K-SEMISTABLE FANO MANIFOLDS WITH THE SMALLEST ALPHA INVARIANT

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ABSTRACT. In this short note, we show that K-semistable Fano manifolds with the smallest alpha invariant are projective spaces. Singular cases are also investigated.

1. INTRODUCTION

Throughout the article, we work over the complex number field $\mathbb{C}$. A $\mathbb{Q}$-*Fano variety* is a normal projective variety X with log terminal singularities such that the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. It has been known that a *Fano manifold* X (i.e., a smooth $\mathbb{Q}$-Fano variety) admits Kähler–Einstein metrics if and only if X is *K-polystable* by the works [DT92, Tia97, Don02, Don05, CT08, Sto09, Mab08, Mab09, Ber16] and [CDS15a, CDS15b, CDS15c, Tia15].

On the other hand, the existence of Kähler–Einstein metrics and K-stability are related to the *alpha invariants* $\alpha(X)$ of X defined by Tian [Tia87] (see also [TY87, Zel98, Lu00, Dem08]). Tian [Tia87] proved that for a Fano manifold X, if $\alpha(X) > \dim X/(\dim X + 1)$, then X admits Kähler–Einstein metrics. Odaka and Sano [OS12, Theorem 1.4] (see also its generalizations [Der16, BHJ15, FO16, Fuj16c]) proved a variant of Tian's theorem: if a $\mathbb{Q}$-Fano variety X satisfies that $\alpha(X) > \dim X/(\dim X + 1)$ (resp. $\geq \dim X/(\dim X + 1)$), then X is K-stable (resp. K-semistable). We are interested in the relation of alpha invariants and K-semistability.

Recall that Fujita and Odaka proved that there exists a lower bound of alpha invariants for K-semistable $\mathbb{Q}$-Fano varieties.

Theorem 1.1 ([FO16, Theorem 3.5]). *Let X be a K-semistable $\mathbb{Q}$-Fano variety of dimension n.*

Then $\alpha(X) \geq \frac{1}{n+1}$.

It is natural and interesting to ask when the equality holds. For example, it is well-known that $\mathbb{P}^n$ is K-semistable with $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$. The main theorem of this paper is the following.

Theorem 1.2. *Let X be a K-semistable Fano manifold of dimension n.*

Then $\alpha(X) = \frac{1}{n+1}$ if and only if $X \cong \mathbb{P}^n$.

This is an application of Birkar's answer to Tian's question [Bir16, Theorem 1.5], and Fujita–Li's criterion for K-semistability [Li15, Fuj16b].

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It is natural to ask whether the same statement holds true for K-semistable $\mathbb{Q}$-Fano varieties instead of manifolds. However, this is on longer true even in dimension 2. We are grateful to Kento Fujita for kindly providing the following example:

Example 1.3. Consider the cubic surface $X = (x_0^3 = x_1x_2x_3) \subset \mathbb{P}^3$, which is a toric *log del Pezzo surface* (i.e, a $\mathbb{Q}$-Fano variety of dimension 2) with 3 du Val singularities of type A_2. On one hand, it is well-known that X admits a Kähler–Einstein metric (cf. [DT92]), hence is K-semistable. On the other hand, $\alpha(X) = \frac{1}{3}$ (cf. [PW10]).

In fact, by the classification of possible du Val singularities of K-semistable log del Pezzo surfaces (cf. [Liu16, Corollary 6]) and explicit computation of alpha invariants (cf. [Par03, PW10, CK14]), we have the following theorem.

Theorem 1.4. *Let X be a K-semistable log del Pezzo surface with at worst du Val singularities. Then $\alpha(X) = \frac{1}{3}$ if and only if $X \cong \mathbb{P}^2$ or $X \subset \mathbb{P}^3$ is a cubic surface with at least 2 singularities of type A_2.*

Moreover, by classification of $\mathbb{Q}$-Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$ with large Fano index due to Prokhorov [Pro10, Pro13], we prove the following:

Theorem 1.5. *Let X be a K-semistable $\mathbb{Q}$-Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$. Assume that $h^0(-K_X) \geq 22$. Then $\alpha(X) = \frac{1}{4}$ if and only if $X \cong \mathbb{P}^3$.*

Finally, we propose the following much stronger conjecture. For some evidence in dimension 3, we refer to [CS08] and [Fuj16a].

