Higher Kac–Moody algebras and moduli spaces of G-bundles

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Abstract

We provide a generalization to the higher dimensional case of the construction of the current algebra $\mathfrak{g}((z))$, its Kac-Moody extension $\widetilde{\mathfrak{g}}$ and of the classical results relating them to the theory of G-bundles over a curve. For a reductive algebraic group G with Lie algebra $\mathfrak{g}$, we define a dg-Lie algebra $\mathfrak{g}_n$ of n-dimensional currents in $\mathfrak{g}$. For any symmetric G-invariant polynomial P on $\mathfrak{g}$ of degree $n+1$, we get a higher Kac–Moody algebra $\widetilde{\mathfrak{g}}_{n,P}$ as a central extension of $\mathfrak{g}_n$ by the base field $\mathbf{k}$. Further, for a smooth, projective variety X of dimension $n \geqslant 2$, we show that $\mathfrak{g}_n$ acts infinitesimally on the derived moduli space $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ of G-bundles over X trivialized at the neighborhood of a point $x \in X$. Finally, for a representation $\phi : G \to GL_r$ we construct an associated determinantal line bundle on $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ and prove that the action of $\mathfrak{g}_n$ extends to an action of $\widetilde{\mathfrak{g}}_{n,P_\phi}$ on such bundle for P_ϕ the $(n+1)^{\mathrm{th}}$ Chern character of ϕ.

Contents

0 Introduction

(0.1) Let G be a reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. The formal current algebra $\mathfrak{g}((z)) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ and its central extension $\widetilde{\mathfrak{g}}$ (the Kac-Moody algebra) play a fundamental role in many fields. It can be considered as the algebraic completion of the loop algebra $\mathrm{Map}(S^1, \mathfrak{g})$, see [PreS].

In particular, $\mathfrak{g}((z))$ is fundamental in the study of $\mathbf{Bun}_G(X)$, the moduli stack of principal G-bundles on a smooth projective curve X over $\mathbb{C}$. More precisely, let $x \in X$ be a point. We then have the scheme (of infinite type) $\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$ parametrizing bundles P together with a trivialization on $\widehat{x}$, the formal neighborhood of x. The ring of functions on $\widehat{x}$ is $\widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[z]]$, the completed local ring and its field of fractions $K_x \simeq \mathbb{C}((z))$ corresponds to the punctured formal neighborhood $\widehat{x}^\circ$.

The key result [KNTY] is that the Lie algebra $\mathfrak{g}_x = \mathfrak{g} \otimes K_x$ acts on the scheme $\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$ by vector fields. Moreover, any representation ϕ of G gives rise to the determinantal line bundle $\det^\phi$ on $\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$; the action of $\mathfrak{g}_x$ extends to the action, on $\det^\phi$, of the central extension $\widetilde{\mathfrak{g}}_x$ with central charge given by a local version of the Riemann-Roch theorem for curves.

(0.2) Our goal in this paper is to generalize these results from curves to n-dimensional varieties X over $\mathbb{C}$, $n \geqslant 1$ (one can replace $\mathbb{C}$ by any field of characteristic 0). The first question in this direction is what should play the role of $\mathfrak{g}((z))$. In the analytic (as opposed to the formal series) theory, natural generalizations of $\mathrm{Map}(S^1, \mathfrak{g})$ are provided by the current Lie algebras $\mathrm{Map}(\Sigma, \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} C^\infty(\Sigma)$ where Σ is a compact C^∞-manifold of dimension > 1. Our approach can be seen as extending this idea to the derived category.

More precisely, the role of $\mathfrak{g}((z))$ will be played by the dg-Lie algebra $\mathfrak{g}_n^\bullet = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{A}_n^\bullet$, where $\mathfrak{A}_n^\bullet = R\Gamma(D_n^\circ, \mathcal{O})$ is the commutative dg-algebra of derived global sections of the sheaf $\mathcal{O}$ on the n-dimensional *punctured formal disk* $D_n^\circ = \mathrm{Spec}\big(\mathbb{C}[[z_1, \ldots, z_n]]\big) - \{0\}$. More invariantly, we have the punctured formal disk $\widehat{x}^\circ \simeq D_n^\circ$ associated to a point $x \in X$ and the corresponding current algebra $\mathfrak{g}_x^\bullet \simeq \mathfrak{g}_n^\bullet$. For $n > 1$, passing from the non-punctured formal disk $\widehat{x}$, to $\widehat{x}^\circ$ does not increase the ring of functions (Hartogs' theorem) but one gets new elements in the higher cohomology of the sheaf $\mathcal{O}$, so $\mathfrak{A}_n^\bullet$ can be regarded as a "higher" generalization of the Laurent series field $\mathbb{C}((z))$, to which it reduces for $n = 1$.

Principal bundles on X form an Artin stack $\mathbf{Bun}_G(X)$ and we can still form a scheme $\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$ as above. However these objects are, for $n > 1$, highly singular because deformation theory can be obstructed. The correct object to consider is the *derived moduli stack* $\mathbb{R}\mathbf{Bun}_G(X)$ obtained, informally, by taking the non-abelian derived functor of $\mathbf{Bun}_G$, i.e., extending the moduli functor to test rings which are commutative dg-algebras [TV]. When X is a curve, $\mathbb{R}\mathbf{Bun}_G(X) \simeq \mathbf{Bun}_G(X)$, but for $n > 1$ there is a difference. Most importantly, the tangent complex of $\mathbb{R}\mathbf{Bun}_G(X)$ is perfect (a smoothness property). We can similarly construct the *derived scheme* $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$, (an object which locally looks like the spectrum of a commutative dg-algebra) which should also be intuitively considered as being smooth.

We show, first of all, (Theorem 5.3.8) that $\mathfrak{g}_n^\bullet$ acts on $\mathbb{R}\mathbf{Bun}^{\mathrm{rig}}(X, x)$ by vector fields, in the derived sense. At the level of cohomology, the action gives, in particular, a map

$$H^{n-1}_{\overline{\partial}}(\mathfrak{g}_n^\bullet) \longrightarrow \mathbb{H}^{n-1}\big(\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X, x), \mathbb{T}\big)$$

(here $\mathbb{T}$ is the tangent complex and $\overline{\partial}$ is the differential of $\mathfrak{g}_n^\bullet$). When $n = 1$, it is the action by vector fields in the usual sense. In the first new case $n = 2$, after restricting to the non-obstructed smooth part of the moduli space, on which $\mathbb{T}$ is the usual tangent sheaf, the target of this map becomes the *space of deformations* of the (part of the) moduli space. Deforming the moduli space can be understood as changing the cocycle condition defining G-bundles (Remark 5.3.9).

Further, each invariant polynomial P on $\mathfrak{g}$ of degree $(n+1)$ gives rise to a central extension $\widetilde{\mathfrak{g}}_{n,P}^\bullet$ (the *higher Kac-Moody algebra*). Note that unlike the case $n = 1$, we now have many non-proportional classes, even for $\mathfrak{g}$ simple. Intuitively, they correspond to degree $n+1$ characteristic classes for principal G-bundles. As before, let ϕ be a representation of G. We prove (Theorem 5.5.9) that the determinantal line bundle $\det^\phi$ on $\mathbb{R}\mathbf{Bun}^{\mathrm{rig}}(X, x)$ is acted upon by $\widetilde{\mathfrak{g}}_{n,P_\phi}^\bullet$ where $P_\phi(x) = \mathrm{tr}(\phi(x)^{n+1})/(n+1)!$ is the "$(n+1)$-th component of the Chern character" of ϕ.

These results suggest that representations of the dg-Lie algebra $\mathfrak{g}_n^\bullet$ should produce geometric data on the derived moduli spaces of G-bundles on n-dimensional manifolds.

(0.3) The stack $\mathbf{Bun}_G(X)$ can be seen as a version of the non-abelian first cohomology $H^1(X, G(\mathcal{O}_X))$. When X is a curve, the above classical results can be seen as forming a part of the "adelic approach" to the geometry of curves. This approach consists in using the idealized "Čech covering" of X formed by $\widehat{x}$ and $X^\circ = X - \{x\}$, with "intersection" $\widehat{x}^\circ$, to calculate the H^1. If X is a curve and G is semi-simple, then G-bundles on X° (and certainly on $\widehat{x}$) are trivial, and we can write $\mathbf{Bun}_G(X) = G(\widehat{x})\backslash G(\widehat{x}^\circ)/G(X^\circ)$ (stack-theoretic quotient on the left). We then similarly represent $\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$ as the coset space $G(\widehat{x}^\circ)/G(X^\circ)$, with $G(\widehat{x}^\circ) = G(K_x)$ being a group ind-scheme with Lie algebra $\mathfrak{g}_x$.

A generalization of the adelic formalism to varieties X of dimension $n > 1$ was proposed by Parshin and Beilinson [Be1] [Hu] [Os]. In this approach the completed local fields (analogs of K_x for curves) are parametrized not by points, but by flags $\{x\} = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X$ of irreducible subvarieties in X. If all the X_i's are smooth, then the completion is isomorphic to $\mathbf{k}((z_1))\cdots((z_n))$, the iterated Laurent series field. As before, it can be seen as a version of the Čech formalism for an idealized open covering formed by certain formal neighborhoods. However, the manipulations with iterated Laurent series fields are quite complicated: in order to capture all the "adic topologies", they should be considered as n-fold iterated ind-pro-objects (n-Tate spaces) [BGW] and every step involves many levels of technical work.

In a sense, our approach can be seen as a "simplified version" of the flag adele formalism, in which we keep track only of points $x \in X$ (just like for curves) and package all the combinatorial data involving subvarieties of dimensions $\neq 0, n$, into a "black box" using the cohomological formalism. This allows us to avoid working with iterated ind-pro-objects and deal instead with classical Tate spaces (just like for curves) at the small price of having to pass to the derived category of such spaces, i.e., to study *Tate dg-spaces* (or *Tate complexes*), see §4.1 for details. For example, $\mathfrak{A}_n^\bullet$ is a Tate complex for each n. Our treatment is an adaptation and development of the approaches of [Dr] [He3].

(0.4) To relate our approach to the idea of $\mathrm{Map}(\Sigma, \mathfrak{g})$, we can use a particular model $A_n^\bullet$ of the "abstract" commutative dg-algebra $\mathfrak{A}_n^\bullet = R\Gamma(D_n^\circ, \mathcal{O})$. This model is formed by relative differential forms on the Jouanolou torsor, see §1.2B. Such torsors have been used in [BD] as a general tool. In our case, $A_n^\bullet$ provides a very precise algebraic analog of $\Omega^{0,\bullet}(\mathbb{C}^n - \{0\})$, the $\overline{\partial}$-algebra of Dolbeault forms on $\mathbb{C}^n - \{0\}$. In particular, such features of classical complex analysis as the Martinelli-Bochner form or its "multipole" derivatives, have direct incarnations in $A_n^\bullet$, see Proposition 1.4.7. Our algebraic approach allows us to include these features in the formal setting of Tate (dg-)spaces. It also lends itself to a representation-theoretic analysis providing the analog of representing elements of $\mathbf{k}((z))$ as infinite sums of monomials (Theorem 1.4.2).

Restricting from $\mathbb{C}^n - \{0\}$ to the unit sphere S^{2n-1}, we can see $A_n^\bullet$ as an algebraic analog of $\Omega_b^{0,\bullet}(S^{2n-1})$, the tangential Cauchy-Riemann complex (the $\overline{\partial}_b$-complex [BeGr][DT]) of the sphere. The degree 0 part of $\Omega_b^{0,\bullet}(S^{2n-1})$ is $C^\infty(S^{2n-1})$, the algebra of smooth complex functions on S^{2n-1}. This means that the degree 0 part of $\mathfrak{g}_n^\bullet$ can be seen as an algebraic version of the current Lie algebra $\mathrm{Map}(S^{2n-1}, \mathfrak{g})$, and the entire $\mathfrak{g}_n^\bullet$ as a natural derived thickening of this current algebra.

Usually, considering $\mathrm{Map}(\Sigma, \mathfrak{g})$ with $\dim(\Sigma) > 1$, produces Lie algebras which, instead of interesting central extensions (classes in H^2) have interesting higher cohomology classes. These classes are typically given by versions of the formula

$$\gamma(f_0, \dots, f_n) \;=\; \int \mathrm{tr}(f_0\, df_1 \cdots df_n). \tag{0.5}$$

In our case (Theorem 3.2.1), we still use a version of this formula (with integration over S^{2n-1}, done algebraically) but the classes we get have total degree 2 and so give central extensions, regardless of n. This happens because we take into account the grading on the dg-algebra, In this sense our derived approach embeds $\mathrm{Map}(S^{2n-1}, \mathfrak{g})$ into an object whose properties are closer to those of $\mathrm{Map}(S^1, \mathfrak{g})$.

All this suggests that our higher Kac-Moody algebras should have an interesting representation theory.

(0.6) As in the 1-dimensional case, a key intermediate step for us is a local analog of the Riemann-Roch theorem (Corollary 4.3.10). It has the form of comparison of two central extensions of $\mathfrak{g}_n^\bullet$ for $\mathfrak{g} = \mathfrak{gl}(r)$: one given by a version of (0.5), the other induced from the "Tate class" of the the endomorphism dg-algebra of the Tate complex $(\mathfrak{A}_n^\bullet)^{\oplus r}$. For $r = 1$ this statement can be seen as an analog, in our simplified adelic formalism, of the main result of Beilinson [Be1]. Since we deal with the current algebras only, we detect only the Chern character; the Todd genus will naturally appear, as in [FT2], after we include the dg-algebra $R\Gamma(D_n^\circ, \mathbb{T})$, see (0.8) below.

(0.7) We use three main technical tools. The first one is the general theory of derived stacks [TV]. It is necessary for us to work freely with quite general derived stacks and even prestacks in order to study, for instance, the group object corresponding to $\mathfrak{g}_n^\bullet$. This is an (infinite dimensional) derived group $G(D_n^\circ)$, see Proposition 5.3.6. In particular, for dealing with various infinitesimal constructions (even such seemingly simple ones as "passing from a group to its Lie algebra") we need to use Lurie's formalism of formal moduli problems [Lu3].

The second technical tool is cyclic homology of dg-categories, a concept of great flexibility and invariance. It includes, in particular cyclic (and de Rham) homology for schemes and at the same time, is related to the Lie algebra homology of endomorphism dg-algebras of objects in a dg-category.

Another important tool is the GL_n-invariance of our cohomology classes. In exploiting this invariance, the Jouanolou model for $\mathfrak{A}_n^\bullet$ is more convenient in that it allows an explicit GL_n-action which can be analyzed in detail at the level of complexes. This level of detail is not available for more abstract models, e.g., for the flag-adelic one.

(0.8) This paper is related to the idea, mentioned already in [BD] and developed in [FG], of generalizing the theory of chiral and holomorphic factorization algebras to higher dimensions. In this approach we are dealing with finite collections (x_i) of points moving in an n-dimensional variety X, with "singularities" developing when $x_i = x_j$ for some $i \neq j$. These singularities are of cohomological nature, representing classes in $H^{n-1}(X^N - \{\text{diag}\}, \mathcal{O})$.

Note that the standard quantum field theory approach deals with collections of points in the Minkowski space with singularities developing when some $x_i - x_j$ lies on the light cone, see, e.g., [Tam] for a discussion from the factorization algebra point of view. However, our cohomological formulas in the Jouanolou model are, up to "details" involving the cohomological grading, algebraically similar to this. For example, the role of the standard "propagator" $1/\|z\|^2$ is played by the Martinelli-Bochner form $\Omega(z, z^*)$.

The next natural step in this direction would be to study the dg-Lie algebra $R\Gamma(D_n^\circ, \mathbb{T})$, the analog of the Witt algebra of formal vector fields on the circle (as well as its central extensions). It should act on the derived moduli stack of n-dimensional rigidified complex manifolds $(X, x, (z_1, \dots, z_n))$ where $x \in X$ is a marked point and $(z_1, \dots, z_n)$ is a formal coordinate system near x. There is a natural combined version involving the derived Atiyah algebra ($R\Gamma$ of the semidirect product of matrix functions and vector fields). We plan to address these issues in a future paper. The additional technical difficulty here is the need to work with (quite general) derived stacks in the analytic context, as not all deformations of an algebraic variety are algebraic.

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1 Derived analogs of functions and series

1.1 Derived adelic formalism

A. Local part. We fix a base field $\mathbf{k}$ of characteristic 0. For $n \geqslant 1$ we have the n-dimensional formal disk $D_n = \operatorname{Spec} \mathbf{k}[[z_1, \dots, z_n]]$ and the punctured formal disk $D_n^\circ = D_n - \{0\}$. We consider them as the completion of the affine space $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}[z_1, \dots, z_n]$ and of the punctured affine space $\mathring{\mathbb{A}}^n = \mathbb{A}^n - \{0\}$.

Fundamental for us will be the commutative dg-algebras

$$(\mathfrak{A}_n^\bullet, \overline{\partial}) = R\Gamma(D_n^\circ, \mathcal{O}), \quad (\mathfrak{A}_{[n]}^\bullet, \overline{\partial}) = R\Gamma(\mathring{\mathbb{A}}^n, \mathcal{O}).$$

defined uniquely up to quasi-isomorphism. The cohomology of these dg-algebras is well known and can be obtained using the covering of D_n° by n affine open subsets $\{z_i \neq 0\}$.

Proposition 1.1.1. *For $n = 1$ the scheme D_1° is affine with ring of functions*

$\mathbf{k}((z))$ *and* $\mathring{\mathbb{A}}^1$ *is affine with ring of functions* $\mathbf{k}[z, z^{-1}]$*. For* $n > 1$ *we have*

$$H^i(D_n^\circ, \mathcal{O}) \;=\; \begin{cases} \mathbf{k}[[z_1, \ldots, z_n]], & i = 0; \\ z_1^{-1} \cdots z_n^{-1} \mathbf{k}[z_1^{-1}, \ldots, z_n^{-1}], & i = n-1; \\ 0, & \textit{otherwise.} \end{cases}$$

Here the notation $z_1^{-1} \cdots z_n^{-1} \mathbf{k}[z_1^{-1}, \ldots, z_n^{-1}]$ *can be seen as encoding the action of the* n*-dimensional torus* $\mathbb{G}_m^n$ *on* $H^{n-1}(D_n^\circ, \mathcal{O})$*.*

The cohomology $H^i(\mathring{\mathbb{A}}^n, \mathcal{O})$ *differs from the above by replacing* $\mathbf{k}[[z_1, \ldots, z_n]]$ *by* $\mathbf{k}[z_1, \ldots, z_n]$*.*

Thus, although for $n > 1$, the scheme D_n°. resp. $\mathring{\mathbb{A}}^n$, is not affine and its global functions are the same as for D_n, resp. $\mathbb{A}^n$, the missing "polar parts" are recovered in the higher cohomology of the sheaf $\mathcal{O}$. The dg-algebras $\mathfrak{A}_{[n]}$ and $\mathfrak{A}_n$ are, therefore, correct n-dimensional generalizations of the the rings of Laurent polynomials and Laurent series in one variable.

We will also use the doubly graded dg-algebras

$$(\mathfrak{A}_n^{\bullet\bullet}, \partial, \overline{\partial}) = \bigoplus_p R\Gamma(D_n^\circ, \Omega^p), \quad (\mathfrak{A}_{[n]}^{\bullet\bullet}, \partial, \overline{\partial}) = \bigoplus_p R\Gamma(\mathring{\mathbb{A}}^n, \Omega^p),$$

with ∂ being induced by the de Rham differential on forms (and increasing the first grading) and $\overline{\partial}$ being the differential on $R\Gamma$ (and increasing the second grading).

B. Global part. Let X be a smooth n-dimensional variety over $\mathbf{k}$ and $x \in X$ a $\mathbf{k}$-point. We then have the completed local ring $\widehat{\mathcal{O}}_{X,x}$ which is isomorphic (non-canonically) to $\mathbf{k}[[z_1, \ldots, z_n]]$. We denote $\widehat{x} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ the formal disk near x and by $\widehat{x}^\circ = \widehat{x} - \{x\}$ the punctured formal disk near x. We then form the commutative dg-algebras

$$\mathfrak{A}_x^\bullet = R\Gamma(\widehat{x}^\circ, \mathcal{O}), \quad \mathfrak{A}_x^{\bullet\bullet} = \bigoplus_p R\Gamma(\widehat{x}^\circ, \Omega^p).$$

In particular, the Grothendieck duality defines a canonical linear functional

$$\text{(1.1.2)} \qquad \operatorname{Res}_x : R\Gamma(\widehat{x}^\circ, \Omega^n) \xrightarrow{\delta} R\Gamma_{\{x\}}(\widehat{x}, \Omega^n)[1] \longrightarrow \mathbf{k}[1-n].$$

Let now $\mathbf{x} = \{x_1, \ldots, x_m\} \subset X$ be a finite set of disjoint $\mathbf{k}$-points. We denote $X^\circ = X - \mathbf{x}$ the complement of $\mathbf{x}$, and write $\widehat{\mathbf{x}} = \bigsqcup \widehat{x}_i$ and $\widehat{\mathbf{x}}^\circ = \bigsqcup \widehat{x}_i^\circ$.

In particular, we have the commutative dg-algebra $R\Gamma(X^\circ, \mathcal{O})$. Elements of this dg-algebra can be seen as n-dimensional analogs of rational functions on a curve with poles in $x_1, \dots, x_m$. Similarly, if E is a vector bundle on X, we have the dg-modules $\mathfrak{A}^\bullet_{\mathbf{x}}(E) = R\Gamma(\widehat{\mathbf{x}}^\circ, E)$ over $\mathfrak{A}^\bullet_{\mathbf{x}} = R\Gamma(\widehat{\mathbf{x}}^\circ, \mathcal{O})$ and $R\Gamma(X^\circ, E)$ over $R\Gamma(X^\circ, \mathcal{O})$. We have the canonical morphism of complexes (commutative dg-algebras for $E = \mathcal{O}$) given by the restriction:

$$\delta : R\Gamma(X^\circ, E) \oplus \Gamma(\widehat{\mathbf{x}}, E) \longrightarrow \mathfrak{A}_{\mathbf{x}}(E) \;=\; \bigoplus_{i=0}^m \mathfrak{A}^\bullet_{x_i}(E). \tag{1.1.3}$$

This morphism can be seen as the dg-version of the adelic complex of Beilinson [Be1] [Hu] , in which the dependence on schemes of dimensions $1, \dots, n-1$ has been "integrated away" and hidden in the cohomological formalism. The proof of the next proposition can be seen as an explicit comparison.

Proposition 1.1.4. *The cone of δ is identified with $R\Gamma(X, E)$.*

Proof: For any Noetherian scheme Y of dimension n over $\mathbf{k}$ (proper or not) and any coherent sheaf $\mathcal{F}$ on Y, the construction of [Be1][Hu] provides an explicit model $C^\bullet(\mathcal{F})$ for $R\Gamma(Y, \mathcal{F})$. By definition,

$$C^p(\mathcal{F}) \;=\; \bigoplus_{0 \leqslant i_0 < \cdots < i_p \leqslant n} C_{i_0, \dots, i_p}(\mathcal{F}),$$

where

$$C_{i_0, \dots, i_p}(\mathcal{F}) \;=\; \prod_{\substack{Y_0 \subset \cdots \subset Y_p \subset Y \\ \dim(Y_\nu) = i_\nu}}{}^{'} \; C_{Y_0, \dots, Y_p}(\mathcal{F})$$

is the appropriate restricted product, over all flags $Y_0 \subset \cdots \subset Y_p \subset Y$ of irreducible subschemes, of the completions $C_{Y_0, \dots, Y_p}(\mathcal{F})$ as described in [Be1][Hu]. In particular, for $p = 0$ and $i_0 = 0$, the summand $C_0(\mathcal{F})$ is the usual product of $C_y(\mathcal{F}) = \Gamma(\widehat{y}, \mathcal{F})$ over all 0-dimensional points $y \in Y$. We now take $Y = X$, $\mathcal{F} = E$ and represent

$$C^\bullet(\mathcal{F}) \;=\; \mathrm{Cone}\big\{ C_1^\bullet \oplus C_2^\bullet \xrightarrow{d} C_3^\bullet \big\},$$

identifying the three summands with the corresponding summands in (1.1.3) and d with δ. Explicitly, we take C_2 to be the direct sum of $C_y(\mathcal{F})$ over $y \in \mathbf{x}$, we take $C_1^\bullet$ to be the direct sum of restricted products of $C_{Y_0, \dots, Y_p}(\mathcal{F})$ over all flags $Y_0 \subset \cdots \subset Y_p$ such that Y_0 is any subvariety (of any dimension) other than some $y \in \mathbf{x}$. Then $C_2^\bullet$ is the adelic complex for the restriction of E $Y = X - \mathbf{x}$. Similarly, we take C_3 to be the direct sum of restricted products of $C_{Y_0,\dots,Y_p}(\mathcal{F})$ over all flags $Y_0 \subset \cdots \subset Y_p$ such that Y_0 equals some $y \in \mathbf{x}$. Then $C_3^\bullet$ is the adelic complex for the restriction of E to $\widehat{\mathbf{x}}$. This proves the statement. □

In particular, for $E = \Omega_X^n$, we have morphisms

$$\mathrm{Res}_{x_i,X} : R\Gamma(X^\circ, \Omega^n) \to \mathbf{k}[1-n], \quad i = 1, \dots, m,$$

which satisfy the *residue theorem*:

Proposition 1.1.5. $\sum_{i=1}^m \mathrm{Res}_{x_i,X} = 0.$ □

Proof: This is a standard feature of Grothendieck duality, cf. [Con]. By degree considerations it suffices to look at the behavior on the $(n-1)$-st cohomology only. That is, we consider, for each i, the map $H^{n-1}(\mathrm{Res}_i) : H^{n-1}(\widehat{x}_i^\circ, \Omega^n) \to \mathbf{k}$ induced by Res_{x_i} on the $(n-1)$st cohomology, and prove that the compositions of these maps with the $H^{n-1}(X^\circ, \Omega^n) \to H^{n-1}(\widehat{x}_i^\circ, \Omega^n)$ sum to zero.

Indeed, $H^{n-1}(\mathrm{Res}_i)$ can be represented as the composition

$$H^{n-1}(\widehat{x}_i^\circ, \Omega^n) \simeq H^n_{\{x_i\}}(\widehat{x}_i, \Omega^n) = H^n_{\{x_i\}}(X, \Omega^n) \xrightarrow{\iota_{x_i}} H^n(X, \Omega^n) \xrightarrow{\mathrm{tr}} \mathbf{k},$$

where tr is the global Serre duality isomorphism. Now the statement follows from the fact that in the long exact sequence relating cohomology with and without support in $\mathbf{x}$,

$$\cdots \to H^{n-1}(X^\circ, \Omega^n) \xrightarrow{\delta} H^n_{\mathbf{x}}(X, \Omega^n) \xrightarrow{\iota_{\mathbf{x}} = \sum \iota_{x_i}} H^n(X, \Omega^n) \to \cdots$$

the composition of any two consecutive arrows is zero. □

1.2 Explicit models

We start with the "polynomial" dg-algebra $\mathfrak{A}^\bullet_{[n]}$. By considering the fibration $\mathring{\mathbb{A}}^n \to \mathbb{P}^{n-1}$, we can write

$$\mathfrak{A}^\bullet_{[n]} \sim \bigoplus_{i \in \mathbb{Z}} R\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(i)). \tag{1.2.1}$$

From here, passing to the completion is easy: it is similar to passing from Laurent polynomials in one variable to Laurent series. More precisely, for a graded vector space $\bigoplus_{i\in\mathbb{Z}} V^i$ we denote

$$\sum_{i\gg-\infty} V^i \;=\; \varinjlim_a \varprojlim_b \bigoplus_{i=-a}^{b} V^i$$

the vector space formed by Laurent series $\sum_{i\gg-\infty} v_i$ with $v_i \in V^i$. Then

$$\mathfrak{A}_n^\bullet \;\sim\; \sum_{i\gg-\infty} R\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(i)) \tag{1.2.2}$$

Applying various way of calculating the cohomology of $\mathbb{P}^{n-1}$, we get various explicit models for $\mathfrak{A}_n$ and related dg-algebras and modules.

A. The Čech model. Covering $\mathbb{P}^{n-1}$ with open sets $\{z_i \neq 0\}$, or, what is the same, covering D_n° with similar open sets right away, we get a model for $\mathfrak{A}_n^\bullet$ as the Čech complex of this covering. The Alexander-Whitney multiplication makes this complex into an *associative but not commutative* dg-algebra.

We can use Thom-Sullivan forms to replace this by a commutative dg-algebra model for $\mathfrak{A}_n^\bullet$.

B. The Jouanolou model. Introduce another set of variables $z_1^*, \dots, z_n^*$ which we think of as dual to the z_ν, i.e., as the coordinates in the dual affine space $\check{\mathbb{A}}^n$. We write

$$zz^* \;=\; \sum z_\nu z_\nu^*, \quad z \in \mathbb{A}^n,\ z^* \in \check{\mathbb{A}}^n.$$

We form the corresponding "dual" projective space $\check{\mathbb{P}}^n = \operatorname{Proj} \mathbf{k}[z_1^*, \dots, z_n^*]$ and consider the incidence quadric

$$Q \;=\; \left\{(z, z^*) \in \mathbb{A}^n \times \check{\mathbb{P}}^n \,\middle|\, zz^* = 0\right\} \;\subset\; \mathbb{A}^n \times \check{\mathbb{P}}^n.$$

We denote the complement $(\mathbb{A}^n \times \check{\mathbb{P}}^n) - Q$ by J and note that the projection to the first factor gives a morphism

$$\pi : J \longrightarrow \mathring{\mathbb{A}}^n$$

whose fibers are affine spaces of dimension $n-1$. We refer to J as the *Jouanolou torsor* for $\mathring{\mathbb{A}}^n$. For further reference let us point out that

$$J \;\simeq\; \big\{(z_1,\dots,z_n,z_1^*,\dots,z_n^*)\in \mathbb{A}^n\times(\check{\mathbb{A}}^n-\{0\})\mid zz^*=1\big\}, \tag{1.2.3}$$

the isomorphism given by the projection $\check{\mathbb{A}}^n-\{0\}\to\check{\mathbb{P}}^n$ on the second factor.

For any quasi-coherent sheaf E on $\mathring{\mathbb{A}}^n$ we then have the global relative de Rham complex

$$A^\bullet_{[n]}(E) \;=\; \Gamma\big(J,\Omega^\bullet_{J/\mathring{\mathbb{A}}^n}\otimes\pi^*E\big).$$

The differential in $A^\bullet_{[n]}(E)$ (given by the relative de Rham differential) will be denoted $\overline{\partial}$.

