TORSION EXCEPTIONAL SHEAVES ON WEAK DEL PEZZO SURFACES OF TYPE A

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Abstract. We investigate torsion exceptional sheaves on a weak del Pezzo surface of degree greater than two whose anticanonical model has at most A_n-singularities. We show that every torsion exceptional sheaf can be obtained from a line bundle on a (-1)-curve by acting spherical twist functors successively.

1. Introduction

Let X be a smooth projective variety and $D(X)(= D^b(\mathrm{Coh}X))$ the bounded derived category of coherent sheaves on X. The category $D(X)$ carries a lot of geometric information on X and has drawn many interests in the study of algebraic varieties. An object $\alpha \in D(X)$ is called *exceptional* if

$$\mathrm{Hom}(\alpha, \alpha[i]) \cong \begin{cases} \mathbb{C} & i = 0; \\ 0 & i \neq 0. \end{cases}$$

Exceptional objects are related to semi-orthogonal decompositions of derived categories and appear in many context (see, for example, [4]). Hence it is natural to consider the classification of exceptional objects.

Exceptional objects on *del Pezzo surfaces* (i.e., smooth projective surfaces with ample anticanonical bundles) were investigated by Kuleshov and Orlov in [8] where they proved that any exceptional object on a del Pezzo surface is isomorphic to a shift of an exceptional vector bundle or a line bundle on a (-1)-curve.

As exceptional objects on del Pezzo surfaces are well-understood, it is natural to consider *weak del Pezzo surfaces* (i.e., smooth projective surfaces with nef and big anticanonical bundles). In this case something interesting happens since twist functors (see Definition 2.3) are involved due to the existence of (-2)-curves on weak del Pezzo surfaces. We could not expect that exceptional objects are as such simple as those on del Pezzo surfaces (see Section 6), but still we expect that they are so after acting autoequivalences of the derived category.

Conjecture 1.1 (cf. [9, Conjecture 1.3]). *Let X be a weak del Pezzo surface. For any exceptional object $\mathcal{E} \in D(X)$, there exists an autoequivalence*

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$\Phi \in \mathrm{Auteq}(D(X))$ *such that* $\Phi(\mathcal{E})$ *is an exceptional vector bundle, or a line bundle on a* (-1)*-curve on* X.

Recently, Okawa and Uehara [9] considered Hirzebruch surface $\mathbb{F}_2$, the simplest weak del Pezzo surface. They classified exceptional sheaves on $\mathbb{F}_2$ and confirmed Conjecture 1.1 for those sheaves. Note that on $\mathbb{F}_2$, there is no torsion exceptional sheaf due to the absence of (-1)-curves. Motivated by Okawa–Uehara's work and this observation, we are interested in torsion exceptional sheaves (and objects) on weak del Pezzo surfaces. In [1], the first author treated torsion exceptional sheaves and objects on a weak del Pezzo surface of degree greater than one whose anticanonical model has at most A_1-singularities.

On the other hand, one may compare Conjecture 1.1 to [5, Proposition 1.6], where Ishii and Uehara showed that a spherical object on the minimal resolution of an A_n-singularity on a surface can be obtained from a line bundle on a (-2)-curve by autoequivalence. But the situation for torsion exceptional objects seems to be more complicated since its scheme theoretic support might be non-reduced (see Example 6.2) while the support of such a spherical object is always reduced (see [5, Corollary 4.10]).

In this article, we give an affirmative answer to Conjecture 1.1 in the case of torsion exceptional sheaves on weak del Pezzo surfaces of degree greater than two of *Type A* (i.e., those whose anticanonical model has at most A_n-singularities). Namely, we prove the following theorem.

Theorem 1.2. *Let* X *be a weak del Pezzo surface of degree* $d > 2$ *of Type A, and* $\mathcal{E}$ *a torsion exceptional sheaf on* X. *Then there exist a* (-1)*-curve* D, *an integer* d, *and a sequence of spherical twist functors* $\Phi_1, \ldots, \Phi_n$ *associated to line bundles on chains of* (-2)*-curves such that*

$$\mathcal{E} \cong \Phi_1 \circ \cdots \circ \Phi_n(\mathcal{O}_D(d)).$$

In fact, we can prove the following slightly general theorem on torsion exceptional sheaves.

Theorem 1.3. *Let* X *be a smooth projective surface and* $\mathcal{E}$ *a torsion exceptional sheaf on* X. *Assume the following conditions hold:*

1. $\mathrm{supp}(\mathcal{E})$ *only contains one* (-1)*-curve* D *and* (-2)*-curves;*
2. *The restriction of* $\mathcal{E}$ *in* D *is a line bundle (see Definition-Proposition 2.8);*
3. (-2)*-curves in* $\mathrm{supp}(\mathcal{E})$ *forms disjoint union of* A_n*-configurations with* $n \leq 6$*;*
4. *The intersection of* D *with any chain of* (-2)*-curves in* $\mathrm{supp}(\mathcal{E})$ *is at most one.*

Then there exist an integer d, *and a sequence of spherical twist functors* $\Phi_1, \ldots, \Phi_n$ *associated to line bundles on chains of* (-2)*-curves such that*

$$\mathcal{E} \cong \Phi_1 \circ \cdots \circ \Phi_n(\mathcal{O}_D(d)).$$

The idea of the proof is based on one observation that, under some good conditions, we can "factor" a spherical sheaf out of a torsion exceptional sheaf to get another one (see Lemmas 2.7 and 5.1), and this step actually corresponds to acting a spherical twist functor. After this factorization, we get an exceptional sheaf with "smaller" support. Keeping factoring spherical sheaves out, eventually we get an exceptional sheaf supported on a (-1)-curve. To check the conditions that allow us to factor out spherical sheaves, we need a detailed investigation on the classification of certain torsion rigid sheaves supported in (-2)-curves (See Section 4). We expect that this idea also works for torsion exceptional sheaves on arbitrary weak del Pezzo surfaces.

Notation and Conventions. We work over the complex number field $\mathbb{C}$. Let X be a smooth projective surface. For a coherent sheaf $\mathcal{E}$ on X, we denote by $\operatorname{supp}(\mathcal{E})$ the support of $\mathcal{E}$ with reduced induced scheme structure. For $\mathcal{E}, \mathcal{F} \in D(X)$, we denote

$$h^i(\mathcal{E}, \mathcal{F}) := \dim \operatorname{Ext}^i(\mathcal{E}, \mathcal{F}) = \dim \operatorname{Hom}(\mathcal{E}, \mathcal{F}[i]),$$

and the Euler characteristic

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i h^i(\mathcal{E}, \mathcal{F}).$$

A (-1)-curve (resp. (-2)-curve) is a smooth rational curve on X with self-intersection number -1 (resp. -2). We say $Z = C_1 \cup \cdots \cup C_n$ is a chain of (-2)-curves on X if C_i is a (-2)-curve and

$$C_i \cdot C_j = \begin{cases} 1 & |i-j| = 1; \\ 0 & |i-j| > 1. \end{cases}$$

We regard Z as a closed subscheme of X with respect to the reduced induced structure. Sometimes we also regard Z as its fundamental cycle $C_1 + \cdots + C_n$. For a coherent sheaf $\mathcal{R}$ on Z, we denote by $\deg_{C_l} \mathcal{R}$ the degree of the restriction $\mathcal{R}|_{C_l}$ on $C_l \cong \mathbb{P}^1$. We denote by

$$\mathcal{R}_0 = \mathcal{O}_{C_1 \cup \cdots \cup C_n}(a_1, \ldots, a_n)$$

the line bundle on Z such that $\deg_{C_l} \mathcal{R}_0 = a_l$ for all l. Sometimes we also consider

$$\mathcal{R}_1 = \mathcal{O}_{r_1 C_1 \cup \cdots \cup r_n C_n}(a_1, \ldots, a_n)$$

for $r_l \in \{1, 2\}$ for all l. Here $\mathcal{R}_1$ is the line bundle on $r_1 C_1 \cup \cdots \cup r_n C_n$ such that $\deg_{C_l} \mathcal{R}_1 = a_l$ for all l. In other words, $\mathcal{R}_1|_{r_l C_l} \cong \mathcal{O}_{r_l C_l}(a_l)$, where $\mathcal{O}_{2C}(a)$ is the unique non-trivial extension of $\mathcal{O}_C(a)$ by $\mathcal{O}_C(a+2)$ for a (-2)-curve C on X.

2. Preliminaries

2.1. **Exceptional and spherical objects.** We recall the definition of exceptional and spherical objects.

Definition 2.1. Let X be a smooth projective variety. We say that an object $\alpha \in D(X)$ is *exceptional* if

$$\operatorname{Hom}(\alpha, \alpha[i]) \cong \begin{cases} \mathbb{C} & i = 0; \\ 0 & i \neq 0. \end{cases}$$

Example 2.2. (1) Let X be a smooth projective variety with

$$H^i(X, \mathcal{O}_X) = 0$$

for $i > 0$ (e.g. Fano manifolds). Then every vector bundle on X is an exceptional object.

(2) Let X be a smooth projective surface and C a (-1)-curve on X. Then any line bundle on C is an exceptional object.

Definition 2.3 ([10])**.** Let X be a smooth projective variety.

