

STABLE PAIRS ON NODAL $K3$ FIBRATIONS

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ABSTRACT. We study Pandharipande-Thomas's stable pair theory on $K3$ fibrations over curves with possibly nodal fibers. We describe stable pair invariants of the fiberwise irreducible curve classes in terms of Kawai-Yoshioka's formula for the Euler characteristics of moduli spaces of stable pairs on $K3$ surfaces and Noether-Lefschetz numbers of the fibration. Moreover, we investigate the relation of these invariants with the perverse (non-commutative) stable pair invariants of the $K3$ fibration. In the case that the $K3$ fibration is a projective Calabi-Yau three-fold, by means of wall-crossing techniques, we write the stable pair invariants in terms of the generalized Donaldson-Thomas invariants of 2-dimensional Gieseker semistable sheaves supported on the fibers.

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1. INTRODUCTION

1.1. Overview. The Fourier-Mukai transform is an equivalence of derived categories of coherent sheaves of algebraic varieties, which was first studied for dual pairs of abelian varieties in the seminal paper of Mukai [Muk81]. When a Calabi-Yau 3-fold admits an elliptic fibration, its relative Fourier-Mukai transform turned out to be an effective tool to prove the correspondence between one dimensional sheaf theory and two dimensional sheaf theory [ACRY00], [AYCR01]. In particular, it provides a recipe to relate two kinds of invariants: *Pandharipande-Thomas stable pair invariants* [PT09] which count curves in Calabi-Yau 3-folds, and *Donaldson-Thomas invariants* which count two dimensional semistable sheaves on them [Tho00].

The above correspondence is also important in string theory. The relative Fourier-Mukai transform is a mathematical interpretation of relative T-duality in string theory, which was used by physicists [KMW12, MNVW98, BDF07] to discuss the correspondence between the BPS theory of D2 branes wrapped on the base and the D4 branes which also wrap the elliptic fibers. The PT stable pairs and the two dimensional semistable sheaves are the

mathematical analogues of the D6-D2-D0 branes, and the D4 branes respectively, and the relative T-duality provides a recipe to relate to the BPS state counting of these D-brane systems.

In a more general setting, without assuming that the threefold admits an elliptic fibration, we can still ask the question regarding the correspondence of PT stable pair theory and the DT theory of two dimensional semistable sheaves (i.e. D6-D2-D0 BPS theory and D4 theory in string theory). This question shapes the main motivation behind the current articles. In [Tod13a], the third author derived the formula relating the low degree terms of the generating series of the latter invariants with those of the former invariants, using the wall-crossing technique and some ideas from string theory [DM]. In [GS13], the first and the second authors studied the generating series of DT invariants counting two dimensional stable sheaves on $K3$ fibered threefolds with possibly nodal fibers, and proved their modularity property.

The purpose of this article is to obtain an analogue of the results in [GS13] for PT stable pair theory on $K3$ fibered threefolds. Namely we investigate the modularity property of the stable pair invariants on $K3$ fibrations. We first treat the case of a smooth $K3$ fibration, and relate the stable pair invariants with the generating series satisfying the modularity property. When the $K3$ fibration has possibly nodal fibers, we relate their stable pair invariants with those on smooth $K3$ fibrations via conifold transition. In a more general situation, we also prove a formula relating stable pair invariants with the DT invariants counting two dimensional semistable sheaves, where the latter invariants are expected to have the modular invariance property.

1.2. Main result I: smooth $K3$ fibration. Let X be a smooth projective threefold over \mathbb{C} . For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of stable pairs $\mathcal{P}_n(X, \beta)$ and the associated invariants were introduced by Pandharipande and Thomas [PT09]. The moduli space $\mathcal{P}_n(X, \beta)$ parametrizes pairs

$$(1) \quad s: \mathcal{O}_X \rightarrow F$$

where F is a pure 1-dimensional sheaf on X with $[F] = \beta$ and $\chi(F) = n$, and the cokernel of s is 0-dimensional. They constructed a perfect obstruction theory on $\mathcal{P}_n(X, \beta)$ (see Theorem 2.1) by interpreting it as the moduli space of two term complexes (1) in the derived category. When the virtual dimension of $\mathcal{P}_n(X, \beta)$ is zero, the stable pair invariant $P_{n, \beta}$ is defined by taking the degree of the virtual cycle obtained from this obstruction theory.

We assume that X admits a morphism

$$(2) \quad \pi: X \rightarrow C$$

onto a smooth projective curve C , whose generic fiber is a smooth $K3$ surface. The morphism π is called a *$K3$ fibration*. We always take the curve class β to be in the kernel of π_* denoted by $H_2(X, \mathbb{Z})^\pi$. The virtual dimension of $\mathcal{P}_n(X, \beta)$ is zero for such curve classes¹. We define the following

¹Note that X is not required to be a Calabi-Yau 3-fold here.

generating series:

$$(3) \quad \text{PT}(X)_\beta = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n, \quad \text{PT}(X) = \sum_{\beta \in H_2(X, \mathbb{Z})^\pi} \text{PT}(X)_\beta t^\beta.$$

In Section 2, we study the case when π is a smooth morphism. The majority of this section (Subsections 2.1-2.6) is devoted to the case where β is an irreducible curve class. In this case, we relate $P_{n,\beta}$ with the Euler characteristics of the moduli space of stable pairs on a nonsingular $K3$ surface S and the Noether-Lefschetz numbers $NL_{h,\beta}^\pi$ of the fibration having modular properties [MP13, Bor99, Bor98, KM90]. We denote by $\mathcal{P}_n(K3, h)$ the moduli space of stable pairs on a $K3$ surface containing an irreducible curve class γ satisfying $\gamma^2 = 2h - 2$. It is known that the topological type of $\mathcal{P}_n(K3, h)$ only depends on (n, h) (cf. [MPT10]).

Theorem 1. (Theorem 2.16) *Let $\pi: X \rightarrow C$ be a smooth $K3$ fibration, and $\beta \in H_2(X, \mathbb{Z})^\pi$ be an irreducible curve class. Then we have the following formula:*

$$\text{PT}(X)_\beta = \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n-1} \chi(\mathcal{P}_n(K3, h)) \cdot NL_{h,\beta}^\pi q^n.$$

The proof of Theorem 1 involves studying the restriction of the obstruction theory constructed in [PT09] to isolated and non-isolated components of $\mathcal{P}_n(X, \beta)$. The Euler characteristics $\chi(\mathcal{P}_n(K3, h))$ in Theorem 1 can be read off from Kawai-Yoshioka's formula having modular properties [KY00, Theorem 5.80]:

$$(4) \quad \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n-1} \chi(\mathcal{P}_n(K3, h)) y^n q^h = - \left(\sqrt{-y} - \frac{1}{\sqrt{-y}} \right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{20} (1 + yq^n)^2 (1 + y^{-1}q^n)^2}.$$

In Subsection 2.7, we study a special analog of Theorem 1 in which the class $\beta \in H_2(X, \mathbb{Z})^\pi$ is allowed to be reducible.

1.3. Main result II: nodal $K3$ fibration. We next study the case that the $K3$ fibration (2) has a finite number of nodal fibers. In this case, we call a morphism (2) a *nodal $K3$ fibration*. For a nodal $K3$ fibration, we find a conifold transition formula, which relate the stable pair invariants on nodal $K3$ fibrations with those on smooth $K3$ fibrations. The following result is based on assuming some foundation of stable pair theory of algebraic spaces that has not been established yet (see Remarks 3.3 and 3.4):

Theorem 2. (Theorem 3.11) *Let $\pi: X \rightarrow C$ be a nodal $K3$ fibration. Then there is a smooth $K3$ fibration² $\tilde{\pi}: \tilde{X} \rightarrow \tilde{C}$ and morphisms*

$$\tilde{\epsilon}: \tilde{X} \xrightarrow{h} X_0 \xrightarrow{\epsilon} X$$

²Here the situation is a little more general than (2), and we allow \tilde{X} to be a proper algebraic space.

where h is a small resolution and ϵ is a double cover, such that the following formula holds:

$$(5) \quad \tilde{\epsilon}_* \frac{\text{PT}(\tilde{X})}{\text{PT}^h(\tilde{X})} = \text{PT}(X)^2.$$

Here $\tilde{\epsilon}_*$ is the variable change $t^\beta \mapsto t^{\tilde{\epsilon}_*\beta}$, and $\text{PT}^h(\tilde{X})$ is the subseries of $\text{PT}(\tilde{X})$ of stable pair invariants of curve classes $\tilde{\beta}$ with $h_*\tilde{\beta} = 0$.

Let us consider the invariant $P_{n,\beta}$ with an irreducible curve class β on a nodal $K3$ fibration $\pi: X \rightarrow C$. The above result leads us to working with the stable pair invariants $P_{\tilde{n},\tilde{\beta}}$ for possibly reducible classes $\tilde{\beta}$ on a smooth $K3$ fibration $\tilde{X} \rightarrow \tilde{C}$ lifting β . In Subsection 3.6, we present an approach to compute the invariants $P_{\tilde{n},\tilde{\beta}}$ by studying the relationship between the invariants $P_{n,\beta}$ and the invariants of the moduli spaces of perverse (non-commutative) stable pairs.