Let also

$$\widehat{J} \;=\; J\times_{\mathbb{A}^n} D_n \;=\; J\times_{\mathring{\mathbb{A}}^n} D_n^\circ \xrightarrow{\widehat{\pi}} D_n^\circ$$

be the restriction of J to the punctured formal disk. As before, $\widehat{J}$ is an affine scheme and an $\mathbb{A}^{n-1}$-torsor over D_n°. For any quasi-coherent sheaf E on D_n° we denote

$$A^\bullet_n(E) \;=\; \Gamma\big(\widehat{J},\Omega^\bullet_{\widehat{J}/D_n^\circ}\otimes\widehat{\pi}^*E\big).$$

Proposition 1.2.4. *(a) $A^\bullet_{[n]}(E)$ is a model for $R\Gamma(\mathring{\mathbb{A}}^n,E)$, and $A^\bullet_n(E)$ is a model for $R\Gamma(D_n^\circ,E)$.*

(b) The functor $E\mapsto A^\bullet_{[n]}(E)$ (resp. $E\mapsto A^\bullet_n(E)$) is a lax symmetric monoidal functor from the category of quasi-coherent sheaves on $\mathring{\mathbb{A}}^n$ (resp. on D_n°) to the category of complexes of $\mathbf{k}$-vector spaces. In particular, if E is a quasi-coherent commutative $\mathcal{O}_{\mathring{\mathbb{A}}^n}$-algebra (resp. $\mathcal{O}_{D_n^\circ}$-algebra), then $A^\bullet_{[n]}(E)$ (resp. $A^\bullet_n(E)$) is a commutative dg-algebra.

Proof: (a) This is a classical argument. We consider the only case of $A^\bullet_{[n]}(E)$. Because J is affine, we have quasi-isomorphisms

$$A^\bullet_{[n]}(E) \;\sim\; R\Gamma(J,\Omega^\bullet_{J/\mathbb{P}^{n-1}}\otimes\pi^*E) \;\sim\; R\Gamma(\mathring{\mathbb{A}}^n,R\pi_*(\Omega^\bullet_{J/\mathring{\mathbb{A}}^n}\otimes\pi^*E)).$$

Because π is a Zariski locally trivial fibration with fiber $\mathbb{A}^{n-1}$, the Poincaré lemma for differential forms on $\mathbb{A}^{n-1}$ implies that the embeddings

$$E\hookrightarrow\pi_*(\Omega^\bullet_{J/\mathring{\mathbb{A}}^n}\otimes E)\hookrightarrow R\pi_*\,(\Omega^\bullet_{J/\mathring{\mathbb{A}}^n}\otimes E)$$

are quasi-isomorphisms of complexes of sheaves on $\mathring{\mathbb{A}}^n$, whence the statement.

(b) Obvious by using the multiplication of differential forms.

□

By the above, the dg-algebras

$$A^\bullet_{[n]} = A^\bullet_{[n]}(\mathcal{O}_{\mathring{\mathbb{A}}^n}), \quad A^\bullet_n = A^\bullet_n(\mathcal{O}_{D_n^\circ}) \tag{1.2.5}$$

are commutative dg-models for $\mathfrak{A}^\bullet_{[n]}$ and $\mathfrak{A}^\bullet_n$ respectively. Their grading is situated in degrees $[0, n-1]$. Let us reformulate their definition closer to (1.2.1) and (1.2.2). For this, let

$$\overline{J} \;=\; \big\{(z,z^*) \in \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1} \,\big|\, zz^* \neq 0\big\} \xrightarrow{\overline{\pi}} \mathbb{P}^{n-1}$$

be the classical Jouanolou torsor for $\mathbb{P}^{n-1}$. For a quasi-coherent sheaf F on $\mathbb{P}^{n-1}$ we define

$$R\Gamma^{(\overline{J})}(\mathbb{P}^{n-1}, F) \;=\; \Gamma\big(\overline{J}, \Omega^\bullet_{\overline{J}/\mathbb{P}^{n-1}} \otimes \overline{\pi}^* F\big).$$

As before, this is a model of $R\Gamma(\mathbb{P}^{n-1}, F)$, depending on F in a way compatible with the symmetric monoidal structures.

Proposition 1.2.6. *We have isomorphism of commutative dg-algebras*

$$A^\bullet_{[n]} \;=\; \bigoplus_i R\Gamma^{(\overline{J})}(\mathbb{P}^{n-1}, \mathcal{O}(i)), \quad A^\bullet_n \;=\; \sum_{i \gg \infty} R\Gamma^{(\overline{J})}(\mathbb{P}^{n-1}, \mathcal{O}(i)).$$

C. The Jouanolou model, explicitly. Let

$$\mathbf{k}[z,z^*] = \mathbf{k}[z_1,\dots,z_n,z_1^*,\dots,z_n^*], \quad \mathbf{k}[[z]][z^*]] = \mathbf{k}[[z_1,\dots,z_n]][z_1^*,\dots,z_n^*]$$

be the algebras of regular functions on $\mathbb{A}^n \times \check{\mathbb{A}}^n$ and $D_n \times \check{\mathbb{A}}^n$ respectively.

Proposition 1.2.7. *Let $m = 0,\dots,n-1$. The m-th graded component $A^m_{[n]}$ (resp. $A^\bullet_n$) is identified with the vector space formed by differential forms*

$$\omega \;=\; \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} f_{i_1,\dots,i_m}(z,z^*) dz^*_{i_1} \cdots dz^*_{i_m}$$

where each $f_{i_1,\dots,i_m}$ is an element of the localized algebra $\mathbf{k}[z,z^*][(zz^*)^{-1}]$ (resp. $\mathbf{k}[[z]][z^*][(zz^*)^{-1}]$) such that:

(1) ω is homogeneous in the z_ν^*, dz_ν^* of total degree 0, that is, each $f_{i_1,\dots,i_m}$ is homogeneous of degree $(-m)$.

(2) The contraction $\iota_\xi(\omega)$ of ω with the Euler vector field $\xi = \sum z_\nu^* \partial/\partial z_\nu^*$ vanishes.

The differential $\overline{\partial}$ is given by

$$\overline{\partial} \;=\; \sum_{\nu=1}^{n} dz_\nu^* \frac{\partial}{\partial z_\nu^*}.$$

Proof: We prove the statement about $A^m_{[n]}$; the statement about A^m_n is proved in the same way.

Consider the product $\mathbb{A}^n \times \check{\mathbb{A}}^n$ and the incidence quadric $\widetilde{Q}$ inside it given by the same equation $zz^* = 0$ as Q. Let $U = (\mathbb{A}^n \times \check{\mathbb{A}}^n) - \widetilde{Q}$. All forms ω as in the proposition (not necessarily satisfying the conditions (1) and (2)) form the space $\Gamma(\widetilde{J}, \Omega^m_{U/\mathring{\mathbb{A}}^n})$.

Now, we have a projection $p : U \to J$ of $\mathring{\mathbb{A}}^n$-schemes with the multiplicative group $\mathbb{G}_m$ acting simply transitively on the fibers (i.e., U is a $\mathbb{G}_m$-torsor over J). The infinitesimal generator of this action is the Euler vector field ξ. Therefore relative forms from $\Omega^m_{J/\mathring{\mathbb{A}}^n}$ are identified with sections ω of $\Omega^m_{U/\mathring{\mathbb{A}}^n}$ which satisfy

$$\partial_\xi(\omega) = 0, \;\; \iota_\xi(\omega) = 0,$$

where ∂_ξ is the Lie derivative, see, e.g., [GKZ]. These conditions translate precisely into the conditions (1) and (2) of the proposition. $\square$

Corollary 1.2.8. *The dg-algebra $A^\bullet_{[n]}$ carries a natural filtration "by the order of poles"*

$$F_r A^m_n \;:=\; \big\{\omega \;\big|\; (zz^*)^{r+m} f_{i_1,\dots,i_m} \in \mathbf{k}[z, z^*],\; \forall i_1,\dots,i_m\big\}.$$

compatible with differential and product:

$$\overline{\partial}(F_r A^\bullet) \subset F_r A^\bullet, \;\; (F_r A^\bullet)\cdot(F_{r'} A^\bullet) \subset F_{r+r'} A^\bullet.$$

1.3 Comparison with the Dolbeault and $\overline{\partial}_b$-complexes.

A. Comparison with the Dolbeault complex. In this section we assume $\mathbf{k} = \mathbb{C}$. The notation $\overline{\partial}$ for the differential in $A^\bullet_{[n]}$ is chosen to suggest the analogy with the Dolbeault differential in complex analysis. In fact, we have the following.

Proposition 1.3.1. *Let $\Omega^{0,\bullet}(\mathbb{C}^n - \{0\})$ be the smooth Dolbeault complex of the complex manifold $\mathbb{C}^n - \{0\}$. We have a (unique, injective) morphism of commutative dg-algebras $\varepsilon : A^\bullet_{[n]} \to \Omega^{0,\bullet}(\mathbb{C}^n - \{0\})$ which sends z^*_ν to $\overline{z}_\nu$ and dz^*_ν to $d\overline{z}_\nu$, i.e.,*

$$f(z, z^*)dz^*_{i_1} \cdots dz^*_{i_m} \quad \mapsto \quad f(z, \overline{z})|_{\mathbb{C}^n - \{0\}} d\overline{z}_{i_1} \cdots d\overline{z}_{i_m}.$$

The proof is obvious once we notice that zz^* is being sent to $z\overline{z} = |z|^2$ which does not vanish on $\mathbb{C}^n - \{0\}$. $\square$

Remark 1.3.2. The morphism ε is not a quasi-isomorphism: it identifies $H^\bullet(A^\bullet_n)$ with the "meromorphic part" of $H^\bullet(\Omega^{0,\bullet}(\mathbb{C}^n - \{0\})) = H^\bullet(\mathbb{C}^n - \{0\}, \mathcal{O}_{\mathrm{hol}})$.

We also notice that $A^\bullet_{[n]}$ is concentrated in degrees $[0, n-1]$ while $\Omega^{0,\bullet}(\mathbb{C}^n - \{0\})$ is situated in degrees $[0, n]$. To exhibit a better analytic fit for $A^\bullet_{[n]}$, we recall some constructions from complex analysis.

B. Reminder on the $\overline{\partial}_b$-complex. Let X be an n-dimensional complex manifold and $S \subset X$ a C^∞ real hyper surface (of real dimension $2n-1$). The embedding $S \subset X$ induces on S a differential geometric structure known as the *CR-structure* [BeGr][DT].

More precisely, let $x \in S$. The $2n-1$-dimensional real subspace T_xS in the n-dimensional complex space T_xX has the maximal complex subspace

$$T^{\mathrm{com}}_x S \;=\; T_xS \,\cap\, i(T_xS) \;\subset\; T_xX$$

of complex dimension $(n-1)$. We get a complex vector bundle T^{com}_S on S embedded into the real vector bundle T_S. Its complexification splits, in the standard way, as

$$(T^{\mathrm{com}}_S) \otimes \mathbb{C} \;=\; T^{1,0}_S \oplus T^{0,1}_S \;\subset\; T_S \otimes \mathbb{C}.$$

So $T^{1,0}_S$ and $T^{0,1}_S$ are complex vector sub-bundles in $T_S \otimes \mathbb{C}$, of complex dimension $(n-1)$. Integrability of the complex structure on X implies that these sub-bundles are integrable in the sense of Frobenius, i.e., their sections are closed under the Lie bracket of the sections of $T_S \otimes \mathbb{C}$.

Let ${}^b\Omega^{0,q}_S$ be the sheaf of C^∞-sections of the complex vector bundle $\Lambda^q(T^{0,1}_S)^*$. Integrability of $T^{0,1}_S$ gives, by the standard Cartan formulas, the exterior differentials "along" $T^{0,1}_S$

$$\overline{\partial}_b : {}^b\Omega^{0,q}_S \longrightarrow {}^b\Omega^{0,q+1}_S$$

making ${}^b\Omega_S^{0,\bullet}$ into a sheaf of commutative dg-algebras known as the *tangential Cauchy-Riemann complex* (or *$\overline{\partial}_b$-complex*) of S. It is concentrated in degrees $[0, n-1]$. The complex of global C^∞-sections of ${}^b\Omega_S^{0,\bullet}$ is traditionally denoted by $\Omega_b^{0,\bullet}(S)$ and is also called the $\overline{\partial}_b$-complex.

C. Generalities on real forms. Let $Y = \operatorname{Spec} R$ be a smooth irreducible affine variety over $\mathbb{C}$ of dimension m. A *real structure* on Y is a $\mathbb{C}$-antilinear involution $f \mapsto \overline{f}$ on R. Such a datum defines an antiholomorphic involution $\sigma : Y(\mathbb{C}) \to Y(\mathbb{C})$ on $\mathbb{C}$-points. The fixed point locus of σ is denoted $Y(\mathbb{R})$. If nonempty, $Y(\mathbb{R})$ has a structure of a C^∞-manifold of dimension m. In this case we have an embedding $\epsilon : R \to C^\infty(Y(\mathbb{R}))$. Moreover, any $f \in C^\infty(Y(\mathbb{R}))$ can be approximated by functions from $\epsilon(R)$ on compact subsets (Weierstrass' theorem). In particular, if $Y(\mathbb{R})$ is compact, then $\epsilon(R)$ is dense in $C^\infty(Y(\mathbb{R}))$ in any of the standard metrics of the functional analysis (e.g., in the L_2-metric).

If, further, E is a vector bundle on Y (not necessarily equipped with a real structure), it gives a C^∞-bundle $E|_{Y(\mathbb{R})}$ on $Y(\mathbb{R})$ and an embedding $\Gamma(E) \to \Gamma_{C^\infty}(E|_{Y(\mathbb{R})})$ with approximation properties similar to the above. A differential operator $D : E \to F$ between vector bundles on Y gives rise to a C^∞ differential operator $D_{Y(\mathbb{R})} : E|_{Y(\mathbb{R})} \to F|_{Y(\mathbb{R})}$.

D. The Jouanolou model and the $\overline{\partial}_b$-complex for S^{2n-1}. We specialize part B to $X = \mathbb{C}^n$ with the standard coordinates z_ν. We take S to be the unit sphere S^{2n-1} with equation $\|z\|^2 = 1$. We have therefore the $\overline{\partial}_b$-complex $({}^b\Omega^{0,\bullet}_{S^{2n-1}}, \overline{\partial}_b)$.

At the same time, we introduce, on $\mathbb{A}^n \times \check{\mathbb{A}}^n$ (with coordinates z_ν, z_ν^*), a real structure by putting $\overline{z}_\nu = z_\nu^*$ and $\overline{z_\nu^*} = z_\nu$. The Jouanolou torsor J is realized in $\mathbb{A}^n \times \check{\mathbb{A}}^n$ as the hypersurface $zz^* = 1$, so it inherits the real structure.

Proposition 1.3.3. *(a) We have $J(\mathbb{R}) = S^{2n-1}$. In other words, $A^0_{[n]} = \mathbb{C}[J]$ is identified with the algebra of (real analytic) polynomial $\mathbb{C}$-valued functions on S^{2n-1}.*

(b) The C^∞-vector bundle $\Omega^q_{J/\mathring{\mathbb{A}}^n}|_{J(\mathbb{R})}$ on $J(\mathbb{R}) = S^{2n-1}$ is identified with ${}^b\Omega^{0,q}_{S^{2n-1}}$, and the differential induced by $d_{J/\mathring{\mathbb{A}}^n}$, is identified with $\overline{\partial}_b$. In other

words, $A^\bullet_{[n]}$ is identified with the dg-algebra of polynomial (in the same sense as in (a)) sections of ${}^b\Omega^{0,\bullet}_{S^{2n-1}}$.

Proof: Part (a) is obvious, as $zz^* = 1$ translates to $z\overline{z} = 1$. To prove (b), we notice that the sub-bundle

$$T_{J/\mathring{\mathbb{A}}^n}|_{S^{2n-1}} \ \subset\ T_J|_{S^{2n-1}} = T_{S^{2n-1}} \otimes \mathbb{C}$$

is equal to $T^{0,1}_{S^{2n-1}}$. □

Corollary 1.3.4. *$A^q_{[n]}$ is dense in the L_2-completion of $\Omega^{0,q}_b(S^{2n-1})$. In particular, $A^0_{[n]}$ is dense in the $L_2(S^{2n-1})$.* □

1.4 Representation-theoretic analysis

A. The GL_n-spectrum of $A^\bullet_n$. The Jouanolou model gives commutative dg-algebras with a natural action of the algebraic group GL_n. We now study this action.

Let V be a $\mathbf{k}$-vector space of finite dimension n. Recall [FH] that irreducible representations of $GL(V)$ are labelled by their highest weights which are sequences of non-increasing integers $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_n)$, possibly negative (dominant weights). We will denote the underlying space of the irreducible representation with highest weight α by $\Sigma^\alpha V$ and regard Σ^α as a functor (known as the *Schur functor*) from the category of n-dimensional $\mathbf{k}$-vector spaces and their isomorphisms, to $\mathrm{Vect}_{\mathbf{k}}$. Here are some tie-ins with more familiar constructions.

Examples 1.4.1. (a) For $d \geqslant 0$ we have $\Sigma^{d,0,\ldots,0}(V) = S^d(V)$ is the dth symmetric power of V. For $0 \leqslant p \leqslant n$ let $1_p = (1,\ldots,1,0,\ldots,0)$ (with p occurrences of 1). Then $\Sigma^{1_p}(V) = \Lambda^p(V)$ is the pth exterior power of V.

(b) We have canonical identifications

$$\Sigma^{\alpha_1,\ldots,\alpha_n}(V)^* \ \simeq\ \Sigma^{\alpha_1,\ldots,\alpha_n}(V^*) \ \simeq\ \Sigma^{-\alpha_n,\ldots,-\alpha_1}(V).$$

In particular, $V^* = \Sigma^{0,\ldots,0,-1}(V)$. Further,

$$\Sigma^{1,0,\ldots,0,-1}(V) \ \simeq\ sl(V) \ =\ \big\{A \in \mathrm{End}(V) : \mathrm{tr}(A) = 0\big\}.$$

We now denote by $V = \mathbf{k}^n$ the space of linear combinations of the coordinate functions z_i on $\mathbb{A}^n$, so $\mathbb{A}^n = \operatorname{Spec} S^\bullet(V)$, and let $GL_n = GL(V)$. The Jouanolou torsor $J \to \mathbb{P}^{n-1}$ is acted upon by GL_n, and so the dg-algebra $A^\bullet_{[n]}$ as well as its completion $A^\bullet_n$, inherit the GL_n-action. The following fact can be seen as a higher-dimensional generalization of the representation of elements of $\mathbf{k}[z, z^{-1}]$ as linear combinations of Laurent monomials (irreducible representations of GL_1).

Theorem 1.4.2. *(a) As a GL_n-module, each $A^p_{[n]}$ has simple spectrum, that is, each $\Sigma^\alpha(V)$ enters into the irreducible decomposition of $A^p_{[n]}$ no more than once.*

(b) More precisely, $\Sigma^\alpha(V)$ enters into $A^p_{[n]}$ if and only if

$$\alpha_1 \geqslant 0 \geqslant \alpha_2 \geqslant 0 \geqslant \cdots \geqslant 0 \geqslant \alpha_{n-p} \geqslant -1 \geqslant \alpha_{n-p+1} \geqslant -1 \geqslant \cdots \geqslant -1 \geqslant \alpha_n.$$

Examples 1.4.3. (a) For $n = 1$ the only possible p is $p = 0$. In this case the condition on $\alpha = (\alpha_1) \in \mathbb{Z}$ is vacuous and the theorem says that $A^0_{[1]} = \bigoplus_{\alpha \in \mathbb{Z}} \Sigma^\alpha(\mathbf{k}) = \mathbf{k}[z, z^{-1}]$.

(b) Let $n = 2$. In this case the theorem says that we have an identification of complexes of GL_2-modules

$$A^\bullet_{[2]} = \left\{ \bigoplus_{\substack{\alpha_1 \geqslant 0 \\ \alpha_2 \leqslant 0}} \Sigma^{\alpha_1, \alpha_2}(V) \xrightarrow{\overline{\partial}} \bigoplus_{\substack{\alpha_1 \geqslant -1 \\ \alpha_2 \leqslant -1}} \Sigma^{\alpha_1, \alpha_2}(V) \right\}.$$

From this we see the identifications

$$\begin{aligned} \operatorname{Ker}(\overline{\partial}) &= \bigoplus_{\alpha_1 \geqslant 0} \Sigma^{\alpha_1, 0}(V) = \mathbf{k}[z_1, z_2], \\ \operatorname{Coker}(\overline{\partial}) &= \bigoplus_{\alpha_2 \leqslant -1} \Sigma^{-1, \alpha_2}(V) = z_1^{-1} z_2^{-1} \mathbf{k}[z_1^{-1}, z_2^{-1}], \end{aligned}$$

the other irreducible representations, common to $A^0_{[2]}$ and $A^1_{[2]}$, are cancelled by the action of $\overline{\partial}$.

(c) For $n = 3$ the theorem identifies $A^\bullet_{[3]}$, as a complex of GL_3-modules, with

$$\bigoplus_{\substack{\alpha_1 \geqslant 0 \\ \alpha_3 \leqslant 0}} \Sigma^{\alpha_1, 0, \alpha_3} \xrightarrow{\overline{\partial}} \bigoplus_{\alpha_1 \geqslant 0 \geqslant \alpha_2 \geqslant -1 \geqslant \alpha_3} \Sigma^{\alpha_1, \alpha_2, \alpha_3} \xrightarrow{\overline{\partial}} \bigoplus_{\substack{\alpha_1 \geqslant -1 \\ \alpha_3 \leqslant -1}} \Sigma^{\alpha_1, -1, \alpha_3}.$$

One can see, for example, the reason why the complex is exact in the middle: if $\alpha_2 = 0$, then $\Sigma^{\alpha_1,\alpha_2,\alpha_3}$ lies in $\mathrm{Im}(\overline{\partial})$, while if $\alpha_2 = -1$, then $\Sigma^{\alpha_1,\alpha_2,\alpha_3}$ is mapped by $\overline{\partial}$ isomorphically to its image.

The proof of Theorem 1.4.2 is based on the following observation. We use the definition of $A_{[n]}$ in terms of the torsor J.

Proposition 1.4.4. *As a variety with GL_n-action, $J \simeq GL_n/GL_{n-1}$. Further, $\Omega^p_{J/\mathring{\mathbb{A}}^n}$ is identified with the homogeneous vector bundle on GL_n/GL_{n-1} associated to the representation $\Lambda^p((\mathbf{k}^{n-1})^*)$ of GL_{n-1}.*

Proof: The identification (1.2.3) exhibits $\widetilde{J}$ as a homogeneous space under GL_n. The stabilizer of the point (z, z^*) where $z = (1, 0, \ldots, 0)$ and $z^* = (1, 0, \ldots, 0)$, consists of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad A \in GL_{n-1},$$

whence the first statement of the proposition. To see the second statement, look at the action of the stabilizer subgroup GL_{n-1} on the fiber of $\Omega^p_{J/\mathring{\mathbb{A}}^n}$ over the chosen point (z, z^*) above. By definition, this fiber is the pth exterior power of the relative cotangent space at this point. It remains to notice that the relative *tangent* space is, as a GL_{n-1}-module, nothing but $\mathbf{k}^{n-1}$. $\square$

We now prove Theorem 1.4.2. By Proposition 1.4.4,

$$A^p_{[n]} = \Gamma(J, \Omega^p_{J/\mathring{\mathbb{A}}^n}) \;\simeq\; \mathrm{Ind}_{GL_{n-1}}^{GL_n} \Lambda^p(\mathbf{k}^{n-1})^*$$

(induction in the sense of algebraic groups, i.e., via regular sections of the homogeneous bundle). So we can apply Frobenius reciprocity and obtain:

$$\mathrm{mult}\big(\Sigma^\alpha \mathbf{k}^n, \mathrm{Ind}_{GL_{n-1}}^{GL_n} \Lambda^p(\mathbf{k}^{n-1})^*\big) \;=\; \mathrm{mult}\big(\Lambda^p(\mathbf{k}^{n-1})^*, \Sigma^\alpha \mathbf{k}^n|_{GL_{n-1}}\big),$$

where mult means the multiplicity of an irreducible representation. Now, $\Lambda^p(\mathbf{k}^{n-1})^* = \Sigma^{0,\ldots,0,-1,\ldots,-1}(\mathbf{k}^{n-1})$ (with p occurrences of (-1)). It remains to apply the following fact [Wey].

Proposition 1.4.5. *For any dominant weight α for GL_n the restriction of $\Sigma^\alpha(\mathbf{k}^n)$ to GL_{n-1} has simple spectrum, explicitly given by:*

$$\Sigma^\alpha(\mathbf{k}^n)|_{GL_{n-1}} \;\simeq\; \bigoplus_{\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{n-1} \geqslant \alpha_n} \Sigma^{\beta_1,\ldots,\beta_{n-1}}(\mathbf{k}^{n-1}). \quad \square$$

Theorem 1.4.2 is proved.

B. The Martinelli-Bochner form and its multipoles. We use the analogy with the Dolbeault complex as a motivation for the following.

Example 1.4.6. The *Martinelli-Bochner form*

$$\Omega = \Omega(z, z^*) \;=\; \frac{\sum_{\nu=1}^{n} (-1)^{\nu-1} z_\nu^* dz_1^* \wedge \cdots \wedge \widehat{dz_\nu^*} \wedge \cdots \wedge dz_n^*}{(zz^*)^n}$$

is an element of $A_{[n]}^{n-1}$. For $n = 1$ it reduces to $1/z_1$.

Proposition 1.4.7. *Let* $n > 1$.

(a) The class of Ω *in* $H^{n-1}(A_{[n]}^\bullet) = H^{n-1}(\mathring{\mathbb{A}}^n, \mathcal{O})$ *is a generator of the 1-dimensional subspace* $z_1^{-1} \cdots z_n^{-1}$ *of weight* $(-1, \ldots, -1)$ *under the coordinate torus, see Proposition 1.1.1.*

(b) The top degree part $A_{[n]}^{n-1}$ *contains precisely one 1-dimensional irreducible representation of* GL_n*, which is* $\Sigma^{-1,\ldots,-1}(V) = \Lambda^n(V^*)$*. This subspace is spanned by the Martinelli-Bochner form* $\Omega(z, z^*)$*.*

(c) Further, every element of $H^{n-1}(A_{[n]}^\bullet)$ *can be represented as the class of a "multipole"*

$$P(\partial_{z_1}, \ldots, \partial_{z_n})\Omega(z, z^*)$$

for a unique polynomial $P(y_1, \ldots, y_n)$*.*

Proof: (b) follows from Theorem 1.4.2. To deduce (a) from (b), note that the class $[z_1^{-1} \cdots z_n^{-1}] \in H^{n-1}(\mathring{\mathbb{A}}^n, \mathcal{O})$ spans a 1-dimensional representation of GL_n, and so does $\Omega(z, z^*)$ (direct calculation). At the same time $\Sigma^{-1,\ldots,-1}(V)$ is not present in $A_{[n]}^{n-2}$, so $[\Omega(z, z^*)]$ is a nonzero scalar multiple of $[z_1^{-1} \cdots z_n^{-1}]$.

To prove (c), we notice the following fact which complements Proposition 1.1.1 and is proved using the same standard affine covering $\{z_i \neq 0\}$. Note that the ring $\mathbf{k}[\partial_{z_1}, \ldots, \partial_{z_n}]$ of differential operators with constant coefficients acts naturally on $\mathcal{O}_{\mathring{\mathbb{A}}^n}$ and therefore on $H^{n-1}(\mathring{\mathbb{A}}^n, \mathcal{O})$.

Proposition 1.4.8. *As a* $\mathbf{k}[\partial_{z_1}, \ldots, \partial_{z_n}]$*-module, the space* $H^{n-1}(\mathring{\mathbb{A}}^n, \mathcal{O})$ *is free of rank 1, with generator* $\delta = [z_1^{-1} \cdots z_n^{-1}] \in H^{n-1}(\mathring{\mathbb{A}}^n, \mathcal{O})$*.* □

Proposition 1.4.7 is proved.

Remark 1.4.9. The fact that $A_{[n]}^0$ has simple spectrum, allows us to define a canonical GL_n-equivariant projection $S : A_{[n]}^0 \to \mathbf{k}[z_1, \ldots, z_n]$ along all the irreducible representations which do not enter into $\mathbf{k}[z_1, \dots, z_n]$. This is the algebraic analog of the classical *Szegö projection* from complex Hilbert space $L_2(S^{2n-1})$ to the Hardy space formed by the boundary values of functions holomorphic in the ball $\|z\|^2 < 1$. See, e.g., [BdMG].

1.5 Residues and duality

A. Jouanolou model for forms. We denote

$$A^{p,q}_{[n]} = A^q_{[n]}(\Omega^p_{\mathring{\mathbb{A}}^n}), \quad A^{\bullet\bullet}_{[n]} = \bigoplus_{p=0}^{n} \bigoplus_{q=0}^{n-1} A^{p,q}_{[n]}. \tag{1.5.1}$$

Elements of $A^{p,q}_{[n]}$ can be viewed as differential forms on $\mathbb{A}^n \times \check{\mathbb{P}}^n$ with poles on the quadric Q given by $zz^* = 0$. Let $\widetilde{Q} = \{zz^* = 0\}$ be the hypersurface in $\mathbb{A}^n \times \check{\mathbb{A}}^n$ lifting Q. By pulling back from $\check{\mathbb{P}}^n$ to $\check{\mathbb{A}}^n - \{0\}$, we can view elements of $A^{p,q}_{[n]}$ as differential forms on $(\mathbb{A}^n \times \check{\mathbb{A}}^n) - \widetilde{Q}$ of the form

$$\omega = \sum f^{i_1,\dots,i_p}_{j_1,\dots,j_q}(z, z^*) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz^*_{j_1} \wedge \cdots \wedge dz^*_{j_q}, \tag{1.5.2}$$

which have total degree 0 in the z^*_ν, dz^*_ν and are annihilated by contraction with the vector field $\sum z^*_\nu \partial/\partial z^*_n$.

It follows that the bigraded vector space $A^{\bullet\bullet}_{[n]}$ is a graded commutative algebra, with respect to multiplication of forms, with grading situated in degrees $[0, n] \times [0, n-1]$. It is equipped with two anticommuting differentials: $\overline{\partial} = \sum dz^*\nu\partial/\partial z^*_\nu$ of degree $(0, 1)$ and $\partial = \sum dz_\nu \partial/\partial z_\nu$ of degree $(1, 0)$ which correspond to exterior differentiation along the two factors in $\mathbb{A}^n \times \check{\mathbb{P}}^n$. One can say that $\overline{\partial}$ is induced by the relative de Rham differential in $\Omega_{J/\mathring{\mathbb{A}}^n} \otimes \pi^*\Omega^p$ and ∂ corresponds to the de Rham differential $d : \Omega^p_{\mathring{\mathbb{A}}^n} \to \Omega^{p+1}_{\mathring{\mathbb{A}}^n}$. Part (a) of Proposition 1.2.4 implies:

Proposition 1.5.3. *The bigraded dg-algebra $A^{\bullet\bullet}_{[n]}$ is a commutative dg-model for $\mathfrak{A}^{\bullet\bullet}_{[n]}$.* □

Example 1.5.4. Let $\mathbf{k} = \mathbb{C}$. Then we have an embedding of commutative bigraded bidifferential algebras

$$\varepsilon : A^{\bullet\bullet}_n \hookrightarrow \Omega^{\bullet\bullet}(\mathbb{C}^n - \{0\}),$$

where on the right we have the algebra of C^∞ Dolbeault forms on $\mathbb{C}^n - \{0\}$ with its standard differentials ∂ and $\overline{\partial}$. The value of ε on a form ω given by (1.5.2), is

$$\varepsilon(\omega) \;=\sum f^{i_1,\dots,i_p}_{j_1,\dots,j_q}(z,\overline{z})|_{\mathbb{C}^n-\{0\}} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$$

B. The residue map. Since $\Omega^n_{\mathring{\mathbb{A}}^n}$ is identified, as a GL_n-equivariant coherent sheaf on $\mathring{\mathbb{A}}^n$, with $\mathcal{O}_{\mathring{\mathbb{A}}^n}\otimes_{\mathbf{k}}\Lambda^n(\mathbf{k}^n)$, we have a GL_n-invariant identification

$$A^{n,\bullet}_{[n]} \;\simeq\; A^\bullet_{[n]}\otimes_{\mathbf{k}}\Lambda^n(\mathbf{k}^n).$$

We define the *residue map*

$$\text{Res}: A^{n,n-1}_{[n]} \longrightarrow \mathbf{k} \tag{1.5.5}$$

as the composition

$$A^{n,n-1}_{[n]} \;=\; A^\bullet_{[n]}\otimes_{\mathbf{k}}\Lambda^n(\mathbf{k}^n) \longrightarrow \Lambda^n(\mathbf{k}^n)^*\otimes\Lambda^n(\mathbf{k}^n) = \mathbf{k},$$

where $A^{n-1}_{[n]}\to\Lambda^n(\mathbf{k}^n)^*$ is the unique GL_n-invariant projection which takes the Martinelli-Bochner form $\Omega(z,z^*)$ to $dz_1^*\wedge\cdots\wedge dz_n^*$, see Proposition 1.4.7(b). Thus, by definition,

$$\text{Res}\big(\Omega(z,z^*)dz_1\wedge\cdots\wedge dz_n\big) \;=\; 1. \tag{1.5.6}$$

Note that $A^{n,n-1}_{[n]}$ is the last graded component of $A^\bullet_{[n]}(\Omega^n_{\mathring{\mathbb{A}}^n})$ which is a dg-model for $R\Gamma(\mathring{\mathbb{A}}^n,\Omega^n)$.