(1) We say that an object $\alpha \in D(X)$ is *spherical* if $\alpha \otimes \omega_X \cong \alpha$ and

$$\mathrm{Hom}(\alpha, \alpha[i]) \cong \begin{cases} \mathbb{C} & i = 0, \dim X; \\ 0 & i \neq 0, \dim X. \end{cases}$$

(2) Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$\mathcal{C} = \mathrm{Cone}(\pi_1^*\alpha^\vee \otimes \pi_2^*\alpha \to \mathcal{O}_\Delta)$$

of the natural evaluation $\pi_1^*\alpha^\vee \otimes \pi_2^*\alpha \to \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal and π_i is the projection from $X \times X$ to the i-th factor. Then the integral functor $T_\alpha := \Phi^{\mathcal{C}}_{X \to X}$ defines an autoequivalence of $D(X)$, called the *twist functor* associated to the spherical object α. By definition, for $\beta \in D(X)$, we have an exact triangle

$$\mathbb{R}\mathrm{Hom}(\alpha, \beta) \otimes \alpha \xrightarrow{\text{evaluation}} \beta \to T_\alpha \beta.$$

Example 2.4 (cf. [5, Example 4.7])**.** Let X be a smooth projective surface and Z a chain of (-2)-curves. Then any line bundle on Z is a spherical object in $D(X)$.

2.2. **Rigid sheaves.** In this subsection, we assume that X is a smooth projective surface. All sheaves are considered to be coherent on X.

A coherent sheaf $\mathcal{R}$ is said to be *rigid* if $h^1(\mathcal{R}, \mathcal{R}) = 0$.

Kuleshov [7] systematically investigated rigid sheaves on surfaces with anticanonical class without base components. We collect some interesting properties for rigid sheaves in this subsection for application. We will use the following easy lemma without mention.

Lemma 2.5. *Consider an extension of coherent sheaves*

$$0 \to \mathcal{G}_2 \to \mathcal{R} \to \mathcal{G}_1 \to 0$$

such that $\mathcal{R}$ is rigid. Then $h^1(\mathcal{G}_1, \mathcal{G}_2) > 0$ if and only if this extension is non-trivial.

Proof. The 'if' part is trivial. For the 'only if' part, assume that $h^1(\mathcal{G}_1, \mathcal{G}_2) > 0$ and this extension is trivial, then

$$\mathcal{R} \cong \mathcal{G}_1 \oplus \mathcal{G}_2,$$

which implies that

$$h^1(\mathcal{R}, \mathcal{R}) \geq h^1(\mathcal{G}_1, \mathcal{G}_2) > 0,$$

a contradiction □

We have the following Mukai's lemma for rigid sheaves.

Lemma 2.6 (Mukai's lemma, [8, 2.2 Lemma])**.** *For each exact sequence*

$$0 \to \mathcal{G}_2 \to \mathcal{R} \to \mathcal{G}_1 \to 0$$

of coherent sheaves such that

$$h^1(\mathcal{R},\mathcal{R}) = h^0(\mathcal{G}_2,\mathcal{G}_1) = h^2(\mathcal{G}_1,\mathcal{G}_2) = 0,$$

the following hold:

(1) $h^1(\mathcal{G}_1,\mathcal{G}_1) = h^1(\mathcal{G}_2,\mathcal{G}_2) = 0$;
(2) $h^0(\mathcal{R},\mathcal{R}) = h^0(\mathcal{G}_1,\mathcal{G}_1) + h^0(\mathcal{G}_2,\mathcal{G}_2) + \chi(\mathcal{G}_1,\mathcal{G}_2)$;
(3) $h^2(\mathcal{R},\mathcal{R}) = h^2(\mathcal{G}_1,\mathcal{G}_1) + h^2(\mathcal{G}_2,\mathcal{G}_2) + \chi(\mathcal{G}_2,\mathcal{G}_1)$;
(4) $h^1(\mathcal{G}_1,\mathcal{G}_2) \leq h^0(\mathcal{G}_1,\mathcal{G}_1) + h^0(\mathcal{G}_2,\mathcal{G}_2) - 1$.

Proof. (1)-(3) are from [8, 2.2 Lemma]. We prove (4) here. In the proof of [8, 2.2 Lemma], we know that the natural map

$$\mathrm{Hom}(\mathcal{G}_1,\mathcal{G}_1) \oplus \mathrm{Hom}(\mathcal{G}_2,\mathcal{G}_2) \xrightarrow{d_1} \mathrm{Ext}^1(\mathcal{G}_1,\mathcal{G}_2)$$

is surjective. Note that the image of $(\mathrm{id}_{\mathcal{G}_1}, \mathrm{id}_{\mathcal{G}_2})$ is zero. Hence we get the inequality by comparing the dimensions. □

Lemma 2.7. *Consider an exact sequence of coherent sheaves*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{S} \to 0,$$

where $\mathcal{E}$ is rigid, $\mathcal{S}$ is spherical, $h^0(\mathcal{E}',\mathcal{S}) = 0$, and $\chi(\mathcal{S},\mathcal{E}') = -1$. Then $h^i(\mathcal{E}',\mathcal{E}') = h^i(\mathcal{E},\mathcal{E})$ for $i = 0, 1, 2$. In particular, $\mathcal{E}$ is exceptional if and only if so is $\mathcal{E}'$, and in this case, $\mathcal{E} \cong T_{\mathcal{S}}(\mathcal{E}')$.

Proof. Since $\mathcal{S}$ is spherical, $h^2(\mathcal{S},\mathcal{E}') = 0$ and $\chi(\mathcal{E}',\mathcal{S}) = -1$ by Serre duality. By Lemma 2.6, $h^i(\mathcal{E}',\mathcal{E}') = h^i(\mathcal{E},\mathcal{E})$ for $i = 0, 1, 2$. In particular, $\mathcal{E}$ is exceptional if and only if so is $\mathcal{E}'$.

Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are exceptional, then by Lemma 2.6(4),

$$h^1(\mathcal{S},\mathcal{E}') \leq h^0(\mathcal{S},\mathcal{S}) + h^0(\mathcal{E}',\mathcal{E}') - 1 = 1.$$

Since $\chi(\mathcal{S},\mathcal{E}') = -1$, we have $h^1(\mathcal{S},\mathcal{E}') = 1$ and $h^0(\mathcal{S},\mathcal{E}') = h^2(\mathcal{S},\mathcal{E}') = 0$. By definition of twist functor, we have a distinguished triangle

$$\mathcal{S}[-1] \to \mathcal{E}' \to T_{\mathcal{S}}(\mathcal{E}'),$$

which corresponds to the exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{S} \to 0.$$

Hence $\mathcal{E} \cong T_{\mathcal{S}}(\mathcal{E}')$. □

We can say more about torsion rigid sheaves.

Definition-Proposition 2.8 (Restriction in curves)**.** Let $\mathcal{R}$ be a torsion rigid sheaf, then $\mathcal{R}$ is pure one-dimensional by [7, Corollary 2.2.3]. Suppose that $\mathrm{supp}(\mathcal{R}) = Z \cup Z'$ where Z and Z' are unions of curves with no common components. Then there exists an exact sequence

$$0 \to \mathcal{R}_{Z'} \to \mathcal{R} \to \mathcal{R}_Z \to 0$$

where $\mathcal{R}_{Z'} = \mathcal{H}^0_{Z'}(\mathcal{R})$ is the subsheaf with supports (see [3, II, Ex. 1.20]) in Z' and $\mathcal{R}_Z$ is the quotient sheaf. Then $\mathrm{supp}(\mathcal{R}_{Z'}) = Z'$, $\mathrm{supp}(\mathcal{R}_Z) = Z$, and

$$h^0(\mathcal{R}_{Z'},\mathcal{R}_Z) = h^2(\mathcal{R}_Z,\mathcal{R}_{Z'}) = 0$$

by the support condition. By Lemma 2.6, $\mathcal{R}_{Z'}$ and $\mathcal{R}_Z$ are torsion rigid (pure one-dimensional) sheaves. Moreover, if we write $Z = \cup_i C_i$ and $Z' = \cup_j C'_j$, then we can write the first Chern class of $\mathcal{R}$ uniquely as

$$c_1(\mathcal{R}) = \sum_i r_i C_i + \sum_j s_j C'_j$$

in the sense that $c_1(\mathcal{R}_Z) = \sum_i r_i C_i$ and $c_1(\mathcal{R}_{Z'}) = \sum_j s_j C'_j$ for some positive integers r_i and s_j. We say that $\mathcal{R}_Z$ is *the restriction of* $\mathcal{R}$ *in* Z.

Here we remark that for a chain of (-2) curves $Z = C_1 \cup \cdots \cup C_n$ and a torsion rigid sheaf $\mathcal{R}$, $\mathrm{supp}(\mathcal{R}) \subset Z$ if and only if $c_1(\mathcal{R})$ can be written as $\sum_i r_i C_i$ for non-negative integers r_i. In this case, r_i is uniquely determined for all i.

