GROMOV-WITTEN THEORY OF QUOTIENT OF FERMAT CALABI-YAU VARIETIES

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1. Introduction

Gromov-Witten theory started as an attempt to provide a rigorous mathematical foundation for the so-called A-model topological string theory of Calabi-Yau varieties. Even though it can be defined for all the Kähler/symplectic manifolds, the theory on Calabi-Yau varieties remains the most difficult one. In fact, a great deal of techniques were developed for non-Calabi-Yau varieties during the last twenty years. These techniques have only limited bearing on the Calabi-Yau cases. In a certain sense, Calabi-Yau cases are very special too. There are two outstanding problems for the Gromov-Witten theory of Calabi-Yau varieties and they are the focus of our investigation.

More than twenty years ago, physicists Bershadsky-Cecotti-Ooguri-Vafa [7] studied the higher genus B-model theory. One of the consequences of their investigation is the following mathematical conjecture.

Modularity Conjecture: *Suppose that X is a Calabi-Yau manifold/orbifold and $\mathcal{F}_g^{GW}$ is the genus g generating function of its Gromov-Witten theory. Then $\mathcal{F}_g^{GW}$ is a quasi-modular form in an appropriate sense (see Definition 2.1, Definition 5.8 and Remark 2.2).*

One of main intellectual advances of the field during the last several years was the realization that the modularity conjecture should be extended to orbifold quotients $[X/G]$ of a Calabi-Yau manifold/orbifold X.

When X is a Calabi-Yau hypersurface of weighted projective space, there is another famous duality from physics as follows. Suppose $X_W = \{W = 0\} \subset \mathbb{P}(c_1, \cdots, c_n)$ is a degree d hypersurface. X_W is a Calabi-Yau orbifold iff $d = \sum_i c_i$. Let G_W be the group of diagonal matrices preserving W. G_W contains a special matrix $J = exp(2\pi i c_1/d, \cdots, 2\pi i c_n/d)$ and is always nontrivial. J acts trivially on X_W. In addition to W, we can choose a so-called *admissible group* $\langle J\rangle \subset G \subset G_W$. Then, $\tilde{G} = G/\langle J\rangle$ acts faithfully on X_W. There are two curve counting theories built out of data (W, G): the Gromov-Witten theory of an orbifold $[X_W/\tilde{G}]$ and the FJRW theory of (W, G) [28, 29]. Let $\mathcal{F}_g^{\mathrm{GW}}, \mathcal{F}_g^{\mathrm{FJRW}}$ be the generating functions of each theory. Define *partition functions*

$$\mathcal{D}_{\mathrm{GW}} = \sum_g \hbar^{g-1}\mathcal{F}_g^{\mathrm{GW}},\ \mathcal{D}_{\mathrm{FJRW}} = \sum_g \hbar^{g-1}\mathcal{F}_g^{\mathrm{FJRW}}.$$

The second outstanding problem for Calabi-Yau varieties is the following conjecture [74, 59].

Landau-Ginzburg/Calabi-Yau correspondence Conjecture: *There is a differential operator $\mathcal{U}$ built out of genus zero data (the quantization of symplectic transformation in the sense of Givental) such that up to an analytic continuation*

$$\mathcal{D}_{\mathrm{GW}} = \mathcal{U}(\mathcal{D}_{\mathrm{FJRW}}).$$

The above two conjectures are central for our understanding the GW-theory of Calabi-Yau varieties. For example, they are at the heart of a recent spectacular advance in physics [40] to compute higher genus Gromov-Witten invariants of the quintic 3-fold up to genus 51!

It is clear that both conjectures are difficult. In [15], it was proposed to put both conjectures into a single framework using *global mirror symmetry*. Here, the word *global* refers to the global property of the B-model. The traditional version of mirror symmetry is *local* in the sense that we study a neighborhood of so-called *a large complex structure limit*. Global mirror symmetry emphasizes the idea of moving away from a large complex structure limit. In fact, we want to move around the entire B-model moduli space and study all the interesting limits including (not exclusively) the large complex limit. One of the special ones is the Gepner limit, corresponding to FJRW theory. Therefore, the knowledge of the Gepner limit (FJRW theory) will yield a wealth of information at the large complex structure limit (GW theory). This provides an effective way to compute higher-genus Gromov-Witten invariants of Calabi-Yau hypersurfaces, which is a central and yet difficult problem in geometry and physics. Furthermore, one can study global properties of the entire family. The global properties of B-model naturally lead to the modularity of Gromov-Witten theory, a remarkable bonus of *global mirror symmetry*. This was exactly the way that BCOV discovered the modularity more than twenty years ago. Since then, there has been steady progress in physics on the modularity conjecture by Klemm and others [3, 35, 40, 41]. In a sense, the mathematicians are finally catching up! However, the recent mathematical development did not follow the physical blueprint. Recall that the physical discussion for last 15 years focused on the Calabi-Yau B-model (see a mathematical formulation in [23]). An unexpected twist of recent events in mathematics is the development of the above framework in the set-up of the Landau-Ginzburg model over $[X/G]$, a related but much larger model.

The main result of this article is to prove both conjectures for (W, G_W) (Theorem 7.9) in the case that W is a Fermat polynomial.

Theorem 1.1. *Suppose that W is a Fermat polynomial with $d = \sum_i c_i$ (hence X_W defines a Calabi-Yau hypersurface). Then,*

(1) *LG/CY correspondence conjecture holds for the pair (W, G_W).*
(2) *The modularity conjecture holds for $[X_W/\tilde{G}_W]$.*

We would like to mention that there are two other parts of LG/CY correspondences, *cohomological corespondence* and *genus zero correspondence*. The cohomological correspondence was solved for an arbitrary admissible pair (W, G) by Chiodo-Ruan [17]. The genus zero correspondence for Fermat polynomial W was solved by Chiodo-Iritani-Ruan [16, 18] for the pair $(W, \langle J \rangle)$ (see wall-crossing proof in [60]) and by Lee-Priddis-Shoemaker [58, 57] for the pair (W, SL_W). The all-genus correspondence for simple elliptic singularities was solved by Krawitz-Shen [46] and Milanov-Ruan [52]. There are also very interesting versions for complete intersections by Clader [19] and non-Calabi-Yau cases by Acosta [1]. Our focus is the higher genus correspondence as we stated in the theorem. However, an intermediate step is a proof of the genus zero correspondence for the pair (W, G_W). The modularity conjecture was solved in dimension one[52, 53, 68] (see [20] for a related work on compact toric orbifolds).

Let's spell out our general strategy. The original version of the LG/CY correspondence is a conjectural statement connecting the GW theory of X_W and the FJRW theory of $(W, \langle J \rangle)$. The computation of higher-genus Gromov-Witten invariants is a very difficult problem, which we hope to solve using the LG/CY correspondence. However, we can improve the situation by taking a certain maximal quotient $[X_W/\tilde{G}_W]$. By the Berglund-Hübsch-Krawitz LG-to-LG mirror symmetry [6, 45], $[X_W/\tilde{G}_W]$ should be mirror to the large complex structure limit of the B-model family of the dual polynomial W^T (a Fermat polynomial is self-dual). Its Gepner limit corresponds to the FJRW theory of (W, G_W). The B-model family of W^T corresponds to miniversal deformation of W^T. Its genus zero theory is known as Saito's Frobenius manifold theory [63]. Saito's Frobenius manifold is generically semi-simple and Givental has defined a higher-genus potential function on the semi-simple locus [31]. Namely, we have a rigorous mathematical definition of the B-model theory in this case for all genera. Using Teleman's solution of the Givental conjecture [72], the higher genus theory of a semi-simple GW-theory is determined by the genus zero theory. Therefore, the all-genus LG/CY correspondence is reduced to the genus zero correspondence. On the other hand, there is no such reduction for CY cases such as X_W. We should mention that the extension of the Givental-Teleman higher genus function to non-semisimple locus is a well-known difficult problem and has been solved recently by Milanov [50].

We shall implement our strategy in two steps: (i) a construction of the global LG B-model of W^T, and (ii) two mirror symmetry theorems connecting the B-model at the large complex structure limit to GW-theory and the B-model at the Gepner limit to FJRW-theory. We have applied the above strategy successfully for quotients of elliptic curves by $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ [46, 52]. But the B-model construction in [52] does not generalize to higher dimensions. In this article, we develop the higher dimensional theory using a different approach.

The main results of this article have been reported in various conferences during last five years. We apologize for the long delay.

The article is organized as follows. In the section 2, we will review the global CY-B-model to motivate our global LG-B-model construction and the appearance of quasi-modular forms in Gromov-Witten theory. Sections 3-5 form the technical core of the paper where we construct the global LG-B-model. We should mention that many ingredients were already in the literature [38]. The two mirror symmetric theorems as well as the proof of the main theorem will be presented in sections 6 and 7. The proof of the main theorem (Theorem 7.9) will be presented in the section 7.

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2. Global CY-B-model and quasi-modular form

We are primarily working in the LG-setting. In this section, we review the some basic properties expected for the Calabi-Yau B-model to motivate our construction. In the process, quasi-modular forms appear naturally in GW-theory.

Let X be a n-dimensional Calabi-Yau manifold. The B-model of X corresponds to the moduli space of complex structures (possibly with a marking) on X. Traditionally, it is studied by its Hodge structure. Let's start from the algebraic set-up. An *abstract Hodge structure of weight k (AHS)* on a real vector space V is a decomposition into direct sum of complex subspaces

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. A *polarization* of AHS on V is a non-degenerate bilinear form Q on V which is symmetric if k is even, and skew-symmetric otherwise. It satisfies the conditions

(i) $Q(x,y) = 0$ for $x \in V^{p,q}, y \in V^{p',q'}, (p,q) \neq (q',p')$;
(ii) $i^{p-q}(-1)^{k(k-1)/2}Q(x,\bar{x}) > 0$.

We can associate the *Hodge filtration*

$$0 \subset F^k \subset F^{k-1} \subset \cdots \subset F^0 = V_{\mathbb{C}},$$

given by $F^p = \sum_{p' \geq p} H^{p',q}$. The above Hodge filtration defines a *flag* of $V_{\mathbb{C}}$. Using the polarization, we can reconstruct the Hodge decomposition from the flag by

$$H^{p,q} = \{x \in F^p : Q(x,\bar{y}) = 0, y \in F^{p+1}\}.$$

Let $m_p = \dim F^p, \mathbf{m} = (m_1, \cdots, m_k)$. Let $Fl(\mathbf{m}, V_{\mathbb{C}})$ be the variety of flags of linear subspaces F^p of dimension m_p, $p = 0, \ldots, k$. It is a closed algebraic subvariety of the product of Grassmann varieties $G(m_p, V_{\mathbb{C}})$. It carries a sequence of tautological bundles of rank m_p pulled back from that of $G(m_p, V_{\mathbb{C}})$. A polarized AHS of weight k defines a point (F^p) in $Fl(\mathbf{m}, V_{\mathbb{C}})$. It satisfies the following conditions

(i) $V_{\mathbb{C}} = F^p \bigoplus \overline{F^{k-p+1}}$;
(ii) $Q(F^p, F^{k-p+1}) = 0$;
(iii) $(-1)^{k(k-1)/2}Q(Cx,\bar{x}) > 0$, where C acts on $H^{p,q}$ as multiplication by i^{p-q}.

The subset of flags in $Fl(\mathbf{m}, V_{\mathbb{C}})$ satisfying the previous conditions is denoted by $\mathbb{D}_{\mathbf{m}}(V,Q)$ and is called *the period space* of (V,Q) of type $\mathbf{m}$. Fix a basis of V with respect to Q to identify V with the space $\mathbb{R}^r$, where $r = m_0$. F^p can be identified with a complex matrix Π_p of size $r \times m_p$, which is called a *period matrix.* Another important structure is the integral structure. An integral structure of an AHS is an abelian subgroup $\Lambda \subset V$ of rank equal to $\dim V$ (a lattice) such that $Q(\Lambda \times \Lambda) \subset \mathbb{Z}$.

Suppose that X is a Kähler manifold of dimension k. The Hodge decomposition and cup product Q on the middle dimensional cohomology $V = H^k(X,\mathbb{R})$ define an AHS of weight k. The integral structure is given by $\Lambda = H^k(X,\mathbb{Z})$. Now, suppose we have a family of compact connected complex manifolds. It is a holomorphic smooth map $f\colon \mathcal{X} \to T$ of complex manifolds with connected base T. For any $t \in T$, let $X_t = f^{-1}(t)$

be the fiber and $V_t = H^k(X_t, \mathbb{R})$ equipped with a Hodge structure. The cup product defines a polarization Q_t of the Hodge structure on V_t. Fix an isomorphism

$$\varphi_t \colon (V, Q_0) \to (H^k(X_t, \mathbb{R}), \cup)$$

called a *marking* of X_t; we assume that the dependence of the marking φ_t on t is locally constant (flat with respect to the Gauss-Manin connection). Then, the pre-image of the Hodge flag (F^p_t) is a Q_0-polarized AHS of weight k on (V, Q_0). Let $\mathbb{D}_{\mathbf{m}}$ be the period space (V, Q_0) of type $\mathbf{m} = (m_p)$ for $m_p = \dim F^p$. We have a multi-valued holomorphic period map

$$\phi \colon T \to \mathbb{D}_{\mathbf{m}}, \ t \mapsto (\varphi_t^{-1}(F^p_t)).$$

To define the single valued period map, we need to consider the universal cover $\tilde{T}$ of T. Let $\tilde{\mathcal{X}} \to \tilde{T}$ be the pull-back family. Then, we can fix a basis of $(H^k(\tilde{X}_t, \mathbb{Z}), \tilde{Q}_t)$ for each t which depends holomorphically on t and hence a single valued period map $\tilde{\phi} \colon \tilde{T} \to \mathbb{D}_{\mathbf{m}}$. Furthermore, the monodromy defines a homomorphism

$$\alpha \colon \pi_1(T, t_0) \to G_\Lambda = \mathrm{Aut}(\Lambda, Q|_\Lambda)$$

called the *monodromy representation*. Let Γ be the image of α. We obtain a single valued period map

$$\bar{\phi} \colon T \to \mathbb{D}_{\mathbf{m}}/\Gamma \to \mathbb{D}_{\mathbf{m}}/\operatorname{Aut}(\Lambda, Q|_\Lambda).$$

The global Torelli theorem is a statement that $\mathbb{D}_{\mathbf{m}}/\operatorname{Aut}(\Lambda, Q|_\Lambda)$ describes the moduli space of complex structures, which is basically true in dimension one and two. It is unknown if the global Torelli theorem holds in higher dimension. Another important property is whether or not $\mathbb{D}_{\mathbf{m}}/\operatorname{Aut}(\Lambda, Q|_\Lambda)$ is a hermitian symmetry space, which makes the connection to number theory. Again, this is the case in dimension one and two and false in higher dimension.

When X is Calabi-Yau, $F^k = H^{k,0}$ is one-dimensional. An element of $H^{k,0}$ is called a *holomorphic* $(k, 0)$ *form or a Calabi-Yau form.* F^k induces a holomorphic line bundle

$$\mathcal{L} \to \mathbb{D}_{\mathbf{m}}.$$

In physical literature, $\mathcal{L}$ is called a *vacuum* line bundle. It is invariant under G_Λ action and hence descends to $\mathbb{D}_{\mathbf{m}}/G_\Lambda$. We use the same $\mathcal{L}$ to denote its pull back to T. Using $\mathcal{L}$, we can define the *modular form.*

Definition 2.1. We call an analytic (holomorphic) section Ψ of $\mathcal{L}^k$ a *(holomorphic) modular form of weight* k of T. Alternatively, Ψ can be viewed as an analytic function on the total space of $\mathcal{L}$ such that $\psi(zv) = z^{-k}\psi(v)$. We call a holomorphic function ψ on $\mathbb{D}_{\mathbf{m}}$ a *quasi-modular form* if it is the holomorphic part of a "non-holomorphic" modular form. In other words, there is a (non-holomorphic) modular form Ψ and functions $h_1, \ldots, h_k$ (anti-holomorphic generators) such that Ψ is a polynomial of $h_1, \cdots, h_k$ with holomorphic functions as coefficients and ψ as the constant term.

Remark 2.2. The above definition is unsatisfactory since it also includes other objects such as mock modular forms. We use it as the working definition of this paper because of the lack of a better definition. The main point of BCOV's paper is that B-model GW-theory generating function should be a *almost holomorphic* section of $\mathcal{L}^k$ and hence almost holomorphic modular form. Here, the almost holomorphic means that its anti-holomorphic generators satisfy the so-called *holomorphic anomaly equation.* The A-model GW-theory generating function corresponds to the holomorphic part of B-model generating function. An important future problem is to study these anti-holomorphic generators, which will lead to a definition closer to that in number theory.

Example 2.3. An abstract Hodge structure of weight 1 on a real vector space V is a decomposition into direct sum of complex linear subspaces

$$V_{\mathbb{C}} = V^{1,0} + V^{0,1}$$

such that

(i) $\overline{V^{1,0}} = V^{0,1}$.

A polarization of AHS of weight 1 is a non-degenerate skew-symmetric form Q on V (a symplectic form) such that

(ii) $Q|_{V^{1,0}} = 0, Q|_{V^{0,1}} = 0$;
(iii) $iQ(x, \bar{x}) > 0, \vee x \in V^{1,0} - \{0\}$.

Then, we extend Q to a skew-symmetric form on $V_{\mathbb{C}}$ by linearity.

$H(x, y) = -iQ(x, \bar{y})$ (or $Q(x, y) = iH(x, \bar{y})$) defines a hermitian form on $V_{\mathbb{C}}$ of signature (g, g). Let $G(g, V_{\mathbb{C}})$ be the Grassmann variety of g-dimensional subspaces of $V_{\mathbb{C}}$. Set

$$G(g, V_{\mathbb{C}})_H = \{W \in G(g, V_{\mathbb{C}}) : Q|_W = 0, H|_W > 0\}.$$

There is a natural bijection between Hodge structures on V of weight 1 with polarization form Q and points in $G(g, V_{\mathbb{C}})$, where H is the associated hermitian form of Q. By choosing a standard symplectic basis in V, $G(g, V_{\mathbb{C}})$ can be described as a set of complex $2g \times g$-matrices satisfying certain condition. Furthermore, we can find a unique basis of W such that the last g rows of the matrix form the identity matrix. Therefore, we identify W with a unique $g \times g$-matrix Z. The matrix Z satisfies the conditions

$$Z^T = Z, \quad \mathrm{Im}(Z) = \frac{1}{2i}(Z - \bar{Z}) > 0.$$

The period space $\mathbb{D}_{(2g,g)}$ parametrizing polarized AHS of weight 1 is isomorphic to the complex manifold

$$\mathcal{Z}_g = \{Z \in \mathrm{Mat}_g(\mathbb{C}) : Z^T = Z, \mathrm{Im}(Z) > 0\}$$

called the *Siegel upper half plane of degree g.* Its dimension is equal to $g(g+1)/2$. The monodromy group Γ is a subgroup of $Sp(2g, \mathbb{Z})$, which acts on $\mathbb{D}_{(2g,g)}$ by

$$M(Z) = (AZ + B)/(CZ + D)$$

where M is written as a block-matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Suppose that $\mathcal{X} \to T$ is a one-dimensional family of elliptic curves. It induces a weight 1 AHS of $g = 1$. Therefore, the period space is $\mathcal{Z}_1$-the upper half plane $\mathbf{H}$. The monodromy group Γ is a subgroup of $SL(2, \mathbb{Z})$. Suppose that $\mathcal{X}$ is not a constant family. Then, the universal cover $\tilde{T} = \mathbf{H}$ and $T = \mathbf{H}/\Gamma$. We would like to consider its modular form. Note that we can consider $\mathbb{D}_{2,1}$ as a sub-domain of $P(V_{\mathbb{C}})$. Then, $\mathcal{L}$ is the pull-back of the tautological line bundle of $P(V_{\mathbb{C}})$.

Suppose that ω is a holomorphic $(1,0)$-form. Choose a symplectic basis or marking A, B. The periods

$$\alpha = \int_A \omega, \quad \beta = \int_B \omega$$

define a homogeneous coordinate system on $\mathbb{D}_{(2,1)}$. The inhomogeneous coordinate is $\tau = \beta/\alpha \in \mathbf{H}$. Moreover, the total space of $\mathcal{L}$ (minus the zero-section) can be identified as $(V_{\mathbb{C}} - \{0\})/\Gamma$, where Γ acts

$$(\alpha, \beta) \to (a\beta + b\alpha, c\beta + d\alpha),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. By the definition, a modular form of weight k is a holomorphic function

$$f \colon V_{\mathbb{C}} - \{0\} \to \mathbb{C}$$

such that

(i): f is invariant under Γ-action;
(ii): $f(zv_0, zv_1) = z^k f(v_0, v_1)$.

Choose A such that $\omega(A) = 1$. Then,

$$\tau \to (1, \tau)$$

defines a section of $\mathcal{L}$. Let $F(\tau) = f(1, \tau)$. Under a fractional linear transformation $\tau \to \frac{a\tau+b}{c\tau+d}$, $F(\tau)$ changes as

$$\begin{aligned} F\left(\frac{a\tau + b}{c\tau + d}\right) &= f\left(1, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-k} f(a\tau + b, c\tau + d) = (c\tau + d)^{-k} f(1, \tau) \\ &= (c\tau + d)^{-k} F(\tau) \end{aligned}$$

which agrees with the usual definition of modular form.

The above example can be generalized to higher rank cases. $V^{1,0}$ defines a rank g bundle $\mathcal{V}$ over the Siegel upper half space $\mathcal{Z}_g$. Let $\mathcal{L} = \det(\mathcal{V})$. A *Seigel modular form* f of weight k is a section of $\mathcal{L}^k$. Similarly, we can work out its inhomogeneous presentation. It corresponds to a function $F \colon \mathcal{Z}_g \to \mathbb{C}$ such that

$$F\left(\frac{A\tau + B}{C\tau + D}\right) = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}^k F(\tau)$$

for $\tau \in \mathcal{Z}_g$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{Z})$.

Example 2.4. Next, we consider the B-model moduli space of a K3-surface. Let X be an algebraic K3-surface. $H^2(X, \mathbb{Z})$ is a free abelian group of rank 22 and carries a unimodular even bilinear form of signature $(3, 19)$. Let ω be an ample class. We consider the Hodge structure on cohomology $V = H^2_{\text{prim}}(X, \mathbb{Z})$ which is the orthogonal complement of ω. The Hodge structure on $V_{\mathbb{C}}$ is an AHS of weight 2 of type $(1, 19, 1)$, where we take the polarization defined by the restriction of intersection form Q. The Hodge flag is

$$0 \subset F^2 = H^{20}(X) \subset F^1 = H^{20}(X) + H^{11}_{\text{prim}}(X) \subset F^0 = H^2_{\text{prim}}(X, \mathbb{C}).$$

The flag (F^p) is completely determined by F^2 since $F^1 = (F^2)^\perp$. This implies that the period space $\mathbb{D}_{\mathbf{m}}(V, Q)$ is isomorphic, as a complex manifold, to

$$\mathbb{D}_h(V) = \{\mathbb{C}v \in P(V_{\mathbb{C}}); Q(v, v) = 0, Q(v, \bar{v}) > 0\}.$$

The integral structure Λ is again given by $H^2_{\text{prim}}(X, \mathbb{Z})$. The monodromy group Γ is a subgroup of $\text{Aut}(\Lambda, Q|_\Lambda)$.

One can generalize this to a so-called lattice K3-surface. Let M be an even lattice of signature $(1, r-1)$. A M-polarized K3-surface is a pair (X, j) consisting of an algebraic K3-surface X and a primitive embedding of lattices $j\colon M \to Pic(X)$ such that the image of j contains an ample class. If M is rank one, it reduces to the previous case. A family of M-polarized K3-surfaces is $\pi\colon Y \to T$, a family of K3-surfaces such that there is an embedding $j_t\colon M \to H^2(X_t, \mathbb{Z})$.

Let $N = M^\perp$ be the orthogonal complement of M in L_{K3}. The period domain is

$$\mathbb{D}_M = \{\mathbb{C}v \in P(N_{\mathbb{C}}); Q(v, v) = 0, Q(v, \bar{v}) > 0\}.$$

$\mathbb{D}_M$ is a hermitian symmetry space and of great interest to number theorist. The modular form in this context is referred as automorphic form in literature.

There is an inhomogeneous description of $\mathbb{D}_M$ similar to that of the upper half plane. Suppose that e and f span a hyperbolic lattice; i.e., $Q(e, e) = Q(f, f) = 0, Q(e, f) = 1$. Consider the decomposition

$$V = V_0 \oplus \mathbb{R}f \oplus \mathbb{R}e.$$

We can identify

$$\mathbb{D}_M = \{z \in V_0(\mathbb{C}) : \operatorname{Im} Q(z, z) > 0\}$$

via the map

$$z \to w(z) = z + f + Q(z, z)e.$$

Using the above map, we can figure out the automorphic factor—the generalization of $(c\tau + d)^{-k}$.

Example 2.5. Suppose that X is a Calabi-Yau 3-fold. We obtain a weight 3 AHS

$$0 \subset F^3 = H^{30} \subset F^2 = H^{30} + H^{21} \subset F^1 = H^{30} + H^{21} + H^{12} \subset F^0 = V_{\mathbb{C}}.$$

on $V = H^3(X, \mathbb{R})$. The polarization Q is symplectic in this case. The moduli space of complex structures M_X on X is smooth of dimension $h = H^{21}$. The period domain $\mathbb{D}_{\mathbf{m}}$ for $\mathbf{m}= (2h + 2, 2h + 1, h + 1, 1)$ is not a hermitian symmetry space in general. The relation to number theory is not clear. However, we can define a modular form formally as a section of $\mathbb{L}^k$. What is lacking is a inhomogeneous description similar to the upper half plane. However, we can again use the periods to define a convenient coordinate system. The monodromy group Γ can be viewed as a subgroup of $Sp(h + 1, \mathbb{Z})$.

One can conveniently forget about F^3, F^2. Then, we obtain a weight 1 AHS

$$0 \subset F^2 \subset F^0.$$

This defines an embedding of the moduli space of complex structures into the Siegel upper half plane

$$i\colon M_X \to \mathcal{Z}_{h+1}/\Gamma.$$

3. Global Landau-Ginzburg B-model at genus zero

In this section we construct the genus-zero data (Saito structure) of the global B-model over a deformation space of quasi-homogeneous polynomials. This is given as a vector bundle formed by the twisted de Rham cohomology, equipped with the Gauss-Manin connection and the higher residue pairing. In many ways, the material in this section is already standard to the experts (see, e.g. [63, 66, 61, 38]); the (only) novel point in our construction is that we restrict ourselves to relevant and marginal deformations so that the resulting structure is global and algebraic.

3.1. **A family of polynomials.** Let $x_1, \dots, x_n$ be variables of degrees $c_1, \dots, c_n$ with $0 < c_i < 1$, $c_i \in \mathbb{Q}$. Let $\mathcal{M}_{\rm mar}$ denote the space of all weighted homogeneous polynomials of degree one.

$$\mathcal{M}_{\rm mar} = \{f \in \mathbb{C}[x_1, \dots, x_n] : \deg f = 1\}.$$

Here the subscript "mar" means marginal deformations following the terminology in physics. We recall the following standard fact:

Proposition 3.1 ([25]). *For a weighted homogeneous polynomial $f \in \mathcal{M}_{\rm mar}$, the following conditions are equivalent:*

(1) *$f(x) = 0$ has an isolated singularity at the origin;*
(2) *$\partial_{x_1} f(x), \dots, \partial_{x_n} f(x)$ form a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$.*

For such f, the dimension of the Jacobi ring

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, \dots, x_n]/(\partial_{x_1} f, \dots \partial_{x_n} f)$$

is independent of f and is given by

$$N := \frac{(1-c_1)(1-c_2)\cdots(1-c_n)}{c_1 c_2 \cdots c_n}.$$

Moreover polynomials f satisfying the conditions (1) and (2) form a (possibly empty) Zariski open subset of $\mathcal{M}_{\rm mar}$.

Definition 3.2. We say that a weighted homogeneous polynomial $f \in \mathcal{M}_{\rm mar}$ is *regular* if one of the equivalent conditions in Proposition 3.1 holds. Let $\mathcal{M}^\circ_{\rm mar} \subset \mathcal{M}_{\rm mar}$ denote the Zariski open subset consisting of regular homogeneous polynomials. We denote by $\mathcal{M}$ the space of polynomials of degree ≤ 1 with regular leading terms:

$$\mathcal{M} := \left\{ f \in \mathbb{C}[x_1, \dots, x_n] : f = \sum_{0<d\le 1} f_d,\ \deg(f_d) = d,\ f_1 \in \mathcal{M}^\circ_{\rm mar} \right\}.$$

We will henceforth assume that $\mathcal{M}^\circ_{\rm mar}$ is nonempty. For a point $t \in \mathcal{M}$, we write $f(x;t) \in \mathbb{C}[x_1, \dots, x_n]$ for the polynomial represented by $t \in \mathcal{M}$. Setting $X := \mathbb{C}^n \times \mathcal{M}$, we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "f(x;t)"] \arrow[d, "\pi"'] & \mathbb{C} \\ \mathcal{M} & \end{tikzcd} \tag{1}$$

where $\pi\colon X = \mathbb{C}^n \times \mathcal{M} \to \mathcal{M}$ is the projection to the second factor. The space $\mathcal{O}(\mathcal{M})$ of regular functions on $\mathcal{M}$ is graded as follows. A finite cover of $\mathbb{C}^\times$ acts on $\mathcal{M}$ by $\lambda \cdot f := \lambda^{-1} f(\lambda^{c_1} x_1, \dots, \lambda^{c_n} x_n)$; this action induces the action on functions $\varphi \in \mathcal{O}(\mathcal{M})$ by $(\lambda \cdot \varphi)(f) = \varphi(\lambda^{-1} \cdot f)$. We say that $\varphi \in \mathcal{O}(\mathcal{M})$ is of degree $d \in \mathbb{Q}$ if $\lambda \cdot \varphi = \lambda^d \varphi$. The grading on $\mathcal{O}(\mathcal{M})$ and $\deg x_i = c_i$ together define a grading on $\mathcal{O}(X) = \mathcal{O}(\mathcal{M} \times \mathbb{C}^n)$. The universal polynomial $f(x;t) \in \mathcal{O}(X)$ is of degree one with respect to this grading. What is important for us is the fact that $\mathcal{O}(\mathcal{M})$ and $\mathcal{O}(X)$ are *non-negatively* graded.

We introduce the *critical scheme* $C \subset X$ as follows:

$$\mathcal{O}_C = \mathcal{O}_X / (\partial_{x_1} f(x;t), \dots, \partial_{x_n} f(x;t)).$$

Proposition 3.1 implies that $(\pi_* \mathcal{O}_C)|_{\mathcal{M}^\circ_{\mathrm{mar}}}$ is a locally free cohrerent sheaf of rank N; we will see that $\pi_* \mathcal{O}_C$ is also locally free in Corollary 3.6 below.

Remark 3.3. A group of coordinate changes on $\mathbb{C}^n$ acts on the parameter space $\mathcal{M}$ and our global B-model is equivariant with respect to the group. Let G be the group of ring automorphisms of $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ preserving the degree filtration $\mathbb{C}[x]^{\le d} = \{f \in \mathbb{C}[x] : \deg f \le d\}$. Then G acts on $\mathcal{M}$ and the diagram (1). The quotient stack $[\mathcal{M}/G]$ should be viewed as a genuine moduli space.

Remark 3.4. In practice, it is convenient to work with a family of polynomials of the following form: for a weighted homogeneous polynomial $f_0(x)$ of degree one and a set of homogeneous polynomials $\phi_e(x)$, we can consider a family $f(x;t) = f_0(x) + \sum_e t_e \phi_e(x)$. For such a family, we say that the deformation parameter t_e is *relevant* (resp. *marginal*, *irrelevant*) if $\deg \phi_e < 1$ (resp. $\deg \phi_e = 1$, $\deg \phi_e > 1$). We can assign the degree of parameters as $\deg t_e := 1 - \deg \phi_e(x)$. The above space $\mathcal{M}$ includes only *relevant and marginal* deformations. When we construct a miniversal deformation (see §5.2), we choose homogeneous polynomials $\{\phi_e\}_{e=1}^N$ such that $[\phi_1], \dots, [\phi_N]$ form a basis of the Jacobi ring $\mathrm{Jac}(f_0)$; in this case the deformation family may also contain irrelevant directions.

3.2. **The twisted de Rham cohomology.** We are interested in the hypercohomology of the twisted de Rham complex:

$$\mathcal{F} = \mathbb{R}^n \pi_* \left(\Omega^\bullet_{X/\mathcal{M}}[z], z d_{X/\mathcal{M}} + df(x;t) \wedge \right)$$

where $f(x;t)\colon X \to \mathbb{C}$ is the universal polynomial in the diagram (1). Since π is affine, this is:

$$\mathcal{F} \cong \Omega^n_{X/\mathcal{M}}[z] / (z d_{X/\mathcal{M}} + df(x;t) \wedge) \Omega^{n-1}_{X/\mathcal{M}}[z].$$

The fiber of $\mathcal{F}$ at a single polynomial f is called the *Brieskorn lattice* [9] of f. A presentation of the Brieskorn lattice as a twisted de Rham cohomology group was given in [64]; this is also called the *filtered de Rham cohomology* (see [66]). We introduce the grading on $\mathcal{F}$ given by the grading on $\mathcal{O}(X)$ together with $\deg(dx_i) = c_i$, $\deg z = 1$. This is well-defined since the differential $z d_{X/\mathcal{M}} + df(x;t)\wedge$ is of degree one. The module of global sections of $\mathcal{F}$ is again non-negatively graded.

Proposition 3.5. *The sheaf $\mathcal{F}$ is a locally free $\mathcal{O}_{\mathcal{M}}[z]$-module of rank N.*

Proof. Let us fix an affine open subset $U \subset \mathcal{M}^\circ_{\mathrm{mar}}$ such that $(\pi_* \mathcal{O}_C)|_U$ is a free $\mathcal{O}_U$-module. Choose quasi-homogeneous polynomials $\psi_i \in \mathcal{O}(U)[x_1, \dots, x_n]$, $1 \le i \le N$

which induce a basis of $(\pi_* \mathcal{O}_C)|_U$. We claim that $\psi_i dx$, $1 \le i \le N$ form a basis of $\mathcal{F}$ over $\mathcal{M}_{\mathrm{rel}} \times U$, i.e. the map

$$\phi \colon (\mathcal{O}_{\mathcal{M}_{\mathrm{rel}} \times U}[z])^{\oplus N} \to \mathcal{F}|_{\mathcal{M}_{\mathrm{rel}} \times U}$$

sending $(v_i)_{i=1}^N$ to the class of $\sum_{i=1}^N v_i \psi_i dx$ is an isomorphism, where we set $dx = dx_1 \wedge \cdots \wedge dx_n$.

First we prove the surjectivity. Suppose by induction that the submodule $\mathcal{F}(\mathcal{M}_{\mathrm{rel}} \times U)^{\le k}$ of degree less than or equal to k is contained in the image of ϕ. Every homogeneous element $\omega \in \Omega^n_{X/\mathcal{M}}[z]$ of degree $\le (k+1)$ can be written as $\omega = \sum_{i=1}^n v_i \psi_i dx + df \wedge \alpha$ for some $\alpha \in \Omega^{n-1}_{X/\mathcal{M}}[z]$. By taking the homogeneous component if necessary, we may assume that α is homogeneous of degree $\le k$. Then $[\omega] = [\omega - (zd + df\wedge)\alpha] = \sum_{i=1}^N v_i[\psi_i dx] - z[d\alpha]$. By induction hypothesis, $[d\alpha]$ is in the image of ϕ and thus ω is also in the image.

Let us prove the injectivity of ϕ. Suppose that $\phi(v) = 0$ for some $v = (v_1, \dots, v_N)$. By definition there exists $\alpha \in \Omega^{n-1}_{X/\mathcal{M}}[z]$ such that $\sum_{i=1}^N v_i \psi_i dx = (zd + df\wedge)\alpha$. Expand v_i and α in powers of z:

$$v_i = \sum_{k \ge 0} v_{i,k} z^k, \quad \alpha = \sum_{k \ge 0} \alpha_k z^k$$

where the sum is finite. Comparing with the coefficient of z^0, we have $\sum_{i=1}^N v_{i,0}\psi_i = df \wedge \alpha_0$. Since ψ_i, $1 \le i \le N$ form a basis of the Jacobi ring, we have $v_{i,0} = 0$ for all i. Therefore $df \wedge \alpha_0 = 0$. Because $\partial_{x_1} f(x;t), \dots, \partial_{x_n} f(x;t)$ form a regular sequence, there exists $\beta_0 \in \Omega^{n-2}_{X/\mathcal{M}}$ such that $\alpha_0 = df \wedge \beta_0$. Setting $\alpha' = \alpha - (zd + df\wedge)\beta_0 = \sum_{k \ge 1} \alpha'_k z^k$, we have

$$\sum_{i=1}^N \sum_{k \ge 1} z^k v_{i,k} \psi_i = (zd + dF\wedge)\alpha'.$$

Comparing with the coefficient of z^1, we obtain $v_{i,1} = 0$ for all i. We can repeat this argument inductively to show that $v_{i,0} = v_{i,1} = \cdots = 0$. ☐

Since we can identify the restriction $\mathcal{F}|_{z=0}$ with $(\pi_* \mathcal{O}_C) dx$, we obtain:

Corollary 3.6. *The sheaf $\pi_* \mathcal{O}_C$ is a locally free $\mathcal{O}_\mathcal{M}$-module of rank N.*

3.3. **The Gauss-Manin connection and the higher residue pairing.** Here we introduce two important structures on the twisted de Rham cohomology $\mathcal{F}$: the Gauss-Manin connection ∇ and the higher residue pairing K. The *Gauss-Manin connection* ∇ is a map

$$\nabla \colon \mathcal{F} \to z^{-1}\Omega^1_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{F} \oplus z^{-2}\mathcal{O}_\mathcal{M}[z]dz$$

defined by the formula:

$$\nabla_{\vec{v}}[\phi(x,t,z)dx] = \left[\vec{v}\phi(x,t,z) + \frac{\vec{v}(f(x;t))}{z}\phi(x,t,z)dx\right]$$
$$\nabla_{\partial_z}[\phi(x,t,z)dx] = \left[\left(\frac{\partial \phi(x,t,z)}{\partial z} - \frac{f(x;t)}{z^2}\phi(x,t,z) - \frac{n}{2}\frac{\phi(x,t,z)}{z}\right)dx\right]$$

where $\phi(x,t,z) \in \mathcal{O}_X[z]$, $dx = dx_1 \wedge \cdots \wedge dx_n$ and $\vec{v}$ is a vector field on $\mathcal{M}$. One can easily check that this is well-defined; moreover it satisfies the Leibnitz rule:

$$\nabla(g(t,z)\omega) = dg(t,z)\omega + g(t,z)\nabla\omega, \qquad g(t,z) \in \mathcal{O}_{\mathcal{M}}[z],\ \omega \in \mathcal{F}$$

and the flatness condition $\nabla^2 = 0$. The *higher residue pairing* of K. Saito [64] is a map

$$K \colon \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{F} \to \mathcal{O}_{\mathcal{M}}[z]$$

which we expand in the form

$$K(\omega_1, \omega_2) = \sum_{p=0}^{\infty} z^p K^{(p)}(\omega_1, \omega_2).$$

The higher residue pairing is uniquely characterized by the following properties:

(1) $zK(\omega_1,\omega_2) = K(z\omega_1,\omega_2) = -K(\omega_1, z\omega_2)$;
(2) $K^{(0)}(\omega_1,\omega_2)$ is the residue pairing on the Jacobi algebra of f:

$$K^{(0)}(\omega_1,\omega_2) = \mathrm{Res}_{X/\mathcal{M}} \left[\frac{\phi_1(x,t,0)\phi_2(x,t,0)dx}{\partial_{x_1} f(x;t), \ldots, \partial_{x_n} f(x;t)} \right]$$

where $\omega_i = \phi_i(x,t,z)dx$;
(3) $K(\omega_1,\omega_2)(z) = K(\omega_2,\omega_1)(-z)$; i.e., $K^{(p)}$ is skew symmetric for p odd and symmetric for p even;
(4) K is flat with respect to the Gauss-Manin connection:

$$\xi K(\omega_1,\omega_2) = K(\nabla_\xi \omega_1, \omega_2) - K(\omega_1, \nabla_\xi \omega_2),$$

where $\xi = z\vec{v}$ (with $\vec{v}$ a vector field on $\mathcal{M}$) or $z^2\partial/\partial z$.