Conjecture 1.6. *Let X be a K-semistable Fano manifold.*
Then $\alpha(X) < \frac{1}{n}$ if and only if $X \cong \mathbb{P}^n$.

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2. Preliminaries

We adopt the standard notation and definitions in [KM98] and will freely use them.

Definition 2.1. Let X be a $\mathbb{Q}$-Fano variety. The *alpha invariant* $\alpha(X)$ of X is defined by the supremum of positive rational numbers α such that the pair $(X, \alpha D)$ is log canonical for any effective $\mathbb{Q}$-divisor D with $D \sim_{\mathbb{Q}} -K_X$. In other words,

$$\alpha(X) := \inf\{\text{lct}(X; D) | 0 \leq D \sim_{\mathbb{Q}} -K_X\}.$$

Tian [Tia90] asked whether whether the infimum is a minimum, which is answered by Birkar affirmatively.

Theorem 2.2 ([Bir16, Theorem 1.5])**.** *Let X be a $\mathbb{Q}$-Fano variety. Assume that $\alpha(X) \leq 1$. Then there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} -K_X$ and* $\mathrm{lct}(X; D) = \alpha(X)$.

Definition 2.3 ([Fuj16b])**.** Let X be a $\mathbb{Q}$-Fano variety of dimension n. Take any projective birational morphism $\sigma : Y \to X$ with Y normal and any prime divisor F on Y , that is, F is a prime divisor *over* X.

(1) Define the *log discrepancy* of F as $A(F) := \mathrm{mult}_F(K_Y - \sigma^* K_X) + 1$;
(2) Define $\mathrm{vol}_X(-K_X - xF) := \mathrm{vol}_Y(-\sigma^* K_X - xF)$;
(3) Define

$$\beta(F) := A(F) \cdot (-K_X)^n - \int_0^\infty \mathrm{vol}_X(-K_X - xF)\,\mathrm{d}x.$$

Note that the definitions do not depend on the choice of birational model Y.

Instead of recalling the original definition, we use the following criterion to define K-semistability.

Definition-Proposition 2.4 ([Fuj16b, Corollary 1.5], [Li15, Theorem 3.7])**.** Let X be a $\mathbb{Q}$-Fano variety. X is *K-semistable* if $\beta(F) \geq 0$ for any divisor F over X.

Note that K-semistability is known to be equivalent to *Ding-semistability* by [BBJ15].

3. Proof of main theorem

Proposition 3.1. *Let X be a K-semistable $\mathbb{Q}$-Fano variety of dimension n. Assume that $\alpha(X) = \frac{1}{n+1}$, then there exists a prime divisor E on X such that $-K_X \sim_{\mathbb{Q}} (n+1)E$ and (X, E) is plt.*

Proof. Let X be a K-semistable $\mathbb{Q}$-Fano variety of dimension n with $\alpha(X) = \frac{1}{n+1}$. By Theorem 2.2, there is a divisor $D \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{lct}(X; D) = \frac{1}{n+1}$. Take F to be a non-klt place of $(X, \frac{1}{n+1}D)$, then there is a resolution $\sigma : Y \to X$ such that F is a divisor on Y.