Lemma 2.9. *Let* $\mathcal{R}$ *be a torsion rigid sheaf on* X*, then any irreducible component of* $\mathrm{supp}(\mathcal{R})$ *is a curve with negative self-intersection.*

Proof. Let C be an irreducible component of $\mathrm{supp}(\mathcal{R})$, then take the restriction in C, we have an exact sequence

$$0 \to \mathcal{R}' \to \mathcal{R} \to \mathcal{R}_C \to 0.$$

By Definition-Proposition 2.8, $\mathcal{R}_C$ is rigid and in particular, $\chi(\mathcal{R}_C, \mathcal{R}_C) > 0$. On the other hand, by Riemann–Roch formula (see Subsection 2.4), we have $\chi(\mathcal{R}_C, \mathcal{R}_C) = -c_1(\mathcal{R}_C)^2$. Hence $c_1(\mathcal{R}_C)^2 < 0$, which implies $C^2 < 0$. □

2.3. **Weak del Pezzo surfaces.** A smooth projective surface X is a *weak del Pezzo surface* if $-K_X$ is nef and big. The *degree* d of X is the self-intersection number $(-K_X)^2$. We say a weak del Pezzo surface is of *Type A* if its anticanonical model has at most A_n-singularities. We collect some basic facts on weak del Pezzo surfaces.

Lemma 2.10 (cf. [2, Theorem 8.3.2])**.** *Let* X *be a weak del Pezzo surface. Then* $|-K_X|$ *has no base components.*

Lemma 2.11. *Let* X *be a weak del Pezzo surface of degree* $d > 1$*. Then the intersection of a* (-1)*-curve with a chain of* (-2)*-curves is at most one.*

Proof. Take a chain of (-2)-curves $C_1 \cup \cdots \cup C_n$ and a (-1)-curve D. Assume that $\sum_{i=1}^n C_i \cdot D \geq 2$, then

$$\Big(\sum_{i=1}^n C_i + D\Big)^2 = \Big(\sum_{i=1}^n C_i\Big)^2 + 2\sum_{i=1}^n C_i \cdot D + D^2 \geq 1.$$

By Hodge index theorem,

$$(-K_X)^2 \cdot \Big(\sum_{i=1}^n C_i + D\Big)^2 \leq \Big((-K_X) \cdot \Big(\sum_{i=1}^n C_i + D\Big)\Big)^2 = 1.$$

This implies that $(-K_X)^2 = 1$, which is a contradiction. □

We remark that on weak del Pezzo surfaces of degree one, it is possible that one (-1)-curve intersects with a chain of (-2)-curves at two points (cf. [6, Lemma 2.8]).

2.4. **Riemann–Roch formula.** We recall Riemann–Roch formula on surfaces.

Theorem 2.12 (Riemann–Roch formula)**.** *For two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on a smooth projective surface X, the Euler characteristic can be calculated by*

$$\chi(\mathcal{E},\mathcal{F}) = r(\mathcal{E})r(\mathcal{F})\Big(\chi(\mathcal{O}_X) + \frac{1}{2}(\mu(\mathcal{F}) - \mu(\mathcal{E})) + q(\mathcal{F}) + q(\mathcal{E}) - \frac{1}{r(\mathcal{E})r(\mathcal{F})}(c_1(\mathcal{E})\cdot c_1(\mathcal{F}))\Big),$$

where $r(\mathcal{E})$ is the rank of $\mathcal{E}$ and

$$\mu(\mathcal{E}) = \frac{1}{r(\mathcal{E})}(-K_X\cdot c_1(\mathcal{E})), \quad q(\mathcal{E}) = \frac{c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})}{2r(\mathcal{E})}.$$

In this paper, we are only interested in the case of torsion sheaves, that is, $r(\mathcal{E}) = r(\mathcal{F}) = 0$. In this case, the formula is quite simple.

Corollary 2.13. *For two torsion sheaves $\mathcal{E}$ and $\mathcal{F}$ on a smooth projective surface X, the Euler characteristic can be calculated by*

$$\chi(\mathcal{E},\mathcal{F}) = -c_1(\mathcal{E})\cdot c_1(\mathcal{F}).$$

2.5. **A polynomial inequality.** In this subsection, we treat a special polynomial which naturally appears in self-intersection numbers of a union of negative curves.

For positive integers $r_1, r_2, \dots, r_n$, and $1 \le k \le n$, define the polynomial

$$f(r_1, r_2, \dots, r_n; k) = \sum_{i=1}^{n} r^2 - \sum_{i=1}^{n-1} r_i r_{i+1} - r_k.$$

Proposition 2.14. *For positive integers $r_1, r_2, \dots, r_n$, and k,*

$$f(r_1, r_2, \dots, r_n; k) \ge 0$$

always holds. Moreover, $f(r_1, r_2, \dots, r_n; k) = 0$ if and only if the following conditions hold:

1. *$r_1 = r_n = 1$,*
2. *$0 \le r_{i+1} - r_i \le 1$ if $i < k$,*
3. *$0 \le r_i - r_{i+1} \le 1$ if $i \ge k$.*

Proof. We use induction on n. The case $n = 1$ is trivial. For convenience, set $r_0 = r_{n+1} = 0$.

Suppose $n > 1$ and $f(r_1, r_2, \dots, r_n; k) \le 0$. Reversing the order of $\{r_i\}$ if necessary, we may assume that $r_n \ge r_1$. Take $l \ge 2$ to be the maximal integer such that $r_l = \max\{r_i\}$. We have

$$f(r_1, r_2, \dots, r_n; l) = f(r_1, r_2, \dots, r_n; k) + r_k - r_l \le 0.$$

On the other hand, by definition of l, $r_{l-1} \le r_l > r_{l+1}$. Hence

$$\begin{aligned} &f(r_1, r_2, \dots, r_n; l) - f(r_1, r_2, \dots, r_{l-1}, r_{l+1}, \dots, r_n; l-1) \\ &= (r_l^2 - r_{l-1}r_l - r_l r_{l+1} - r_l) + (r_{l-1}r_{l+1} + r_{l-1}) \\ &= (r_l - r_{l-1})(r_l - r_{l+1} - 1) \ge 0. \end{aligned}$$

Hence

$$f(r_1, r_2, \ldots, r_{l-1}, r_{l+1}, \ldots, r_n; l-1) \leq 0.$$

By induction,

$$f(r_1, r_2, \ldots, r_{l-1}, r_{l+1}, \ldots, r_n; l-1) = 0 \tag{2.1}$$

and all the inequalities above become equalities. That is,

$$f(r_1, r_2, \ldots, r_n; k) = 0; \tag{2.2}$$

$$r_k = r_l = \max\{r_i\}; \tag{2.3}$$

$$r_l = r_{l-1} \text{ or } r_l = r_{l+1} + 1. \tag{2.4}$$

By induction, (2.1) is equivalent with

$$r_1 = r_n = 1;$$
$$0 \leq r_{i+1} - r_i \leq 1 \text{ if } i < l-1;$$
$$0 \leq r_{l-1} - r_{l+1} \leq 1;$$
$$0 \leq r_i - r_{i+1} \leq 1 \text{ if } i \geq l+1.$$

Combining with (2.3) and (2.4), we conclude that

(1) $r_1 = r_n = 1$,
(2) $0 \leq r_{i+1} - r_i \leq 1$ if $i < k$,
(3) $0 \leq r_i - r_{i+1} \leq 1$ if $i \geq k$.

This proves that $f(r_1, r_2, \ldots, r_n; k) \geq 0$ and $f(r_1, r_2, \ldots, r_n; k) = 0$ only if conditions (1)-(3) hold. The 'if' part can be also checked by induction on k easily. We omit the proof. □

3. Factorizations of rigid sheaves

In this section, we assume that X is a smooth projective surface. All sheaves are considered to be coherent on X. We will define factorizations of rigid sheaves and give basic properties.

Definition 3.1. A coherent sheaf $\mathcal{R}$ has a *factorization*

$$\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$$

if there exists a filtration of coherent sheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathcal{R},$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{G}_i$ for $1 \leq i \leq n$ and we write this factorization as

$$\mathcal{R} \equiv \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}.$$

This factorization is said to be *perfect* if $h^0(\mathcal{G}_i, \mathcal{G}_j) = 0$ for all $i < j$.

Example 3.2 ([9, Lemma 2.4])**.** Let C be a (-2)-curve on a smooth projective surface X. Let $\mathcal{F}$ be a pure one-dimensional sheaf on the scheme mC. Then the subquotients of the Harder–Narasimhan filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^n = \mathcal{F}$$

of $\mathcal{F}$ are of the form

$$\{\mathcal{F}^1/\mathcal{F}^0, \mathcal{F}^2/\mathcal{F}^1, \ldots, \mathcal{F}^n/\mathcal{F}^{n-1}\} = \{\mathcal{O}_C(a_1)^{\oplus r_1}, \mathcal{O}_C(a_2)^{\oplus r_2}, \ldots, \mathcal{O}_C(a_n)^{\oplus r_n}\}$$

with $a_1 > a_2 > \cdots > a_n$ and $r_i > 0$, which gives a perfect factorization of $\mathcal{F}$.

The following lemma is a direct consequence of Lemma 2.6.

Lemma 3.3. *Let $\mathcal{R}$ be a rigid sheaf with a perfect factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$$

and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$ the corresponding filtration. Then $\mathcal{F}_j/\mathcal{F}_i$ is rigid for all $i < j$.

In application, we usually need to get new factorizations from old ones. Here we give some lemmas about operations on factorizations.