Remark 3.7. The Gauss-Manin connection is defined in such a way that oscillatory integrals

$$\mathcal{F} \ni [\phi dx] \longmapsto (-2\pi z)^{-n/2} \int_\Gamma \phi(x) e^{f(x;t)/z} dx \tag{2}$$

define solutions (i.e. intertwine the Gauss-Manin connection with the standard differential), where Γ is a cycle in $H_n(\mathbb{C}^n, \{x \in \mathbb{C}^n : \mathrm{Re}(f(x)/z) \ll 0\}; \mathbb{Z})$. The prefactor $(-2\pi z)^{-n/2}$ here should be viewed as a shift of weights by $n/2$; this is introduced in order to make the Gauss-Manin connection compatible with the Dubrovin connection on the A-side under mirror symmetry. This in turn results in the shift of the higher residue pairing K_f by the factor of z^n.

Definition 3.8. We call the triple $(\mathcal{F}, \nabla, K)$ consisting of the twisted de Rham cohomology, the Gauss-Manin connection and the higher residue pairing the *Saito structure* of the family (1) of polynomials.

Remark 3.9. The Saito structure gives a *TEP structure* in the sense of Hertling [39].

4. Opposite subspaces

In this section, we introduce opposite subspaces for the Saito structure $(\mathcal{F}, \nabla, K)$. For a marginal polynomial f, we obtain a one-to-one correspondence between homogeneous opposite subspaces and splittings (opposite filtration) of the Hodge filtration on the vanishing cohomology. We also observe that the complex-conjugate opposite subspace yields a positive-definite Hermitian bundle with connection, called the Cecotti-Vafa structure. The notion of opposite filtrations were originally used in the work of M. Saito [66] to construct a flat structure (Frobenius structure) [63, 26] on the base of miniversal deformations (see also §5.2). Most of the materials in this section are again not new; similar (and in fact more general) results have been obtained by Saito [66] and Hertling [39]. Since we restrict ourselves to weighted homogeneous polynomials, our presentation has the advantage of being more explicit and elementary.

4.1. **Symplectic vector space and semi-infinite VHS.** Let us recall that we sometimes identify the points $t \in \mathcal{M}$ with the corresponding polynomials $f = f(x;t)$, so the points in $\mathcal{M}$ are functions. Recall the sheaf $\mathcal{F}$ of twisted de Rham cohomology groups from §3.2. Proposition 3.5 implies that $\mathcal{F}$ is the sheaf of sections of a vector bundle $\mathbb{H}_+$ on $\mathcal{M}$, whose fiber over a deformation $f \in \mathcal{M}$ is given by the infinite-dimensional vector space

$$\mathbb{H}_+(f) := H_{\mathrm{twdR}}(f) := \Omega^n_{\mathbb{C}^n}[z]/(zd + df\wedge)\Omega^{n-1}_{\mathbb{C}^n}[z] \cong \mathrm{Jac}(f)[z].$$

We introduce the free $\mathbb{C}[z, z^{-1}]$-module

$$\mathbb{H}(f) := H_{\mathrm{twdR}}(f) \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}]$$

and its completion

$$\widehat{\mathbb{H}}(f) := H_{\mathrm{twdR}}(f) \otimes_{\mathbb{C}[z]} \mathbb{C}(\!(z)\!).$$

The spaces $\mathbb{H}(f)$, $\widehat{\mathbb{H}}(f)$ are equipped with the symplectic form

$$\Omega(\omega_1, \omega_2) = \mathrm{Res}_{z=0}\, K_f(\omega_1, \omega_2) dz$$

where K_f is the restriction of the higher residue pairing to the fiber of the sheaf $\mathcal{F}$ at f. Note that $\mathbb{H}_+(f)$ is a Lagrangian (i.e. maximally isotropic) with respect to Ω.

The spaces $\mathbb{H}(f)$, $\widehat{\mathbb{H}}(f)$ are the B-model analogues of Givental's symplectic space [33]. The Gauss-Manin connection induces a flat connection ∇ on the bundle $\mathbb{H} = \bigcup_f \mathbb{H}(f)$ and the symplectic form Ω is flat with respect to ∇. Over a contractible subset U of the marginal locus $\mathcal{M}^\circ_{\mathrm{mar}}$, we can identify all fibers $\mathbb{H}(f)$ via parallel transport[1] with a single symplectic space $\mathcal{H}$; then we can regard $f \mapsto \mathbb{H}_+(f)$ as a family of Lagrangian subspaces in $\mathcal{H}$ parametrized by $f \in U$. This is an example of the *semi-infinite variation of Hodge structure* (semi-infinite VHS) in the sense of Barannikov [5] (see also [21]). The main property of the semi-infinite VHS is the Griffiths transversality:

$$\nabla_{\vec{v}} \mathbb{H}_+(f) \subset z^{-1}\mathbb{H}_+(f) \qquad \text{for } \vec{v} \in T\mathcal{M}$$

[1]The parallel transport is well-defined only over the marginal locus; the parallel transport along relevant deformations involves infinitely many negative powers of z, and only makes sense after tensoring $\mathbb{H}$ with the ring of holomorphic functions on $\{z \in \mathbb{C}^\times\}$ over $\mathbb{C}[z, z^{-1}]$.

for the semi-infinite flag $\cdots \subset z\mathbb{H}_+(f) \subset \mathbb{H}_+(f) \subset z^{-1}\mathbb{H}_+(f) \subset \cdots$. In the z-direction, we also have $\nabla_{z\partial_z}\mathbb{H}_+(f) \subset z^{-1}\mathbb{H}_+(f)$.

4.2. Definition and first properties.

Definition 4.1 ([5, 21])**.** We say that a Lagrangian subspace $P \subset \mathbb{H}(f)$ is *opposite* if $\mathbb{H}(f) = \mathbb{H}_+(f) \oplus P$ and $z^{-1}P \subset P$.

The vector space $\mathbb{H}(f)$ can be identified with the space of sections of a vector bundle over $\{z \in \mathbb{C}^\times\}$ and the subspace $\mathbb{H}_+(f)$ corresponds to the extension of the vector bundle across 0. In this viewpoint, the data of an opposite subspace P corresponds to an extension of the bundle across ∞ such that the resulting bundle over $\mathbb{P}^1$ is trivial.

Proposition 4.2. *If P is an opposite subspace, then the following properties hold:*

(1) *The vector space $\mathbb{H}_+(f) \cap zP$ has dimension N.*
(2) *If $\{\omega_i\}_{i=1}^N$ is a basis of $\mathbb{H}_+(f) \cap zP$, then $K(\omega_i, \omega_j) \in \mathbb{C}$.*
(3) *Let $\{\omega_i\}$ and $\{\omega^i\}$ be dual bases of $\mathbb{H}_+(f) \cap zP$ with respect to the residue pairing $K_f^{(0)}$. Then*

$$\{\omega_i z^k\}_{i=1,\dots,N}^{k=0,1,\dots} \quad \text{and} \quad \{\omega^i(-z)^{-k-1}\}_{i=1,\dots,N}^{k=0,1,\dots}$$

are bases of respectively $\mathbb{H}_+(f)$ and P dual with respect to the symplectic pairing.

The proof of the above proposition is straightforward, so it will be omitted. Motivated by Proposition 4.2, for a given opposite subspace P we will refer to a basis of $\mathbb{H}_+(f) \cap zP$ as a *good basis*. Note that a $\mathbb{C}[z]$-basis $\{\omega_i\}$ of $\mathbb{H}_+(f)$ is good if and only if $K(\omega_i, \omega_j) \in \mathbb{C}$. Similarly, one can define the notion of an opposite subspace and a good basis for the completion $\widehat{\mathbb{H}}(f)$ and its Lagrangian subspace $\widehat{\mathbb{H}}_+(f) := \mathbb{H}_+(f) \otimes_{\mathbb{C}[z]} \mathbb{C}[\![z]\!]$. Proposition 4.2 still holds, except for property (3), which takes the following form. Put $H := \widehat{\mathbb{H}}_+(f) \cap zP$, then

$$\widehat{\mathbb{H}}_+(f) = H[\![z]\!], \quad P = H[z^{-1}]z^{-1}. \tag{3}$$

An opposite subspace $P \subset \mathbb{H}(f)$ at $f \in \mathcal{M}$ can be extended to a family of opposite subspaces in a neighbourhood U of $f \in \mathcal{M}$ by parallel transport (see the discussion in §5.2 and [21, §2.2]). We regard this family of opposite subspaces as a subbundle of $\mathbb{H} = \bigcup_f \mathbb{H}(f)$ and denote it again by P. The Gauss-Manin connection induces a flat connection on the finite-dimensional bundle zP/P and the identification

$$\mathbb{H}_+ \cap zP \cong zP/P$$

induces a trivialization $\mathbb{H}_+ \cong (zP/P)[z]$ over U by a flat bundle zP/P. With respect to this trivialization, the Gauss-Manin connection is of the form:

$$\nabla = d + \frac{1}{z}C$$

with $C \in \mathrm{End}(zP/P) \otimes \Omega^1_{\mathcal{M}}$ independent of z. This fact is crucial in the construction of a Frobenius (flat) structure. See §5.2 for more details.

4.3. **Homogeneous opposite subspaces over the marginal moduli.** In this subsection we assume that f lies in the marginal moduli $\mathcal{M}^\circ_{\rm mar}$, i.e. f is a weighted homogeneous polynomial of degree 1. The operator

$$z\partial_z + \mathrm{Lie}_\xi, \quad \xi := \sum_{i=1}^n c_i x_i \partial_{x_i}, \tag{4}$$

where Lie denotes the Lie derivative, defines a grading on $\Omega^n_{\mathbb{C}^n}[z,z^{-1}]$. Since the twisted de Rham differential $zd + df(x;t)\wedge$ is homogeneous (of degree 1), the twisted de Rham cohomology $\mathbb{H}(f)$ inherits the grading. We say that an opposite subspace $P \subset \mathbb{H}(f)$ is *homogeneous* if $(z\partial_z + \mathrm{Lie}_\xi)P \subset P$. We would like to establish one-to-one correspondence between homogeneous opposite subspaces and splittings of the Steenbrink's Hodge filtration of the vanishing cohomology $\mathfrak{h} := H^{n-1}(f^{-1}(1);\mathbb{C})$.

Remark 4.3. An opposite subspace P is homogeneous if and only if P is preserved by the Gauss-Manin connection in the z-direction, i.e. $\nabla_{z\partial_z} P \subset P$. This implies that the Gauss-Manin connection has a logarithmic singularity at $z=\infty$ with respect to the extension of the bundle $\mathbb{H}_+(f)$ across ∞ defined by P; this corresponds to the notion of TLEP structure [38].

4.3.1. *Steenbrink's Hodge structure for weighted-homogeneous singularities.* Given a holomorphic form $\omega \in \Omega_{\mathbb{C}^n}(\mathbb{C}^n)$ we recall the so-called *geometric section* (see [4])

$$s(\omega,\lambda) := \int \frac{\omega}{df} \quad \in \quad H^{n-1}(f^{-1}(\lambda);\mathbb{C}),$$

where ω/df denotes a holomorphic $(n-1)$-form η defined in a tubular neighborhood of $f^{-1}(\lambda)$ such that $\omega = df\wedge\eta$; the restriction of η to $f^{-1}(\lambda)$ is well-defined. By definition, Steenbrink's Hodge filtration [70, 71] on $\mathfrak{h}$ is given by

$$F^p\mathfrak{h} := \{A \in \mathfrak{h} \mid A = s(\omega,1) \text{ for some } \omega \text{ such that } \deg(\omega) \le n-p\},$$

where $\deg(\omega)$ denotes the maximal degree of a homogeneous component of ω. This is an exhaustive filtration; in particular every cohomology class of $f^{-1}(1)$ can be represented by a geometric section.

The vector space $\mathfrak{h} = H^{n-1}(f^{-1}(1);\mathbb{C})$ is equipped with a linear transformation $M \in \mathrm{End}(\mathfrak{h})$, called the *classical monodromy*, which corresponds to the monodromy of the Gauss–Manin connection around $\lambda = 0$. Using the fact that f is weighted-homogeneous, it is easy to see that if $A = s(\omega,1)$ for some homogeneous form ω, then $M(A) = e^{-2\pi\sqrt{-1}\deg(\omega)}A$. Let us decompose $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_{\neq 1}$, where $\mathfrak{h}_1$ is the invariant subspace of M and $\mathfrak{h}_{\neq 1}$ is the remaining part of the spectral decomposition of $\mathfrak{h}$ with respect to M. Following Hertling (see [38, Ch. 10]) we introduce the non-degenerate bilinear form

$$S(A,B) = (-1)^{(n-1)(n-2)/2}\langle A, \mathrm{Var}\circ\nu(B)\rangle, \quad A,B \in \mathfrak{h},$$

where ν is a linear operator such that $\nu = (M-1)^{-1}$ on $\mathfrak{h}_{\neq 1}$ and $\nu = -1$ on $\mathfrak{h}_1$, and the variation operator

$$\mathrm{Var}\colon H^n(f^{-1}(1);\mathbb{C}) \to H_n(f^{-1}(1);\mathbb{C})$$

is an isomorphism constructed via the composition of the Lefschetz duality

$$H^n(f^{-1}(1);\mathbb{C}) \cong H_n(f^{-1}(1), \partial f^{-1}(1); \mathbb{C})$$

and the isomorphism $H_n(f^{-1}(1), \partial f^{-1}(1)) \cong H_n(f^{-1}(1))$ mapping a relative cycle γ to an absolute cycle $\widetilde{M}(\gamma) - \gamma$, where $\widetilde{M} : f^{-1}(1) \to f^{-1}(1)$ is the geometric monodromy fixing the boundary (see [4, Ch. 1.1, 2.3]).

Combining the results of Hertling (see [38, Ch. 10]) and Steenbrink (see [71]) we get the following: the filtration $\{F^p\mathfrak{h}\}_{p=0}^{n-1}$, the form S, and the real subspace $\mathfrak{h}_{\mathbb{R}} := H^{n-1}(f^{-1}(1), \mathbb{R})$ give rise to a (pure) Polarized Hodge Structures on $\mathfrak{h}_{\neq 1}$ and $\mathfrak{h}_1$ of weights respectively $n-1$ and n. More precisely, put $\mathfrak{h}_s = \mathrm{Ker}(M - s\,\mathrm{Id})$; then $F^p\mathfrak{h} = \bigoplus_{s\in S^1} F^p\mathfrak{h}_s$ with $F^p\mathfrak{h}_s = F^p\mathfrak{h} \cap \mathfrak{h}_s$ and

$$\begin{array}{ll} \text{(a)} & \mathfrak{h}_s = F^p\mathfrak{h}_s \oplus \overline{F^{m+1-p}\mathfrak{h}_{\bar{s}}}, \quad \forall p \in \mathbb{Z}, \\ \text{(b)} & S(u,v) = (-1)^m S(v,u), \\ \text{(c)} & S(F^p\mathfrak{h}, F^{m+1-p}\mathfrak{h}) = 0, \\ \text{(d)} & \sqrt{-1}^{2p-m} S(u, \overline{u}) > 0 \quad \text{for} \quad u \in F^p\mathfrak{h}_s \cap \overline{F^{m-p}\mathfrak{h}_{\bar{s}}} \setminus 0, \end{array}$$

where $m = n-1$ for $s \neq 1$ and $m = n$ for $s = 1$. Note that $S(\mathfrak{h}_s, \mathfrak{h}_t) = 0$ unless $t = \bar{s}$ and that $\overline{\mathfrak{h}_s} = \mathfrak{h}_{\bar{s}}$.

4.3.2. *The polarizing form and the higher-residue pairing.* We will identify the vector space $\mathfrak{h} = H^{n-1}(f^{-1}(1); \mathbb{C})$ with a fiber of the local system underlying the Gauss-Manin connection $(\mathbb{H}_+(f) = \mathcal{F}|_f, \nabla_{z\partial_z})$. Then we describe the higher residue pairing in terms of the polarizing form S on $\mathfrak{h}$.

Recall that oscillatory integrals (2) give solutions of the Gauss-Manin connection, and therefore the local system underlying the Gauss-Manin connection is dual to the space

$$V_{f,z} := \varprojlim_{M} H_n(\mathbb{C}^n, \{x \in \mathbb{C}^n : \mathrm{Re}(f(x)/z) \leq -M\}) \tag{5}$$

of Lefschetz thimbles (twisted by $(-2\pi z)^{-n/2}$). By the relative homology exact sequence, we can easily see that this is isomorphic to $H_{n-1}(f^{-1}(1))$; hence fibers of the Gauss-Manin local system should be identified with $\mathfrak{h}$. To make this identification explicit, we use the Laplace transformation. When $z < 0$ and the integration cycle Γ in (2) is a Lefschetz thimble of f lying over the straight ray $[0, \infty)$, we may rewrite the oscillatory integral (2) as the Laplace transform of a period

$$(-2\pi z)^{-n/2} \int_0^\infty e^{\lambda/z} \int_{\Gamma_\lambda} s(\omega, \lambda)$$

where Γ_λ is a vanishing cycle in $f^{-1}(\lambda)$ such that $\Gamma = \bigcup_{\lambda\in[0,\infty)} \Gamma_\lambda$. This can be viewed as the pairing of the vanishing cycle $\Gamma_1 \subset f^{-1}(1)$ and the cohomology class $\widehat{s}(\omega, z)$ of $f^{-1}(1)$ given by:

$$\widehat{s}(\omega, z) := (-2\pi z)^{-n/2} \int_0^\infty e^{\lambda/z} s(\omega, \lambda) d\lambda$$

where we identified $H^{n-1}(f^{-1}(\lambda); \mathbb{C}) \cong \mathfrak{h}$ via the parallel transport along the integration path (with respect to the Gauss-Manin connection), so that $s(\omega, \lambda)$ takes values in $\mathfrak{h}$.

Thus the map $[\omega] \mapsto \widehat{s}(\omega, z)$ defines a flat identification between $\mathbb{H}_+(f)|_z$ and $\mathfrak{h}$. For a homogeneous form $\omega \in \Omega^n_{\mathbb{C}^n}(\mathbb{C}^n)$, the geometric section $s(\omega,\lambda)$ satisfies the homogeneity $s(\omega,\lambda) = \lambda^{\deg(\omega)-1} s(\omega,1)$, and therefore we find

$$\widehat{s}(\omega, z) = (-2\pi z)^{-n/2}(-z)^{\deg(\omega)}\Gamma(\deg\omega)s(\omega,1). \tag{6}$$

Thus $\widehat{s}(\omega,z)$ makes sense as a Laurent polynomial of z (with fractional exponents) taking values in $\mathfrak{h}$. We verify the following lemma directly.

Lemma 4.4. *The map*

$$\mathbb{H}_+(f) = \mathcal{F}|_f \longrightarrow \mathfrak{h}[z^{\pm 1/d}], \quad [\omega] \longmapsto \widehat{s}(\omega, z)$$

is well-defined and intertwines the Gauss-Manin connection $\nabla_{z\partial_z}$ with the standard differential $z\partial_z$, where d is a common denominator of $c_1, \dots, c_n$ and $n/2$. This induces an isomorphism $\mathbb{H}_+(f)|_z \cong \mathfrak{h}$ between fibers for every $z \in \mathbb{C}^\times$.

Proof. Let us first check that the map passes to the quotient $\mathbb{H}_+(f) = \Omega^n_{\mathbb{C}^n}(\mathbb{C}^n)[z]/(zd + df\wedge)\Omega^{n-1}_{\mathbb{C}^n}(\mathbb{C}^n)[z]$. If ω is a homogeneous $(n-1)$-form of degree m, then the image of $zd\omega + df\wedge\omega$ is

$$\Gamma(m)s(d\omega,1) - \Gamma(m+1)s(df\wedge\omega,1). \tag{7}$$

multiplied by $(-2\pi z)^{-n/2}(-z)^m z$. On the other hand

$$s(d\omega,\lambda) = \int \frac{d\omega}{df} = \partial_\lambda \int \omega = \partial_\lambda s(df\wedge\omega,\lambda).$$

Using homogeneity, $s(df\wedge\omega,\lambda) = \lambda^m s(df\wedge\omega,1)$. Hence $s(d\omega,1) = m\, s(df\wedge\omega,1)$, so the expression (7) vanishes. This proves that the map in the Lemma passes to the quotient.

Next we show that the map intertwines the Gauss-Manin connection with $z\partial_z$. For a homogeneous form ω of degree m, the image of $\nabla_{z\partial_z}[\omega] = [-(f/z + n/2)\omega]$ is

$$(-2\pi z)^{-n/2}(-z)^m \left(\Gamma(m+1)s(f\omega,1) - \frac{n}{2}\Gamma(m)s(\omega,1)\right)$$

which equals $z\partial_z \widehat{s}(\omega,z) = (m - n/2)\widehat{s}(\omega,z)$ since $s(f\omega,1) = s(\omega,1)$.

The last statement follows by comparing the ranks: the map is surjective since every class on $f^{-1}(1)$ is represented by a geometric section, and Proposition 3.5 shows that the rank of $\mathbb{H}_+(f)$ equals the Milnor number $N = \dim\mathfrak{h}$. □

Let us denote by $\widehat{s}(\omega,z)^* := \widehat{s}(\omega, e^{-\pi\sqrt{-1}}z)$ the analytic continuation along the semi-circle $\theta \mapsto e^{-\sqrt{-1}\theta}z$, $0 \le \theta \le \pi$. The relation between the polarizing form S and the higher residue pairing K has been determined by Hertling (see [38, Ch. 10], [39, §7.2(f); §8, Step 2]). We follow the presentation in [51].

Theorem 4.5 ([38, 39], [51, Lemma 3.3])**.** *The polarizing form S and the higher residue pairing K_f are related by the formula:*

$$K_f(\omega_2,\omega_1) = -S(\widehat{s}(\omega_1,z)^*, \nu^{-1}\widehat{s}(\omega_2,z)) \tag{8}$$

where in the right-hand side we use the determination of $\widehat{s}(\omega_i,z)$ given canonically for $z \in \mathbb{R}_{<0}$ via formula (6).

Remark 4.6. We give a brief explanation for the above formula (8). Pham [56, 2ème, §4] identified the higher residue pairing with the dual of the intersection pairing between the relative homologies $V_{f,z}$ and $V_{f,-z}$ from (5) (see also [39, §8], [21, Definition 2.18]). This intersection pairing can then be identified with the Seifert form $\langle A, \mathrm{Var}(B)\rangle$ on $\mathfrak{h}$ by a topological argument in [4, Ch. 2.3], [56, 2ème, §3.2]. Note that the right-hand side of (8) is induced by the Seifert form.

4.3.3. *Splitting of the Hodge structure.* Following M. Saito [66], we use opposite filtrations on $\mathfrak{h}$ to construct homogeneous opposite subspaces for $\mathbb{H}_+(f)$ (see also [38, Theorem 7.16]).

Definition 4.7. An *opposite filtration* on $\mathfrak{h} = H^{n-1}(f^{-1}(1);\mathbb{C})$ is an increasing M-invariant filtration $\{U_p\mathfrak{h}\}_{p\in\mathbb{Z}}$ such that

$$\text{(a)}\quad U_p\mathfrak{h} = 0 \quad \text{for} \quad p \ll 0 \quad \text{and} \quad U_p\mathfrak{h} = \mathfrak{h} \quad \text{for} \quad p \gg 0,$$

$$\text{(b)}\quad \mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} F^p\mathfrak{h} \cap U_p\mathfrak{h},$$

$$\text{(c)}\quad S(U_p\mathfrak{h}, U_{m-1-p}\mathfrak{h}) = 0, \quad \forall p \in \mathbb{Z},$$

where $m = n-1$ for $s \neq 1$ and $m = n$ for $s = 1$.

Let $\psi\colon \mathfrak{h} \to \mathbb{H}(f)[z^{\pm 1/d}]$ denote the map inverse to the map $[\omega] \mapsto \widehat{s}(\omega, z)$ in Lemma 4.4. This is given by

$$\psi(A) = \frac{(2\pi)^{n/2}}{\Gamma(\deg\omega)}(-z)^{-\deg\omega + n/2}[\omega] \tag{9}$$

when $A = s(\omega, 1) \in \mathfrak{h}$ for some homogeneous form $\omega \in \Omega^n_{\mathbb{C}^n}(\mathbb{C}^n)$. The image $\psi(A)$ is homogeneous of degree $n/2$ and is flat with respect to the Gauss-Manin connection. Take $s = e^{2\pi\sqrt{-1}\alpha}$ with $0 \le \alpha < 1$. Note that $\psi(A) \in z^{\alpha - n/2}\mathbb{H}(f)$ for $A \in \mathfrak{h}_s$. The Hodge filtration $F^p\mathfrak{h}_s$ can be described in terms of the Lagrangian subspace $\mathbb{H}_+(f)$ as

$$F^p\mathfrak{h}_s = \{A \in \mathfrak{h}_s : \psi(A) \in z^{p+\alpha-n/2}\mathbb{H}_+(f)\}$$

since, for $A = s(\omega, 1) \in \mathfrak{h}_s$, every homogeneous component ω_j of ω satisfies $\langle -\deg\omega_j\rangle = \alpha$ and we have

$$-\deg\omega_j + \frac{n}{2} \ge p + \alpha - \frac{n}{2} \iff \lceil \deg\omega_j \rceil \le n - p.$$

Conversely, $\mathbb{H}_+(f)$ can be reconstructed from $F^p\mathfrak{h}$ as

$$\mathbb{H}_+(f) = \sum_{0\le\alpha<1}\sum_{p\in\mathbb{Z}} z^{-p-\alpha+n/2}\psi(F^p\mathfrak{h}_s)[z] \qquad \text{with } s = e^{2\pi\sqrt{-1}\alpha}.$$

The correspondence between homogeneous opposite subspaces $P \subset \mathbb{H}(f)$ and opposite filtrations $U_p\mathfrak{h}$ is given similarly. We require that P and $U_p\mathfrak{h}$ are related by

$$\begin{aligned} U_p\mathfrak{h}_s &= \{A \in \mathfrak{h}_s : \psi(A) \in z^{p+\alpha-n/2} zP\} \\ P &= \sum_{0\le\alpha<1}\sum_{p\in\mathbb{Z}} z^{-p-\alpha+n/2} z^{-1}\psi(U_p\mathfrak{h}_s)[z^{-1}] \qquad \text{with } s = e^{2\pi\sqrt{-1}\alpha}. \end{aligned} \tag{10}$$

Proposition 4.8 ([66], [38, Ch 7.4])**.** *The formulas* (10) *establish one-to-one correspondence between homogeneous opposite subspaces P and opposite filtrations $\{U_p\mathfrak{h}\}_{p\in\mathbb{Z}}$.*

Proof. Given an opposite filtration $\{U_p\mathfrak{h}\}$, we prove that the subspace P defined by the second formula of (10) is an opposite subspace. It is obvious that P is homogeneous. Using the decomposition $\mathfrak{h}_s = \bigoplus_{p\in\mathbb{Z}}(F^p\mathfrak{h}_s \cap U_p\mathfrak{h}_s)$, we can rewrite

$$\begin{aligned} \mathbb{H}_+(f) &= \bigoplus_{0\le\alpha<1}\bigoplus_{p\in\mathbb{Z}} z^{-p-\alpha+n/2}\psi(F^p\mathfrak{h}_s\cap U_p\mathfrak{h}_s)[z] \\ P &= \bigoplus_{0\le\alpha<1}\bigoplus_{p\in\mathbb{Z}} z^{-p-\alpha+n/2}z^{-1}\psi(F^p\mathfrak{h}_s\cap U_p\mathfrak{h}_s)[z^{-1}] \end{aligned} \tag{11}$$

where the summand indexed by α is generated by elements $\omega\in\mathbb{H}(f)$ with $\alpha=\langle-\deg\omega\rangle$. This decomposition clearly shows $\mathbb{H}(f)=\mathbb{H}_+(f)\oplus P$. The Lagrangian property of P follows from the property (c) of the opposite filtration: using the fact that ψ is inverse to $[\omega]\mapsto\widehat{s}(\omega,z)$ and equation 8, we have

$$\begin{aligned} &K(z^{-p-\alpha+\frac{n}{2}}z^{-1}\psi(U_p\mathfrak{h}_s), z^{-q-\beta+\frac{n}{2}}z^{-1}\psi(U_q\mathfrak{h}_t)) \\ &= z^{-p-q-\alpha-\beta+n-2}S(U_q\mathfrak{h}_t,\nu^{-1}U_p\mathfrak{h}_s) = z^{-p-q-\alpha-\beta+n-2}S(U_q\mathfrak{h}_t,U_p\mathfrak{h}_s) \end{aligned}$$

when $s=e^{2\pi\sqrt{-1}\alpha}$ and $t=e^{2\pi\sqrt{-1}\beta}$ with $\alpha,\beta\in[0,1)$, and this is nonzero only if $-p-q-\alpha-\beta+n\le 0$ by the property (c). This means $K(P,P)\subset z^{-2}\mathbb{C}[z^{-1}]$ and thus P is Lagrangian.

In the opposite direction, we start from a homogeneous opposite subspace $P\subset\mathbb{H}(f)$. The filtration $U_p\mathfrak{h}_s$ defined by the first formula of (10) is an increasing filtration since $P\subset zP$. The homogeneity of P implies that the finite-dimensional space $\mathbb{H}_+(f)\cap zP$ is spanned by homogeneous elements. The map $A\mapsto(-z)^{-p-\alpha+n/2}\psi(A)$ identifies $U_p\mathfrak{h}_s\cap F^p\mathfrak{h}_s$ with the homogeneous component of $zP\cap\mathbb{H}_+(f)$ of degree $n-p-\alpha$ (when $s=e^{2\pi\sqrt{-1}\alpha}$, $\alpha\in[0,1)$). Setting $z=-1$ and using the isomorphy of the map $\mathfrak{h}\cong\mathbb{H}_+(f)\cap zP\cong\mathbb{H}(f)|_{z=-1}$, $A\mapsto\psi(A)|_{z=-1}$ in Lemma 4.4, we conclude the decomposition $\mathfrak{h}_s=\bigoplus_{p\in\mathbb{Z}}F^p\mathfrak{h}_s\cap U_p\mathfrak{h}_s$, i.e. property (b) for opposite filtrations holds. The property (a) follows from (b), and the property (c) follows from the Lagrangian property of P by reversing the above argument. $\square$

An opposite filtration $\{U_p\mathfrak{h}\}_{p\in\mathbb{Z}}$ induces a homogeneous opposite subspace P (10) and an isomorphism $\sigma=\sigma_{U_\bullet}\colon\mathfrak{h}\xrightarrow{\cong}\mathbb{H}_+(f)\cap zP$ by the formula (cf. (11))

$$\sigma(A) = (-z)^{-p-\alpha+\frac{n}{2}}\psi(A) \qquad \text{for } A\in F^p\mathfrak{h}_s\cap U_p\mathfrak{h}_s \tag{12}$$

where $s=e^{2\pi\sqrt{-1}\alpha}$, $\alpha\in[0,1)$ and ψ is given in (9). The map $\sigma_{U_\bullet}$ gives a splitting of the projection $\mathbb{H}_+(f)\to\mathbb{H}_+(f)|_{z=-1}\cong\mathfrak{h}$.

The higher residue pairing takes values in $\mathbb{C}$ on the image of σ. We compute the precise values for later purposes.

Lemma 4.9. *Let σ be the splitting* (12) *associated to an opposite filtration $U_\bullet$. If $A\in F^p\mathfrak{h}_s\cap U_p\mathfrak{h}_s$ with $s=e^{2\pi\sqrt{-1}\alpha}$, $\alpha\in[0,1)$ and $B\in\mathfrak{h}$ is arbitrary, then*

$$K_f(\sigma(A),\sigma(B)) = C(s)\sqrt{-1}^{2p-m}S(A,B)$$

where we set $m=n-1$ if $s\ne1$ and $m=n$ if $s=1$, and

$$C(s)=\begin{cases}2\sin(\pi\alpha) & \text{if } \alpha\ne0;\\ 1 & \text{otherwise.}\end{cases}$$

Proof. We may assume that $B \in F^q\mathfrak{h}_t \cap U_q\mathfrak{h}_t$ with $t = e^{2\pi\sqrt{-1}\beta}$ and $\beta \in [0,1)$. Combining the fact that ψ is inverse to $[\omega] \mapsto \widehat{s}(\omega, z)$ and Theorem 4.5, we find

$$K_f(\sigma(A), \sigma(B)) = -S(e^{-\pi\sqrt{-1}(-q-\beta+\frac{n}{2})}(-z)^{-q-\beta+\frac{n}{2}}B, \nu^{-1}(-z)^{-p-\alpha+\frac{n}{2}}A).$$

This pairing vanishes unless $\alpha + \beta \equiv 0 \mod \mathbb{Z}$ and $p + q + \alpha + \beta = n$. If $\alpha \neq 0$, this equals

$$-e^{\pi\sqrt{-1}(m-p+1-\alpha)}(\sqrt{-1})^{-m-1}(e^{2\pi\sqrt{-1}\alpha} - 1)(-1)^m S(A, B)$$

by $\nu^{-1}A = (e^{2\pi\sqrt{-1}\alpha} - 1)A$, $\beta = 1 - \alpha$ and $p + q = n - 1 = m$; if $\alpha = 0$, this equals

$$-e^{\pi\sqrt{-1}(m-p)}(\sqrt{-1})^{-m}(-1)(-1)^m S(A, B)$$

by $\nu^{-1}A = -A$, $\alpha = \beta = 0$, $p + q = n = m$. The conclusion follows easily. □

Remark 4.10. When we regard $\mathbb{H}_+(f)$ as a vector bundle over $\mathbb{C}$ and zP as the extension data across ∞, the filtration $U_p\mathfrak{h}$ (10) on the space $\mathfrak{h}$ of flat sections is determined by pole orders at $z = \infty$.

4.4. **The complex conjugate opposite subspace.** Over the marginal locus, the vector bundle $\mathbb{H} \to \mathcal{M}^\circ_{\mathrm{mar}}$ has a natural real structure coming from the space of real semi-infinite cycles

$$V_{f,z} = \varprojlim_{M \in \mathbb{R}_+} H_n(\mathbb{C}^n, \{x : \mathrm{Re}(f(x)/z) < -M\}; \mathbb{R}) \cong \mathbb{R}^N,$$

where the homology groups form an inverse system with respect to the natural order on $\mathbb{R}_+$ and the limit is the projective (or inverse) limit of vector spaces. The vector spaces $V_{f,z}$ form a real vector bundle on $\mathcal{M}^\circ_{\mathrm{mar}} \times \mathbb{C}^\times$ equipped with a flat Gauss–Manin connection. For each $f \in \mathcal{M}_{\mathrm{mar}}$, let us denote by $\mathbb{H}(f;\mathbb{R}) \subset \mathbb{H}(f)$ the real vector subspace consisting of ω such that

$$(-2\pi z)^{-n/2} \int_\alpha e^{f/z}\omega \in \mathbb{R} \qquad \forall \alpha \in V_{f,z}, \quad \forall z \in S^1 \tag{13}$$

or equivalently,

$$\widehat{s}(\omega, z) \in \mathfrak{h}_\mathbb{R} \qquad \forall z \in S^1$$

where $S^1 = \{|z| = 1\}$ is the unit circle. Let $\kappa\colon \mathbb{H}(f) \to \mathbb{H}(f)$ be the complex conjugation corresponding to the real subspace $\mathbb{H}(f;\mathbb{R})$. The main properties of the complex conjugation κ can be summarized as follows (see [42] for generalities on the real structure in a semi-infinite VHS). Let us denote by

$$\gamma\colon \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}], \quad \gamma(g)(z) := \overline{g(\overline{z}^{-1})}$$

the complex conjugation corresponding to the real subspace consisting of Laurent polynomials that take real values on $|z| = 1$. By definition, we have

$$\kappa(g\omega) = \gamma(g)\kappa(\omega).$$

Remark 4.11. The definition (13) for $[\omega]$ to be real extends to non-marginal polynomials $f \in \mathcal{M} \setminus \mathcal{M}^\circ_{\mathrm{mar}}$; however our algebraic model $\mathbb{H}(f)$ is not necessarily closed under the real involution κ. In general, κ is defined on the analytification $\mathbb{H}(f)^{\mathrm{an}} = \mathbb{H}(f) \otimes_{\mathbb{C}[z,z^{-1}]} \mathcal{O}^{\mathrm{an}}(\mathbb{C}^\times)$ and $\mathbb{H}(f;\mathbb{R})$ can be only defined as a subspace of $\mathbb{H}(f)^{\mathrm{an}}$, where $\mathcal{O}^{\mathrm{an}}(\mathbb{C}^\times)$ denotes the ring of holomorphic functions on $\mathbb{C}^\times$.