1.4. Main result III: product expansion formula. Finally let X be a smooth projective Calabi-Yau 3-fold, which admits a $K3$ fibration $\pi: X \rightarrow \mathbb{P}^1$ such that every fiber is an integral scheme. In Section 4, using the wall-crossing techniques mostly developed by the third author [Tod13b, Tod10a, Tod10b, Tod12a, Tod12b, Tod], we express the generating series (3) in terms of the generalized Donaldson-Thomas invariants

$$(6) \quad J(r, \beta, n) \in \mathbb{Q}$$

which count semistable sheaves E supported on the fibers of π satisfying

$$(7) \quad \text{ch}(E) = (0, r[F], \beta, n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X).$$

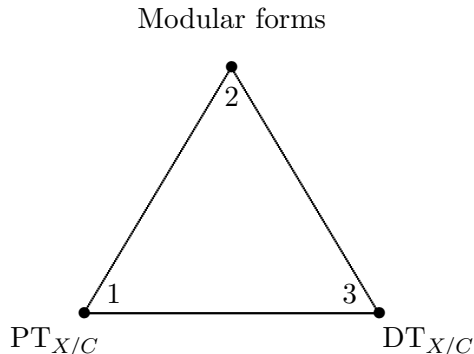
Here $[F]$ is the class of a fiber of π in X . The invariants (6) are defined by the method of Joyce-Song [JS12]. These invariants were studied by the first and second authors in [GS13] when there are no strictly semistable sheaves F satisfying (7). The modularity property of the invariants (6) holds in the situation of [GS13], and we expect such a property in general. The following result gives a complete answer on describing the relationship between stable pair invariants and DT type invariants counting two dimensional sheaves:

Theorem 3. (Theorem 4.4) *We have the following formula*

$$\begin{aligned} \text{PT}(X) = & \prod_{r \geq 0, \beta > 0, n \geq 0} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^n t^\beta \right)^{n+2r} \\ & \cdot \prod_{r > 0, \beta > 0, n > 0} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^{-n} t^\beta \right)^{n+2r}. \end{aligned}$$

The above result is proved by generalizing the argument in [Tod12b] showing a similar result when $X = S \times \mathbb{C}$ for a $K3$ surface S . Namely, we apply Joyce-Song's wall-crossing formula [JS12] in a certain space of weak stability conditions. As the arguments are the almost same as in [Tod12b] except a few modifications, we will just outline its proof in Section 4.

1.5. Summary. The main purpose of the current article is to find the connection between the vertices of the following triangle:



Here by $\text{PT}_{X/C}$ and $\text{DT}_{X/C}$ we mean the fiberwise stable pair and Donaldson-Thomas theories of the Calabi-Yau $K3$ fibration $X \rightarrow C$ discussed above. As mentioned earlier, the connection $1 \leftrightarrow 3$ is motivated by T -duality and we provide a complete answer to it in Section 4. The connection $3 \leftrightarrow 2$ is motivated by S -duality, and the first two authors have provided partial results on it in [GS13]. In Section 2 and Section 3 of this paper, we provide partial results on the connection between $1 \leftrightarrow 2$. More recently, Pandharipande-Thomas [PT14] studied the connection $1 \leftrightarrow 2$ in more generality by a different method (degeneration method) and for a different purpose. A refined (motivic) version of connection $1 \leftrightarrow 2$ is conjectured in [KKP]. We hope the combination of the established bridges $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, sheds some lights on the connection $3 \leftrightarrow 2$ in full generality, which we hope we can work out in the future.

1.6. Acknowledgment. We would like to thank Jan Manschot and Richard Thomas for helpful discussions. The first author was partially supported by NSF grant DMS-1406788. The second author would like to thank MIT and the Institute for the Physics and Mathematics of the Universe (IPMU) for hospitality and providing the opportunity to discuss about this project during his visits. Special thanks to Max Planck Institut für Mathematik for hospitality during the second author's stay in Bonn. The second and third authors are supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan. The third author is also supported by Grant-in Aid for Scientific Research grant (No. 26287002) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

1.7. Notation and convention. In this paper, all the varieties or schemes are defined over \mathbb{C} . For a morphism $\pi: X \rightarrow Y$ of schemes and coherent sheaves E, F on X , we write the sheaf $\pi_* \mathcal{H}om_X(E, F)$ as $\mathcal{H}om_\pi(E, F)$. Its right derived functor is denoted by $\mathbf{R}\mathcal{H}om_\pi(E, F)$, which coincides with $\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(E, F)$. If π is smooth, we denote by ω_π the relative canonical line bundle on X . For a scheme X , we denote by \mathbb{L}_X^\bullet the cotangent complex of X .

2. STABLE PAIRS ON SMOOTH $K3$ FIBRATIONS

2.1. Stable pairs on $K3$ fibrations. Let X be a smooth projective three-fold over \mathbb{C} . By definition, a $K3$ fibration is a morphism

$$\pi: X \rightarrow C$$

onto a smooth projective curve C whose generic fiber is a smooth $K3$ surface. If π is a smooth morphism, then it is called a *smooth $K3$ fibration*. We set

$$H_2(X, \mathbb{Z})^\pi := \{\beta \in H_2(X, \mathbb{Z}) : \pi_*\beta = 0\}.$$

We always take curve classes contained in the above subgroup of $H_2(X, \mathbb{Z})$. An element $\beta \in H_2(X, \mathbb{Z})$ is called *irreducible* if it is not written as a sum $\beta_1 + \beta_2$ for non-zero effective curve classes β_1, β_2 .

For $\beta \in H_2(X, \mathbb{Z})^\pi$, we study the moduli space of stable pairs $\mathcal{P}_n(X, \beta)$ in the sense of [PT09]. It parametrizes the pairs

$$(8) \quad s: \mathcal{O}_X \rightarrow F$$

where F is a pure 1-dimensional sheaf on X with $[F] = \beta$ and $\chi(F) = n$, and the cokernel of s is 0-dimensional. Here $[F]$ is the fundamental homology class determined by F , which is the Poincare dual of $\text{ch}_2(F)$. For simplicity we set $\mathcal{P} = \mathcal{P}_n(X, \beta)$, and denote by

$$(9) \quad \mathbb{I}^\bullet = (\mathcal{O}_{X \times \mathcal{P}} \rightarrow \mathbb{F})$$

the universal pairs, which we interpret as an object in $D^b(X \times \mathcal{P})$. Note that we have the exact triangle on $X \times \mathcal{P}$

$$(10) \quad \mathbb{I}^\bullet \rightarrow \mathcal{O}_{X \times \mathcal{P}} \rightarrow \mathbb{F} \rightarrow \mathbb{I}^\bullet[1].$$

Let

$$\pi_{\mathcal{P}}: X \times \mathcal{P} \rightarrow \mathcal{P}, \quad \pi_X: X \times \mathcal{P} \rightarrow X$$

be the natural projections. In [PT09], $\mathcal{P}_n(X, \beta)$ is equipped with a perfect obstruction theory by studying the deformations of the two term complex (8) in the derived category:

Theorem 2.1. [PT09] *There is a perfect obstruction theory over \mathcal{P} given by the following morphism in the derived category:*

$$\mathcal{E}^\bullet := \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega_{\pi_{\mathcal{P}}})_0[2] \rightarrow \mathbb{L}_{\mathcal{P}}^\bullet.$$

Here $(-)_0$ means taking the traceless part.

Since $\omega_X \cdot \beta = 0$ for $\beta \in H_2(X, \mathbb{Z})^\pi$, the Riemann-Roch calculation easily implies that the virtual dimension of \mathcal{P} is zero. The stable pair invariants are then defined by

$$P_{n, \beta} := \int_{[\mathcal{P}_n(X, \beta)]^{\text{vir}}} 1.$$

We define the generating series $\text{PT}(X)_\beta$, $\text{PT}(X)$ to be

$$(11) \quad \text{PT}(X)_\beta := \sum_{n \in \mathbb{Z}} P_{n, \beta} q^n, \quad \text{PT}(X) := \sum_{\beta \in H_2(X, \mathbb{Z})^\pi} \text{PT}(X)_\beta t^\beta.$$

2.2. Stable pairs on K3 surfaces. Let S be a smooth projective K3 surface over \mathbb{C} . For $\gamma \in H_2(S, \mathbb{Z})$, we can similarly define the moduli space of stable pairs $\mathcal{P}_n(S, \gamma)$ on S . We review several results on the moduli space $\mathcal{P}_n(S, \gamma)$. In [PT10], Pandharipande and Thomas identify $\mathcal{P}_n(S, \gamma)$ by a relative Hilbert scheme of points:

Proposition 2.2. [PT10, Proposition B.8] *The moduli space of stable pairs $\mathcal{P}_n(S, \gamma)$ on S is isomorphic to the relative Hilbert scheme $\text{Hilb}^{n+\gamma^2/2}(\mathcal{C}/\mathcal{M})$ where \mathcal{M} is the moduli space of pure one dimensional subschemes of S in class γ and \mathcal{C} denotes the universal curve over $S \times \mathcal{M}$. \square*

We next consider the case that γ is irreducible. In this case, the following result is proved in [MPT10]:

Proposition 2.3. [MPT10, Proposition 5] *If γ is irreducible, the moduli space $\mathcal{P}_n(S, \gamma)$ is non-singular of dimension $n + \gamma^2 + 1$. It depends only upon (n, γ^2) up to deformation equivalence.*

By the above proposition, we may write

$$(12) \quad \mathcal{P}_n(S, h) := \mathcal{P}_n(S, \gamma)$$

for an irreducible curve class $\gamma \in H_2(S, \mathbb{Z})$ with $\gamma^2 = 2h - 2$. We sometimes write $\mathcal{P}_n(\text{K3}, h)$ if we do not want to specify a particular K3 surface with this property. The generating series of $\chi(\mathcal{P}_n(\text{K3}, h))$ is computed by Kawai-Yoshioka's formula (4).