Lemma 3.4. *Let $\mathcal{R}$ be a coherent sheaf with a factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$$

and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$ the corresponding filtration. If $\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ has another factorization $\{\mathcal{G}'_i, \mathcal{G}'_{i+1}\}$ for some i, then $\mathcal{R}$ has another factorization

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_{i-1}, \mathcal{G}'_i, \mathcal{G}'_{i+1}, \mathcal{G}_{i+2}, \ldots, \mathcal{G}_n\}.$$

In particular, if $h^1(\mathcal{G}_{i+1}, \mathcal{G}_i) = 0$, we are free to change the order of $\mathcal{G}_{i+1}$ and $\mathcal{G}_i$ in a factorization.

Lemma 3.5. *Let $\mathcal{R}$ be a coherent sheaf with a perfect factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$$

and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$ the corresponding filtration. Then

$$\{\mathcal{F}_{n-1}, \mathcal{G}_n\}$$

is a perfect factorization of $\mathcal{R}$.

Lemma 3.6. *Let $\mathcal{G}$ and $\mathcal{S}$ be two coherent sheaves and r a positive integer. Assume that $\mathcal{S}$ is simple.*

1. *Assume $h^1(\mathcal{S}, \mathcal{G}) = 1$. Denote by $\mathcal{G}'$ the unique non-trivial extension of $\mathcal{S}$ by $\mathcal{G}$. Then $h^0(\mathcal{S}, \mathcal{G}) = h^0(\mathcal{S}, \mathcal{G}')$, and any non-trivial extension of $\mathcal{S}^{\oplus r}$ by $\mathcal{G}$ is isomorphic to $\mathcal{S}^{\oplus(r-1)} \oplus \mathcal{G}'$.*
2. *Assume $h^1(\mathcal{G}, \mathcal{S}) = 1$. Denote by $\mathcal{G}''$ the unique non-trivial extension of $\mathcal{G}$ by $\mathcal{S}$. Then $h^0(\mathcal{G}, \mathcal{S}) = h^0(\mathcal{G}'', \mathcal{S})$, and any non-trivial extension of $\mathcal{G}$ by $\mathcal{S}^{\oplus r}$ is isomorphic to $\mathcal{S}^{\oplus(r-1)} \oplus \mathcal{G}''$.*

Proof. (1) Consider the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{S}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{S}, \mathcal{G}') \to \mathrm{Hom}(\mathcal{S}, \mathcal{S}) \xrightarrow{\delta} \mathrm{Ext}^1(\mathcal{S}, \mathcal{G})$$

induced by the extension

$$0 \to \mathcal{G} \to \mathcal{G}' \to \mathcal{S} \to 0.$$

Since the extension is non-trivial and $\mathcal{S}$ is simple, the map δ is injective. And hence $h^0(\mathcal{S}, \mathcal{G}) = h^0(\mathcal{S}, \mathcal{G}')$.

Consider a non-trivial extension $\mathcal{R}$ corresponding to

$$\eta = (\eta_1, \ldots, \eta_r) \in \mathrm{Ext}^1(\mathcal{S}^{\oplus r}, \mathcal{G}) \cong \mathbb{C}^r.$$

Since $\mathcal{S}$ is simple, $\mathrm{Aut}(\mathcal{S}^{\oplus r}) = \mathrm{GL}(r, \mathbb{C})$. Since $\mathrm{Aut}(\mathcal{S}^{\oplus r})$ acts on

$$\mathrm{Ext}^1(\mathcal{S}^{\oplus r}, \mathcal{G}) \cong \mathbb{C}^r$$

through the natural action of $\mathrm{GL}(r, \mathbb{C})$, after taking an automorphism of $\mathcal{S}^{\oplus r}$, we may assume that $\eta_i = 0$ except for one index i_0. Hence $\mathcal{R} \cong \mathcal{S}^{\oplus(r-1)} \oplus \mathcal{G}'$.

(2) can be proved similarly. □

Lemma 3.7. *Let $\mathcal{R}$ be a rigid sheaf with a perfect factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{S}^{\oplus r}, \mathcal{H}_1, \ldots, \mathcal{H}_m\}.$$

Assume that $\mathcal{S}$ is spherical.

(1) *Suppose $h^0(\mathcal{S}, \mathcal{G}_n) = 0$ and $\chi(\mathcal{S}, \mathcal{G}_n) = -1$, then there is a new perfect factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_{n-1}, \mathcal{S}^{\oplus(r-1)}, \mathcal{G}'_n, \mathcal{H}_1, \ldots, \mathcal{H}_m\}.$$

Here $\mathcal{G}'_n$ is the (unique) non-trivial extension of $\mathcal{S}$ by $\mathcal{G}_n$.

(2) *Suppose $h^0(\mathcal{H}_1, \mathcal{S}) = 0$ and $\chi(\mathcal{H}_1, \mathcal{S}) = -1$, then there is a new perfect factorization*

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{H}'_1, \mathcal{S}^{\oplus(r-1)}, \mathcal{H}_2, \ldots, \mathcal{H}_m\}.$$

Here $\mathcal{H}'_1$ is the (unique) non-trivial extension of $\mathcal{H}_1$ by $\mathcal{S}$.

Proof. (1) Since the factorization is perfect, $h^0(\mathcal{G}_n, \mathcal{S}) = 0$. Since $\mathcal{S}$ is spherical, $h^2(\mathcal{S}, \mathcal{G}_n) = 0$ by Serre duality. Hence $\chi(\mathcal{S}, \mathcal{G}_n) = -1$ implies that $h^1(\mathcal{S}, \mathcal{G}_n) = 1$. The unique non-trivial extension $\mathcal{G}'_n$ of $\mathcal{S}$ by $\mathcal{G}_n$ is well-defined. Note that the perfect factorization

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{S}^{\oplus r}, \mathcal{H}_1, \ldots, \mathcal{H}_m\}.$$

induces another perfect factorization

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_{n-1}, \mathcal{F}', \mathcal{H}_1, \ldots, \mathcal{H}_m\}.$$

where $\mathcal{F}'$ is an extension of $\mathcal{S}^{\oplus r}$ by $\mathcal{G}_n$. By Lemma 3.3, $\mathcal{F}'$ is rigid, hence the extension is non-trivial. By Lemma 3.6(1), $\mathcal{F}' \cong \mathcal{S}^{\oplus(r-1)} \oplus \mathcal{G}'_n$. Hence there exists a factorization

$$\{\mathcal{G}_1, \ldots, \mathcal{G}_{n-1}, \mathcal{S}^{\oplus r}, \mathcal{G}'_n, \mathcal{H}_1, \ldots, \mathcal{H}_m\}.$$

It is easy to check that this factorization is perfect, since $h^0(\mathcal{S}, \mathcal{G}'_n) = h^0(\mathcal{S}, \mathcal{G}_n) = 0$ by Lemma 3.6(1).

(2) can be proved similarly. □

4. Torsion rigid sheaves supported in (-2)-curves

In this section, we assume that X is a smooth projective surface. All sheaves are considered to be coherent on X. We will classify certain torsion rigid sheaves supported in (-2)-curves.

Proposition 4.1. *Let $C_1 \cup C_2$ be a chain of (-2)-curves and $\mathcal{R}$ a torsion rigid sheaf with $c_1(\mathcal{R}) = C_1 + 2C_2$. Then $\mathcal{R}$ has one of the following perfect factorizations:*

(1) $\{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1 \cup C_2}(a_1, a_2)\}$;
(2) $\{\mathcal{O}_{C_1 \cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2 - 1)\}$;
(3) $\{\mathcal{O}_{C_1 \cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2 - 2)\}$.

Here a_1, a_2 are integers.

Proof. Taking the restriction in C_2, we have an exact sequence

$$0 \to \mathcal{R}_1 \to \mathcal{R} \to \mathcal{R}_2 \to 0.$$

By Definition-Proposition 2.8, $\mathcal{R}_1$ and $\mathcal{R}_2$ are rigid. Note that $c_1(\mathcal{R}_1) = C_1$ and $c_1(\mathcal{R}_2) = 2C_2$. Hence $\mathcal{R}_1 = \mathcal{O}_{C_1}(a_1 - 1)$ is a line bundle on C_1 for some integer a_1, and $\mathcal{R}_2$ has a perfect factorization induced by Harder–Narasimhan filtration, which is

Case 1. $\{\mathcal{O}_{C_2}(a_2)^{\oplus 2}\}$ for some integer a_2, or

Case 2. $\{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_2}(b_2)\}$, for integers $a_2 > b_2$.

In Case 1, $\mathcal{R}$ has a perfect factorization

$$\{\mathcal{O}_{C_1}(a_1 - 1), \mathcal{O}_{C_2}(a_2)^{\oplus 2}\},$$

for which we can apply Lemma 3.7 to get a new perfect factorization

$$\{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1 \cup C_2}(a_1, a_2)\}.$$

This gives (1).