Complex conjugation in the vanishing cohomology $\mathfrak{h}$ gives a natural splitting of the Steenbrink's Hodge filtration:

$$U_p\mathfrak{h}_s := \overline{F^{m-p}\mathfrak{h}_{\bar{s}}} \qquad \text{for } p \in \mathbb{Z},\ |s| = 1, \tag{14}$$

where $m = n-1$ for $s \neq 1$ and $m = n$ for $s = 1$.

Proposition 4.12. *Let $f \in \mathcal{M}^\circ_{\mathrm{mar}}$ be a marginal deformation. The subspace $z^{-1}\kappa(\mathbb{H}_+(f))$ is opposite and corresponds to the complex conjugate opposite filtration* (14) *under* (10).

Proof. Let $\sigma\colon \mathfrak{h} \to \mathbb{H}(f)$ be the splitting (12) defined by the complex conjugate filtration (14). It suffices to show that $\sigma(\mathfrak{h})$ is κ-invariant; indeed this implies $\sigma(\mathfrak{h})[z^{-1}] = \kappa(\sigma(\mathfrak{h})[z]) = \kappa(\mathbb{H}_+(f))$. We will prove that

$$\kappa(\sigma(A)) = \sigma(\overline{A}) \qquad \forall A \in \mathfrak{h}. \tag{15}$$

Recall from the definition of $\widehat{s}(\omega, z)$ in §4.3.2 that we have

$$(-2\pi z)^{-n/2} \int_\Gamma e^{f/z}\omega = \int_{\Gamma_1} \widehat{s}(\omega, z)$$

and thus the map $[\omega] \mapsto \widehat{s}(\omega, z)$ intertwines the complex conjugation κ on $\mathbb{H}(f)|_{S^1}$ with the standard complex conjugation on $\mathfrak{h}$. This implies $\psi(\overline{A}) = \kappa(\psi(A))$ since ψ is inverse to the map $[\omega] \mapsto \widehat{s}(\omega, z)$. For $A \in F^p\mathfrak{h}_s \cap U_p\mathfrak{h}_s$ with $s = e^{2\pi\sqrt{-1}\alpha}$, $0 \le \alpha < 1$, we have

$$\kappa(\sigma(A)) = \kappa(z^{-p-\alpha+\frac{n}{2}}\psi(A)) = z^{p+\alpha-\frac{n}{2}}\psi(\overline{A}).$$

Here $\overline{A} \in F^q\mathfrak{h}_{\bar{s}} \cap U_q\mathfrak{h}_{\bar{s}}$ for q with $p + \alpha - \frac{n}{2} = -q - \langle -\alpha \rangle + \frac{n}{2}$. Therefore, the above quantity equals $z^{-q-\langle-\alpha\rangle+\frac{n}{2}}\psi(\overline{A}) = \sigma(\overline{A})$. The proposition is proved. □

4.5. **Opposite subspaces for Fermat polynomials.** In this subsection we will assume that

$$f(x) = x_1^{N_1+1} + \cdots + x_n^{N_n+1}$$

is a Fermat polynomial. The higher residue pairing K_f factorizes into a tensor product of the higher residue pairings of the summands $x_i^{N_i+1}$. Using a simple degree count it is easy to see that the forms

$$x_1^{i_1} \cdots x_n^{i_n} dx_1 \cdots dx_n, \quad 0 \le i_s \le N_s - 1 \tag{16}$$

form a good basis, i.e., if we denote by $H \subset \mathbb{H}_+(f)$ the subspace spanned by the above forms, then $P = z^{-1}H[z^{-1}]$ is an opposite subspace.

Proposition 4.13. *The complex conjugate subspace $\kappa(\mathbb{H}_+(f))$ equals the subspace zP spanned by the forms* (16) *over $\mathbb{C}[z^{-1}]$.*

Proof. Since P is an opposite subspace, we have $\mathbb{H}_+(f) = H[z]$. Therefore, it is enough to prove that $\kappa(H) \subset H$. On the other hand, note that if $f = f_1 \oplus f_2$ is a direct sum of the quasi-homogeneous functions $f_i\colon \mathbb{C}^{n_i} \to \mathbb{C}$, then the direct product of cycles defines an isomorphism

$$H_{n_1}(\mathbb{C}^{n_1}, \{\mathrm{Re}(f_1/z) \ll 0\}) \otimes H_{n_2}(\mathbb{C}^{n_2}, \{\mathrm{Re}(f_2/z) \ll 0\}) \cong H_n(\mathbb{C}^n, \{\mathrm{Re}(f/z) \ll 0\}).$$

Moreover, if $\omega_i \in \mathbb{H}_+(f_i)$, $i = 1, 2$, then $\omega_1 \wedge \omega_2 \in \mathbb{H}_+(f)$ and

$$\kappa(\omega_1 \wedge \omega_2) = \kappa_{f_1}(\omega_1) \wedge \kappa_{f_2}(\omega_2),$$

where on the RHS we use the index f_i ($i = 1, 2$) to indicate that the conjugation is in the corresponding twisted de Rham cohomology. This observation reduces the proof of our Proposition to the case $n = 1$, i.e., we may assume that $f(x) = x^{N+1}$. The oscillatory integrals are very easy to compute explicitly and we can verify directly that

$$\kappa\left(\frac{x^i dx}{\Gamma\left(\frac{i+1}{N+1}\right)}\right) = \frac{x^{N-1-i}dx}{\Gamma\left(\frac{N-i}{N+1}\right)}.$$

Alternatively, we could argue that the opposite subspace P corresponds to a splitting of the Steenbrink's Hodge filtration of f. However, in this case $\mathfrak{h}_1 = 0$ and $F^p\mathfrak{h} = 0$ for $p > 0$ and $F^p\mathfrak{h} = \mathfrak{h}$ for $p \leq 0$. Note that there is a unique monodromy invariant filtration $U_\bullet$ which gives a splitting: $U_p\mathfrak{h} = \mathfrak{h}$ for $p \geq 0$ and $U_p\mathfrak{h} = 0$ for $p < 0$. Using Proposition 4.12, we get that $P = \kappa(\mathbb{H}_+(f))z^{-1}$. □

Remark 4.14. The results of this section can be generalised to any invertible polynomial.

4.6. **The Cecotti-Vafa structure.** Cecotti and Vafa introduced tt^*-geometry for $N = 2$ supersymmetric quantum field theories [10, 11]. This structure has been studied in mathematics by Dubrovin [27], Hertling [39] and many others. The Cecotti-Vafa structure for isolated hypersurface singularities has been introduced in [39]. We describe the structure for weighted homogeneous polynomials using the complex conjugate opposite subspaces.

Proposition 4.15. *If $f \in \mathcal{M}^\circ_{\mathrm{mar}}$, then the subspace $z^{-1}\kappa(\mathbb{H}_+(f))$ is an opposite subspace and*

$$h(\omega_1, \omega_2) = K^{(0)}(\kappa(\omega_1), \omega_2)$$

is a positive-definite Hermitian pairing on

$$\mathbb{K}(f) := \mathbb{H}_+(f) \cap \kappa(\mathbb{H}_+(f)).$$

Proof. According to Proposition 4.12, $z^{-1}\kappa(\mathbb{H}_+(f))$ is an opposite subspace. Thus we have the corresponding splitting (see (12))

$$\sigma\colon \mathfrak{h} \to \mathbb{H}_+(f) \cap \kappa(\mathbb{H}_+(f)).$$

Moreover, using formula (15), we have

$$h(\omega, \omega) = K^{(0)}(\sigma(\overline{A}), \sigma(A)),$$

where $\omega = \sigma(A)$. Let us assume that $A \in F^p\mathfrak{h}_s \cap U_p\mathfrak{h}_s$. Then using Lemma 4.9, we get that the above pairing is

$$C(s)\sqrt{-1}^{2p-m} S(\overline{A}, A).$$

Recalling that $F^p\mathfrak{h}$ is a Polarized Hodge Structure (see property (d) in Section 4.3.1), we get that the above number is a positive real number. □

By Remark 4.11, we can extend the real structure κ over the whole space $\mathcal{M}$ by extending scalars. Following the notation there, we write $\mathbb{H}_+(f)^{\mathrm{an}} = \mathbb{H}_+(f) \otimes_{\mathbb{C}[z]} \mathcal{O}^{\mathrm{an}}(\mathbb{C})$.

Because the oppositeness is an open condition, the subspace $z^{-1}\kappa(\mathbb{H}_+(f)^{\mathrm{an}})$ is opposite to $\mathbb{H}_+(f)^{\mathrm{an}}$ for f in a neighborhood of $\mathcal{M}^\circ_{\mathrm{mar}}$. Moreover, the Hermitian form $h(\omega_1,\omega_2) = K(\kappa(\omega_1),\omega_2)$ on the vector space

$$\mathbb{K}(f) := \mathbb{H}_+(f)^{\mathrm{an}} \cap z\kappa(\mathbb{H}_+(f)^{\mathrm{an}})$$

is positive definite for f in a neighborhood of $\mathcal{M}^\circ_{\mathrm{mar}}$. On the other hand, Sabbah [62, §4] proved that the Brieskorn lattice of any cohomologically tame function on a smooth affine manifold with only isolated critical points satisfies these properties, i.e. the oppositeness and the positive-definiteness of h. Therefore we have a globally defined Hermitian C^∞ vector bundle $\mathbb{K} \to \mathcal{M}$ whose fiber at $f \in \mathcal{M}$ is $\mathbb{K}(f)$. The Gauss–Manin connection ∇ on $\mathbb{H}$ induces a family of flat connections of $\mathbb{K}$ depending on a parameter $z \in \mathbb{C}^\times$, which will be called the *Cecotti–Vafa connection*. Namely, let us pick a C^∞-frame $\{\omega_i\}_{i=1}^N$ for $\mathbb{K}$. The deformation space $\mathcal{M}$, being a Zariski open subset of a standard complex vector space, has a natural holomorphic coordinate system $\sigma = (\sigma_1,\dots,\sigma_{N'})$ and the vector fields

$$\partial/\partial\sigma_1,\dots,\partial/\partial\sigma_{N'},\partial/\partial\overline{\sigma}_1,\dots,\partial/\partial\overline{\sigma}_{N'}$$

give a frame for the complexified tangent bundle $T^{\mathbb{C}}\mathcal{M} := T\mathcal{M}\otimes_{\mathbb{R}}\mathbb{C}$. The properties $\nabla_X\mathbb{H}_+(f) \subset z^{-1}\mathbb{H}_+(f)$, $\nabla_{z\partial_z}\mathbb{H}_+(f) \subset z^{-1}\mathbb{H}_+(f)$ imply that

$$\begin{aligned}\nabla_X(\kappa\mathbb{H}_+(f)) &= \kappa(\nabla_{\overline{X}}\mathbb{H}_+(f)) \subset z\kappa(\mathbb{H}_+(f))\\ \nabla_{z\partial_z}\kappa(\mathbb{H}_+(f)) &= \kappa(\nabla_{-z\partial_z}\mathbb{H}_+(f)) = z\kappa(\mathbb{H}_+(f))\end{aligned}$$

for a complexified vector field $X \in T^{\mathbb{C}}\mathcal{M}$, where we used $\nabla_X\kappa = \kappa\nabla_{\overline{X}}$ and $\kappa(z^{-1}\omega) = z\kappa(\omega)$. From these properties we get that in the frame $\{\omega_i\}$ the Gauss–Manin connection takes the form

$$\begin{aligned}\nabla_i\omega_a &= \sum_{b=1}^N \Big(\Gamma^b_{ia}(\sigma) + z^{-1}C^b_{ia}(\sigma)\Big)\omega_b\\ \nabla_{\overline{\imath}}\omega_a &= \sum_{b=1}^N \Big(\Gamma^b_{\overline{\imath}a}(\sigma) + z\widetilde{C}^b_{\overline{\imath}a}(\sigma)\Big)\omega_b\\ \nabla_{z\partial_z}\omega_a &= \sum_{b=1}^N \Big(-U^b_a(\sigma)z^{-1} + Q^b_a(\sigma) + \widetilde{U}^b_a(\sigma)z\Big)\omega_b\end{aligned}$$

where $\nabla_i := \nabla_{\partial/\partial\sigma_i}$, $\nabla_{\overline{\imath}} := \nabla_{\partial/\partial\overline{\sigma}_i}$, and the connection matrices are C^∞ functions in σ. It is easy to prove that

$$D = d + \sum_{i=1}^{N'}\Big(\Gamma_i d\sigma_i + \Gamma_{\overline{\imath}}d\overline{\sigma}_i\Big)$$

is compatible with the Hermitian metric h and its $(0,1)$ part defines the holomorphic structure on $\mathbb{K} \cong \mathbb{H}_+/z\mathbb{H}_+$ and its $(1,0)$ part defines the anti-holomorphic structure on $\mathbb{K} \cong \kappa(\mathbb{H}_+)/z^{-1}\kappa(\mathbb{H}_+)$. Here $\Gamma_i,\Gamma_{\overline{\imath}} \in \mathrm{End}(\mathbb{K})$ are defined by $\Gamma_i\omega_a = \sum_b \Gamma^b_{ia}\omega_b$ and $\Gamma_{\overline{\imath}}\omega_a = \sum_b\Gamma^b_{\overline{\imath}a}\omega_b$. Using the compatibility of the Gauss–Manin connection with the complex conjugation

$$\nabla_{\overline{X}} = \kappa\nabla_X\kappa, \quad \nabla_{z\partial_z} = -\kappa\nabla_{z\partial_z}\kappa$$

we get the following relations between the connection matrices

$$\widetilde{C}_{\bar{i}} = \kappa C_i \kappa, \quad \widetilde{U} = \kappa U \kappa, \quad \widetilde{Q} = -\kappa Q \kappa.$$

Remark 4.16. When we choose $\{\omega_i\}$ to be a frame that is holomorphic under the identification $\mathbb{K} \cong \mathbb{H}_+/z\mathbb{H}_+ \cong \bigcup_f \mathrm{Jac}(f) \cdot dx$, then we have $\Gamma_{\bar{i}} = 0$. Moreover the operators C_i and U above correspond to the multiplication on $\mathrm{Jac}(f) \cdot dx$ by $\partial_{\sigma_i} f(x;\sigma)$ and f, respectively. In particular, $U = 0$ along the marginal locus $\mathcal{M}^\circ_{\mathrm{mar}}$.

5. Quantization and Fock bundle

Using Givental's quantization formalism [32], we define a vector bundle of Fock spaces on the moduli space $\mathcal{M}^\circ_{\mathrm{mar}}$. The main motivation for our definition is to provide a convenient language to state mirror symmetry as well as to investigate the transformation properties under analytic continuation of Givental's total ancestor potential.

5.1. **Givental's quantization formalism.** Let H be a complex vector space of dimension N equipped with a non-degenerate bi-linear pairing $(\ ,\ )$. Givental's quantization is based on the vector space $\mathcal{H} = H(\!(z)\!)$ equipped with the following symplectic structure:

$$\Omega(\mathbf{f}_1(z), \mathbf{f}_2(z)) = \mathrm{Res}_{z=0}(\mathbf{f}_1(-z), \mathbf{f}_2(z))dz.$$

The Lie algebra of infinitesimal symplectic transformations A of $\mathcal{H}$ is naturally identified with the Poisson Lie algebra of quadratic Hamiltonians via

$$A \mapsto h_A(\mathbf{f}) := \frac{1}{2}\Omega(A\mathbf{f}, \mathbf{f}).$$

Note that $\mathbf{f} \mapsto A\mathbf{f}$ can be interpreted as a vector field on $\mathcal{H}$. This vector field is Hamiltonian with Hamiltonian h_A if and only if A is an infinitesimal symplectic transformation.

The symplectic vector space has a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ := H[\![z]\!]$ and $\mathcal{H}_- := H[z^{-1}]z^{-1}$ are Lagrangian subspaces. The polarization allows us to use the so-called *canonical quantization* to represent quadratic Hamiltonians by differential operators. In coordinates, the representation can be constructed as follows. Let $\{\phi_i\}_{i=1}^N$ and $\{\phi^i\}_{i=1}^N$ be bases of H dual with respect to the pairing $(\ ,\ )$. Then the linear functions on $\mathcal{H}$ defined by

$$p_{k,i}(\mathbf{f}) = \Omega(\mathbf{f}, \phi_i z^k), \quad q_{k,i}(\mathbf{f}) = \Omega(\phi^i(-z)^{-k-1}, \mathbf{f}), \quad 1 \le i \le N, k \ge 0,$$

form a Darboux coordinate system. We define $\widehat{A} := \widehat{h}_A$, where a function in $p_{k,i}$ and $q_{k,i}$ is quantized by the rules

$$p_{k,i} \mapsto \hbar^{1/2}\frac{\partial}{\partial q_{k,i}}, \quad q_{k,i} \mapsto \hbar^{-1/2} q_{k,i}$$

and normal ordering, i.e., all differentiation operations should preceed the multiplication ones.

If R is a symplectic transformation of $\mathcal{H}$ of the form $1 + R_1 z + R_2 z^2 + \cdots$, where $R_k \in \mathrm{End}(H)$, then we can formally define $A = \log R$ and $\widehat{R} = e^{\widehat{A}}$. Let us introduce the quadratic differential operator

$$V_R := \sum_{k,\ell=0}^{\infty} \sum_{i,j=1}^{N} (V_{k\ell}\phi^j, \phi^i) \frac{\partial^2}{\partial q_{k,i} \partial q_{\ell,j}},$$

where $V_{k\ell} \in \mathrm{End}(H)$ are defined by

$$\sum_{k,\ell=0}^{\infty} V_{k\ell} z^k w^\ell = \frac{1 - R(z)R^t(w)}{z+w}.$$

.

Proposition 5.1. *Let $\mathcal{F} = \mathcal{F}(\hbar; \mathbf{q})$ be a formal power series in $\mathbf{q} = (q_{k,i})$ with coefficients in $\mathbb{C}_\hbar = \mathbb{C}(\!(\hbar)\!)$ such that $\widehat{R}^{-1}\mathcal{F}$ is well defined. Then*

$$\widehat{R}^{-1}\mathcal{F} = \left(e^{\frac{\hbar}{2} V_R} \mathcal{F} \right)\Big|_{\mathbf{q} \mapsto R(z)\mathbf{q}},$$

where $R(z)\mathbf{q}$ is defined by identifying $\mathbf{q}$ with a vector $\sum_{k=0}^{\infty} \sum_{i=1}^{N} q_{k,i} \phi_i z^k \in H[\![z]\!]$.

5.2. **From an opposite subspace to a Frobenius structure.** Let us denote by $\mathcal{L} \to \mathcal{M}^\circ_{\mathrm{mar}}$ the line bundle whose fiber over a point $f \in \mathcal{M}^\circ_{\mathrm{mar}}$ is the space of elements in $\mathbb{H}_+(f)$ of minimal degree; i.e., $\mathcal{L}_f = \mathbb{C}\, dx_1 \cdots dx_n$. We refer to $\mathcal{L}$ as the *vacuum line bundle*.

Assume now that $f \in \mathcal{M}_{\mathrm{mar}}$ is a given point, $\omega \in \mathcal{L}_f$ is a non-zero form, and P is a homogeneous opposite subspace of $\mathbb{H}_+(f)$. Let us choose a good basis of homogeneous forms $\{\omega\}_{i=1}^N \subset \mathbb{H}_+(f) \cap Pz$ and define $\phi_i \in \mathrm{Jac}(f)$ such that[2]

$$\omega_i \equiv \phi_i\, \omega \mod z\mathbb{H}_+(f), \quad 1 \le i \le N.$$

We construct a miniversal unfolding of f by

$$F(x,t) = f(x) + \sum_{i=1}^{N} t_i \phi_i(x), \quad t = (t_1, \ldots, t_N) \in B_f,$$

where $B_f \subset \mathbb{C}^N$ is a sufficiently small ball representing the holomorphic germ at 0 of $\mathbb{C}^N$ and $\phi_i(x)$ is a homogeneous polynomial representing $\phi_i \in \mathrm{Jac}(f)$. Let us assign a degree to t_i such that $F(x,t)$ is weighted homogeneous of degree 1, and let us split the deformation parameters t into 3 groups: relevant $t^{\mathrm{rel}} = (t_1, \ldots, t_{N_{\mathrm{rel}}})$, marginal $t^{\mathrm{mar}} = (t_{N_{\mathrm{rel}}+1}, \ldots, t_{N_{\mathrm{rel}}+N_{\mathrm{mar}}})$, and irrelevant $t^{\mathrm{irr}} = (t_{N_{\mathrm{rel}}+N_{\mathrm{mar}}+1}, \ldots, t_N)$, depennding on whether their degrees are respecively > 0, $= 0$, or < 0. There is a natural way to construct a Frobenius structure on B_f. Let us outline the construction referring for more details to [38, 63, 65]. To begin with, we choose an appropriately small Stein domain $X_f \subset \mathbb{C}^n \times B_f$ around 0 (see [4]). Let us denote by $F\colon X_f \to \mathbb{C}$ the miniversal unfolding of f and put $\widehat{\mathcal{F}} := \mathbb{R}^n \varphi_*(\widehat{\Omega}_{\mathrm{twdR}}, \widehat{d}_{\mathrm{twdR}})$ for the hypercohomology of the twisted de Rham complex

$$(\widehat{\Omega}^\bullet_{\mathrm{twdR}}, \widehat{d}_{\mathrm{twdR}}) := (\Omega^\bullet_{X_f/B_f}[\![z]\!], z d_{X_f/B_f} + dF \wedge),$$

[2] The differential form $\phi_i\omega$ depends on the choice of a representative of $\phi_i \in \mathrm{Jac}(f)$, but the class of $\phi_i\omega$ in $\mathbb{H}_+(f)/z\mathbb{H}_+(f)$ does not.

where $\varphi\colon X_f \subset \mathbb{C}^n \times B_f \to B_f$ is the natural projection. Following the argument in Section 3 it is easy to prove that after decreasing B_f if necessary, $\widehat{\mathcal{F}}$ is a trivial vector bundle on B_f, whose fiber over a point $s \in B_f$

$$\widehat{\mathbb{H}}_+(F) := \Omega^n_{X_f}[\![z]\!]/(zd + dF\wedge)\Omega^{n-1}_{X_f}[\![z]\!], \quad F = F(x,s),$$

is a free $\mathbb{C}[\![z]\!]$-module of rank the Milnor number N. Put

$$\widehat{\mathbb{H}}(F) := \widehat{\mathbb{H}}_+(F) \otimes_{\mathbb{C}[\![z]\!]} \mathbb{C}(\!(z)\!).$$

This way we obtain vector bundles $\widehat{\mathbb{H}}_+ \subset \widehat{\mathbb{H}}$ on B_f. For a given holomorphic function $g(x,z) \in \mathcal{O}_{X_f}(X_f) \otimes \mathbb{C}(\!(z)\!)$ let us denote by

$$[g(x,z)dx]_F =: \int e^{F(x)/z} g(x,z)dx$$

the equivalence class of the form $g(x,z)dx_1 \cdots dx_n$ in the de Rham cohomology group $\widehat{\mathbb{H}}(F)$.

5.2.1. *Extension in the relevant and marginal directions.* There is a unique way to extend $\omega_i = [g_i(x,z)dx]_f$ to sections $\widetilde{\omega}_i$ of $\widehat{\mathbb{H}}_+|_{\{t^{\mathrm{irr}}=0\}}$ so that they give a good basis in each fiber $\widehat{\mathbb{H}}_+(F)$ for $F \in \{t^{\mathrm{irr}} = 0\} \subset B_f$. The extension can be constructed by Birkhof's factorization as follows. Let us denote by G_i the section of $\widehat{\mathbb{H}}_+$ obtained by flat extension of ω_i with respect to the Gauss–Manin connection. The Gauss–Manin connection ∇ gives rise to a system of differential equations

$$z\nabla_{\partial/\partial t_i}[g_j(x,z)dx]_F = \sum_{k=1}^N \Gamma_{ij}^k(t,z)[g_k(x,z)dx]_F, \quad 1 \le i \le N_{\mathrm{rel}} + N_{\mathrm{mar}}.$$

Since f is weighted-homogeneous, the functions $g_i(x,z)$ are polynomials in z. In particular, the connection matrix Γ is holomorphic at $(t,z) = (0,0)$. Let us pick a fundamental solution $\Phi(t,z)$; i.e., a $N \times N$ non-degenerate matrix solving the differential equations

$$z\partial_{t_i}\Phi(t,z) = \Gamma_i(t,z)\Phi(t,z), \quad 1 \le i \le N_{\mathrm{rel}} + N_{\mathrm{mar}}, \tag{17}$$

where $\Gamma_i(t,z)$ is the matrix whose (j,k)-entry is $\Gamma_{ij}^k(t,z)$ and satisfying $\Phi(0,z) = 1$. We have

$$[g_i(x,z)dx]_F = \sum_{j=1}^N \Phi_{ij}(t,z)G_j.$$

Note that $\Phi(t,z)$ is a holomorphic matrix for $z \in \mathbb{C}^* := \mathbb{P}^1 \setminus \{0,\infty\}$ that has a Birkhof factorization at $t=0$, so $\Phi(t,z)$ must have a Birkhof factorization for all $t \in B_f$ provided we choose B_f sufficiently small. Put $\Phi(t,z) = \Phi_+(t,z)^{-1}\Phi_-(t,z)$, where $\Phi_-(t,z)$ is holomorphic for $z \in \mathbb{P}^1 \setminus \{0\}$ (with $\Phi_-(t,\infty) = 1$) and $\Phi_+(t,z)$ is holomorphic for $z \in \mathbb{P}^1 \setminus \{\infty\}$. One can check that the forms

$$\widetilde{\omega}_i = \sum_{j=1}^N (\Phi_+(t,z))_{ij}[g_j(x,z)dx]_F, \quad 1 \le i \le N,$$

give rise to a good basis. Moreover, the good basis $\widetilde{\omega}_i = [\widetilde{g}_i(x, t^{\mathrm{rel}}, t^{\mathrm{mar}}; z)dx]_F$ depends polynomially on x, t^{rel} and z (because these variables have positive degrees) and analytically in t^{mar}.

5.2.2. *Extension in the irrelevant directions.* If we want to extend in the irrelevant directions, then the above argument becomes much more involved, because the system (17) might fail to be convergent and holomorphic in z. To offset this difficulty one can take the formal Laplace transform, solve the resulting system, and then obtain $\widetilde{\omega}_i$ via the inverse Laplace transform. The details are quite delicate, so we refer to [38, 66]. An alternative way to proceed is the perturbative approach of [47]. The main idea is to look for a good basis that depends formally on the irrelevant parameters, i.e., we are looking for a good basis of the form

$$\widetilde{\omega}_i = [\widetilde{g}_i(x, t, z)dx]_F, \quad \widetilde{g}_i \in \mathbb{C}\{t^{\mathrm{mar}}\}[x, t^{\mathrm{rel}}][\![t^{\mathrm{irr}}, z]\!],$$

where $\mathbb{C}\{a\}$ is the ring of convergent power series in a. According to [47], first we have to find the extension $\widetilde{\omega} = \widetilde{g}(x, t, z)dx$ of the volume form $\omega \in \mathcal{L}_f - \{0\}$ by solving the following equation in $\widehat{\mathbb{H}}_+(f)$:

$$J(t, z) := e^{(F(x,t) - f(x, t_{\mathrm{rel}}, t_{\mathrm{mar}}))/z}\widetilde{g}(x, t, z)dx \in \omega + H[\![z^{-1}]\!]z^{-1},$$

where $H = \mathbb{H}_+(f) \cap zP$ is the vector space spanned by the good basis $\{\omega_i\}_{i=1}^N$. The extension of the remaining forms is obtained from the period map

$$TB_f[\![z]\!] \to \widehat{\mathbb{H}}_+, \quad \partial_{t_i} \mapsto z\nabla_{\partial_{t_i}}[\widetilde{\omega}] \tag{18}$$

as the image of the *flat* vector fields. The latter are the vector fields corresponding to the coordinate system on B_f given by the coefficients in front of z^{-1} of $J(t, z)$. We define a Frobenius structure on B_f for which a basis of flat vector fields corresponds via the period map (18) to the good basis $\{\widetilde{\omega}_i\}_{i=1}^N$ and the flat pairing corresponds to $K_F^{(0)}$. Let us point out that the extension $\widetilde{\omega}$ of the volume form ω is a *primitive form* in the sense of K. Saito. Slightly abusing the terminology we will sometimes refer to the sections of $\mathcal{L}$ as primitive forms, keeping in mind that they do become primitive only after an appropriate extension.

5.3. **The total ancestor potential.** Given $f \in \mathcal{M}_{\mathrm{mar}}$, $\omega \in \mathcal{L}_f \setminus \{0\}$, an opposite subspace $P \subset \mathbb{H}(f)$, and a good basis $\{\omega_i\}_{i=1}^N \subset \mathbb{H}_+(f) \cap Pz$, let us construct a miniversal unfolding space B_f equipped with a Frobenius structure as explained above. Using the flat structure we trivialize the tangent and the co-tangent bundle

$$T^*B_f \cong TB_f \cong B_f \times T_0B_f \cong B_f \times \mathrm{Jac}(f),$$

where the first isomorphism uses the non-degenerate pairing, the 2nd one uses the flat Levi-Civita connection, and the last one is induced from the period isomorphism $T_0B_f \cong \mathbb{H}_+(f)/z\mathbb{H}_+(f)$ and the isomorphism

$$\mathrm{Jac}(f) \cong \mathbb{H}_+(f)/z\mathbb{H}_+(f), \quad \phi(x) \longmapsto \phi(x)\omega \mod z\mathbb{H}_+(f). \tag{19}$$

Note that $\phi_i \in \mathrm{Jac}(f)$ are the elements corresponding to the good basis ω_i via the isomorphism (19). Let us introduce the Fock space

$$\mathbb{C}_\hbar[\![q_0, q_1 + \mathbf{1}, q_2, \dots]\!] = \mathbb{C}(\!(\hbar)\!)[\![q_0, q_1 + \mathbf{1}, q_2, \dots]\!]$$

of formal power series in $\mathbf{q} = (q_{k,i})_{i=1,\dots,N}^{k=0,1,\dots}$. We denote $q_k = \sum_{i=1}^N q_{k,i}\phi_i$. The shift $q_1 + \mathbf{1}$ means that the element $\mathbf{1} \in \mathrm{Jac}(f)$ should be written as $\mathbf{1} = \sum_i a_i \phi_i$ and in the formal power series the variables $q_{1,i}$ are shifted to $q_{1,i} + a_i$.

On the other hand, if F is a generic deformation of f, then the critical values of F give rise to the so-called *canonical* coordinate system $u = (u_1, \dots, u_N)$, defined locally near F, in which the Frobenius multiplication and the pairing are diagonal

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{i,j}/\Delta_i, \quad \partial_{u_i} \bullet \partial_{u_j} = \delta_{i,j}\partial_{u_j}.$$

Let us denote by

$$\Psi_F \colon \mathbb{C}^N \to T_F B \cong \mathrm{Jac}(f), \quad \Psi_F(e_i) = \sqrt{\Delta_i}\partial_{u_i}$$

the trivialization of the tangent bundle at a generic F. The total ancestor potential is an element of the Fock space defined by

$$\mathcal{A}_F(\hbar; \mathbf{q}) := \widehat{\Psi}_F \widehat{R}_F \prod_{i=1}^N \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i; {}^i\mathbf{q}\sqrt{\Delta_i}),$$

where R_F is a symplectic transformation of $\mathbb{C}^N(\!(z)\!)$ of the type $1 + R_1 z + \cdots$, which will be defined below. We have a different set of formal variables ${}^i\mathbf{q} = ({}^i q_0, {}^i q_1, \dots)$ which is related to the previous one by

$$\sum_{i=1}^N {}^i q_k \Psi_F(e_i) = q_k, \quad k \geq 0.$$

By definition the quantization $\widehat{\Psi}_F$ acts by the above substitution. Finally, $\mathcal{D}_{\mathrm{pt}}$ is the Witten–Kontsevich tau function:

$$\mathcal{D}_{\mathrm{pt}}(\hbar; \mathbf{q}) = \exp\Big(\sum_{g,n=0}^\infty \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n (\mathbf{q}(\psi_i) + \psi_i) \Big),$$

where $\mathbf{q} = (q_k)_{k\geq 0}$ is a sequence of formal variables and $\mathbf{q}(\psi_i) = \sum_{k=0}^\infty q_k \psi_i^k$ (with ψ_i the 1st Chern class of the line bundle of i-th marked point cotangent lines) is a cohomology class on $\overline{\mathcal{M}}_{g,n}$. Note that the dilaton shift is incorporated here, so the function is an element of $\mathbb{C}_\hbar[\![q_0, q_1 + 1, q_2, \dots]\!]$.

The operator R_F is in general defined as a formal solution of the Gauss–Manin connection near the irregular singular point $z = 0$. In the case of singularity theory, however, we have an alternative description in terms of a stationary phase asymptotic. Namely, let $\beta_i \subset \mathbb{C}^n$ be the cycle swept by the vanishing cycle vanishing at the critical point of F corresponding to the critical value u_i, then the stationary phase asymptotic

$$(-2\pi z)^{-1/2} \int_{\beta_i} e^{F(x)/z} \widetilde{\omega}_a \sim (\Psi_F R_F(z) e_i, \phi_a) e^{u_i/z}, \quad z \to 0, \tag{20}$$

where $\phi_a \in \mathrm{Jac}(f)$ corresponds to the flat vector field determined by $\widetilde{\omega}_a$.

According to Milanov [50], the total ancestor potential $\mathcal{A}_F$ extends analytically for all $F \in B_f$. In particular, we have a well defined limit

$$\mathcal{A}_{f,\omega}^{\omega_1,\dots,\omega_N}(\hbar; \mathbf{q}) := \lim_{F\to f} \mathcal{A}_F(\hbar; \mathbf{q}).$$

Let us describe the dependence of the total ancestor potential on the choices of ω and $\omega_1, \ldots, \omega_N$. Assume that $\omega' \in \mathcal{L}_f - \{0\}$, $P' \subset \mathbb{H}(f)$ is an opposite subspace, and $\{\omega_i'\}_{i=1}^N \subset \mathbb{H}_+(f) \cap P'z$ is a good basis. It is convenient to split the general formula into two cases. The first case is the following: if $P' = P$, then

$$\omega_j' = \sum_{i=1}^N \omega_i B_{ij}, \quad \omega' = c\omega \tag{21}$$

for some invertible matrix $B = (B_{ij})_{i,j=1}^N$ and some non-zero constant c. The second case is the following: if $\omega' = \omega$ and

$$\omega_i' \equiv \omega_i \equiv \phi_i \omega \mod z\mathbb{H}_+(f), \quad 1 \le i \le N,$$

where $\{\phi_i\}_{i=1}^N \subset \mathrm{Jac}(f)$. Let us denote by $R(f,z)$ the linear operator in $\mathrm{Jac}(f)(\!(z)\!)$ whose matrix $(R_{ij}(f,z))_{i,j=1}^N$ with respect to the basis $\{\phi_i\}_{i=1}^N$ is defined by

$$\omega_j'(f) = \sum_{i=1}^N \omega_i(f) R_{ij}(f,z), \quad 1 \le j \le N. \tag{22}$$

Let us recall Givental's quantization formalism for

$$H = \mathrm{Jac}(f), \quad (\psi_1, \psi_2) := K^{(0)}(\psi_1\omega, \psi_2\omega), \quad \psi_1, \psi_2 \in \mathrm{Jac}(f).$$

Note that $R(f,z)$ is a symplectic transformation of $\mathcal{H} = H(\!(z^{-1})\!)$ of the type $1 + R_1(f)z + R_2(f)z^2 + \cdots$.