2.3. Stable pairs with irreducible curve classes. In this subsection, we assume that $\beta \in H_2(X, \mathbb{Z})^\pi$ is an irreducible curve class. Let $\mathcal{P} = \mathcal{P}_n(X, \beta)$ be the moduli space of stable pairs, and consider the perfect obstruction theory $\mathcal{E}^\bullet \rightarrow \mathbb{L}_{\mathcal{P}}^\bullet$ in Theorem 2.1. We have the following proposition:

Proposition 2.4. *In the above situation, we have the canonical isomorphism*

$$\mathcal{H}^1(\mathcal{E}^{\bullet\vee}) \xrightarrow{\cong} \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee.$$

Proof. We write $\omega = \omega_{\pi_{\mathcal{P}}}$ for simplicity. Applying $\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, - \otimes \omega)$ to the exact triangle (10), we obtain the following exact triangle on \mathcal{P} :

$$\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega) \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega)[1] \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \omega)[1].$$

By the above triangle and the natural morphism

$$\mathbf{R}\pi_{\mathcal{P}*}\omega \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega)$$

we can form the following commutative diagram of vertical and horizontal exact triangles:

$$(13) \quad \begin{array}{ccccc} & & \mathbf{R}\pi_{\mathcal{P}*}\omega[1] & \xrightarrow{=} & \mathbf{R}\pi_{\mathcal{P}*}\omega[1] \\ & & \downarrow & & \downarrow \\ \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega) & \rightarrow & \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega)[1] & \rightarrow & \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \omega)[1] \\ \downarrow id & & \downarrow & & \downarrow \\ \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega) & \rightarrow & \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \omega)_0[1] & \rightarrow & \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{F}, \omega)[2] \end{array}$$

After dualizing, the bottom row of the diagram gives the exact triangle

$$\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathcal{O}, \mathbb{F})[1] \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0[2] \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{F}, \mathbb{I}^{\bullet})[3].$$

The 0-th cohomology of the sequence above gives the exact sequence

$$\mathcal{E}xt_{\pi_{\mathcal{P}}}^2(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0 \xrightarrow{g} \mathcal{E}xt_{\pi_{\mathcal{P}}}^3(\mathbb{F}, \mathbb{I}^{\bullet}) \rightarrow \mathcal{E}xt_{\pi_{\mathcal{P}}}^2(\mathcal{O}, \mathbb{F})$$

Note that the first term in the sequence above is $\mathcal{H}^1(\mathcal{E}^{\bullet\vee})$. Since the fibers of $\pi_{\mathcal{P}}|_{\text{Supp}(\mathbb{F})}$ are one dimensional, we obtain $\mathcal{E}xt_{\pi_{\mathcal{P}}}^2(\mathcal{O}, \mathbb{F}) = 0$ and hence the map g is surjective. Now we show that the map g is also injective. It is enough to prove this over any closed point $p \in \mathcal{P}$. Here we will use part of the strategy of the proof of [PT09, Proposition 4.4]. Over the point p and after dualizing and taking cohomology, the diagram (13) gives the following diagram of the vertical and horizontal exact sequences

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
\text{Ext}^2(F, I^{\bullet}) & \longrightarrow & H^1(F) & \longrightarrow & \text{Ext}^2(I^{\bullet}, I^{\bullet})_0 & & \\
\downarrow = & & \downarrow & & \downarrow & \searrow g_p & \\
\text{Ext}^2(F, I^{\bullet}) & \longrightarrow & \text{Ext}^2(\mathcal{O}_X, I^{\bullet}) & \longrightarrow & \text{Ext}^2(I^{\bullet}, I^{\bullet}) & \longrightarrow & \text{Ext}^3(F, I^{\bullet}) \\
& & \downarrow & & \downarrow tr & & \\
& & H^2(\mathcal{O}_X) & \xrightarrow{=} & H^2(\mathcal{O}_X) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

where the map g_p is induced by the map g described above. As in [PT09, Proposition 4.4], in order to show that $\text{Ext}^2(I^{\bullet}, I^{\bullet})_0 \xrightarrow{g_p} \text{Ext}^3(F, I^{\bullet})$ is injective, it is enough to show that the composition

$$(14) \quad \text{Ext}^1(F, F) \rightarrow \text{Ext}^2(F, I^{\bullet}) \rightarrow H^1(F)$$

is surjective. Here the first map is induced by the exact triangle $I^{\bullet} \rightarrow \mathcal{O}_X \rightarrow F$ for the stable pair $I^{\bullet} = (\mathcal{O}_X \rightarrow F)$.

Suppose that F is supported on a curve $D \subset X$ which is irreducible by our assumption. We claim the inclusion and the isomorphism

$$(15) \quad \text{Ext}^1(F, F) \supseteq H^1(\mathcal{H}om(F, F)) \cong H^1(\mathcal{O}_D).$$

The first inclusion follows from the local to global spectral sequence. As for the second isomorphism, let us consider the trace map $\mathcal{H}om(F, F) \xrightarrow{tr} \mathcal{O}_D$. The above map is obviously surjective, and the stability of pairs implies that its kernel is at most zero dimensional. Therefore we have the second isomorphism in (15). On the other hand, using the fact that the cokernel of the map $\mathcal{O}_D \xrightarrow{s} F$ defining the stable pair is zero dimensional, we can see that $H^1(\mathcal{O}_D) \xrightarrow{s} H^1(F)$ is surjective. Hence combined with (15), we conclude that the map (14) is also surjective.

Now we have proved that the map g is an isomorphism:

$$g: \mathcal{H}^1(\mathcal{E}^{\bullet\vee}) \xrightarrow{\cong} \mathcal{E}xt_{\pi_{\mathcal{P}}}^3(\mathbb{F}, \mathbb{I}^{\bullet}).$$

The desired result follows from the above isomorphism and the relative duality $\mathcal{E}xt_{\pi_{\mathcal{P}}}^3(\mathbb{F}, \mathbb{I}^{\bullet}) \cong \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^{\bullet}, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^{\vee}$. \square

We next distinguish two kinds of components of \mathcal{P} . We use the following lemma:

Lemma 2.5. *Suppose that β is irreducible. Then for any $(\mathcal{O}_X \rightarrow F) \in \mathcal{P}_n(X, \beta)$, there exists a unique point $p \in C$ such that F is scheme theoretically supported on $\pi^{-1}(p)$.*

Proof. Since β is irreducible, the sheaf F is set theoretically supported on $\pi^{-1}(p)$ for some $p \in C$. Let $m_p \subset \mathcal{O}_{C,p}$ be the maximal ideal, and take $s \in m_p$. Then the sheaf homomorphism $\cdot s: F \rightarrow F$ must be a zero map as β is irreducible and F is pure. Hence F is an $\mathcal{O}_{\pi^{-1}(p)}$ -module. \square

The map sending $(\mathcal{O}_X \rightarrow F)$ to $p \in C$ in Lemma 2.5 defines the morphism

$$\rho: \mathcal{P} \rightarrow C.$$

Definition 2.6. *We call a connected component \mathcal{P}_c of $\mathcal{P} = \mathcal{P}_n(X, \beta)$ a type I component if $\rho(\mathcal{P}_c) = C$. Otherwise we call \mathcal{P}_c a type II component.*

Below we denote by $\mathcal{P}_I \subset \mathcal{P}$ the union of type I connected components, and $\mathcal{P}_{II} \subset \mathcal{P}$ the union of type II connected components. We set

$$i_{\mathcal{S}}: \mathcal{S} := X \times_C \mathcal{P} \hookrightarrow X \times \mathcal{P}.$$

We have the Cartesian square:

$$(16) \quad \begin{array}{ccc} \mathcal{S} & \xrightarrow{i_{\mathcal{S}}} & X \times \mathcal{P} \\ \rho_{\mathcal{S}} \downarrow & \square & \downarrow (\pi, \rho) \\ C & \longrightarrow & C \times C. \end{array}$$