Denote μ to be the multiplicity of F in $\sigma^* D$. Note that $\mu > 0$ since X is klt. By assumption,

$$\mathrm{mult}_F\left(K_Y - \sigma^*\left(K_X + \frac{1}{n+1}D\right)\right) = -1,$$

which means that

$$A(F) = \frac{\mu}{n+1}.$$

By Definition-Proposition 2.4, $\beta(F) \geq 0$, which means that

$$\begin{aligned}\frac{1}{n+1}(-K_X)^n &= \frac{A(F)}{\mu}(-K_X)^n \\ &\geq \frac{1}{\mu}\int_0^\infty \mathrm{vol}_X(-K_X - xF)\,\mathrm{d}x \\ &= \int_0^\infty \mathrm{vol}_X(-K_X - x\mu F)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^{\infty} \operatorname{vol}_X(-K_X - xD)\,\mathrm{d}x\\
&= \int_0^{1} (1-x)^n(-K_X)^n\,\mathrm{d}x\\
&= \frac{1}{n+1}(-K_X)^n.
\end{aligned}
$$

The second equality holds since $\sigma^*D \geq \mu F$. Hence all inequalities become equalities. In particular,

$$\operatorname{vol}_X(-K_X - x\mu F) = \operatorname{vol}_X(-K_X - xD) = (1-x)^n(-K_X)^n$$

for almost all x. By differentiability of volume functions ([BFJ09, Corollary C]),

$$
\begin{aligned}
&\mu \cdot \operatorname{vol}_{Y|F}(-\sigma^*K_X)\\
= &-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} \operatorname{vol}_Y(-\sigma^*K_X - x\mu F)\\
= &-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} (1-x)^n(-K_X)^n\\
= &\,(-K_X)^n.
\end{aligned}
$$

Here $\operatorname{vol}_{Y|F}$ is the *restricted volume*, we refer to [ELMNP09] for definition and properties. Since $\operatorname{vol}_{Y|F}(-\sigma^*K_X) > 0$, $F \not\subseteq \mathbf{B}_+(-\sigma^*K_X)$ by [ELMNP09, Theorem C]. Hence by [ELMNP09, Corollary 2.17],

$$\operatorname{vol}_{Y|F}(-\sigma^*K_X) = (-\sigma^*K_X)^{n-1} \cdot F = (-K_X)^{n-1} \cdot \sigma_*F.$$

In other words, we have

$$(-K_X)^{n-1}(D - \mu\sigma_*F) = (-K_X)^n - \mu \cdot \operatorname{vol}_{Y|F}(-\sigma^*K_X) = 0.$$

This implies that $D = \mu\sigma_*F$ since $D \geq \mu\sigma_*F$ and $-K_X$ is ample. In particular, F is not σ-exceptional and σ_*F is a prime divisor on X. Denote $E := \sigma_*F$. Moreover, since F is a non-klt place of $(X, \frac{1}{n+1}D)$, $\operatorname{mult}_E \frac{1}{n+1}D = 1$, that is, $\mu = n+1$. In particular, $-K_X \sim_{\mathbb{Q}} D = (n+1)E$. Finally, this argument shows that F is the only non-klt place of (X, E), which means that (X, E) is plt. □

Corollary 3.2. *Let (X, E) as in Proposition 3.1. Then $X \simeq \mathbb{P}^n$ if one of the following condition holds:*

1. *(1) X is factorial;*
2. *(2) $(E)^n \geq 1$;*
3. *(3) E is Cartier in codimension two and $E \simeq \mathbb{P}^{n-1}$.*

Proof. (1) If X is factorial, then E is a Cartier divisor. In particular, $(E)^n \geq 1$. Hence this is a special case of (2).

(2) If $(E)^n \geq 1$, then

$$(-K_X)^n = (n+1)^n(E)^n \geq (n+1)^n.$$

By [Liu16, Theorem 1.1] or [LZ16, Theorem 9], $X \simeq \mathbb{P}^n$.

(3) If E is Cartier in codimension two and $E \simeq \mathbb{P}^{n-1}$, then by adjunction, $(K_X + E)|_E = K_E$, and

$$(-K_X)^n = \frac{(n+1)^n}{n^{n-1}}(-(K_X+E))^{n-1} \cdot E = \frac{(n+1)^n}{n^{n-1}}(-K_E)^{n-1} = (n+1)^n.$$

Again by [Liu16, Theorem 1.1] or [LZ16, Theorem 9], $X \simeq \mathbb{P}^n$. □

Proof of Theorem 1.2. It follows directly from Proposition 3.1 and Corollary 3.2(1) (or [KO73]). □

4. Singular surfaces

Recall the following theorem on classification of possible du Val singularities of a K-semistable log del Pezzo surface.

