \pi^{-1}(V)$ and $\mathscr{I}_{\pi^{-1}(V)} = \mathscr{I}_{\widetilde{V}}$. Moreover

$$N_{\widetilde{V}/Bl_Z Y} \cong \pi^* N_{V/Y}$$

(b) if $V \supset Z$, then $\mathscr{I}_{\pi^{-1}(V)} = \mathscr{I}_{\widetilde{V}} \cdot \mathscr{I}_E$. Moreover

$$N_{\widetilde{V}/Bl_Z Y} \cong \pi^* N_{V/Y} \otimes \mathscr{O}(E)$$

Also, if Z has codimension 1 in V, the projection from $\widetilde{V}$ to V is an isomorphism.

(2) Let Z_1, Z_2 be smooth subvarieties of Y intersecting transversally.

(a) Assume $Z_1 \cap Z_2 \supseteq Z$. Then their strict transforms $\widetilde{Z}_1$ and $\widetilde{Z}_2$ intersect transversally and $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \widetilde{Z_1 \cap Z_2}$; in particular, if $Z_1 \cap Z_2 = Z$, then $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \emptyset$.

(b) Assume Z intersects transversally with Z_1 and Z_2, as well as with their intersection $Z_1 \cap Z_2$. Then their strict transforms $\widetilde{Z}_1$ and $\widetilde{Z}_2$ intersect transversally and $\widetilde{Z}_1 \cap \widetilde{Z}_2 =$

$\widetilde{Z_1 \cap Z_2}$.

(c) *If $Z_1 \supseteq Z$ and Z_2 intersects transversally with Z, then the intersections*

$$\widetilde{Z}_1 \cap \widetilde{Z}_2 \text{ and } (E \cap \widetilde{Z}_1) \cap \widetilde{Z}_2$$

are transversal. Moreover, $\widetilde{Z}_1 \cap \widetilde{Z}_2 = \widetilde{Z_1 \cap Z_2}$.

Proof. (1) is standard; the proof of (2) follows from [Li09, Lemma 2.9]. □

In order to prove Lemma 4.25, we give a description of the iterated strict transform of δ_I in $\mathbf{P}^{[3]}$. To this end, we consider the intersections of the iterated strict transforms of δ_I with the centers of each of the blowups in the sequence $\mathbf{P}^{[3]} \to \mathbb{P}^{d(n-1)-1}$, that is intersections of the form $\widetilde{\delta}_I \cap \widetilde{\delta}_J$, where $\delta_J \in \mathscr{H}_i$, $i = 1,2,3$ (here the $\widetilde{}$ sign above a variety stands for its strict transform). In addition, we distinguish two *types of subvarieties* of δ_I under the identification of δ_I with

$$\mathbb{P}^{d(n-|I|)-1} = [x_{11}; x_{12}; \ldots; x_{1d}; \ldots; x_{21}; x_{22}; \ldots x_{2d}; \ldots; x_{(n-|I|)1}; x_{(n-|I|)2}; \ldots; x_{(n-|I|)d}]$$

as follows

(a) Let $\delta_J \in \mathscr{H}_1$ (i.e J contains I). In this case $\delta_J \cap \delta_I = \delta_J \cong V(\{x_{ij}\} | i \in J, j = 1, \ldots d) \cong \mathbb{P}^{d(n-|I|-|J|)-1}$

(b) Let $\delta_J \in \mathscr{H}_2$ (i.e J is disjoint from I). In this case $\delta_J \cap \delta_I \cong V(\{x_{ik} - x_{jk}\} | i, j \in J, k = 1, \ldots d) \cong \mathbb{P}^{d(n-|I|-|J|)-1}$

Now consider the set of the above subvarieties of $\delta_I = \mathbb{P}^{d(n-|I|)-1}$ of type (a) and (b) with partial order compatible with the partial order $(\prec)$ above, that is:

- subvarieties of the same type are given an ascending dimension order and
- any subvariety of type (a) is smaller than any subvariety of type (b).

Remark 6.2. It is clear that the subvarieties of $\delta_I = \mathbb{P}^{d(n-|I|)-1}$ of type (a) and (b) are precisely the *loci of coincident points* in $\mathbb{P}^{d(n-|I|)-1}$ introduced in Definition 4.15, for input data $d, (N \setminus I) \cup \{n+1\}$ and $\mathscr{A}_+(I^c)$ as in Theorem 1.2 and Definition 2.1. In other words, they are the elements of the set $\mathscr{H}_{\mathscr{A}_+(I^c)}$, however listed in a different order compatible with $(\prec)$, which we also denote by $(\prec)$.

Lemma 6.3. *The ordered set $(\mathscr{H}_{\mathscr{A}_+(I^c)}, \prec)$ satisfies the condition of Theorem 3.4 (2).*

Proof. The proof is almost identical to the proof of Lemma 4.17, so we omit it. □

Proof Lemma 4.25 Consider the sequence of blowups $\mathbf{P}^{[1]} \to \mathbb{P}^{d(n-1)-1}$ of $\mathbb{P}^{d(n-1)-1}$ along all varieties in $\mathscr{H}_1$ (cf definition 4.23) and let $\widetilde{V}$ denote the dominant transform of an arbitrary variety $V \subset \mathbb{P}^{d(n-1)-1}$ in $\mathbf{P}^{[1]}$. In order to prove the claim, it suffices, by the above remark and Lemma 6.3, to show that

(i) $\widetilde{\delta_I \cap \delta_J} = \widetilde{\delta}_I \cap \widetilde{\delta}_J$ for any $\delta_J \in \mathscr{H}_2$ and