Here the bottom arrow is the diagonal. The universal one dimensional sheaves \mathbb{F} in (9) is written as $i_{\mathcal{S}*}\mathbb{G}$ for a sheaf \mathbb{G} on \mathcal{S} . We denote by $\mathbb{I}_{\mathcal{S}}^{\bullet}$ the universal pairs on \mathcal{S} :

$$\mathbb{I}_{\mathcal{S}}^{\bullet} := (\mathcal{O}_{\mathcal{S}} \rightarrow \mathbb{G}).$$

Also we denote by $\pi'_{\mathcal{P}}$ and π'_X the compositions of $i_{\mathcal{S}}$ with $\pi_{\mathcal{P}}$ and π_X respectively, i.e. they are projections:

$$\pi'_{\mathcal{P}}: \mathcal{S} \rightarrow \mathcal{P}, \quad \pi'_X: \mathcal{S} \rightarrow X.$$

By [KT14, Proposition 3.4] and the standard techniques of [HT10], the type I (resp. II) components can be equipped with a relative (absolute) obstruction theory:

Theorem 2.7. (i) *There exists a ρ -relative perfect obstruction theory over \mathcal{P}_I given by*

$$\mathcal{G}_I^{\bullet} := \mathbf{R}\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}, \mathbb{I}_{\mathcal{S}}^{\bullet} \otimes \omega_{\pi'_{\mathcal{P}}})|_{\mathcal{P}_I}[2] \rightarrow \mathbb{L}_{\mathcal{P}_I/C}^{\bullet}.$$

(ii) *There exists a perfect obstruction theory over \mathcal{P}_{II} given by*

$$\mathcal{G}_{II}^{\bullet} := \mathbf{R}\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}, \mathbb{I}_{\mathcal{S}}^{\bullet} \otimes \omega_{\pi'_{\mathcal{P}}})|_{\mathcal{P}_{II}}[2] \rightarrow \mathbb{L}_{\mathcal{P}_{II}}^{\bullet}.$$

2.4. Type I component. Now we assume that $\pi: X \rightarrow C$ is a smooth $K3$ fibration and β an irreducible curve class. Here we investigate the contributions of the type I components to the virtual classes. We have the following statement:

Proposition 2.8. [PT10, Proposition C.2 and Lemma C.7] *The type I components \mathcal{P}_I is non-singular of dimension $n + 2g$, where g is the arithmetic genus of the support of a stable pair in \mathcal{P}_I . Moreover, we have the isomorphism of the tangent bundles:*

$$\mathcal{T}_{\mathcal{P}_I} \cong \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F})|_{\mathcal{P}_I}, \quad \mathcal{T}_{\mathcal{P}_I/C} \cong \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G})|_{\mathcal{P}_I}.$$

Let \mathcal{K} be the line bundle on C given by

$$\mathcal{K} := \pi_* \omega_{X/C}.$$

Proposition 2.9. *We have the exact sequence of vector bundles on \mathcal{P}_I :*

$$\begin{aligned} 0 \rightarrow \rho^* \mathcal{K}^\vee \rightarrow \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \\ \rightarrow \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \otimes \rho^* T_C \rightarrow 0. \end{aligned}$$

Proof. Since $\mathbb{F} = i_{\mathcal{S}*} \mathbb{G}$, we have the exact triangle

$$(17) \quad \mathbb{G}(-\mathcal{S}) \rightarrow \mathbf{L}i_{\mathcal{S}}^* \mathbb{I}^\bullet \rightarrow \mathbb{I}_{\mathcal{S}}^\bullet.$$

We apply $\mathbf{R}\mathcal{H}om_{\pi'_{\mathcal{P}}}(-, \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})$ to the above exact triangle. By taking the cohomologies and restricting to \mathcal{P}_I , we get the exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})|_{\mathcal{P}_I} \rightarrow \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})|_{\mathcal{P}_I} \\ \rightarrow \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}(-\mathcal{S}), \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})|_{\mathcal{P}_I}. \end{aligned}$$

By Proposition 2.8, all the sheaves in the above sequences are vector bundles. We investigate the fibers of the vector bundles in the sequence above. By the fiberwise stability of \mathbb{G} , the bundle $\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}(-\mathcal{S}), \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})|_{\mathcal{P}_I}$ is a line bundle on \mathcal{P}_I . Therefore by Proposition 2.8 and the definition of type I component, the above sequence must be a short exact sequence. After dualizing, we obtain the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{E}xt_{\pi'_{\mathcal{P}}}^2(\mathbb{G}, \mathbb{G})|_{\mathcal{P}_I} \rightarrow \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \\ \rightarrow \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \rightarrow 0. \end{aligned}$$

Here the first term is obtained by applying the Grothendieck duality and the adjunction formula $\omega_{\pi'_{\mathcal{P}}} \cong i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}} \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{S})$. Now we have

$$\mathcal{E}xt_{\pi'_{\mathcal{P}}}^2(\mathbb{G}, \mathbb{G})|_{\mathcal{P}_I} \cong R^2 \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}, \mathbb{G})|_{\mathcal{P}_I}$$

which is identified with $R^2 \pi'_{\mathcal{P}*} \mathcal{O}_{\mathcal{S}}|_{\mathcal{P}_I}$ via the trace map using the fiberwise stability of \mathbb{G} again. By the Grothendieck duality and the adjunction formula again and noting that $\omega_{\pi'_{\mathcal{P}}} = \pi_X'^* \omega_{X/C}$, we obtain

$$R^2 \pi'_{\mathcal{P}*} \mathcal{O}_{\mathcal{S}}|_{\mathcal{P}_I} \cong (R^0 \pi'_{\mathcal{P}*} \omega_{\pi'_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \cong \rho^* \mathcal{K}^\vee.$$

Finally, again using the adjunction and noting $\mathcal{O}_{\mathcal{S}}(\mathcal{S}) \cong \rho^* T_C$ by the diagram (16), we have the isomorphism

$$\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes i_{\mathcal{S}}^* \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \cong \mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \otimes \rho^* T_C.$$

Therefore we obtain the desired exact sequence. \square

Using the above proposition, we have the following statement:

Proposition 2.10. *The restriction of the virtual class of the obstruction theory \mathcal{E}^\bullet in Theorem 2.1 to the type I components \mathcal{P}_I is given by*

$$[\mathcal{P}_I, \mathcal{E}^\bullet]^{\text{vir}} = [\mathcal{P}_I] \cap (c_1(\rho^* \mathcal{K}^\vee) \cup c_{\text{top}}(\Omega_{\mathcal{P}_I/C})).$$

Proof. By Propositions 2.4 and 2.8, \mathcal{P}_I is smooth with the obstruction sheaf $\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I}$. Using the short exact sequence in Proposition 2.9, we can write

$$(18) \quad \begin{aligned} c_{\text{top}}(\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_I}) \\ = c_1(\rho^* \mathcal{K}^\vee) \cup c_{\text{top}}(\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}}})^\vee|_{\mathcal{P}_I} \otimes \rho^* T_C). \end{aligned}$$

By the following lemma and using the fact that the intersection product of any two classes on $A_{\mathbb{Q}}^1(C)$ is zero, we can see that only

$$c_1(\rho^* \mathcal{K}^\vee) \cup c_{\text{top}}(\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G})^\vee|_{\mathcal{P}_I}) = c_1(\rho^* \mathcal{K}^\vee) \cup c_{\text{top}}(\Omega_{\mathcal{P}_I/C})$$

contributes to the formula (18). Here the above identity is due to Proposition 2.8. Therefore, the desired equality follows from [BF97, Proposition 5.6]. \square

Lemma 2.11. *Suppose that $\mathcal{P}_0 \subset \mathcal{P}_I$ is a type I component of dimension l . Then, we have the following relation in $A_{\mathbb{Q}}^*(\mathcal{P}_0)$:*

$$c_{\text{top}}(\mathcal{H}om_{\pi_{\mathcal{P}_0}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}_0}})^\vee) = c_l(\mathcal{H}om_{\pi'_{\mathcal{P}_0}}(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G})^\vee) + A \cdot \rho^*(B)$$

for some $A \in A_{\mathbb{Q}}^{l-1}(\mathcal{P}_0)$ and $B \in A_{\mathbb{Q}}^1(C)$.