In Case 2, since $\mathcal{R}_2$ is rigid, by Lemma 2.6(4), we have

$$h^1(\mathcal{O}_{C_2}(b_2), \mathcal{O}_{C_2}(a_2)) \leq 1,$$

which implies that $a_2 \leq b_2 + 2$, that is, $b_2 = a_2 - 1$ or $a_2 - 2$. In this case, $\mathcal{R}$ has a perfect factorization

$$\{\mathcal{O}_{C_1}(a_1 - 1), \mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_2}(b_2)\},$$

for which we can apply Lemma 3.7 to get a new perfect factorization

$$\{\mathcal{O}_{C_1 \cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(b_2)\}.$$

This gives (2) and (3). □

Proposition 4.2. *Let $C_1 \cup C_2 \cup C_3$ be a chain of (-2)-curves and $\mathcal{R}$ a torsion rigid sheaf with $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3$. Then $\mathcal{R}$ has one of the following perfect factorizations:*

(1-1) $\{\mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_2 \cup C_3}(a_2, a_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3)\}$;
(1-2) $\{\mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_2 \cup C_3}(a_2 - 1, a_3)\}$;
(1-3) $\{\mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_2 \cup C_3}(a_2 - 2, a_3)\}$;
(2-1) $\{\mathcal{O}_{C_2 \cup C_3}(a_2, a_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_3}(b_3)\}$;
(2-2) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_2 \cup C_3}(a_2 - 1, a_3), \mathcal{O}_{C_3}(b_3)\}$;
(2-3) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_2 \cup C_3}(a_2 - 2, a_3), \mathcal{O}_{C_3}(b_3)\}$;
(3-1) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_2 \cup C_3}(a_2, b_3)\}$;
(3-2) $\{\mathcal{O}_{C_2 \cup C_3}(a_2, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2 + 1, b_3)\}$;
(3-3) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_2 \cup C_3}(a_2 - 1, b_3)\}$;
(3-4) $\{\mathcal{O}_{C_1 \cup 2C_2 \cup C_3}(a_1, a_2, a_3) \oplus \mathcal{O}_{C_3}(a_3 - 1)^{\oplus 2}\}$;
(4-1) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, a_3), \mathcal{O}_{C_2 \cup C_3}(a_2, b_3), \mathcal{O}_{C_3}(c_3)\}$;
(4-2) $\{\mathcal{O}_{C_2 \cup C_3}(a_2, a_3), \mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2 + 1, b_3), \mathcal{O}_{C_3}(c_3)\}$;
(4-3) $\{\mathcal{O}_{C_1 \cup C_2 \cup C_3}(a_1, a_2, b_3 + 1), \mathcal{O}_{C_2 \cup C_3}(a_2 - 1, b_3), \mathcal{O}_{C_3}(c_3)\}$;
(4-4) $\{\mathcal{O}_{C_1 \cup 2C_2 \cup C_3}(a_1, a_2, b_3 + 1) \oplus \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\}$.

Here a_i, b_i, c_i are integers and $a_3 > b_3 > c_3$.

Proof. Taking the restriction in C_3, we have an exact sequence

$$0 \to \mathcal{R}_{12} \to \mathcal{R} \to \mathcal{R}_3 \to 0.$$

By Definition-Proposition 2.8, $\mathcal{R}_{12}$ and $\mathcal{R}_3$ are rigid. Note that $c_1(\mathcal{R}_{12}) = C_1 + 2C_2$ and $c_1(\mathcal{R}_3) = 3C_3$. $\mathcal{R}_3$ has a perfect factorization induced by Harder–Narasimhan filtration, we have 4 cases:

Case 1. $\{\mathcal{O}_{C_3}(a_3)^{\oplus 3}\}$ for some integer a_3;

Case 2. $\{\mathcal{O}_{C_3}(a_3)^{\oplus 2}, \mathcal{O}_{C_3}(b_3)\}$, for integers $a_3 > b_3$;

Case 3. $\{\mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}$, for integers $a_3 > b_3$;

Case 4. $\{\mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\}$, for integers $a_3 > b_3 > c_3$.

Each case can be divided in to 3 subcases according to the perfect factorization $\mathcal{R}_{12} \equiv \{\mathcal{G}_1, \mathcal{G}_2\}$ in Proposition 4.1.

In Case 1, applying Lemma 3.7 twice with $\mathcal{S} = \mathcal{O}_{C_3}(a_3)$ to the perfect factorization

$$\mathcal{R} \equiv \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{O}_{C_3}(a_3)^{\oplus 3}\},$$

we get a new perfect factorization, which gives (1-1), (1-2), or (1-3) by changing a_2 properly.

In Case 2, applying Lemma 3.7 twice with $\mathcal{S} = \mathcal{O}_{C_3}(a_3)$ to the perfect factorization

$$\mathcal{R} \equiv \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{O}_{C_3}(a_3)^{\oplus 2}, \mathcal{O}_{C_3}(b_3)\},$$

we get a new perfect factorization, which gives (2-1), (2-2), or (2-3) by changing a_2 properly.

In Case 3, we have 3 subcases:

Subcase 3.1. $\mathcal{R}_{12} \equiv \{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1\cup C_2}(a_1, a_2)\}$;

Subcase 3.2. $\mathcal{R}_{12} \equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2 - 1)\}$;

Subcase 3.3. $\mathcal{R}_{12} \equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2 - 2)\}$.

In Subcase 3.1, $\mathcal{R}$ has a perfect factorization

$$\{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Applying Lemma 3.7, we get a new perfect factorization

$$\{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Note that Hom's and χ between the first two factors are trivial, we get

$$h^1(\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_2}(a_2)) = 0,$$

and we can exchange the first two factors to get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Applying Lemma 3.7(1), we get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_2\cup C_3}(a_2+1, b_3)\}.$$

This gives (3-1) by changing a_2 properly.

In Subcase 3.2, $\mathcal{R}$ has a perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2 - 1), \mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Applying Lemma 3.7, we get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Note that Hom's and χ between the first two factors are trivial, we get

$$h^1(\mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_1\cup C_2}(a_1, a_2)) = 0,$$

and we can exchange the first two factors to get a new perfect factorization

$$\{\mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Applying Lemma 3.7(1), we get a new perfect factorization

$$\{\mathcal{O}_{C_2\cup C_3}(a_2,a_3),\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1,a_2+1,b_3)\}.$$

This gives (3-2).

In Subcase 3.3, $\mathcal{R}$ has a perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_2}(a_2-2),\mathcal{O}_{C_3}(a_3),\mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Applying Lemma 3.7, we get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_2\cup C_3}(a_2-1,a_3),\mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Note that

$$h^1(\mathcal{O}_{C_2\cup C_3}(a_2-1,a_3),\mathcal{O}_{C_1\cup C_2}(a_1,a_2))=1$$

since

$$\chi(\mathcal{O}_{C_2\cup C_3}(a_2-1,a_3),\mathcal{O}_{C_1\cup C_2}(a_1,a_2))=0,$$
$$h^0(\mathcal{O}_{C_2\cup C_3}(a_2-1,a_3),\mathcal{O}_{C_1\cup C_2}(a_1,a_2))=1,$$
$$h^0(\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_2\cup C_3}(a_2-1,a_3))=0,$$

and the unique non-trivial extension is $\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3)$, we get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3),\mathcal{O}_{C_3}(b_3)^{\oplus 2}\}.$$

Note that $\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3)$ can be also viewed as the extension of $\mathcal{O}_{C_2}(a_2-1)$ by $\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1,a_2+1,a_3-1)$.

Now if $a_3>b_3+1$, then $\mathcal{R}$ has a new factorization

$$\{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1,a_2+1,a_3-1),\mathcal{O}_{C_2}(a_2-1),\mathcal{O}_{C_3}(b_3)^{\oplus 2}\},$$

which is perfect by checking Hom's. Applying Lemma 3.7, we get a new perfect factorization

$$\{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1,a_2+1,a_3-1),\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_2\cup C_3}(a_2,b_3)\}.$$

This gives (3-3) by changing a_2,a_3 properly.

If $a_3=b_3+1$, then

$$h^1(\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3))=0$$

since

$$\chi(\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3))=0,$$
$$h^0(\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3))=0,$$
$$h^0(\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3),\mathcal{O}_{C_3}(b_3))=0.$$

Hence

$$\mathcal{R}\cong\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1,a_2-1,a_3)\oplus\mathcal{O}_{C_3}(a_3-1)^{\oplus 2},$$

which gives (3-4) by changing a_1,a_2 properly.

Finally we consider Case 4. Again we have 3 subcases:

Subcase 4.1. $\mathcal{R}_{12}\equiv\{\mathcal{O}_{C_2}(a_2),\mathcal{O}_{C_1\cup C_2}(a_1,a_2)\}$;

Subcase 4.2. $\mathcal{R}_{12}\equiv\{\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_2}(a_2-1)\}$;

Subcase 4.3. $\mathcal{R}_{12}\equiv\{\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_2}(a_2-2)\}$.

In Subcase 4.1, arguing as Subcase 3.1, we have

$$\mathcal{R}\equiv\{\mathcal{O}_{C_2}(a_2),\mathcal{O}_{C_1\cup C_2}(a_1,a_2),\mathcal{O}_{C_3}(a_3),\mathcal{O}_{C_3}(b_3),\mathcal{O}_{C_3}(c_3)\}$$

$$
\begin{aligned}
&\equiv \{\mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_2}(a_2), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3), \mathcal{O}_{C_2\cup C_3}(a_2+1, b_3), \mathcal{O}_{C_3}(c_3)\}.
\end{aligned}
$$

This gives (4-1) by changing a_2 properly.

In Subcase 4.2, arguing as Subcase 3.2, we have perfect factorizations

$$
\begin{aligned}
\mathcal{R} &\equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2-1), \mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_2\cup C_3}(a_2, a_3), \mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, b_3), \mathcal{O}_{C_3}(c_3)\}.
\end{aligned}
$$

This gives (4-2).