Proposition 1.5.7. *For any $f(z)\in\mathbf{k}[z_1,\dots,z_n]$ we have*

$$\text{Res}\big(f(z)\Omega(z,z^*)dz_1\wedge\cdots\wedge dz_n\big) \;=\; f(0)$$

(the algebraic Martinelli-Bochner formula).

Proof: Note that both sides of the proposed equality are GL_n-invariant functionals of $f\in\mathbf{k}[z_1,\dots,z_n] = \bigoplus_{d\geqslant 0}S^d(V)$. For $d>0$ the space $S^d(V)$ does not admit any GL_n-equivariant functionals, so the LHS factors through the projection to $d=0$ which is nothing but the evaluation at 0. So our statement reduces to (1.5.6). $\square$.

Proposition 1.5.8. *Let $\mathbf{k} = \mathbb{C}$. Then for any $\omega \in A^{n,n-1}_{[n]}$ we have*

$$\mathrm{Res}(\omega) \;=\; \frac{(n-1)!}{(2\pi i)^n} \oint\limits_{S^{2n-1}} \varepsilon(\omega)$$

where the integral is taken over any sphere $\|z\| = R$ in $\mathbb{C}^n - \{0\}$.

Proof: Each $\varepsilon(\omega)$, $\omega \in A^{n,n-1}_{[n]}$, is a closed $(n, n-1)$-form on $\mathbb{C}^n - \{0\}$, so its integrals over all spheres as above are equal. We see that the RHS of the proposed equality is a $GL_n(\mathbb{C})$-invariant functional on $A^{n,n-1}_{[n]}$, and any such functional, by Theorem 1.4.2, should factor through the projection to $\Lambda^n(\mathbf{k}^n)^* \otimes \Lambda^n(\mathbf{k}^n)$, i.e., through the residue map. This means the statement holds up to a universal constant depending only on n. To see that this constant is 1, we invoke the classical Martinelli-Bochner formula [GH], which gives

$$\oint\limits_{S^{2n-1}} \Omega(z, \overline{z}) dz_1 \cdots dz_n \;=\; \frac{(2\pi i)^n}{(n-1)!}.$$

$\square$

Proposition 1.5.9. *Each irreducible representation of GL_n enters into $A^{p,q}_{[n]}$ with at most finite multiplicity.*

Proof: As a GL_n-module, $A^{p,q}_{[n]} = \Lambda^p(V) \otimes A^q_{[n]}$. By the Pieri formula [GH], the irreducible components $\Sigma^\beta(V)$ of $\Lambda^p(V) \otimes \Sigma^\alpha(V)$ all satisfy $\beta = \alpha + e_{i_1} + \cdots + e_{i_p}$ for some $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Here e_i is the ith basis vector. So the allowed β situated in some fixed radius neighborhood of α in $\mathbb{Z}^n \subset \mathbb{R}^n$. This means that tensoring a simple spectrum representation with $\Lambda^p(V)$ gives a representation with finite multiplicities. So our statement follows from Theorem 1.4.2. $\square$

Let $E \simeq \bigoplus_\alpha C_\alpha \otimes \Sigma^\alpha(V)$ be a representation of $GL(V)$ with finite multiplicities (so C_α are finite-dimensional vector spaces). We define the *restricted dual* of E to be

$$E^\star \;:=\; \bigoplus_\alpha C^*_\alpha \otimes \Sigma^\alpha(V)^*.$$

Proposition 1.5.10. *The (GL_n-invariant) residue pairing*

$$(\alpha, \beta) \mapsto \mathrm{Res}(\alpha \cdot \beta) : A^{p,q}_{[n]} \otimes_{\mathbf{k}} A^{n-p,n-1-q}_{[n]} \longrightarrow \mathbf{k}$$

gives an isomorphism $A^{n-p,n-1-q}_{[n]} \to (A^{p,q}_{[n]})^\star$.

Proof: We first prove a weaker statement: $(A_{[n]}^{p,q})^\star$ is isomorphic, as a GL_n-module, to $A_{[n]}^{n-p,n-1-q}$. Because of the non-degenerate pairing $\Lambda^p(V) \otimes \Lambda^{n-p}(V) \to \Lambda^n(V)$, the statement reduces to the isomorphism

$$(A_{[n]}^q)^\star \;\simeq\; A_{[n]}^{n-1-q} \otimes \Lambda^n(V)^*.$$

This isomorphism follows at once from inspecting the irreducible components of $A_{[n]}^q$ and $A_{[n]}^{n-1-q}$ given by Theorem 1.4.2. They are in bijection $\Sigma^\alpha \leftrightarrow \Sigma^\beta$, so that

$$\beta_i = -1 - \alpha_{n-i}, \;\; i = 1, \ldots, n, \tag{1.5.11}$$

which means that $\Sigma^\beta(V) \simeq \Lambda^n(V)^* \otimes (\Sigma^\alpha(V))^*$.

We now prove that the residue pairing is actually an isomorphism as claimed. As before, we reduce to considering the pairing

$$A_{[n]}^q \otimes A_{[n]}^{n-1-q} \longrightarrow A_{[n]}^{n-1} \longrightarrow \Lambda^n(V)^*.$$

Only the summands $\Sigma^\alpha(V) \subset A_{[n]}^q$ and $\Sigma^\beta(V) \subset A_{[n]}^{n-1-q}$ where α, β correspond to each other as in (1.5.11), can pair in a non-trivial way. It remains to show that they indeed pair non-trivially. If they pair trivially, then the subspace $\Sigma^\alpha(V) \subset A_{[n]}^q$ is orthogonal to the entire $A_{[n]}^{n-1-q}$. The easiest way to see why this is impossible, is to reduce (by the Lefschetz principle) to $\mathbf{k} = \mathbb{C}$. In this case we can use the fact that $A_{[n]}^{n-1-q}$ is L_2-dense in $\Omega_b^{0,n-1-q}(S^{2n-1})$, see Corollary 1.3.4, and the non-degeneracy of the L_2-pairing on $\Omega_b^{0,\bullet}(S^{2n-1})$. □

Proposition 1.5.12 (Algebraic Stokes formula)**.** *The residue functional* Res : $A_{[n]}^{n,n-1} \to \mathbf{k}$ *vanishes on* $\partial(A_{[n]}^{n-1,n-1}) + \overline{\partial}(A_{[n]}^{n,n-2})$.

Proof: As before, the easiest proof is to reduce to $\mathbf{k} = \mathbb{C}$ and to embed $A_{[n]}^{\bullet\bullet}$ into $\Omega^{\bullet\bullet}(\mathbb{C}^n - \{0\})$. Then we can use the classical Stokes formula for $d = \partial + \overline{\partial}$, noticing that elements of $A_{[n]}^{n-1,n-1}$ are annihilated by $\overline{\partial}$, while elements of $A_{[n]}^{n,n-2}$ are annihilated by ∂.

A purely algebraic proof, by inspection of the possible relevant irreducible components, is left to the reader. In this inspection we find that $A_{[n]}^{n,n-2}$ does not contain the trivial representation, while $A_{[n]}^{n-1,n-1}$ does contain it but the corresponding subspace is annihilated by ∂ (it represents $H^{n-1}(\mathbb{P}^{n-1}, \Omega^{n-1})$). □

2 The residue class in cyclic cohomology

2.1 Cyclic homology of dg-algebras and categories

In this subsection we compare various definitions of cyclic homology of associative dg-algebras without any restriction on the grading. Care is needed, since existing treatments of some issues apply only to $\mathbb{Z}_{\leqslant 0}$-graded algebras and rely on spectral sequences which may not converge in the general case.

For general background on cyclic homology (ungraded and $\mathbb{Z}_{\leqslant 0}$-graded cases), see [Lo].

A. General definitions. Our basic approach is that of Keller [Ke1]. That is, let $(A, \overline{\partial})$ be any associative dg-algebra over $\mathbf{k}$, possibly without unit. The Hochschild and the cyclic complexes of A, denoted $C_\bullet^{\mathrm{Hoch}}(A)$ and $CC_\bullet(A)$, are defined similarly to what is described in [Lo] 5.2.2 for the $\mathbb{Z}_{\leqslant 0}$-graded case. That is, we form the total complex of the double or triple complex obtained when we take into account the grading and differential on A. For example,

$$C_\bullet^{\mathrm{Hoch}}(A) \;=\; \mathrm{Tot}\left\{\cdots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A\otimes A \xrightarrow{b} A\right\}.$$

The total complex here and elsewhere is always understood in the sense of *direct sums*. The new phenomenon compared to the $\mathbb{Z}_{\leqslant 0}$-graded case is that direct sums can be infinite.

The definition of $CC_\bullet(A)$ is similar. More precisely, let V be a cochain complex of $\mathbf{k}$-vector spaces with a $\mathbb{Z}/(n+1)$-action, the generator of the action denoted by t. We denote

$$V_C \;=\; \mathrm{Tot}\left\{\cdots \xrightarrow{1-t} V \xrightarrow{1+\cdots+t^n} V \xrightarrow{1-t} V\right\},$$

(the horizontal grading situated in degrees $\leqslant 0$). Then

$$CC_\bullet(A) \;=\; \mathrm{Tot}\left\{\cdots \xrightarrow{b} A_C^{\otimes 3} \xrightarrow{b} (A\otimes A)_C \xrightarrow{b} A_C\right\}.$$

where the $\mathbb{Z}/(n+1)$-action on $A^{\otimes(n+1)}$ is given as in [Lo] (2.1.0), understood via the Koszul sign rule.

The homology of $C_\bullet^{\mathrm{Hoch}}(A)$ and $CC_\bullet(A)$ will be denoted $HH_\bullet(A)$ and $HC_\bullet(A)$. Each of these complexes has an exhaustive increasing filtration by the number of tensor factors. This gives a convergent spectral sequence

$$E_2 \;=\; HC_\bullet(H^\bullet_{\overline{\partial}}(A)) \;\Rightarrow\; HC_\bullet(A),$$

and similarly for $HH_\bullet(A)$. It follows that the functors $HC_\bullet$ and $HH_\bullet$ take quasi-isomorphisms of dg-algebras to isomorphisms and so descend to functors on the homotopy category of associative dg-algebras.

Lemma 2.1.1. *Let V be a cochain complex of* $\mathbf{k}$*-vector spaces with a* $\mathbb{Z}/(n+1)$*-action. Then morphism* $V_C \to V_{\mathbb{Z}/(n+1)}$ *from the last term to the cokernel of* $(1-t)$*, is a quasi-isomorphism.*

Proof: This is well known if V in ungraded (Tate resolution). When V is $\mathbb{Z}_{\leqslant 0}$-graded, it follows from a spectral sequence argument, as V_C has an increasing exhaustive filtration with quotients V_C^i. To prove the general case, it suffices to consider the case when $\mathbf{k}$ is algebraically closed. Assuming this, we consider the abelian category $\mathrm{dgVect}_{\mathbf{k}}{}^{\mathbb{Z}/(n+1)}$ formed by cochain complexes with $\mathbb{Z}/(n+1)$-action.

Lemma 2.1.2. *Each object of* $\mathrm{dgVect}_{\mathbf{k}}{}^{\mathbb{Z}/(n+1)}$ *is isomorphic to a direct sum of (possibly infinitely many copies) of the following indecomposable objects:*

(1) A 1-term complex $\mathbf{k}$ *(situated in some degree) on which* $\mathbb{Z}/(n+1)$ *acts via some character.*

(2) A 2-term complex $\mathbf{k} \xrightarrow{\mathrm{Id}} \mathbf{k}$ *(situated in a pair of adjacent degrees) on which* $\mathbb{Z}/(n+1)$ *acts via the same character.*

Lemma 2.1.2 implies Lemma 2.1.1 because the indecomposable objects are bounded and so 2.1.1 holds for them.

Proof of Lemma 2.1.2: This is well known if $n = 0$ (i.e., if we consider just cochain complexes with no group action). Now, given V with action of $\mathbb{Z}/(n+1)$, we first split it (as a complex) into a direct sum of eigencomplexes V_χ corresponding to the characters χ of $\mathbb{Z}/(n+1)$. Then we decompose each complex V_χ into indecomposables. □

Corollary 2.1.3. *The complex* $CC_\bullet(A)$ *is quasi-isomorphic to the Connes' complex*

$$C_\bullet^\lambda(A) \;=\; \mathrm{Tot}\left\{\cdots \xrightarrow{b} A^{\otimes 3}_{\mathbb{Z}/3} \xrightarrow{b} (A\otimes A)_{\mathbb{Z}/2} \xrightarrow{b} A\right\}.$$

B. Morita invariance. For unital dg-algebras, Keller [Ke1] proved that cyclic homology is Morita-invariant. To formulate the results compactly, it is convenient to extend the definition of Hochschild and cyclic complexes and cohomology to small dg-categories $\mathcal{A}$ (a unital dg-algebra is the same as a dg-category with one object). That is, we define $C^{\mathrm{Hoch}}_\bullet(A)$ to be the total complex of the double complex

$$\cdots \xrightarrow{b} \bigoplus_{a_0,\dots,a_n\in \mathrm{Ob}(\mathcal{A})} \bigotimes_{i=0}^{n} \mathrm{Hom}^\bullet_{\mathcal{A}}(a_i,a_{i+1}) \xrightarrow{b} \cdots \to \bigotimes_{a_0\in \mathrm{Ob}(\mathcal{A})} \mathrm{Hom}^\bullet_{\mathcal{A}}(a_0,a_0),$$

(here we put $a_{n+1} := a_0$) and similarly for $CC_\bullet(\mathcal{A})$.

We recall [To4] that a functor $f : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence*, if $f_* : \mathrm{Perf}_{\mathcal{A}} \to \mathrm{Perf}_{\mathcal{B}}$ is a quasi-equivalence. Recall further that $\mathrm{Perf}_{\mathcal{A}}$ is essentially small, if $\mathcal{A}$ is small, and the canonical (Yoneda) embedding $\upsilon : \mathcal{A} \to \mathrm{Perf}_{\mathcal{A}}$ is a Morita equivalence.

Proposition 2.1.4 (Keller, [Ke2])**.** *(a) If $f : \mathcal{A} \to \mathcal{B}$ is a Morita equivalence of small dg-categories, then $HC_\bullet(f) : HC_\bullet(\mathcal{A}) \to HC_\bullet(\mathcal{B})$ is an isomorphism.*

(b) In particular, if $\mathcal{A}$ is an essentially small dg-category, then $HC_\bullet(\mathcal{A}')$ where $\mathcal{A}' \subset \mathcal{A}$ is an equivalent small dg-subcategory, are canonically identified, and denoted $HC_\bullet(\mathcal{A})$.

(c) It follows that υ induces an isomorphism $HC_\bullet(\mathcal{A}) \to HC_\bullet(\mathrm{Perf}_{\mathcal{A}})$.

(d) Therefore any dg-functor $\phi : \mathrm{Perf}_{\mathcal{A}} \to \mathrm{Perf}_{\mathcal{B}}$ induces a morphism $\phi_* : HC_\bullet(\mathcal{A}) \to HC_\bullet(\mathcal{B})$ which is an isomorphism, if ϕ is a quasi-equivalence.

Let us note a more elementary instance of this proposition, cf. [Lo] 1.2.4 and 2.2.9.

Proposition 2.1.5. *Let A be a unital dg-algebra and $r \geqslant 1$. The collection of the trace maps*

$$\begin{aligned} \mathrm{Mat}_r(A)^{\otimes(n+1)} \simeq \big(\mathrm{Mat}_r(\mathbf{k}) \otimes_{\mathbf{k}} A\big)^{\otimes(n+1)} &\xrightarrow{\mathrm{tr}_{A,n}} A^{\otimes(n+1)}, \\ \mathrm{tr}_{A,n}\big(u_0a_0 \otimes \cdots \otimes u_na_n\big) \;&=\; \mathrm{tr}(u_0\cdots u_n)a_0 \otimes \cdots \otimes a_n, \end{aligned}$$

defines quasi-isomorphisms of complexes

$$C^{\mathrm{Hoch}}_\bullet(\mathrm{Mat}_r(A)) \xrightarrow{\mathrm{tr}} C^{\mathrm{Hoch}}_\bullet(A), \quad CC_\bullet(\mathrm{Mat}_r(A)) \xrightarrow{\mathrm{tr}} CC_\bullet(A).$$

The morphisms on $C_\bullet^{\mathrm{Hoch}}$ and $CC_\bullet$, induced by the embedding $A \hookrightarrow \mathrm{Mat}_r(A)$, are quasi-inverse to tr. □

C. Localization sequence.

Definition 2.1.6. A localization sequence of perfect dg-categories is a an homotopy cofiber sequence

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$$

in the Morita model category of dg-categories (see Appendix), such that the functor $\mathcal{A} \to \mathcal{B}$ is (quasi-)fully faithful.

Remark 2.1.7. Given such a localization sequence, the homotopy category $[\mathcal{A}]$ is a thick subcategory of the triangulated category $[\mathcal{B}]$ and $[\mathcal{C}]$ is equivalent to the Verdier quotient $[\mathcal{B}]/[\mathcal{A}]$.

Theorem 2.1.8 (Keller, [Ke2])**.** *A localization sequence of perfect dg-categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ induces a cofiber sequence*

$$CC(\mathcal{A}) \longrightarrow CC(\mathcal{B}) \longrightarrow CC(\mathcal{C})$$

in the category of complexes. In particular we get a long exact sequence

$$\cdots \longrightarrow HC_p(\mathcal{A}) \longrightarrow HC_p(\mathcal{B}) \longrightarrow HC_p(\mathcal{C}) \longrightarrow HC_{p-1}(\mathcal{A}) \longrightarrow \cdots$$

2.2 Cyclic homology of schemes

A. Sheaf-theoretic definition and Hodge decomposition. We recall the basic constructions and results of Weibel.

Definition 2.2.1 [W1]. Let X be a $\mathbf{k}$-scheme. Denote by $\mathcal{C}_{\bullet,X}^{\mathrm{Hoch}}$ the complex of sheaves on the Zariski topology of X obtained by sheafifying (term by term) the complex of presheaves $U \mapsto C_\bullet^{\mathrm{Hoch}}(\mathcal{O}(U))$. We define the Hochschild homology of X as the hypercohomology of this complex:

$$\mathbf{HH}(X) = \mathbb{H}^k(X, \mathcal{C}_\bullet^{\mathrm{H}}(X)).$$

We also define the bicomplex of sheaves $\mathcal{CC}_{\bullet,X}$ as the sheafification of the complex of presheaves $U \mapsto CC_\bullet(\mathcal{O}(U))$, and the cyclic homology of X as

$$\mathbf{HC}(X) = \mathbb{H}^k(X, \mathrm{Tot}(\mathcal{CC}_{\bullet,X})).$$

So **HH** and **HC** fit into Mayer-Vietoris sequences by definition. Weibel proved that for an affine scheme $X = \mathrm{Spec}(A)$, one has

$$\mathbf{HH}(X) = HH(A), \quad \mathbf{HC}(X) = HC(A).$$

Let us also recall here that the cyclic homology of a commutative algebra has a Hodge decomposition (also called λ-decomposition, see [Lo]):

$$HH_\bullet(A) = \bigoplus_i HH_\bullet^{(i)}(A) \qquad \text{and} \qquad HC_\bullet(A) = \bigoplus_i HC_\bullet^{(i)}(A)$$

Weibel extends this decomposition to the case of schemes and describes it in term of HKR-isomorphism in the smooth case.

Theorem 2.2.2 ([W2]). *Let X be a qcqs (quasi-compact and quasi-separated) $\mathbf{k}$-scheme. There are Hodge decompositions*

$$\mathbf{HH}_\bullet(X) = \bigoplus_i \mathbf{HH}_\bullet^{(i)}(X) \qquad \textit{and} \qquad \mathbf{HC}_\bullet(X) = \bigoplus_i \mathbf{HC}_\bullet^{(i)}(X).$$

Moreover, if X is smooth, then we have for any i and k

$$\mathbf{HH}_k^{(i)}(X) = H^{i-k}(X, \Omega_X^i) \qquad \textit{and} \qquad \mathbf{HC}_k^{(i)}(X) = \mathbb{H}^{2i-k}(X, \Omega_X^{\leqslant i}).$$

B. Relation to HC of dg-algebras and categories. We first recall the following result of Keller.

Theorem 2.2.3 ([Ke3]). *For any qcqs scheme X we have*

$$\mathbf{HH}_\bullet(X) \simeq HH_\bullet(\mathrm{Perf}(X)) \qquad \textit{and} \qquad \mathbf{HC}_\bullet(X) \simeq HC_\bullet(\mathrm{Perf}(X)),$$

where Perf *denotes the dg-category of perfect complexes.*

Next, we use this result to relate cyclic homlogy of schemes with that of dg-algebras of derived functions.

Definition 2.2.4. A scheme U is called quasi-affine if it is isomorphic to a qcqs open subscheme of an affine scheme.

Theorem 2.2.5. *For any quasi-affine scheme U, we have*

$$\mathbf{HC}_\bullet(U) \simeq HC_\bullet(R\Gamma(U, \mathcal{O})).$$

Proof: By Theorem 2.2.3 and Proposition 2.1.4, it is enough to show that $\mathrm{Perf}(X)$ is equivalent to $\mathrm{Perf}_{R\Gamma(U,\mathcal{O})}$. Because perfect complexes (resp. dg-modules) are intrinsically characterized as compact objects in the derived category of all quasicoherent sheaves (resp. all dg-modules), we reduce to:

Proposition 2.2.6. *Let U be a quasi-affine scheme. The derived category of quasi-coherent sheaves on U is equivalent to the derived category of dg-modules on the cdga $R\Gamma(U,\mathcal{O})$.*

Note that this statement is not true for more general schemes (e.g., for $U = \mathbb{P}^n$).

Proof of the proposition: Let us denote by A any model for the cdga $R\Gamma(U,\mathcal{O})$. Let $u\colon U \to X = \operatorname{Spec} B$ be an open immersion. In particular, we get a map of cdgas $p\colon B \to A$. It induces an adjunction of ∞-categories

$$p^*\colon \mathrm{dgMod}_B \leftrightarrows \mathrm{dgMod}_A \ :\ p_*$$

We will prove the functor p_* to be fully faithful and its essential image to coincide with that of u_*.

The element A is a compact generator of dgMod_A and both p_* and p^* preserve small colimits. To show that p_* is fully faithful, we consider the adjunction map $p^*p_*A \to A$. This map is equivalent to the multiplication map $A \otimes_B^L A \to A$.

Lemma 2.2.7. *The multiplication map $A \otimes_B^L A \to A$ is a quasi-isomorphism.*

Proof of the lemma: Let $(f_i)_{i\in I}$ denote a finite family of elements of B such that $U = \bigcup X_{f_i} \subset X$, where $X_{f_i} = \mathrm{Spec}(B[f_i^{-1}])$. For any non-empty subset $J \subset I$, let B_J denote the (derived) tensor product

$$B_J = \bigotimes_{B}^{i\in J}{}^{L}\, B[f_i^{-1}].$$

By definition, the cdga A is equivalent to the homotopy limit $\underleftarrow{\mathrm{holim}}_{\varnothing\neq J\subset I} B_J$. The derived tensor product $A \otimes_B^L A \to A$ is then equivalent to

$$\begin{aligned} A \otimes_B^L A \ &\simeq\ \underleftarrow{\mathrm{holim}}_J B_J \otimes_B^L \underleftarrow{\mathrm{holim}}_{J'} B_{J'} \ \simeq\ \underleftarrow{\mathrm{holim}}_J \underleftarrow{\mathrm{holim}}_{J'} (B_J \otimes_A B_{J'}) \ \simeq \\ &\simeq\ \underleftarrow{\mathrm{holim}}_J \underleftarrow{\mathrm{holim}}_{J'} B_{J\cup J'} \ \simeq\ \underleftarrow{\mathrm{holim}}_J \underleftarrow{\mathrm{holim}}_{J\subset J''} B_{J''} \ \simeq\ \underleftarrow{\mathrm{holim}}_J B_J \ \simeq\ A. \end{aligned}$$

Lemma 2.2.7 is proved.

To continue with Proposition 2.2.6, we can now identify dgMod_A with the full stable and presentable ∞-subcategory of dgMod_B generated by A. We remark that the derived category of quasi-coherent sheaves on U identifies, through the functor u_*, with a full stable and presentable ∞-subcategory of dgMod_B containing $u_*\mathcal{O}_U \simeq \mathbb{R}\Gamma(U, \mathcal{O}) \simeq A$. We hence get an adjunction

$$F := p^* u_* \colon \mathrm{D}_{\mathrm{qcoh}}(U) \rightleftarrows \mathrm{dgMod}_A \, : \, G = u^* p_*$$

where the functor G is fully faithful. It therefore suffices to prove that F is conservative. Let then $E \in \mathrm{D}_{\mathrm{qcoh}}(U)$ such that $F(E) = 0$. For any i, if we denote by $v_i \colon \mathrm{Spec}(B_i) \to U$ the Zariski embedding, we get

$$v_i^*(E) = F(E) \otimes_A B_i = 0.$$

As $\{\mathrm{Spec}(B_i)\}_{i \in I}$ is a cover of U, we deduce that E is acyclic and hence that the functor F is conservative. Proposition 2.2.6 and therefore Theorem 2.2.5, is proved.

2.3 The residue class $\rho \in HC^1(A_n^\bullet)$

A. Definition via a cyclic cocycle. Let $(A_n^\bullet, \overline{\partial}) \simeq R\Gamma(\mathring{\mathbb{A}}^n, \mathcal{O})$ be the commutative dg-algebra from (1.2.5). We recall that $A_n^\bullet$ is included into the doubly graded differential algebra $(A_n^{\bullet\bullet}, \partial, \overline{\partial}) \simeq R\Gamma(\mathring{\mathbb{A}}^n, \Omega^\bullet)$: we have $A_n^\bullet = A_n^{0,\bullet}$.

Consider the $(n+1)$-linear functional

$$r : (A_n^\bullet)^{\otimes(n+1)} \longrightarrow \mathbf{k}, \quad r(f_0, \ldots, f_n) = \mathrm{Res}(f_0 \partial f_1 \cdots \partial f_n). \tag{2.3.1}$$

Here $\partial f_i \in A_n^{1,\bullet}$, and $\mathrm{Res} : A_n^{n-1,n} \to \mathbf{k}$ is the residue map from (1.5.5). By definition $r(f_0, \ldots, f_n)$ is assumed to be equal to 0 unless $f_0 \partial f_1 \cdots \partial f_n$ lies in $A_n^{n,n-1}$.

Proposition 2.3.2. *The functional r is a degree 1 cocycle in the Connes cochain complex $C_\lambda^\bullet(A_n^\bullet) = \mathrm{Hom}_{\mathbf{k}}(C_\bullet^\lambda(A_n^\bullet), \mathbf{k})$.*

Proof: *Degree 1:* The expression $f_0 \partial f_1 \cdots \partial f_n$ always lies in $A_n^{n,\bullet}$. For it to lie in $A_n^{n,n-1}$, we must have $\sum \deg(f_i) = n - 1$. The horizontal grading of $(A_n^\bullet)^{\otimes(n+1)}$ in the Hochschild chain complex of A, is $(-n)$. So r, as an element of the dual complex, has degree $+1$.

Cyclic symmetry: With the Koszul sign rule taken into account, the condition for r to lie in $C^\bullet_\lambda(A^\bullet_n)$ is

$$r(f_0,\dots,f_n) \;=\; (-1)^{n+\deg(f_0)(\deg(f_1)+\cdots+\deg(f_n))} r(f_1,\dots,f_n,f_0).$$

This follows at once from the Leibniz rule for ∂ and the fact that Res vanishes on the image of ∂.

r is a cocycle: As $C^\bullet_\lambda(A^\bullet_n)$ is a subcomplex in $C^\bullet_{\mathrm{Hoch}}(A^\bullet_n)$, we need to show that r is closed under the differential in $C^\bullet_{\mathrm{Hoch}}$. This differential is the sum $\beta+\overline{\partial}^*$, where β is the Hochschild cochain differential, and $\overline{\partial}^*$ is induced by the differential $\overline{\partial}$ in $A^\bullet_n$. We claim that both $\beta(r)$ and $\overline{\partial}^*(r)$ vanish. These statements follow from the Leibniz rule for ∂ and $\overline{\partial}$ respectively and the fact that Res vanishes on the images of both ∂ and $\overline{\partial}$. □

We denote by $\rho \in HC^1(A^\bullet_n)$ the class of r and call it the *residue class*. By construction, ρ is GL_n-invariant.

B. Definition via the Hodge decomposition. By Theorems 2.2.5 and 2.2.2, we have the Hodge decomposition

$$HC_1(A^\bullet_n) \;\simeq\; \mathbf{HC}_1(\mathring{\mathbb{A}}^n) \;=\; \bigoplus_i \mathbf{HC}_1^{(i)}(\mathring{\mathbb{A}}^n) \;\simeq\; \bigoplus_{i=1}^n \mathbb{H}^{2i-1}(\mathring{\mathbb{A}}^n,\Omega^{\leqslant i}). \tag{2.3.3}$$

We consider the projection to the summand

$$\mathbf{HC}_1^{(n)}(\mathring{\mathbb{A}}^n) \;=\; \mathbb{H}^{2n-1}(\mathring{\mathbb{A}}^n,\Omega^{\leqslant n}) \;=\; \mathbb{H}^{2n-1}(\mathring{\mathbb{A}}^n,\Omega^{\bullet}) \;=\; \mathbf{k}, \tag{2.3.4}$$

which is the $(2n-1)$st de Rham cohomology of $\mathring{\mathbb{A}}^n$ (e.g., for $\mathbf{k}=\mathbb{C}$, the $(2n-1)$-st topological cohomology of $\mathbb{C}^n\backslash\{0\}\sim S^{2n-1}$).