Proof. Since $\omega_{X/C}$ is fiberwise trivial, it must be the pullback of a line bundle M on C . So we can write $\omega_{\pi'_{\mathcal{P}_0}} = \pi_X^* \omega_{X/C} = \pi_{\mathcal{P}_0}^* \circ \rho^* M$. The Grothendieck-Riemann-Roch formula gives:

$$\begin{aligned} & \text{ch} \left(\sum_{j=0}^1 (-1)^j \mathcal{E}xt_{\pi'_{\mathcal{P}_0}}^j(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}_0}})^\vee \right) \\ &= \pi_{\mathcal{P}*}' \left(\text{ch}(\mathbb{I}_{\mathcal{S}}^\bullet) \cdot \text{ch}(\mathbb{G})^\vee \cdot \text{ch}(\omega_{\pi'_{\mathcal{P}_0}})^\vee \cdot \text{td}(X_\pi \times_\rho \mathcal{P}_0) \right) \\ &= (1 - \rho^* c_1(M)) \cdot \text{ch} \left(\sum_{j=0}^1 (-1)^j \mathcal{E}xt_{\pi'_{\mathcal{P}_0}}^j(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G})^\vee \right). \end{aligned}$$

By (C.6) in the proof of [PT10, Proposition C.2], the fibers of the sheaves

$$\mathcal{E}xt_{\pi'_{\mathcal{P}_0}}^1(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G} \otimes \omega_{\pi'_{\mathcal{P}_0}})^\vee \quad \text{and} \quad \mathcal{E}xt_{\pi'_{\mathcal{P}_0}}^1(\mathbb{I}_{\mathcal{S}}^\bullet, \mathbb{G})^\vee$$

at any closed point $I_{\mathcal{S}}^\bullet = (\mathcal{O}_S \rightarrow G) \in \mathcal{P}_0$ is naturally identified with

$$\text{Ext}^1(I_{\mathcal{S}}^\bullet, G) \cong H^2(\mathcal{O}_S)$$

for a K3 fiber $S \subset X$. Therefore, both sheaves are line bundles pulled back from the base curve C , thus their Chern characters are the pull backs of the classes from $A_{\mathbb{Q}}^*(C)$. The lemma is then proven by an inductive argument on n . \square

2.5. Type II component. Let $\mathcal{P}_c \subset \mathcal{P}_{\text{II}}$ be a type II component. Then there exists a point $p \in C$ such that $\rho(\mathcal{P}_c) = p$. Let us set $S = \pi^{-1}(p)$ and $i: S \hookrightarrow X$ is the inclusion. Then there exists unique $\gamma \in H_2(S, \mathbb{Z})$ with $i_*\gamma = \beta$ and a closed embedding

$$(19) \quad \mathcal{P}_n(S, \gamma) \subset \mathcal{P}_c.$$

Definition 2.12. A type II component \mathcal{P}_c is called isolated if the embedding (19) is an isomorphism.

Remark 2.13. Since $\mathcal{P}_n(S, \gamma)$ is non-singular by Proposition 2.3, the embedding (19) is an isomorphism if and only if (19) induces isomorphisms of tangent spaces at every points in $\mathcal{P}_n(S, \gamma)$.

We have the following lemma:

Lemma 2.14. Suppose that any component of \mathcal{P}_{II} is isolated. Then we have the isomorphism of vector bundles:

$$\Omega_{\mathcal{P}_{\text{II}}} \xrightarrow{\cong} \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}, \mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}})^\vee|_{\mathcal{P}_{\text{II}}}.$$

Proof. By the definition of type II components and Remark 2.13, the following morphism is an isomorphism

$$T_{\mathcal{P}_{\text{II}}} \cong \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}_S, \mathbb{G})|_{\mathcal{P}_{\text{II}}} \xrightarrow{\cong} \mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}, \mathbb{F})|_{\mathcal{P}_{\text{II}}}.$$

Here the first isomorphism is due to Theorem 2.7 (ii). Since ω_X restricted to any $K3$ fiber is trivial, and for any component $\mathcal{P}_c \subset \mathcal{P}_{\text{II}}$, $\mathbb{F}|_{\mathcal{P}_c \times X}$ is supported on $\mathcal{P}_c \times S$ for some $K3$ fiber $S \subset X$, we have the isomorphism

$$\mathbb{F} \otimes \omega_{\pi_{\mathcal{P}}}|_{\mathcal{P}_{\text{II}} \times X} \cong \mathbb{F}|_{\mathcal{P}_{\text{II}} \times X}.$$

Therefore we obtain the lemma. \square

By the above lemma and Proposition 2.4, the contributions to virtual classes from the isolated type II components are given as follows:

Corollary 2.15. In the situation of Lemma 2.14, the restriction of the virtual class of the obstruction theory \mathcal{E}^\bullet in Theorem 2.1 to the type II components \mathcal{P}_{II} is given by

$$[\mathcal{P}_{\text{II}}, \mathcal{E}^\bullet]^{\text{vir}} = [\mathcal{P}_{\text{II}}] \cap c_{\text{top}}(\Omega_{\mathcal{P}_{\text{II}}}).$$

2.6. Generating series of stable pair invariants. Following [MP13], let

$$(20) \quad \mathcal{V} = R^2\pi_*(\mathbb{Z}) \rightarrow C$$

be the rank 22 local system determined by the $K3$ fibration π . We denote by \mathcal{V}_c the fiber of (20) at $c \in C$. Let $\mathcal{H}^\mathcal{V}$ denote the π -relative moduli space of Hodge structures as in [MP13, Section 1.4]³, i.e. there is a map

$$\mathcal{H}^\mathcal{V} \rightarrow C$$

such that each fiber $\mathcal{H}^{\mathcal{V}_c}$ at $c \in C$ is the moduli space of weight two Hodge structures on $\mathcal{V}_c \otimes \mathbb{C} = H^2(S, \mathbb{C})$ for $S = \pi^{-1}(c)$. There exists a section map

$$(21) \quad \sigma: C \rightarrow \mathcal{H}^\mathcal{V}$$

³In [MP13, Section 1.4] this is denoted by $\mathcal{M}^\mathcal{V}$.

which is determined by the Hodge structures of the fibers of π :

$$(22) \quad \sigma(c) = [H^{2,0}(S) \subset H^2(S, \mathbb{C})] \in \mathcal{H}^{\mathcal{V}_c}, \quad S = \pi^{-1}(c).$$

For $c \in C$, an irreducible class $\beta \in H^2(X, \mathbb{Z})^\pi$ and $h \in \mathbb{Z}$, we define

$$\mathcal{V}_c(h, \beta) = \{0 \neq \gamma \in \mathcal{V}_c \mid \gamma^2 = 2h - 2, i_*\gamma = \beta\}.$$

Also let $B_c(h, \beta) \subseteq \mathcal{V}_c(h, \beta)$ be the subset of $\gamma \in \mathcal{V}_c(h, \beta)$ where γ is a $(1, 1)$ class on $S = \pi^{-1}(c)$. Then $B_c(h, \beta)$ is finite by [MP13, Proposition 1]. The subset

$$B(h, \beta) := \bigcup_{c \in C} B_c(h, \beta) \subset \mathcal{V}$$

can be decomposed into

$$B(h, \beta) = B_I(h, \beta) \coprod B_{II}(h, \beta)$$

where the first component defines a finite local system $\epsilon : B_I(h, \beta) \rightarrow C$, and the second component is an isolated set. Let \mathcal{P}_ϵ be the connected component of the stable pair moduli space $\mathcal{P} = \mathcal{P}_n(X, \beta)$ corresponding to the local system ϵ , and let

$$P_{n, \epsilon} = \int_{[\mathcal{P}_\epsilon, \mathcal{E}^\bullet]^{\text{vir}}} 1$$

be the contribution of this component to $P_{n, \beta}$. Note that \mathcal{P}_ϵ is a type I component of $\mathcal{P}_n(X, \beta)$ in the sense of Definition 2.6. By Proposition 2.10, we have

$$(23) \quad [\mathcal{P}_\epsilon, \mathcal{E}^\bullet]^{\text{vir}} = [\mathcal{P}_\epsilon] \cap \left(c_1(\rho^* \mathcal{K}^\vee) \cup c_{\text{top}}(\Omega_{\mathcal{P}_\epsilon/C}) \right).$$

On the other hand, suppose that $\alpha \in B_{II}(h, \beta)$ supported on the fiber S is a result of a *transversal intersection* (i.e. with intersection multiplicity 1) of $\sigma(C)$ with a Noether-Lefschetz divisor. Then the corresponding connected component \mathcal{P}_α of $\mathcal{P}_n(X, \beta)$ is an isolated type II component in the sense of Definition 2.6, and it is isomorphic to $\mathcal{P}_n(S, h)$ given in (12). By Corollary 2.15, we have

$$(24) \quad [\mathcal{P}_\alpha, \mathcal{E}^\bullet]^{\text{vir}} = [\mathcal{P}_\alpha] \cap c_{\text{top}}(\Omega_{\mathcal{P}_\alpha}).$$

We define

$$P_{n, \alpha} = \int_{[\mathcal{P}_\alpha, \mathcal{E}^\bullet]^{\text{vir}}} 1$$

to be the contribution to $P_{n, \beta}$ of this component. For any integer $h \in \mathbb{Z}$, the Noether-Lefschetz number

$$NL_{h, \beta}^\pi \in \mathbb{Z}$$

was defined in [MP13] by intersecting $\sigma(C)$ with the π -relative Noether-Lefschetz divisor in \mathcal{H}^\vee associated to h and β . Informally, $NL_{h, \beta}^\pi$ is the number of the fibers S of π for which there exists a $(1, 1)$ class $\gamma \in H^2(S, \mathbb{Z})$ such that

$$\gamma^2 = 2h - 2 \quad \text{and} \quad i_*\gamma = \beta.$$

The following theorem expresses the stable pair invariants of X in terms of the Euler characteristics of the moduli spaces of stable pairs on the fibers and the Noether-Lefschetz numbers:

Theorem 2.16. *For a smooth K3 fibration $\pi : X \rightarrow C$ and an irreducible class $\beta \in H_2(X, \mathbb{Z})^\pi$, we have*

$$\mathrm{PT}(X)_\beta = \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n-1} \chi(\mathcal{P}_n(\mathbf{K}3, h)) \cdot NL_{h,\beta}^\pi q^n.$$

Here, the Euler characteristics $\chi(\mathcal{P}_n(\mathbf{K}3, h))$ are determined by Kawai-Yoshioka's formula (4).