Proposition 5.2. *a) The transformation of the total ancestor potential under the change* (21) *is given by*

$$\mathcal{A}_{f,\omega'}^{\omega_1',\ldots,\omega_N'}(\hbar;\mathbf{q}) = \mathcal{A}_{f,\omega}^{\omega_1,\ldots,\omega_N}(\hbar c^{-2}; c^{-1}B\mathbf{q}),$$

where $(c^{-1}B\mathbf{q})_{k,i} = \sum_{j=1}^N c^{-1}B_{ij}q_{k,j}$.

b) The transformation of the total ancestor potential under the change (22) *is given by*

$$\mathcal{A}_{f,\omega}^{\omega_1',\ldots,\omega_N'}(\hbar;\mathbf{q}) = (R(f,z)^t)^\wedge \mathcal{A}_{f,\omega}^{\omega_1,\ldots,\omega_N}(\hbar;\mathbf{q}),$$

where $R(f,z)^t$ *is the transpose of* $R(f,z)$ *with respect to the residue pairing.*

Proof. Let us denote by

$$\mathcal{A}_F'(\hbar;\mathbf{q}) = \widehat{\Psi}_F' \widehat{R}_F' \prod_{i=1}^N \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i'; {}^i\mathbf{q}\sqrt{\Delta_i'})$$

the total ancestor potential corresponding to the Frobenius structure determined by the primitive form ω' and the good basis $\omega_1', \ldots, \omega_N'$. Let $\tau = (\tau_1, \ldots, \tau_N)$ and $\tau' = (\tau_1', \ldots, \tau_N')$ be the flat coordinates on B_f corresponding respectively to the good bases $\{\omega_i\}_{i=1}^N$ and $\{\omega_i'\}_{i=1}^N$. By definition

$$z\nabla_{\partial/\partial\tau_i}\omega = \omega_i, \quad z\nabla_{\partial/\partial\tau_i'}\omega' = \omega_i', \quad 1 \le i \le N.$$

a) Recalling the change (21) we get the following relations

$$\tau_i = \sum_{j=1}^{N} c^{-1} B_{ij} \tau'_j, \quad \frac{\partial}{\partial \tau'_j} = \sum_{i=1}^{N} c^{-1} B_{ij} \frac{\partial}{\partial \tau_i}.$$

The matrix of $\Psi'_F \colon \mathbb{C}^N \to \mathrm{Jac}(f)$ with respect to the bases $\{e_i\}_{i=1}^N \subset \mathbb{C}^N$ and $\{\phi'_i\}_{i=1}^N \subset \mathrm{Jac}(f)$, $\phi'_i = \partial/\partial\tau'_i$ has entries $(\Psi'_F)_{ki}$ defined by

$$\Psi'_F e_i = \sum_{k=1}^{N} \phi'_k \, (\Psi'_F)_{ki}, \quad (\Psi'_F)_{ki} = \sqrt{\Delta_i}\, \partial\tau'_k/\partial u_i.$$

Similarly, Ψ_F is represented by a matrix with entries $(\Psi_F)_{ki} = \sqrt{\Delta_i}\, \partial\tau_k/\partial u_i$. Note that $\sqrt{\Delta'_i} = c^{-1}\sqrt{\Delta_i}$, $R'_F = R_F$, and $\Psi'_F = B^{-1}\,\Psi_F$. Using also that the quantized action $\widehat{R}_F$ commutes with the rescaling

$$\hbar \mapsto \hbar c^{-2}, \quad {}^i q_k \mapsto {}^i q_k c^{-1},$$

we get

$$\mathcal{A}'_F(\hbar; \mathbf{q}) = \mathcal{A}_F(\hbar c^{-2}, c^{-1} B\mathbf{q}).$$

Taking the limit $F \to f$ completes the proof of part a).

b) The entries of the matrix of the linear operator $\Psi'_F R'_F$ with respect to the bases $\{e_i\}_{i=1}^N \subset \mathbb{C}^N$ and $\{\phi_a\}_{a=1}^N$ are by definition the stationary phase asymptotics

$$(-2\pi z)^{-n/2} \int_{\beta_i} e^{(F-u_i)/z} \widetilde{\omega}'_b \, \eta^{ab} \;\sim\; (\Psi'_F R'_F)_{ai}, \quad z \to 0,$$

where $\{\widetilde{\omega}'_a\}$ is the extension of the good basis $\{\omega'_a\}_{a=1}^N$ to a good basis on B_f and $(\eta^{ab})_{a,b=1}^N$ is the inverse matrix of the matrix of the residue pairing $(\eta_{ab})_{a,b=1}^N$, $\eta_{ab} = (\phi_a, \phi_b)$. Similarly,

$$(-2\pi z)^{-n/2} \int_{\beta_i} e^{(F-u_i)/z} \widetilde{\omega}_b \, \eta^{ab} \;\sim\; (\Psi_F R_F)_{ai}, \quad z \to 0.$$

Let us denote by $\widetilde{R}(F, z)$ the symplectic transformation of $\mathrm{Jac}(f)(\!(z)\!)$ whose entries $\widetilde{R}_{ab}(F, z)$ with respect to the basis $\{\phi_a\}_{a=1}^N$ are given by

$$\widetilde{\omega}'_b = \sum_{a=1}^{N} \widetilde{\omega}_a \, \widetilde{R}_{ab}(F, z).$$

By definition $\lim_{F\to f} \widetilde{R}_{ab}(F, z) = R_{ab}(f, z)$. Comparing the above asymptotic expansions, we get

$$(\Psi'_F R'_F)_{ai} = \sum_{\mu,\nu=1}^{N} \eta^{a\mu} \widetilde{R}_{\nu\mu}(F, z) \eta_{\nu b} \, (\Psi_F R_F)_{bi}.$$

Note that the matrix of the transpose $\widetilde{R}(F,z)^t$ with respect to the residue pairing has entries

$$(\widetilde{R}(F,z)^t)_{ab} = \sum_{\mu,\nu=1}^{N} \eta^{a\mu}\widetilde{R}_{\nu\mu}(F,z)\eta_{\nu b}.$$

We get that $\Psi'_F R'_F = \widetilde{R}(F,z)^t \Psi_F R_F$, so

$$\mathcal{A}'_F(\hbar;\mathbf{q}) = (\widetilde{R}(F,z)^t)^\wedge\, \mathcal{A}_F(\hbar;\mathbf{q}).$$

Taking the limit $F \to f$ completes the proof. □

5.4. **The abstract Fock bundle.** Recall that a series of the form

$$\sum_{g\in\mathbb{Z}} \sum_{\kappa=(k_1,i_1),\ldots,(k_r,i_r)} c_\kappa^{(g)}\hbar^{g-1} t_{k_1,i_1}\cdots t_{k_r,i_r}, \quad t_{k,i} = q_{k,i} + \delta_{k,1}a_i,$$

is called *tame* if $c_\kappa^{(g)} \neq 0$ only for κ satisfying the $(3g-3+r)$-jet constraint

$$k_1 + \cdots + k_r \leq 3g-3+r.$$

It is known that Givental's total ancestor potential $\mathcal{A}_{f,\omega}^{\omega_1,\ldots,\omega_N}(\hbar;\mathbf{q})$ is tame (see [34]). Motivated by Proposition 5.2 we define a vector bundle $\widehat{\mathbb{V}}_{\mathrm{tame}}$ on $\mathcal{M}^\circ_{\mathrm{mar}}$ whose fibers are the Fock spaces $\mathbb{C}_\hbar[\![q_0, q_1+\mathbf{1}, q_2, \ldots]\!]_{\mathrm{tame}}$ of tame series[3] and the transition functions are given by the transformation laws of Proposition 5.2 (with $c = 1$). Following Costello-Li's interpretation [23] of Givental's quantization formalism, we will identify each fiber of $\widehat{\mathbb{V}}_{\mathrm{tame}}$ with a highest weight module of a certain Weyl algebra, which in particular yields an intrinsic definition of $\widehat{\mathbb{V}}_{\mathrm{tame}}$.

5.4.1. *The Weyl algebra and the Fock space.* Let us fix $f \in \mathcal{M}^\circ_{\mathrm{mar}}$. The Weyl algebra of $\mathbb{H}(f)$ is defined by

$$\mathcal{W}(f) = \bigoplus_{n=0}^{\infty} \Big(\mathbb{H}(f)^{\otimes n} \otimes \mathbb{C}_\hbar\Big)/I,$$

where $\mathbb{C}_\hbar = \mathbb{C}(\!(\hbar)\!)$ and I is the two sided ideal generated by the elements

$$a\otimes b - b\otimes a - \hbar\,\Omega(a,b), \quad a,b \in \mathbb{H}(f).$$

The Lagrangian subspace $\mathbb{H}_+(f)$ determines the following Fock space

$$\mathbb{V}(f) := \mathcal{W}(f)/\mathcal{W}(f)\mathbb{H}_+(f).$$

Lemma 5.3. *If P is an opposite subspace, then the natural map*

$$\phi_P\colon \bigoplus_{n=0}^{\infty} \Big(\mathrm{Sym}^n(P)\otimes\mathbb{C}_\hbar\Big) \to \mathbb{V}(f), \quad a_1\cdots a_n \mapsto a_1\otimes\cdots\otimes a_n,$$

induces an isomorphism.

[3]Note that elements of $\mathbb{C}[\![\hbar]\!]$ are tame, but $\hbar^{-1}$ is not tame; hence $\mathbb{C}_\hbar[\![q_0, q_1+\mathbf{1}, q_2, \ldots]\!]_{\mathrm{tame}}$ is not a $\mathbb{C}_\hbar$-algebra (only a $\mathbb{C}[\![\hbar]\!]$-algebra).

Proof. The map is well defined and injective because P is Lagrangian. The surjectivity follows from the Wick's formula (see [43]). Namely, given $a_1, \dots, a_n \in \mathbb{H}(f)$ we have the following identity in $\mathbb{V}(f)$:

$$a_1 \otimes \cdots \otimes a_n = \sum \Big(\prod_{s=1}^{n'} \Omega(a^+_{i'_s}, a^-_{i''_s}) \Big) a^-_{j_1} \otimes \cdots \otimes a^-_{j_{n''}},$$

where the sum is over all possible ways to select pairs $(i'_1, i''_1), \dots, (i'_{n'}, i''_{n'}) \subset \{1, 2, \dots, n\}$ such that $i'_s < i''_s$ and $i'_1 < \cdots < i'_{n'}$ and $\{j_1, \dots, j_{n''}\} = \{1, 2, \dots, n\} \backslash \{i'_1, \dots, i'_{n'}, i''_1, \dots, i''_{n'}\}$, and where $a^- \in P$ (resp. $a^+ \in \mathbb{H}_+(f)$) denotes the projection of a on P (resp. $\mathbb{H}_+(f)$) along $\mathbb{H}_+(f)$ (resp. P). □

5.4.2. *The tame Weyl algebra and the tame Fock space.* If $P \subset \mathbb{H}(f)$ is an opposite subspace, then using the vector spaces isomorphism

$$\mathcal{W}(f) = \bigoplus_{r,s=0}^{\infty} P^{\otimes r} \otimes \mathbb{H}_+(f)^{\otimes s} \otimes \mathbb{C}_\hbar$$

we can write an element of $\mathcal{W}(f)$ as a finite sum of terms of the form

$$c^{(g)}_{I,J} \hbar^{g-1} \omega^{i_1}(-z)^{-k_1-1} \cdots \omega^{i_r}(-z)^{-k_r-1} \otimes \omega_{j_1} z^{\ell_1} \cdots \omega_{j_s} z^{\ell_s}, \tag{23}$$

where $\{\omega^i\}$ and $\{\omega_j\}$ are dual bases of $\mathbb{H}_+(f) \cap zP$, and the coefficient $c^{(g)}_{I,J}$ is a constant depending on g and the multi-indexes $I = \{(i_1, j_1), \dots, (i_r, k_r)\}$ and $J = \{(j_1, \ell_1), \dots, (j_s, \ell_s)\}$. The *tame* Weyl algebra $\mathcal{W}_{\text{tame}}(f)$ is defined as the vector subspace of $\mathcal{W}(f)$ spanned by monomials of the type (23) such that

$$k_1 + \cdots + k_r - r \le 3(g - 1 + s/2). \tag{24}$$

Finally, we need to introduce the completion of $\mathcal{W}_{\text{tame}}(f)$

$$\widehat{\mathcal{W}}_{\text{tame}}(f) := \varprojlim_m \mathcal{W}_{\text{tame}}(f)/\mathcal{W}^m_{\text{tame}}(f),$$

where the decresing filtration $\{\mathcal{W}^m_{\text{tame}}(f)\}_{k=0}^\infty$ is defined as the span of all terms of the type (23) such that

$$k_1 + \cdots + k_r + \ell_1 + \cdots + \ell_s + r + s \ge m.$$

Equivalently, the elements of $\widehat{\mathcal{W}}_{\text{tame}}(f)$ are arbitrary infinite sums of terms of the type (23) satisfying the tameness condition (24). We can prove the following proposition by an argument similar to the proof of the fact that tame functions are preserved by the upper-triangular Givental group action [34].

Proposition 5.4. *The tame Weyl algebra $\mathcal{W}_{\text{tame}}(f)$ and its completion $\widehat{\mathcal{W}}_{\text{tame}}(f)$ are independent of the choices of an opposite subspace and a good basis. Moreover, the multiplication induced from $\mathcal{W}(f)$ is well defined, so both $\mathcal{W}_{\text{tame}}(f)$ and $\widehat{\mathcal{W}}_{\text{tame}}(f)$ are associative algebras.*

Let $\{\phi_i\}_{i=1}^N \subset \text{Jac}(f)$ be a fixed basis, let $P \subset \mathbb{H}(f)$ be an opposite subspace, and let $\omega \in \mathcal{L}_f - \{0\}$. Given these data, we can uniquely construct a good basis $\{\omega_i\}_{i=1}^N \subset \mathbb{H}_+(f) \cap Pz$ such that

$$\omega_i \equiv \phi_i \omega \mod z\mathbb{H}_+(f)$$

and so that there is an isomorphism

$$\Phi_{\omega_1,\dots,\omega_N}\colon \mathbb{C}_\hbar[q_0,q_1,\dots]\to \mathbb{V}(f)$$

defined by

$$\Phi_{\omega_1,\dots,\omega_N}(q_{k_1,i_1}\cdots q_{k_1,i_1}) := (\omega^{i_1}z^{-k_1-1})\otimes\cdots\otimes(\omega^{i_n}z^{-k_n-1}),$$

where $\{\omega^i\}_{i=1}^N$ is a basis of $\mathbb{H}_+(f)\cap zP$ dual to $\{\omega_i\}_{i=1}^N$ with respect to the residue pairing $K_f^{(0)}$. Let us define

$$\sigma_{\omega,P}^{\phi_1,\dots,\phi_N} := e^{-\omega z/\hbar}\Phi_{\omega_1,\dots,\omega_N}\colon \mathbb{C}_\hbar[q_0,q_1,\dots]\to \mathbb{V}(f),$$

where we use that $\mathbb{V}(f)$ is a $\mathcal{W}(f)$-module on which ωz acts locally nilpotently. Note that

$$[-\omega z,\omega^i(-z)^{-k-1}] = \hbar\,\mathrm{Res}_{z=0}\,K_f(-\omega z,\omega^i(-z)^{-k-1})dz = -\hbar\delta_{k,1}a_i,$$

where a_i are the coordinates of $\mathbf{1}$, i.e., $\mathbf{1}=\sum_{i=1}^N a_i\phi_i$. Therefore the operator $e^{-\omega z/\hbar}$ acts as the shift $q_1\mapsto q_1-\mathbf{1}$, and we have

$$\sigma_{\omega,P}^{\phi_1,\dots,\phi_N}(g(q_0,q_1,q_2,\dots)) = \Phi_{\omega_1,\dots,\omega_N}(g(q_0,q_1-\mathbf{1},q_2,\dots))$$

for any $g\in\mathbb{C}_\hbar[q_0,q_1,\dots]$. It follows that $\sigma_{\omega,P}^{\phi_1,\dots,\phi_N}$ induces an isomorphism between the completed tame Fock spaces

$$\sigma_{\omega,P}^{\phi_1,\dots,\phi_N}\colon \mathbb{C}_\hbar[\![q_0,q_1+\mathbf{1},q_2\dots]\!]_{\mathrm{tame}}\to \widehat{\mathbb{V}}_{\mathrm{tame}}(f), \tag{25}$$

where $\widehat{\mathbb{V}}_{\mathrm{tame}}(f):=\widehat{\mathcal{W}}_{\mathrm{tame}}/\widehat{\mathcal{W}}_{\mathrm{tame}}\mathbb{H}_+(f)$.

5.4.3. *The total ancestor potential and the abstract Fock space.* It turns out that the dependence of the isomorphism $\Phi_{\omega_1,\dots,\omega_N}$ on the choice of an opposite subspace and a good basis is controlled by Givental's symplectic loop group quantization. Let us assume that we have two opposite filtrations P' and P and $\{\omega_i'\}_{i=1}^N\subset\mathbb{H}_+(f)\cap P'z$ and $\{\omega_i\}_{i=1}^N\subset\mathbb{H}_+(f)\cap Pz$ are corresponding good bases.

Lemma 5.5. *If $P'=P$ and the transition between the good bases is given by* (21)*, then*

$$\Phi^{-1}_{\omega_1',\dots,\omega_N'}\circ\Phi_{\omega_1,\dots,\omega_N}\mathcal{F}(\hbar;\mathbf{q}) = \mathcal{F}(\hbar;B\mathbf{q}).$$

It remains only to investigate the case when P' and P'' are different. Let us choose $\omega\in\mathcal{L}_f-\{0\}$. Using Lemma 5.5 we may reduce the general case to the case when $\omega_i\equiv\omega_i'\equiv\phi_i\omega \mod z\mathbb{H}_+(f)$. In order to compare with Givental's formalism put $H:=\mathrm{Jac}(f)$, and denote by $(\ ,\ )$ the pairing on H induced by the residue pairing $K_f^{(0)}$. Let $R(f,z)$ be the symplectic transformation of $H(\!(z)\!)$ defined by (22).

Lemma 5.6. *The following formula holds*

$$\Phi^{-1}_{\omega_1',\dots,\omega_N'}\circ\Phi_{\omega_1,\dots,\omega_N}(\mathcal{F}) = (R(f,z)^t)^\wedge\mathcal{F},$$

where $R(f,z)^t$ is the transopse of $R(f,z)$ with respect to the residue pairing.

Proof. It enough to prove that if the Lemma is true for some $\mathcal{F}$, then it is true for $q_{k,i}\mathcal{F}$ for all $k \geq 0$, $1 \leq i \leq N$. Recalling Proposition 5.1 and that $R(f,z)^t = R(f,-z)^{-1}$, we get

$$(R(f,z)^t)^\wedge \mathcal{F}(\mathbf{q}) = \left(e^{\frac{\hbar}{2}V(\partial_\mathbf{q},\partial_\mathbf{q})}\,\mathcal{F}\right)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}},$$

where the 2nd order differential operator

$$V(\partial_\mathbf{q},\partial_\mathbf{q}) = \sum_{k,\ell=0}^{\infty} (V_{k\ell}\phi^j,\phi^i)\frac{\partial^2}{\partial q_{k,i}\partial q_{\ell,j}}$$

is given by

$$V_{k\ell} = \sum_{a=0}^{\ell}(-1)^{a+k+\ell+1}R_{k+1+a}R^t_{\ell-a}.$$

By definition,

$$\Phi_{\omega_1,\ldots,\omega_N}(q_{k,i}\mathcal{F}) = \omega^i(-z)^{-k-1}\Phi_{\omega_1,\ldots,\omega_N}(\mathcal{F})$$

and $\omega^i(-z)^{-k-1}$ is given by

$$\sum_{j=1}^{N}\Big(\sum_{a=0}^{k}R_{ij;a}(f)(-1)^a\omega'^j(-z)^{-(k-a)-1} + \sum_{a=k+1}^{\infty}R_{ij;a}(f)(-1)^a\omega'^j(-z)^{a-k-1}\Big).$$

By definiton, if $k-a\geq 0$, then

$$\Phi^{-1}_{\omega'_1,\ldots,\omega'_N}\circ\omega'^j(-z)^{-(k-a)-1} = q_{k-a,j}\circ\Phi^{-1}_{\omega'_1,\ldots,\omega'_N}$$

and if $a \geq k+1$, then

$$\Phi^{-1}_{\omega'_1,\ldots,\omega'_N}\circ\omega'^j(-z)^{a-k-1} = (-1)^{a+k+1}\hbar\sum_{j'=1}^{N}(\phi^j,\phi^{j'})\frac{\partial}{\partial q_{a-k-1,j'}}\circ\Phi^{-1}_{\omega'_1,\ldots,\omega'_N}.$$

Therefore $\Phi^{-1}_{\omega'_1,\ldots,\omega'_N}\Phi_{\omega_1,\ldots,\omega_N}(q_{k,i}\mathcal{F})$ can be written as the sum of

$$\sum_{j=1}^{N}\sum_{a=0}^{k}R_{ij;a}(f)(-1)^a q_{k-a,j}\,\left(e^{\frac{\hbar}{2}V(\partial_\mathbf{q},\partial_\mathbf{q})}\,\mathcal{F}\right)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}}$$

and

$$\sum_{j=1}^{N}\sum_{a=k+1}^{\infty}R_{ij;a}(f)(-1)^{k+1}\hbar\sum_{j'=1}^{N}(\phi^j,\phi^{j'})\frac{\partial}{\partial q_{a-k-1,j'}}\quad\left(e^{\frac{\hbar}{2}V(\partial_\mathbf{q},\partial_\mathbf{q})}\,\mathcal{F}\right)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}}.$$

Note that

$$\sum_{j=1}^{N}\sum_{a=0}^{k}R_{ij;a}(f)(-1)^a q_{k-a,j} = (R(f,-z)\mathbf{q})_{k,i}.$$

A straightforward computation shows that

$$\sum_{j=1}^{N}\sum_{a=k+1}^{\infty}R_{ij;a}(f)(-1)^{k+1}\hbar\sum_{j'=1}^{N}(\phi^j,\phi^{j'})\frac{\partial}{\partial q_{a-k-1,j'}}\quad\left(\mathcal{G}(\mathbf{q})\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}}\right)$$

equals

$$\Big(\frac{\hbar}{2}[V(\partial_{\mathbf{q}},\partial_{\mathbf{q}}),q_{k,i}]\mathcal{G}\Big)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}}.$$

Finally, for $\Phi^{-1}_{\omega_1',\dots,\omega_N'}\Phi_{\omega_1,\dots,\omega_N}(q_{k,i}\mathcal{F})$ we get

$$\Big(\Big(q_{k,i}+\frac{\hbar}{2}[V(\partial_{\mathbf{q}},\partial_{\mathbf{q}}),q_{k,i}]\Big)e^{\frac{\hbar}{2}V(\partial_{\mathbf{q}},\partial_{\mathbf{q}})}\,\mathcal{F}\Big)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}} = \Big(e^{\frac{\hbar}{2}V(\partial_{\mathbf{q}},\partial_{\mathbf{q}})}\,(q_{k,i}\mathcal{F})\Big)\Big|_{\mathbf{q}\mapsto R(f,-z)\mathbf{q}}.$$

The above expression is precisely $(R(f,z)^t)^\wedge(q_{k,i}\mathcal{F})$. □

Comparing the transformation laws for the total ancestor potential (see Proposition 5.2) and the transformation laws in Lemma 5.5 and Lemma 5.6, we get that

$$\mathcal{A}(\hbar,f,\omega):=\sigma^{\phi_1,\dots,\phi_N}_{\omega,P}(\mathcal{A}^{\omega_1,\dots,\omega_N}_{f,\omega}(\hbar;\mathbf{q}))$$

is a vector in $\widehat{\mathbb{V}}_{\mathrm{tame}}(f)$ independent of the choice of the basis $\{\phi_i\}_{i=1}^N$ and the choice of the opposite subspace P. We refer to $\mathcal{A}(\hbar,f,\omega)$ as the *global ancestor potential* of f. Let us denote by $\widehat{\mathbb{V}}_{\mathrm{tame}}$ the vector bundle on $\mathcal{M}^\circ_{\mathrm{mar}}$ whose fiber over a point $f\in\mathcal{M}_{\mathrm{mar}}$ is the completed tame Fock space $\widehat{\mathbb{V}}_{\mathrm{mar}}(f)$. We call it the *completed tame Fock bundle* or simply the *abstract Fock bundle*. The global ancestor potential may be viewed as a holomorphic function

$$\mathcal{A}\colon \mathcal{L}\setminus\{0\}\to\widehat{\mathbb{V}}_{\mathrm{tame}},\quad (f,\omega)\mapsto\mathcal{A}(\hbar,f,\omega).$$

Note that the above map is not a map of vector bundles. Nevertheless, we have the following symmetry, which in some sense allows us to think of $\hbar$ as a coordinate along the fiber of $\mathcal{L}$.

Corollary 5.7. *The global ancestor potential has the following scaling property:*

$$\mathcal{A}(\hbar,f,c\,\omega)=\mathcal{A}(\hbar c^{-2},f,\omega),\quad \forall c\in\mathbb{C}\setminus\{0\}.$$

Proof. The statement is a Corollary of Proposition 5.2, a) and Lemma 5.5. □

5.5. **Abstract modular forms.** Motivated by Corollary 5.7 and the generalized definition of a quasi-modular form in the theory of the period maps (see Definition 2.1), we would like to introduce the notion of a quasi-modular form for the moduli space $\mathcal{M}^\circ_{\mathrm{mar}}$.

Definition 5.8. We say that a function

$$\mathcal{A}:\mathcal{L}-\{0\}\to\widehat{\mathbb{V}}_{\mathrm{tame}},$$

is an *abstract modular form* if $\mathcal{A}(\hbar,f,c\,\omega)=\mathcal{A}(\hbar c^{-2},f,\omega)$.

Remark 5.9. According to Corollary 5.7, the global ancestor potential is an abstract modular form.

In order to compare Definitions 5.8 and 2.1, let us trivialize the abstract Fock bundle over an open subset $\mathcal{U}\subset\mathcal{M}^\circ_{\mathrm{mar}}$. Let us denote by Jac the vector bundle over $\mathcal{M}^\circ_{\mathrm{mar}}$ whose fiber over f is the Jacobi algebra of f. Assume that $\{\phi_i\}_{i=1}^N$ is a frame for $\mathrm{Jac}\,|_{\mathcal{U}}$, ω is a frame for $\mathcal{L}|_{\mathcal{U}}$, and $P\subset\mathbb{H}|_{\mathcal{U}}$ is a sub-bundle such that $P(f)\subset\mathbb{H}(f)$ is an opposite subspace for all $f\in\mathcal{U}$. The isomorphism (25) is a trivialization of the abstract Fock bundle.

Let $\mathcal{A}(\hbar, f, \omega)$ be an abstract modular form. Put

$$\mathcal{A}(\hbar; f, \omega, \mathbf{q}) := (\sigma^{\phi_1,\dots,\phi_N}_{\omega,P(f)})^{-1}(\mathcal{A}(\hbar, f, \omega)).$$

Suppose that we have the following genus expansion

$$\mathcal{A}(\hbar; f, \omega, \mathbf{q}) = \exp\left(\sum_{g=0}^{\infty} \hbar^g \mathcal{F}_g(f, \omega, \mathbf{q})\right).$$

The scaling property of $\mathcal{A}$ is equivalent to

$$\mathcal{F}_g(f, c\,\omega, c\,\mathbf{q}) = c^{2-2g}\mathcal{F}_g(f, \omega, \mathbf{q}),$$

so the coefficients of $\mathcal{F}_g(f, \omega, \mathbf{q})$ in front of the $\mathbf{q}$-monomials can be interpreted as sections of $\mathcal{L}^{2g-2+\deg}|_{\mathcal{U}}$, where deg is the degree of the corresponding $\mathbf{q}$-monomial.

The notion of quasi-modularity is contained in Definition 5.8 as follows. Given an abstract modular form $\mathcal{A}$ and a chart $\mathcal{U}$ such that $\mathrm{Jac}\,|_{\mathcal{U}}$ is trivial, we can choose the complex conjugate opposite sub-bundle $\kappa(\mathbb{H}_+)z^{-1}$ to obtain coordinate expressions $\widetilde{\mathcal{F}}_g(f, \omega, \mathbf{q})$. The latter depends non-holomorphically on f, but its coefficients transform as modular forms in the sense of Definition 2.1. Choosing a different opposite sub-bundle $P \subset \mathbb{H}$ with a holomorphic trivialization of $\mathbb{H}_+ \cap Pz$, we can obtain another coordinate expression $\mathcal{F}_g(f, \omega, \mathbf{q})$ for $\mathcal{A}$. The change of the opposite subspace is given by a matrix $B(f)$ and a symplectic transformation $R(f, z) = 1 + R_1(f)z + R_2(f)z^2 + \cdots$ (see Section 5.4.3). Recalling Lemma 5.5 and Lemma 5.6, we get that the coefficients of $\widetilde{\mathcal{F}}_g(f, \omega, \mathbf{q})$ are polynomial expressions of the coefficients of $\mathcal{F}_l(f, \omega, \mathbf{q})$, $l \leq g$, with coefficients in $\mathbb{C}[B, R_1, R_2, \dots]$, where $\mathbb{C}[B, R_1, R_2, \dots]$ is the polynomial ring on the entries of the matrices $B(f), R_1(f), R_2(f), \dots$. The entries of these matrices depend non-holomorphically on f, so following the terminology in Definition 2.1, we call them *anti-holomorphic generators*. Let us point out that finding explicit formulas for the anti-holomorphic generators is in general a difficult problem (see the example in Section 5.7).

Remark 5.10. We explain a relationship of the Fock bundle in the present paper to the Fock sheaf in [20]. Given an opposite subspace P, a section of the Fock sheaf in [20] can be locally identified with a function on the Givental Lagrangian cone of the form $Z = \exp(\sum_{g=0}^{\infty} \hbar^{g-1} F_g)$ (this is similar to the trivialization (25)). The total space of $\mathcal{L} - \{0\}$ over $\mathcal{M}^{\circ}_{\mathrm{mar}}$ can be identified with a finite-dimensional slice of the Givental cone, and a coordinate expression $\mathcal{A}(\hbar; f, \omega, \mathbf{q}) = (\sigma^{\phi_1,\dots,\phi_N}_{\omega,P})^{-1}\mathcal{A}(\hbar, f, \omega)$ of the abstract ancestor potential corresponds to the jet of the potential Z at the point $(f, -z\omega)$. This is related to the *jetness* in [20].

5.6. **The holomorphic anomaly equations.** Let us pick local holomorphic frames $\{\phi_i\}_{i=1}^N$ and ω for respectively Jac and $\mathcal{L}$. Let us choose a local holomorphic coordinate system $\sigma = (\sigma_1, \dots, \sigma_{N'})$ on $\mathcal{M}^{\circ}_{\mathrm{mar}}$, so that each $\phi_i = \phi_i(x, \sigma)$ is a weighted-homogeneous polynomial depending holomorphically on σ. We define the *hybrid ancestor potential* $\mathcal{A}_f(\hbar, \mathbf{q})$ to be the coordinate expression $\mathcal{A}^{\omega_1,\dots,\omega_N}_{f,\omega}(\hbar, \mathbf{q})$ of the global ancestor potential with respect to the opposite subspace $\kappa(\mathbb{H}_+)z^{-1}$ (see Section 5.4.3). The hybrid ancestor potential depends non-holomorphically on f. We would like to derive differential equations for $\mathcal{A}_f(\hbar, \omega)$, which following the physics literature will be called *holomorphic anomaly equations*, that govern the non-holomorphic dependence on f.

Let us begin with several remarks about the Cecotti-Vafa connection from Section 4.6. For given $f \in \mathcal{M}^\circ_{\mathrm{mar}}$, we have a vector spaces isomorphism

$$\mathrm{Jac}(f) \cong \mathbb{K}(f), \quad \phi_a \mapsto \omega_a,$$

which allows us to interpret all connection matrices and the complex conjugation κ as endomorphisms and a complex conjugation of $\mathrm{Jac}(f)$. Note that C_i is the operator of multiplication by $\partial f/\partial \sigma_i$ and U is the operator of multiplication by f. In particular, $\widetilde{U} = U = 0$ because f is weighted-homogeneous, so it vanishes in $\mathrm{Jac}(f)$. The next observation is that $\{\omega_a\}_{a=1}^N$ is a holomorphic frame for $\mathbb{K}$, because the transition functions in this frame are the same as the transition functions of $\mathrm{Jac} \otimes \mathcal{L}$ in the frame $\{\phi_a \otimes \omega\}$ and the latter is by definition holomorphic. This observation implies that the connection matrices satisfy $\Gamma_{\bar{\imath}} = 0$ and $\Gamma_i = h^{-1}\partial h$, where ∂ is the holomorphic de Rham differential on $\mathcal{M}^\circ_{\mathrm{mar}}$ and $h = (h_{ab})$ is the matrix of the Hermitian pairing

$$h_{ab} = K^{(0)}(\kappa(\omega_a), \omega_b).$$

Finally, let us point out that the operators $z\widetilde{C}_{\bar{\imath}}$, $1 \le i \le N'$ are infinitesimal symplectic transformations of $\mathrm{Jac}(f)(\!(z^{-1})\!)$. This can be proved by using the compatibility of the Gauss–Manin connection with the higher residue pairings and the fact that up to a holomorphic factor, the quantity

$$K(\omega_i, \omega_j) = K^{(0)}(\omega_i, \omega_j)$$

is the Grothendick residue of $\phi_i(x,\sigma)\phi_j(x,\sigma)$, so it must be holomorphic in $\sigma \in \mathcal{M}^\circ_{\mathrm{mar}}$.

Proposition 5.11. *The hybrid ancestor potential satisfies the following differential equations*

$$\partial_{\overline{\sigma}_i}\mathcal{A}_f(\hbar, \mathbf{q}) = (z\widetilde{C}^t_{\bar{\imath}})^\wedge \mathcal{A}_f(\hbar, \mathbf{q}), \quad 1 \le i \le N',$$

where ^{t} *is conjugation with respect to the residue pairing.*

Proof. Let us fix $f \in \mathcal{M}^\circ_{\mathrm{mar}}$ and denote by $P(F)$ ($F \in B_f$) the holomorphic extension of the opposite subspace $\kappa(\mathbb{H}_+(f))$ to a family of opposite subspaces on the parameter space B_f of a miniversal unfolding of f. Let us fix arbitrary holomorphic frames $\{\phi_a\}_{a=1}^N$ and ω of Jac and $\mathcal{L}$ on $\mathcal{M}^\circ_{\mathrm{mar}} \cap B_f$. For every $f' \in \mathcal{M}^\circ_{\mathrm{mar}} \cap B_f$ we have two opposite subspaces in $\mathbb{H}(f')$: the complex conjugate subspace $\kappa(\mathbb{H}_+(f'))z^{-1}$ and the holomorphic opposite subspace $P(f')$. Let us denote by $\{\omega_a\}_{a=1}^N \subset \mathbb{K}(f')$ and $\{\widetilde{\omega}_a\}_{a=1}^N \subset \mathbb{H}_+(f') \cap P(f')z$ the good bases such that

$$\omega_a \equiv \widetilde{\omega}_a \equiv \phi_a \omega \mod z\mathbb{H}_+(f').$$

According to Proposition 5.2, b), the ancestor potentials

$$\mathcal{A}_{f'}(\hbar, \mathbf{q}) := \mathcal{A}^{\omega_1,\dots,\omega_N}_{f',\omega}(\hbar; \mathbf{q}) \quad \text{and} \quad \widetilde{\mathcal{A}}_{f'}(\hbar, \mathbf{q}) := \mathcal{A}^{\widetilde{\omega}_1,\dots,\widetilde{\omega}_N}_{f',\omega}(\hbar; \mathbf{q})$$

are related by

$$\mathcal{A}_{f'}(\hbar; \mathbf{q}) = (R(f', z)^t)^\wedge\, \widetilde{\mathcal{A}}_{f'}(\hbar; \mathbf{q}),$$

where the symplectic transformation $R(f',z)$ of $\mathrm{Jac}(f')(\!(z^{-1})\!)$ is represented in the basis $\{\phi_a\}_{a=1}^N$ of $\mathrm{Jac}(f')$ by the matrix $(R_{ab}(f',z))_{a,b=1}^N$ that describes the change

$$\omega_b(f') = \sum_{a=1}^N \widetilde{\omega}_a(f')R_{ab}(f',\sigma), \quad 1 \le b \le N. \tag{26}$$

Since $\widetilde{\mathcal{A}}_{f'}$ depends holomorphically on f', we just need to find the derivatives of $R(f',z)$ with respect to $\overline{\sigma}'_i$, where $\sigma' = (\sigma'_1,\dots,\sigma'_{N'})$ denotes the coordinates of f'. Differentiating (26) with respect to $\overline{\sigma}'_i$ we get

$$\sum_{k=1}^N \omega_k(f')(\Gamma^k_{\bar{\imath}b} + \widetilde{C}^k_{\bar{\imath}b}z) = \sum_{a,k=1}^N \widetilde{\omega}_a(f')R_{ak}(f',\sigma)(\Gamma^k_{\bar{\imath}b} + \widetilde{C}^k_{\bar{\imath}b}z)$$

for the LHS and

$$\sum_{a=1}^N \widetilde{\omega}_a(f')\partial_{\overline{\sigma}_i}R_{ab}(f',z)$$

for the RHS. Comparing the coefficients in front of $\widetilde{\omega}_a$ we get

$$\partial_{\overline{\sigma}_i}R_{ab}(f',z) = \sum_{k=1}^N R_{ak}(f',\sigma)(\Gamma^k_{\bar{\imath}b} + \widetilde{C}^k_{\bar{\imath}b}z).$$

In matrix form the above equation becomes $\partial_{\overline{\sigma}_i}R(f',z) = R(f',z)(\Gamma_{\bar{\imath}} + z\widetilde{C}_{\bar{\imath}})$. Although quantization of symplectic transformations is only a projective representation, when restricted to the subgroup of symplectic transformations of the type $R_0 + R_1 z + \cdots$ the quantization becomes a representation. Therefore

$$\partial_{\overline{\sigma}_i}(R(f',z)^t)^\wedge = (\Gamma^t_{\bar{\imath}} + z\widetilde{C}^t_{\bar{\imath}})^\wedge (R(f',z)^t)^\wedge.$$

It remains only to use that the frame $\{\omega_a\}_{a=1}^N$ is holomorphic, so $\Gamma_{\bar{\imath}} = 0$. ☐

Remark 5.12. Note that by definition, the quantization of the infinitesimal symplectic transformation $z\widetilde{C}^t_{\bar{\imath}}$ is the following differential operator

$$\sum_{a,b=1}^N \Big(\frac{\hbar}{2}\widetilde{C}^{ab}_{\bar{\imath}}\frac{\partial^2}{\partial q_{0,a}\partial q_{0,b}} - \sum_{k=0}^\infty \widetilde{C}^b_{\bar{\imath}a} q_{k,a}\frac{\partial}{\partial q_{k+1,b}}\Big),$$

where

$$\widetilde{C}^{ab}_{\bar{\imath}} := (\widetilde{C}_{\bar{\imath}}\phi^a, \phi^b) = \overline{(C_i\kappa(\phi^a), \kappa(\phi^b))}$$

is symmetric in a and b. Here $\{\phi^a\}_{a=1}^N$ is a basis of $\mathrm{Jac}(f) \cong \mathbb{K}(f)$ dual to $\{\phi_a\}_{a=1}^N$ with respect to the residue pairing. From this explicit formula we see that our differential equations have the same form as the BCOV holomorphic anomaly equations (see [7], formula (3.17)).

Remark 5.13. The holomorphic anomaly equation was also studied in [20, §9.3].