Theorem 2.3.5. *The class in $HC^1(A^\bullet_n)$ given by the projection of* (2.3.3) *to* (2.3.4)*, is the unique, up to scalar, GL_n-invariant element in $HC^1(A^\bullet_n)$. In particular, it is proportional to the residue class ρ.*

Proof: We analyze the GL_n-module structure of the $\mathbb{H}^{2i-1}(\mathring{\mathbb{A}}^n,\Omega^{\leqslant i})$. Applying the spectral sequence starting from $H^p(\mathring{\mathbb{A}}^n,\Omega^q)$ and converging to $\mathbb{H}^{2i-1}(\mathring{\mathbb{A}}^n,\Omega^{\leqslant i})$, we reduce to the following statement.

Proposition 2.3.6. *The vector space $H^p(\mathring{\mathbb{A}}^n,\Omega^q)$ has no GL_n-invariants unless $p=q=0$ or $p=n-1$, $q=n$, in which case the space of invariants is 1-dimensional.*

3 Higher current algebras and their central extensions

3.1 Cyclic homology and Lie algebra homology

A. Reminder on the dg-Lie algebra (co)homology. Let $\mathfrak{l}^\bullet$ be a dg-Lie algebra with differential $\overline{\partial}$ and bracket $c : \Lambda^2\mathfrak{l} \to \mathfrak{l}$. The *Chevalley-Eilenberg cochain complex* of $\mathfrak{l}^\bullet$ is the symmetric algebra $\mathrm{CE}^\bullet(\mathfrak{l}) = (S^\bullet(\mathfrak{l}^\bullet[1]))^*$ equipped with the algebra differential $d_{\mathrm{Lie}} + D$, where:

- D is the algebra differential on the symmetric algebra which extends $(\overline{\partial}[1])^*$ by the Leibniz rule.
- d_{Lie} is the algebra differential given on the generators by the map $(s \circ c[2])^* : (\mathfrak{l}^\bullet[1])^* \to (S^2(\mathfrak{l}^\bullet[1]))^*$ (i.e., extended from the generators by the Leibniz rule). Here we suppressed the notation for dec_2 from (A.2).

The cohomology of $\mathrm{CE}^\bullet(\mathfrak{l})$ (equipped with the total grading) will be denoted $\mathbb{H}^\bullet_{\mathrm{Lie}}(\mathfrak{l})$.

In fact, $\mathrm{CE}^\bullet(\mathfrak{l})$ is dual to the *Chevalley-Eilenberg chain complex* $\mathrm{CE}_\bullet(\mathfrak{l}) = S^\bullet(\mathfrak{l}^\bullet[1])$ whose differential is defined by obvious "undualizing" of the formulas above. In other words, it can be described in terms of the coalgebra structure on $\mathrm{CE}_\bullet(\mathfrak{l})$. We have an increasing exhaustive filtration of $\mathrm{CE}_\bullet(\mathfrak{l})$ by the number of tensor factors which gives a convergent spectral sequence $H^{\mathrm{Lie}}_\bullet(H^\bullet_{\overline{\partial}}(\mathfrak{l})) \Rightarrow H^{\mathrm{Lie}}_\bullet(\mathfrak{l})$. This implies that $H^{\mathrm{Lie}}_\bullet$, and therefore $H^\bullet_{\mathrm{Lie}}$, are quasi-isomorphism invariant, in particular, descend to functors on the homotopy category $[\mathrm{dgLie}_{\mathbf{k}}]$. The following is then standard.

Proposition 3.1.1. *We have a canonical identification*

$$\mathrm{Hom}_{[\mathrm{dgLie}_{\mathbf{k}}]}(\mathfrak{l}, \mathbf{k}[n]) \;=\; \mathbb{H}^{n+1}_{\mathrm{Lie}}(\mathfrak{l}). \quad \square$$

In particular, $\mathbb{H}^2_{\mathrm{Lie}}(\mathfrak{l})$ classifies central extensions of $\mathfrak{l}$.

B. Loday's homomorphism θ. Let A be an associative dg-algebra (possibly without unit). The graded commutator

$$[a,b] \;=\; ab - (-1)^{\deg(a)\cdot\deg(b)} ba$$

makes A into a dg-Lie algebra which we denote A_{Lie}. If A is ungraded, then

$$H_1^{\mathrm{Lie}}(A_{\mathrm{Lie}}) \;=\; HC_0(A) \;=\; A/[A,A],$$

Extending [Lo] (10.2.3) straightforwardly from the ungraded case, we include this in the following.

Proposition 3.1.2. *For any associative dg-algebra A there is a natural morphism of complexes*

$$\theta^A : \mathrm{CE}_{\bullet+1}(A_{\mathrm{Lie}}) \longrightarrow C^{\lambda}_{\bullet}(A),$$
$$a_0 \wedge \cdots \wedge a_n \;\mapsto\; \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)[(Id,\sigma)^*(a_0\otimes\cdots\otimes a_n)].$$

Here, for $\sigma \in S_n$, we denote by $(Id,\sigma)\in S_{n+1}$ the permutation of $\{0,1,\dots,n\}$ fixing 0 and acting on $1,2,\dots,n$ as σ. The notation $[x]$ means the class of x in the coinvariant space of $\mathbb{Z}/(n+1)$. In particular, this gives natural maps

$$\theta^A = \theta^A_i : \mathbb{H}^{\mathrm{Lie}}_{i+1}(A_{\mathrm{Lie}}) \to HC_i(A).$$

3.2 Higher current algebras and their central extensions

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over $\mathbf{k}$. We consider the dg-Lie algebra

$$\mathfrak{g}_n^\bullet \;=\; \mathfrak{g}\otimes_{\mathbf{k}} A_n^\bullet \;\simeq\; R\Gamma(\mathring{\mathbb{A}}^n, \mathfrak{g}\otimes_{\mathbf{k}}\mathcal{O}).$$

We call $\mathfrak{g}_n^\bullet$ the nth *derived current algebra* associated to $\mathfrak{g}$. For $n=1$ we get the Laurent polynomial algebra $\mathfrak{g}_1 = \mathfrak{g}[z,z^{-1}]$.

Let $P \in S^{n+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ be an invariant polynomial on $\mathfrak{g}$, homogeneous of degree $(n+1)$. We will also consider P as a symmetric $(n+1)$-linear form $(x_0,\dots,x_n)\mapsto P(x_0,\dots,x_n)$ on $\mathfrak{g}$.

Theorem 3.2.1. *(a) Consider the functional $\gamma_P : (\mathfrak{g}_n^\bullet[1])^{\otimes(n+1)} \to \mathbf{k}$ given by*

$$\gamma_P\big((x_0\otimes f_0)\otimes(x_2\otimes f_1)\otimes\cdots\otimes(x_n\otimes f_n)\big) \;=\; P(x_0,\dots,x_n)\cdot\mathrm{Res}\big(f_0\cdot\partial f_1\wedge\cdots\wedge\partial f_n\big).$$

Here $x_\nu\in\mathfrak{g}$, $f_\nu\in A_n^\bullet$, ∂ is the degree $(1,0)$ differential in $A_n^{\bullet\bullet}\supset A_n^\bullet$, and we consider Res *as a functional on the entire $A_n^{\bullet\bullet}$ vanishing on all $A_n^{p,q}$ with $(p,q)\neq(n,n-1)$.*

This functional is of total degree 2, is symmetric and is annihilated by both differentials d_{Lie} and D. Therefore it is a cocycle in $\mathrm{CE}^2(\mathfrak{g}_n^\bullet)$ and defines a class $[\gamma_P] \in \mathbb{H}^2_{\mathrm{Lie}}(\mathfrak{g}_n^\bullet)$.

(b) Assume $\mathfrak{g}$ semisimple. The correspondence $P \mapsto [\gamma_P]$ given an embedding $S^{n+1}(\mathfrak{g}^*)^{\mathfrak{g}} \subset \mathbb{H}^2_{\mathrm{Lie}}(\mathfrak{g}_n^\bullet)$.

Example 3.2.2. let $\mathfrak{g} = \mathfrak{gl}_r$ and $P_{\mathrm{tr}}(x) = \mathrm{tr}(x^{n+1})$ or, in the polarized form,

$$P_{\mathrm{tr}}(x_0, \ldots, x_n) \;=\; \frac{1}{(n+1)!} \sum_{s \in S_{n+1}} \mathrm{tr}(x_{s(0)} \cdots x_{s(n)}).$$

In this case $\gamma_{P_{\mathrm{tr}}}$ is the image of the residue cocycle $\rho \in C^1_\lambda(A_n^\bullet)$ under the composite map

$$C^1_\lambda(A_n^\bullet) \xrightarrow{\mathrm{tr}^*} C^1_\lambda(\mathfrak{gl}_r(A_n^\bullet)) \xrightarrow{\theta_{A_n^\bullet}} \mathrm{CE}^2(\mathfrak{gl}_r(A_n^\bullet)),$$

where tr^* is dual to the trace map $\mathrm{tr}_{A_n^\bullet,n}$ from Proposition 2.1.5, and $\theta_{A_n^\bullet}$ is dual to the map $\theta^{A_n^\bullet}$ from Proposition 3.1.2. In particular, we see that $\gamma_{P_{\mathrm{tr}}}$ is a Chevalley-Eilenberg cocycle satisfying all the conditions of part (a) of Theorem 3.2.1.

Proof of Theorem 3.2.1: (a) Let $\phi : \mathfrak{g} \to \mathfrak{gl}_r$ be a representation of $\mathfrak{g}$. Take $P_\phi(x) = \mathrm{tr}(\phi(x)^{n+1})$. In this case γ_{P_ϕ} is induced from $\gamma_{P_{\mathrm{tr}}} \in \mathrm{CE}^2(\mathfrak{gl}_r(A_n^\bullet))$ by ϕ and therefore satisfies the conditions of (a). Further, notice that γ_P depends on P in a linear way. Now, the statement follows from the next lemma, to be proved further below.

Lemma 3.2.3. *Fix $m \geqslant 0$. Any $P \in S^m(\mathfrak{g}^*)^{\mathfrak{g}}$ is a linear combination of polynomials of the form $P_{\phi,m} : x \mapsto \mathrm{tr}(\phi(x)^m)$.*

(b) Note that $\mathrm{CE}^\bullet(\mathfrak{g}_n^\bullet)$ is the total complex of a bicomplex $\mathrm{CE}^{\bullet\bullet}(\mathfrak{g}_n^\bullet)$ with

$$\mathrm{CE}^{pq}(\mathfrak{g}_n^\bullet) \;=\; \mathrm{Hom}^p_{\mathbf{k}}(\Lambda^q \mathfrak{g}_n^\bullet, \mathbf{k}),$$

and with differentials d_{Lie} and D. We consider the corresponding spectral sequence

$$E_1^{pq} \;=\; H_D^{p,q}\big(\mathrm{CE}^{\bullet\bullet}(\mathfrak{g}_n^\bullet)\big) \;=\; \mathrm{Hom}^p_{\mathbf{k}}\big(\Lambda^q H^\bullet_{\bar\partial}(\mathfrak{g}_n^\bullet), \mathbf{k}\big) \;\Rightarrow\; \mathbb{H}^{p+q}_{\mathrm{Lie}}(\mathfrak{g}_n^\bullet). \tag{3.2.4}$$

Since $\mathfrak{g}_n^\bullet$ has $\overline{\partial}$-cohomology only in degrees 0 and $(n-1)$, the spectral sequence is supported in the fourth quadrant, on the horizontal lines

$$q \geqslant 0,\ p = l(1-n),\ l \geqslant 0.$$

Since γ_P is annihilated by both differentials, it gives a permanent cycle in $E_1^{-n+1,n+1}$, denote it (γ_P). For the class $[\gamma_P]$ in $\mathbb{H}^2_{\mathrm{Lie}}(\mathfrak{g}_n^\bullet)$ to be zero, (γ_P) must be killed by some differential of the spectral sequence. From the shape of it, the only possible such differential is $d_n : E_n^{0,1} \to E_n^{-n+1,n+1}$. But, denoting $\mathfrak{g}[z] = \mathfrak{g}[z_1, \ldots, z_n]$, we have

$$E_2^{0,1} \;=\; H^1_{\mathrm{Lie}}(H^0_{\overline{\partial}}(\mathfrak{g}_n^\bullet)) \;=\; H^1_{\mathrm{Lie}}(\mathfrak{g}[z]) \;=\; \big(\mathfrak{g}[z]/\big[\mathfrak{g}[z], \mathfrak{g}[z]\big]\big)^*$$

and this vanishes for a semi-simple $\mathfrak{g}$. Therefore $E_n^{0,1} = 0$ and (γ_P) cannot be killed. Theorem 3.2.1 is proved.

Proof of the lemma 3.2.3: Consider the completion $\widehat{S}^\bullet(\mathfrak{g}^*)$, i.e., the ring of formal power series on $\mathfrak{g}$ near 0, with its natural adic topology. To any representation ϕ of $\mathfrak{g}$ we associate the invariant series

$$Q_\phi(x) \;=\; \mathrm{tr}(e^{\phi(x)}) \;=\; \sum_{m=0}^{\infty} \frac{1}{m!} P_{\phi,m}(x) \;\in\; \widehat{S}^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}.$$

By separating the series into homogeneous components, the lemma is equivalent to the following statement.

Lemma 3.2.5. *The* $\mathbf{k}$*-linear space spanned by series* $Q_\phi(x)$ *for finite dimensional representations* ϕ *of* $\mathfrak{g}$*, is dense in* $\widehat{S}^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$*.*

Proof of Lemma 3.2.5: Let G be a reductive algebraic group with Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ identifies $\widehat{S}^\bullet(\mathfrak{g}^*)$ with $\widehat{\mathcal{O}}_{G,1}$, the completed local ring of G at 1, in a way compatible with the adjoint $\mathfrak{g}$-action on both spaces. Now, $\mathbf{k}[G]$, the coordinate algebra of G, is dense in $\widehat{\mathcal{O}}_{G,1}$. Let $\Phi : G \to GL_r$ be an algebraic representation of G. Then the character $\mathrm{tr}(\Phi)$ is a $\mathfrak{g}$-invariant element in $\mathbf{k}[G]^{\mathfrak{g}}$, and such elements span $\mathbf{k}[G]^{\mathfrak{g}}$. Therefore the space spanned by the $\mathrm{tr}(\Phi)$, is dense in $\widehat{\mathcal{O}}^{\mathfrak{g}}_{G,1}$. If now ϕ is the representation of $\mathfrak{g}$ tangent to Φ, then the image of $\mathrm{tr}(\Phi)$ under the above identification $\exp^* : \widehat{\mathcal{O}}^{\mathfrak{g}}_{G,1} \to \widehat{S}^\bullet(\mathfrak{g}^*)^{\mathfrak{g}}$, is precisely the series $Q_\phi(x)$. So the space of such series is dense in $\widehat{S}^\bullet(\mathfrak{g}^*)$, as claimed. $\square$

Example 3.2.6 (Heisenberg dg-Lie algebra). Let $\mathfrak{g} = \mathfrak{gl}_1$ (abelian) and $P(x) = x^{n+1}$. In this case the (2-)cocycle

$$\gamma(f_0, \ldots, f_n) \ = \ \mathrm{Res}\big(f_0\, \partial f_1 \cdots \partial f_n\big)$$

defines a central extension $\mathcal{H}_n$ of the abelian dg-Lie algebra $\mathfrak{g}_n^\bullet = A_n^\bullet$. The dg-Lie algebra $\mathcal{H}_n$ is the n-dimensional analog of the Heisenberg Lie algebra associated to the vector space $\mathbf{k}((z))$ equipped with the skew-symmetric form $[f_0, f_1] = \mathrm{Res}(f_0 df_1)$.

4 The Tate extension and local Riemann-Roch

4.1 Background on Tate complexes

A. The quasi-abelian category of linearly topological spaces. We will use the concept of a *quasi-abelian category* [Sch], a weakening of that of an abelian category. In particular, in a quasi-abelian category $\mathcal{A}$ any morphism $a : V \to W$ has categorical kernel $\mathrm{Ker}(u)$, cokernel, image $\mathrm{Im}(a) = \mathrm{Ker}\{W \to \mathrm{Coker}(a)\}$ and the coimage $\mathrm{Coim}(a) = \mathrm{Coker}\{\mathrm{Ker}(a) \to V\}$ but the canonical morphism $\mathrm{Coim}(a) \to \mathrm{Im}(a)$ need not be an isomorphism. If it is, a is called *strict*.

As pointed out in [Sch], any quasi-abelian category $\mathcal{A}$ has an intrinsic structure of exact category in the sense of Quillen.

Example 4.1.1. Let $\mathrm{TopVect}_{\mathbf{k}}$ be the category of Hausdorff linearly topological $\mathbf{k}$-vector spaces with a countable base of neighborhoods of 0. Here $\mathbf{k}$ is considered with discrete topology. See [Le], Ch. 2 for background; in this paper we additionally impose the countable base assumption.

For a morphism $a : V \to W$, the kernel $\mathrm{Ker}(a) \subset V$ is the usual kernel with induced topology, $\mathrm{Im}(a)$ is the closure of the set-theoretic image, and $\mathrm{Coim}(a)$ is the set-theoretical quotient of V by $\mathrm{Ker}(a)$ with the quotient topology.

The category $\mathrm{TopVect}_{\mathbf{k}}$ is analogous to several categories treated in the literature [Pro1] [ProS] [Pro2]. In particular, arguments similar to [Pro2] Cor. 3.1.5 and Prop. 3.1.8, give:

Proposition 4.1.2. *(a) The category* $\mathrm{TopVect}_{\mathbf{k}}$ *is quasi-abelian.*

(b) A morphism $a : V \to W$ in $\mathrm{TopVect}_{\mathbf{k}}$ *is strict, if its set-theoretical image is closed and a is quasi-open. That is, for any open subspace $U \subset V$ there is an open subspace $U' \subset W$ such that $a(U) \supset a(V) \cap U'$.* $\square$

Every quasi-abelian category $\mathcal{A}$ gives rise to the (bounded) derived category $D^b(\mathcal{A})$ equipped with a canonical t-structure whose heart ${}^\heartsuit\mathcal{A}$ (called *left heart* in [Sch]), is a natural abelian envelope of $\mathcal{A}$. It is equipped with a fully faithful left exact embedding $h : \mathcal{A} \to {}^\heartsuit\mathcal{A}$ (so h preserves kernels but not cokernels).

Objects of ${}^\heartsuit\mathcal{A}$ can be thought of as formal "true cokernels" of monomorphisms a in $\mathcal{A}$ and in fact have the form $\mathrm{Coker}_{{}^\heartsuit\mathcal{A}}(a)$ (actual cokernels in ${}^\heartsuit\mathcal{A}$). See [Sch] Cor. 1.2.21 or, in the more general framework of exact categories, [La], Def. 1.5.7.

Example 4.1.3. The formal quotient $\mathbf{k}[[t]]/\mathbf{k}[t]$ represents an object of the abelian category ${}^\heartsuit\mathrm{TopVect}_\mathbf{k}$. The short exact sequence

$$0 \to \mathbf{k}[t] \longrightarrow \mathbf{k}[[t]] \longrightarrow \mathbf{k}[[t]]/\mathbf{k}[t] \to 0$$

represents a nontrivial extension in ${}^\heartsuit\mathrm{TopVect}_\mathbf{k}$.

One can identify $D^b(\mathcal{A})$ with the localization

$$D^b(\mathcal{A}) \;\simeq\; K^b(\mathcal{A})[\mathrm{qis}^{-1}] \tag{4.1.4}$$

of $K^b(\mathcal{A})$, the homotopy category of bounded complexes over $\mathcal{A}$, with respect to quasi-isomorphisms (understood in the sense of complexes over ${}^\heartsuit\mathcal{A}$), see [La] [Sch]. Further, the natural functor $D^b(\mathcal{A}) \to D^b({}^\heartsuit\mathcal{A})$ into the usual bounded derived category of $\mathcal{A}$, is an equivalence ([Sch] Prop. 1.2.32).

Remarks 4.1.5. (a) We note that $K^b(\mathcal{A})$ has a natural dg-enhancement: it is the H^0-category of the dg-category of bounded complexes over $\mathcal{A}$. Therefore (4.1.4) can be used to represent $D^b(\mathcal{A})$ as the H^0-category of a dg-category, by using the Gabriel-Zisman localization for dg-categories ([To5], §2.1) which is the dg-analog of the Dwyer-Kan simplicial localization for categories [DK1][DK2].

(b) The concept of a quasi-abelian category is self-dual. Therefore there exist another abelian envelope $\mathcal{A}^\heartsuit = ({}^\heartsuit(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$ (the right heart) whose objects can be thought as formal "true kernels" of epimorphisms in $\mathcal{A}$, and a right exact embedding $\mathcal{A} \to {}^\heartsuit\mathcal{A}$. In our example $\mathcal{A} = \mathrm{TopVect}_\mathbf{k}$, the categorical kernels coincide with set-theoretical ones so it is natural to use the left heart to keep the kernels unchanged.

Proposition 4.1.6. *Let $\mathcal{A}$ be quasi-abelian. A bounded complex $V^\bullet$ over $\mathcal{A}$ has all $H^i(V^\bullet) \in \mathcal{A} \subset {}^\heartsuit\mathcal{A}$, if and only if all the differentials are strict. In particular, for a monomorphism $a : V \to W$ in $\mathcal{A}$, the cokernel* $\mathrm{Coker}_{\heartsuit\mathcal{A}}(a)$ *lies in $\mathcal{A}$ if and only if a is strict.*

Proof: See [Pro1], Cor. 1.13. □

B. Tate spaces and Tate complexes. For $V \in \mathrm{TopVect}_{\mathbf{k}}$ we have the *topological dual* $V^\vee = \underline{\mathrm{Hom}}_{\mathbf{k}}(V, \mathbf{k})$ (continuous linear functionals, with weak topology). The functor $V \mapsto V^\vee$ is not a perfect duality on $\mathrm{TopVect}_{\mathbf{k}}$; however, the canonical morphism $V \to V^{\vee\vee}$ is an isomorphism on the following full subcategories in $\mathrm{TopVect}_{\mathbf{k}}$:

(1) The category $\mathrm{Vect}_{\mathbf{k}}$ of discrete (at most countably dimensional) vector spaces $V \simeq \bigoplus_{i\in I} \mathbf{k}$.

(2) The category $\mathrm{LC}_{\mathbf{k}}$ of linearly compact spaces $V \simeq \prod_{i\in I} \mathbf{k}$.

(3) The category $\mathrm{Ta}_{\mathbf{k}}$ of locally linearly compact spaces ([Le], Ch. 2, §6) which we will call *Tate spaces*. Each Tate space V can be represented as $V \simeq V^d \oplus V^c$ with $V^d \in \mathrm{Vect}_{\mathbf{k}}$ and $V^c \in \mathrm{LC}_{\mathbf{k}}$.

Thus, the topological dual identifies

$$\mathrm{Vect}_k^{\mathrm{op}} \simeq \mathrm{LC}_{\mathbf{k}}, \quad \mathrm{LC}_{\mathbf{k}}^{\mathrm{op}} \simeq \mathrm{Vect}_{\mathbf{k}}, \quad \mathrm{Ta}_{\mathbf{k}}^{\mathrm{op}} \simeq \mathrm{Ta}_{\mathbf{k}}.$$

In particular, since $\mathrm{Vect}_{\mathbf{k}}$ is abelian, so is $\mathrm{LC}_{\mathbf{k}}$, while $\mathrm{Ta}_{\mathbf{k}}$ is a self-dual quasi-abelian (in particular, exact) category, cf. [Be2] [BGW]. Let us add two more examples to the above list:

(4) The quasi-abelian category $\mathrm{ILC}_{\mathbf{k}}$ formed by *inductive limits of linearly compact spaces.*

(5) The quasi-abelian category $\mathrm{PVect}_{\mathbf{k}}$ formed by *projective limits of discrete vector spaces* or, what is the same, objects in TopVect which, considered as topological vector spaces, are *complete.*

Example 4.1.7. The space of Laurent series $\mathbf{k}((z)) = \mathbf{k}[[z]][z^{-1}]$ is an object of $\mathrm{Ta}_{\mathbf{k}}$. The localized ring $\mathbf{k}[[z_1, z_2]][(z_1 z_2)^{-1}]$ is an object of $\mathrm{ILC}_{\mathbf{k}}$ but not of $\mathrm{Ta}_{\mathbf{k}}$. Similarly, the ring $\mathbf{k}[[z]][z^*][(zz^*)^{-1}]$, see §1.2 C, is an object of $\mathrm{ILC}_{\mathbf{k}}$.

Proposition 4.1.8. *We have*

$$\mathrm{ILC}_k^{\mathrm{op}} \simeq \mathrm{PVect}_{\mathbf{k}}, \quad \mathrm{PVect}_{\mathbf{k}}^{\mathrm{op}} \simeq \mathrm{ILC}_{\mathbf{k}}, \quad \mathrm{Ta}_{\mathbf{k}} = \mathrm{ILC}_{\mathbf{k}} \cap \mathrm{PVect}_{\mathbf{k}},$$

the first two identifications given by forming the topological dual.

Proof: Let us prove the third statement. It is clear that $\mathrm{Ta}_{\mathbf{k}} \subset \mathrm{ILC}_{\mathbf{k}} \cap \mathrm{PVect}_{\mathbf{k}}$. Let us prove the inverse inclusion. Suppose

$$V \;=\; \varprojlim \left\{ \cdots \xrightarrow{q_3} V_2 \xrightarrow{q_2} V_1 \xrightarrow{q_1} V_0 \right\} \;\in \mathrm{PVect}_{\mathbf{k}}$$

is represented as the projective limit of a diagram (V_i) of discrete vector spaces and surjections q_i. Then V is Tate, if and only if $\mathrm{Ker}(q_i)$ is finite-dimensional for all but finitely many i. Suppose this is not so. Then we can, without loss of generality, assume that all $\mathrm{Ker}(q_i)$ are inifinite-dimensional, by composing finite strings of the arrows in (V_i) and getting a diagram with the same projective limit.

With this assumption, suppose that $V = \varinjlim_j L_j$ where $(L_j)_{j \geqslant 0}$ is an increasing chain of linearly compact subspaces. Then, for each j,

$$L_j \;=\; \varprojlim \left\{ \cdots \longrightarrow V_2^{(j)} \longrightarrow V_1^{(j)} \longrightarrow V_0^{(j)} \right\}$$

where $V_i^{(j)}$ is the image of L_j in V_i, a finite-dimensional subspace in V_i. We now construct an element $v \in V$, i.e., a compatible system $(v_i \in V_i)$, by a version of the Cantor diagonal process. That is, we take $v_0 = 0$, then take v_1 from $\mathrm{Ker}(q_1)$ (an infinite-dimensional space) not lying in $V_1^{(1)}$ (a finite-dimensional space). Then take $v_2 \in V_2$ with $q_2(v_2) = v_1$ in such a way that $v_2 \notin V_2^{(2)}$ (this is possible since $\mathrm{Ker}(q_2)$ is infinite-dimensional), and so on. We get an element (v_i) of the projective limit with $v_i \notin V_i^{(i)}$ for all i. Such an element cannot lie in the union of the L_i. □

Definition 4.1.9. (a) A *Tate complex* over $\mathbf{k}$ is a bounded complex $V^\bullet$ over $\mathrm{ILC}_{\mathbf{k}}$ whose cohomology groups $H^i(V^\bullet) \in {}^\heartsuit\mathrm{ILC}_{\mathbf{k}}$ belong to ${}^\heartsuit\mathrm{Ta}_{\mathbf{k}}$. We denote by $\mathrm{Tate}_{\mathbf{k}}$ the full dg-subcategory in $C^b(\mathrm{ILC}_{\mathbf{k}})$ formed by Tate complexes.

(b) A Tate complex $V^\bullet$ is called *strict*, if all the $H^i(V^\bullet)$ belong to $\mathrm{Ta}_{\mathbf{k}} \subset {}^\heartsuit\mathrm{Ta}_{\mathbf{k}}$ or, what is the same, if its differentials are strict (Prop. 4.1.6).

Note that $\mathrm{Tate}_{\mathbf{k}}$ is a perfect dg-subcategory, i.e., $[\mathrm{Tate}_{\mathbf{k}}]$, the corresponding cohomology category, is triangulated and closed under direct summands.

Example 4.1.10. The Jouanolou complex $A_n^\bullet$ is a strict Tate complex. More precisely, the topology on each A_n^p is given by the convergence of series. An explicit representation of A_n^p as an inductive limit of linearly compact spaces is given by the filtration of Corollary 1.2.8. Thus $A_n^\bullet$ is a complex over $\mathrm{ILC}_{\mathbf{k}}$. As we have seen, its cohomology groups are Tate vector spaces. For $n > 0$, we have only two cohomology spaces: H^0, linearly compact and H^{n-1}, discrete.

More generally, for any finite dimensional vector bundle E on D_n°, the complex $A_n^\bullet(E)$ is naturally made into a strict Tate complex.

C. Tate complexes, algebraically. For a category $\mathcal{C}$ we denote by $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ the category of countable ind- and pro-objects in $\mathcal{C}$, see [AM], [KS] for general background. In particular, we will use the notation "$\varinjlim$"$_{i\in I} C_i$ for an object of $\mathrm{Ind}(\mathcal{C})$ represented by a filtering inductive system $(C_i)_{i\in I}$ over $\mathcal{C}$. Similarly for "$\varprojlim$"$_{i\in I} C_i$, an object of $\mathrm{Pro}(\mathcal{C})$.

Assume $\mathcal{C}$ is abelian. Then so are $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$. In this case we denote by $\mathrm{Ind}^s(\mathcal{C}), \mathrm{Pro}^s(\mathcal{C})$ the full subcategories formed by ind- and pro-objects which are *essentially strict* i.e., isomorphic to objects "$\varinjlim$"$_{i\in I} C_i$, resp. "$\varprojlim$"$_{i\in I} C_i$ where (C_i) is an filtering inductive (resp. projective) system formed by monomorphisms (resp. epimorphisms). These are quasi-abelian but not, in general, abelian categories.

Let $\mathrm{Vect}_{\mathbf{k}}^f$ be the category of finite-dimensional $\mathbf{k}$-vector spaces.

Proposition 4.1.11. *(a) We have*

$$\mathrm{Ind}^s(\mathrm{Vect}_{\mathbf{k}}^f) = \mathrm{Ind}(\mathrm{Vect}_{\mathbf{k}}^f) \simeq \mathrm{Vect}_{\mathbf{k}}, \quad \mathrm{Pro}^s(\mathrm{Vect}_{\mathbf{k}}^f) = \mathrm{Pro}(\mathrm{Vect}_{\mathbf{k}}^f) \simeq \mathrm{LC}_{\mathbf{k}}.$$

(b) Further, the taking of inductive or projective limits in $\mathrm{TopVect}_{\mathbf{k}}$ *gives identifications*

$$\mathrm{Ind}^s(\mathrm{LC}_{\mathbf{k}}) \simeq \mathrm{ILC}_{\mathbf{k}}, \quad \mathrm{Pro}^s(\mathrm{Vect}_{\mathbf{k}}) \simeq \mathrm{PVect}_{\mathbf{k}}.$$

(c) The abelian envelopes of the semi-abelian categories in (b) are identified as

$$^\heartsuit\mathrm{ILC}_{\mathbf{k}} \simeq \mathrm{Ind}(\mathrm{LC}_{\mathbf{k}}), \quad ^\heartsuit\mathrm{PVect}_{\mathbf{k}} \simeq \mathrm{Pro}(\mathrm{Vect}_{\mathbf{k}})$$

(the categories of all, not necessarily strict, ind- and pro-objects).