Proof. The proof follows the same ideas as the proof of [MP13, Theorem 1]. We compare the contributions of B_I and B_{II} to $P_{n,\beta}$ and the Noether-Lefschetz numbers. Suppose that ϵ is a local system giving rise to $B_I(h, \beta)$ as above. Then, we can write

$$\begin{aligned} P_{n,\epsilon} &= \int_{\mathcal{P}_\epsilon} c_{top}(\Omega_{\mathcal{P}_\epsilon/C}) \cdot c_1(\rho^* \mathcal{K}^\vee) \\ &= (-1)^{n-1} \chi(P_n(\mathbf{K}3, h)) \cdot \int_{B_I(h,\beta)} c_1(\epsilon^* \mathcal{K}^\vee). \end{aligned}$$

By virtue of (22), it is shown in [MP13, Theorem 1] that the integration

$$\int_{B_I(h,\beta)} c_1(\epsilon^* \mathcal{K}^\vee)$$

gives the contribution of $B_I(h, \beta)$ to $NL_{h,\beta}^\pi$.

Using the deformation invariance of the stable pair invariants, we may assume that any $\alpha \in B_{II}(h, \beta)$ is a result of a transversal intersection of $\sigma(C)$ with a Noether-Lefschetz divisor⁴. As a result, the contribution of α to $NL_{h,\beta}^\pi$ is exactly 1, and moreover, the corresponding component \mathcal{P}_α is an isolated type II component. Therefore by Corollary 2.15, we have

$$P_{n,\alpha} = \int_{\mathcal{P}_\alpha} c_{top}(\Omega_{\mathcal{P}_\alpha}) = (-1)^{n-1} \chi(\mathcal{P}_n(\mathbf{K}3, h)).$$

The above arguments prove the desired identity. \square

2.7. Reducible curve classes. In this section we study a special analog of Theorem 2.16 in which the class $\beta \in H_2(X, \mathbb{Z})^\pi$ is allowed to be reducible.

Let \mathcal{P}_c be an isolated type II component of $\mathcal{P}_n(X, \beta)$. This means that we have an isomorphism $\mathcal{P}_c \cong \mathcal{P}_n(S, \gamma)$ where $\beta = i_* \gamma$ and $i : S \hookrightarrow X$ is a K3 fiber of X . Here, γ is no longer needed to be irreducible, but we always assume that⁵

$$(25) \quad H^1(L) = 0 \text{ for any line bundle on } S \text{ with } c_1(L) = \gamma.$$

⁴Here as in [MP13, pg. 23], one may need to make a local holomorphic perturbation of the section σ to make it transversal to the Noether-Lefschetz divisor (i.e. with the local intersection multiplicity 1). The K3 fibration after perturbation will be over an analytic curve, but the stable pairs under consideration will remain on the K3 fibers, which are algebraic. The corresponding type II component of the moduli space of stable pairs is always algebraic and compact and lies over the interior of a small (analytic) open set in the curve that is being perturbed. So the usual deformation invariance of the intersection numbers gives the deformation invariance of the contribution of the stable pair invariants of this type II component.

⁵This is the case for example if γ is big and nef (see [Huy16, Proposition 3.1]).

As γ is not an irreducible class, the moduli space $\mathcal{P}_n(S, \gamma)$ may be singular and have several irreducible components. However we have the following lemma:

Lemma 2.17. *Suppose that γ satisfies (25), then the dimension of each irreducible component of $\mathcal{P}_n(S, \gamma)$ does not exceed $v := n + \gamma^2 + 1$.*

Proof. In the notation of Proposition 2.2, the dimension of \mathcal{M} is $\gamma^2/2 + 1$, and for a fixed curve D in class γ the dimension of the components of the Hilbert scheme are at most $n + \gamma^2/2$. These two claims prove the lemma. The first claim follows because in this case \mathcal{M} is the linear system of curves in class γ on S satisfying (25). The second claim can be verified by analyzing the fibers of the Hilbert to Chow morphism from the Hilbert scheme of k points on D

$$HC : \text{Hilb}^k(D) \rightarrow \text{Sym}^k(D).$$

We know that the dimension of the punctual Hilbert scheme of a points supported on a fixed point of S is equal to $a - 1$ (see [Bri77]). Since $D \subset S$ the dimension of the punctual Hilbert scheme of a points supported on a fixed point of D does not exceed $a - 1$. Now given a d -dimensional diagonal $\Delta_d \subset \text{Sym}^k(D)$ corresponding to the partition $k = a_1 + \dots + a_d$ and a point $p \in \Delta_d$, by what we said above, the dimension of $HC^{-1}(p)$ does not exceed $\sum_i (a_i - 1) = k - d$. From this the claim follows. \square

By [KT14, Proposition 3.4] (also see Theorem 2.7 (ii)), the component $\mathcal{P}_c = \mathcal{P}_n(S, \gamma)$ is equipped with a perfect obstruction theory

$$\mathcal{G}^\bullet := \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}'}(\mathbb{G}, \mathbb{I}_S^\bullet \otimes \omega_{\pi_{\mathcal{P}}})|_{\mathcal{P}_c}[2] \rightarrow \mathbb{L}_{\mathcal{P}_c}^\bullet.$$

Note that the obstruction sheaf $\mathcal{H}^1(\mathcal{G}^{\bullet\vee})$ of \mathcal{G}^\bullet admits a surjection to the trivial vector bundle $\mathcal{O}_{\mathcal{P}_c}$, given by the natural surjection

$$(26) \quad \text{Hom}(I_S^\bullet, G[1]) \twoheadrightarrow \text{Ext}_S^2(G, G) \twoheadrightarrow \mathbb{C}$$

at the fiber of $I_S^\bullet = (\mathcal{O}_S \rightarrow G) \in \mathcal{P}_c$. The second morphism of (26) is dual to $\mathbb{C} \cdot \text{id} \subset \text{Hom}(G, G)$. By removing the trivial factor $\mathcal{O}_{\mathcal{P}_c}$ from the obstruction theory \mathcal{G}^\bullet , Kool-Thomas [KT14] constructs a v -dimensional reduced virtual cycle

$$[\mathcal{P}_n(S, \gamma)]^{\text{red}} \in A_v(\mathcal{P}_n(S, \gamma)).$$

Here v is given in Lemma 2.17. Let \mathcal{E}^\bullet be the obstruction theory in Theorem 2.1 restricted to \mathcal{P}_c . We define the following element in the K -group

$$\mathcal{V} = -\mathcal{E}^{\bullet\vee} + \mathcal{G}^{\bullet\vee} + \mathcal{O}_{\mathcal{P}_c} \in K(\mathcal{P}_c).$$

Then \mathcal{V} is of constant rank v .

Proposition 2.18. *Suppose that γ satisfies (25) and v is as in Lemma 2.17, then we have the following identity:*

$$(27) \quad [\mathcal{P}_c, \mathcal{E}^\bullet]^{\text{vir}} = c_v(\mathcal{V}) \cap [\mathcal{P}_n(S, \gamma)]^{\text{red}}.$$

Proof. The obstruction theories \mathcal{E}^\bullet and \mathcal{G}^\bullet are related by the following two natural exact triangles:

$$(28) \quad \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{F}) \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \rightarrow \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{F}, \mathcal{O}_{X \times \mathcal{P}})[2],$$

and

$$(29) \quad \mathbf{R}\mathcal{H}om_{\pi'_P}(\mathbb{I}_S^\bullet, \mathbb{G}) \rightarrow \mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}^\bullet, \mathbb{F}) \rightarrow \mathbf{R}\mathcal{H}om_{\pi'_P}(\mathbb{G}(-S), \mathbb{G}).$$

Here the triangle (28) is obtained similarly to the bottom triangle of the diagram (13), replacing ω by $\mathcal{O}_{X \times \mathcal{P}}$. The triangle (29) follows from applying $\mathbf{R}\mathcal{H}om_{\pi'_P}(-, \mathbb{G})$ to the triangle (17). The complexes $\mathcal{E}^{\bullet \vee}$ and $\mathcal{G}^{\bullet \vee}$ can be identified in the derived category with 2-term complexes

$$\mathcal{E}_0 \rightarrow \mathcal{E}_1 \quad \text{and} \quad \mathcal{G}_0 \rightarrow \mathcal{G}_1$$

of vector bundles. Their K -group classes are then respectively $\mathcal{E}_0 - \mathcal{E}_1$ and $\mathcal{G}_0 - \mathcal{G}_1$. We define the element \mathcal{V}' of the K -group by

$$\mathcal{V}' = \mathcal{E}_1 - \mathcal{E}_0 - (\mathcal{G}_1 - \mathcal{G}_0).$$

Then we have $\mathcal{V} = \mathcal{V}' + \mathcal{O}_{\mathcal{P}_c}$. By the exact triangles (28), (29) above, the fiber of \mathcal{V}' over a closed point $(\mathcal{O}_S \rightarrow G) \in \mathcal{P}_c$ is naturally given by $\chi(G) - \chi(G, G)$.