In Subcase 4.3, arguing as Subcase 3.3, we have perfect factorizations

$$
\begin{aligned}
\mathcal{R} &\equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2}(a_2-2), \mathcal{O}_{C_3}(a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup C_2}(a_1, a_2), \mathcal{O}_{C_2\cup C_3}(a_2-1, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1, a_2-1, a_3), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\}.
\end{aligned}
$$

If $a_3 > b_3 + 1$, then $a_3 = b_3 + 2$ in this case since the extension of $\mathcal{O}_{C_3}(b_3)$ by $\mathcal{O}_{C_3}(a_3)$ is rigid (see Case 2 of proof of Proposition 4.1). Arguing as Subcase 3.3, we have perfect factorizations

$$
\begin{aligned}
\mathcal{R} &\equiv \{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, a_3-1), \mathcal{O}_{C_2}(a_2-1), \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\} \\
&\equiv \{\mathcal{O}_{C_1\cup C_2\cup C_3}(a_1, a_2+1, b_3+1), \mathcal{O}_{C_2\cup C_3}(a_2, b_3), \mathcal{O}_{C_3}(c_3)\}.
\end{aligned}
$$

This gives (4-3) by changing a_2 properly.

If $a_3 = b_3 + 1$, then as Subcase 3.3, we have a perfect factorization

$$
\mathcal{R} \equiv \{\mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1, a_2-1, b_3+1) \oplus \mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_3}(c_3)\}
$$

since

$$
h^1(\mathcal{O}_{C_3}(b_3), \mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1+1, a_2-1, b_3+1)) = 0.
$$

This gives (4-4) by changing a_1, a_2 properly. □

We get the following corollary directly.

Corollary 4.3. *Let $C_1 \cup C_2 \cup C_3$ be a chain of (-2)-curves and $\mathcal{R}$ a torsion rigid sheaf with $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3$. Then one of the following holds*

(1) *$\mathcal{R}$ has a perfect factorization $\{\mathcal{G}, \mathcal{L}\}$ where $\mathcal{L}$ is a line bundle supported on the chain $C_i \cup \cdots \cup C_3$ for some $1 \le i \le 3$, or*
(2) *$\mathcal{R} \cong \mathcal{O}_{C_1\cup 2C_2\cup C_3}(a_1, a_2, a_3) \oplus \mathcal{O}_{C_3}(a_3-1)^{\oplus 2}$ for some integers a_1, a_2, a_3.*

Corollary 4.4. *Let $C_1 \cup \cdots \cup C_5$ be a chain of (-2)-curves and $\mathcal{R}$ a torsion rigid sheaf with $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3 + 2C_4 + C_5$. Then $\mathcal{R}$ has a perfect factorization $\{\mathcal{G}, \mathcal{L}\}$ where $\mathcal{L}$ is a line bundle supported on either the chain $C_i \cup \cdots \cup C_3$ for some $1 \le i \le 5$, or the chain $C_1 \cup \cdots \cup C_5$.*

Proof. Taking the restriction in $C_1 \cup C_2 \cup C_3$ and $C_3 \cup C_4 \cup C_5$, we have exact sequences

$$
\begin{aligned}
0 \to \mathcal{R}_{45} \to \mathcal{R} \to \mathcal{R}_{123} \to 0, \\
0 \to \mathcal{R}_{12} \to \mathcal{R} \to \mathcal{R}_{345} \to 0.
\end{aligned}
$$

Note that $c_1(\mathcal{R}_{123}) = C_1 + 2C_2 + 3C_3$ and $c_1(\mathcal{R}_{345}) = 3C_3 + 2C_4 + C_5$. If one of $\mathcal{R}_{123}$ and $\mathcal{R}_{345}$ satisfies Corollary 4.3(1), then we can get the desired perfect factorization.

Suppose that $\mathcal{R}_{123}$ and $\mathcal{R}_{345}$ satisfy Corollary 4.3(2), note that their restriction on C_3 are the same, for simplicity and without loss of generality, we may write

$$\mathcal{R}_{123} \cong \mathcal{O}_{C_1 \cup 2C_2 \cup C_3} \oplus \mathcal{O}_{C_3}(-1)^{\oplus 2},$$
$$\mathcal{R}_{345} \cong \mathcal{O}_{C_3 \cup 2C_4 \cup C_5} \oplus \mathcal{O}_{C_3}(-1)^{\oplus 2}.$$

In this case we have an exact sequence

$$0 \to \mathcal{O}_{C_1 \cup 2C_2}(0,-1) \oplus \mathcal{O}_{2C_4 \cup C_5}(-1,0) \to \mathcal{R} \to \mathcal{O}_{C_3} \oplus \mathcal{O}_{C_3}(-1)^{\oplus 2} \to 0.$$

This gives perfect factorizations

$$\begin{aligned}\mathcal{R} &\equiv \{\mathcal{O}_{C_1 \cup 2C_2}(0,-1) \oplus \mathcal{O}_{2C_4 \cup C_5}(-1,0), \mathcal{O}_{C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_4}, \mathcal{O}_{C_1 \cup C_2}(0,-1) \oplus \mathcal{O}_{C_4 \cup C_5}(-1,0), \mathcal{O}_{C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_4}, \mathcal{O}_{C_1 \cup C_2}(0,-1), \mathcal{O}_{C_4 \cup C_5}(-1,0), \mathcal{O}_{C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_4}, \mathcal{O}_{C_1 \cup C_2}(0,-1), \mathcal{O}_{C_3 \cup C_4 \cup C_5}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_4}, \mathcal{O}_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\}.\end{aligned}$$

Here we apply Lemma 3.7 in the last two steps. Note that

$$h^1(\mathcal{O}_{C_3}(-1), \mathcal{O}_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5}) = 0$$

by computing Hom's and χ. Hence we exchange the last two factors and get a perfect factorization

$$\mathcal{R} \equiv \{\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_4}, \mathcal{O}_{C_3}(-1)^{\oplus 2}, \mathcal{O}_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5}\},$$

and the proof is completed. □

Corollary 4.5. *Let $C_1 \cup \cdots \cup C_6$ be a chain of (-2)-curves and $\mathcal{R}$ a torsion rigid sheaf with $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3 + 3C_4 + 2C_5 + C_6$. Then $\mathcal{R}$ has a perfect factorization $\{\mathcal{G}, \mathcal{L}\}$ where $\mathcal{L}$ is a line bundle supported on one of the following chains:*

(1) *$C_i \cup \cdots \cup C_3$ for some $1 \leq i \leq 3$;*
(2) *$C_2 \cup \cdots \cup C_j$ for some $4 \leq j \leq 6$;*
(3) *$C_3 \cup C_4$.*

Proof. Taking the restriction in $C_1 \cup C_2 \cup C_3$, we have an exact sequence

$$0 \to \mathcal{R}_{456} \to \mathcal{R} \to \mathcal{R}_{123} \to 0.$$

Note that $c_1(\mathcal{R}_{123}) = C_1 + 2C_2 + 3C_3$. If $\mathcal{R}_{123}$ satisfies Corollary 4.3(1), then we get the first case.

Now suppose that $\mathcal{R}_{123}$ satisfies Corollary 4.3(2). For simplicity and without loss of generality, we may assume that

$$\mathcal{R}_{123} \cong \mathcal{O}_{C_1 \cup 2C_2 \cup C_3} \oplus \mathcal{O}_{C_3}(-1)^{\oplus 2},$$

and we have a perfect factorization

$$\mathcal{R} \equiv \{\mathcal{R}_{456}, \mathcal{O}_{C_1 \cup C_2}(-1,1), \mathcal{O}_{C_2 \cup C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\}.$$

On the other hand, $c_1(\mathcal{R}_{456}) = 3C_4 + 2C_5 + C_6$.

Suppose that $\mathcal{R}_{456}$ has a perfect factorization $\{\mathcal{G}', \mathcal{L}'\}$ where $\mathcal{L}'$ is a line bundle supported on the chain $C_4 \cup \cdots \cup C_j$ for some $4 \leq j \leq 6$. For simplicity and without loss of generality, we may assume that $\mathcal{L}' \cong \mathcal{O}_{C_4 \cup \cdots \cup C_j}$. Hence we have perfect factorizations

$$\begin{aligned}\mathcal{R} &\equiv \{\mathcal{G}', \mathcal{O}_{C_4\cup\cdots\cup C_j}, \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_2\cup C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{G}', \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_4\cup\cdots\cup C_j}, \mathcal{O}_{C_2\cup C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{G}', \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_2\cup C_3\cup C_4\cup\cdots\cup C_j}(0,0,1,\ldots), \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{G}', \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_3}(-1)^{\oplus 2}, \mathcal{O}_{C_2\cup C_3\cup C_4\cup\cdots\cup C_j}(0,0,1,\ldots)\}.\end{aligned}$$

We apply Lemma 3.7 in the second step, and the last step is because

$$h^1(\mathcal{O}_{C_3}(-1), \mathcal{O}_{C_2\cup C_3\cup C_4\cup\cdots\cup C_j}(0,0,1,\ldots)) = 0$$

by computing Hom's and χ. This gives the second case of this corollary.