Proof: (a) This is well known. For instance, because of the Noetherian and Artinian property of $\mathrm{Vect}_{\mathbf{k}}^f$, and ind- or pro-object in this category is essentially strict. Part (b) is also clear.

(c) Let us prove the first statement, the second one is dual. Consider the abelian category $\mathcal{A}$ of arbitrary chains of morphisms (inductive systems)

$$V = \{V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots\} \tag{4.1.12}$$

in $\mathrm{LC}_{\mathbf{k}}$, i.e., of graded $\mathbf{k}[t]$-modules in $\mathrm{LC}_{\mathbf{k}}$. Let $\mathcal{A}^s \subset \mathcal{A}$ be the semi-abelian subcategory formed by chains of monomorphisms, i.e., torsion-free $\mathbf{k}[t]$-mod-ules. Every object V of $\mathcal{A}$ can be represented as the cokernel of a monomorphism i in $\mathcal{A}^s$. More precisely, we have a short exact sequence

$$0 \to K \xrightarrow{i} V \otimes_{\mathbf{k}} \mathbf{k}[t] \xrightarrow{c} V \to 0. \tag{4.1.13}$$

Here $V \otimes_{\mathbf{k}} \mathbf{k}[t]$ is the free $\mathbf{k}[t]$-module generated by V as a graded vector space, c is the canonical map given by the $\mathbf{k}[t]$-module structure and $K = \mathrm{Ker}(c)$. This implies that any object in $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$ is the cokernel of a monomorphism in $\mathrm{Ind}^s(\mathrm{LC}_{\mathbf{k}})$. □

Proposition 4.1.14. *(a) The objects of* $\mathrm{ILC}_k \subset {}^{\heartsuit}\mathrm{ILC}_k = \mathrm{Ind}(\mathrm{LC}_k)$ *are projective. In particular, Tate spaces are projective objects in the abelian category* ${}^{\heartsuit}\mathrm{Ta}_{\mathbf{k}}$.

(b) The abelian categories $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$, $\mathrm{Pro}(\mathrm{Vect}_{\mathbf{k}})$ *have homological dimension 1.*

Proof: We start with three lemmas. As in the proof of Proposition 4.1.11(c), let $\mathcal{A}$ be the category of diagrams as in (4.1.12). We have a functor

$$\text{“}\varinjlim\text{”} : \mathcal{A} \longrightarrow \mathrm{Ind}(\mathrm{LC}_{\mathbf{k}}), \quad V \mapsto \text{“}\varinjlim\text{”} V_i.$$

Lemma 4.1.15. *“$\varinjlim$” is an exact, essentially surjective functor. Furthermore, any epimorphism in* $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$ *is isomorphic (in the category of arrows) to an image of an epimorphism in $\mathcal{A}$.*

Proof of the lemma: Exactness of “$\varinjlim$” and the statement about epimorphisms are general properties of ind-categories of abelian categories, see [KS], Lemmas 8.6.4 and 8.6.7. Next, any countable filtering category admits a cofinal map from the poset $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. This means that any countably indexed ind-object is isomorphic to the object of the image of “$\varinjlim$” as above. □

Lemma 4.1.16. *Objects of $\mathcal{A}^s$ are projective in $\mathcal{A}$.*

Proof of the lemma: A diagram V as in (4.1.12) formed by embeddings, is a free graded $\mathbf{k}[t]$-module in $\mathrm{LC}_{\mathbf{k}}$. Indeed, since $\mathrm{LC}_{\mathbf{k}} = \mathrm{Vect}_{\mathbf{k}}^{\mathrm{op}}$ is a semisimple abelian category, the embedding $V_{i-1} \to V_i$ admits a direct sum complement W_i. Considering $W = (W_i)$ as a $\mathbb{Z}_+$-graded object in $\mathrm{LC}_{\mathbf{k}}$, we see that $V \simeq W \otimes_{\mathbf{k}} \mathbf{k}[t]$ is free. This, and semisimplicity of $\mathrm{LC}_{\mathbf{k}}$, implies projectivity of V.

□

Let us call an object $V \in \mathcal{A}$ essentially strict, if "$\varinjlim$"(V) is an essentially strict object of $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$, i.e., is isomorphic to "$\varinjlim$"(M) where $M \in \mathcal{A}^s$.

Lemma 4.1.17. *If $V \in \mathcal{A}$ is essentially strict, then there is $M \in \mathcal{A}^s$ and an epimorphism $q : V \to M$ in $\mathcal{A}$ such that "$\varinjlim$"(q) is an isomorphism in* $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$.

Proof of the lemma: For each i consider the diagram of epimorphisms

$$V_i \longrightarrow \mathrm{Im}\{V_i \to V_{i+1}\} \longrightarrow \mathrm{Im}\{V_i \to V_{i+2}\} \longrightarrow \cdots .$$

If V is essentially strict, these epimorphisms eventually become isomorphisms, so the terms of the diagram stabilize to some $M_i \in \mathrm{LC}_{\mathbf{k}}$, and we get the diagram of monomorphisms

$$M = \{M_0 \longrightarrow M_1 \longrightarrow \cdots\}.$$

We see that $M \in \mathcal{A}^s$ equipped with a natural epimorphism $q : V \to M$ in $\mathcal{A}$ which induces an isomorphism on the "$\varinjlim$". □

We now prove part (a) of Proposition 4.1.14. Let $f : A \to B$ be an epimorphism in $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$ with B essentially strict. We prove that f splits. By Lemma 4.1.15, f is isomorphic to $f' =$ "$\varinjlim$"(g) where $g : N \to V$ is a surjection in $\mathcal{A}$. It is enough to prove that f' splits. Now, $V \in \mathcal{A}$ is essentially strict, so by Lemma 4.1.17, there is a surjection $q : V \to M$ in $\mathcal{A}$ with $M \in \mathcal{A}^s$ and such that "$\varinjlim$"(q) is an isomorphism. It is enough therefore to prove that the epimorphism $f'' =$ "$\varinjlim$"(qg) splits in $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$. But $qg : N \to M$ is a surjection in $\mathcal{A}$ with M projective. So qg splits in $\mathcal{A}$ and therefore f'' splits in $\mathrm{Ind}(\mathrm{LC}_{\mathbf{k}})$. This proves part (a).

Part (b), follows by considering the 2-term resolution (4.1.13). Proposition 4.1.14 is proved.

Corollary 4.1.18. *(a) Every object of* $D^b(\mathrm{ILC}_{\mathbf{k}}) \simeq D^b({}^{\heartsuit}\mathrm{ILC}_{\mathbf{k}})$ *is quasi-isomorphic to its graded object of cohomology equipped with zero differential.*

(b) The triangulated category $\mathrm{Tate}_{\mathbf{k}}$ *is equivalent to the bounded derived category of the abelian category* ${}^{\heartsuit}\mathrm{Ta}_{\mathbf{k}}$*.*

(c) Any strict Tate complex $V^\bullet$ *can be split into a direct sum of complexes (over* $\mathrm{ILC}_{\mathbf{k}}$*)* $V^\bullet = H^\bullet \oplus E^\bullet$*, where* $E^\bullet$ *is exact and* $H^\bullet$ *has zero differential (and is thus a graded Tate space isomorphic to* $H^\bullet(V^\bullet)$*.)* □

Inside $\mathrm{Tate}_{\mathbf{k}}$ we have the full dg-subcategories: $\mathrm{Perf}_{\mathbf{k}}$, complexes with finite-dimensional cohomology; $D_{\mathbf{k}}$, complexes with discrete cohomology; $C_{\mathbf{k}}$, complexes with linearly compact cohomology. Thus $\mathrm{Perf}_{\mathbf{k}} = C_{\mathbf{k}} \cap D_{\mathbf{k}}$ and the associated cohomology categories are identified as

$$[\mathrm{Perf}_{\mathbf{k}}] \simeq D^b(\mathrm{Vect}^f_{\mathbf{k}}), \quad [D_{\mathbf{k}}] \simeq D^b(\mathrm{Vect}_k), \quad [C_k] \simeq D^b(\mathrm{LC}_{\mathbf{k}}).$$

Proposition 4.1.19. $[\mathrm{Tate}_{\mathbf{k}}]$*is the smallest strictly full triangulated subcategory in* $D^b(\mathrm{ILC}_{\mathbf{k}})$ *which contains* $[D_{\mathbf{k}}], [C_{\mathbf{k}}]$ *and is closed under direct summands.*

Proof: Let $\mathcal{T}$ be the smallest subcategory in question. Then $[\mathrm{Tate}_{\mathbf{k}}] \subset \mathcal{T}$, because any object of the quasi-abelian category $\mathrm{Ta}_{\mathbf{k}}$ is a direct sum of an object from $C_{\mathbf{k}}$ and an object from $D_{\mathbf{k}}$. Conversely, ${}^{\heartsuit}\mathrm{Ta}_{\mathbf{k}} \subset {}^{\heartsuit}\mathrm{ILC}_{\mathbf{k}}$ is closed under extensions. Therefore forming cones and direct summands, starting with $\mathrm{Ob}(C_{\mathbf{k}}) \cup \mathrm{Ob}(D_{\mathbf{k}})$ will always give complexes whose cohomology objects lie in ${}^{\heartsuit}\mathrm{Ta}_{\mathbf{k}}$. □

4.2 Tate objects in dg-categories

A. Ind- and pro-objects in dg-categories. For background on ind- and pro-objects in a (stable) ∞-category $\mathcal{C}$ we refer to [Lu7], §5.3. In this paper we consider only ind- and pro-objects represented by countable filtering diagrams. Such objects form new ∞-categories $\mathrm{Ind}(\mathcal{C})$, $\mathrm{Pro}(\mathcal{C})$. Thus, objects of $\mathrm{Ind}(\mathcal{C})$ can be represented by symbols "$\underrightarrow{\mathrm{holim}}_{i\in I}$" x_i where I is a countable filtering ∞-category and $(x_i)_{i\in I}$ is an ∞-functor $I \to \mathcal{C}$. Similarly, objects of $\mathrm{Pro}(\mathcal{C})$ can be represented by symbol "$\underleftarrow{\mathrm{holim}}_{i\in I}$" y_i, where I is as before and $(y_i)_{i\in I}$ is an ∞-functor $I^{\mathrm{op}} \to \mathcal{C}$.

We apply these concepts to dg-categories by converting them to $\mathbf{k}$-linear ∞-categories by using the dg-nerve construction [C][Fao]. The resulting (countable) ind- and pro-categories associated to a dg-category $\mathcal{A}$ will be denoted $\mathrm{Ind}(\mathcal{A})$ and $\mathrm{Pro}(\mathcal{A})$. They are still $\mathbf{k}$-linear ∞-categories that we can and will see as dg-categories. We note that, with this understanding, $\mathrm{Ind}(\mathcal{A})$ and $\mathrm{Pro}(\mathcal{A})$ are perfect, whenever $\mathcal{A}$ is perfect. Let us point out the following more explicit description.

Proposition 4.2.1. *Let $\mathcal{A}$ be a dg-category. Then:*

(I0) $\mathrm{Ind}(\mathcal{A})$ *is quasi-equivalent to its full dg-subcategory whose objects are* "$\underrightarrow{\mathrm{holim}}$"$x_i$, *where*

$$(x_i) \;=\; \left\{x_0 \xrightarrow{f_{01}} x_1 \xrightarrow{f_{12}} x_2 \xrightarrow{f_{23}} \cdots \right\}$$

is a diagram consisting of objects $x_i \in \mathcal{A}$, $i \geqslant 0$ and closed degree 0 *morphisms $f_{i,i+1} : x_i \to x_{i+1}$.*

(I1) We have quasi-isomorphisms ("ϵ-δ formula"):

$$\mathrm{Hom}^\bullet_{\mathrm{Ind}(\mathcal{A})}\big(\text{“}\underrightarrow{\mathrm{holim}}\text{”}x_i, \text{“}\underrightarrow{\mathrm{holim}}\text{”}y_j\big) \;\simeq\; \underleftarrow{\mathrm{holim}}_i \; \underrightarrow{\mathrm{holim}}_j \, \mathrm{Hom}^\bullet_{\mathcal{A}}(x_i, y_j),$$

where the homotopy limits on the right are taken in the model category $\mathrm{dgVect}_{\mathbf{k}}$ *of complexes.*

Remarks 4.2.2. (a) The $\underrightarrow{\mathrm{holim}}_j$ in (I1) is the same as the naive inductive limit in $\mathrm{dgVect}_{\mathbf{k}}$, see Proposition A.3(a).

(b) By duality we get a description of $\mathrm{Pro}(\mathcal{A}) = \mathrm{Ind}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ in terms of symbols "$\underleftarrow{\mathrm{holim}}$"$x_i$ where (x_i) is a diagram of objects and closed degree 0 morphisms in the order opposite to that of (I0).

Proof of Proposition 4.2.1: By definition, objects of $\mathrm{Ind}(\mathcal{A})$ are represented by ∞-functors $I \to \mathcal{A}$ where I is a filtering ∞-category. In our setting we assume that $\mathrm{Ob}(I)$ is at most countable. As in the classical case, any filtering I admits a cofinal ∞-functor from a filtering poset ([Lu7], Prop. 5.3.1.16), and in the countable case we can take this poset to be $\mathbb{Z}_+$. Now, the category corresponding to the poset $\mathbb{Z}_+$, is freely generated by the elementary arrows $i \to i+1$. Therefore any ∞-functor $x : \mathbb{Z}_+ \to \mathcal{A}$ can be replaced by an honest functor obtained by extending x from these elementary arrows. Such functors are precisely the data in (I0). Finally, the appearance of the "ϵ-δ formula" in (I1) from conceptual properties of $\mathrm{Ind}(\mathcal{A})$ is explained in [Lu7], p.378. □

B. Tate A-modules. Let A be a $\mathbb{Z}_{\leqslant 0}$-graded commutative dg-algebra. For any $m \in \mathbb{Z}$ we then have the full dg-category $\mathrm{Perf}_A^{\leqslant m} \subset \mathrm{Perf}_A$ formed by those perfect dg-modules over A which, as complexes, are situated in degrees $\leqslant m$. Recall that the duality functor $M \mapsto M^\vee = \mathrm{RHom}_A(M, A)$ identified Perf_A with its dual. We define the full subcategories in Perf_A:

$$\mathrm{Perf}_A^{\geqslant l} = (\mathrm{Perf}_A^{\leqslant -l})^\vee, \quad \mathrm{Perf}_A^{[l,m]} = \mathrm{Perf}_A^{\geqslant l} \cap \mathrm{Perf}_A^{\leqslant m}, \quad l \leqslant m.$$

We define then the ∞ (or dg-) categories

$$\mathbf{D}_A = \bigcup_{l,m} \mathrm{Ind}(\mathrm{Perf}_A^{[l.m]}) \subset \mathrm{Ind}(\mathrm{Perf}_A), \quad \mathbf{C}_A = \bigcup_{l,m} \mathrm{Pro}(\mathrm{Perf}_A^{[l,m]}) \subset \mathrm{Pro}(\mathrm{Perf}_A)$$

formed by ind- or pro-diagrams of which all terms belong to $\mathrm{Perf}_A^{[l,m]}$ for some l, m (depending on the diagram). These categories are dual to each other.

The category $\mathbf{D}_A$ is tensored over the category $\mathrm{Vect}_{\mathbf{k}}$ of all (possibly infinite-dimensional) $\mathbf{k}$-vector spaces. In particular, with each object M it contains $\mathbf{k}[z] \otimes M = \bigoplus_{i=0}^\infty M$, the direct sum of infinitely many copies of M. The category $\mathbf{C}_A$ is tensored over the category of linearly compact topological $\mathbf{k}$-vector spaces. In particular, with each object M it contains $\mathbf{k}[[z]] \widehat{\otimes} M = \prod_{i=0}^\infty M$, the direct product of infinitely many copies of M.

We define the dg-category Tate_A of *Tate A-modules* as the perfect envelope of the full dg-subcategory in $\mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_A))$ whose class of objects is $\mathrm{Ob}(\mathbf{C}_A) \cup \mathrm{Ob}(\mathbf{D}_A)$. Since $\mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_A))$ is a perfect dg-category, we can view Tate_A as the minimal dg-subcategory in $\mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_A))$ containing $\mathbf{C}_A, \mathbf{D}_A$ and closed under forming shifts, cones and homotopy direct summands.

Remark 4.2.3. This is a slight modification of the definition in [He3] in that, besides restricting to countable ind- or pro-diagrams, we force all objects to be "bounded complexes". In particular, an infinite resolution of a non-perfect A-module M (and thus such an M itself) would be an object in $\mathrm{Ind}(\mathrm{Perf}_A)$ but not in $\mathbf{D}_A$. In particular, it is, a priori, not an object of Tate_A.

Proposition 4.2.4. *Let $A = \mathbf{k}$. Then, in comparison with the constructions from §4.1:*

(a) $\mathbf{D}_{\mathbf{k}}$ is quasi-equivalent to $D_{\mathbf{k}} \simeq C^b(\mathrm{Vect}_{\mathbf{k}})$ and $\mathbf{C}_{\mathbf{k}}$ to $C_{\mathbf{k}} \simeq C^b(\mathrm{LC}_{\mathbf{k}})$;

(b) Tate_A (for $A = \mathbf{k}$) is quasi-equivalent to the category of Tate complexes from §4.1.

Proof: (a) Let us show the first identification, the second one follows by duality. It is known that $\mathrm{Ind}(\mathrm{Perf}_{\mathbf{k}}) = \mathrm{dgVect}_{\mathbf{k}}$. (Both homotopy limits in the RHS of (I1) can be replaced by the ordinary projective resp. inductive limits because of Proposition A.3(b1).) Then $D_{\mathbf{k}}$ and $C^b(\mathrm{Vect}_{\mathbf{k}})$ are full subcategories of $\mathrm{dgVect}_{\mathbf{k}}$ and have the same objects.

We now prove part (b). Let us denote temporarily by $\mathrm{Tate}_{A=\mathbf{k}}$ the dg-category obtained by specializing the above definition of Tate_A to $A = \mathbf{k}$, as distinguished from the dg-category $\mathrm{Tate}_{\mathbf{k}}$ defined in §4.1. Let

$$IP_{\mathbf{k}} \;=\; \bigcup_{l,m} \mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}}^{[l,m]})) \;\subset\; \mathrm{Ind}(\mathbf{C}_{\mathbf{k}}).$$

Then $\mathrm{Tate}_{A=\mathbf{k}}$ is the perfect envelope of $\mathbf{CD}_{\mathbf{k}}$, the full dg-subcategory in $IP_{\mathbf{k}}$ on the class of objects $\mathrm{Ob}(\mathbf{C}_{\mathbf{k}}) \cup \mathrm{Ob}(\mathbf{D}_{\mathbf{k}})$. Similarly, since $C^b(\mathrm{ILC}_{\mathbf{k}})$ is a perfect dg-category, Proposition 4.1.19 implies that $\mathrm{Tate}_{\mathbf{k}}$ is identified with the perfect envelope of $CD_{\mathbf{k}}$, the full subcategory in $C^b(\mathrm{ILC}_{\mathbf{k}})$ on the class of objects $\mathrm{Ob}(C_{\mathbf{k}}) \cup \mathrm{Ob}(D_{\mathbf{k}})$. Let us denote both identifications $\mathbf{D}_{\mathbf{k}} \to D_{\mathbf{k}}$ and $\mathbf{C}_{\mathbf{k}} \to C_{\mathbf{k}}$ from part (a) by the same letter λ (taking the limit). Then it is enough to prove the following,

Lemma 4.2.5. *Let $V = $ "$\underrightarrow{\mathrm{holim}}$" $V_i^\bullet$ be an object of $\mathbf{D}_{\mathbf{k}}$ and $W = $ "$\underleftarrow{\mathrm{holim}}$" $W_i^\bullet$ be an object of $\mathbf{C}_{\mathbf{k}}$, with $V_i^\bullet$, resp. $W_i^\bullet$ being an inductive resp. projective system over $\mathrm{Perf}_{\mathbf{k}}^{[l,m]}$. Then the natural morphisms of complexes*

$$\mathrm{Hom}^\bullet_{IP_{\mathbf{k}}}(V, W) \longrightarrow \mathrm{Hom}^\bullet_{C^b(\mathrm{ILC}_{\mathbf{k}})}(\lambda(V), \lambda(W)),$$
$$\mathrm{Hom}^\bullet_{IP_{\mathbf{k}}}(W, V) \longrightarrow \mathrm{Hom}^\bullet_{C^b(\mathrm{ILC}_{\mathbf{k}})}(\lambda(W), \lambda(V))$$

are quasi-isomorphisms.

Proof of the lemma: We can assume that $(V_i^\bullet)$ consists of injective morphisms and $(W_i^\bullet)$ consists of surjective morphisms of perfect complexes. Then, applying the formula (I1) from Proposition 4.2.1 twice, we realize $\mathrm{Hom}^\bullet_{IP_{\mathbf{k}}}(V, W)$ as double $\underleftarrow{\mathrm{holim}}$ of a diagram of perfect complexes and surjective maps. Applying Proposition A.3(b) once, we replace the first $\underleftarrow{\mathrm{holim}}$ by $\underleftarrow{\lim}$ and get a $\underleftarrow{\mathrm{holim}}$ of a diagram of complexes and surjective maps. Applying Proposition A.3(b) once again, we replace the second $\underleftarrow{\mathrm{holim}}$ by $\underleftarrow{\lim}$. After this the result reduces to the set of continuous morphisms of complexes $\lambda(V) \to \lambda(W)$.

Similarly, we realize $\mathrm{Hom}^\bullet_{IP_{\mathbf{k}}}(W, V)$ as double $\underrightarrow{\mathrm{holim}}$ of a diagram of perfect complexes and injective maps. Applying Proposition A.3(a), we reduce it to double $\varinjlim$ which again gives the space of continuous morphisms of complexes $\lambda(W) \to \lambda(V)$. This proves the lemma and Proposition 4.2.4.

$\square$

Proposition 4.2.6. *The construction* $E \mapsto A_n^\bullet(E) = \Gamma(\widehat{J}, \Omega^\bullet_{\widehat{J}/D_n^\circ} \otimes \widehat{\pi}^* E)$ *defines an exact functor*

$$R\Gamma\colon \operatorname{Perf}_{D_n^\circ} \longrightarrow \operatorname{Tate}_k.$$

This statement, as well as the stronger Proposition 5.5.1 later, can be seen as analogs, in our setting, of Theorem 7.2 of [Dr].

Proof. The functor $\Gamma(\widehat{J}, \Omega^\bullet_{J/D_n^\circ} \otimes -)$ is naturally made into a (strict) functor

$$\operatorname{Mod}_B \longrightarrow \operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_k$$

where B is the ring of function of the affine scheme $\widehat{J}$. Deriving this functor, we get an exact dg-functor

$$\mathrm{D}_{\mathrm{qcoh}}(\widehat{J}) \simeq \operatorname{dgMod}_B \longrightarrow \operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_{\mathbf{k}}$$

which restricts along $\widehat{\pi}^*$ to the announced functor

$$R\Gamma\colon \operatorname{Perf}_{D_n^\circ} \longrightarrow \operatorname{Tate}_{\mathbf{k}}.$$

$\square$

4.3 The Tate class, the residue class and the local Riemann–Roch

A. The Tate class in cyclic cohomology. We start with a delooping result.

Theorem 4.3.1. *There is a canonical isomorphism*

$$HC_*(\operatorname{Tate}_{\mathbf{k}}) \simeq HC_{*-1}(\operatorname{Perf}_{\mathbf{k}}) \simeq HC_{*-1}(\mathbf{k}).$$

Proof. Let us consider the morphism of localization sequences (of perfect dg-categories):

$$\begin{tikzcd}
\operatorname{Perf}_{\mathbf{k}} \arrow[r] \arrow[d] & \mathbf{C}_{\mathbf{k}} \arrow[r] \arrow[d] & \mathbf{C}_{\mathbf{k}}/\operatorname{Perf}_{\mathbf{k}} \arrow[d, "\alpha"] \\
\mathbf{D}_{\mathbf{k}} \arrow[r] & \operatorname{Tate}_{\mathbf{k}} \arrow[r] & \operatorname{Tate}_{\mathbf{k}}/\mathbf{D}_{\mathbf{k}}
\end{tikzcd}$$

It follows from [He3, Proposition 4.2] that the functor α is an equivalence. We will deduce the result from the following lemma.

Lemma 4.3.2. *If $\mathcal{A}$ is a perfect dg-category with infinite direct sums, then the cyclic complex $CC(\mathcal{A})$ is acyclic. In particular $HC_*(\mathcal{A}) \simeq 0$.*

Let us postpone the proof of the lemma for now. The categories $\mathbf{C_k}$ and $\mathbf{D_k}$ both admit infinite sums, and therefore have vanishing cyclic homology. Using the localization invariance of HC (see Theorem 2.1.8), we get quasi-isomorphisms

$$CC(\mathrm{Tate}_{\mathbf{k}}) \xrightarrow{\sim} CC(\mathrm{Tate}_{\mathbf{k}}/\mathbf{D_k}) \xleftarrow{\sim} CC(\mathbf{C_k}/\,\mathrm{Perf}_{\mathbf{k}}) \xrightarrow{\sim} CC(\mathrm{Perf}_{\mathbf{k}})[1].$$

This concludes the proof of Theorem 4.3.1. □

Proof (of Lemma 4.3.2): We use the following fact (see [Ke2]): if f, g are two functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{B}$ is perfect (in particular, has direct sums), then the action of $f \oplus g$ on HC is the sum of the action of f and of g. Specifying to the case $f = \bigoplus_{i=0}^{\infty} \mathrm{Id}$ and $g = \mathrm{Id}$, then $f \oplus g \simeq f$ and we get $HC(g) = HC(\mathrm{Id}) = 0$. □

Definition 4.3.3. Let us denote by τ the class $\tau \in HC^1(\mathrm{Tate}_{\mathbf{k}})$ given by the image of the trace class through the isomorphism $HC^0(\mathbf{k}) \simeq HC^1(\mathrm{Tate}_{\mathbf{k}})$. For any object $V \in \mathrm{Tate}_{\mathbf{k}}$ we have a class $\tau_V \in HC^1(\mathrm{End}(V))$ induced by τ.

Remark 4.3.4. The class τ_V vanishes (by construction) as soon as V belongs to either $\mathbf{C_k}$ or $\mathbf{D_k}$.

B. Comparison with the residue class and local Riemann-Roch. The Tate complex $A_n^\bullet$ is, as the same time, a commutative dg-algebra . We note that (left) action of $A_n^\bullet$ on itself gives rise to a morphism of associative dg-algebras which we call the *regular representation*

$$l : A_n^\bullet \longrightarrow \mathrm{End}_{\mathrm{Tate}_{\mathbf{k}}}(A_n^\bullet).$$

In other words, for each p and each $a \in A_n^p$, the multiplication operator $l(a) : A_n^\bullet \to A_n^\bullet[p]$, is a continuous morphism of graded objects of $\mathrm{ILC}_{\mathbf{k}}$. This follows from the fact that the filtration on $A_n^\bullet$ from Corollary 1.2.8 which, by Example 4.1.10, gives the inductive limit representation of $A_n^\bullet$, is compatible with multiplication.

Theorem 4.3.5. *There exists a non-zero constant $\lambda \in \mathbf{k}$ with the following property. The pullback $l^*\tau_{\mathrm{End}(A_n^\bullet)}$ of the Tate class $\tau_{\mathrm{End}(A_n^\bullet)}$ with respect to l is a class in $HC^1(A_n^\bullet)$ equal to $\lambda \cdot \rho$, where ρ is the residue class, see §2.3.*

Proof: For any associative dg-algebra $R^\bullet$ let us denote by $BR^\bullet$ the corresponding dg-category with one object. The global section functor

$$R\Gamma \colon \operatorname{Perf}_{A_n^\bullet} \simeq \operatorname{Perf}_{D_n^\circ} \longrightarrow \operatorname{Tate}_{\mathbf{k}}$$

from Proposition 4.2.6 is compatible with the map l. In other words, we have a commutative diagram of dg-categories

$$\begin{tikzcd} BA_n^\bullet \arrow[r] \arrow[d, "l"] & \operatorname{Perf}_{A_n^\bullet} \arrow[d, "R\Gamma"] \\ B(\operatorname{End}(A_n^\bullet)) \arrow[r] & \operatorname{Tate}_{\mathbf{k}} \end{tikzcd}$$

where the horizontal functors map the one object onto $A_n^\bullet$. In particular we have

$$l^*\tau_{\mathrm{End}(A_n^\bullet)} = R\Gamma^*\tau \ \in \ HC^1(\operatorname{Perf}(A_n^\bullet)) \simeq HC^1(A_n^\bullet).$$

Recall that the dg-category $\operatorname{Perf}_{A_n^\bullet} \simeq \operatorname{Perf}_{D_n^\circ}$ is the dg-quotient of $\operatorname{Perf}_{D_n}$ by the full subcategory $\operatorname{Perf}_{D_n,\{0\}}$ spanned by perfect complexes supported at 0. The global section functor $\Gamma_{D_n} \colon \operatorname{Perf}_{D_n} \to \mathbf{C}_{\mathbf{k}}$ hence induces a functor G and a commutative diagram

$$\begin{tikzcd} \operatorname{Perf}_{\{0\}}(D) \arrow[r] \arrow[d, "\Gamma_{\{0\}}"] & \operatorname{Perf}(D) \arrow[r, "P"] \arrow[d, "\Gamma_{D_n}"] & \operatorname{Perf}(D^\circ) \arrow[d, "G"] \\ \operatorname{Perf}_k \arrow[r] \arrow[d] & \mathbf{C}_{\mathbf{k}} \arrow[r] \arrow[d, "i"] & \mathbf{C}_{\mathbf{k}}/\operatorname{Perf}_{\mathbf{k}} \arrow[d, "\alpha", "\simeq"'] \\ \mathbf{D}_{\mathbf{k}} \arrow[r] & \operatorname{Tate}_k \arrow[r, "Q"] & \operatorname{Tate}_k/\mathbf{D}_{\mathbf{k}}. \end{tikzcd}$$

Lemma 4.3.6. *The composite functor $Q \circ R\Gamma$ is equivalent to the composite $\alpha \circ G$.*

Proof: From the universal property of $\operatorname{Perf}(D_n^\circ)$ as a quotient of $\operatorname{Perf}(D_n)$, it suffices to compare $Q \circ R\Gamma \circ P$ and $\alpha \circ G \circ P \simeq Q \circ i \circ g$. The inclusion $H^0(A_n^\bullet) \to A_n^\bullet$ induces a canonical natural transformation $\alpha \circ g \to R\Gamma \circ P$.