5.7. **Fermat simple elliptic singularity of type $E_6^{(1,1)}$.** We would like to give an example in which our construction gives an elegant way to investigate the modular properties of the total ancestor potential. Put

$$f(x,Q)=x_1^3+x_2^3+x_3^3-\frac{1}{Q}x_1x_2x_3,\quad Q(1-3^3Q^3)\neq 0.$$

This family of polynomials represents a transversal slice to the orbits of the group of coordinate changes, so it could be viewed as an open chart in the marginal moduli space (see Remark 3.3). Let us introduce the functions

$$\begin{aligned} f_0(Q) &= 1+\sum_{k=1}^{\infty}\frac{(3k)!}{(k!)^3}Q^{3k},\\ f_1(Q) &= \log Q+\sum_{k=1}^{\infty}\frac{(3k)!}{(k!)^3}Q^{3k}(\log Q+h_{3k}-h_k),\\ g(Q,\overline{Q}) &= -\frac{(Df_1)\overline{f}_0+(Df_0)\overline{f}_1}{f_1\overline{f}_0+f_0\overline{f}_1}, \end{aligned}$$

where $D=Q\partial_Q Q=(Q+Q^2\partial_Q)$ and $h_\ell=1+\frac{1}{2}+\cdots+\frac{1}{\ell}$ are the harmonic numbers. Note that f_0 and f_1 are solutions to a 2nd order Fuchsian equation defined by the differential operator

$$(Q\partial_Q)^2-3^3Q^3(Q\partial_Q+1)(Q\partial_Q+2). \tag{27}$$

We are going to choose two sets of good bases $\{\omega_e^{\mathrm{KS}}\}$ and $\{\omega_e^{\mathrm{GW}}\}$, where the index set for e is splitted into the following 4 groups:

$$\{(0,0,0),\,(1,1,1)\}\sqcup\{(1,0,0),\,(2,0,0)\}\sqcup\{(0,1,0),\,(0,2,0)\}\sqcup\{(0,0,1),\,(0,0,2)\}.$$

The following set of forms is a good basis for the complex conjugate opposite subspace:

$$\begin{aligned} \omega_{0,0,0}^{\mathrm{KS}} &= dx/Q,\\ \omega_{1,1,1}^{\mathrm{KS}} &= (x_1x_2x_3+g(Q,\overline{Q})z)dx/Q\\ \omega_{m\,e_i}^{\mathrm{KS}} &= x_i^m dx/Q,\quad 1\le m\le 2,\quad 1\le i\le 3, \end{aligned}$$

where e_i is the ith coordinate vector in $\mathbb{Z}^3$. Another good basis is computed from mirror symmetry at the large radius limit point $Q=0$

$$\begin{aligned} \omega_{0,0,0}^{\mathrm{GW}} &= \sqrt{-1}\,\frac{dx}{Qf_0(Q)},\\ \omega_{1,1,1}^{\mathrm{GW}} &= \sqrt{-1}\left(x_1x_2x_3-z\frac{Df_0}{f_0}\right)f_0(Q)\det(I_0)^{-1}\frac{dx}{Q},\\ \omega_{me_i}^{\mathrm{GW}} &= \sqrt{-1}\,(1-3^3Q^3)^{m/3}\,x_i^m\frac{dx}{Q},\quad m=1,2, \end{aligned}$$

where

$$I_0=\begin{bmatrix} f_0(Q) & Df_0(Q)\\ f_1(Q) & Df_1(Q)\end{bmatrix}$$

is the Wronskian matrix of the differential equation (27). The details of both computations will be presented in a future investigation.

Let us pick $\omega = \omega_{0,0,0}^{\mathrm{GW}}$ to be a frame for the vacuum line bundle and denote by $\mathcal{A}_f^{\mathrm{KS}}(\hbar;\mathbf{q})$ and $\mathcal{A}_f^{\mathrm{GW}}(\hbar;\mathbf{q})$ the total ancestor potentials corresponding respectively to the good bases $\{\omega_i^{\mathrm{KS}}\}$ and $\{\omega_i^{\mathrm{GW}}\}$. Let us define

$$t := 2\pi\sqrt{-1}\tau/3 := f_1(Q)/f_0(Q).$$

Using the positive definite Hermitian pairing on $\mathbb{H}_+(f)\cap\kappa(\mathbb{H}_+(f))$, it is easy to check that $t+\bar{t}<0$, i.e., $\mathrm{Im}(\tau)>0$. Note that

$$(\omega_{0,0,0}^{\mathrm{KS}},\dots,\omega_{0,0,2}^{\mathrm{KS}}) = (\omega_{0,0,0}^{\mathrm{GW}},\dots,\omega_{0,0,2}^{\mathrm{GW}})\, R(Q,\overline{Q},z)\, B(Q),$$

where the matrices R and B are block-diagonal

$$R = \mathrm{Diag}(R^{(1)},R^{(2)},R^{(3)},R^{(4)}),$$
$$B = \mathrm{Diag}(B^{(1)},B^{(2)},B^{(3)},B^{(4)}),$$

where the blocks $R^{(i)}$ and $B^{(i)}$ are given by the following formulas:

$$R^{(1)} = \begin{bmatrix} 1 & z/(t+\bar{t}) \\ 0 & 1 \end{bmatrix}, \quad R^{(2)} = R^{(3)} = R^{(4)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B^{(1)} = -\sqrt{-1}\begin{bmatrix} f_0 & 0 \\ 0 & \det(I_0)/f_0 \end{bmatrix},$$
$$B^{(2)} = B^{(3)} = B^{(4)} = -\sqrt{-1}\begin{bmatrix} (1-3^3Q^3)^{-1/3} & 0 \\ 0 & (1-3^3Q^3)^{-2/3} \end{bmatrix}.$$

Recalling Proposition 5.2, we get that

$$\widetilde{\mathcal{A}}_\tau(\hbar;\mathbf{q}) := (R(Q,\overline{Q},z)^t)^\wedge\,\mathcal{A}_f^{\mathrm{GW}}(\hbar;\mathbf{q}) = \mathcal{A}_f^{\mathrm{KS}}(\hbar;B^{-1}\mathbf{q}). \tag{28}$$

Let $\Sigma = \mathbb{P}^1 - \{Q(1-3^3Q^3)=0\}$ be the domain of the deformation parameter Q. The function τ gives an identification between the unniversal covering of Σ and the upper-half plane $\mathbf{H}$. It is easy to check that the monodromy transformations of τ are given by

$$\tau \mapsto g(\tau) = \frac{a\tau+b}{c\tau+d}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(3).$$

Under the analytic continuation the primitive form ω and B^{-1} are transformed respectively to

$$\omega \mapsto \omega(c\tau+d)^{-1} \quad \text{and} \quad B^{-1} \mapsto B^{-1}J(g,\tau)(c\tau+d)^{-1},$$

where

$$J(g,\tau) = \mathrm{Diag}(1,(c\tau+d)^2,\underbrace{c\tau+d,\dots,c\tau+d}_{6\text{ times}}).$$

The analytic continuation of the identity (28) yields

$$\widetilde{\mathcal{A}}_{g(\tau)}(\hbar;\mathbf{q}) = \widetilde{\mathcal{A}}_\tau(\hbar(c\tau+d)^2;J(g,\tau)\mathbf{q}).$$

Comparing the coefficients in front of the monomials in $\mathbf{q}$ we get that the coefficients of the total ancestor potential $\widetilde{\mathcal{A}}_\tau(\hbar;\mathbf{q})$ transform as modular forms on $\Gamma(3)$.

Remark 5.14. The potential $\widetilde{\mathcal{A}}_\tau(\hbar;\mathbf{q})$ coincides with the anti-holomorphic completion constructed in an ad hoc way in [52]. Recalling also the mirror symmetry established in Theorem 6.15, we recover the main result of [52]: the Gromov–Witten invariants of the elliptic orbifold $\mathbb{P}^1_{3,3,3}$ are quasi-modular forms.

Remark 5.15. Slightly generalizing the above argument, we would like to investigate the total ancestor potential $\mathcal{A}_f^{\mathrm{GW}}(\hbar;\mathbf{q})$ as a formal series in $q_{k,i}$, $k>0$, whose coefficients are analytic functions of $\tau_i := q_{0,i}$, $1\le i\le N$. Recalling the results of Looijenga (see [49]), we may identify each relevant deformation parameter τ_i with an appropriate θ-function. We expect that under this identification, the GW invariants will turn into quasi-Jacobi forms. We will address this problem in a future investigation.

6. Mirror symmetry for orbifold Fermat hypersurfaces

In the last three sections, we have constructed a global B-model generating function which is modular in an appropriate generalized sense, but non-holomorphic. In the remaining part of this paper we will prove two mirror theorems for Fermat type polynomials satisfying a CY condition. Namely, we will prove that the generating functions of certain GW-theory/FJRW-theory invariants are holomorphic limits of the global B-model generating function. In this section, we will establish the mirror theorem for GW-theory.

Let $d_1,\dots,d_n\in\mathbb{Z}$ be positive integers satisfying

$$\frac{1}{d_1}+\cdots+\frac{1}{d_n}=1.$$

Let G be the group

$$G=\{t\in(\mathbb{C}^*)^n \mid t_1^{d_1}=\cdots=t_n^{d_n}\}.$$

We define two orbifolds

$$\mathbf{P}:=[(\mathbb{C}^n\setminus\{0\})/G]$$

and a suborbifold Calabi–Yau (CY) hypersurface

$$Y=[Z/G],\quad Z=\{z_1^{d_1}+\cdots+z_n^{d_n}=0\}\subset\mathbb{C}^n\setminus\{0\}.$$

The above quotients are taken in the category of orbifold groupoids or equivalently in the category of smooth Deligne–Mumford stacks.

6.1. Orbifold Gromov–Witten theory.

Let Y be an orbifold groupoid whose coarse moduli space $|Y|$ is a projective variety. Let us denote by $H:=H_{\mathrm{CR}}(Y,\mathbb{C})$ the Chen–Ruan cohomology of Y. Our main interest is in the orbifold Gromov–Witten (GW) invariants of Y

$$\langle\phi_{i_1}\psi^{k_1},\dots,\phi_{i_n}\psi^{k_n}\rangle_{g,n,d},\tag{29}$$

where $\{\phi_i\}_{i=1}^N$ is a basis of H and $d\in\mathrm{Eff}(Y)\subset H_2(|Y|;\mathbb{Z})$ is an effective curve class. The invariants are defined through the intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}(Y,d)$. The latter is the moduli space of degree-d stable orbifold maps $f\colon (C,(z_1,g_1),\dots,(z_n,g_n))\to Y$, where C is a genus-g nodal Riemann surface equipped with an orbifold structure, n marked points z_i, and a choice of a generator $g_i\in\mathrm{Aut}_C(z_i)$. The evaluation at the i-th marked point $(f(z_i),f(g_i))$ determines an evaluation map $\mathrm{ev}_i\colon\overline{\mathcal{M}}_{g,n}(Y,d)\to IY$, where

IY is the inertia orbifold of Y. Let us denote by $\psi_i = c_1(L_i)$, $1 \le i \le n$, the descendant classes, where L_i is the line bundle on $\overline{\mathcal{M}}_{g,n}(Y,d)$ formed by the cotangent lines of the coarse space of C at the i-th marked point (see [73]). By definition the GW invariant (29) is obtained by pairing the cohomology class

$$\mathrm{ev}_1^*(\phi_{i_1})\psi_1^{k_1} \cup \cdots \cup \mathrm{ev}_1^*(\phi_{i_n})\psi_n^{k_n}$$

with the virtual fundamental cycle $[\overline{\mathcal{M}}_{g,n}(Y,d)]^{\mathrm{virt}}$. The invariant takes its value in the Novikov ring $\mathbb{C}[\![Q]\!]$. Let us fix an ample $\mathbb{Z}$-basis $\{L^{(i)}\}_{i=1}^r$ of $\mathrm{Pic}(|Y|)$. We embed

$$\mathbb{C}[\![Q]\!] \subset \mathbb{C}[\![Q_1,\dots,Q_r]\!], \quad Q^d \mapsto Q_1^{\langle c_1(L^{(1)}),d\rangle} \cdots Q_r^{\langle c_1(L^{(r)}),d\rangle}$$

For further details on orbifold GW theory we refer to [2] for the algebraic approach and to [13] for the analytic approach.

6.1.1. *Givental's cone.* Let us introduce a set of formal variables $\mathbf{t} = \{t_{k,i}\}$, $1 \le i \le N$, $k \ge 0$. The generating function

$$\mathcal{F}_Y^{(g)}(\mathbf{t}) = \sum_{n=0}^{\infty} \sum_{d\in \mathrm{Eff}(Y)} \frac{Q^d}{n!} \langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle_{g,n,d}$$

is called the *genus-g total descendant potential.* Here we write $\mathbf{t}(\psi) = \sum_{k=0}^{\infty}\sum_{i=1}^N t_{k,i}\phi_i\psi^k$ and expand the correlator multiniearly as a formal power series in $\mathbf{t}$ whose coefficients are the GW invariants (29).

Following Givental [32] we introduce the symplectic vector space

$$\mathcal{H}_Y = H_{\mathrm{CR}}(Y;\mathbb{C}[\![Q]\!])(\!(z^{-1})\!), \quad \Omega(f,g) = \mathrm{Res}_{z=0}(f(-z),g(z))dz,$$

where $(\ ,\ )$ is the orbifold Poincare pairing. The subspaces $\mathcal{H}_Y^+ := H_{\mathrm{CR}}(Y;\mathbb{C}[\![Q]\!])[z]$ and $\mathcal{H}_Y^- := H_{\mathrm{CR}}(Y;\mathbb{C}[\![Q]\!])[\![z^{-1}]\!]z^{-1}$ are Lagrangian subspaces and define a polarization $\mathcal{H}_Y = \mathcal{H}_Y^+ \oplus \mathcal{H}_Y^-$, which allows us to identify $\mathcal{H}_Y \cong T^*\mathcal{H}_Y^+$. By definition, Givental's cone $\mathcal{L}_Y$ is the graph of the differential $d\mathcal{F}_Y^{(0)}$. Explicitly,

$$\mathcal{L}_Y = \left\{ -z + \mathbf{t} + \sum_{k=0}^{\infty}\sum_{i=1}^{N} \frac{\partial \mathcal{F}_Y^{(0)}}{\partial t_{k,i}}(\mathbf{t})\, \phi^i(-z)^{-k-1} \;\Big|\; \mathbf{t}(z) \in \mathcal{H}_Y^+ \right\},$$

where $\{\phi^i\} \subset H$ is a basis dual to $\{\phi_i\}$ with respect to the Poincare pairing. The above definition should be understood in the formal sense, i.e., $\mathcal{L}_Y$ is the formal germ at $\mathbf{t} = 0$ of a cone in $\mathcal{H}_Y$.

6.1.2. *The J-function and the calibration.* Let us fix $\tau \in H$. It is convenient to introduce the notation

$$\left\langle\!\!\left\langle \alpha_1,\dots,\alpha_k \right\rangle\!\!\right\rangle_{g,n}(\tau) = \sum_{\ell=0}^{\infty}\sum_{d\in\mathrm{Eff}(Y)} \frac{Q^d}{\ell!} \langle \alpha_1,\dots,\alpha_k,\tau,\dots,\tau\rangle_{g,n+\ell,d},$$

where $\alpha_s = \phi_{i_s}\psi^{k_s}$, $1 \le s \le n$, are arbitrary insertions. By definition, Givental's J-function of Y is

$$\widetilde{J}_Y(\tau,Q,z) = z + \tau + \sum_{k=0}^{\infty}\sum_{i=1}^{N}\sum_{d\in\mathrm{Eff}(Y)} Q^d \left\langle\!\!\left\langle \phi_i\psi^k \right\rangle\!\!\right\rangle_{0,1,d}(\tau)\, \phi^i z^{-k-1}.$$

Note that

$$\widetilde{J}_Y(\tau, Q, -z) = -z + \tau + d_{-z+\tau}\mathcal{F}_Y^{(0)} \;\in\; \mathcal{L}_Y.$$

Recall also the calibration series $S(\tau, Q, z) = 1 + S_1(\tau, Q)z^{-1} + \cdots$ defined by

$$(S(\tau, Q, z)\phi_i, \phi_j) = (\phi_i, \phi_j) + \sum_{k=0}^{\infty} \sum_{d \in \mathrm{Eff}(Y)} Q^d \left\langle\!\!\left\langle \phi_i \psi^k, \phi_j \right\rangle\!\!\right\rangle_{0,2,d}(\tau)\, z^{-1-k}.$$

Let us denote by $\widetilde{L}_\tau$ the tangent space to $\mathcal{L}_Y$ at $\widetilde{J}(\tau, Q, -z)$, then we have

$$\widetilde{J}(\tau, Q, -z) = -z\, S(\tau, Q, z)^{-1}\, 1, \quad \widetilde{L}_\tau = S(\tau, Q, z)^{-1}\mathcal{H}_Y^+.$$

6.1.3. *Quantum cohomology.* The quantum cup product $\bullet_\tau$ is defined by

$$(\phi_a \bullet_\tau \phi_b, \phi_c) = \left\langle\!\!\left\langle \phi_a, \phi_b, \phi_c \right\rangle\!\!\right\rangle_{0,3}(\tau).$$

Let us fix $\epsilon \ll 1$, assume that the Novikov variables $|Q_i| \le \epsilon$ $(1 \le i \le r)$, and denote by $B \subset H$ the open subset of $\tau \in H$ for which the quantum cup product is convergent. In the absence of convergence, we think of B as a formal analytic germ at $Q = 0$ and $\tau = 0$. Let us introduce also the *Euler vector field*

$$E = \sum_{i=1}^{N} (1 - \deg_{\mathrm{CR}}(\phi_i)) t_i \partial_{t_i} + c_1(Y),$$

where $\deg_{\mathrm{CR}}$ denotes the Chen–Ruan degree. Finally, let

$$\theta : H \to H, \quad \theta(\phi_i) = \Big(\frac{\dim_{\mathbb{C}}(Y)}{2} - \deg_{\mathrm{CR}}(\phi_i)\Big)\phi_i$$

be the so-called *Hodge grading operator*.

By definition the quantum (or Dubrovin's) connection is the connection ∇ on the trivial H-bundle with base $B \times \mathbb{C}^*$ defined by

$$\nabla = d + \Big(-z^{-1}\theta + z^{-2} E\, \bullet \Big) dz - \sum_{i=1}^{N} z^{-1}(\phi_i \bullet) dt_i.$$

It is known that the gauge transformation defined by the calibration $S(\tau, Q, z)$ acts on ∇ as follows:

$$S(\tau, Q, z)^{-1} \nabla S(\tau, Q, z) = d + \Big(-z^{-1}\theta + z^{-2}\rho \Big) dz,$$

where $\rho := c_1(Y)\cup$ is the operator of classical cup product multiplication by $c_1(Y)$. In particular ∇ is a flat connection.

6.1.4. *The total ancestor potential.* Let $\tau \in H$ be an arbitrary parameter. The ancestor GW invariants

$$\left\langle\!\!\left\langle \phi_{i_1}\overline{\psi}^{k_1}, \ldots, \phi_{i_n}\overline{\psi}^{k_n} \right\rangle\!\!\right\rangle_{g,n,d}(\tau) := \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \langle \phi_{i_1}\overline{\psi}^{k_1}, \ldots, \phi_{i_n}\overline{\psi}^{k_n}, \tau, \ldots, \tau \rangle_{g,n+\ell,d}$$

are defined in the same way as the descendant ones except that instead of the descendant classes ψ_i $(1 \le i \le n)$ we use $\overline{\psi}_i := \mathrm{ft}^*\psi_i$, where $\mathrm{ft}\colon \overline{\mathcal{M}}_{g,n+\ell}(Y, \beta) \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the map to Y, the orbifold structure on the domain curve, the last ℓ marked points, and it contracts all unstable components. If $\overline{\mathcal{M}}_{g,n} = \emptyset$, i.e., $2g - 2 + n \le 0$, then the ancestor invariant is by definition 0. Let us point out that in general the dependence on τ is only formal, i.e., the ancestor invariant is a formal power series in τ.

The generating function

$$\overline{\mathcal{F}}^{(g)}_{\tau,Q}(\mathbf{t}) = \sum_{n=0}^{\infty} \sum_{d\in \mathrm{Eff}(Y)} \frac{Q^d}{n!} \left\langle\!\!\left\langle \mathbf{t}(\overline{\psi}),\dots,\mathbf{t}(\overline{\psi}) \right\rangle\!\!\right\rangle_{g,n,d}(\tau)$$

is called the *genus-g total ancestor potential* and

$$\mathcal{A}^Y_{\tau,Q}(\hbar;\mathbf{t}) := \exp\Big(\sum_{g=0}^{\infty} \overline{\mathcal{F}}^{(g)}_{\tau,Q}(\mathbf{t})\,\hbar^{g-1}\Big)$$

is called the *total ancestor potential* of Y. The relation between ancestors and descendants is completely determined by the calibration $S(\tau,z)$ and the genus-1 primary potential of Y (see [32] for more details). Thanks to the divisor equation we have the following symmetry

$$\mathcal{A}^Y_{\tau,Q} = \mathcal{A}^Y_{\tau-\sum_{i=1}^r P_i \log Q_i,1},$$

where $P_i = c_1(L^{(i)})$ and $Q=1$ means $Q_i=1$ for all i. Therefore, without loss of generality we may set $Q_i = 1$ for all i and work with $\mathcal{A}^Y_\tau := \mathcal{A}^Y_{\tau,1}$

6.2. ***I*-function.** Let us return to the case when Y is the orbifold Fermat CY hypersurface.

6.2.1. *Combinatorics of the inertia orbifold.* Let

$$\begin{aligned} b_i &= (0,\dots,d_i,\dots,0)\in \mathbb{Z}^{n-1} \quad (1\le i\le n-1)\\ b_n &= (-d_n,\dots,-d_n)\in\mathbb{Z}^{n-1}. \end{aligned}$$

Let Σ be the fan consisting of all subcones of $\tau_1,\dots,\tau_n$, where τ_i is the cone in $\mathbb{R}^{n-1}$ spanned by the $(n-1)$ rays

$$b_1,\dots,\hat{b}_i,\dots,b_n.$$

Note that $\mathbf{P}$ is the toric orbifold corresponding to the fan Σ, so according to the general theory (see [8]) the connected components of $\mathbf{P}$ are parametrized by the set

$$\mathrm{Box}(\Sigma) = \{c\in\mathbb{Q}^n \mid 0\le c_i<1,\ \mathrm{supp}(c)\subset\sigma,\ \sum_{i=1}^n c_i b_i \in \mathbb{Z}^{n-1} \text{ for some } \sigma\in\Sigma\}$$

where $\mathrm{supp}(c)$ is the set of all b_i such that $c_i\neq 0$. We have

$$I\mathbf{P} = \sqcup_{c\in \mathrm{Box}(\Sigma)} \mathbf{P}_c,$$

where

$$\mathbf{P}_c = [\{z_1c_1 = \cdots = z_nc_n = 0\}/G].$$

The dimension of $\mathbf{P}_c$ is 1 less than the number of i such that $c_i = 0$. The orbifold $\mathbf{P}_c$ is non-reduced and it has a generic stabilizer

$$G_c := \prod_{i:c_i\neq 0} \mu_{d_i}.$$

In particular, the order $|G_c| = \prod_{i:c_i\neq 0} d_i$.

The inertia orbifold IY is a suborbifold of $I\mathbf{P}$, we have

$$\dim(Y_c) = \dim(\mathbf{P}_c) - 1,$$

so the twisted sectors are parametrized by $c \in \mathrm{Box}(\Sigma)$ such that $\dim(\mathbf{P}_c) > 0$; i.e., c has at least 2 entries that are 0. We denote the set of all such c by $\mathrm{Box}_Y(\Sigma)$.

6.2.2. *Cohomology and Poincaré pairing.* The coarse moduli spaces of Y and $\mathbf{P}$ are respectively $\mathbb{P}^{n-2}$ and $\mathbb{P}^{n-1}$. Indeed, the map

$$\pi\colon (\mathbb{C}^n \setminus \{0\}) \times G \to (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^*, \quad (z,t) \mapsto (z, t_1^{d_1}) \tag{30}$$

induces a map of orbifolds that maps the pair $(|Y|, |\mathbf{P}|)$ homeomorphically onto $(\mathbb{P}^{n-2}, \mathbb{P}^{n-1})$. Let us define $L = \pi^*\mathcal{O}_{\mathbb{P}^{n-2}}(1)$ and $p = c_1(L) \in H^2(Y,\mathbb{Z})$. A basis in $H_{\mathrm{CR}}(Y;\mathbb{C})$ can be fixed as follows:

$$p^k \mathbf{1}_c, \quad 0 \le k \le \dim(Y_c), \quad c \in \mathrm{Box}_Y(\Sigma),$$

where $\mathbf{1}_c$ is the unit in $H(Y_c;\mathbb{C})$. Given a cohomology class $p^k\mathbf{1}_c \in H^{2k}(Y_c;\mathbb{C})$, the orbifold Poincaré pairing $(p^k\mathbf{1}_c, p^{k'}\mathbf{1}_{c'})$ is non-zero if and only if

$$k + k' = \dim(Y_c), \quad c_i + c_i' = 0 (\mathrm{mod}\ d_i), \quad 1 \le i \le n.$$

If (k,c) and (k',c') satisfy the above conditions, then we have

$$(p^k\mathbf{1}_c, p^{k'}\mathbf{1}_{c'}) = \frac{1}{|G_c|}.$$

Finally,

$$H_2(|Y|;\mathbb{Z}) \cong \mathbb{Z}, \quad \beta \mapsto d := \langle c_1(\mathcal{O}(1)), \beta \rangle.$$

6.2.3. *The J-function of Y.* Given $d \in \mathbb{Z}_{\ge 0}$ and $\nu \in (\mathbb{Z}_{\ge 0})^n$ we define

$$I_{d,\nu}(z) = \frac{\Gamma(1+d+pz^{-1})}{\Gamma(1+pz^{-1})} \prod_{i=1}^n \frac{\Gamma(1-c_i+(p/d_i)z^{-1})}{\Gamma(1-c_i+k_i+(p/d_i)z^{-1})} \mathbf{1}_c z^{d-|\nu|-|k|},$$

where $k_i \in \mathbb{Z}$ and $0 \le c_i < 1$ are defined uniquely by the identity

$$\frac{\nu_i - d}{d_i} = -k_i + c_i$$

and we put $|\nu| = \nu_1 + \cdots + \nu_n$ and $|k| = k_1 + \cdots + k_n$. Put

$$I_Y(t,Q,z) = e^{p\log Q/z} \sum_{d=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n_{\ge 0}} I_{d,\nu}(z)\, Q^d \frac{t^\nu}{\nu!},$$

where

$$t^\nu = t_1^{\nu_1}\cdots t_n^{\nu_n}, \quad \nu! = \nu_1!\cdots\nu_n!.$$

Note that

$$I_Y(t,Q,z) = f_0(Q)\mathbf{1} + z^{-1}f_1(t,Q) + z^{-2}f_2(t,Q) + \cdots ,$$

where $f_k(t,Q) \in H_{\mathrm{CR}}(Y;\mathbb{C})$ $(k \ge 1)$ and

$$f_0(Q) = 1 + \sum_{d=1}^{\infty} \frac{(d\ell)!}{(dw_1)!\cdots(dw_n)!} Q^d,$$

where $\ell = \mathrm{lcm}(d_1, \ldots, d_n)$ and $w_i = \ell/d_i$.

It will be convenient for our purposes to modify slightly Givental's J-function and to work with

$$J_Y(\tau, Q, z) = e^{p \log Q/z} \widetilde{J}_Y(\tau, Q, z) = \widetilde{J}_Y(\tau + P \log Q, 1, z),$$

where the 2nd equality is a consequence of the *divisor equation*. The suborbifold $Y \subset \mathbf{P}$ is cut out by the section $z_1^{d_1} + \cdots + z_n^{d_n}$ of the convex line bundle $\pi^*\mathcal{O}(1)$, where $\pi\colon \mathbf{P} \to \mathbb{P}^{n-1}$ is the map induced from (30). We may recall Theorem 25 in [22] and get the following formula for the J-function of Y.

Proposition 6.1. *If $\tau = f_1(t, Q)/f_0(Q)$, then $J_Y(\tau, 1, z) f_0(Q) = z I_Y(t, Q, z)$.*

The I-function is known to be a solution to a Picard–Fuchs differential equation in Q, so it has a non-zero radius of convergence as a power series at $Q = 0$. Let Δ be the disc of convergence, $\Delta^* := \Delta - \{0\}$, and $\pi\colon \widetilde{\Delta^*} \to \Delta^*$ be the universal cover of Δ^*. The function τ in Proposition 6.1 defines a map

$$\tau\colon \mathbb{C}^n \times \widetilde{\Delta^*} \to H, \quad (t, Q) \mapsto \tau(t, Q) := f_1(t, Q)/f_0(Q) \tag{31}$$

which will be called the *mirror map.*

6.3. **Mirror symmetry for $\mathcal{D}$-modules.** Let us introduce the following family of polynomials:

$$f(x, t, Q) = \sum_{i=1}^{n} (x_i^{d_i} + t_i x_i) - \frac{1}{Q} x_1 \cdots x_n,$$

where

$$(t, Q) \in \mathbb{C}^n \times (\mathbb{P}^1 \setminus \{0, a_1, \ldots, a_r, \infty\}),$$

where a_i are the values of Q for which the polynomial has a non-isolated singularity.

Remark 6.2. The radius of the disc Δ is precisely $\max_{1 \le i \le r} |a_i|$.

6.3.1. *The twisted de Rham cohomology and the quantum $\mathcal{D}$-module.* The main goal of this section is to construct an isomorphism between the sheaf $\mathcal{F}$ of the twisted de Rham cohomology and quantum $\mathcal{D}$-module. Let $\mathcal{D}$ be the sheaf of differential operators

$$\mathcal{O}_{\mathbb{C}^n \times \Delta^*}[z] \Big\langle z \frac{\partial}{\partial t_1}, \ldots, z \frac{\partial}{\partial t_n}, zQ \frac{\partial}{\partial Q} \Big\rangle.$$

We would like to construct a $\mathcal{D}$-module isomorphism

$$\pi^*(\mathcal{F}|_{\mathbb{C}^n \times \Delta^*}) \cong \tau^*(\mathcal{O}_B \otimes H[z]),$$

where $\pi\colon \mathbb{C}^n \times \widetilde{\Delta^*} \to \mathbb{C}^n \times \Delta^*$ is the universal covering, τ is the mirror map, and $B \subset H$ is the domain of convergence for the quantum cohomology. The $\mathcal{D}$-module structures on the LHS and the RHS of the above isomorphism are induced respectively from the Gauss–Manin connection and the Dubrovin's connection, i.e.,

$$z\partial_{t_a} \mapsto z\partial_{t_a} - \sum_{i=1}^{N} \frac{\partial \tau_i}{\partial t_a}(t, Q)\, \phi_i \bullet, \quad zQ\partial_Q \mapsto zQ\partial_Q - \sum_{i=1}^{N} Q \frac{\partial \tau_i}{\partial Q}(t, Q)\, \phi_i \bullet,$$

where $\tau(t,Q) =: \tau_1(t,Q)\phi_1 + \cdots + \tau_N(t,Q)\phi_N$. Put

$$I_Y^e(t,Q,z) := (z\partial_t)^e I_Y(t,Q,z), \quad e \in \mathbb{Z}^n_{\geq 0},$$

where

$$(z\partial_t)^e = (z\partial_{t_1})^{e_1} \cdots (z\partial_{t_n})^{e_n}.$$

The Taylor's series of $I_Y^e(t,Q,z)$ at $Q = t = 0$ takes the form

$$I_Y^e(t,Q,z) = e^{p\log Q/z} \sum_{d=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n_{\geq 0}} Q^d \frac{t^\nu}{\nu!} I_{d,\nu+e}(z) z^{|e|}.$$

We will need the following lemma.

Lemma 6.3. *Let $d \geq 0$ and $\nu \in \mathbb{Z}^n_{\geq 0}$.*

a) If $m = (1,\ldots,1) \in \mathbb{Z}^n$, then

$$I_{d+1,\nu+e+m}(z)\, z^{|e|+n} = (1 + d + pz^{-1}) I_{d,\nu+e}(z)\, z^{|e|+1}.$$

b) If $e_i \in \mathbb{Z}^n$ is the vector with a non-zero entry equal to 1 only at the i-th place, then

$$I_{d,\nu+e+d_ie_i}(z) = d_i^{-1}(d - \nu_i - e^i + pz^{-1}) I_{d,\nu+e}(z) z^{-d_i+1},$$

where e^i is the i-th entry of e.

The proof follows immediately from the definitions and it is omitted. We also need the following Lemma, which is a corollary of Proposition 3.5.

Lemma 6.4. *The sheaf $\mathcal{F}|_{\mathbb{C}^n\times\Delta^*}$ is a free $\mathcal{O}_{\mathbb{C}^n\times\Delta^*}[z]$-module of rank*

$$N := (d_1 - 1)\cdots(d_n - 1).$$

The main result of this subsection can be stated as follows.

Proposition 6.5. *The assignment*

$$x^e dx/Q \mapsto S(\tau, 1, -z)\, I_Y^e(t,Q,z),$$

where $\tau = \tau(t,Q)$ is the mirror map, induces an isomorphism of $\mathcal{D}$-modules

$$\mathrm{Mir}\colon \pi^*(\mathcal{F}|_{\mathbb{C}^n\times\Delta^*}) \to \tau^*(\mathcal{O}_B \otimes H[z]).$$

Proof. According to Proposition 6.1, $S(\tau,1,-z)\, I_Y^e(t,z) \in H[z]$. To prove that we have an induced map Mir we have to prove that Mir maps

$$(zd + df\wedge) x^e dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

to 0 for all $i = 1,2,\ldots,n$ and $e \in \mathbb{Z}^n_{\geq 0}$. If $e = (e^1,\ldots,e^n)$, then the above form takes the form

$$(ze^i x^{e-e_i} + d_i x^{e+(d_i-1)e_i} + t_i x^e - Q^{-1} x^{e+m-e_i})dx,$$

where $m, e_i \in \mathbb{Z}^n$ are the same as in Lemma 6.3. Shifting $e \mapsto e + e_i$ we get

$$(z(e^i+1)x^e + d_i x^{e+d_ie_i} + t_i x^{e+e_i} - Q^{-1} x^{e+m})dx.$$

It is enough to prove that Mir maps the above form to 0. Recalling the definition of Mir we get that the above form is mapped to

$$\sum_{d,\nu}\Big(z(e^i+1)I_{d,\nu+e}z^{|e|}+d_iI_{d,\nu+e+d_ie_i}z^{|e|+d_i}+t_iI_{d,\nu+e+e_i}z^{|e|+1}-Q^{-1}I_{d,\nu+e+m}z^{|e|+n}\Big)Q^d\frac{t^\nu}{\nu!}.$$

The terms in the brackets are transformed as follows: for the 2nd one we apply Lemma 6.3, b); for the 3rd term we shift the index $\nu\mapsto\nu-e_i$ and use that

$$t_i\frac{t^{\nu-e_i}}{(\nu-e_i)!}=\nu_i\frac{t^\nu}{\nu!};$$

for the 4th term we first shift the index $d\mapsto d+1$ and then recall Lemma 6.3, a). We get

$$\sum_{d,\nu}((e^i+1)+(d-\nu_i-e^i+pz^{-1})+\nu_i-(1+d+pz^{-1}))I_{d,\nu+e}(z)\,z^{|e|+1}Q^d\frac{t^\nu}{\nu!}=0.$$

To prove that Mir is a $\mathcal{D}$-module morphism we need only to verify that

$$\mathrm{Mir}(zQ\partial_Q[x^e dx/Q])=zQ\partial_Q\,\mathrm{Mir}([x^e dx/Q]).$$

Since

$$\mathrm{Mir}(zQ\partial_Q[x^e dx/Q])=\sum_{d,\nu}(I_{d+1,\nu+e+m}z^{|e|+n}-I_{d,\nu+e}z^{|e|+1})Q^d\frac{t^\nu}{\nu!}$$

the above identity follows from Lemma 6.3, a).

Finally, since

$$I_Y(t,Q,z)=f_0(Q)S(\tau,1,-z)^{-1}\,\mathbf{1}, \tag{32}$$

the map Mir is surjective. Since both sheaves are free $\mathcal{O}_{\mathbb{C}^n\times\widetilde{\Delta^*}}[z]$-modules of rank N, the map must be an isomorphism. $\square$

6.4. **Pairing matches.** Let us introduce the following pairing

$$\widetilde{K}(\omega_1,\omega_2)=-(\mathrm{Mir}(\pi^*\omega_1),\mathrm{Mir}(\pi^*\omega_2)^*),\quad \omega_1,\omega_2\in\mathcal{F}|_{\mathbb{C}^n\times\Delta^*}, \tag{33}$$

where $*$ is the involution in $H[z]$ induced from $z\mapsto -z$, and $(\ ,\ )$ is the Poincaré pairing. It is convenient to expand in the powers of z

$$\widetilde{K}(\omega_1,\omega_2)=:\sum_{p\in\mathbb{Z}}^{\infty}\widetilde{K}^{(p)}(\omega_1,\omega_2)z^p.$$

The main goal in this section, which is one of the key ingredients in the proof of our mirror symmetry theorem, is the following proposition.

Proposition 6.6. *The pairing $\widetilde{K}$ coincides with K. Saito's higher resdiue pairing.*

The main idea of the proof is to use Hertling's formula (8) in order to obtain a formula for Saito's pairing similar to (33).

6.4.1. *The mirror map for vanishing cohomology.* Recall the notation from Sections 4.3.1 and 4.3.2. Let us fix a polynomial f_0 corresponding to a reference point in $\mathbb{C}^n\times\Delta^*$ and define for all f sufficiently close to f_0 a linear isomorphism

$$\Psi:\mathfrak{h}=H^{n-1}(f^{-1}(1);\mathbb{C})\to H=H_{\rm CR}(Y;\mathbb{C}), \tag{34}$$

such that

$$\Psi(\widehat{s}(\omega_e,z))=(-z)^{-\theta}I^e_Y(t,Q,z), \tag{35}$$

where $\omega_e=x^e dx/Q$ is a fixed set of weighted-homogeneous forms that induces a trivialization of $\mathcal{F}|_{\mathbb{C}^n\times\Delta^*}$. Using that the isomorphism in Proposition 6.5 is a $\mathcal{D}$-module isomrphism we can check that Ψ is independent of t and Q. Note that the homogeneity of the I-function can be written in the form

$$(z\partial_z+\sum_{i=1}^n(1-1/d_i)t_i\partial_{t_i}+\deg_{\rm CR})\,I^e_Y(t,z)=(\deg(\omega_e)-1)\,I^e_Y(t,z).$$

From this equation we get that Ψ is independent of z as well.