(ii) $\widetilde{\delta}_I \cap \widetilde{\delta}_J = \emptyset$ for any $\delta_J \in \mathscr{H}_3$

For (i), let $k > 0$ and consider the center of the k-th blowup P_k in the sequence of blowups $\mathbf{P}^{[1]} \to \cdots \to P_k \to \ldots \mathbb{P}^{d(n-1)-1}$; we denote by $V^{(k)}$ the dominant transform of an arbitrary variety $V \subset \mathbb{P}^{d(n-1)-1}$ in P_k. Then the center of is a minimal element of the set $(\mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3)^{(k-1)} :=$

$\{\delta_J{}^{(k-1)} \subset P_{k-1} \mid \delta_J \in \mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3\}$ with order $(\prec)$. Therefore, by repeated use of [Li09, Proposition 2.8], we infer that the set $(\mathscr{H}_1 \cup \mathscr{H}_2 \cup \mathscr{H}_3)^{(k)}$ with partial order $(\prec)$ is a building set for all the above k. Then, by [Li09, Lemma 2.6(i)], each iterated strict transform of $\delta_J \in \mathscr{H}_2$ in P_k must either intersect the corresponding center transversally or it must contain that center. Also, observe that for $\delta_J \in \mathscr{H}_2$ we have that the intersection $\delta_I \cap \delta_J$ is transversal. Moreover, each iterated strict transform of δ_I in P_k contains the corresponding center by the assumptions. Therefore, by repeated use of Lemma 6.1(2)(a) and (c) we deduce the validity of (i).

Now we prove (ii). For any $\delta_J \in \mathscr{H}_3$, by definition, the set J overlaps with I. Consider an element $j \in J \cap I$ and set $J' := j \cup (J \setminus I)$. Since $\widetilde{\delta_{J'}} \supseteq \widetilde{\delta}_J$, it is enough to show that $\widetilde{\delta}_I \cap \widetilde{\delta_{J'}} = \emptyset$. While $\delta_{J'}$ may not be an element of $\mathscr{H}_{\mathscr{A}}$, the argument as in Lemma 4.17 shows that the set $\mathscr{H}_1 \cup \{\delta_{J'}\}$, with $\mathscr{H}_1$ given an ascending dimension order and δ_J listed last, is a building set. Now the intersection $\delta_I \cap \delta_J' = \delta_{I \cup J'}$ is in $\mathscr{H}_1$; assume it is the m-th element of that set with respect to $(\prec)$ for some $m > 0$. Consider the $(m-1)$-th blowup $P_{m-1} \to \mathbb{P}^{d(n-1)-1}$ and the iterated strict transforms $\delta_I^{(m-1)}$ and $\delta_{J'}^{(m-1)}$ of δ_I and $\delta_{J'}$ respectively. Since the intersection $\delta_I \cap \delta_J'$ is transversal, by using the argument of the previous paragraph we see that $\delta_I^{(m-1)} \cap \delta_{J'}^{(m-1)} = \delta_{I \cup J'}^{(m-1)}$. Now consider the m-th blowup $P_m \to P_{m-1}$. By Lemma 6.1(2), we deduce that $\delta_I^{(m)} \cap \delta_{J'}^{(m)} = \emptyset$, so $\widetilde{\delta}_I \cap \widetilde{\delta_{J'}} = \emptyset$ in $\mathbf{P}^{[1]}$. $\square$

Proof of Lemma 4.26 First, note that the normal bundle of $\delta_I \cong \mathbb{P}^{d(n-|I|)-1}$ in $\mathbb{P}^{d(n-1)-1}$ is isomorphic to

$$\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\mathbb{P}^{d(n-|I|)-1}}(-1)$$

Consider the sequence of blowups $\mathbf{P}^{[1]} \to \mathbb{P}^{d(n-1)-1}$ along all varieties in $\mathscr{H}_1$ (with the order $(\prec)$ given above) and let us also denote the strict transform of δ_I in $\mathbf{P}^{[1]}$ by $\widetilde{\delta}_I$. Clearly $\mathscr{O}_{\mathbb{P}^{d(n-1)-1}}(-1)$ pulls back to $\mathscr{O}_{\mathbf{P}^{[1]}}(-1)$ on $\mathbf{P}^{[1]}$, so $\mathscr{O}_{\mathbb{P}^{d(n-|I|)-1}}(-1)$ pulls back to $\mathscr{O}_{\widetilde{\delta}_I}(-1)$ on $\widetilde{\delta}_I$. Therefore, by a repeated application of Lemma 6.11(b) we see that the normal bundle of the iterated strict transform of δ_I in $\mathbf{P}^{[1]}$ is isomorphic to

$$\left(\bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\widetilde{\delta}_I}(-1)\right) \otimes \underbrace{\mathscr{O}_{\widetilde{\delta}_I}(1) \otimes \cdots \otimes \mathscr{O}_{\widetilde{\delta}_I}(1)}_{p\,\text{times}} = \bigoplus_{i=1}^{d(|I|-1)} \mathscr{O}_{\widetilde{\delta}_I}(p-1)$$

Now, as we saw in the proof of Lemma 4.25, each stepwise strict transform of δ_I in the sequence of blowups $\mathbf{P}^{[3]} \to \mathbf{P}^{[1]}$ meets each of the centers (corresponding to $\mathscr{H}_2 \cup \mathscr{H}_3$) transversally. Therefore, by applying Lemma 6.1(1)(a) we complete the proof.

(2) The proof of (2) is identical to the proof of (1) and is therefore omitted. $\square$

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