In turn, it induces the required equivalence (as the pointwise kernel of the natural transformation is killed by the projection Q). $\square$

It follows from this lemma that $R\Gamma^*\tau$ equals the composite

$$HC_1(\mathrm{Perf}(D_n^\circ)) \xrightarrow{\delta} HC_0(\mathrm{Perf}_{D_n,\{0\}}) \xrightarrow{\Gamma_{\{0\}}} HC_0(\mathrm{Perf}_{\mathbf{k}}).$$

Let us write $z_\bullet$ for the coordinate system $z_1, \cdots, z_n$ on D_n and denote, as in §1.4, by $V = \bigoplus \mathbf{k} z_i$ the space spanned by the z_i. In what follows we will pay attention to the GL_n-action on various spaces arising.

The dg-category $\mathrm{Perf}_{D_n,\{0\}}$ is compactly generated by $\mathbf{k}$ (considered as a trivial $\mathbf{k}[[z_\bullet]]$-module), and we have a GL_n-equivariant identification

$$R\,\mathrm{End}_{\mathbf{k}[[z_\bullet]]}(\mathbf{k}) \;\simeq\; \mathrm{Ext}^\bullet_{\mathbf{k}[[z_\bullet]]}(\mathbf{k},\mathbf{k}) \;\simeq\; S^\bullet(V^*[-1]) \tag{4.3.7}$$

with the exterior algebra of V graded by its degree. Using [SS], we get a Morita equivalence between $\mathrm{Perf}_{D_n,\{0\}}$ and $R\,\mathrm{End}_{\mathbf{k}[[z_\bullet]]}(\mathbf{k})$ (a version of the classical $S-\Lambda$ duality [GM]). Under this Morita equivalence, the functor $\Gamma_{\{0\}}$ amounts to the augmentation morphism $S^\bullet(V^*[-1]) \to \mathbf{k}$. The induced map $HC_0(\mathrm{Perf}_{D_n,\{0\}}) \to HC_0(\mathrm{Perf}_{\mathbf{k}})$ is thus non-trivial. It is also GL_n-invariant.

Lemma 4.3.8. *$HC_0(\mathrm{Perf}_{D_n,\{0\}})$ admits a unique GL_n-coinvariant class.*

Proof: Consider the Hodge decomposition of $HC_\bullet(S^\bullet V^*)$. The part of weight p is computed by the complex $S^\bullet(V^*[-1]) \otimes \Lambda^{\leqslant p}(V^*[-1])[2p]$ with the de Rham differential. The de Rham differential is GL_n-equivariant. Therefore, to compute the GL_n-coinvariant elements in the cyclic homology at hand, it suffices to understand the action on each degree of the graded vector space

$$S^\bullet(V^*[-1]) \otimes \Lambda^{\leqslant p}(V^*[-1])[2p] \simeq \bigoplus_{j=0}^{n} \bigoplus_{i=0}^{p} \Lambda^j V^* \otimes S^i V^*[2p-(2i-j)]$$

The GL_n-representation $\Lambda^j V^* \otimes S^i V^*$ has simple spectrum and admits coinvariants if and only if $i = j = 0$. We get

$$HC_m^{(p)}(S^\bullet(V^*[-1]))_{GL_n} = \begin{cases} \mathbf{k} & \text{if } m = 2p \\ 0 & \text{else.} \end{cases}$$

In particular $HC_0(S^\bullet(V^*[-1]))$ has exactly one invariant 1-dimensional subspace.

□

We now finish the proof of Theorem 4.3.5. The residue $HC_1(A_n) \to HC_0(\mathbf{k})$ is GL_n-invariant and vanishes when restricted to $HC_1(\mathbf{k}[[z_\bullet]])$. The long exact sequence

$$\cdots \longrightarrow HC_1(\mathbf{k}[[z_\bullet]]) \longrightarrow HC_1(A_n) \xrightarrow{\delta} HC_0(S^\bullet(V^*[-1])) \longrightarrow \cdots$$

implies that δ maps isomorphically the unique invariant line of $HC_1(A_n)$ onto the unique invariant line of $HC_1(S^\bullet(V^*[-1]))$. This concludes the proof of Theorem 4.3.5.

Remark 4.3.9. Since $\mathbf{k}$ is allowed to be an arbitrary field of characteristic 0, we have $\lambda \in \mathbb{Q}^*$. We expect $\lambda = \pm 1$. This can be possibly proved either by direct calculation or by upgrading some of the considerations of this paper to fields of arbitrary characteristic.

Recall (Proposition 3.1.2) that the class $\tau_V \in HC^1(\mathrm{End}(V))$, $V \in \mathrm{Tate}_{\mathbf{k}}$ gives a Lie algebra cohomology class $\theta^*\tau_V$ in $\mathbb{H}^2_{\mathrm{Lie}}(\mathrm{End}(V))$. We will also call $\theta^*\tau_V$ the Tate class. Let us consider the following particular case.

Let $r \geqslant 1$ and $V = (A_n^\bullet)^{\oplus r}$. As before, we have then the regular representation (morphism of dg-algebras) $l_r : \mathrm{Mat}_r(A_n^\bullet) \to \mathrm{End}_{\mathrm{Tate}_{\mathbf{k}}}((A_n^\bullet)^{\oplus r})$ which we can also see as a morphism of dg-Lie algebras.

Corollary 4.3.10 (Local Riemann–Roch)**.** *The pullback $l_r^*(\theta^*\tau_{(A_n^\bullet)^{\oplus r}})$ is equal to the class of the cocycle $\lambda \cdot \gamma_{P_r}$, where λ is the same as in Theorem 4.3.5 and γ_{P_r} is the special case, for $P(x) = P_r(x) = \mathrm{tr}(x^{n+1})/(n+1)!$, of the cocycle defined in Theorem 3.2.1.* □

We note that $P_r(x)$ is the degree $(n+1)$ component of $\mathrm{tr}(e^x)$, the "Chern character" of x. Corollary 4.3.10 can be therefore seen as a simplified version of a local Riemann-Roch-type theorem where we take into account the infinitesimal symmetries of a vector bundle but not of the underlying manifold.

5 Action on derived moduli spaces

In the rest of the paper we will relate the dg-Lie algebra $\mathfrak{g}_n^\bullet$ and its central extensions, with derived moduli spaces of principal bundles on n-dimensional manifolds. We start with recalling the general setup.

5.1 Background on derived geometry

A. General conventions. We will work in the framework of derived algebraic geometry. For general results on the subject, we refer to [TV]. For a comprehensive survey, the reader may look at [To2].

Derived algebraic geometry can be though as algebraic geometry, where rings are being replaced by "homological rings". Namely, the category of $\mathbf{k}$-algebras will be replaced the category $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ formed by $\mathbb{Z}_{\leqslant 0}$-graded commutative dg-algebras over $\mathbf{k}$ (up to quasi-isomorphisms). It is naturally made into a model category. Moreover, the usual notions of Zariski open or closed immersions, flat, smooth or étale morphisms extend to morphisms in $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$. In particular, one can form an étale Grothendieck topology.

Given any commutative algebra, one can consider it as an object in $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ concentrated in degree 0. On the other hand, for any object $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, the cohomology space H^0A is a commutative algebra. We also get a canonical morphism $A \to H^0A$.

Let also $s\mathcal{S}et$ be the category of simplicial sets. Given two objects $A, B \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, we get a simplicial set of morphisms $\mathrm{Map}(A, B)$. In particular, we get a Yoneda functor mapping A to a covariant functor $\mathrm{Spec}\, A \colon \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0} \to s\mathcal{S}et$.

A derived prestack is a covariant functor $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0} \to s\mathcal{S}et$. A derived stack is a derived prestack satisfying the natural homotopy étale descent condition. A derived stack representable by a cdga is called a derived affine scheme. We will denote by $\mathrm{dAff}_{\mathbf{k}}$ the category of derived affine schemes. It is equivalent to the opposite category of $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$.

B. Derived stacks and derived categories. The category of derived stacks will be denoted by $\mathcal{S}t$. It will be considered either as a model category, or as an ∞-category. Given $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, the category of A-dg-modules is endowed with a standard model structure. We denote by dgMod_A the ($\mathbf{k}$-linear) dg-category of fibrant-cofibrant A-dg-modules. Given a derived stack Y, we define, following [To3]

$$\mathrm{D}_{\mathrm{qcoh}}(Y) = \underleftarrow{\mathrm{holim}}_{\mathrm{Spec}\, A \to Y} \mathrm{dgMod}_A$$

(homotopy limit in the model category of dg-categories). An object in $\mathrm{D}_{\mathrm{qcoh}}(Y)$ can be informally described as the data of a A-dg-module $M_{A,\phi}$ for any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ and any map $\phi : \mathrm{Spec}\, A \to Y$, together with natural homotopy glueing data. We also define the $\mathbb{Z}_{\leqslant 0}$-graded derived category $\mathrm{D}_{\mathrm{qcoh}}^{\leqslant 0}(Y) \subset \mathrm{D}_{\mathrm{qcoh}}(Y)$ formed by $(M_{A,\phi})$ consisting of $\mathbb{Z}_{\leqslant 0}$-graded dg-modules.

Example 5.1.1. Any $M \in \mathrm{D}^{\leqslant 0}_{\mathrm{qcoh}}(Y)$ gives rise to the *dual number stack* $Y[M]$ defined by gluing $\mathrm{Spec}(A \oplus M_{A,\phi})$ (the trivial square zero extension).

Note that for any map $f\colon Y \to Z$, one gets an adjunction

$$Lf^*\colon \mathrm{D}_{\mathrm{qcoh}}(Z) \rightleftarrows \mathrm{D}_{\mathrm{qcoh}}(Y) \;:\; Rf_*$$

For example, for $\phi : \mathrm{Spec}(A) \to Y$ and $M \in \mathrm{D}_{\mathrm{qcoh}}(Y)$, the object $L\phi^*M$ is just the structure datum $M_{A,\phi}$ (we identify $\mathrm{D}_{\mathrm{qcoh}}(\mathrm{Spec}\, A)$ with dgMod_A).

Another example: if $Z = \mathrm{Spec}\,\mathbf{k}$ and f is the canonical projection, then $Rf_*\colon \mathrm{D}_{\mathrm{qcoh}}(Y) \to \mathrm{dgMod}_k$ computes the cohomology of Y with values in a given object in $\mathrm{D}_{\mathrm{qcoh}}(Y)$.

We will also need not necessarily quasi-coherent sheaves. The functor $A \mapsto \mathrm{dgMod}_A$ lands in dg-categories, hence in ($\mathbf{k}$-linear) ∞-categories. Let $\xi\colon \int \mathrm{dgMod} \to \mathrm{dAff}_{\mathbf{k}}$ denote the associated Cartesian Grothendieck construction (see [Lu7, Chap. 3]).

Definition 5.1.2. Let Y be a derived stack. We define its derived category of $\mathcal{O}_Y$-complexes $\mathrm{D}(Y)$ as the ∞-category of sections $\mathrm{dAff}_{\mathbf{k}}/Y \to \int \mathrm{dgMod}$ of ξ over Y.

Note that by [Lu7, 3.3.3.2], the ∞-category $\mathrm{D}_{\mathrm{qcoh}}(Y)$ is the full subcategory of $\mathrm{D}(Y)$ spanned by Cartesian sections. Informally, an object in $M \in \mathrm{D}(Y)$ is the data of A-dg-modules $M_{A,\phi}$ for any map $\phi\colon \mathrm{Spec}\, A \to Y$, together with coherence maps $\zeta_f\colon M_{A,\phi} \otimes^L_A B \to M_{B,\phi\circ f}$ for any map $f\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ and higher coherence data. The module M is then quasi-coherent if and only if all the maps ζ_f are quasi-isomorphisms.

The category $\mathrm{D}(Y)$ admits internal homs that we will denote by $R\,\mathrm{Hom}_{\mathcal{O}_Y}$.

C. Geometric objects and tangent complexes. For the definition of geometric derived stacks (or, what is the same, derived Artin stacks) we refer to [TV].

This class includes, first all derived schemes, that is, derived stacks that are Zariski locally equivalent to derived affine schemes. Following [Lu1], one can represent derived schemes in terms of "homotopically" ring spaces. Namely, a derived scheme X is a topological space together with a sheaf (up to homotopy) of $\mathbb{Z}_{\leqslant 0}$-graded cdga's $\mathcal{O}_X$ such that $(X, H^0(\mathcal{O}_X))$ is a scheme.

In fact, a derived Artin stack is a derived stack that can be obtained from derived affine schemes by a finite number of smooth quotients.

The *cotangent complex* $\mathbb{L}_Y$ of a derived stack Y is an object of $\mathrm{D}_{\mathrm{qcoh}}(Y)$ defined (when it exists) by the universal property

$$\mathrm{Map}_{\mathrm{D}_{\mathrm{qcoh}}(Y)}(\mathbb{L}_Y, M) \;\simeq\; \mathrm{Map}_{{}^{Y/}\mathcal{S}t}(Y[M], Y), \quad M \in \mathrm{D}^{\leqslant 0}_{\mathrm{qcoh}}(Y).$$

Here ${}^{Y/}\mathcal{S}t$ is the comma category of derived stacks under Y. The object $\mathbb{L}_Y$ is known to exist [TV] when Y is geometric (no smoothness assumption).

The *tangent complex* $\mathbb{T}_Y$ is defined as the dual

$$\mathbb{T}_Y \;=\; R\,\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{L}_Y, \mathcal{O}_Y) \in \mathrm{D}(Y).$$

If Y is locally of finite presentation [TV], then $\mathbb{L}_Y$ is a perfect complex and hence so is $\mathbb{T}_Y$. In particular $\mathbb{T}_Y$ is an object of $\mathrm{D}_{\mathrm{qcoh}}(Y)$.

For a $\mathbf{k}$-point $i_y : y \hookrightarrow Y$ we will write

$$\mathbb{T}_{Y,y} = Li_y^*(\mathbb{T}_Y) = R\,\mathrm{Hom}_{\mathcal{O}_Y}(\mathbb{L}_Y, \mathcal{O}_y)$$

for the tangent complex of Y at y. This is a complex of $\mathbf{k}$-vector spaces.

D. Derived intersection: Given a diagram $X \to Z \leftarrow Y$, we have the *derived (or homotopy) fiber product* $X \times^h_Z Y$. If X, Y, Z are affine, so our diagram is represented by a diagram $A \leftarrow C \to B$ in $\mathcal{C}dga^{\leqslant 0}_{\mathbf{k}}$, then

$$X \times^h_Z Y \;=\; \mathrm{Spec}\big(A \otimes^L_C B\big).$$

We will be particularly interested in the following situation. Let $f : X \to Y$ be a morphism of derived stacks, and $y \in Y$ be a $\mathbf{k}$-point. Then we have the derived stack (a derived (affine) scheme, if both X and Y are derived (affine) schemes)

$$Rf^{-1}(y) \;=\; X \times^h_Y \{y\}.$$

It will be called the *derived preimage of* y. It is the analog of the homotopy fiber of a map between spaces in topology.

5.2 The Kodaira–Spencer homomorphism

A. Group objects and actions. By a *group stack* we will mean a stack G together with simplicial stack $G_\bullet$ such that $G_0 \simeq \mathrm{Spec}\,\mathbf{k}$, $G_1 \simeq G$ and which satisfies the *Kan condition*: the morphisms corresponding to the inclusions of horns are equivalences. Intuitively, $G_\bullet$ is the nerve of the group structure on G, see [Lu6, §4.2.2] for more details.

Similarly, an *action* of a group stack G (given by $G_\bullet$) on a stack Y is a simplicial stack $Y_\bullet$ together with a morphism $q : Y_\bullet \to G_\bullet$ with an identification $Y_0 \simeq Y$ such that, for any m, the morphism

$$(q_m, \partial_{\{m\}\hookrightarrow\{0,1,\dots,m\}}) : Y_m \longrightarrow G_m \times Y_0$$

is an equivalence. In this case $Y_\bullet$ satisfies the Kan condition. Intuitively, $Y_\bullet$ is the nerve of the "action groupoid". The "realization" of $Y_\bullet$, i.e., the derived stack associated to the prestack $A \mapsto |Y_\bullet(A)|$, is the quotient derived stack $[Y/G]$. In particular, we have the stack $BG = [*/G]$, the *classifying stack* of G.

Examples 5.2.1. (a) Let Y be a derived stack and $y \in Y$ be a $\mathbf{k}$-point. The *pointed loop stack*

$$\Omega_y Y = \{y\} \times^{\mathrm{h}}_Y \{y\} : A \mapsto \Omega(Y(A), y)$$

is a group stack. The corresponding simplicial stack $(\underline{\Omega}_y Y)_\bullet$ is the (homotopy) nerve of the morphism $\{y\} \to Y$, i.e.,

$$(\underline{\Omega}_y Y)_m \;=\; \{y\} \times^{\mathrm{h}}_Y \{y\} \times^{\mathrm{h}}_Y \cdots \times^{\mathrm{h}}_Y \{y\} \;\simeq\; (\Omega_y Y)^m$$

($(m+1)$-fold product).

(b) Let Y be any derived stack. Its *automorphism stack* is the group stack

$$\mathbb{R}\mathbf{Aut}(Y) : A \mapsto \mathrm{Map}^{\mathrm{eq}}_{\mathcal{S}t/\operatorname{Spec} A}(Y \times \operatorname{Spec} A, Y \times \operatorname{Spec} A)$$

Here the superscript "eq" means the union of connected components of the mapping space formed by vertices which are equivalences. Alternatively, we can describe it as the functor

$$A \mapsto \Omega(\mathcal{S}t/\operatorname{Spec} A, Y \times \operatorname{Spec} A),$$

the based loop space of the nerve of the category of derived stacks over $\operatorname{Spec} A$ with the base point being the object $Y \times \operatorname{Spec} A$. By construction, we have an action of $\mathbb{R}\mathbf{Aut}(Y)$ on Y; an action of a group stack G on Y gives a morphism of group stacks $G \to \mathbb{R}\mathbf{Aut}(Y)$.

Proposition 5.2.2. *Let $f\colon X \to Y$ be a map of derived stacks and $y \in Y$ be a* $\mathbf{k}$*-point. Then the group stack $\Omega_y Y$ has a natural action on the derived preimage $Rf^{-1}(y)$.*

Proof: We define the simplicial stack $\underline{R}f^{-1}(y)_\bullet$ as the nerve of the morphism $Rf^{-1}(y) \to Y$, i.e.,

$$\begin{aligned} \underline{R}f^{-1}(y)_m \;&=\; Rf^{-1}(y) \times^{\mathrm{h}}_X Rf^{-1}(y) \times^{\mathrm{h}}_X \cdots \times^{\mathrm{h}}_X Rf^{-1}(y) \;\simeq \\ &\simeq\; \{y\} \times^{\mathrm{h}}_Y \{y\} \times^{\mathrm{h}}_Y \cdots \times^{\mathrm{h}}_Y X \;\simeq\; (\Omega_y Y)^m \times Rf^{-1}(y). \end{aligned}$$

All the required data and properties come from contemplating the commutative diagram

$$\begin{tikzcd} Rf^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d] \\ \{y\} \arrow[r] & Y. \end{tikzcd}$$

□

Example 5.2.3 (Eilenberg-MacLane stacks). Let Π be a commutative algebraic group (in our applications $\Pi = \mathbb{G}_m$). For each $r \geqslant 0$ we then have group stack $\mathrm{EM}(\Pi, n)$, known as the *Eilenberg-MacLane stack*. It is defined in the standard way using the Eilenberg-MacLane spaces for abelian groups $\Pi(A)$ for commutative $\mathbf{k}$-algebras A.

Thus $\mathrm{EM}(\Pi, 0) = \Pi$ as a group stack, i.e., the corresponding simplicial stack $\mathrm{EM}(\Pi, 0)_\bullet = \Pi_\bullet$ is the simplicial classifying space of Π. Similarly (the underlying stack of the group stack) $\mathrm{EM}(\Pi, 1)$ is identified with $B\Pi$. In general, if we denote $\mathrm{EM}(\Pi, n)_\bullet$ the simplicial stack describing the group structure on $\mathrm{EM}(\Pi, n)$, then $|\mathrm{EM}(\Pi, n)| = \mathrm{EM}(\Pi, n+1)$.

Definition 5.2.4. Let G be a group stack and Π a commutative algebraic group. A *central extension* of G by Π is a morphism of group stacks $\phi : G \to B\Pi$ or, what is the same, a morphism of stacks $BG \to \mathrm{EM}(\Pi, 2)$.

A central extension ϕ gives, in a standard way, a fiber and cofiber sequence of group stacks

$$1 \to \Pi \longrightarrow \widetilde{G} \longrightarrow G \to 1,$$

where $\widetilde{G}$ is the fiber of ϕ.

B. Formal moduli problems. We recall Lurie's work [Lu3] on formal moduli problems which serve as infinitesimal analogs of derived stacks.

Definition 5.2.5. A cdga $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ is called Artinian, if:

- The cohomology of A is finite dimensional (over $\mathbf{k}$);
- The ring $H^0 A$ is local and the unit induces an isomorphism between $\mathbf{k}$ and the residue field of $H^0 A$.

In particular, any Artinian cdga admits a canonical augmentation (the unique point of $\operatorname{Spec} A$). Artinian cdga's form an ∞-category which we will denote by $\mathbf{dgArt_k}$.

Definition 5.2.6. A formal moduli problem is a functor (of ∞-categories)

$$F\colon \mathbf{dgArt_k} \to s\mathcal{S}et$$

such that:

(1) $F(\mathbf{k}) \simeq *$ is contractible.

(2) (Schlessinger condition): For any diagram $A \to B \leftarrow C$ in $\mathbf{dgArt_k}$ with both maps surjective on H^0, the canonical map $F(A \times^{\mathrm{h}}_B C) \to F(A) \times^{\mathrm{h}}_{F(B)} F(C)$ is an equivalence.

We denote by $\mathrm{Fun}_*(\mathrm{dgArt}_{\mathbf{k}}, s\mathcal{S}et)$ the (∞-)category of functors from $\mathrm{dgArt}_{\mathbf{k}}$ to simplicial sets satisfying the condition (1) of Definition 5.2.6, and by **FMP** the full subcategory of formal moduli problems. General criteria for representability of functors imply (see [Lu3], 1.1.17) that we have a left adjoint (the "formal moduli envelope")

$$\mathcal{L} : \mathrm{Fun}_*(\mathrm{dgArt}_{\mathbf{k}}, s\mathcal{S}et) \longrightarrow \mathbf{FMP} \tag{5.2.7}$$

to the embedding functor.

For a formal moduli problem F we define its *tangent complex* $\mathbb{T}_F$ (at the only point $*$ of F). This is a complex of $\mathbf{k}$-vector spaces defined as follows. First, we define

$$\mathcal{T}_F^{(p)} = F(\mathbf{k}[\epsilon_p]/\epsilon_p^2), \quad \deg(\epsilon_p) = -p, \quad p \geqslant 0.$$

These simplicial sets are actually simplicial vector spaces, forming a spectrum in the sense of homotopy theory, that is, connected by morphisms (in fact, by equivalences) $\mathcal{T}_F^{(p)} \to \Omega \mathcal{T}_F^{(p+1)}$. We define the complex $\mathbb{T}_F$ to correspond to the spectrum $(\mathcal{T}_F^{(p)})$ by the Dold-Kan equivalence.

Theorem 5.2.8 (Lurie)**.** *For any formal moduli problem F, the complex $\mathbb{T}_F[-1]$ has a homotopy Lie structure. Moreover, the data of F is equivalent (up to homotopy) to the data of a complex $\mathbb{T} = \mathbb{T}_F[-1]$ and of a dg-Lie structure on this complex.*

For future reference we recall from [Lu3] §2, the characterization of the Lie algebra structure on $\mathbb{T}_F[-1]$. For a graded $\mathbf{k}$-vector space $M^\bullet$ let $\mathrm{FL}(M^\bullet)$ be the free graded Lie algebra (with zero differential) generated by $M^\bullet$. Denote by $\mathrm{FLie}_{\mathbf{k}}^{\geqslant 1}$ be the full (∞-) subcategory in $\mathrm{dgLie}_{\mathbf{k}}$ whose objects are $\mathrm{FL}(M^\bullet)$ where $M^\bullet$ is finite-dimensional and situated in degrees $\geqslant 1$. For any $L \in \mathrm{FLie}_{\mathbf{k}}^{\geqslant 1}$ the Chevalley-Eilenberg cochain algebra $\mathrm{CE}^\bullet(L)$ is an object of $\mathrm{dgArt}_{\mathbf{k}}$. In fact, $\mathrm{CE}^\bullet(L)$ is quasi-isomorphic to the dual number algebra $\mathbf{k} \oplus (M^\bullet)^*[-1]$, see [Lu3] (2.2.15). We then have

$$\mathrm{Map}_{\mathrm{dgLie}_{\mathbf{k}}}(L, \mathbb{T}_F[-1]) \;=\; F(\mathrm{CE}^\bullet(L)), \quad L \in \mathrm{FLie}_{\mathbf{k}}^{\geqslant 1} \tag{5.2.9}$$

(identification of functors on $\mathrm{FLie}_{\mathbf{k}}^{\geqslant 1}$). Note that this defines the dg-Lie algebra structure uniquely.

We will also need the following global analog of the Schlessinger condition, see [TV, Def. 1.4.2.1].

Definition 5.2.10. A derived stack Y is *infinitesimally cartesian* if the following condition holds. Let $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, $M \in \mathrm{dgMod}_A^{\leqslant -1}$ and

$$M \longrightarrow B \longrightarrow A \xrightarrow{\delta} M[1]$$

be a square zero extension of A by M. Then the square

$$\begin{array}{ccc} F(B) & \longrightarrow & F(A) \\ \big\downarrow & & \big\downarrow{\scriptstyle F(1+\delta)} \\ F(A) & \xrightarrow[F(1)]{} & F(A \oplus M[1]) \end{array}$$

is cartesian.

It is known that any derived Artin stack is infinitesimally cartesian [TV, Prop. 1.4.2.5].

Example 5.2.11. Any infinitesimally cartesian stack Y and any $\mathbf{k}$-point $y \in Y$ defines a formal moduli problem

$$\widehat{Y}_y\colon A \mapsto Y(A) \times^{\mathrm{h}}_{Y(\mathbf{k})} \{y\}.$$

Its tangent complex $\mathbb{T}_{\widehat{Y}_y}$ is identified with $\mathbb{T}_{Y,y} = \mathrm{Hom}(\mathbb{L}_Y, \mathbf{k}_y)$, if $\mathbb{L}_Y$ exists (e.g., if Y is geometric). In other cases it can be considered as the definition of $\mathbb{T}_{Y,y}$. By Theorem 5.2.8, the shifted complex $\mathbb{T}_{Y,y}[-1]$ carries a homotopy Lie structure. It is an analog of the fundamental group $\pi_1(Y, y)$ of a topological space Y at a point y.

For any derived stacks Y and X we define the mapping stack

$$\mathbb{R}\mathbf{Map}(Y, X) : A \mapsto \mathrm{Map}_{\mathcal{S}t/\operatorname{Spec} A}(Y \times \operatorname{Spec} A, X \times \operatorname{Spec} A).$$

Notice that $\mathbb{R}\mathbf{Aut}(Y)$ is an open substack in $\mathbb{R}\mathbf{Map}(Y, Y)$, that is, the pullback of it under any morphism $U \to \mathbb{R}\mathbf{Map}(Y, Y)$ from an affine derived scheme U, is an open subscheme in U.

Proposition 5.2.12. *(a) If Y is an infinitesimally cartesian derived stack, then so is $\mathbb{R}\mathbf{Map}(X, Y)$ for any X. Moreover, suppose Y is geometric and $f : X \to Y$ is any morphism, representing a $\mathbf{k}$-point $[f]$ of $\mathbb{R}\mathbf{Map}(X, Y)$. Then the tangent complex $\mathbb{T}_{\mathbb{R}\mathbf{Map}(X,Y),[f]}$ (defined as the tangent complex of the associated formal moduli problem) is identified with $R\Gamma(X, f^*\mathbb{T}_Y)$.*

(b) Further, if Y is infinitesimally cartesian, then so is $\mathbb{R}\mathbf{Aut}(Y)$. In this case $\mathbb{T}_{\mathbb{R}\mathbf{Aut}(X),\mathrm{Id}}$ is $R\Gamma(X, \mathbb{T}_X)$.

Proof. (a) Consider the full subcategory $\mathcal{C}$ of $\mathcal{S}t$ spanned by those $X \in \mathcal{S}t$ such that $\mathbb{R}\mathbf{Map}(X, Y)$ is infinitesimally cartesian. Since Y is infinitesimally cartesian, $\mathcal{C}$ contains all derived affine schemes. The category $\mathcal{C}$ is moreover stable by (homotopy) colimits. We get $\mathcal{C} = \mathcal{S}t$. The identification of the tangent complex is the standard argument using the universal property of $\mathbb{L}_Y$ and the appropriate adjunctions.

(b) Follows from (a) because $\mathbb{R}\mathbf{Aut}(X)$ is a open in $\mathbb{R}\mathbf{Map}(X, X)$ and so their completions are identified. □

C. The Lie dg-algebra of a group stack. Let G be a group stack (represented by a simplicial stack $G_\bullet$, as in part A). Assume that the completion $\widehat{G}_1$ is a formal moduli problem (this is true, for example, if G is infinitesimally cartesian). In this case we have the tangent complex $\mathrm{Lie}(G) = \mathbb{T}_{G,1}$, naturally made into a dg-Lie algebra. Explicitly, we form the "classifying formal moduli problem" $B(\widehat{G}_1)$ as the formal moduli envelope (5.2.7) of the functor

$$A \mapsto \underrightarrow{\operatorname{holim}}\, (G_n)\widehat{_1}\,(A), \quad \mathrm{dgArt}_{\mathbf{k}} \longrightarrow s\mathcal{S}et.$$

Then $\mathrm{Lie}(G)$ is, as a dg-Lie algebra, identified with $\mathbb{T}_{B(\hat{G}_1)}[-1]$.

Examples 5.2.13. (a) This construction extends the classical correspondence between group schemes and Lie algebras. Further, if Π is a commutative algebraic group, then $\mathrm{Lie}(\mathrm{EM}(\Pi, n)) \simeq \mathrm{Lie}(\Pi)[n]$, an abelian Lie algebra $\mathrm{Lie}(\Pi)$ put in degree n.

(b) For the group stack $\Omega_{Y,y}$ of loop spaces, we have $\mathrm{Lie}(\Omega_y Y) = \mathbb{T}_{Y,y}[-1]$, because $B((\Omega_{Y,y})\widehat{_1})$ is identified with $\widehat{Y}_y$.

(c) Let G be a group stack such that $\widehat{G}_1$ is a formal moduli problem, and $\mathfrak{g} = \mathrm{Lie}(G)$. A central extension of G by $\mathbb{G}_m$, i.e., a morphism of group stacks $\phi : G \to B(\mathbb{G}_m)$ gives, after passing to Lie dg-algebras, a morphism $\mathrm{Lie}(\phi) : \mathfrak{g} \to \mathbf{k}[1]$, since $\mathrm{Lie}(\mathbb{G}_m) = \mathbf{k}$. By Proposition 3.1.1, $\mathrm{Lie}(\phi)$ gives rise to a cohomology class $\gamma_\phi \in \mathbb{H}^2_{\mathrm{Lie}}(\mathfrak{g}, \pi)$.