Theorem 4.1 ([Liu16, Theorem 23, Proof of Corollary 6])**.** *Let X be a K-semistable log del Pezzo surface with at worst du Val singularities.*

(1) *If $(-K_X)^2 = 1$, then X has at worst singularities of type A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, or D_4;*
(2) *If $(-K_X)^2 = 2$, then X has at worst singularities of type A_1, A_2, or A_3;*
(3) *If $(-K_X)^2 = 3$, then X has at worst singularities of type A_1 or A_2;*
(4) *If $(-K_X)^2 = 4$, then X has at worst singularities of type A_1;*
(5) *If $(-K_X)^2 \geq 5$, then X is smooth.*

We remark that in [Liu16, Corollary 6], log del Pezzo surfaces are assumed to be admitting Kähler–Einstein metrics, but the proof works well for K-semistable log del Pezzo surfaces. The only part that the existence of Kähler–Einstein metrics is needed is to exclude the case that $(-K_X)^2 = 1$ and X has singularities of type A_8.

Recall the following theorem on explicit computation of alpha invariants.

Theorem 4.2 ([Par03], [PW10, Theorems 1.4, 1.5, and 1.6], [CK14, Theorem 1.26, Example 1.27])**.** *Let X be a log del Pezzo surface with at worst du Val singularities. Assume that X is singular, then $\alpha(X) = \frac{1}{3}$ if and only if one of the following holds:*

(1) $(-K_X)^2 = 6$ *and* $\mathrm{Sing}(X) = \{A_1\}$*;*
(2) $(-K_X)^2 = 5$ *and* $\mathrm{Sing}(X) = \{A_2\}$ *or* $\{2A_1\}$*;*
(3) $(-K_X)^2 = 4$ *and* $\mathrm{Sing}(X) = \{A_3\}$ *or* $\mathrm{Sing}(X) \supseteq \{A_1 + A_2\}$*;*
(4) $(-K_X)^2 = 3$ *and* $\mathrm{Sing}(X) \supseteq \{A_4\}$, $\{2A_2\}$, *or* $\mathrm{Sing}(X) = \{D_4\}$*;*
(5) $(-K_X)^2 = 2$ *and* $\mathrm{Sing}(X) \supseteq \{D_5\}, \{(A_5)'\}$, *or* $\{A_7\}$*;*
(6) $(-K_X)^2 = 1$ *and* $\mathrm{Sing}(X) \supseteq \{D_8\}$ *or* $\{E_6\}$.

Proof of Theorem 1.4. Let X be a K-semistable log del Pezzo surface with at worst du Val singularities and $\alpha(X) = \frac{1}{3}$. If X is smooth, then $X \simeq \mathbb{P}^2$ by Theorem 1.2. If X is singular, then $(-K_X)^2 = 3$ and $\mathrm{Sing}(X) \supseteq \{2A_2\}$ by Theorems 4.1 and 4.2. To see the "if" part, one just notice that any cubic surface with at worst singularities of type A_1 or A_2 is K-semistable (cf. [OSS16, Theorem 4.3]). □

5. Singular threefolds

In this section, we prove Theorem 1.5. Recall the following theorem on the upper bound of volumes.

Theorem 5.1 (cf. [Liu16, Theorem 25]). *Let X be a K-semistable $\mathbb{Q}$-Fano 3-fold with at worst terminal singularities. Let $p \in X$ be an isolated singularity with local index r. Then*

$$(-K_X)^3 \leq \frac{(r+2)(4+4r)^3}{(3r)^3}.$$

Proof. Denote by $\mathfrak{m}_p$ the maximal ideal at p. We may take a log resolution of $(X, \mathfrak{m}_p)$, namely $\pi : Y \to X$ such that π is an isomorphism away from p and $\pi^{-1}\mathfrak{m}_p \cdot \mathcal{O}_Y$ is an invertible ideal sheaf on Y. Let E_i be exceptional divisors of π. We define the numbers a_i and b_i by

$$K_Y = \pi^* K_X + \sum_i a_i E_i$$

and

$$\pi^{-1}\mathfrak{m}_p \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_i b_i E_i).$$