Remark 6.7. The map Ψ is multivalued in f. It can be viewed as a trivialization of the pullback via $\pi:\mathbb{C}^n\times\widetilde{\Delta^*}\to\mathbb{C}^n\times\Delta^*$ of the vanishing cohomology bundle.

6.4.2. *The polarization form and the Poincare pairing.* Let us introduce the following bilinear form on H:

$$\chi(a,b):=S(\Psi^{-1}(a),\nu^{-1}\Psi^{-1}(b)),$$

where S is the polarization form of Steenbrink's Hodge structure (see Section 4.3.1).

Lemma 6.8. *The claim in Proposition 6.6 is equivalent to the identity*

$$\chi(a,e^{-\pi\sqrt{-1}\theta}b)=(a,b),\quad a,b\in H.$$

Proof. Let us first establish that $p\cup$ is an infinitesimal symmetry of χ. Let us denote by $M_{\rm mar}$ the monodromy transformation of $\mathfrak{h}$ of the Gauss–Manin connection around $Q=0$ in counter clockwise direction. The analytic continuation around $Q=0$ transforms the RHS of (35) into

$$(-z)^{-\theta}\;e^{2\pi\sqrt{-1}p/z}\,I^e_Y(t,Q,z)=e^{-2\pi\sqrt{-1}p}\,(-z)^{-\theta}I^e_Y(t,Q,z).$$

Therefore,

$$\Psi\circ M_{\rm mar}^{-1}=e^{-2\pi\sqrt{-1}p}\circ\Psi.$$

In particular, $M_{\rm mar}$ is unipotent and there is uniquely defined nilpotent operator $N_{\rm mar}:=-\frac{1}{2\pi\sqrt{-1}}\log M_{\rm mar}$. By definition

$$\chi(a,b)=(-1)^{(n-1)(n-2)/2}\langle\Psi^{-1}(a),\operatorname{Var}\circ\Psi^{-1}(b)\rangle.$$

Since the form $\langle\cdot,\operatorname{Var}(\cdot)\rangle$ is $M_{\rm mar}$-invariant and

$$\Psi(N_{\rm mar}\,A)=p\,\Psi(A),\quad A\in\mathfrak{h},$$

we get that $\chi(p\cup a,b)+\chi(a,p\cup b)=0$. Using this property we can complete the proof as follows. By definition

$$\widehat{s}(\omega_e,z)=\Psi^{-1}((-z)^{-\theta}I^e_Y(t,Q,z))=e^{\log Q\,N_{\rm mar}}\,\Psi^{-1}((-z)^{-\theta}\widetilde{I}^e_Y(t,Q,z)),$$

where $I^e_Y(t,Q,z) =: e^{p\log Q/z}\widetilde{I}^e_Y(t,Q,z)$. Recalling formula (8) we get

$$K(\omega_{e'},\omega_{e''}) = -\chi(e^{\pi\sqrt{-1}\theta}(-z)^{-\theta}\widetilde{I}^{e''}_Y(t,Q,-z),(-z)^{-\theta}\widetilde{I}^{e'}_Y(t,Q,z)), \tag{36}$$

In order to complete the proof, we just have to notice that (33) can be written as

$$\widetilde{K}(\omega_{e'},\omega_{e''}) = -((-z)^{-\theta}\widetilde{I}^{e''}_Y(t,Q,-z),(-z)^{-\theta}\widetilde{I}^{e'}_Y(t,Q,z)),$$

where we used that $S(\tau,1,z)$ is a symplectic transformation, so

$$S(\tau,1,z)^t S(\tau,1,-z) = 1. \quad \square$$

We will compute χ by specializing (36) to $t=Q=0$, i.e., we will express χ in terms of the limit of the higher residue pairing at $t=Q=0$. To begin with, let us compute the Chen–Ruan product of Y. Put $\phi_i = \mathbf{1}_{(0,\dots,1/d_i,\dots,0)}$, where the non-zero entry is on the i-th place. Let

$$\phi_e := \phi_1^{e_1}\cdots\phi_n^{e_n},$$

where $e=(e_1,\dots,e_n)$ is a sequence of non-negative integers and the monomial on the RHS is defined via the Chen–Ruan cup product.

Lemma 6.9. *The following formula holds*

$$\phi_e = (d_1^{-\ell_1}\cdots d_n^{-\ell_n})\, p^\ell \mathbf{1}_c, \quad c=(c_1,\dots,c_n),$$

where the numbers $\ell := \ell_1+\cdots+\ell_n$ *and* c_i *are defined by*

$$\frac{e_i}{d_i} = \ell_i + c_i, \quad 0\le c_i<1, \quad \ell_i\in\mathbb{Z}.$$

Proof. Note that at $Q=0$ the J-function of Y is

$$\widetilde{J}_Y(\widetilde{\tau},0,z) = z e^{\widetilde{\tau}\cup_{\mathrm{CR}}/z}, \quad \widetilde{\tau} := t_1\phi_1+\cdots+t_n\phi_n,$$

where $\cup_{\mathrm{CR}}$ denotes the Chen–Ruan product. Note also that $\tau(t,Q) = p\log Q + \widetilde{\tau} + O(Q)$ and that the calibration $S(0,0,z)=1$. Recalling Proposition 6.1 we get that the vector $\widetilde{I}^e_Y(0,0,z)$ is polynomial in z and that its free term

$$\widetilde{I}^e_Y(0,0,0) = (d_1^{-\ell_1}\cdots d_n^{-\ell_n})\, p^\ell \mathbf{1}_c$$

must coincide with $(z\partial_t)^e \widetilde{J}_Y(t,0,z)|_{t=0} = \phi_e$. $\square$

Choosing $e=(e_1,\dots,e_n)$ appropriately we can arrange that the vectors ϕ_e give a basis of the Chen–Ruan cohomology. Note that the Chen–Ruan degree of ϕ_e is $\deg(e) := \sum_{i=1}^n e_i/d_i$.

Lemma 6.10. *We have*

$$\chi(\phi_{e''}, e^{-\pi\sqrt{-1}\theta}\phi_{e'}) = (\phi_{e'},\phi_{e''}).$$

Proof. Recalling (36) we get

$$K(\omega_{e'},\omega_{e''})|_{t=Q=0} = -z^{-n+\deg(e')+\deg(e'')+2}\chi(\phi_{e''}, e^{-\pi\sqrt{-1}\theta}\phi_{e'}). \tag{37}$$

We need to check that the higher residues $-K^{(p)}(\omega_{e'},\omega_{e''})|_{t=Q=0}$ vanish for $p>0$ and coincide with the Poinare pairing $(\phi_{e'},\phi_{e''})$ for $p=0$. Using that $\chi(p\cup a,b)+\chi(a,p\cup b)=0$, we may reduce the proof to the case when $\phi_{e'}$ is not divisible by p, i.e., $0\le e'_i\le d_i-1$. Finally, let us assume also that $t=0$. The rest of the proof is splitted into four steps.

Step 1. We claim that if $K(\omega_{e'},\omega_{e''})|_{Q=0}\neq 0$, then $e'_i+e''_i\equiv 0 \mod d_i$. Put $\eta_i := e^{2\pi\sqrt{-1}/d_i}$, then the rescaling

$$(x_1,\dots,x_i,\dots,x_n)\mapsto (x_1,\dots,\eta_i x_i,\dots,x_n),\quad Q\mapsto \eta_i Q,$$

defines an automorphism of $\mathbb{H}_+(f)$. It is easy to see that the higher residue pairing K is invariant under the rescaling. We have

$$K(\omega_{e'},\omega_{e''}) = z^r\sum_{m=0}^{\infty} K_{e',e'',m}Q^m,$$

where $r:=\deg(\omega_{e'})+\deg(\omega_{e''})-n$. Rescaling the above identity we get that

$$\eta_i^{e'_i+e''_i}K(\omega_{e'},\omega_{e''}) = z^p\sum_{m=0}^{\infty} K_{e',e'',m}Q^m\eta_i^m.$$

It follows that $e'_i+e''_i\equiv m \mod d_i$, so if the pairing $K(\omega_{e'},\omega_{e''})|_{Q=0}\neq 0$, then $e'_i+e''_i\equiv 0 \mod d_i$.

Step 2. We claim that if $K(\omega_{e'},\omega_{e''})|_{Q=0}\neq 0$, then $\deg(\phi_{e'})+\deg(\phi_{e''})=n-2$. To prove this, recall that $\phi_{e''}=(d_1^{-\ell''_1}\cdots d_n^{-\ell''_n})\,p^{\ell''}\mathbf{1}_{c''}$, where $0\le \ell''\le \dim(Y_{c''})$, $0\le c''_i<1$ are defined by

$$\ell''=\sum_{i=1}^n \ell''_i,\quad e''_i/d_i=\ell''_i+c''_i,\quad \ell_i\in\mathbb{Z}.$$

Using that $e'_i+e''_i\equiv 0 \mod d_i$ we get that

$$\frac{e'_i}{d_i}+\frac{e''_i}{d_i}=\begin{cases}\ell''_i+1, & \text{if } c''_i\neq 0,\\ \ell''_i, & \text{otherwise.}\end{cases}$$

Recall that $\dim(Y_{c''})+2$ is the number of i such that $c''_i=0$. Therefore,

$$\deg(\phi_{e'})+\deg(\phi_{e''})=\deg(e')+\deg(e'')=n-2+\ell''-\dim(Y_{c''})\le n-2.$$

This proves that $K^{(r)}(\omega_{e'},\omega_{e''})|_{Q=0}$ could be non-zero only if $r=0$, $\ell''=\dim(Y_{c''})$, and $\deg(\omega_{e'})+\deg(\omega_{e''})=n$. It remains only to check that

$$K^{(0)}(\omega_{e'},\omega_{e''})=-(\phi_{e'},\phi_{e''}). \tag{38}$$

Step 3. We claim that it is enough to verify (38) in the case when $e'_i+e''_i=d_i$ for $n-2$ values of i and $e'_i=e''_i=0$ for the remaining two other values. Indeed, we may assume again that $0\le e'_i\le d_i-1$ and write $\phi_{e''}=(d_1^{-\ell''_1}\cdots d_n^{-\ell''_n})\,p^{\ell''}\mathbf{1}_{c''}$ as we did above. Since the set $I(c''):=\{i\mid c''_i=0\}$ contains $\dim(Y_{c''})+2$ elements we can choose a subset $J\subset I(c'')$ with ℓ'' elements. Note that $e'_j=0$ for all $j\in J$, because $e'_j+d_jc''_j=0 \mod d_j$. Let us define $\widetilde{e}''_i=d_ic''_i$ $(1\le i\le n)$ and

$$\widetilde{e}'_j=\begin{cases}d_j, & \text{if } j\in J\\ e'_j, & \text{otherwise.}\end{cases}$$

Using the relation $d_j x_j^{d_j} = Q^{-1} x_1 \cdots x_n$ in $\mathrm{Jac}(f)$ we get

$$K^{(0)}(\omega_{e'}, \omega_{e''}) = \Big(\prod_{i \notin J} d_i^{-\ell''_i} \Big) K^{(0)}(\omega_{\widetilde{e}'}, \omega_{\widetilde{e}''}).$$

Similarly, using the relation $d_i \phi_i^{d_i} = p$ in the Chen–Ruan cohomology, we get

$$(\phi_{e'}, \phi_{e''}) = \Big(\prod_{i \notin J} d_i^{-\ell''_i} \Big) (\phi_{\widetilde{e}'}, \phi_{\widetilde{e}''}),$$

which completes the proof of our claim.

Step 4. Due to permutation symmetry of our computation, it is enough to consider only the case $e'_i + e''_i = d_i$ for $1 \le i \le n-2$. By definition

$$K^{(0)}(\omega_{e'}, \omega_{e''}) = \mathrm{Res} \frac{Q^{n-2} x_1^{d_1} \cdots x_{n-2}^{d_{n-2}} dx_1 \cdots dx_n}{(Q d_1 x_1^{d_1 - 1} - x_2 \cdots x_n) \cdots (Q d_n x_n^{d_n - 1} - x_1 \cdots x_{n-1})}.$$

Since $x_i^{d_i} = (x_1 \cdots x_n)/(d_i Q)$ in the Jacobi ring of f, the above residue turns into

$$\frac{1}{d_1 \ldots d_{n-2}} \mathrm{Res} \frac{(x_1 \cdots x_n)^{n-2} dx_1 \cdots dx_n}{(Q d_1 x_1^{d_1 - 1} - x_2 \cdots x_n) \cdots (Q d_n x_n^{d_n - 1} - x_1 \cdots x_{n-1})}.$$

Since the Poincare pairing $(\phi_{e'}, \phi_{e''})$ equals $1/(d_1 \cdots d_{n-2})$, to complete the proof we have to verify that the above residue is -1 when $Q = 0$.

In order to compute the residue, recall that

$$\mathrm{Res} \Big(\frac{df_{x_1}}{f_{x_1}} \wedge \cdots \wedge \frac{df_{x_n}}{f_{x_n}} \Big) = \mathrm{Res} \Big(\frac{\det(\mathrm{Hess}(f))}{f_{x_1} \cdots f_{x_n}} dx_1 \cdots dx_n \Big) = N. \tag{39}$$

On the other hand $df_{x_1} \wedge \cdots \wedge df_{x_n}$ is given by

$$(d_1 x_1^{d_1 - 1} dx_1 - Q^{-1} d(x_2 \cdots x_n)) \wedge \cdots \wedge (d_n x_n^{d_n - 1} dx_1 - Q^{-1} d(x_1 \cdots x_{n-1})).$$

This wedge product can be computed explicitly as follows:

$$\begin{aligned} \sum_{m=0}^{n} \sum_{1 \le i_1 < \cdots < i_m \le n} (1 + m - n)\, Q^{-n+m} \Big(\prod_{s=1}^{m} d_{i_s}(d_{i_s} - 1) x_{i_s}^{d_{i_s} + n - m - 2} \Big) \times \\ \times (x_{j_1} \cdots x_{j_{n-m}})^{n-m-2}\, dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{40}$$

where $\{j_1, \ldots, j_{n-m}\} = \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. In the derivation of the above formula we used the following simple fact: if $g(y_1, \ldots, y_k) = y_1 \cdots y_k$, then

$$\det(\mathrm{Hess}(g)) = (-1)^k (1 - k)\, (y_1 \cdots y_k)^{k-2}.$$

This formula is applied for $k = n - m$ and $(y_1 \cdots y_k) = (x_{j_1}, \ldots, x_{j_{n-m}})$. Note that the term with $m = n - 1$ in (40) vanishes, while the term with $m = n$ reads

$$\prod_{i=1}^{n} d_i (d_i - 1) x_i^{d_i - 2} dx_1 \wedge \cdots \wedge dx_n.$$

The contribution of this term to the residue (39) is analytic at $Q = 0$ and it vanishes at $Q = 0$ of order at least 2. Therefore, we may assume that the summation range for m in (40) is up to $n - 2$. Using the relations in the Jacobi ring of f we get

$$d_i(d_i - 1)x_i^{d_i+n-m-2} = (d_i - 1)x_i^{n-m-2}\,(x_1 \cdots x_n)\,Q^{-1}.$$

The sum (40) is transformed into

$$\sum_{m=0}^{n-2} \sum_{1\le i_1<\cdots<i_m\le n} (1+m-n)\left(\prod_{s=1}^{m}(d_{i_s}-1)\right)\times \\ \times Q^{-n}\,(x_1\cdots x_n)^{n-2}\,dx_1\wedge\cdots\wedge dx_n,$$

Note that the sum on the first line can be computed explicitly. We have the following identity:

$$\sum_{m=0}^{n} \sum_{1\le i_1<\cdots<i_m\le n} (d_{i_1}-1)\cdots(d_{i_m}-1)x^{1-n+m} = x^{1-n}(x(d_1-1)+1)\cdots(x(d_n-1)+1).$$

Differentiating with respect to x and setting $x = 1$ we get

$$\sum_{m=0}^{n} \sum_{1\le i_1<\cdots<i_m\le n} (d_{i_1}-1)\cdots(d_{i_m}-1)(1-n+m) = 0;$$

therefore,

$$\sum_{m=0}^{n-2} \sum_{1\le i_1<\cdots<i_m\le n} (d_{i_1}-1)\cdots(d_{i_m}-1)(1-n+m) = -(d_1-1)\cdots(d_n-1) = -N.$$

Restricting the identity (39) to $Q = 0$ gives

$$\mathrm{Res}\ \frac{Q^{-n}\,(x_1\cdots x_n)^{n-2}}{f_{x_1}\cdots f_{x_n}}\,dx_1\wedge\cdots\wedge dx_n\bigg|_{Q=0} = -1. \quad \square$$

6.5. **Mirror symmetry in genus 0.** We enumerate the elements of the basis $\{p^k\mathbf{1}_c\}$ of $H^*(Y,\mathbb{C})$ in an arbitrary way and denote by ϕ_i the ith element. It is convenient to enumerate in such a way that

$$\phi_a = \mathbf{1}_{e_a/d_a}, \quad 1 \le a \le n,$$

where e_a is the a-th coordinate vector in $\mathbb{Z}^n$. Let $\tau = (\tau_1,\dots,\tau_N)$ be the linear coordinates on $H^*(Y,\mathbb{C})$ corresponding to the basis $\{\phi_i\}$ and put $\partial_i := \partial/\partial\tau_i$ $(1 \le i \le N)$.

6.5.1. *The big quantum cohomology of Y.* Let us fix a constant $Q \in \Delta^*$ and specify a value of $\log Q$, so that the mirror map $\mathbb{C}^n \to H, t \mapsto \tau(t,Q)$ is analytic. In this way $\mathbb{C}^n$ is identified with an analytic subvariety Σ of $H^*(Y,\mathbb{C})$. The linear coordinates $(\tau_1,\dots,\tau_n)$ form a coordinate system on Σ, because on Σ we have

$$\tau_a = t_a \pmod{Q}, \quad 1 \le a \le n.$$

Lemma 6.11. *There are differential operators*

$$P_i(z,t,Q;z\partial_{t_1},\dots,z\partial_{t_n})\in\mathbb{C}\{Q\}[z,t]\langle z\partial_{t_1},\dots,z\partial_{t_n}\rangle,\quad 1\leq i\leq N,$$

such that

$$P_i\,J_Y(\tau,1,z)\quad\in\quad z\,\phi_i+H[\![z^{-1}]\!],$$

where $\tau=\tau(t,Q)$. Moreover, for any choice of such differential operators, we have

$$P_i\,J_Y(\tau,1,-z)=-zS(\tau,1,z)^{-1}\phi_i.$$

Proof. Put

$$J_Y^e(t,Q,z):=(z\partial_t)^eJ_Y(\tau(t,Q),1,z)=zI_Y^e(t,Q,z)/f_0(Q).$$

Using the quantum differential equations, we get

$$J_Y^e(t,Q,z)=z(z\partial_t)^eS(\tau,1,-z)^{-1}1=zS(\tau,1,-z)^{-1}\prod_{a=1}^n(z\partial_{t_a}-M_a)^{e^a}1,$$

where $M_a=\partial_{t_a}\bullet_t$ is the operator of quantum multiplication by ∂_{t_a}. Let us choose a set of indexes $e=(e^1,\dots,e^n)$ such that the cohomology classes $\phi_e:=\widetilde{I}_Y^e(0,0,0)$ form a basis of $H^*(Y,\mathbb{C})$. For example, for a given basis vector $p^k\mathbf{1}_c$ if we define

$$e^1=(k+c_1)d_1,e^2=c_2d_2,\dots e^n=c_nd_n,$$

then $\phi_e=d_1^{-k}p^k\mathbf{1}_c$. In order to prove the existance of the differential operators P_i, it is enough to prove that the determinant of the matrix C whose columns are $\prod_{a=1}^n(z\partial_{t_a}-M_a)^{e^a}1$ is independent of z and t. Under this assumption, the inverse of the matrix C has entries in $\mathbb{C}\{Q\}[z,t]$, so the columns of $zS(\tau,Q,-z)^{-1}$ can be written as linear combinations of $J_Y^e(t,Q,z)$ with coefficients in $\mathbb{C}\{Q\}[z,t]$, which is what we have to prove.

Note that

$$(z\partial_z+E+\deg_{\mathrm{CR}})\widetilde{I}_Y^e(t,Q,z)=\deg(e)\widetilde{I}_Y^e(t,Q,z),$$

where $\deg(e):=\sum e^i/d_i$. The determinant $\Delta_I(t,z)$ of the matrix with columns $\widetilde{I}_Y^e(t,Q,z)$ satisfies the following differential equation

$$(z\partial_z+E)\Delta_I(t,z)=\Big(\sum_e\deg(e)-\mathrm{Tr}(\deg_{\mathrm{CR}})\Big)\Delta_I(t,z)=0.$$

Similarly, the calibration $S(\tau,Q,-z)$ is known to satisfy the differential equation

$$(z\partial_z+E+\deg_{\mathrm{CR}})S(\tau,Q,-z)=S(\tau,Q,-z)\deg_{\mathrm{CR}}.$$

Hence, the determinant also satisfies

$$(z\partial_z+E)\det(S(\tau,Q,-z))=0.$$

We get $(z\partial_z+E)\det(C)=0$. However, the matrix C depends holomorphically on t and z at $t=z=0$, so $\det(C)$ is a constant independent of t and z.

Let us assume that $\widetilde{P}_i$ $(1\leq i\leq N)$ is another set of differential operators such that $\widetilde{P}_i\,J_Y(\tau(t,Q),1,z)\in z\phi_i+H[\![z^{-1}]\!]$. Using that the calibration solves the quantum differential equations we get

$$\widetilde{P}_i\,J_Y(\tau,1,z)z^{-1}=S(\tau,1,-z)^{-1}g(t,Q,z),\quad g\in H[z].$$

The projection of the LHS of the above identity on $H[z]$ is by definition ϕ_i, so on the RHS we must have $g = \phi_i$. $\square$

Note that in the proof of Lemma 6.11, we obtained an explicit algorithm to find differential operators P_i from the I-function. Namely, let us choose a set of N indices e such that the vectors $\widetilde{I}^e_Y(0,0,0)$ give a basis of H. The matrix $A(t,Q,z)$ whose columns are the vectors $\widetilde{I}^e(t,Q,z)/f_0(Q)$ has a Birkhof factorization $A_-(t,Q,z)A_+(t,Q,z)$ with $A_- = 1+O(z^{-1})$. The entries of the i-th column of the matrix $A_+(t,Q,z)^{-1}$ determine the coefficients of a differential operator P_i that has the required properties. In particular, the I-function determines explicitly $S(\tau,1,-z) = A_-(t,Q,z)^{-1}$ for all $\tau \in \Sigma$ and the operators $M_a(t,Q)$, $(1 \le a \le n)$ of quantum multiplication by ∂_{t_a}.

Lemma 6.12. *The big quantum cohomology of Y is uniquely determined by the polynomials P_i, $1 \le i \le N$ and the flatness of the Dubrovin's connection.*

Proof. Let us denote by $\Omega_i(\tau,Q)$ the linear operator of quantum multiplication by $\phi_i \bullet_{\tau,Q}$. Lemma 6.11 implies that

$$\Omega_i(\tau,Q) = P_i(0,t,Q;-M_1,\ldots,-M_n), \quad \tau = \tau(t,Q) \in \Sigma.$$

so the restriction of the multiplication operators to Σ is also uniquely determined. Note that $\Omega_i(0,0)$ generate the orbifold cohomology. In fact, using that the J-function at $Q=0$ is $e^{\tau/z}$ we get that

$$\Omega_1(0,0)^{\nu_1}\cdots\Omega_n(0,0)^{\nu_n} = \Big(\prod_{i=1}^n d_i^{-\ell_i}\Big)\, p^\ell \mathbf{1}_c,$$

where the numbers $c = (c_1,\ldots,c_n)$, $\ell_1,\ldots,\ell_n$, and $\ell := \ell_1+\cdots+\ell_n$ are uniquely defined by

$$\nu_i/d_i = \ell_i + c_i, \quad \ell_i \in \mathbb{Z}, \quad 0 \le c_i < 1.$$

The matrix $A = A(t,Q;\Omega_1,\ldots,\Omega_n)$ with columns

$$P_i(0,t,Q;-\Omega_1,\ldots,-\Omega_n)\,\mathbf{1}, \quad 1 \le i \le N$$

is non-degenerate, because at $t=Q=0$ it reduces to the identity matrix. The quantum multiplication is commutative; therefore, $\Omega_i(\tau,Q)A$ coincides with the matrix $B = B(t,Q;\Omega_1,\ldots,\Omega_n)$ whose columns are given by

$$P_i(0,t,Q;-\Omega_1,\ldots,-\Omega_n)\,\phi_i, \quad 1 \le i \le N.$$

Here A and B are viewed as functions in t,Q and the entries of the matrices $\Omega_1,\ldots,\Omega_n$. It follows that $\Omega_i(\tau,Q) = BA^{-1}$ is a rational function $R_i(\tau,Q;\Omega_1,\ldots,\Omega_n)$ in the entries of Ω_a $(1 \le a \le n)$. Using the flatness of Dubrovin's connection we get

$$\partial_i\Omega_a = \partial_a\Omega_i = \partial_a R_i(\Omega_1,\ldots,\Omega_n),$$

and we get that the restriction of all higher order derivatives in τ of $\Omega_a(\tau,Q)$, $1 \le a \le n$ to Σ are uniquely determined. In particular, we can express the higher order derivatives in τ of $\Omega_a(\tau,Q)$ at $\tau = 0$ in terms of the polynomials P_i, which completes the proof. $\square$

6.5.2. *The mirror isomorphism.* Let us fix $Q \in \Delta^*$ and define

$$f := f(x, 0, Q) = \sum_{i=1}^{n} x_i^{d_i} - \frac{1}{Q} x_1 \cdots x_n.$$

Let us embed $\mathbb{C}^N \subset B_f$ via $t \mapsto f(x, t, Q)$. Put

$$\omega_i = \sqrt{-1}\ \mathrm{Mir}^{-1}(\phi_i) \quad \in \quad \mathcal{F}|_{\mathbb{C}^n \times \{Q\}},$$

where the scalar $\sqrt{-1}$ is chosen, so that the Poincare pairing matches the residue pairing $K^{(0)}$ (see Proposition 6.6). According to Proposition 6.6, the forms ω_i form a good basis, i.e., $K(\omega_i, \omega_j) \in \mathbb{C}$. Let us assume that $\phi_1 = 1$ and define $\omega := \omega_1$ to be the primitive form. The good basis $\{\omega_i\}_{i=1}^N$ extends uniquely to a good basis over the space B_f of miniversal deformations of f and it determines a flat coordinate system $\tau = (\tau_1, \ldots, \tau_N)$ on B_f such that

$$\sum_{i=1}^{N} \tau_i(t)\phi_i = \tau(t, Q), \quad \forall t \in \mathbb{C}^n.$$

We can use the map Mir to obtain a reconstruction of the Frobenius structure on B_f similar to the reconstruction of the big quantum cohomology given by Lemmas 6.11 and 6.12. Namely, using Proposition 6.5, it is easy to verify that the statements of both lemmas remain the same if we replace $J_Y(\tau, 1, z)$ with the primitive form ω and Dubrovin's connection with the Gauss–Manin connection. Note that since Mir is a $\mathcal{D}$-module isomorphism, we can use the same set of differential operators P_i for both reconstructions. Therefore, we can uniquely extend the mirror map $\mathbb{C}^n \to H$, $t \mapsto \tau(t, Q)$ to an isomorphism of Frobenius structures, i.e., we have the following proposition.

Proposition 6.13. *The map*

$$B_f \to H := H^*_{\mathrm{CR}}(Y; \mathbb{C}), \quad \tau = (\tau_1, \ldots, \tau_N) \mapsto \sum_{i=1}^{N} \tau_i \phi_i \tag{41}$$

induces an isomorphism of the germ of the Frobenius structure of B_f at $\tau = 0$ and the quantum cohomology of Y .

Remark 6.14. The map Mir is defined in terms of an extended I-function of Y depending on the relevant deformation parameters $t_1, \ldots, t_n$. Using the results of [22] we can extend the I-function even further to include all deformation parameters. This would give us an appropriate extension of the map Mir, which will provide us directly with a trivialization of $TB_f[\![z]\!] \cong B \times H[\![z]\!]$ that intertwines the Gauss–Manin connection and the Dubrovin connection. The advantage of using a reconstruction argument is that, after analyzing the reconstruction scheme more carefully, we can prove the convergence of the Frobenius multiplication on TB_f in the irrelevant direction.

6.6. **Mirror symmetry in higher genus.** Proposition 6.13 implies that the quantum cohomology of Y is semi-simple. Therefore, we can recall Givental's higher genus reconstruction [31, 32] proved by Teleman [72]. Let $\tau = \tau(0, Q) \in H$ be the value of the mirror map 31 at $t=0$, put $f := f(x,0,Q)$, and recall the good basis

$$\omega_i = \sqrt{-1}\ \mathrm{Mir}^{-1}(\phi_i)|_{t=0} \quad \in \quad \mathbb{H}_+(f), \quad 1 \le i \le N,$$

where $\{\phi_i\}_{i=1}^N \subset H$ is a fixed basis with $\phi_1 = \mathbf{1}$. The higher genus mirror symmetry for Y can be stated as follows.

Theorem 6.15. *The total ancestor potentials of Y and f are related by the following formula:*

$$\mathcal{A}^Y_\tau(\hbar;\mathbf{q}) = \mathcal{A}^{\omega_1,\ldots,\omega_N}_{f,\omega}(\hbar;\mathbf{q}),$$

where

$$\omega := \omega_1 = \sqrt{-1}\,\frac{dx_1\cdots dx_n}{Qf_0(Q)}.$$

6.7. **A-model opposite subspace.** Let $f = f(x,0,Q)$, $Q \in \Delta^*$. Recall also the notation $\mathfrak{h}$ and $\mathfrak{h}^*$ for respectively the middle cohomology and homology of f. The opposite subspace $P \subset \mathbb{H}_+(f)$ that corresponds to the good basis of GW theory can be characterized as follows. The group μ_{d_i} of d_ith roots of 1 acts naturally on $\mathbb{C}^n \times \Delta^*$ via

$$\eta.((x_1,\ldots,x_n),Q) := ((x_1,\ldots,\eta x_i,\ldots,x_n),\eta Q).$$

The function f is invariant under this action, so the vanishing homology and cohomology bundles on Δ^* become naturally μ_{d_i}-equivariant bundles. Let us a define a linear map

$$L_i : \mathfrak{h} \to \mathfrak{h}, \langle L_i(A),\alpha\rangle = \langle A, \eta_i^{-1}\cdot \alpha_{\eta_i Q}\rangle,$$

where $\eta_i = e^{2\pi\sqrt{-1}/d_i}$, $\eta_i^{-1}\cdot$ is the μ_{d_i}-equivariant action, and $\alpha_{\eta_i Q}$ is the parallel trasnport of the cycle α along the arc $Qe^{\sqrt{-1}\theta}$, $0 \le \theta \le 2\pi/d_i$. Note that $L_i^{d_i} = M_{\mathrm{mar}}^{-1}$, where M_{mar} is the monodromy transformation of $\mathfrak{h}$ corresponding to a closed loop around $Q=0$ in counter clockwise direction. Recall that $M_{\mathrm{mar}} = e^{-2\pi\sqrt{-1}N_{\mathrm{mar}}}$ (see Lemma 6.8), where N_{mar} is a nilpotent operator. In particular, we can define $M_{\mathrm{mar}}^{1/d_i} := e^{-(2\pi\sqrt{-1}/d_i)N_{\mathrm{mar}}}$. Note that the linear operators L_i and M_{mar} pairwise commute for $1 \le i \le n$. Therefore, the map $\eta_i \mapsto L_i \circ M_{\mathrm{mar}}^{1/d_i}$ gives a representation of μ_{d_i} on $\mathfrak{h}$ for each $i = 1,2,\ldots,n$. Since the operators defininig the representations pairwise commute we have a joint spectrum decomposition

$$\mathfrak{h} = \bigoplus_{e=(e_1,\ldots,e_n)} \mathfrak{h}_e, \quad \mathfrak{h}_e = \{v \in \mathfrak{h}_e \ :\ L_i v = \eta_i^{-e_i}v,\ 1 \le i \le n\}. \tag{42}$$

where the direct sum is over $e = (e_1,\ldots,e_n)$ such that $0 \le e_i \le d_i - 1$ and $\mathfrak{h}_e \ne \{0\}$.

Let us recall the definition of a *weight filtration* (see [67], Lemma 6.4). Given a triple (V,m,N) consisting of a vector space V, a positive integer m (called *weight*), and a nilpotent operator N such that $N^m = 0$, there is a unique increasing filtration $0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = V$, called a *weight filtration*, such that $N(W_\ell) \subset W_{\ell-2}$ and $N^l : \mathrm{Gr}^W_{m+\ell} \to \mathrm{Gr}^W_{m-\ell}$ is an isomorphism for all ℓ.

Put

$$N_e = N_{\mathrm{mar}}|_{\mathfrak{h}_e}, \quad m_e := n + |\{i \ :\ e_i \ne 0\}| - 2\lceil \iota(e)\rceil,$$

where $|S|$ denotes the number of elements in a set S and $\iota(e) := \sum_{i=1}^n e_i/d_i$. Let us define $W^e_\bullet$ to be the weight filtration corresponding to the tripple $(\mathfrak{h}_e, m_e, N_e)$.

Proposition 6.16. *The opposite filtration $\{U_\bullet\}$ of $\mathfrak{h}$, which corresponds to the GW opposite subspace P via* (10) *is given by the formula*

$$U_\ell = \bigoplus_e W^e_{2\ell},$$

where the direct sum over e is the same as in (42).

Proof. Our argument is based on mirror symmetry. Recall the isomorphism (35)

$$\Psi : \mathfrak{h} \to H.$$

We will find the images of the opposite filtration and the various weight filtrations under Ψ and see that the desired relation is obvious.

By definition the one-to-one correspondence, (10) associates to every $A \in F^\ell\mathfrak{h}_s \cap U_\ell\mathfrak{h}_s$ with $s = e^{2\pi\sqrt{-1}\alpha}$, $0 \le \alpha < 1$, a homogeneous form $\omega \in \mathbb{H}_+(f) \cap Pz$ of degree $\deg(\omega) = n - \ell - \alpha$, such that

$$\widehat{s}(\omega, z) = (-z)^{-\ell-\alpha+\frac{n}{2}}\, A,$$

where we used that for every fixed z the map ψ in (10) is the inverse to $\widehat{s}(\ ,z)$. By definition

$$\Psi(\widehat{s}(\omega, z)) = (-z)^{-\theta} S(\tau, 1, -z)^{-1}\ \mathrm{Mir}(\omega),$$

where $\tau = \tau(0, Q)$ is the image of the mirror map. On the other hand, due to homogeneity

$$(-z)^{-\theta} S(\tau, 1, -z)^{-1}(-z)^{\theta}$$

is independent of z and

$$(-z)^{-\theta}\,\mathrm{Mir}(\omega) = (-z)^{\deg(\omega)-\frac{n}{2}}\,\mathrm{Mir}(\omega).$$

We get

$$\Psi(A) = S(\tau, 1, 1)^{-1}\ \ \mathrm{Mir}(\omega).$$

Therefore, $\Psi(F^\ell\mathfrak{h}_s \cap U_\ell\mathfrak{h}_s)$ is the span of all $S(\tau, 1, 1)^{-1}\phi \in H$ such that ϕ is a homogeneous class satisfying

$$\lceil \deg_{\mathrm{CR}}(\phi) \rceil = n - 1 - \ell, \quad \deg_{\mathrm{CR}}(\phi) + \alpha \in \mathbb{Z}.$$

Recall the homogeneity condition for $S(\tau, Q, z)$:

$$(z\partial_z + E) S(\tau, Q, z) = [\theta, S(\tau, Q, z)].$$

It follows that $S(\tau, 1, 1) = 1 + \sum_{k=1}^\infty S_k(\tau, 1)$, where each operator $S_k(\tau, 1)$ increases the degree by k. Since

$$U_\ell = \bigoplus_{\ell' \ge \ell} \bigoplus_s F^{\ell'}\mathfrak{h}_s \cap U_{\ell'}\mathfrak{h}_s,$$

we get that $\Psi(U_\ell)$ is spanned by homogeneous classes ϕ in H such that

$$\lceil \deg_{\mathrm{CR}}(\phi) \rceil \ge n - 1 - \ell. \tag{43}$$

Let us determine the images of the weight filtrations. We already proved that $\Psi \circ N_{\mathrm{mar}} = p \cup \Psi$ (see Lemma 6.8). Using the identity

$$L_i \widehat{s}(\omega_e, z) = \eta_i^{-e_i} \widehat{s}(\omega_e, z)|_{Q \mapsto \eta_i Q},$$

where $\omega_e = x^e dx/Q$ and the definition of Ψ (see (35)), it is easy to check that $\Psi \circ (L_i M_{\mathrm{mar}}^{1/d_i}) = J_i \circ \Psi$, where $J_i : H \to H$ is the linear operator defined by

$$J_i(p^k \mathbf{1}_c) = e^{-2\pi\sqrt{-1}c_i}\, p^k \mathbf{1}_c.$$

Therefore, we have $\Psi(\mathfrak{h}_e) = H(Y_e;\mathbb{C})$, where Y_e is the twisted sector corresponding to $c = (c_1, \dots, c_n)$, with $c_i = e_i/d_i$. The weight filtration of the triple $(H(Y_e;\mathbb{C}), m_e, p\cup)$ is straightforward to find. We get that $\Psi(W^e_{2\ell})$ is spanned by homogeneous classes $\phi \in H$, such that

$$\deg(\phi) \geq \frac{1}{2}\Big(\dim(Y_e) + m_e\Big) - \ell,$$

where if ϕ is a class of (usual) real degree $2i$, then $\deg(\phi) = i$. Note that

$$\deg_{\mathrm{CR}}(\phi) = \deg(\phi) + \iota(e). \tag{44}$$

Comparing the inequalities (43) and (44) we see that they are equivalent when $m_e = n + |\{i \ : \ e_i \neq 0\}| - 2\lceil \iota(e) \rceil$. ☐

7. Mirror symmetry for Fermat CY singularities

Now we discuss the mirror symmetry on the LG side. In [37], He-Li-Shen-Webb identified the FJRW ancestor potential (LG A-model) of invertible quasi-homogeneous polynomial singularities to the Saito-Givental ancestor potential (LG B-model) of the mirror polynomials, by using Givental-Teleman's [31, 72] unique higher genus formula for semisimple Frobenius manifolds and matching Frobenius manifolds on both sides via WDVV equations and a perturbative formula in [47]. In this section, we establish a mirror symmetry statement of $\mathcal{D}$-module structures and opposite subspaces between FJRW theory and Saito's theory for Fermat CY singularities. For Fermat CY singularities, our result recovers He-Li-Shen-Webb's result. More general cases remain unknown due to the lack of a toric model.