D. The Kodaira-Spencer morphism. Let $f : X \to Y$ be a morphism of derived stacks and $y \in Y$ be a $\mathbf{k}$-point. Assume that $\widehat{Y}_y$ is a formal moduli problem and the homotopy fiber $Rf^{-1}(y)$ is geometric. In particular, we have the cotangent complex $\mathbb{L}_{Rf^{-1}(y)}$.

Definition 5.2.14. The Kodaira-Spencer morphism of f is the morphism of homotopy Lie algebras

$$\kappa : \mathbb{T}_{Y,y}[-1] \longrightarrow R\Gamma(Rf^{-1}(y), \mathbb{T}_{Rf^{-1}(y)})$$

obtained as the differential, at the identity element, of the action

$$a : \Omega_y Y \longrightarrow \mathbb{R}\mathbf{Aut}(Rf^{-1}(y)) \tag{5.2.15}$$

from Proposition 5.2.2. Here we use Proposition 5.2.12(c) to identify the tangent to $\mathbb{R}\mathbf{Aut}$.

Proposition 5.2.16. *Assume X is geometric, so we have the normal fiber sequence in* $\mathrm{D}(Rf^{-1}(y))$

$$\mathbb{T}_{Rf^{-1}(y)} \longrightarrow \mathbb{T}_X \otimes^L_{\mathcal{O}_X} \mathcal{O}_{Rf^{-1}(y)} \longrightarrow \mathbb{T}_{Y,y} \otimes_{\mathbf{k}} \mathcal{O}_{Rf^{-1}(y)}.$$

Let δ be the coboundary map of this triangle. Then κ is identified, as a morphism of complexes, with the composite of $\delta[-1]$ with the adjunction map $\mathbb{T}_{Y,y} \to R\Gamma(Rf^{-1}(y), \mathbb{T}_{Y,y} \otimes \mathcal{O}_{Rf^{-1}(y)})$.

Proof: We deduce the claim from a more general local statement. Let $i : F \to X$ be any morphism of geometric stacks. We then have the relative tangent complex $\mathbb{T}_{F/X}$ on X fitting into the exact triangle

$$\mathbb{T}_{F/X} \xrightarrow{c} \mathbb{T}_F \longrightarrow i^*\mathbb{T}_X \longrightarrow \mathbb{T}_{F/X}[1]. \tag{5.2.17}$$

Let us form the "groupoid stack" $Z_\bullet$ with $Z_n = F \times^h_X \cdots \times^h_X F$ ($(n+1)$ factors). Thus $Z_0 = F$ (the objects), $Z_1 = Z := F \times^h_X F$ (the morphisms) and so on. The second projection $p_2 : Z \to F$ defines an action of the groupoid $Z_\bullet$ on F which we write as

$$\underline{a} : Z \times_F F \longrightarrow F.$$

The tangent, at the identities of the groupoid (which are represented by the diagonal embedding $e : F \to Z$) of $\underline{a}$ is a map

$$d\underline{a} : e^*(\mathbb{T}_{Z/F}) \longrightarrow \mathbb{T}_F$$

of which the source is identified, via the projection $p_1 : Z \to F$, with $\mathbb{T}_{F/X}$. The following is then obvious.

Lemma 5.2.18. *In the above situation, $d\underline{a}$ is identified with the canonical morphism $c : \mathbb{T}_{F/X} \to \mathbb{T}_F$.* □

We now deduce Proposition 5.2.16 from the lemma. Let $F = Rf^{-1}(y) \xrightarrow{i} X$ be the inclusion of the homotopy fiber. By definition, the action a from (5.2.15) comes from the action $\underline{a}$, while $\delta[-1]$ class from the normal sequence, as in the proposition. We now note that the normal sequence is canonically identified with the shift of the triangle (5.2.17), that is, $\mathbb{T}_{Y,y} \otimes \mathcal{O}_F$ (the "normal bundle" to F) is the same as $\mathbb{T}_{F/X}[1]$. This finishes the proof.

5.3 Derived current groups and moduli of G-bundles

A. Derived moduli spaces. Let G be a reductive algebraic group over $\mathbf{k}$ with Lie algebra $\mathfrak{g}$. Let X be a smooth irreducible algebraic variety over

$\mathbf{k}$ and $x \in X$ be a $\mathbf{k}$-point. Recall the notations $\widehat{x} = \operatorname{Spec} \mathcal{O}_{X,x} \simeq D_n$ and $\widehat{x}^\circ = \widehat{x} - \{x\} \simeq D_n^\circ$ for the (punctured) formal neighborhood of x in X.

Recall that principal G-bundles are classified by the 1-stack $BG = [*/G]$: for any scheme Y, the (nerve of the) groupoid of principal G-bundles on Y is equivalent to the simplicial set of maps $Y \to BG$. This allows us to define the notion of principal G-bundles over a derived scheme Y: first, we denote by $\mathbb{R}\mathrm{Bun}_G(Y)$ the simplicial set of morphisms $Y \to BG$. A vertex in that simplicial set is called a principal G-bundle on Y. We then define the derived moduli stack of principal G-bundles over Y as the following functor from $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ to simplicial sets:

$$\mathbb{R}\mathbf{Bun}_G(Y) = \mathbb{R}\mathbf{Map}(Y, BG) \colon \;\; A \mapsto \mathbb{R}\mathrm{Bun}_G(Y \times \operatorname{Spec} A).$$

Fundamental for us will be the derived stack $\mathbb{R}\mathbf{Bun}_G(X)$.

Proposition 5.3.1. *Let P be a principal G-bundle on X and $[P] \in \mathbb{R}\mathbf{Bun}_G(X)$ be the corresponding $\mathbf{k}$-point. Then*

$$\mathbb{T}_{[P]}\, \mathbb{R}\mathbf{Bun}_G(X) \;\; \simeq \;\; R\Gamma(X, \mathrm{Ad}(P))[1].$$

Proof: Follows from Proposition 5.2.12 with $Y = BG$.

□

We then define $\mathbb{R}\mathbf{Bun}_G(\widehat{x})$ as the functor

$$A \;\mapsto \mathbb{R}\mathbf{Bun}_G(\operatorname{Spec}(A \,\widehat{\otimes}_{\mathbf{k}}\, \widehat{\mathcal{O}}_{X,x})), \quad A \,\widehat{\otimes}_{\mathbf{k}}\, \widehat{\mathcal{O}}_{X,x} = \underleftarrow{\mathrm{holim}}(A \otimes_{\mathbf{k}} \mathcal{O}_X/\mathcal{I}^n). <$$

Here $\mathcal{I}$ is the ideal of x.

Remark 5.3.2. Note that the definition of $\mathbb{R}\mathbf{Bun}_G(\widehat{x})$ is not the result of applying the construction $\mathbb{R}\mathbf{Bun}_G(Y)$ to the scheme $Y = \widehat{x}$. More precisely, for any integer n, denote by $x^{(n)} = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}^n)$, the n^{th} infinitesimal neighborhood of x in X. Then

$$\mathbb{R}\mathbf{Bun}_G(\widehat{x}) \simeq \underleftarrow{\mathrm{holim}}_n\, \mathbb{R}\mathbf{Bun}_G(x^{(n)}) \;\; = \;\; \mathbb{R}\mathbf{Bun}_G(\operatorname{Spf} \widehat{\mathcal{O}}_{X,x})$$

is a particular case of $\mathbb{R}\mathbf{Bun}$ construction but for a formal scheme.

We have the restriction map

$$\lambda : \mathbb{R}\mathbf{Bun}_G(X) \to \mathbb{R}\mathbf{Bun}_G(\widehat{x}).$$

Definition 5.3.3. The derived stack of G-bundles on X rigidified at x is defined to be

$$\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x) = \mathbb{R}\mathbf{Bun}_G(X) \times^h_{\mathbb{R}\mathbf{Bun}_G(\hat{x})} \{\mathrm{Triv}\} \;=\; R\lambda^{-1}(\mathrm{Triv}),$$

where $\mathrm{Triv} = X \times G$ denotes the trivial bundle. For any cdga A, the simplicial set of A-points of $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ is

$$\mathbb{R}\mathrm{Bun}_G(X \times \operatorname{Spec} A) \times^h_{\mathbb{R}\mathrm{Bun}_G(\operatorname{Spec}(A \widehat{\otimes} \widehat{\mathcal{O}}_{X,x}))} \{\mathrm{Triv}\}.$$

In other words, this is the groupoid formed by G-bundles on $X \times \operatorname{Spec} A$ endowed with a trivialization on $\operatorname{Spec}(A \,\widehat{\otimes}\, \widehat{\mathcal{O}}_{X,x})$.

Proposition 5.3.4. *(a) The derived stack $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ is represented by a derived Artin stack.*

(b) If X is projective, then $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ is represented by a derived scheme of amplitude $[0, n-1]$.

Proof: We first deduce from [He1, Lemma 4.2.4] that the inclusion of the trivial bundle $\{\mathrm{Triv}\} \to \mathbb{R}\mathbf{Bun}_G(\widehat{x})$ is a smooth atlas. In particular, we get that the derived stack $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ is representable by a derived Artin stack.

Using Lurie's representability criterion for derived schemes (see [Lu5, Theorem 3.1.1]), we can hence reduce to the non-derived moduli problem. It is known that $\mathbf{Bun}_G(X)$ is a geometric stack of amplitude $[-1, n-1]$, which locally is represented as the quotient of a scheme of finite type by an algebraic group. When we introduce the rigidification, we kill the stack structure. $\square$

We next define the derived stack $\mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)$ as the functor

$$A \mapsto \mathbb{R}\mathrm{Bun}_G\left(\operatorname{Spec}\big(A \,\widehat{\otimes}_{\mathbf{k}}\, \widehat{\mathcal{O}}_{X,x}\big) - \big(\operatorname{Spec}(A) \times \{x\}\big)\right).$$

Let us fix the notations $X^\circ = X - \{x\}$.

Proposition 5.3.5. *The natural morphisms*

$$\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x) \to \mathbb{R}\mathbf{Bun}_G(X^\circ) \times^h_{\mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)} \{\mathrm{Triv}\}$$

is an equivalence

This is proved in [HPV, Theorem 6.20].

B. Derived current groups and their action. We define $G(\widehat{x}^\circ)$ as the functor

$$A \mapsto \mathrm{Map}\left(\mathrm{Spec}(A\widehat{\otimes}\widehat{\mathcal{O}}_{X,x}) - (\mathrm{Spec}(A)\times\{x\}),\, G\right).$$

This is group object in derived stacks.

Proposition 5.3.6. *(a) $G(\widehat{x}^\circ)$ is a derived affine ind-scheme, identified with the group of automorphisms of the point* Triv *in the derived stack* $\mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)$.

(b) The Lie algebra of $G(\widehat{x}^\circ)$ is identified with

$$\mathrm{Lie}(G(\widehat{x}^\circ)) \;=\; \mathbb{T}_e G(\widehat{x}^\circ) \;=\; \mathfrak{g}\otimes R\Gamma(\widehat{x}^\circ,\mathcal{O}) \;\simeq\; \mathfrak{g}_x^\bullet \;\simeq\; \mathfrak{g}_n^\bullet$$

studied earlier.

Proof:

(a) The first claim follows from [He1]. The second is obvious.

(b) We use (a) and compare $\mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)$ with the mapping stack

$$\mathbb{R}\mathbf{Map}\,(\widehat{x}^\circ, BG) :\, A \mapsto \mathrm{Map}\,(\widehat{x}^\circ \times \mathrm{Spec}\, A, BG)\,.$$

The group of automorphisms of Triv in $\mathbb{R}\mathbf{Map}\,(\widehat{x}^\circ, BG)$ has Lie algebra identified with $\mathfrak{g}\otimes R\Gamma(\widehat{x}^\circ,\mathcal{O})$ by Proposition 5.2.12(a).

We now consider the morphism of derived stacks

$$f : \mathbb{R}\mathbf{Map}\,(\widehat{x}^\circ, BG) \longrightarrow \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)$$

induced by the canonical maps

$$A\otimes_{\mathbf{k}} \widehat{\mathcal{O}}_{X,x} \to A\,\widehat{\otimes}_{\mathbf{k}}\,\widehat{\mathcal{O}}_{X,x}. \tag{5.3.7}$$

We note that (5.3.7) is a quasi-isomorphism whenever A has finite dimensional cohomology. This implies that f is formally étale at the point Triv and therefore the induced morphism of tangent Lie algebras at Triv is a quasi-isomorphism.

$\square$

Consider the projection of derived stacks

$$g : \mathbb{R}\mathbf{Bun}_G(X^\circ) \longrightarrow \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ).$$

Proposition 5.3.5 identifies $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ with the homotopy fiber $Rg^{-1}(\mathrm{Triv})$. Therefore Propositions 5.3.6 and 5.2.2 imply:

Theorem 5.3.8. *(a) The derived group stack $G(\widehat{x}^\circ)$ acts on $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$ by changing the trivialization.*

(b) The Kodaira-Spencer map of g gives rise to a morphism of dg-Lie algebras

$$\beta : \mathfrak{g}_x^\bullet \longrightarrow R\Gamma(\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x), \mathbb{T}).$$

□

Remark 5.3.9. In particular, β induces a morphism on the $(n-1)$-st cohomology of the dg-Lie algebra $\mathfrak{g}_x^\bullet$:

$$H_{\overline{\partial}}^{n-1}(\mathfrak{g}_x^\bullet) \longrightarrow \mathbb{H}^{n-1}\big(\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x), \mathbb{T}\big).$$

Consider the first non-classical case $n = 2$, when X is a surface. In this case we are dealing with $\mathbb{H}^1$ of the tangent complex which has the meaning of the space of deformation of $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$. Theorem 5.3.8 produces therefore a class of deformations of the rigidified derived moduli space labelled by the space of "polar parts" $H_{\overline{\partial}}^1(\mathfrak{g}_x^\bullet) = H_{\{x\}}^2(\mathfrak{g} \otimes \mathcal{O}_X)$.

A natural way of deforming the moduli space would be to "twist" the cocycle condition $g_{ik}^{-1} g_{jk} g_{ij} = 1$ defining G-bundles by replacing it with $g_{ik}^{-1} g_{jk} g_{ij} = \lambda_{ijk}$ with some "curvature data" $\lambda = (\lambda_{ijk})$. Considering such twisted bundles is standard when $G = GL_r$ and λ consists of scalar functions (we then get modules over an Azumaya algebra). In our case, elements of $H_{\{x\}}^2(\mathfrak{g} \otimes \mathcal{O}_X)$ can be seen as providing infinitesimal germs of more general (non-abelian) twistings and thus deformations of the rigidified moduli space.

5.4 Central extensions associated to Tate complexes

A. The group stack $GL(V)$ and its Lie algebra. The categories of Tate modules assemble into two prestacks

$$\mathbf{Tate} : A \mapsto \mathrm{Tate}_A, \quad \underline{\mathbf{Tate}} : A \mapsto \mathrm{Tate}_A^{\mathrm{equiv}}$$

where "equiv" means the maximal ∞-groupoid in the ∞-categorical envelope (nerve) of the dg-category.

In particular, for any object V of $\mathrm{Tate}_{\mathbf{k}}$ we have a group prestack of automorphisms $GL(V) = \Omega_V \underline{\mathbf{Tate}}$.

Example 5.4.1. If V is the Tate space $\mathbf{k}((z))$ in degree 0, then $GL(V)$ is the non-derived group ind-scheme $GL(\infty)$ studied in [Ka]. For a general V, we get a derived analog of $GL(\infty)$.

Proposition 5.4.2. *(a) Each prestack $GL(V)$ is an infinitesimally cartesian stack.*

(b) The Lie algebra $\mathrm{Lie}(GL(V))$ *is identified with* $\mathrm{End}(V)$*, the algebra of endomorphism of V in* $\mathrm{Tate}_{\mathbf{k}}$ *made into a Lie algebra.*

Proof: (a) let $A \to B$ be a homotopy étale cover in $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$. Denote $B_n = B_A^{\otimes n}$, so that $B_\bullet = (B_n)$ is a cosimplicial object in $\mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$. The natural functor $F : \mathrm{Tate}_A \to \underleftarrow{\mathrm{holim}}\, \mathrm{Tate}_{B_n}$ is fully faithful by embedding into the functor

$$\mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_A)) \longrightarrow \underleftarrow{\mathrm{holim}}_n\, \mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_{B_n})).$$

Fully faithfulness of this last functor follows from étale descent for the categories of dg-modules. Now, fully faithfulness of F implies that each $GL(V)$ is a stack.

Next, we show that $GL(V)$ is infinitesimally cartesian. This goes in the similar way, using that the stack of right bounded dg-modules is infinitesimally cartesian [Lu2, Theorem 7.2].

(b) We construct a morphism of dg-Lie algebras

$$\phi : \mathrm{Lie}(GL(V)) \longrightarrow \mathrm{End}(V). \tag{5.4.3}$$

For this we construct a natural transformation Φ of the functors represented by both Lie algebras on $L \in \mathrm{FLie}_{\mathbf{k}}^{\geqslant 1}$. For such L we have

$$\mathrm{Map}(L, \mathrm{Lie}(GL(V)) \;=\; \mathcal{L}(\underline{\mathbf{Tate}}_V^{\wedge})(\mathrm{CE}^\bullet(L)),$$

where $\mathcal{L}$ is the formal moduli envelope functor (5.2.7). We will construct, first, a natural morphism of simplicial sets

$$\Psi_L : \underline{\mathbf{Tate}}_V^{\wedge}(\mathrm{CE}^\bullet(L)) \to \mathrm{Map}_{\mathrm{dgLie}_{\mathbf{k}}}(L, \mathrm{End}(V)).$$

By definition, an object of $\underline{\mathbf{Tate}}_V^{\wedge}(\mathrm{CE}^\bullet(L))$ is a Tate complex W over $\mathrm{CE}^\bullet(L)$ together with an identification $\mathbf{k} \otimes^L_{\mathrm{CE}^\bullet(L)} W \to V$. Note that we have a canonical identification (Koszul duality)

$$U(L) \;\simeq\; \mathrm{RHom}_{\mathrm{CE}^\bullet(L)}(\mathbf{k}, \mathbf{k}). \tag{5.4.4}$$

Therefore, for any such W, the associative dg-algebra $U(L)$, and therefore, the dg-Lie algebra L, acts on V, giving a morphism $L \to \mathrm{End}(V)$. This defines Ψ_L on vertices, and the standard arguments of Morita theory extend it to a morphism of simplicial sets, in fact to a natural transformation Ψ of functors $\mathrm{FLie}^{\geqslant 1} \to s\mathcal{S}et$.

Let $e^{\mathrm{End}(V)}$ be the formal moduli problem corresponding to the dg-Lie algebra $\mathrm{End}(V)$ by Theorem 5.2.8. We then can write each Ψ_L as a morphism

$$\Psi_L : \underline{\mathbf{Tate}}_V^{\widehat{}}(\mathrm{CE}^\bullet(L)) \longrightarrow e^{\mathrm{End}(V)}(\mathrm{CE}^\bullet(L)).$$

Next, we notice that each Ψ_L descends to a natural morphism of simplicial sets

$$\Phi_L : \mathcal{L}(\underline{\mathbf{Tate}}_V^{\widehat{}})(\mathrm{CE}^\bullet(L)) \longrightarrow e^{\mathrm{End}(V)}(\mathrm{CE}^\bullet(L)).$$

This is just because $e^{\mathrm{End}(V)}$ is a formal moduli problem. Now, the collection of the Φ_L extends, in a canonical way, to the natural transformation Φ and therefore to a morphism ϕ of dg-Lie algebras as claimed in (5.4.3).

We now prove that ϕ is an equivalence or, what this the same, that Ψ_L is an equivalence whenever L is free on one generator of degree $d \geqslant 1$. In this case $\mathrm{CE}^\bullet(L) = \mathbf{k}[\epsilon]$ is the algebra of dual numbers with generator $\epsilon = \epsilon_{d-1}$ of degree $1-d$.

Applying the Morita theory associated with (5.4.4), we include Ψ_L into an adjunction

$$\begin{gathered}\psi_L : \mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}[\epsilon]})) \leftrightarrow \{\text{Representations of } L \text{ in } \mathrm{Ind}(\mathrm{Pro}(\mathrm{Perf}_{\mathbf{k}}))\} : \eta_L \\ \psi_L(W) = \mathbf{k} \otimes^L_{\mathbf{k}[\epsilon]} W, \quad \eta_L(V, f : V \to V[d]) = \mathrm{CE}^\bullet(L, V) \simeq \mathrm{Cone}(f)[-1].\end{gathered}$$

Here f is the action of (the generator of) L on V. We note that $\mathrm{Cone}(f)[-1]$ can be written as $V[\epsilon] = V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ (free $\mathbf{k}[\epsilon]$-module) with an additional differential given by f.

Note that ψ_L is fully faithful. This is a formal consequence of the fact that

$$\psi_L^{\mathrm{Perf}} : \mathrm{Perf}_{\mathbf{k}[\epsilon]} \leftrightarrow \text{ Representations of } L \text{ in } \mathrm{Perf}_{\mathbf{k}}$$

is fully faithful. In fact, ψ_L^{Perf} is an equivalence (Koszul duality).

Therefore Ψ_L is fully faithful. Let us prove that it is essentially surjective.

Lemma 5.4.5. *Let (V, f) be a representation of L in* $\mathrm{Tate}_{\mathbf{k}}$*. Then $\eta_L(V, f) \in$* $\mathrm{Tate}_{\mathbf{k}[\epsilon]}$*.*

Proof: We can assume that V is a graded Tate space with a zero differential, and so f is a morphism of graded Tate spaces. So V has two lattices $V_1^c \subset V_2^c$

so that f induces a morphism of pro-finite-dimensional spaces $f^c : V_1^c \to V_2^c[d]$. We make $\mathrm{Cone}(f^c)[-1]$ into a (pro-perfect) $\mathbf{k}[\epsilon]$ module by making ϵ acts by the embedding $V_1^c \to V_2^c$.

Similarly, f induces a morphism of ind-finite-dimensional spaces $f^d : V_1^d \to V_2^d$ where $V_i^d = V/V_i^c$. Note that we have the quotient map $V_1^d \to V_2^d$ and so $\mathrm{Cone}(f^d)[-1]$ is made into an (ind-perfect) $\mathbf{k}[\epsilon]$-module. Now we have a short exact sequence

$$0 \to \mathrm{Cone}(f^c)[-1] \longrightarrow \mathrm{Cone}(f)[-1] \longrightarrow \mathrm{Cone}(f^d)[-1] \to 0$$

which implies that $\eta_L(V, f) = \mathrm{Cone}(f)[-1] \in \mathrm{Tate}_{\mathbf{k}[\epsilon]}$. □

We now prove that the canonical map $c : \psi_L(\eta_L(V, f)) \to (V, f)$ is a quasi-isomorphism in the category of representations of L. i.e., that $C = (\mathrm{Cone}(c), g)$ is contractible. By the above, $\eta_L(c)$ is a quasi-isomorphism, i.e., $\eta_L(C, g) = \mathrm{Cone}(g)$ is contractible. So g is a degree d quasi-isomorphism of C to itself. But C is bounded by our assumption. So C is contractible. This finishes the proof of Proposition 5.4.2. □

B. K-theoretic extensions. Recall the stack of categories **Perf** defined by

$$\mathbf{Perf}(A) = \mathrm{Perf}_A$$

for any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$.

For a perfect dg-category $\mathcal{A}$ we denote by $K(\mathcal{A})$ the space of K-theory of $\mathcal{A}$, so that $\pi_i K(\mathcal{A}) = K_i(\mathcal{A})$. Explicitly, we can define $K(\mathcal{A}) = \Omega|S_\bullet(\mathcal{A})|$ as the loop space of the Waldhausen S-construction (in which all $S_n(\mathcal{A})$ are understood as ∞-groupoids).

We now make K-theory into a prestack

$$\mathbf{K} = K \circ \mathbf{Perf} : A \mapsto K(\mathrm{Perf}(A)).$$

By composing K with **Tate** we get the prestack

$$\mathbf{KTate} : A \mapsto K(\mathrm{Tate}_A).$$

We have the morphism of prestacks

$$\underline{\mathbf{Tate}} \longrightarrow \mathbf{KTate}$$

induced by the identification

$$\mathrm{Tate}_A^{\mathrm{grp}} \longrightarrow S_1(\mathrm{Tate}_A).$$

The following is proven in [He3].

Theorem 5.4.6. $\mathbf{KTate}$ *is identified with* $B(\mathbf{K}) = |S_\bullet(\mathbf{Perf})|$ *(after stackification on Nisnevich topology).*

Remark 5.4.7. Theorem 5.4.6 is a geometric analog of Theorem 4.3.1. In particular, as Theorem 4.3.1 allows us to build central Lie algebra extensions, Theorem 5.4.6 gives us group central extensions. The construction goes as follows.

We have the determinantal $\mathbb{G}_m$-torsor $\mathrm{Det} \to \mathbf{K}$ or, equivalently, a group morphism $\mathbf{K} \to B\mathbb{G}_m$. Applying the classifying stack on both ends, we get the *determinantal gerbe*

$$\mathrm{Det}^{(2)} : \mathbf{KTate} \longrightarrow \mathrm{EM}(\mathbb{G}_m, 2). \tag{5.4.8}$$

This gerbe gives, for any $V \in \mathrm{Tate}_{\mathbf{k}}$, a central extension of group prestacks

$$1 \to \mathbb{G}_m \longrightarrow \widetilde{\underline{\mathrm{Aut}}}(V) \longrightarrow \underline{\mathrm{Aut}}(V) \to 1.$$

Recall that using Theorem 4.3.1, we built in Definition 4.3.3 a cyclic class $\tau_V \in HC^1(\mathrm{End}(V))$ for each $V \in \mathrm{Tate}_{\mathbf{k}}$. With Loday's map from Proposition 3.1.2, we get Lie algebra cohomology classes $\theta(\tau_V) \in H^2_{\mathrm{Lie}}(\mathrm{End}(V))$. Those classes give central extensions of Lie algebras

$$0 \to \mathbf{k} \longrightarrow \widetilde{\mathrm{End}}(V) \to \mathrm{End}(V) \to 0.$$

Theorem 5.4.9. *Let V be a strict Tate complex. The Lie algebra of* $\widetilde{\underline{\mathrm{Aut}}}(V)$ *is identified with* $\widetilde{\mathrm{End}}(V)$.

Remark 5.4.10. It is very natural to expect that the cyclic homology of a dg-category $\mathcal{A}$ can be recovered from its K-theory by some functorial procedure ("taking the tangent space") so that, in particular, the trace class $\mathrm{tr} \in HC^0(\mathbf{k})$ corresponds to the determinantal character (the identification $\det : K_1(\mathbf{k}) \to \mathbf{k}^*$). Then one could argue that the Tate class (the delooping of tr) is similarly "tangent" to the determinantal gerbe (the delooping of det), thus obtaining a very natural proof of Theorem 5.4.9. This would also justify the name "additive K-theory" for cyclic homology.

However, such a direct construction seems to be unknown. The closest statement in this direction is the recovery (due to L. Hesselholt) of the Hochschild homology of a (dg-)algebra R in terms of the rational K-theory of the ring of dual numbers $R[\epsilon]/\epsilon^2$, see [DGM].

We therefore dedicate the rest of this section to a proof of Theorem 5.4.9 by a series of reductions.

C. Primitivity of the Lie cohomology classes. For $V \in \mathrm{Tate}_{\mathbf{k}}$ we denote by $\gamma_V \in \mathbb{H}^2_{\mathrm{Lie}}(\mathrm{End}(V))$ the class corresponding to the Lie algebra of $\widetilde{\underline{\mathrm{Aut}}}(V)$. We need to prove the equality

$$\gamma_V = \theta^*(\tau_V) \tag{5.4.11}$$

where $\tau_V \in HC^1(\mathrm{End}(V))$ is induced by the Tate class $\tau \in HC^1(\mathrm{Tate}_{\mathbf{k}})$ and θ is the Loday homomorphism, see §3.1B.

Note that the statement is known (and classical) in the case when $V = \mathbf{k}((z))$ is the most standard example of a Tate space. We will now reduce to this case by showing that the system of classes γ_V satisfies compatibilities that hold for the system of $\theta^*(\tau_V)$.

Definition 5.4.12. Let $\eta_V \in \mathbb{H}^2_{\mathrm{Lie}}(\mathrm{End}(V))$, $V \in \mathrm{Tate}_{\mathbf{k}}$ be a system of Lie algebra cohomology classes. We say that (η_V) is a *primitive system* if, for any direct sum decomposition $V \simeq V_1 \oplus V_2$ in the abelian category of strict Tate complexes we have

$$\eta_V|_{\mathrm{End}(V_1)\oplus\mathrm{End}(V_2)} \;=\; p_1^*\eta_{V_1} + p_2^*\eta_{V_2}.$$

Here $p_\nu : \mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \to \mathrm{End}(V_\nu)$ is the projection.

We note that direct sum decompositions with $V_2 = 0$ are given by isomorphisms $\phi : V \to V_1$, so a primitive system satisfies, in particular, the compatibility condition: $\mathrm{Ad}_\phi^*(\eta_{V_1}) = \eta_V$. Here

$$\mathrm{Ad}_\phi : \mathrm{End}(V) \to \mathrm{End}(V_1), \quad u \mapsto \phi \circ u \circ \phi^{-1}.$$

Lemma 5.4.13. *The classes γ_V form a* primitive system*.*

Proof: This is because the determinantal gerbe, being a K-theory datum, is "group-like", i.e., gives a local system on the Waldhausen space of Tate_A for any A. That is, for any triangle (simplest cell on the Waldhausen space)

$$V_1 \to V \to V_2$$

in Tate_A we have an isomorphism of $\mathbb{G}_m$-gerbes over $\mathrm{Spec}(\mathbf{k})$

$$\mathrm{Det}^{(2)}(V_1) \otimes \mathrm{Det}^{(2)}(V_2) \to \mathrm{Det}^{(2)}(V)$$

satisfying coherent compatibilties. In particular, for $V \simeq V_1 \oplus V_2$, a direct sum decomposition in $\mathrm{Tate}_{\mathbf{k}}$, we have

$$\widetilde{\underline{\mathrm{Aut}}}(V_1 \oplus V_2)_{\underline{\mathrm{Aut}}(V_1)\times \underline{\mathrm{Aut}}(V_2')} \;\simeq\; \widetilde{\underline{\mathrm{Aut}}}(V_1) \star \widetilde{\underline{\mathrm{Aut}}}(V_2)$$

(Baer sum). This, by differentiation (passing to the Lie algebras of group stacks), implies that the system (γ_V) is primitive. $\square$.