Now suppose that $\mathcal{R}_{456}$ satisfies Corollary 4.3. For simplicity and without loss of generality, we may assume that

$$\mathcal{R}_{456} \cong \mathcal{O}_{C_4\cup 2C_5\cup C_6} \oplus \mathcal{O}_{C_4}(-1)^{\oplus 2}.$$

Then we have perfect factorizations

$$\begin{aligned}\mathcal{R} &\equiv \{\mathcal{O}_{C_4\cup 2C_5\cup C_6}, \mathcal{O}_{C_4}(-1)^{\oplus 2}, \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_2\cup C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_4\cup 2C_5\cup C_6}, \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_4}(-1)^{\oplus 2}, \mathcal{O}_{C_2\cup C_3}, \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_4\cup 2C_5\cup C_6}, \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_2\cup C_3\cup C_4}, \mathcal{O}_{C_4}(-1), \mathcal{O}_{C_3}(-1)^{\oplus 2}\} \\ &\equiv \{\mathcal{O}_{C_4\cup 2C_5\cup C_6}, \mathcal{O}_{C_1\cup C_2}(-1,1), \mathcal{O}_{C_2\cup C_3\cup C_4}, \mathcal{O}_{C_3}(-1), \mathcal{O}_{C_3\cup C_4}(-1,0)\}.\end{aligned}$$

Here we apply Lemma 3.7 in the last two steps. This gives the third case of this corollary. □

5. Classification of torsion exceptional sheaves

In this section, we prove Theorems 1.2 and 1.3.

Lemma 5.1. *Let $\mathcal{E}$ be a torsion exceptional sheaf on a smooth projective surface X satisfying conditions in Theorem 1.3. Assume that there exists at least one (-2)-curve in* $\mathrm{supp}(\mathcal{E})$*. Then there exists a chain of (-2)-curves Z in* $\mathrm{supp}(\mathcal{E})$*, and a line bundle $\mathcal{L}$ on Z, such that $c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) = -1$ and there is an exact sequence*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

with $h^0(\mathcal{E}', \mathcal{L}) = 0$.

Proof. By assumption, we may write

$$\mathrm{supp}(\mathcal{E}) = D \cup \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} C_i^j,$$

where $C_1^j \cup \cdots \cup C_{n_j}^j$ is a chain of (-2)-curves for each j and they are disjoint from each other. Since $\mathcal{E}$ is exceptional, $\mathrm{supp}(\mathcal{E})$ is connected. Hence we assume that D intersects with the chain $C_1^j \cup \cdots \cup C_{n_j}^j$ on the curve $C_{k_j}^j$ at one point. We may write

$$c_1(\mathcal{E}) = D + \sum_{j=1}^{m} \sum_{i=1}^{n_j} r_i^j C_i^j,$$

in the sense of Definition-Proposition 2.8 since the first Chern class of the restriction in every chain of (-2)-curves is uniquely determined. Since $\mathcal{E}$ is exceptional, by Riemann–Roch formula,

$$c_1(\mathcal{E})^2 = -\chi(\mathcal{E}, \mathcal{E}) = -1.$$

On the other hand,

$$\begin{aligned} c_1(\mathcal{E})^2 &= \Big(D + \sum_{j=1}^{m} \sum_{i=1}^{n_j} r_i^j C_i^j\Big)^2 \\ &= D^2 + 2D \cdot \sum_{j=1}^{m} \sum_{i=1}^{n_j} r_i^j C_i^j + \Big(\sum_{j=1}^{m} \sum_{i=1}^{n_j} r_i^j C_i^j\Big)^2 \\ &= D^2 + 2D \cdot \sum_{j=1}^{m} \sum_{i=1}^{n_j} r_i^j C_i^j + \sum_{j=1}^{m} \Big(\sum_{i=1}^{n_j} r_i^j C_i^j\Big)^2 \\ &= -1 + 2\sum_{j=1}^{m} r_{k_j}^j + \sum_{j=1}^{m} \Big(-2\sum_{i=1}^{n_j} (r_i^j)^2 + 2\sum_{i=1}^{n_j-1} r_i^j r_{i+1}^j\Big). \end{aligned}$$

This implies that

$$\sum_{j=1}^{m} f(r_1^j, \dots, r_{n_j}^j; k_j) = 0,$$

where f is the polynomial defined in Subsection 2.5. By Proposition 2.14, $f(r_1^j, \dots, r_{n_j}^j; k_j) = 0$ for each j and $\{r_1^j, \dots, r_{n_j}^j, k_j\}$ satisfies the conditions in Proposition 2.14.

For convenience, we write $n_1 = n$, $r_i^1 = r_i$, $C_i^1 = C_i$, $k_1 = k$. Note that $n \leq 6$ by assumption, and hence $r_k \leq 3$.

Reversing the order of $\{C_i\}$ if necessary, by the conditions in Proposition 2.14, we only have the following 6 cases:

(1) $k = n = 1$, $r_1 = 1$;
(2) $k \geq 2$ and $r_1 = r_2 = 1$;
(3) $k = 2$ and $r_1 = 1, r_2 = 2, r_3 = 1$;
(4) $k \geq 3$ and $r_1 = 1, r_2 = r_3 = 2$;
(5) $k = 3$, $n = 5$ and $(r_1, \dots, r_5) = (1, 2, 3, 2, 1)$;
(6) $k = 4$, $n = 6$ and $(r_1, \dots, r_6) = (1, 2, 3, 3, 2, 1)$.

In Case (1) and (2), taking the restriction in C_1, we have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{R}_1 \to 0.$$

Then $c_1(\mathcal{R}_1) = C_1$ and hence $\mathcal{R}_1$ is a line bundle on C_1. Moreover, $h^0(\mathcal{E}', \mathcal{R}_1) = 0$ by construction, and

$$c_1(\mathcal{E}) \cdot c_1(\mathcal{R}_1) = \begin{cases} (C_1 + D) \cdot C_1 = -1 & \text{Case (1);} \\ (C_1 + C_2) \cdot C_1 = -1 & \text{Case (2).} \end{cases}$$

Hence we may take $\mathcal{L} = \mathcal{R}_1$.

In Case (3) and (4), taking the restriction in $C_1 \cup C_2$, we have an exact sequence

$$0 \to \mathcal{E}_{12} \to \mathcal{E} \to \mathcal{R}_{12} \to 0.$$

Then $c_1(\mathcal{R}_{12}) = C_1 + 2C_2$. By Proposition 4.1, there exists a line bundle $\mathcal{L}$ supported on C_2 or the chain $C_1 \cup C_2$ with an exact sequence

$$0 \to \mathcal{G} \to \mathcal{R}_{12} \to \mathcal{L} \to 0$$

such that $h^0(\mathcal{G}, \mathcal{L}) = 0$. Consider the exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

given by the surjection $\mathcal{E} \to \mathcal{R}_{12} \to \mathcal{L}$. Then $h^0(\mathcal{E}', \mathcal{L}) = 0$ since $\mathcal{E}'$ is an extension of $\mathcal{G}$ by $\mathcal{E}_{12}$ and $\mathrm{supp}(\mathcal{E}_{12})$ does not contain C_1 or C_2. Note that $c_1(\mathcal{L}) = C_2$ or $C_1 + C_2$, we have

$$c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) = \begin{cases} (C_1 + 2C_2 + C_3 + D) \cdot c_1(\mathcal{L}) = -1 & \text{Case (3);} \\ (C_1 + 2C_2 + 2C_3) \cdot c_1(\mathcal{L}) = -1 & \text{Case (4).} \end{cases}$$

This $\mathcal{L}$ satisfies all conditions we require.

In Case (5), taking the restriction in $C_1 \cup \cdots \cup C_5$, we have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{R} \to 0.$$

Then $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3 + 2C_4 + C_5$. By Corollary 4.4, $\mathcal{R}$ has a perfect factorization $\{\mathcal{G}, \mathcal{L}\}$ where $\mathcal{L}$ is a line bundle supported on either the chain $C_i \cup \cdots \cup C_3$ for some $1 \le i \le 5$, or the chain $C_1 \cup \cdots \cup C_5$. This induces an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{E}'$ is an extension of $\mathcal{G}$ by $\mathcal{E}_1$. In particular, we have $h^0(\mathcal{E}', \mathcal{L}) = 0$. By construction, $c_1(\mathcal{L}) = \sum_{j=i}^{3} C_j$ for some $1 \le i \le 5$ or $\sum_{j=1}^{5} C_j$. Note that D only intersects with C_3, it is easy to compute that

$$c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) = (C_1 + 2C_2 + 3C_3 + 2C_4 + C_5 + D) \cdot c_1(\mathcal{L}) = -1.$$

This $\mathcal{L}$ satisfies all conditions we require.