It is clear that $\operatorname{lct}(X; \mathfrak{m}_p) = \min_i \frac{1+a_i}{b_i}$. Since π is an isomorphism away from p, we have $b_i \geq 1$ for any i. Since X is terminal at p, by [Kaw93], there exists an index i_0 such that $a_{i_0} = \frac{1}{r}$. Hence

$$\operatorname{lct}(X; \mathfrak{m}_p) \leq \frac{1+a_{i_0}}{b_{i_0}} \leq 1 + \frac{1}{r}.$$

On the other hand, by [Kak00] (see also [TW04, Proposition 3.10]), $\operatorname{mult}_p X \leq r + 2$. Hence by [Liu16, Theorem 16],

$$(-K_X)^3 \leq \left(1 + \frac{1}{3}\right)^3 \operatorname{lct}(X; \mathfrak{m}_p)^3 \operatorname{mult}_p X \leq \frac{(r+2)(4+4r)^3}{(3r)^3}.$$

□

Now let X be a K-semistable $\mathbb{Q}$-Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$ with $\alpha(X) = \frac{1}{4}$. By Proposition 3.1, there exists a prime divisor E on X such that $-K_X \sim_{\mathbb{Q}} 4E$.

Recall that we may define ([Pro10])

$$\operatorname{qW}(X) := \max\{q \mid -K_X \sim qA, A \text{ is a Weil divisor}\},$$
$$\operatorname{q\mathbb{Q}}(X) := \max\{q \mid -K_X \sim_{\mathbb{Q}} qA, A \text{ is a Weil divisor}\}.$$

It is known by [Suz04 , Pro10] that

$$\operatorname{qW}(X), \operatorname{q\mathbb{Q}}(X) \in \{1, \dots, 11, 13, 17, 19\}.$$

Moreover, by [Pro10, Lemma 3.2], in our case, $4 | \operatorname{q\mathbb{Q}}(X)$. Hence there are 2 cases: (i) $\operatorname{q\mathbb{Q}}(X) = 8$; (ii) $\operatorname{q\mathbb{Q}}(X) = 4$.

Now assume that $h^0(-K_X) \geq 22$. Define the genus $g(X) := h^0(-K_X) - 2 \geq 20$.

If $\operatorname{q\mathbb{Q}}(X) = 8$, since $g(X) > 10$, then by [Pro13, Theorem 1.2(ii)], either $X \simeq X_6 \subset \mathbb{P}(1, 2, 3, 3, 5)$ or $X \simeq X_{10} \subset \mathbb{P}(1, 2, 3, 5, 7)$. But in either case,

$-K_X \sim 8A$ where A is an effective divisor, which implies that $\alpha(X) \leq \frac{1}{8}$ since (X, A) is not klt, a contradiction.

Now assume that $q\mathbb{Q}(X) = 4$, by [Pro13, Lemma 8.3], $\mathrm{Cl}(X)$ is torsion-free and $qW(X) = q\mathbb{Q}(X)$, hence there is a Weil divisor A such that $-K_X \sim 4A$. If $g(X) \geq 22$, then by [Pro13, Theorem 1.2(vi)], $X \simeq \mathbb{P}^3$ or $X_4 \subset \mathbb{P}(1,1,1,2,3)$. The latter is absurd, since it has a singularity of index 3, and $(-K_{X_4})^3 = 128/3$, which contradicts to Theorem 5.1. If $20 \leq g(X) \leq 21$, then we have the following possibilities due to computer computation (see [GRD], or [BS07, Pro10, Pro13]):

$g(X)$	$\mathbf{B}$	A^3
21	$\{3\}$	$2/3$
20	$\{5,7\}$	$22/35$

Here $\mathbf{B}$ is the set local indices of singular points. It is easy to see that both cases contradict to Theorem 5.1.

In summary, Theorem 1.5 is proved.

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