Using these facts and noting that the summation of $\mathcal{O}_{\mathcal{P}_c}$ does not affect the total Chern class, the result of [Sie04, Theorem 4.6] implies

$$(30) \quad [\mathcal{P}_c, \mathcal{E}^\bullet]^{\text{vir}} = \{c(\mathcal{E}_1 - \mathcal{E}_0) \cap c_F(\mathcal{P}_c)\}_0 \\ = \{(c(\mathcal{V}) \cup c(\mathcal{G}_1 - \mathcal{G}_0)) \cap c_F(\mathcal{P}_n(S, \gamma))\}_0.$$

Here $\{-\}_r$ means the r -dimensional part, and c_F denotes Fulton's canonical class. Similarly, we have

$$[\mathcal{P}_n(S, \gamma)]^{\text{red}} = \{c(\mathcal{G}_1 - \mathcal{G}_0) \cap c_F(\mathcal{P}_n(S, \gamma))\}_v.$$

By the discussion in [Sie04, Section 4.1], we know that

$$(31) \quad \{c(\mathcal{G}_1 - \mathcal{G}_0) \cap c_F(\mathcal{P}_n(S, \gamma))\}_r = 0$$

for $r < v - 1$. It is also 0 for $r = v - 1$ because as mentioned earlier the obstruction theory \mathcal{G}^\bullet contains a trivial factor, hence $[\mathcal{P}_c, \mathcal{G}^\bullet]^{\text{vir}} = 0$. The vanishing (31) also holds for $r > v$ because $\dim \mathcal{P}_n(S, \gamma) \leq v$ by Lemma 2.17. Therefore by the dimension reason, we obtain the desired identity (27). \square

Corollary 2.19. *In the situation of Proposition 2.18, if γ is irreducible then*

$$[\mathcal{P}_n(S, \gamma)] = [\mathcal{P}_n(S, \gamma)]^{\text{red}}, \\ c_v(\mathcal{V}) \cap [\mathcal{P}_n(S, \gamma)]^{\text{red}} = c_v(\Omega_{\mathcal{P}_n(S, \gamma)}) \cap [\mathcal{P}_n(S, \gamma)].$$

Proof. In this case by Proposition 2.3, $\mathcal{P}_n(S, \gamma)$ is smooth of dimension v , which is the same as the virtual dimension of $[\mathcal{P}_n(S, \gamma)]^{\text{red}}$. This proves the first equality. The second equality follows by noting that both sides give $[\mathcal{P}_c, \mathcal{E}^\bullet]^{\text{vir}}$ by Proposition 2.18 and Corollary 2.15. \square

Corollary 2.20. *In the situation of Proposition 2.18, suppose that there is a smooth deformation of S to a K3 surface in which γ becomes irreducible. Then*

$$\int_{[\mathcal{P}_n(S, \gamma)]^{\text{red}}} c_v(\mathcal{V}) = (-1)^v \chi(\mathcal{P}_n(\text{K3}, h)),$$

where $h = \gamma^2/2 + 1$.

Proof. Since we are working with an isolated type II component, the virtual bundle \mathcal{V} depends on S only. To see this first note that by Verdier duality applied to the closed immersion $S \times \mathcal{P}_c \rightarrow X \times \mathcal{P}_c$ and that $\omega_S = \mathcal{O}_S \cong \omega_X|_S$, we have

$$\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{F}, \mathcal{O}_{X \times \mathcal{P}}) \cong \mathbf{R}\mathcal{H}om_{\pi'_{\mathcal{P}}}(\mathbb{G}, \mathcal{O}_{S \times \mathcal{P}})[-1].$$

Using this and the exact triangles (28) and (29), it is clear that the K-group class $[\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0]$ only depends on the data on S , and hence the same is true for \mathcal{V} . The corollary now follows from Corollary 2.19 and deformation invariance of the reduced virtual cycle for $(1, 1)$ classes. \square

In the following special situation we can prove an analog of Theorem 2.16 for possibly reducible classes β :

Theorem 2.21. *For a smooth K3 fibration $\pi : X \rightarrow C$ suppose that the class $\beta \in H_2(X, \mathbb{Z})^\pi$ is such that $\mathcal{P}_n(X, \beta)$ consists of only isolated type II components. For any such component $\mathcal{P}_n(S, \gamma)$ as above, suppose that γ satisfies the condition (25). Then,*

$$(32) \quad P_{n,\beta} = \sum_{h=0}^{\infty} (-1)^{n-1} \chi(\mathcal{P}_n(\mathbf{K3}, h)) \cdot NL_{h,\beta}^\pi.$$

Proof. Since all the components of $\mathcal{P}_n(X, \beta)$ are isolated type II, for any such component $\mathcal{P}_n(S, \gamma)$, we necessarily have $\gamma = \gamma_1 + \dots + \gamma_r$ where γ_i 's are distinct irreducible classes⁶. In particular, γ is a primitive class and hence one can find a deformation of S as in Corollary 2.20. This together with similar argument as in the proof of Theorem 2.16 give the result. \square

Remark 2.22. *The right hand side of the formula (32) can be obtained by Kawai-Yoshioka's formula (4).*

3. STABLE PAIRS ON NODAL K3 FIBRATIONS

3.1. Nodal K3 fibrations. In this section, we aim to prove the compatibility condition for stable pair invariants via conifold transitions. Let X be a smooth projective 3-fold, and π be a K3 fibration

$$\pi : X \rightarrow C$$

onto a smooth projective curve C . A K3 fibration π is called a *nodal K3 fibration* if the singularities of the fibers of π are at worst ordinary double point (ODP) singularities.

Example 3.1. *Let $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ be a generic hyperplane section of bidegree $(4, 2)$, and π the composition*

$$\pi : X \subset \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

where the second morphism is the projection. Then π is a nodal K3 fibration.

⁶If any of γ_i 's is not reduced then one could have a stable pair whose support infinitesimally thickens outside of the supporting K3 fiber S contradicting the assumption that all the components of $\mathcal{P}_n(X, \beta)$ are isolated type II.

For any $\beta \in H_2(X, \mathbb{Z})^\pi$, the stable pair invariants $P_{n,\beta}$ are defined as before by

$$P_{n,\beta} = \int_{[\mathcal{P}_n(X,\beta)]^{\text{vir}}} 1,$$

where $[\mathcal{P}_n(X,\beta)]^{\text{vir}}$ is the 0-dimensional virtual cycle associated to the obstruction theory in Theorem 2.1. We will study the generating series $\text{PT}(X)_\beta$, $\text{PT}(X)$ given by (11).

Remark 3.2. *When the singularities of fibers of π are more general type of rational double points (RDP), then the study of stable pair invariants may be reduced to the nodal case by using the deformation invariance of stable pair invariants.*

3.2. Conifold transition. Let

$$s_1, \dots, s_k \in X, \quad c_1, \dots, c_{k'} \in C$$

are the singular points of the fibers of $\pi: X \rightarrow C$, the points in C over which the fibers have singularities, respectively. By our assumption, each $s_i \in X$ is an ordinary double point (nodal) singularity for any i in the fiber of π . If k' is even, define $c_0 = c_1$ and if k' is odd, define c_0 to be an arbitrary point of C distinct from $c_1, \dots, c_{k'}$. Let

$$\epsilon: \tilde{C} \rightarrow C$$

be the double cover of C branched over the points $c_0, c_1, \dots, c_{k'}$. It can be seen that $X_0 := \epsilon^*(X)$ is a threefold with the conifold singularities. Let $h: \tilde{X} \rightarrow X_0$ be its small resolution with the exceptional nonsingular rational curves

$$(33) \quad e_1, \dots, e_k \subset \tilde{X}, \quad h(e_i) = s_i$$

and $\tilde{\pi}: \tilde{X} \rightarrow \tilde{C}$ the induced morphism. In general, the small resolution \tilde{X} may not be a projective variety, but is realized as an algebraic space by [Art74]. As a summary, we have the commutative diagram

$$(34) \quad \begin{array}{ccccc} & & \tilde{\epsilon} & & \\ & \nearrow h & & \searrow & \\ \tilde{X} & \xrightarrow{\quad} & X_0 & \longrightarrow & X \\ & \searrow \tilde{\pi} & \downarrow & \square & \downarrow \pi \\ & & \tilde{C} & \xrightarrow{\quad \epsilon \quad} & C. \end{array}$$

The normal bundle of e_i in \tilde{X} is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ [Ati58]. Moreover, let $\epsilon_t: \tilde{C}_t \rightarrow C$ be a double cover of C branched at $k + 2\{k/2\}$ generic points⁷ of C when $t \neq 0$, and set $\tilde{C}_0 = \tilde{C}$. Define $X_t = \epsilon_t^*(X)$, so we have the Cartesian square

$$\begin{array}{ccc} X_t & \longrightarrow & X \\ \downarrow & \square & \downarrow \pi \\ \tilde{C}_t & \xrightarrow{\quad \epsilon_t \quad} & C. \end{array}$$

⁷ $\{k/2\} = 0$ if k is even, and $\{k/2\} = 1/2$ if k is odd.