In Case (6), taking the restriction in $C_1 \cup \cdots \cup C_6$, we have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{R} \to 0.$$

Then $c_1(\mathcal{R}) = C_1 + 2C_2 + 3C_3 + 3C_4 + 2C_5 + C_6$. By Corollary 4.5, $\mathcal{R}$ has a perfect factorization $\{\mathcal{G}, \mathcal{L}\}$ where $\mathcal{L}$ is a line bundle supported on the chain $C_i \cup \cdots \cup C_3$ for some $1 \le i \le 3$, or the chain $C_2 \cup \cdots \cup C_j$ for some $4 \le j \le 6$, or the chain $C_3 \cup C_4$. This induces an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{E}'$ is an extension of $\mathcal{G}$ by $\mathcal{E}_1$. In particular, we have $h^0(\mathcal{E}', \mathcal{L}) = 0$. By construction, $c_1(\mathcal{L}) = \sum_{l=i}^{3} C_l$ for some $1 \le i \le 3$, or $\sum_{l=2}^{j} C_l$ for some $4 \le j \le 6$, or $C_3 + C_4$. Note that D only intersects with C_4, it is easy to compute that

$$c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) = (C_1 + 2C_2 + 3C_3 + 3C_4 + 2C_5 + C_6 + D) \cdot c_1(\mathcal{L}) = -1.$$

This $\mathcal{L}$ satisfies all conditions we require. $\square$

Proof of Theorem 1.3. As in the proof of Lemma 5.1, we may write

$$c_1(\mathcal{E}) = D + \sum_{j=1}^{m}\sum_{i=1}^{n_j} r_i^j C_i^j,$$

where $C_1^j \cup \cdots \cup C_{n_j}^j$ is a chain of (-2)-curves for each j and they are disjoint from each other.

Assume that there exists at least one (-2)-curve in $\mathrm{supp}(\mathcal{E})$, then by Lemma 5.1 there exists a chain of (-2)-curves Z in $\mathrm{supp}(\mathcal{E})$, and a line bundle $\mathcal{L}$ on Z, such that $c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) = -1$ and there is an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

with $h^0(\mathcal{E}', \mathcal{L}) = 0$. Note that $\mathcal{L}$ is a spherical object, and

$$\chi(\mathcal{L}, \mathcal{E}') = \chi(\mathcal{L}, \mathcal{E}) - \chi(\mathcal{L}, \mathcal{L}) = -c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) - 2 = -1.$$

By Lemma 2.7, $\mathcal{E}'$ is exceptional and $\mathcal{E} \cong T_{\mathcal{L}}\mathcal{E}'$. Moreover, by the proof of Lemma 5.1,

$$c_1(\mathcal{E}') = c_1(\mathcal{E}) - c_1(\mathcal{L}) = D + \sum_{j=1}^{m}\sum_{i=1}^{n_j} (r_i^j)' C_i^j,,$$

where $(r_i^j)' = \begin{cases} r_i^j - 1 & \text{if } C_i^j \subset \mathrm{supp}(\mathcal{L}); \\ r_i^j & \text{otherwise.} \end{cases}$

By induction on the number $\sum_{j=1}^{m}\sum_{i=1}^{n_j} r_i^j$, after finitely many steps, we may assume that $c_1(\mathcal{E}) = D$. This implies that $\mathcal{E}$ is a line bundle on D and the proof is completed. □

Lemma 5.2. *Let $\mathcal{E}$ be a torsion exceptional sheaf on a weak del Pezzo surface X, then there exists exactly one (-1)-curve D in* $\mathrm{supp}(\mathcal{E})$*, and the restriction of $\mathcal{E}$ in D is a line bundle.*

Proof. Since $|-K_X|$ has no base component by Lemma 2.10, choose a general element in $E \in |-K_X|$ which is not contained in $\mathrm{supp}(\mathcal{E})$. There is a short exact sequence

$$0 \to \omega_X \to \mathcal{O}_X \to \mathcal{O}_E \to 0.$$

Tensoring with $\mathcal{E}$, since $\mathcal{E}$ is pure one-dimensional and $E \not\subset \mathrm{supp}(\mathcal{E})$, we get an exact sequence

$$0 \to \mathcal{E} \otimes \omega_X \to \mathcal{E} \to \mathcal{E}|_E \to 0.$$

Applying $\mathrm{Hom}(\mathcal{E}, -)$ to this sequence, we get an exact sequence

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}|_E) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \omega_X).$$

Since $\mathcal{E}$ is exceptional, $h^0(\mathcal{E}, \mathcal{E}) = 1$ and $h^0(\mathcal{E}, \mathcal{E} \otimes \omega_X) = h^1(\mathcal{E}, \mathcal{E} \otimes \omega_X) = 0$ by Serre duality. Hence

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}|_E) \cong \mathbb{C}. \tag{5.1}$$

By Lemma 2.9, $\mathrm{supp}(\mathcal{E})$ only contains (-1)-curves and (-2)-curves. Note that each (-1)-curve intersects with E at one point and each (-2)-curve does not intersect with E, we conclude that there is only one (-1)-curve D in $\mathrm{supp}(\mathcal{E})$. Moreover, taking restriction in D, we get an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}_D \to 0,$$

where the support of $\mathcal{E}'$ only contains (-2)-curves. Combining with (5.1), we have

$$\mathrm{Hom}(\mathcal{E}_D, \mathcal{E}_D|_E) \cong \mathbb{C}.$$

Since $\mathcal{E}_D$ is pure one-dimensional, $\mathcal{E}_D$ is a line bundle on D. □

Proof of Theorem 1.2. It suffices to check that any torsion exceptional sheaf $\mathcal{E}$ on a weak del Pezzo surface X of $d > 2$ of Type A satisfies conditions (1)-(4) in Theorem 1.3.

By Lemma 2.9, any irreducible component of $\mathrm{supp}(\mathcal{E})$ is a curve with negative self-intersection, hence is a (-1)-curve or (-2)-curve. By Lemma 5.2, conditions (1)-(2) are satisfied. By the assumption $d > 2$, there are at most 6 (-2)-curves on X, hence condition (3) is satisfied since X is of Type A. Again by the assumption $d > 2$ and Lemma 2.11, condition (4) is satisfied. □

6. Examples

In this section, we provide several interesting examples of torsion exceptional sheaves on weak del Pezzo surfaces.

Example 6.1. Let X be a weak del Pezzo surface of degree $d > 1$ whose anticanonical model has at most A_1-singularities. Then by our proof, every torsion exceptional sheaf on X has the form $\mathcal{O}_{D\cup C_1\cup\cdots\cup C_n}(d, a_1, \ldots, a_n)$, where C_i are disjoint (-2)-curves, D is a (-1) curve intersecting with each C_i, and d, a_i are integers. Note that n can be 0 which means there is no (-2)-curve in the support. Similar result holds true for weak del Pezzo surfaces of degree $d > 1$ whose anticanonical model has at most A_2-singularities.

The following example suggests that the scheme theoretic support of a torsion exceptional sheaf can be non-reduced.

Example 6.2. Let X be a smooth projective surface, $C_1 \cup C_2 \cup C_3$ a chain of (-2)-curves, and D a (-1)-curve. Assume that $D \cdot C_2 = 1$ and $D \cdot C_1 = D \cdot C_3 = 0$. Then the structure sheaf $\mathcal{O}_{D\cup C_1\cup 2C_2\cup C_3}$ is a torsion exceptional sheaf on X with non-reduced support. In fact, by applying Lemma 2.7 to the following exact sequences

$$0 \to \mathcal{O}_{D\cup C_1\cup C_2\cup C_3}(-1, -1, 2, -1) \to \mathcal{O}_{D\cup C_1\cup 2C_2\cup C_3} \to \mathcal{O}_{C_2} \to 0,$$

$$0 \to \mathcal{O}_D(-2) \to \mathcal{O}_{D\cup C_1\cup C_2\cup C_3}(-1, -1, 2, -1) \to \mathcal{O}_{C_1\cup C_2\cup C_3}(-1, 2, -1) \to 0,$$

we get

$$\mathcal{O}_{D\cup C_1\cup 2C_2\cup C_3} = T_{\mathcal{O}_{C_2}} \circ T_{\mathcal{O}_{C_1\cup C_2\cup C_3}(-1,2,-1)}(\mathcal{O}_D(-2)).$$

The following example suggests that the support of a torsion exceptional sheaf on a weak del Pezzo surface of degree one can contain loops.

Example 6.3. Let X be a smooth projective surface, $C_1 \cup C_2 \cup C_3$ a chain of (-2)-curves, and D a (-1)-curve. Assume that $D \cdot C_2 = 0$ and $D \cdot C_1 = D\cdot C_3 = 1$, that is, C_1, C_2, C_3, D form a loop (this might happen, for example, on the minimal resolution of a singular del Pezzo surface of degree one with one A_3-singularity, cf. [6, Lemma 2.8]). Then the unique non-trivial extension $\mathcal{E}$ of $\mathcal{O}_{C_2 \cup C_3}$ by $\mathcal{O}_{D \cup C_1 \cup C_2}$ is a torsion exceptional sheaf on X whose support is a loop. In fact,

$$h^1(\mathcal{O}_{C_2 \cup C_3}, \mathcal{O}_{D \cup C_1 \cup C_2}) = 1$$

since

$$\begin{aligned} h^0(\mathcal{O}_{C_2 \cup C_3}, \mathcal{O}_{D \cup C_1 \cup C_2}) &= 0, \\ \chi(\mathcal{O}_{C_2 \cup C_3}, \mathcal{O}_{D \cup C_1 \cup C_2}) &= -1, \\ h^0(\mathcal{O}_{D \cup C_1 \cup C_2}, \mathcal{O}_{C_2 \cup C_3}) &= 0. \end{aligned}$$

By applying Lemma 2.7 twice, we get

$$\mathcal{E} = T_{\mathcal{O}_{C_2 \cup C_3}} \circ T_{\mathcal{O}_{C_1 \cup C_2}}(\mathcal{O}_D(-1)).$$

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