7.1. **FJRW theory of Fermat CY singularities.** As before, we consider the Landau-Ginzburg side of the Fermat polynomial of Calabi-Yau type

$$W = x_1^{d_1} + \cdots + x_n^{d_n}, \quad \sum_{i=1}^n \frac{1}{d_i} = 1.$$

Let G_W be the group of diagonal symmetries of W, so

$$G_W := \Big\{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n \Big| W(\lambda_1 x_1, \dots, \lambda_n x_n) = W(x_1, \dots, x_n)\Big\} \cong \prod_{i=1}^n \boldsymbol{\mu}_{d_i}.$$

For each $\gamma \in G_W$, there exist unique $\{\Theta_\gamma^{(i)} \in [0,1) \cap \mathbb{Q}\}$, such that

$$\gamma = \Big(\exp(2\pi\sqrt{-1}\Theta_\gamma^{(1)}), \cdots, \exp(2\pi\sqrt{-1}\Theta_\gamma^{(n)})\Big).$$

A mathematical construction of the LG model for a generic pair (W, G_W) is given by Fan, Jarvis, and Ruan [28, 29], based on a proposal of Witten [75]. More generally, the group G_W can be replaced by any subgroup that contains the exponential grading element

$$j_W := \left(\exp(\frac{2\pi\sqrt{-1}}{d_1}), \cdots, \exp(\frac{2\pi\sqrt{-1}}{d_n})\right). \tag{45}$$

In this paper, we only focus on the pair

$$\left(W = x_1^{d_1} + \cdots + x_n^{d_n}, \quad G_W\right).$$

Its FJRW theory consists of a graded vector space H_W (called FJRW state space), and a Cohomological Field Theory $\{\Lambda_{g,k}^W\}$. We recall some basics in this section and refer the readers as to [28] for more details.

Each $\gamma \in G_W$ acts on $\mathbb{C}^n$ by homothesis and we denote $\mathrm{Fix}(\gamma) \subset \mathbb{C}^n$ the fix locus of γ. Let W_γ be the restriction of W on $\mathrm{Fix}(\gamma)$. Each γ-twisted sector H_γ consists of G_W-invariant part of the middle-dimensional relative cohomology for W_γ.

$$H_\gamma := \left(H^*(\mathrm{Fix}(\gamma), (\mathrm{Re}\, W_\gamma)^{-1}(-\infty, -M); \mathbb{C})\right)^{G_W}, \quad M \gg 0.$$

Here H_γ is called *narrow* if $\mathrm{Fix}(\gamma) = \mathbf{0} \in \mathbb{C}^n$ and is called *broad* otherwise. Each narrow sector is canonically isomorphic to $\mathbb{C}$,

$$H_\gamma := H^*(\{\mathbf{0}\}, \emptyset; \mathbb{C}) \cong \mathbb{C}.$$

We denote $\mathbf{1}_\gamma$ the canonical generator in $1 \in \mathbb{C} \cong H_\gamma$.

In particular, since W is Fermat CY singularity, the FJRW state space is given by

$$H_W = \bigoplus_{\gamma \in \mathscr{N}} H_\gamma \cong \bigoplus_{\gamma \in \mathscr{N}} \mathbb{C} \cdot \mathbf{1}_\gamma,$$

where γ belongs to the set of narrow elements

$$\mathscr{N} := \left\{\gamma \in G_W \middle| 1 \leq d_j \Theta_\gamma^{(j)} \leq d_j - 1, \forall 1 \leq j \leq n\right\}. \tag{46}$$

The cardinality of $\mathscr{N}$ is $N := \prod_{j=1}^n (d_j - 1)$, hence H_W is a vector space of rank N. Moreover, $(H_W, \deg_W)$ is a graded vector space, where

$$\deg_W \mathbf{1}_\gamma := \sum_{j=1}^n \left(\Theta_\gamma^{(j)} - \frac{1}{d_j}\right). \tag{47}$$

For each $\gamma \in \mathscr{N}$, we define its involution $\gamma' \in \mathscr{N}$ by

$$\Theta_{\gamma'}^{(j)} := 1 - \Theta_\gamma^{(j)}.$$

Let $\delta_{(-)}^{(-)}$ be the Kronecker symbol. Then H_W has a non-degenerate pairing η_W, given by

$$\eta_W(\mathbf{1}_\alpha, \mathbf{1}_\beta) = \delta_\alpha^{\beta'}, \quad \forall \alpha, \beta \in \mathscr{N}. \tag{48}$$

The triple $(H_W, \deg_W, \eta_W)$ has a Cohomological Field Theory $\{\Lambda_{g,k}^W\}$, which consists of multilinear maps

$$\Lambda_{g,k}^W : H_W^{\otimes k} \to H^*(\overline{\mathcal{M}}_{g,k}, \mathbb{C}).$$

Here $\overline{\mathcal{M}}_{g,k}$ is the moduli space of stable k-pointed curves of genus g. Letting $\gamma_j \in \mathscr{N}$, and $\bar\psi_j \in H^*(\overline{\mathcal{M}}_{g,k}, \mathbb{C})$ be the j-th $\bar\psi$-class, we have the following *FJRW invariant*

$$\langle \mathbf{1}_{\gamma_1}\bar\psi_1^{\ell_1}, \cdots, \mathbf{1}_{\gamma_k}\bar\psi_k^{\ell_k}\rangle_{g,k}^W := \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_{g,k}^W(\mathbf{1}_{\gamma_1}, \cdots, \mathbf{1}_{\gamma_k}) \prod_{j=1}^k \bar\psi_j^{\ell_j}. \tag{49}$$

Similarly as in GW theory, for any $\tau \in H_W$, we can define a formal function

$$\left\langle\!\!\left\langle \mathbf{1}_{\gamma_1}\bar\psi_1^{\ell_1}, \cdots, \mathbf{1}_{\gamma_k}\bar\psi_k^{\ell_k} \right\rangle\!\!\right\rangle_{g,k}^W (\tau) := \sum_{m\geq 0} \frac{1}{m!} \langle \mathbf{1}_{\gamma_1}\bar\psi_1^{\ell_1}, \cdots, \mathbf{1}_{\gamma_k}\bar\psi_k^{\ell_k}, \tau, \cdots, \tau\rangle_{g,k+m}^W.$$

The quantum multiplication $\bullet_\tau$ is given by

$$\eta_W\left(\mathbf{1}_\alpha \bullet_\tau \mathbf{1}_\beta, \mathbf{1}_\gamma\right) = \left\langle\!\!\left\langle \mathbf{1}_\alpha, \mathbf{1}_\beta, \mathbf{1}_\gamma \right\rangle\!\!\right\rangle_{0,3}^W(\tau).$$

The product $\bullet_\tau$ has an identity $\mathbf{1} := \mathbf{1}_{j_W}$ with j_W defined in (45).

Again we introduce a set of formal variables $\mathbf{t} = \{t_{k,i}\}$, $1 \leq i \leq N$, $k \geq 0$. We introduce a genus-g generating function

$$\overline{\mathcal{F}}_{\tau,W}^{(g)}(\mathbf{t}) = \sum \frac{1}{k!} \left\langle\!\!\left\langle \mathbf{t}(\bar\psi), \ldots, \mathbf{t}(\bar\psi) \right\rangle\!\!\right\rangle_{g,k}^W (\tau)$$

and the *total ancestor potential*

$$\mathcal{A}_\tau^W(\hbar; \mathbf{t}) := \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1}\, \overline{\mathcal{F}}_{\tau,W}^{(g)}(\mathbf{t})\right).$$

7.1.1. *J-function.* Let $\mathcal{H}_W := H_W(\!(z^{-1})\!)$ be the infinite vector space. Let us consider the Darboux coordinate s

$$\mathbf{p}(z) = \sum_{k\geq 0}\sum_\alpha p_k^\alpha \mathbf{1}_\alpha z^{-k-1}, \quad \mathbf{q}(z) = \sum_{k\geq 0}\sum_\alpha q_k^\alpha \mathbf{1}_\alpha z^k.$$

We may write an element in $\mathcal{H}_W$ as

$$f(z) = \sum_{k\geq 0} q_k^\alpha \mathbf{1}_\alpha z^k + \sum_{k<0} p_k^\alpha \mathbf{1}_\alpha z^k.$$

The infinite dimensional vector space $\mathcal{H}_W$ is equipped with a symplectic pairing

$$\Omega_W\Big(f(z), g(z)\Big) = \mathrm{Res}_{z=0}\, \eta_W(f(-z), g(z))dz.$$

We have $\mathcal{H}_W = \mathcal{H}_W^+ \oplus \mathcal{H}_W^-$ where $\mathcal{H}_W^+ = H_W[\![z]\!]$ and $\mathcal{H}_W^- := z^{-1}H_W[z^{-1}]$ are Lagrangian subspaces. Since $\mathcal{H} \cong T\mathcal{H}_+$, after the *dilation shift* $\mathbf{q}(z) = -z + t(z)$, the graph of the genus zero generating function $\overline{\mathcal{F}}_W^{(0)}$ defines a formal germ of Lagrangian submanifold $\mathcal{L}_W$ in $\mathcal{H}_W$,

$$\mathcal{L}_W := \left\{(\mathbf{p}, \mathbf{q}) \in T\mathcal{H}_W^+ : \mathbf{p} = d_{\mathbf{q}}\overline{\mathcal{F}}_W^{(0)}\right\}.$$

We define the FJRW J-function

$$J_{\mathrm{FJRW}}(\tau, z) := -z + \mathbf{t} + \sum_{m=0}^\infty \sum_{k=0}^\infty \sum_\gamma \frac{1}{m!} \langle \mathbf{1}_\gamma \bar\psi^k, \tau, \cdots, \tau\rangle_{0,m+1}^W\, \mathbf{1}_{\gamma'}\,(-z)^{-k-1}. \tag{50}$$

It is standard to check that

Lemma 7.1. *The J-function $J_{\mathrm{FJRW}}(\tau,z)$ belongs to $\mathcal{L}_W$.*

Similar to GW theory, we define the calibration operator $S_W(\tau,z)$ in FJRW theory by

$$\eta_W(S_W(\tau,z)\mathbf{1}_\alpha,\mathbf{1}_\beta)=\eta_W(\mathbf{1}_\alpha,\mathbf{1}_\beta)+\sum_{k=0}^{\infty}\left\langle\!\!\left\langle \mathbf{1}_\alpha\bar{\psi}^k,\mathbf{1}_\beta\right\rangle\!\!\right\rangle_{0,2}^{W}(\tau)\,z^{-1-k},$$

where the unstable terms vanishes. We can also rewrite the J-function as follows:

$$J_{\mathrm{FJRW}}(\tau,z)=-z\,S_W(\tau,z)^{-1}\mathbf{1}. \tag{51}$$

7.2. **I-function in FJRW theory.** Now we introduce an FJRW I-function $I^0_{\mathrm{LG}}(t,z)$ in (60) via toric geometry. This I-function lies on the FJRW Langrangian cone, and Birkhoff factorization of the I-function will induce a mirror map (64).

7.2.1. *Toric setup and an* FJRW *I-function.* Let $\{b_0,b_1,\cdots,b_n\}$ be vectors in $\mathbb{Z}^n$ such that

$$b_0=(1,\cdots,1),\quad b_i^{(j)}=\delta_i^j\,d_i,\quad \forall j=1,\cdots,n.$$

Let S is be a set of vectors in $\mathbb{Z}^n$,

$$S=R\coprod\mathfrak{B},\quad R=\{b_1,\cdots,b_n\}$$

and $\mathfrak{B}$ is a set of ghost variables

$$\mathfrak{B}:=\left\{b=(b^{(1)},\cdots,b^{(n)})\in\mathbb{Z}^n\middle|0\le b^{(j)}\le d_j-2,\forall j=1,\cdots,n\right\}. \tag{52}$$

In LG side, we have an exact *fan sequence*

$$0\to\mathbb{L}\to\mathbb{Z}^S\xrightarrow{\varphi}\mathbb{Z}^n. \tag{53}$$

For each $b\in S$, the map φ in (53) is defined by

$$\varphi(b)=b\in\mathbb{Z}^n.$$

Let $\Psi_{\mathrm{LG}}:\sigma\cap\mathbb{Z}^n\to\mathbb{Q}^S$, where $\Psi_{\mathrm{LG}}(\mathbf{e})=\{\Psi^b_{\mathrm{LG}}(\mathbf{e})\}$ and the coefficient of $b\in S$ is given by

$$\Psi^b_{\mathrm{LG}}(\mathbf{e})=\begin{cases}\mathbf{e}^{(j)}/d_j, & b=b_j\in R;\\ 0, & b\in\mathfrak{B}.\end{cases}$$

Let $\nu(\mathbf{e})=\sum_{b\in S}\nu_b(\mathbf{e})\,b\in\mathbb{Q}^S$ be defined by

$$\nu(\mathbf{e}):=-\Psi_{\mathrm{LG}}(\mathbf{e}+b_0)+\sum_{b\in\mathfrak{B}}\nu_b(\mathbf{e})\,\xi_b, \tag{54}$$

where for each $b\in\mathfrak{B}$, $\nu_b(\mathbf{e})\in\mathbb{Z}_{\ge0}$ and

$$\xi_b:=b-\sum_{c\in R}\Psi^c_{\mathrm{LG}}(b)c\in\mathbb{L}_{\mathbb{Q}}:=\mathbb{L}\otimes_{\mathbb{Z}}\mathbb{Q}.$$

Thus for each $j=1,\cdots,n$, we have

$$\nu_j(\mathbf{e})=-\frac{1}{d_j}\left(\mathbf{e}^{(j)}+1+\sum_{b\in\mathfrak{B}}\nu_b(\mathbf{e})\,b^{(j)}\right)\in\mathbb{Q}_{<0}. \tag{55}$$

For each $\nu\in\mathbb{Q}^S$, we can assign an element $\gamma_\nu\in G_W$

$$\gamma_\nu=\left(\exp(2\pi\sqrt{-1}\langle-\nu_1\rangle),\cdots,\exp(2\pi\sqrt{-1}\langle-\nu_n\rangle)\right)\in G_W. \tag{56}$$

Let $t^{(-)} : \mathbb{L} \to \mathbb{C}$ be a formal function given by

$$t^{\Psi_{\mathrm{LG}}(\mathbf{e}+b_0)+\nu} := \prod_{b\in\mathfrak{B}} t_b^{\nu_b}.$$

Recalling the definition of ν in (54), we define the box element $\square_\nu$ to be

$$\square_\nu := \frac{\prod_{j=1}^n \prod_{k=1}^{\lfloor -\nu_j \rfloor} (\nu_j + k)\, z}{\prod_{b\in\mathfrak{B}} \prod_{k=1}^{\nu_b} (kz)}. \tag{57}$$

If there exists $j \in \{1, \cdots, n\}$ such that $-\nu_j \in \mathbb{Z}$, then $\square_\nu = 0$. Thus for $\square_\nu \neq 0$, we know $\gamma_\nu \in \mathscr{N}$, and it makes sense to introduce

$$I_{\mathrm{LG}}^{\mathbf{e}}(t,z) = \sum_{\nu\in\mathbb{Q}^S} \left(\prod_{b\in\mathfrak{B}} t_b^{\nu_b}\right) \square_\nu \, \mathbf{1}_{\gamma_\nu} \in H_W[\![t]\!]((z)). \tag{58}$$

Here $H_W[\![t]\!] := H_W \otimes_{\mathbb{C}} \mathbb{C}[\![t_b; b \in \mathfrak{B}]\!]$. In particular, when $t = 0$, i.e., $\nu_b = 0$ for all $b \in \mathfrak{B}$, then according to Equation (55),

$$\nu = \sum_{j=1}^n \left(-\frac{\mathbf{e}^{(j)}+1}{d_j}\right) b_j.$$

A direct calculation shows:

$$I_{\mathrm{LG}}^{\mathbf{e}}(t=0,z) = \mathbf{1}_{\gamma_\nu}. \tag{59}$$

Taking $\mathbf{e} = 0$, we get the I-function in the LG side:

$$I_{\mathrm{LG}}^{0}(t,z) = \sum_{\nu} \left(\prod_{b\in\mathfrak{B}} t_b^{\nu_b}\right) \square_\nu \, \mathbf{1}_{\gamma_\nu} \quad \in H_W[\![t]\!]((z)). \tag{60}$$

Now we assign the following degree:

$$\deg_W t_b = 1 - \sum_{j=1}^n \frac{b^{(j)}}{d_j}, \quad \deg_W z = 1.$$

When we apply (55) and (47), we see that each term in $I_{\mathrm{LG}}^{\mathbf{e}}(t,z)$ has degree

$$\begin{aligned}\deg_W \left(\prod_{b\in\mathfrak{B}} t_b^{\nu_b} \square_\nu \, \mathbf{1}_{\gamma_\nu}\right) &= \sum_{b\in\mathfrak{B}} \nu_b \left(1 - \sum_{j=1}^n \frac{b^{(j)}}{d_j}\right) + \sum_{j=1}^n \lfloor -\nu_j \rfloor - \sum_{b\in\mathfrak{B}} \nu_b + \sum_{j=1}^n \left(\langle -\nu_j \rangle - \frac{1}{d_j}\right) \\ &= \sum_{j=1}^n \frac{\mathbf{e}^{(j)}}{d_j}.\end{aligned}$$

This depends on $\mathbf{e}$ only, so we know $I_{\mathrm{LG}}^{0}(t,z)$ is homogeneous of degree zero; i.e.,

$$\deg_W \left(I_{\mathrm{LG}}^{0}(t,z)\right) = 0. \tag{61}$$

Definition 7.2. Following [22], we say $f(t,-z)$ is an $\mathcal{H}_W[\![t]\!]$-point in the Lagrangian cone $\mathcal{L}_W$ if

$$f(t,-z) = -z\mathbf{1} + t(-z) + \sum_{\gamma} \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{-z-\psi} \right\rangle\!\!\right\rangle_{0,1}^{W} (t)\; \mathbf{1}_{\gamma'}$$

for some $t(z) \in \mathcal{H}_+[\![y]\!]$ such that $t(z)|_{y=0} = 0$.

The following result is known to experts.

Proposition 7.3. *The formal function $I^0_{\mathrm{LG}}(t,-z)$ is an $\mathcal{H}_W[\![t]\!]$-point in the Lagrangian cone $\mathcal{L}_W$.*

For the reader's convenience, we give a proof in Appendix A following the method of Ross and Ruan [60]. It uses a localization computation in GLSM theory [30].

7.2.2. *Convergence.* Let $e_i \in \mathbb{Z}^n$ be the i-th standard vector; i.e. $e_i^{(j)} = \delta_i^j$, $i = 1, \cdots, n$. Let t_i be the parameter of e_i. Let σ be the parameter of $b_0 = (1, \cdots, 1) \in \mathbb{Z}^n$. We restrict $I^0_{\mathrm{LG}}(t,z)$ to the following subspace of $\mathbb{C}^{\mathfrak{B}} \times \mathbb{C} \ni (\{t_b\}_{b\in\mathfrak{B}}, z)$:

$$\mathbb{C}^{n+1} \times \mathbb{C} \ni (t_1, \cdots, t_n, \sigma, z).$$

We denote the restriction by $I^0_{\mathrm{LG}}(t_1, \cdots, t_n, \sigma, z)$. From to (61), we know

$$\deg_W \left(I^0_{\mathrm{LG}}(t_1, \cdots, t_n, \sigma, z)\right) = 0.$$

On the other hand, since

$$\deg_W t_i = 1 - \frac{1}{d_i} > 0, \quad \deg_W(\sigma) = 0, \quad \deg_W z = 1, \quad 0 \leq \deg_W \mathbf{1}_\gamma \leq n-2,$$

we can rewrite the function $I^0_{\mathrm{LG}}(t_1, \cdots, t_n, \sigma, z)$ as

$$I^0_{\mathrm{LG}}(t_1, \cdots, t_n, \sigma, z) = \sum_{k=0} I^W_k(t_1, \cdots, t_n, \sigma) z^{-k} \in H_W[\![t_1, \cdots, t_n, \sigma]\!][\![z^{-1}]\!].$$

Hence $I^W_0(t_1, \cdots, t_n, \sigma)$ is homogeneous of degree zero. If $d = l.c.m(d_1, \cdots, d_n)$, then

$$I^W_0(t_1, \cdots, t_n, \sigma) = f^W_0(\sigma)\, \mathbf{1} := \mathbf{1} + \sum_{m\geq 1} \frac{\sigma^{md}}{(md)!} \prod_{j=1}^{n} \prod_{k=1}^{md/d_j} \left(k - \frac{md+1}{d_j}\right). \tag{62}$$

The ratio test shows that $f^W_0(\sigma)$ is analytic near $\sigma = 0$. Further more, we have the following convergence result.

Corollary 7.4. *For each $k \geq 0$, $I^W_k(t_1, \cdots, t_n, \sigma) \in H_W[t_1, \cdots, t_n]\{\sigma\}$.*

Proof. The polynomiality of $t_1, \cdots, t_n$ follows from degree counting and $\deg_W t_i > 0$ for each $i = 1, \cdots, n$. For any fixed homogeneous element in $H_W[t_1, \cdots, t_n]$, we can use the ratio test to obtain the analyticity of σ near $\sigma = 0$. $\square$

More generally, recall that $W = x_1^{d_1} + \cdots + x_n^{d_n}$ is an element in $\mathcal{M}$ in (1). We define

$$\mathfrak{B}^{\mathrm{rel}} = \{b \in \mathfrak{B} |\, \deg_W t_b > 0\}, \quad \mathfrak{B}^{\mathrm{mar}} = \{b \in \mathfrak{B} |\, \deg_W t_b = 0\}.$$

We may consider a neighborhood of $W \in \mathcal{M}$ consisting of

$$W + \sum_{b\in \mathfrak{B}^{\mathrm{rel}} \cap \mathfrak{B}^{\mathrm{mar}}} t_b x^b, \quad |t_b| < \delta \quad \textit{if} \quad b \in \mathfrak{B}^{\mathrm{mar}}.$$

We denote this neighborhood by $\mathcal{M}_{\mathrm{LG},\delta}$,

$$\mathcal{M}_{\mathrm{LG},\delta} \cong \mathbb{C}^{\mathfrak{B}^{\mathrm{rel}}} \times \Delta_\delta^{\mathfrak{B}^{\mathrm{mar}}}.$$

If δ is sufficiently small, then a discussion similar to that in Corollary (7.4) shows

$$I^0_{\mathrm{LG}}(t,z)|_{\mathcal{M}_{\mathrm{LG},\delta}} \in H_W \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}_{\mathrm{LG},\delta}}[\![z^{-1}]\!].$$

7.2.3. *Mirror map.* The Birkhoff factorization allows us to rewrite $I^0_{\mathrm{LG}}(t,z)$ as

$$I^0_{\mathrm{LG}}(t,z) = L_-(t,z)\Upsilon_{\mathrm{LG}}(t,z), \tag{63}$$

with

$$\Upsilon_{\mathrm{LG}}(t,z) \in \mathcal{H}_+[\![t]\!] \quad \textit{and} \quad L_- := 1 + \sum_{k\geq 1} L_k(t)\, z^{-k} \in \mathrm{End}(H_W)[\![z^{-1}]\!].$$

A mirror map $\tau : \mathbb{C}^{\mathfrak{B}} \to H_W[\![t]\!]$ is given by

$$\tau(t) := L_1(t)(\mathbf{1}) \in H_W[\![t]\!]. \tag{64}$$

Combining Lemma 7.1 and Proposition 7.3, we have the equality

$$J_{\mathrm{FJRW}}(\tau, z) = -zL_-(t,-z)\mathbf{1}.$$

Here we identify the calibration operator $S_W(\tau(t),z)^{-1}$ with the operator $L_-(t,z)$ via the mirror map (64).

The restriction of $I^0_{\mathrm{LG}}(t,z)$ to $\mathcal{M}_{\mathrm{LG},\delta}$ will imply

$$\Upsilon_{\mathrm{LG}}(t,z)|_{\mathcal{M}_{\mathrm{LG},\delta}} \in \mathcal{O}_{\mathcal{M}_{\mathrm{LG},\delta}} \cdot \mathbf{1},$$

and the mirror map restricts to

$$\tau : \mathcal{M}_{\mathrm{LG},\delta} \longrightarrow H_W. \tag{65}$$

In particular, if we restrict to the $(t_1,\cdots,t_n,\sigma)$-plane, then

$$\Upsilon_{\mathrm{LG}}(t_1,\cdots,t_n,\sigma,z) = f^W_0(\sigma)\,\mathbf{1},$$

where $f^W_0(\sigma)$ is given in (62), and the mirror map restricts to

$$\tau(t_1,\cdots,t_n,\sigma) = \frac{I^W_1(t_1,\cdots,t_n,\sigma)}{f^W_0(\sigma)} \in H^{\leq 1}_W[t_1,\cdots,t_n]\{\sigma\}.$$

Here $H^{\leq 1}_W$ are the elements of H_W with $\deg_W \leq 1$.

7.3. **Mirror symmetry to FJRW theory.**

7.3.1. *An isomorphism between $\mathcal{D}$-modules.*

Lemma 7.5. *The set $\{I^{\mathbf{e}}_{\mathrm{LG}}(t,z)|\mathbf{e}\in\mathbb{Z}^n_{\geq 0}\}$ satisfies the following differential equations:*

$$z\frac{\partial}{\partial t_b} I^{\mathbf{e}}_{\mathrm{LG}}(t,z) = I^{\mathbf{e}+b}_{\mathrm{LG}}(t,z), \quad \forall b\in\mathfrak{B}; \tag{66}$$

$$z(\mathbf{e}^{(i)}+1)\, I^{\mathbf{e}}_{\mathrm{LG}}(t,z) + \sum_{b\in\mathfrak{B}} b^{(i)} t_b\, I^{\mathbf{e}+b}_{\mathrm{LG}}(t,z) + \sum_{j=1}^n b^{(i)}_j\, I^{\mathbf{e}+b_j}_{\mathrm{LG}}(t,z) = 0. \tag{67}$$

Proof. Recalling (55), we will simply denote $\nu=\nu(\mathbf{e})$, $\nu'=\nu(\mathbf{e}+b)$ and $\nu''=\nu(\mathbf{e}+b_j)$.

For the first equation, we shall compare the coefficient of $t_b^{\nu_b-1}\prod_{c\neq b} t_c^{\nu_c}$ on both sides. The corresponding vector $\{\nu'_c\}_{c\in\mathfrak{B}}$ on the right hand side should satisfy

$$\nu'_c = \nu_c - \delta^b_c, \quad c\in\mathfrak{B}. \tag{68}$$

Both coefficients are 0 when $\nu_b = 0$, and thus it is enough to match them when $\nu_b \geq 1$.

When $\nu_b \geq 1$, on the right hand side, similar to (55), we have

$$\nu'_j = -\frac{1}{d_j}\left(\mathbf{e}^{(j)} + b^{(j)} + 1 + \sum_{c\in\mathfrak{B}} \nu'_c c^{(j)}\right) \in \mathbb{Q}_{<0}.$$

Equation (68) implies $\nu'_j = \nu_j$ and

$$\Box_{\nu'} = \frac{\prod_{j=1}^n \prod_{k=1}^{\lfloor -\nu'_j \rfloor} (\nu'_j z + kz)}{\prod_{c\in G} \prod_{k=1}^{\nu'_c} (kz)} = (\nu_b z)\Box_\nu.$$

Thus the coefficient of $t_b^{\nu_b - 1}\prod_{c\neq b} t_c^{\nu_c}$ on the right hand side is

$$\Box_{\nu'}\mathbf{1}_{\nu'} = (\nu_b z)\Box_\nu \mathbf{1}_{\gamma_\nu}.$$

Now let us prove the second identity. There are three terms and we will consider the coefficient of $\prod_{c\in\mathfrak{B}} t_c^{\nu_c}$ for a fixed vector $\nu = \{\nu_c\}_{c\in\mathfrak{B}}$. For each $b \in \mathfrak{B}$, the contribution from $I_{\mathrm{LG}}^{\mathbf{e}+b}(t,z)$ comes from the vector $\{\nu'_c\}_{c\in\mathfrak{B}}$ such that

$$\nu'_c = \nu_c - \delta_c^b, \quad \forall c \in S.$$

For each $j = 1, \cdots, n$, the contribution from $I_{\mathrm{LG}}^{\mathbf{e}+b_j}(t,z)$ comes from the vector $\{\nu''_c\}_{c\in\mathfrak{B}}$ such that

$$\nu''_c = \nu_c - \delta_c^j, \quad \forall c \in S.$$

Thus

$$\Box_{\nu'}\mathbf{1}_{\gamma_{\nu'}} = (\nu_b z)\Box_\nu \mathbf{1}_{\gamma_\nu}, \quad \Box_{\nu''}\mathbf{1}_{\gamma_{\nu''}} = (\nu_j z)\Box_\nu \mathbf{1}_{\gamma_\nu}.$$

Put everything together, we know the coefficient of $\prod_{c\in\mathfrak{B}} t_c^{\nu_c}$ of the LHS in (67) is given by

$$\left(z(\mathbf{e}^{(i)}+1) + \sum_{b\in\mathfrak{B}} b^{(i)}\nu_b z + \sum_{j=1}^n b_j^{(i)}\nu_j z\right)\Box_\nu \mathbf{1}_{\gamma_\nu}.$$

Since $b_j^{(i)} = d_j \delta_j^i$, Equation (55) implies that the formula above vanishes.

Finally, we check the constant term in equation (67). The constant in the second term vanishes by definition. The remaining two terms give

$$\left(z(\mathbf{e}^{(i)}+1) + \sum_{j=1}^n b_j^{(i)}\nu_j z\right)\Box_\nu \mathbf{1}_{\gamma_\nu} = 0.$$

This again follows from Equation (55), where now $\nu_b = 0$ for all $b \in \mathfrak{B}$. $\square$

According to this lemma, there exist a $\mathcal{D}$-module on $\mathcal{M}_{\mathrm{LG},\delta}$. We identify this $\mathcal{D}$-module with the $\mathcal{D}$-module $(\widehat{\mathbb{H}}_+(W), \nabla)$ in the B-model using the mirror map in (65).

Proposition 7.6. *The following map*

$$\mathrm{Loc}_W : \widehat{\mathbb{H}}_+(W) \to \mathcal{L}_W, \qquad [x^{\mathbf{e}}dx] \mapsto I_{\mathrm{LG}}^{\mathbf{e}}(t,z). \tag{69}$$

extends to a $\mathcal{D}$-module isomorphism.

Proof. The surjectivity is obvious and the injectivity is a consequence of Equation (67). The result follows since both $\mathcal{D}$-modules has the same rank. $\square$

7.3.2. *Matching pairings.* We extend the pairing η_W in (48) $\mathbb{C}[\![z]\!]$-linearly, and still denote the extension by $\eta_W : H_W[\![z]\!] \times H_W[\![z]\!] \to \mathbb{C}[\![z]\!]$. Via the map Loc^W in (69), we call pull back the pairing η_W to get

$$\widetilde{K}_W : \widehat{\mathbb{H}}_+(W) \times \widehat{\mathbb{H}}_+(W) \to \mathbb{C}[\![z]\!],$$

where

$$\widetilde{K}_W(\omega_1, \omega_2) = \eta_W\left(\mathrm{Loc}_W(\omega_1), \mathrm{Loc}_W^*(\omega_2)\right).$$

For Fermat singularities $W := x_1^{d_1} + \cdots + x_n^{d_n}$, we recall the set in (52). The set $\{[x^b dx] \in \widehat{\mathbb{H}}_+(W) \mid b \in \mathfrak{B}\}$ forms a good basis [37, Theorem 2.10], such that

$$K_W([x^b dx], [x^c dx]) = \delta_b^c.$$

We identify the set $\mathfrak{B}$ with $\mathscr{N}$, the set of indicies of narrow elements in FJRW theory, by a *shifting map*

$$\mathfrak{S} : \mathfrak{B} \to \mathscr{N}, \quad via \quad b \mapsto \gamma \in \mathscr{N}, \quad \Theta_\gamma^{(j)} = \frac{b^{(j)} + 1}{d_j}. \tag{70}$$

Proposition 7.7. *For any $b, c \in \mathfrak{B}$, we have*

$$K_W([x^b dx], [x^c dx]) = \widetilde{K}_W([x^b dx], [x^c dx]).$$

Proof. Since $L(t(\tau), z)^{-1}$ preserves the pairing, in order to prove the mirror map (69) preserves the pairing, we only need to verify

$$\eta_W\left(I_{\mathrm{LG}}^b(t, -z)\mid_{t=0}, I_{\mathrm{LG}}^c(t, z)\mid_{t=0}\right) = \delta_b^c.$$

This equality follows easily from (59). □

7.3.3. *Matching opposite subspaces.* The vector space of the good basis

$$H_{\mathfrak{B}} := \{[x^b dx] \in \widehat{\mathbb{H}}_+(W) \mid b \in \mathfrak{B}\}$$

induces an opposite subspace $P_{\mathfrak{B}}$ in $\widehat{\mathbb{H}}(W)$. Recalling (3) in Section 4.2, we have

$$P_{\mathfrak{B}} = H_{\mathfrak{B}}[z^{-1}]z^{-1}.$$

On the other hand, $H_W(\!(z)\!)$ has a natural opposite subspace $H_W[z^{-1}]z^{-1}$. Then the restriction of Loc_W on $H_{\mathfrak{B}}$ is induced by the shifting map (70)

$$\mathrm{Loc}_W : H_{\mathfrak{B}} \to H_W, \quad [x^b dx] \mapsto \mathbf{1}_{\mathfrak{S}(b)}.$$

It is easy to see that

Proposition 7.8. *The map Loc_W matches the opposite subspaces $P_{\mathfrak{B}}$ with $H_W[z^{-1}]z^{-1}$.*

7.4. **Proof of main theorem.** Recall our main theorem:

Theorem 7.9. *Suppose that W is a Fermat polynomial with $d = \sum_i c_i$ (hence X_W defines a Calabi-Yau hypersurfac). Then,*

1. *LG/CY correspondence conjecture holds for the pair (W, G_W).*
2. *The modularity conjecture holds for $[X_W/\tilde{G}_W]$.*

Proof. The proof of modularity conjecture follows directly from the definition of B-model generating function (Definition 5.8), which is modular but non-holomorphic; and GW-mirror theorems (Theorem 6.15), which express B-model generating function as the anti-holomorphic completion of GW-generating function. Although our main interest is GW-theory, a similar statement holds for FJRW-theory as well.

To prove the LG-CY correspondence, we need to consider the analytic continuation of holomorphic generating function of GW/FJRW-theory. This can be done as follows. Using two mirror theorems (Theorem 6.15, Proposition 7.6, 7.7, and 7.8), we identify GW/FJRW-generating function to the local generating functions near large complex structure/Gepner limits on the B-model moduli space. Now we can use the complex coordinates (not flat coordinates) on the B-model moduli space. The GW/FJRW-generating functions were induced by the GW/FJRW-opposite subspaces. Now, we use the Gauss-Manin connection to parallel transport the GW-opposite subspace at the large complex structure to the Gepner limit along a path. Note that Gauss-Manin connection preserves the Givental symplectic vector space and Givental cone, so the parallel transport of an opposite subspace will remain Lagrangian and opposite. In such a way, we obtain a holomorphic generating function in a neighborhood of a path connecting large complex limit to Gepner limit. Namely, we construct an analytic continuation of the GW-generating function. But the analytic continuation of the GW-generating function to the Gepner limit is not the FJRW-generating function because the parallel transport of GW-opposite subspace is different from FJRW-opposite subspace. By Lemma 5.6, the two generating functions differ by the quantization of the symplectic transformation mapping one opposite subspace to other. □

Appendix A. A proof of Proposition 7.3

A.1. **Weighted invariants and concavity.**

Definition A.1. For any $\epsilon \in \mathbb{Q}_{>0}$, we say $(\mathcal{C}, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ is a (G_W, ϵ)-stable structure if the following conditions are satisfied:

- $\mathcal{C}$ is a connected proper one-dimensional DM stack of genus 0 with weight-1 marked points $x_1, \cdots, x_m$, and weight ϵ points $y_1, \cdots, y_\ell$. The total weight at each point $p \in \mathcal{C}$ is bounded by 1. Stacky point can only occur at marked points and nodal points.
- Let G_p be the local isotropy group at the stacky point p. There is a faithful representation $r_p : G_p \to G_W$.
- Each $\mathcal{L}_j$ is an invertible sheaf over $\mathcal{C}$ and there exists integers $\xi_{i,j} \in [0, d_j)$ such that

$$\mathcal{L}_j^{\otimes d_j} \cong \omega_{\mathcal{C},\log}\left(-\sum_{i=1}^{\ell} \xi_{i,j}[y_i]\right). \tag{71}$$

At each marking x_i, the local representation sends the generator $1 \in G_{x_i}$ to some

$$\gamma_i := \left(\exp(2\pi\sqrt{-1}\Theta_{\gamma_i}^{(1)}), \cdots \exp(2\pi\sqrt{-1}\Theta_{\gamma_i}^{(n)})\right) \in G_W, \quad \Theta_{\gamma_i}^{(j)} \in [0,1) \cap \mathbb{Q}.$$

We fix the decorations $\gamma = (\gamma_1, \cdots, \gamma_m)$ and $\xi = (\xi_1, \cdots, \xi_\ell)$ such that $\Theta^{(j)}_{\xi_i} = \xi_{i,j}/d_j$. We denote the moduli of (G_W, ϵ)-stable structures by $\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)$. $\mathcal{L}_j$ has a desingularization, which is a line bundle L_j on the coarse curve C [14, Prop. 4.1.2]. When $\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)$ is nonempty,

$$\deg L_j = \frac{1}{d_j}\left(-2 + m - \sum_{i=1}^{\ell} \xi_{i,j}\right) - \sum_{i=1}^{m} \Theta^{(j)}_{\gamma_i} \in \mathbb{Z}. \tag{72}$$

According to [28], nonempty $\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)$ is a smooth Deligne-Mumford stack properly fibered over the Hasset moduli space of stable weighted rational curves, which we denoted by $\overline{\mathcal{M}}_{m|\ell}$. Furthermore,

$$\dim_{\mathbb{C}} \overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi) = -3 + m + \ell. \tag{73}$$

In [28], if the vector space consists of narrow sectors and compact sectors only, the authors use cosection technique [19, 12] to construct a virtual fundamental cycle on the moduli space. We denote such a cycle by $[\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)]^{\mathrm{vir}}$. Let $\mathbf{1}_{\gamma_i}$ be the insertion at the marked point x_i and $\mathbf{1}_{\xi_i}$ be the insertion at the marked point y_i, then the following weighted-ϵ invariant is defined

$$\left\langle \mathbf{1}_{\gamma_1}\bar{\psi}_1^{k_1}, \cdots, \mathbf{1}_{\gamma_m}\bar{\psi}_m^{k_m} \middle| \mathbf{1}_{\xi_1}, \cdots, \mathbf{1}_{\xi_\ell} \right\rangle^{\epsilon}_{m|\ell} = \int_{[\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)]^{\mathrm{vir}}} \prod_{i=1}^{m} \bar{\psi}_i^{k_i}. \tag{74}$$

In particular, if $\epsilon > 1$, then there is no weighted point and we get the FJRW invariant up to a sign [12, Theorem 5.6].