Lemma 5.4.14. *The classes $\theta^*\tau_V$ form a primitive system as well.*

Proof: This is a general property of cyclic homology. Let $A = \mathrm{End}(V)$ and $A_i = \mathrm{End}(V_i)$. We then have the enbedding of dg-algebras $A_1 \oplus A_2 \to A$. It is enough to prove that the restriction of the Loday homomorphism θ^A to $\mathbb{H}_2^{\mathrm{Lie}}(A_1 \oplus A_2)$ is equal to the sum of the restrictions on $\mathbb{H}^2_{\mathrm{Lie}}(A_1)$ and $\mathbb{H}^2_{\mathrm{Lie}}(A_2)$. This restriction is the left path in the commutative diagram

$$\begin{tikzcd}
\mathbb{H}_2^{\mathrm{Lie}}(A_1 \oplus A_2) \arrow[r, "\theta^{A_1\oplus A_2}"] \arrow[d] & HC_{r-1}(A_1 \oplus A_2) \arrow[d] \\
\mathbb{H}_2^{\mathrm{Lie}}(A) \arrow[r, "\theta^A"] & HC_{r-1}(A).
\end{tikzcd}$$

Looking at the right path we see that $HC_1(A_1 \oplus A_2)$ being identified with $HC_1(A_1) \oplus HC_1(A_2)$, the composition splits into the direct sum of the two restrictions, as claimed. $\square$

D. Comparison of Lie cohomology classes. It remains now to prove the following statement.

Proposition 5.4.15. *Let η and η' be two primitive system of classes in $\mathbb{H}^2_{\mathrm{Lie}}(\mathrm{End}(V))$, $V \in \mathrm{Tate}_{\mathbf{k}}$. Suppose that $\eta_{\mathbf{k}((z))} = \lambda \cdot \eta'_{\mathbf{k}((z))}$ for some $\lambda \in \mathbf{k}$. Then $\eta_V = \lambda \cdot \eta'_V$ for any strict Tate complex V.*

We notice first:

Proposition 5.4.16. *Let (η_V) be a primitive system. If $V \simeq V_1 \oplus V_2$ as before, then the pullback of η_V to $\mathrm{End}(V_1) \subset \mathrm{End}(V)$, is equal to η_{V_1}.* $\square$

Let now $V^\bullet$ be a strict Tate complex. Decomposing it as $V^\bullet = H^\bullet \oplus E^\bullet$ as in Corollary 4.1.18(c), we have an isomorphism of associative dg-algebras (and hence of dg-Lie algebras) $\mathrm{End}(H^\bullet) \to \mathrm{End}(V^\bullet)$. It implies an isomorphism $\mathbb{H}^\bullet_{\mathrm{Lie}}(\mathrm{End}(V^\bullet)) \simeq \mathbb{H}^\bullet_{\mathrm{Lie}}(\mathrm{End}(H^\bullet))$. Since $H^\bullet$ has no differential, $\mathrm{End}(H^\bullet)$ is a graded Lie algebra without differential.

Proposition 5.4.17. *Let H be a graded Tate space (situated in finitely many degrees) which is neither discrete nor linearly compact. Assume the graded components of H are of dimension either 0 or ∞. Then we have $\mathbb{H}^2_{\mathrm{Lie}}(\mathrm{End}(H)) \simeq \mathbf{k}$.*

Proof: This is a modification of the result of [FT1] which can be considered as corresponding to V being $\mathbf{k}((z))$ in degree 0. We first relate $HC^1(\mathrm{End}(H))$ with $HC^1(\mathrm{Tate}_\mathbf{k}) \simeq \mathbf{k}$. More precisely, we note:

Lemma 5.4.18. *Let H be as above.*

(a) The functor

$$\rho = R\,\mathrm{Hom}(H, -) : \mathrm{Tate}_\mathbf{k} \longrightarrow \mathrm{dgMod}_{\mathrm{End}(H)}\,.$$

takes values in perfect dg-modules over $\mathrm{End}(H)$*.*

(b) This functor gives a quasi-equivalence between $\mathrm{Tate}_\mathbf{k}$ *and* $\mathrm{Perf}_{\mathrm{End}(H)}$*. In particular, $HC_\bullet(\mathrm{End}(H)) \simeq HC_\bullet(\mathrm{Tate}_\mathbf{k})$ is spanned by generators in degrees $1, 3, 5, \ldots$.*

Proof: (a) Since H is neither linearly compact nor discrete, it decomposes as $C \oplus D$, where $C \in C_\mathbf{k}$ and $D \in D_\mathbf{k}$ have at least one graded component infinite-dimensional. That is, D admits a shift of $\bigoplus_{\mathbb{Z}_+} \mathbf{k}$ as a direct summand, while C admits a shift of $\prod_{\mathbb{Z}_+} \mathbf{k}$ as a direct summand. It follows that any Tate complex W can be obtained, up to a quasi-isomorphism, by a finite number of extensions and retracts from H. This means that $R\,\mathrm{Hom}(H, W)$ can be obtained by a finite number of extensions and retracts from $R\,\mathrm{Hom}(H, H) = \mathrm{End}(H)$, so it is perfect.

(b) Denote $A = \mathrm{End}(H)$. We first prove that ρ is fully faithful in the dg-sense, i.e., induces quasi-isomorphisms on Hom-complexes. This is certainly true for the complex $\mathrm{Hom}(H, H)$ which is sent by ρ to $\mathrm{Hom}_A(A, A) = A$. Further, ρ is exact and takes direct summands (retracts) to direct summands. So ρ induces a quasi-isomorphism on $\mathrm{Hom}(W_1, W_2)$, where W_1 and W_2 are any Tate complexes obtained from H by a finite number of extensions and retracts. But by the above, all Tate complex are obtained in such a way.

Next, we show that ρ is essentially surjective. This is immediate since Perf_A is generated, under extensions and retracts by $A = \rho(H)$ itself. $\square$

We now prove Proposition 5.4.17 by the same arguments as in [FT1]. We keep the notation $A = \mathrm{End}(H)$ and apply the dg-algebra analog of the Loday-Quillen-Tsygan theorem [Bur] which gives that $\mathbb{H}^{\mathrm{Lie}}_\bullet(\mathfrak{gl}_\infty(A))$ is the symmetric algebra on the graded space $HC_{\bullet-1}(A)$. Next, because each component of H is infinite-dimensional, $H \simeq (H)^{\oplus r}$ and therefore $A \simeq \mathfrak{gl}_r(A)$ for each $r \geqslant 1$. This allows us to pass from $\mathfrak{gl}_\infty(A)$ to A itself and conclude that $\mathbb{H}^2_{\mathrm{Lie}}(A) \simeq HC^1(A) = \mathbf{k}$. Proposition 5.4.17 is proved.

We now finish the proof of Proposition 5.4.15. If $\eta_{\mathbf{k}((z))} = \lambda \cdot \eta'_{\mathbf{k}((z))}$, we have $\eta_V = \lambda \cdot \eta'_V$ for any $V = \mathbf{k}((z)) \otimes_{\mathbf{k}} F$, where F is a finite-dimensional graded $\mathbf{k}$-vector space. In other words, V is a direct sum of shifts of $\mathbf{k}((z))$. Indeed, up to a shift $\mathbf{k}((z))$ is a direct summand of V and so the statement follows from Proposition 5.4.16. Further, if V is any strict Tate complex, then there exists an F as above such that V is a direct summand of $\mathbf{k}((z)) \otimes_{\mathbf{k}} F$, and so the statement again follows from Proposition 5.4.16. Proposition 5.4.15 and Theorem 5.4.9 are proved.

5.5 Action on determinantal torsors

A. Global sections on D_n°. Let us denote by $\mathbf{Perf}_{D_n^\circ}$ the derived prestack in categories

$$\mathbf{Perf}_{D_n^\circ} \colon A \mapsto \mathrm{Perf}_{\mathrm{Spec}(A[[z_1,\ldots,z_n]])-\{0\}}$$

It is actually a stack [HPV, Theorem 6.10].

Proposition 5.5.1. *The global section functor $R\Gamma$ from Proposition 4.2.6 naturally extends to a morphism of prestacks*

$$R\Gamma \colon \mathbf{Perf}_{D_n^\circ} \longrightarrow \mathbf{Tate}.$$

Proof. For any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ and $p \geqslant 0$, denote by A_p the Kozsul resolution associated to $(z_1^p, \ldots, z_n^p)$ in $A[z_1, \ldots, z_n]$. The cdga $A[[z]]$ is the homotopy limit of the diagram $q \mapsto A[z]/I_q$, where I_q is the ideal generated by degree q monomials. For any p, we have $(z_1^p, \dots, z_n^p) \subset I_p$ and $I_{np} \subset (z_1^p, \dots, z_n^p)$. We get

$$A[[z]] \simeq \underleftarrow{\operatorname{holim}} A_p.$$

Let $\mathbf{Perf}_{D_n}$ denote the functor $A \mapsto \operatorname{Perf}_{A[[z]]}$ and $\mathbf{Perf}_{D_n^{(p)}}$ be the functor $A \mapsto \operatorname{Perf}_{A_p}$. It follows from [Lu4, §5.1] that the canonical morphism of stacks

$$\mathbf{Perf}_{D_n} \longrightarrow \underleftarrow{\operatorname{holim}}_p \mathbf{Perf}_{D_n^{(p)}}$$

is an equivalence. For any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$ and any $p \in \mathbb{N}$, the category $\operatorname{Perf}_{A_p}$ is canonically equivalent to the category of A_p-modules in Perf_A (as A_p is perfect on A). In particular, it embeds fully faithfully into the category of A_p-modules in $\operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_A$. We get

$$\mathbf{Perf}_{D_n} \subset \underleftarrow{\operatorname{holim}}_p \mathbf{Mod}_{\mathcal{O}_p}^{\mathbf{IPP}},$$

where $\mathbf{Mod}_{\mathcal{O}_p}^{\mathbf{IPP}}$ is the functor $A \mapsto \underleftarrow{\operatorname{holim}}_p \operatorname{Mod}_{A_p}(\operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_A)$. Denote by $\mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}}$ the functor $A \mapsto \operatorname{Mod}_{A[[z]]^{\mathrm{top}}}(\operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_A)$ where $A[[z]]^{\mathrm{top}}$ is "$\underleftarrow{\operatorname{holim}}$" A_p, considered as a commutative algebra in $\operatorname{Pro}\operatorname{Perf}_A$. The base change natural transformation

$$\mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}} \longrightarrow \underleftarrow{\operatorname{holim}}_p \mathbf{Mod}_{\mathcal{O}_p}^{\mathbf{IPP}}$$

admits a pointwise right adjoint ψ. However ψ is not a natural transformation as it does not commute with base change. It does once restricted to $\mathbf{Perf}_{D_n}$ though and we get a natural transformation

$$\mathbf{Perf}_{D_n} \longrightarrow \mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}}.$$

We now consider $\mathfrak{A}_n^\bullet = R\Gamma(D_n^\circ, \mathcal{O})$ as a $\mathbf{k}[[z]]^{\mathrm{top}}$-module in $\operatorname{Ind}\operatorname{Pro}\operatorname{Perf}_{\mathbf{k}}$. Tensoring with $\mathfrak{A}_n^\bullet$ defines an endotransformation of $\mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}}$. We then deduce the proposition from the following lemma.

Lemma 5.5.2. *The composite natural transformation*

$$\eta \colon \mathbf{Perf}_{D_n} \longrightarrow \mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}} \xrightarrow{-\otimes \mathfrak{A}_n^\bullet} \mathbf{Mod}_{\mathcal{O}[[z]]}^{\mathbf{IPP}} \xrightarrow{\text{Forget}} \operatorname{Ind}\operatorname{Pro}\mathbf{Perf}$$

has values in **Tate** *and is null-homotopic once restricted to perfect complexes supported at* $0 \in D_n$.

Proof. Let $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$. The functor η_A has by construction values in Tate_A. It now suffices to prove that the image $\eta_A(A)$ vanishes (where A is seen as a $A[[z]]$-modules with the trivial action). Using base-change, we can assume $A = \mathbf{k}$.

The $\mathbf{k}[[z]]^{\mathrm{top}}$-complex $\mathfrak{A}_n^\bullet$ is, by Čech descent, the homotopy limit of modules of the form $\mathbf{k}[[z]]^{\mathrm{top}}[z_I^{-1}]$ for a none empty $I \subset \{1, \ldots, n\}$. Resolving $\mathbf{k}$ as a $\mathbf{k}[[z]]$-module using the natural Kozsul complex, we get

$$\mathbf{k} \otimes^L_{\mathbf{k}[[z]]^{\mathrm{top}}} \mathbf{k}[[z]]^{\mathrm{top}}[z_I^{-1}] \simeq 0$$

for any $I \neq \varnothing$. The functor $\eta_{\mathbf{k}}$ therefore maps k to an acyclic complex. $\square$

We now finish the proof of Proposition 5.5.1. Notice that for any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, the category $\mathbf{Perf}_{D_n^\circ}(A)$ is a quotient of the category $\mathbf{Perf}_{D_n}(A)$ by the stable full subcategory of perfect complexes supported at 0. It follows from the lemma that η factors through the morphism of prestacks

$$R\Gamma \colon \mathbf{Perf}_{D_n^\circ} \longrightarrow \mathbf{Tate}$$

which coincides with the functor from Proposition 4.2.6 over $\mathbf{k}$-points. This concludes the proof of Proposition 5.5.1. $\square$

Remark 5.5.3. Note that the above construction can be mimicked to define a global section morphism

$$R\Gamma \colon \mathbf{Perf}_{\widehat{x}^\circ} \to \mathbf{Tate}$$

for any $\mathbf{k}$-point x in a variety of dimension n.

B. Determinantal torsors and determinantal gerbes. Let $\phi : G \to GL_r$ be a representation of G. Each G-bundle E on any derived stack Z induces a vector bundle $\phi_* E \in \mathrm{Perf}_Z$.

As before, X is a smooth projective variety, $\dim(X) = n$. We construct a $\mathbb{G}_m$-torsor $\det^\phi \in \mathrm{Pic}(\mathbb{R}\mathbf{Bun}_G(X))$ as the morphism $\mathbb{R}\mathbf{Bun}_G(X) \to B\mathbb{G}_m$ defined as the composition

$$\mathbb{R}\mathbf{Bun}_G(X) \xrightarrow{\phi_*} \underline{\mathbf{Perf}}_X \xrightarrow{R\Gamma} \underline{\mathbf{Perf}}_{\mathbf{k}} \longrightarrow \mathbf{KPerf}_{\mathbf{k}} \xrightarrow{\det} B\mathbb{G}_m.$$

We also denote by $\det^\phi$ the pullback of this torsor to $\mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X, x)$.

The determinantal gerbe $\mathrm{Det}^{\phi} : \mathbb{R}\mathbf{Bun}_G(\widehat{x}^{\circ}) \to K(\mathbb{G}_m, 2)$ is the composition

$$\mathbb{R}\mathbf{Bun}_G(\widehat{x}^{\circ}) \xrightarrow{\phi_*} \underline{\mathbf{Perf}}_{\widehat{x}^{\circ}} \xrightarrow{R\Gamma} \underline{\mathbf{Tate}} \longrightarrow \mathbf{KTate} \xrightarrow{\mathrm{Det}^{(2)}} K(\mathbb{G}_m, 2)$$

where $\mathrm{Det}^{(2)}$ defined in (5.4.8).

Proposition 5.5.4. *(a) The determinantal gerbe* Det^{ϕ} *comes with canonical trivializations* $\widehat{\tau}$ *and* τ° *over* $\mathbb{R}\mathbf{Bun}_G(\widehat{x})$ *and* $\mathbb{R}\mathbf{Bun}_G(X^{\circ})$*.*

(b) The determinantal torsor $\det^{\phi}$ *is equivalent to* $\mathrm{Hom}_{\mathrm{Det}^{\phi}}(\widehat{\tau}, \tau^{\circ})$*.*

Proof. (a) Let us first deal with $\widehat{\tau}$. We have a canonical natural transformation

$$\widehat{\alpha} \colon R\Gamma_{\widehat{x}} = R\Gamma(\widehat{x}, -) \longrightarrow R\Gamma_{\widehat{x}^{\circ}} = R\Gamma(\widehat{x}^{\circ}, -)$$

of maps $\mathbf{Perf}_{\widehat{x}} \to \mathbf{Tate}$. For any $A \in \mathcal{C}dga_{\mathbf{k}}^{\leqslant 0}$, the K-theory of Tate_A is equivalent to that of the quotient $\mathrm{Tate}_A/\mathbf{D}_A$. In particular, the natural transformation $\widehat{\alpha}$ induces an equivalence of morphisms between the two composites

$$\underline{\mathbf{Perf}}_{\widehat{x}} \xrightarrow{R\Gamma_{\widehat{x}}} \underline{\mathbf{Tate}} \longrightarrow \mathbf{KTate} \quad \text{and} \quad \underline{\mathbf{Perf}}_{\widehat{x}} \xrightarrow{R\Gamma_{\widehat{x}^{\circ}}} \underline{\mathbf{Tate}} \longrightarrow \mathbf{KTate}.$$

The LHS composite factors through $\mathbf{KC} \colon A \mapsto \mathbf{K}(\mathbf{C}_A)$ which vanishes, as the categories of compact complexes admit infinite sums. The RHS composite appears in the restriction of Det^{ϕ} to $\mathbb{R}\mathbf{Bun}_G(\widehat{x})$ and this identification induces the trivialization $\widehat{\tau}$.

The case of τ° is done similarly, using the natural transformation

$$\alpha^{\circ} \colon R\Gamma(X^{\circ}, -) \longrightarrow R\Gamma(\widehat{x}^{\circ}, -).$$

This concludes the proof of (a).

(b) By construction, the torsor $\mathrm{Hom}_{\mathrm{Det}^{\phi}}(\widehat{\tau}, \tau^{\circ})$ is induced by the map $\mathbf{Perf}_X \to \mathbf{Tate}$ mapping a family of perfect complexes E on X to the homotopy equalizer of the maps

$$R\Gamma(X^{\circ}, E) \oplus R\Gamma(\widehat{x}, E) \underset{\alpha^{\circ}}{\overset{\widehat{\alpha}}{\rightrightarrows}} R\Gamma(\widehat{x}^{\circ}, E).$$

This homotopy equalizer is equivalent to $R\Gamma(X, E)$, by a family version of Proposition 1.1.4 (which follows from [HPV, Corollary 6.13]). The result follows. $\square$

C. The action of central extensions. Let X be a projective variety of dimension n, and $x \in X(\mathbf{k})$ a $\mathbf{k}$-point. Let $\phi\colon G \to GL_r$ be a representation. Pulling back the Tate class from Definition 4.3.3 along the functor $R\Gamma\colon \mathrm{Perf}_{\widehat{x}^\circ} \to \mathrm{Tate}_{\mathbf{k}}$, we get a class

$$\tau_x \in HC^1(\widehat{x}^\circ) \simeq HC^1(\mathfrak{A}_x^\bullet).$$

Recall the definition $\mathfrak{A}_x^\bullet = R\Gamma(\widehat{x}^\circ, \mathcal{O})$.

Definition 5.5.5. The class τ_x induces a central extension of $\mathfrak{g}_x^\bullet = \mathfrak{g} \otimes \mathfrak{A}_x^\bullet$ that we will denote by $\widetilde{\mathfrak{g}}_{x,\phi}^\bullet$

$$\mathbf{k} \longrightarrow \widetilde{\mathfrak{g}}_{x,\phi}^\bullet \longrightarrow \mathfrak{g}_x^\bullet.$$

This extension has a geometric counterpart. Recall the determinantal gerbe $\mathrm{Det}^\phi\colon \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ) \to K(\mathbb{G}_m, 2)$. We denote by $[\mathrm{Det}^\phi]$ its total space

$$[\mathrm{Det}^\phi] = \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ) \times^h_{K(\mathbb{G}_m,2)} \{*\}.$$

The diagonal map and the trivial bundle define a $\mathbf{k}$-point $d \in [\mathrm{Det}^\phi](\mathbf{k})$.

Definition 5.5.6. Let $\widetilde{G}(\widehat{x}^\circ)_\phi$ denote the group stack $\Omega_d[\mathrm{Det}^\phi]$. It comes with a natural projection $\pi\colon \widetilde{G}(\widehat{x}^\circ)_\phi \to G(\widehat{x}^\circ) = \Omega_{\mathrm{Triv}}\, \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ)$. The homotopy fiber of π at the unit is the group scheme $\mathbb{G}_m$, so that we have a central group extension

$$\mathbb{G}_m \longrightarrow \widetilde{G}(\widehat{x}^\circ)_\phi \longrightarrow G(\widehat{x}^\circ).$$

Remark 5.5.7. Note that the extension $\widetilde{G}(\widehat{x}^\circ)_\phi$ is classified by the group morphism $G(\widehat{x}^\circ) \to B\mathbb{G}_m$ obtained by taking the pointed loops of the map $\mathrm{Det}^\phi\colon \mathbb{R}\mathbf{Bun}_G(\widehat{x}^\circ) \to K(\mathbb{G}_m, 2)$.

The following is a direct consequence of the above definitions and of Proposition 5.4.2.

Proposition 5.5.8. *The Lie algebra extensions* $\mathrm{Lie}(\widetilde{G}(\widehat{x}^\circ)_\phi)$ *and* $\widetilde{\mathfrak{g}}_{x,\phi}^\bullet$ *of* $\mathrm{Lie}(G(\widehat{x}^\circ)) \simeq \mathfrak{g}_x^\bullet$ *are equivalent.*

We denote by $[\det^\phi]$ the total space of the determinantal torsor $\det^\phi$ on $\mathbb{R}\mathbf{Bun}^{\mathrm{rig}}(X, x)$.

Theorem 5.5.9. *(a) The group $\widetilde{G}(\widehat{x}^\circ)_\phi$ acts on $[\det^\phi]$ in a way compatible with the projections $\widetilde{G}(\widehat{x}^\circ)_\phi \to G(\widehat{x}^\circ)$ and $[\det^\phi] \to \mathbb{R}\mathbf{Bun}_G^{\mathrm{rig}}(X,x)$, and with the action from Proposition 5.3.8.*

(b) The dg-Lie algebra $\widetilde{\mathfrak{g}}^\bullet_{x,\phi}$ acts infinitesimally on $[\det^\phi]$ in a way compatible with the infinitesimal action of $\mathfrak{g}^\bullet_x$ on $\mathbb{R}\mathbf{Bun}^{\mathrm{rig}}(X,x)$.

Proof. (a) By construction the group $\widetilde{G}(\widehat{x}^\circ)_\phi$ is the pointed loop group of the point d in $[\mathrm{Det}^\phi]$. Moreover, the trivialization τ° and Proposition 5.5.4 give a homotopy cartesian square

$$\begin{tikzcd}
{[\det^\phi]} \arrow[r] \arrow[d] & \mathbb{R}\mathbf{Bun}_G(X^\circ) \arrow[d, "\tau^\circ"] \\
\{d\} \arrow[r] & {[\mathrm{Det}^\phi].}
\end{tikzcd}$$

Furthermore, the inclusion $\{d\} \to [\mathrm{Det}^\phi]$ factors as

$$\{d\} = \operatorname{Spec}\mathbf{k} = \{\mathrm{Triv}\} \longrightarrow \mathbb{R}\mathbf{Bun}_G(\widehat{x}) \xrightarrow{\widehat{\tau}} [\mathrm{Det}^\phi].$$

In particular, Proposition 5.2.2 defines the announced action of $\widetilde{G}(\widehat{x}^\circ)_\phi$ on $[\det^\phi]$. The above diagram being compatible with the various projections, we see that the action is indeed compatible with the one from Proposition 5.3.8.

(b) It follows from (a) and the Kodaira–Spencer morphism. □

A Model categories of dg-algebras and dg-categories

(A.A) Conventions on complexes. We recall that $\mathbf{k}$ is a field of characteristic 0. We follow the usual sign conventions on (differential) graded $\mathbf{k}$-vector spaces, their tensor products, Koszul sign rule for symmetry and so on. The degree of the differential is always assumed to be +1. The degree of a homogeneous element v will be denoted $|v|$. Note, in particular, the convention

$$(f\otimes g)(v\otimes w) \;=\; (-1)^{|g|\cdot|v|} f(v)\otimes g(w) \tag{A.1}$$

for the action of the tensor product of two operators $f : V \to V'$ and $g : W \to W'$.

The *shift of grading* of a graded vector space $V^\bullet$ is defined by $V^\bullet[n] = \mathbf{k}[n] \otimes V^\bullet$, where $\mathbf{k}[n]$ is the field $\mathbf{k}$ put in degree $(-n)$. So the basis of $\mathbf{k}[n]$ is formed by the vector $1[n]$. For $v \in V^\bullet$ we denote $v[n] = (1[n]) \otimes v \in V^\bullet[n]$. This gives the *suspension morphism* (an "isomorphism" of degree n)

$$s^n : V \longrightarrow V[n], \quad v \mapsto v[n].$$

With respect to tensor products, we have the *decalage isomorphism* (of degree 0)

$$\mathrm{dec} : V_1^\bullet[1] \otimes V_2^\bullet[1] \otimes \cdots \otimes V_n^\bullet[1] \to (V_1^\bullet \otimes V_2^\bullet \otimes \cdots \otimes V_n^\bullet)[n]$$

given by:

$$\mathrm{dec}(s(v_1) \otimes \cdots \otimes s(v_n)) = (-1)^{\sum_{i=1}^n (n-i)|v_i|} s^n(v_1 \otimes \cdots \otimes v_n)$$

This isomorphism induces an isomorphism of graded vector spaces

$$\mathrm{dec}_n : S^n\big(V^\bullet[1])\big) \longrightarrow \big(\Lambda^n V^\bullet\big)[n]. \tag{A.2}$$

(A.B) Model structures and categories of dg-algebras. We will freely use the concept of model categories, see, e.g., [Lu7] for background. For a model category $\mathcal{M}$ we denote by $[\mathcal{M}] = \mathcal{M}[W^{-1}]$ the corresponding homotopy category obtained by inverting weak equivalences.

We denote by $\mathrm{dgVect}_\mathbf{k}$ the category of differential graded vector spaces (i.e., cochain complexes) over $\mathbf{k}$ with no assumptions on grading. This is a symmetric monoidal category.

Let $\mathcal{P}$ be a $\mathbf{k}$-linear operad. By a dg-algebra of type $\mathcal{P}$ we will mean a $\mathcal{P}$-algebra in $\mathrm{dgVect}_\mathbf{k}$. They form a category denoted $\mathrm{dgAlg}^\mathcal{P}$. Thus, for $\mathcal{P}$ being one of the three operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$, describing associative, commutative and Lie algebras, we will speak about *associative dg-algebras*, resp. *commutative dg-algebras*, resp. *dg-Lie algebras* (over $\mathbf{k}$). The categories formed by such algebras will be denoted by $\mathrm{dgAlg}_\mathbf{k}$, $\mathrm{Cdga}_\mathbf{k}$ and $\mathrm{dgLie}_\mathbf{k}$ respectively. They have products, given by direct sums over $\mathbf{k}$, and coproducts given by free products of algebras, denoted $A * B$. For commutative dg-algebras, $A * B = A \otimes_\mathbf{k} B$.

The category $\mathrm{dgAlg}^\mathcal{P}$ carries a natural model structure [Hi], in which:

- Weak equivalences are quasi-isomorphisms.
- Fibrations are surjective morphisms of dg-algebras.

Cofibrations are uniquely determined by the axioms of model categories. In particular, any dg-algebra A of type $\mathcal{P}$ and any graded vector space V we can form $F_{\mathcal{P}}(V)$, the free algebra of type $\mathcal{P}$ generated by V, and the embedding $A \to A * F_{\mathcal{P}}(V)$ is a cofibration.

As usual, the model structure allows us to form homotopy limits and colimits in the categories $\mathrm{dgAlg}^{\mathcal{P}}$. They will be denoted by $\underleftarrow{\mathrm{holim}}$ and $\underrightarrow{\mathrm{holim}}$. In particular, if $\mathcal{F}$ is a sheaf of algebras of type $\mathcal{P}$ on a topological space S, then

$$R\Gamma(S, \mathcal{F}) \;=\; \underleftarrow{\mathrm{holim}}_{\substack{U \subset S \\ \text{open}}} \mathcal{F}(U)$$

is canonically defined as an object of the homotopy category $[\mathrm{dgAlg}^{\mathcal{P}}]$. An explicit way of calculating the homotopy limit of a diagram of algebras represented by a cosimplicial algebra (this includes $R\Gamma(S, \mathcal{F})$) is provided by the Thom-Sullivan construction, see [HiS].

The above includes (for $\mathcal{P}$ being the trivial operad) the category $\mathrm{dgVect}_{\mathbf{k}}$ itself. In particular, let us note the following fact about homotopy limits in $\mathrm{dgVect}_{\mathbf{k}}$ indexed by $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$.

Proposition A.3. *(a) Let $(E_i^\bullet)$ be an inductive system over $\mathrm{dgVect}_{\mathbf{k}}$, indexed by $\mathbb{Z}_+$. Then the natural morphism $\underrightarrow{\mathrm{holim}}\, E_i^\bullet \to \underrightarrow{\lim}\, E_i^\bullet$ is a quasi-isomorphism.*

(b) Let $(E_i^\bullet)$ be a projective system over $\mathrm{dgVect}_{\mathbf{k}}$, indexed by $\mathbb{Z}_+$. Then the natural morphism $\underleftarrow{\lim}\, E_i^\bullet \to \underleftarrow{\mathrm{holim}}\, E_i^\bullet$ is a quasi-isomorphism in each of the following two cases:

(b1) Each $E_i^\bullet$ is a perfect complex.

(b2) The morphisms in the projective system $(E_i^\bullet)$ are termwise surjective.

Proof: (a) $\underrightarrow{\mathrm{holim}}$ is the left derived functor of $\underrightarrow{\lim}$. Therefore part (a) follows from the fact that the functor $\underrightarrow{\lim}^{\mathrm{Vect}_{\mathbf{k}}}$ is exact on the category of inductive systems of vector spaces indexed by $\mathbb{Z}_+$.

(b) $\underleftarrow{\mathrm{holim}}$ is the right derived functor of $\underleftarrow{\lim}$. Therefofe we have the spectral sequence

$$(R^q \underleftarrow{\lim}_i)(E_i^q) \;\Rightarrow\; H^{p+q}(\underleftarrow{\mathrm{holim}}_i E_i^\bullet).$$

As well known, the functions $R^q \varprojlim_i$ for countable filtering diagrams can be nonzero only for $q = 0, 1$. Further, $R^1 \varprojlim_i$ vanishes for diagrams of finite-dimensional spaces as well as for any diagrams formed by surjective maps. $\square$

(A.C) Model structure on category of dg-categories. We denote by $\mathrm{dgCat}_{\mathbf{k}}$ the category of $\mathbf{k}$-linear dg-categories. For a dg-category $\mathcal{A}$ we denote by $[\mathcal{A}]$ the corresponding H^0-category: it has the same objects as $\mathcal{A}$, while $\mathrm{Hom}_{[\mathcal{A}]}(x, y) = H^0 \mathrm{Hom}^\bullet_{\mathcal{A}}(x, y)$. This notation, identical with the notation for the homotopy category of a model category, does not cause confusion: when both meanings are possible, the result is the same.

We equip $\mathrm{dgCat}_{\mathbf{k}}$ with the Morita model structure of Tabuada [Tab]. Weak equivalences in this structure are Morita equivalences. Fibrant objects are *perfect dg-categories*, i.e., dg-categories quasi-equivalent to $\mathrm{Perf}_{\mathcal{B}}$ where $\mathcal{B}$ is some small dg-category. We recall two additional characterizations of perfect dg-categories.

First, $\mathcal{A}$ is perfect, if and only if the Yoneda embedding $\mathcal{A} \to \mathrm{Perf}_{\mathcal{A}}$ is a quasi-equivalence.

Second, $\mathcal{A}$ is perfect, if and only if $\mathcal{A}$ is pre-triangulated and $[\mathcal{A}]$ (which is then triangulated) is closed under direct summands.

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