A.1.1. *Concavity.* The following lemma is very useful.

Lemma A.2. *Each geometric fiber $(\mathcal{C}, \mathcal{L}_1, \cdots, \mathcal{L}_n) \in \overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)$ is concave, i.e.,*

$$H^0(\mathcal{C}, \mathcal{L}_j) = 0, \quad \forall \quad 1 \le j \le n. \tag{75}$$

Proof. If $\mathcal{C}$ is smooth, since $d_j\Theta^{(j)}_{\gamma_i} \ge 1$ and $\xi_{i,j} \ge 0$, (72) implies $\deg L_j < 0$ and (75) follows.

If $\mathcal{C}$ is a nodal, then for each $j = 1, \cdots, n$, the normalization induces a long exact sequence:

$$0 \to H^0(\mathcal{C}, \mathcal{L}_j) \to \bigoplus_v H^0(\mathcal{C}_v, \mathcal{L}_j) \xrightarrow{\varrho} \bigoplus_p H^0(p, \mathcal{L}_j) \to H^1(\mathcal{C}, \mathcal{L}_j) \to \bigoplus_v H^1(\mathcal{C}_v, \mathcal{L}_j) \to 0.$$

Here $\mathcal{C}_v$ runs over all the components after normalization and p runs over all the nodes. It is enough to prove $\mathrm{Ker}(\varrho) = 0$. If p is a narrow node, then $H^0(p, \mathcal{L}_j) = 0$ and we can split the exact sequence into two different sequences and then discuss individually. Thus we may assume all the nodes are broad.

We call a broad node *external* if one of the component attached to this node has exactly one node. Otherwise we call a broad node *internal*. We denote the number of external broad nodes by E and the number of internal broad nodes by I. Since C is a genus zero nodal curve, there are $E + I + 1$ components in the normalization, where E

of them contain exactly one node. Also, we must have $E \geq 2$. If we denote the number of broad nodes on the component C_v by B_v, then

$$\sum_v B_v = E + 2I.$$

Moreover, since $d_j \Theta_{\gamma_i}^{(j)} \geq 1$, $\xi_{i,j} \geq 0$, and $d_j \geq 2$, the formula (72) implies

$$\deg_{C_v} L_j \leq \frac{B_v}{2} - 1. \tag{76}$$

By definition of ϱ, any nonzero section σ such that $\varrho(\sigma) = 0$ must vanish on each external node. Thus the total degree of σ is at least E. However, (76) implies the number of zeros of σ is at most

$$\sum_{v;B_v \geq 2} (\frac{B_v}{2} - 1) = \sum_{v;B_v \geq 2} \frac{B_v}{2} - (I+1) < \frac{E}{2} + I - (I+1) = \frac{E}{2} - 1 < E.$$

This is a contradiction. Thus we must have $\mathrm{Ker}(\varrho) = 0$. $\square$

Let $\mathscr{C}$ be the universal curve, $\pi : \mathscr{C} \to \overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)$ be the universal family, and $\mathscr{L}_j$ be the j-th universal line bundle. Lemma A.2 implies

$$\left[\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)\right]^{\mathrm{vir}} = c_{\mathrm{top}} \left(R^1\pi_* \bigoplus_{j=1}^{n} \mathscr{L}_j \right) \cap \left[\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi)\right] \in H_*(\overline{\mathcal{W}}^{\epsilon}_{m|\ell}(\gamma|\xi), \mathbb{Q}). \tag{77}$$

A.2. **Graph spaces and localization.**

Definition A.3. We consider a graph space $(f : \mathcal{C} \to \mathbb{P}^1, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ where

- The rational coarse curve C contains a component $C_1 \cong \mathbb{P}^1$ with $\deg(f)|_{C_1} = 1$.
- $(\overline{\mathcal{C}/\mathcal{C}_1}, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ is (G_W, ϵ)-stable on each component of $\overline{\mathcal{C}/\mathcal{C}_1}$.

For fixed decorations $(\gamma|\xi)$, we denote the moduli space of graph spaces by $\mathcal{G}^{\epsilon}_{m|\ell}(\gamma|\xi)$. For any $x_i, y_j \in \mathcal{C}_1$, there are evaluation morphisms

$$\mathrm{ev}_i, \widetilde{\mathrm{ev}}_j : \mathcal{G}^{\epsilon}_{m|\ell}(\gamma|\xi) \to \mathbb{P}^1$$

which send $(f : \mathcal{C} \to \mathbb{P}^1, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ to $f(x_i)$ and $f(y_j)$ respectively. The moduli of graph spaces has a GLSM model description [30, Example 4.2.22], so the cosection technique [44, 12] allows [30] to construct a perfect obstruction theory on $\mathcal{G}^{\epsilon}_{m|\ell}(\gamma|\xi)$. In particular, since the genus g is 0, the moduli space is concave. We write the obstruction sheaf as

$$\mathrm{Ob}_{\mathcal{G}^{\epsilon}_{m|\ell}(\gamma|\xi)} = R^1\pi_* \bigoplus_{j=1}^{n} \mathscr{L}_j. \tag{78}$$

Now we consider a $\mathbb{C}^*$-action on $\mathbb{P}^1$:

$$\lambda \cdot [x_0 : x_1] = [\lambda x_0 : x_1], \quad \lambda \in \mathbb{C}^*. \tag{79}$$

Let us denote $[0] := c_1^{\mathbb{C}^*}(\mathcal{O}(1)) \in H^2_{\mathbb{C}^*}(\mathbb{P}^1)$ if the weight of the $\mathbb{C}^*$-action is 1 at $0 = [0 : 1] \in \mathbb{P}^1$ and 0 at $\infty = [1 : 0] \in \mathbb{P}^1$. We also denote $[\infty] := c_1^{\mathbb{C}^*}(\mathcal{O}(1)) \in H^2_{\mathbb{C}^*}(\mathbb{P}^1)$ if the weight of $\mathbb{C}^*$-action is 0 at $0 \in \mathbb{P}^1$ and -1 at $\infty \in \mathbb{P}^1$.

Let $(f\colon \mathcal{C} \to \mathbb{P}^1, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ be a graph space such that $f(x_{m+1}) = \infty$, $f(y_{\ell+1}) = 0$. We fix the type of decorations by

$$\big(\gamma(\alpha)|\xi(\beta)\big) := (\gamma_1, \cdots, \gamma_m, \alpha|\xi_1, \cdots, \xi_\ell, \beta).$$

The $(m+1)$-th marked point x_{m+1} is decorated by $\mathbf{1}_\alpha$ and the $(\ell+1)$-th weighted point $y_{\ell+1}$ is decorated by $\mathbf{1}_\beta$. For simplicity, we denote the moduli space by

$$\mathcal{G}^\epsilon := \mathcal{G}^\epsilon_{m+1|\ell+1}\big(\gamma(\alpha)|\xi(\beta)\big).$$

The $\mathbb{C}^*$-action (79) induces a $\mathbb{C}^*$-action on the moduli $\mathcal{G}^\epsilon$ and on its universal bundle. The moduli stack $\mathcal{G}^\epsilon$ has a $\mathbb{C}^*$-equivariant virtual cycle $[\mathcal{G}^\epsilon]^{\mathrm{vir}}_{\mathbb{C}^*} \in A_*^{\mathbb{C}^*}(\mathcal{G}^\epsilon)$ and

$$\widetilde{\mathrm{ev}}^*_{\ell+1}([0]) = z, \quad \mathrm{ev}^*_{m+1}([\infty]) = -z. \tag{80}$$

We label the fixed locus by decorated dual graph Γ. Let m_0, n_0 be the number of marked points and number of weighted points on $f^{-1}(0) = \mathcal{C}_0$. Here neither the node, nor the weighted point with decoration $\mathbf{1}_\beta$ is included. Similarly, we define m_∞ and n_∞ on $f^{-1}(\infty) = \mathcal{C}_\infty$. Thus

$$m_0 + m_\infty = m, \quad n_0 + n_\infty = \ell.$$

Let Γ_0 and Γ_∞ be the decorated dual graph of $\mathcal{C}_0$ and $\mathcal{C}_\infty$ respectively. Let $\overline{\mathcal{W}}^\epsilon(\Gamma_0)$ and $\overline{\mathcal{W}}^\epsilon(\Gamma_\infty)$ be the corresponding moduli spaces of (G_W, ϵ)-structures. Let $\mathcal{F}_\Gamma$ be the fixed locus labeled by Γ. Again it has an obstruction bundle $\mathrm{Ob}_{\mathcal{F}_\Gamma} = \oplus R^1\pi_*\mathcal{L}_j$ and the virtual fundamental cycle

$$[\mathcal{F}_\Gamma]^{\mathrm{vir}} = c_{\mathrm{top}}\left(\bigoplus_{j=1}^n R^1\pi_*\mathcal{L}_j\right) \cap [\mathcal{F}_\Gamma].$$

We have morphisms

$$\iota_\Gamma\colon \mathcal{F}_\Gamma = \overline{\mathcal{W}}^\epsilon(\Gamma_0) \times \overline{\mathcal{W}}^\epsilon(\Gamma_\infty) \longrightarrow \mathcal{G}^\epsilon.$$

Let $\mathcal{N}_\Gamma$ be the normal bundle of $\mathcal{F}_\Gamma$ in $\mathcal{G}^\epsilon$. We use Atiyah-Bott localization to obtain

$$[\mathcal{G}^\epsilon]^{\mathrm{vir}}_{\mathbb{C}^*} = \sum_\Gamma (\iota_\Gamma)_*\left(\frac{[\mathcal{F}_\Gamma]^{\mathrm{vir}}_{\mathbb{C}^*}}{e_{\mathbb{C}^*}(\mathcal{N}_\Gamma^{\mathrm{vir}})}\right).$$

Then the $\mathbb{C}^*$-integral

$$\begin{aligned} &\left\langle \mathbf{1}_{\gamma_1}\bar\psi_1^{k_1}, \cdots, \mathbf{1}_{\gamma_m}\bar\psi_m^{k_m}, \mathbf{1}_\alpha \middle| \mathbf{1}_{\xi_1}, \cdots, \mathbf{1}_{\xi_\ell}, \mathbf{1}_\beta \middle| \mathrm{ev}^*_{m+1}([\infty]) \cup \widetilde{\mathrm{ev}}^*_{\ell+1}([0])\right\rangle^{\epsilon, \mathbb{C}^*}_{m+1|\ell+1} \\ &= \int_{[\mathcal{G}^\epsilon]^{\mathrm{vir}}_{\mathbb{C}^*}} \left(\prod_{i=1}^m \bar\psi_i^{k_i}\right) \mathrm{ev}^*_{m+1}([\infty]) \cap \widetilde{\mathrm{ev}}^*_{\ell+1}([0]) \in \mathbb{C}[\![z]\!] \end{aligned} \tag{81}$$

allows us to define a formal power series with variables t and y.

$$\begin{aligned} &\left\langle\!\left\langle \mathbf{1}_\alpha \middle| \mathbf{1}_\beta \middle| \mathrm{ev}^*([\infty]) \cup \widetilde{\mathrm{ev}}^*([0])\right\rangle\!\right\rangle^{\epsilon, \mathbb{C}^*}_{1+\bullet|1+\bullet}(t, y) \\ &= \sum_{m,\ell} \frac{1}{m!\ell!}\left\langle t, \cdots, t, \mathbf{1}_\alpha \middle| y, \cdots, y, \mathbf{1}_\beta \middle| \mathrm{ev}^*_{m+1}([\infty]) \cup \widetilde{\mathrm{ev}}^*_{\ell+1}([0])\right\rangle^{\epsilon, \mathbb{C}^*}_{m+1|\ell+1}. \end{aligned} \tag{82}$$

The fixed locus is called *stable* if both $p_0 := f^{-1}(0) \cap \mathcal{C}_1$ and $p_\infty := f^{-1}(\infty) \cap \mathcal{C}_1$ are nodes. Otherwise, it is called *unstable*.

A.2.1. *Stable contribution.* Let $(f\colon \mathcal{C} \to \mathbb{P}^1, \mathcal{L}_1, \cdots, \mathcal{L}_n)$ be a geometric point in the stable fixed locus $\mathcal{F}_\Gamma$. Then

$$m_0 + 1 + (n_0+1)\epsilon > 2 \quad \textit{and} \quad m_\infty + 2 + n_\infty \epsilon > 2.$$

The normalization $\widetilde{\mathcal{C}} = C_0 \coprod \mathcal{C}_1 \coprod \mathcal{C}_\infty \to \mathcal{C}$ induces the following long exact sequence:

$$\begin{aligned}0 \to \bigoplus_{j=1}^n H^0(\mathcal{C}, \mathcal{L}_j) \to \bigoplus_{j=1}^n \bigoplus_{a\in\{0,1,\infty\}} H^0(\mathcal{C}_a, \mathcal{L}_j) \to \bigoplus_{j=1}^n \bigoplus_{a\in\{0,\infty\}} H^0(p_a, \mathcal{L}_j) \to \\ \to \bigoplus_{j=1}^n H^1(\mathcal{C}, \mathcal{L}_j) \to \bigoplus_{j=1}^n \bigoplus_{a\in\{0,1,\infty\}} H^1(\mathcal{C}_a, \mathcal{L}_j) \to 0.\end{aligned} \tag{83}$$

Both $\mathcal{C}_0$ and $\mathcal{C}_\infty$ contain at most one broad insertion. Thus we can proceed as in Lemma A.2 to obtain

$$H^0(\mathcal{C}_0, \mathcal{L}_j) = H^0(\mathcal{C}_\infty, \mathcal{L}_j) = 0.$$

On the other hand, for the component $\mathcal{C}_1$, the formula (72) implies

$$\deg_{C_1} L_j = \begin{cases} 0, & \textit{two nodes are broad,} \\ -1, & \textit{two nodes are narrow.} \end{cases}$$

In either case, we have

$$H^1(\mathcal{C}_1, \mathcal{L}_j) = H^1(C_1, L_j) = 0.$$

If both nodes are narrow, then the first line in (83) will vanish. Thus we get isomorphic $\mathbb{C}^*$-equivariant vector spaces

$$H^1(\mathcal{C}, \mathcal{L}_j) \cong H^1(\mathcal{C}_0, \mathcal{L}_j) \bigoplus H^1(\mathcal{C}_\infty, \mathcal{L}_j). \tag{84}$$

Now if both nodes are broad, then there exists some j such that

$$H^0(\mathcal{C}_1, \mathcal{L}_j) \cong \mathbb{C}, \quad H^0(p_0, \mathcal{L}_j) \bigoplus H^0(p_\infty, \mathcal{L}_j) \cong \mathbb{C}^2.$$

The first line in (83) contains a summand of $\mathbb{C} \hookrightarrow \mathbb{C}^2$, with $\mathbb{C}^*$ acting trivially on $\mathbb{C}^2$. Recall (78) and (84), we have

$$c_{\text{top}}^{\mathbb{C}^*}(\text{Ob}_{\mathcal{F}_\Gamma}) = \begin{cases} c_{\text{top}}^{\mathbb{C}^*}(\text{Ob}_{\overline{\mathcal{W}}^\epsilon(\Gamma_0)}) c_{\text{top}}^{\mathbb{C}^*}(\text{Ob}_{\overline{\mathcal{W}}^\epsilon(\Gamma_\infty)}), & \textit{if both } 0 \textit{ and } \infty \textit{ are narrow.} \\ 0, & \textit{if both } 0 \textit{ and } \infty \textit{ are broad.} \end{cases} \tag{85}$$

On the other hand, we have

$$c_1^{\mathbb{C}^*}(T_{p_0}\mathcal{C}_0 \oplus T_{p_0}\mathcal{C}_1) = -z - \psi_{p_0}, \quad c_1^{\mathbb{C}^*}(T_{p_\infty}\mathcal{C}_\infty \oplus T_{p_\infty}\mathcal{C}_1) = z - \psi_{p_\infty}.$$

Since the $\mathbb{C}^*$-equivariant Euler class of deformation of the maps $f : \mathcal{C}_1 \to \mathbb{P}^1$ is $-z^2$, we use (80) to obtain that the stable contribution of (82) is

$$\left\langle\!\!\left\langle \mathbf{1}_\alpha \middle| \mathbf{1}_\beta \middle| \text{ev}^*([0]) \cup \widetilde{\text{ev}}^*([\infty]) \right\rangle\!\!\right\rangle_{\text{stable}}^{\epsilon, \mathbb{C}^*} = \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_{\gamma'}}{-z-\psi} \middle| \mathbf{1}_\beta \right\rangle\!\!\right\rangle_{1|1}^\epsilon \left\langle\!\!\left\langle \mathbf{1}_\alpha, \frac{\mathbf{1}_\gamma}{z-\psi} \right\rangle\!\!\right\rangle_{2|0}^\epsilon. \tag{86}$$

We remark that by definition of (74), the RHS only contains stable terms.

A.2.2. *Unstable contribution.* There are three unstable situations:

(1) Both $f^{-1}(0)$ and $f^{-1}(\infty)$ are unstable.
(2) $f^{-1}(0)$ is stable but $f^{-1}(\infty)$ is unstable.
(3) $f^{-1}(0)$ is unstable but $f^{-1}(\infty)$ is stable.

Before we start to compute the unstable terms, let us introduce a weighted I-function. Recall that when $\mathbf{e}=0$, ν and $\Box_\nu$ are given by (54) and (55). Recall that $\mathfrak{B}$ is the set of ghost variables defined in (52). For any $\epsilon\in\mathbb{Q}_{>0}$, we consider

$$\mathbb{L}_\epsilon := \left\{ \nu=\{\nu_b\}_{b\in\mathfrak{B}} \middle| \nu_b\geq 0, \quad \sum_{b\in\mathfrak{B}}\nu_b\leq\frac{1}{\epsilon}\right\}.$$

In particular, $\lim_{\epsilon\to 0}\mathbb{L}_\epsilon=\mathbb{L}$ in (53). We define an I^ϵ-function

$$I^{0,\epsilon}_{\mathrm{LG}}(y,z)=\sum_{\nu\in\mathbb{L}_\epsilon}\prod_{b\in\mathfrak{B}}t_b^{\nu_b}\Box_\nu\mathbf{1}_{\gamma_\nu} \tag{87}$$

For the first case of unstable terms, $\mathcal{C}=\mathcal{C}_1$, and

$$m_0=m_\infty=n_\infty=0,\quad (n_0+1)\epsilon\leq 1.$$

Thus $m=0$ and $\ell=n_0$. Each $\mathbb{C}^*$-action on $\mathcal{L}_j$ induces an isomorphism $\mathcal{C}_1\cong\mathbb{P}[d_j,1]$ with p_∞ the only orbifold point. All the weighted points $y_1,\cdots,y_\ell,y_{\ell+1}$ stack at p_0. For each y_i, we have some $\xi_{i,j}\in\{0,1,\cdots,d_j-1\}$ such that

$$\mathcal{L}_j^{\otimes d_j}\cong\omega_{\mathcal{C},\log}\left(-\sum_{i=1}^{\ell}\xi_{i,j}[y_i]-d_j\Theta^{(j)}_\beta[y_{\ell+1}]\right).$$

Here $\xi_{\ell+1,j}=d_j\Theta^{(j)}_\beta$ since $y_{\ell+1}$ is decorated with the narrow element $\mathbf{1}_\beta$. Also we have

$$\mathcal{L}_j\cong\mathcal{O}_{\mathbb{P}[d_j,1]}\left((-1-\sum_{i=1}^{\ell+1}\xi_{i,j})[\infty]\right).$$

According to (72), the node p_∞ must be decorated by a narrow $\mathbf{1}_\gamma\in H_W$ where

$$\Theta^{(j)}_\gamma=\langle\frac{1}{d_j}\left(-1-\sum_{i=1}^{\ell+1}\xi_{i,j}\right)\rangle\in(0,1)\cap\mathbb{Q}. \tag{88}$$

According to [48, Example 98], we may choose the $\mathbb{C}^*$ action $\lambda[x_0;x_1]=[\lambda x_0;x_1]$ such that

$$c_1^{\mathbb{C}^*}(T_{p_0}\mathcal{C})=-z,\quad c_1^{\mathbb{C}^*}(T_{p_\infty}\mathcal{C})=\frac{z}{d},\quad c_1^{\mathbb{C}^*}(\mathcal{L}_j|p_\infty)=0,$$

then

$$e_{\mathbb{C}^*}\left(\bigoplus_{j=1}^{n}R^1\pi_*\mathscr{L}_j\right)=\prod_{j=1}^{n}\prod_{k=1}^{\lfloor-\nu_j\rfloor}(-\nu_j-k)z.$$

We reindex $\mathbf{1}_{\xi_i}$ by $b\in\mathfrak{B}$ such that $b^{(j)}=\xi_{i,j}-1$. Let ν_b be the number of $b\in\mathfrak{B}$ and we parametrize such an element by y_b. By definition of (55) and (56), we get an element $\mathbf{1}_{\gamma_\nu}$ such that $\mathbf{1}_{\gamma_\nu}=\mathbf{1}_{\gamma'}$. Here γ satisfies (88) and γ' is the involution of γ.

Since the $\mathbb{C}^*$-equivariant Euler class of deformation of the maps $f : \mathcal{C} \to \mathbb{P}^1$ is $(-z)^{\ell+1}(-z^2)$, we obtain that the first unstable part of (82) is given by

$$\sum_{\sum \nu_b \le \frac{1}{\epsilon}-1} \frac{-z^2}{(-z)^{\ell+1}(-z^2)} \prod_{b\in\mathfrak{B}} \frac{y_b^{\nu_b}}{\nu_b!} \prod_{j=1}^{n} \prod_{k=1}^{\lfloor -v_j \rfloor} (-\nu_j - k)z = \eta_W\left(\frac{\partial}{\partial y^\beta} I_{\mathrm{LG}}^{0,\epsilon}(y,-z), \mathbf{1}_\alpha\right). \tag{89}$$

Similarly, we get the rest of the unstable part of (82) as follows

$$\left\langle\!\!\left\langle \frac{\mathbf{1}_\alpha}{-z-\psi}\middle|\mathbf{1}_\beta \right\rangle\!\!\right\rangle_{1|1}^{\epsilon} + \eta_W\left(\frac{\partial}{\partial y^\beta} I_{\mathrm{LG}}^{0,\epsilon}(y,-z), \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z-\psi}, \mathbf{1}_\alpha \right\rangle\!\!\right\rangle_{2|0}^{\epsilon} \mathbf{1}_{\gamma'}\right). \tag{90}$$

A.3. **A proof of Proposition 7.3.**

A.3.1. *Regularity.* We define a J^ϵ-function:

$$J^\epsilon(t,y,z) = I_{\mathrm{LG}}^{0,\epsilon}(y,z) + z\mathbf{1} + t(z) + \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z-\psi} \right\rangle\!\!\right\rangle_{1|0}^{\epsilon} \mathbf{1}_{\gamma'}. \tag{91}$$

Proposition A.4. *We have the following equality*

$$\eta_W\left(\frac{\partial}{\partial y_\beta} J^\epsilon(t,y,-z), \frac{\partial}{\partial t_0^\alpha} J^\epsilon(t,y,z)\right) = \left\langle\!\!\left\langle \mathbf{1}_\alpha \middle| \mathbf{1}_\beta \middle| \mathrm{ev}^*([\infty]) \cup \widetilde{\mathrm{ev}}^*([0]) \right\rangle\!\!\right\rangle_{1|1}^{\epsilon,\mathbb{C}^*} \in \mathbb{C}[\![z]\!]. \tag{92}$$

Proof. Thus

$$\frac{\partial}{\partial y_\beta} J^\epsilon(t,y,-z) = \frac{\partial}{\partial y^\beta} I_{\mathrm{LG}}^{0,\epsilon}(y,-z) + \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{-z-\psi} \middle| \mathbf{1}_\beta \right\rangle\!\!\right\rangle_{1|1}^{\epsilon} \mathbf{1}_{\gamma'}$$

and each ν_j satisfies

$$\nu_j = -q_j - \nu_\beta \Theta_\beta^{(j)} - \sum_{b\neq\beta} \nu_b b^{(j)} = -q_j - \Theta_\beta^{(j)} - \sum_b \widetilde{\nu}_b b^{(j)}. \tag{93}$$

For the second term, we get

$$\frac{\partial}{\partial t_0^\alpha} J^\epsilon(t,y,z) = \mathbf{1}_\alpha + \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z-\psi}, \mathbf{1}_\alpha \right\rangle\!\!\right\rangle_{2|0}^{\epsilon} \mathbf{1}_{\gamma'}.$$

The identity follows from matching the formula above with (86), (89), and (90). As a consequence of (81), we know the LHS is regular at $z = 0$. □

A.3.2. *Reconstruction.* Let $\left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\ge 0}$ and $\left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{-}$ be the truncation of $I_{\mathrm{LG}}^{0,\epsilon}$ along nonnegative direction and along negative direction respectively, then

$$\left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\ge 0}(y,z) \in \overline{H[\![y]\!][z]}, \quad \left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\ge 0}(0,z) = 0.$$

We can rewrite the J^ϵ-function as

$$J^\epsilon(t,y,z) = \left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\ge 0}(y,z) + z\mathbf{1} + t(z) + \left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{-}(y,z) + \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z-\psi} \right\rangle\!\!\right\rangle_{0,1}^{\epsilon} \mathbf{1}_{\gamma'}. \tag{94}$$

We introduce multi-indices $\mathbf{m}$ and $\mathbf{n}$:

$$\mathbf{m} = (\cdots, m_i^\gamma, \cdots) \in (\mathbb{Z}_{\ge 0})^\infty, \quad \mathbf{n} = (\cdots, n_0^\gamma, \cdots) \in (\mathbb{Z}_{\ge 0})^N, \quad \forall i \ge 0, \gamma \in \mathscr{N}.$$

Here all but finitely many m_i^γ are nonzero. We adopt the notation

$$|\mathbf{m}| = \sum_{i\geq 0}\sum_{\gamma} m_i^\gamma, \quad |\mathbf{n}| = \sum_{\gamma} n_0^\gamma,$$

We define two vectors $\mathbf{m}(\alpha)$ and $\mathbf{n}(\beta)$ such that

$$\mathbf{m}(\alpha)_i^\gamma = m_i^\gamma + \delta_i^0\delta_\alpha^\gamma, \quad \mathbf{n}(\beta)_0^\gamma = n_0^\gamma + \delta_\beta^\gamma.$$

We define coefficients $A_{\mathbf{n},j\geq 0,\gamma}$ and $B_{\mathbf{m},j<0,\gamma}$ by expanding

$$\left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\geq 0}(y,z) := \sum_{\mathbf{n}}\sum_{j\geq 0}\sum_{\gamma} A^\epsilon_{\mathbf{n},j,\gamma}\, y^{\mathbf{n}}\, z^j\, \mathbf{1}_\gamma \tag{95}$$

$$\sum_{\gamma}\left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z-\psi}\right\rangle\!\!\right\rangle^{\epsilon}_{0,1}\Big|_{y=0}\mathbf{1}_{\gamma'} := \sum_{\mathbf{m}}\sum_{j\leq -1}\sum_{\gamma} B^\epsilon_{\mathbf{m},j,\gamma}\, t^{\mathbf{m}}\, z^j\, \mathbf{1}_\gamma \tag{96}$$

Let us write

$$J^\epsilon(t,y,z) := \sum_{\mathbf{n}}\sum_{\mathbf{m}}\sum_{j\in\mathbb{Z}}\sum_{\gamma} C^\epsilon_{\mathbf{m},\mathbf{n},j,\gamma}\, t^{\mathbf{m}}\, y^{\mathbf{n}}\, z^j\, \mathbf{1}_\gamma. \tag{97}$$

Definition A.5. If $C^\epsilon_{\mathbf{m},\mathbf{n},j,\gamma}$ is determined by the coefficients in (95) and (96), then we say

$$C^\epsilon_{\mathbf{m},\mathbf{n},j,\gamma} \in \mathfrak{Y}.$$

Proposition A.6. *The function $J^\epsilon(t,y,z)$ is determined by the coefficients $\{A^\epsilon_{\mathbf{n},j\geq 0,\gamma}\}$ in* (95) *and $\{B^\epsilon_{\mathbf{m},j<0,\gamma}\}$ in* (96).

Proof. Direct calculation shows

$$\begin{aligned}
&\eta_W\left(\frac{\partial}{\partial y_\alpha}J^\epsilon(t,y,-z), \frac{\partial}{\partial t_0^\beta}J^\epsilon(t,y,z)\right)\\
=\ & \sum_{\gamma}\sum_{\mathbf{n},\mathbf{n}'}\sum_{\mathbf{m},\mathbf{m}'}\sum_{j,j'}(n_0^\beta+1)(m'^\alpha_0+1)C^\epsilon_{\mathbf{m},\mathbf{n}(\beta),j,\gamma}C^\epsilon_{\mathbf{m}'(\alpha),\mathbf{n}',j',\gamma'}\ t^{\mathbf{m}+\mathbf{m}'}y^{\mathbf{n}+\mathbf{n}'}z^{j+j'}.
\end{aligned}$$

For any fixed $\mathbf{M},\mathbf{N}$, and a positive integer K, the regularity formula (92) implies

$$\sum_{\gamma}\sum_{\mathbf{n}+\mathbf{n}'=\mathbf{N}}\sum_{\mathbf{m}+\mathbf{m}'=\mathbf{M}}\sum_{j}(n_0^\beta+1)(m'^\alpha_0+1)C^\epsilon_{\mathbf{m},\mathbf{n}(\beta),j,\gamma}C^\epsilon_{\mathbf{m}'(\alpha),\mathbf{n}',-K-j,\gamma'} = 0. \tag{98}$$

We do induction as follows. Starting with $\mathbf{n}=\mathbf{0}:=(0,\cdots,0)$, we know

$$C^\epsilon_{\mathbf{m},\mathbf{0},j,\gamma}\in\mathfrak{Y}$$

since for any $\mathbf{m}$ and γ, we have

$$C^\epsilon_{\mathbf{m},\mathbf{0},j,\gamma} = \begin{cases} B^\epsilon_{\mathbf{m},j,\gamma}, & \textit{if} \quad j<0;\\ 1, & \textit{if} \quad j=1, \mathbf{m}=\mathbf{0}, \mathbf{1}_\gamma=\mathbf{1};\\ 1, & \textit{if} \quad j\geq 0, m_i^\xi = \delta_i^j\delta_\gamma^\xi;\\ 0, & \textit{otherwise}.\end{cases} \tag{99}$$

For any integer $n_0\in\mathbb{Z}_{\geq 0}$, assume that if $|\mathbf{n}|\leq n_0$, then

$$C^\epsilon_{\mathbf{m},\mathbf{n},j,\gamma}\in\mathfrak{Y}, \quad \forall j\in\mathbb{Z}, \gamma\in\mathscr{N}. \tag{100}$$

Now we fix $|\mathbf{N}| = n_0$ and consider the coefficient

$$C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),j,\gamma}, \quad \forall \beta \in \mathscr{N}.$$

We notice that $|\mathbf{N}(\beta)| = n_0 + 1$. By the definitions in (95) and (97),we know that

$$C^{\epsilon}_{\mathbf{m},\mathbf{N}(\beta),j,\gamma} = \begin{cases} 0, & \textit{if} \quad j \geq 0, \mathbf{m} \neq 0. \\ A_{\mathbf{N}(\beta),j,\gamma}, & \textit{if} \quad j \geq 0, \mathbf{m} = 0. \end{cases} \tag{101}$$

We do induction on the positive integer K by assuming

$$C^{\epsilon}_{\mathbf{m},\mathbf{N}(\beta),j,\gamma} \in \mathfrak{Y}, \quad \forall -K - j < 0. \tag{102}$$

This is true when $K = 1$ by (101). Thus it is enough to prove for all $\mathbf{M}$ and $\alpha' \in \mathscr{N}$,

$$C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),-K,\alpha'} \in \mathfrak{Y}.$$

In order to prove this, we rewrite equation (98) as

$$\begin{aligned} 0 \;\; = \;\; & \sum_{\gamma} \sum_{j\in\mathbb{Z}} (N_0^{\beta} + 1)\, C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),j,\gamma} C^{\epsilon}_{\mathbf{0}(\alpha),\mathbf{0},-K-j,\gamma'} \\ & + \sum_{\gamma} \sum_{\mathbf{m}\neq\mathbf{M}} \sum_{j\in\mathbb{Z}} (N_0^{\beta} + 1)({m'}^{\alpha}_0 + 1) C^{\epsilon}_{\mathbf{m},\mathbf{N}(\beta),j,\gamma} C^{\epsilon}_{\mathbf{m}'(\alpha),\mathbf{0},-K-j,\gamma'} \\ & + \sum_{\gamma} \sum_{\mathbf{n}\neq\mathbf{N}} \sum_{\mathbf{m}+\mathbf{m}'=\mathbf{M}} \sum_{j\in\mathbb{Z}} (n_0^{\beta} + 1)({m'}^{\alpha}_0 + 1) C^{\epsilon}_{\mathbf{m},\mathbf{n}(\beta),j,\gamma} C^{\epsilon}_{\mathbf{m}'(\alpha),\mathbf{n}',-K-j,\gamma'}. \end{aligned} \tag{103}$$

Let us analyze the RHS of (103) line by line. From (99), we observe that

$$C^{\epsilon}_{\mathbf{0}(\alpha),\mathbf{0},-K-j,\gamma'} = \begin{cases} B^{\epsilon}_{\mathbf{0}(\alpha),-K-j,\gamma'} \in \mathfrak{Y}, & \textit{if} \quad -K - j < 0, \\ \delta^{\alpha}_{\gamma'}, & \textit{if} \quad -K - j = 0, \\ 0, & \textit{if} \quad -K - j > 0. \end{cases}$$

When $-K - j < 0$, by induction (102), $C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),j,\gamma} \in \mathfrak{Y}$. The first line of the RHS of (103) is a sum the target term $(N_0^{\beta} + 1)\, C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),-K,\alpha'}$ and

$$\sum_{\gamma} \sum_{-K-j<0} (N_0^{\beta} + 1)\, C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),j,\gamma} C^{\epsilon}_{\mathbf{0}(\alpha),\mathbf{0},-K-j,\gamma'} \in \mathfrak{Y}.$$

The second line belongs to $\mathfrak{Y}$, since by (102),

$$C^{\epsilon}_{\mathbf{m},\mathbf{N}(\beta),j,\gamma} \in \mathfrak{Y}, \quad \forall -K - j < 0,$$

and for $\mathbf{m}' \neq \mathbf{0}$,

$$C^{\epsilon}_{\mathbf{m}'(\alpha),\mathbf{0},-K-j,\gamma'} = \begin{cases} 0, & \textit{if} \quad -K - j \geq 0, \\ B^{\epsilon}_{\mathbf{m}'(\alpha),-K-j,\gamma'} \in \mathfrak{Y}, & \textit{if} \quad -K - j < 0. \end{cases}$$

Since $\mathbf{n} \neq \mathbf{N}$, the formula (100) implies that the third line of (103) is in $\mathfrak{Y}$.

Since $N_0^{\beta} + 1 \neq 0$ and $\alpha \in \mathscr{N}$ is arbitrary, we know $C^{\epsilon}_{\mathbf{M},\mathbf{N}(\beta),-K,\alpha'} \in \mathfrak{Y}$. This finishes the induction argument on (102). □

Now we conclude a proof of Proposition 7.3.

Proof. Let

$$\widetilde{t}_\epsilon(z) := t(z) + \left[I_{\mathrm{LG}}^{0,\epsilon}\right]_{\geq 0}(y, z)$$

and then consider

$$J(\widetilde{t}_\epsilon, z) = z\mathbf{1} + \widetilde{t}_\epsilon(z) + \sum_\gamma \left\langle\!\!\left\langle \frac{\mathbf{1}_\gamma}{z - \psi} \right\rangle\!\!\right\rangle_{0,1}^{\infty} (\widetilde{t}_\epsilon)\ \mathbf{1}_\gamma.$$

Using the same method in Proposition A.4, we can check

$$\eta_W\left(\frac{\partial}{\partial y_\beta} J(\widetilde{t}_\epsilon, -z), \frac{\partial}{\partial t_0^\alpha} J(\widetilde{t}_\epsilon, z)\right) \in \mathbb{C}[\![z]\!].$$

Thus the function $J(\widetilde{t}_\epsilon, z)$ satisfies the same reconstruction procedure as the function $J^\epsilon(t, y, z)$ in Proposition A.6. Moreover, the initial reconstruction data (see (95) and (96)) are identical for both functions. This implies that

$$J^\epsilon(t, y, z) = J(\widetilde{t}_\epsilon, z).$$

On the other hand, the function $J(\widetilde{t}_\epsilon, -z)$ is an $\mathcal{H}[\![t, y]\!]$-point on the Lagrangian cone $\mathcal{L}$. Thus when we let $t(z) = 0$ and $\epsilon \to 0$, the last two terms in (91) vanish. In particular, the second term vanishes due to the unstability condition $1 + n\epsilon \leq 2$. The result follows by choosing an appropriate completion. □

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