Our plan is to relate stable pair invariants of \tilde{X} and X_t which differ by the conifold transitions. As in GW theory [LY06], [LR01], this can be done using degeneration techniques.

Remark 3.3. *We will not pursue the foundation of moduli theory of stable pair invariants on the algebraic space \tilde{X} . The Hilbert schemes of curves on algebraic spaces are realized as algebraic spaces [Art69, Corollary 6.2], and the similar argument may be applied to construct the moduli spaces of stable pairs on \tilde{X} as algebraic spaces. Also in the situation that any stable pair on \tilde{X} is scheme theoretically supported on a K3 fiber, then by Proposition 2.2, the moduli space of stable pairs can alternatively be constructed as a relative Hilbert scheme of points over the universal curve of the relative linear system. Hence it is an algebraic space by [Art69, Corollary 6.2].*

Remark 3.4. *If the moduli space of stable pairs on \tilde{X} is an algebraic space, then the arguments in [BF97] show the existence of the zero dimensional virtual fundamental class. Below we will also assume that Li-Wu's degeneration formula [LW15] works for stable pair invariants on algebraic spaces. Of course, we don't have to address the foundational matters in Remark 3.3 and this remark if we can choose the small resolution \tilde{X} to be projective.*

3.3. Relative stable pair invariants. In order to apply the degeneration technique, we will use the notion of relative stable pair theory given by Li-Wu [LW15].

Definition 3.5. *Let W be a smooth projective threefold and $D \subset W$ a smooth divisor. For $\beta \in H_2(W, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by $\mathcal{P}_n(W/D, \beta)$ the moduli stack of relative stable pairs $(\mathcal{O}_W \rightarrow F)$ on W satisfying $[F] = \beta$ and $\chi(F) = n$, in the sense of [LW15].*

There is an open substack of $\mathcal{P}_n(W/D, \beta)$ whose \mathbb{C} -points correspond to pairs $(\mathcal{O}_W \rightarrow F)$ such that F is supported on W and normal to D , i.e. $\text{Tor}_1^{\mathcal{O}_X}(F, \mathcal{O}_D) = 0$. A \mathbb{C} -point of the boundary corresponds to an admissible stable pair (cf. [LW15, Definition 4.8]) supported on an n -step degeneration of $(W[n], D[n])$. By the relativity of stable pairs, the restriction map defines the morphism

$$\text{ev}: \mathcal{P}_n(W/D, \beta) \rightarrow \text{Hilb}(D, |\eta|).$$

Given a cohomology weighted partition η with respect to a basis of $H^*(D, \mathbb{Q})$, we can associate a cohomology class

$$C_\eta \in H^*(\text{Hilb}(D, |\eta|), \mathbb{Q})$$

which forms a basis of $H^*(\text{Hilb}(D, |\eta|), \mathbb{Q})$ called *Nakajima basis* (cf. [Nak99]). If we choose a basis of $H^*(D, \mathbb{Q})$ which is self-dual with respect to the Poincaré pairing, then for each cohomology weighted partition η , there is a dual partition η^\vee such that

$$\int_{\text{Hilb}(D, |\eta|)} C_\eta \cup C_\nu = \frac{(-1)^{|\eta| - l(\eta)}}{a(\eta)} \delta_{\nu, \eta^\vee}$$

for any cohomology weighted partition ν with $|\nu| = |\eta|$. Here $l(\eta)$ is the length of the partition η and $a(\eta)$ is defined by

$$a(\eta) = \prod_i \eta_i |\text{Aut}(\eta)|.$$

Definition 3.6. *The relative stable pair invariant (without insertions) is defined by*

$$P_{n,\beta}(W/D)_\eta := \int_{[\mathcal{P}_n(W/D,\beta)]^{\text{vir}}} \text{ev}^* C_\eta.$$

The virtual dimension of $P_n(W/D,\beta)$ is given by $c_1(W) \cdot \beta$. Therefore the above invariant is zero unless

$$c_1(W) \cdot \beta = \deg C_\eta \geq 0.$$

We define the following generating series

$$\begin{aligned} \text{PT}(W/D)_{\beta,\eta} &= \sum_n P_{n,\beta}(W/D)_\eta q^n \\ \text{PT}(W/D)_\eta &= \sum_\beta \text{PT}(W/D)_{\beta,\eta} t^\beta. \end{aligned}$$

We drop η or D from the notation if respectively $|\eta| = 0$ or $D = \emptyset$.

3.4. Degeneration formula. Let Y be the threefold obtained by blowing up X_0 at all the conifold points, with the exceptional divisors

$$D := D_1 \sqcup \cdots \sqcup D_k, \quad D_i = \mathbb{P}^1 \times \mathbb{P}^1.$$

Note that we have the factorization

$$f: Y \xrightarrow{g} \tilde{X} \xrightarrow{h} X_0$$

such that g is the blowing up at all the exceptional loci (33) of h . We use degenerations of the threefolds \tilde{X} and X_t to respectively

$$(35) \quad Y \bigcup_{D_1, \dots, D_k} \prod_{i=1}^k \mathbf{P}_1 \quad \text{and} \quad Y \bigcup_{D_1, \dots, D_k} \prod_{i=1}^k \mathbf{P}_2.$$

Here \mathbf{P}_1 is given by

$$\mathbf{P}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$$

and \mathbf{P}_2 is a smooth quadric hypersurface in \mathbb{P}^4 .

The first degeneration in (35) is the degeneration to the normal cone [Ful98] in which $D_i \subset Y$ is attached to the divisor at infinity $H_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2)$ in the i -th copy of \mathbf{P}_1 . The second degeneration is called the semi-stable reduction of a conifold degeneration [LY06] in which $D_i \subset Y$ is attached to a smooth hyperplane section H_2 in the i -th copy of \mathbf{P}_2 . We denote by

$$\mathfrak{X}_1 \rightarrow \mathbb{A}^1, \quad \mathfrak{X}_2 \rightarrow \mathbb{A}^1$$

the total spaces of the first and second degenerations above. Let L be an ample line bundle on X . We define line bundles L_t on X_t to be

$$L_t = \epsilon_t^*(L)$$

where we set $\epsilon_0 := \epsilon$.

Now we apply the degeneration formula of stable pair invariants with respect to the above degenerations. Suppose for simplicity that there is only one critical locus for the fibration $X \rightarrow C$, i.e. $k = 1$. By the degeneration formula, we obtain the following identities for $\beta \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}$

$$(36) \quad \text{PT}(\tilde{X})_\beta = \sum_{\eta, \beta_1 + \beta_2 = \beta} \text{PT}(Y/D_1)_{\beta_1, \eta} \frac{(-1)^{|\eta| - l(\eta)} a(\eta)}{q^{|\eta|}} \text{PT}(\mathbf{P}_1/H_1)_{\beta_2, \eta^\vee}$$

and

$$(37) \quad \text{PT}(X_t)_\beta = \sum_{\eta, \beta_1 + \beta_2 = \beta} \text{PT}(Y/D_1)_{\beta_1, \eta} \frac{(-1)^{|\eta| - l(\eta)} a(\eta)}{q^{|\eta|}} \text{PT}(\mathbf{P}_2/H_2)_{\beta_2, \eta^\vee}.$$

Here the sum $\beta_1 + \beta_2 = \beta$ is an equality in $H_2(\mathfrak{X}_1)$ and $H_2(\mathfrak{X}_2)$ respectively.

Proposition 3.7. *The degeneration formula (36) implies that*

$$\text{PT}(\tilde{X})_\beta = \sum_{\substack{\beta_1 \in H_2(X_0), \beta_2 \in H_2(\tilde{X}) \\ h^! \beta_1 + \beta_2 = \beta, h_* \beta_2 = 0}} \text{PT}(Y/D)_{f^! \beta_1} \cdot \text{PT}(\tilde{X})_{\beta_2}.$$

Proof. The degeneration formula of relative rank one DT invariants for the blow-up at $(-1, -1)$ -curves is worked out by Hu-Li [HL12], and we apply the same argument. By the agreement of the virtual dimensions, it is proved in [HL12, Theorem 4.2] that a non-zero term of the RHS of (36) satisfies

$$0 = |\eta| = \beta_1 \cdot D_1 = \beta_2 \cdot H_1.$$

This implies that β_1 is written as $h^! \beta'_1$ for some $\beta'_1 \in H_2(X_0)$, and β_2 is a multiple of the class of the curve $e \subset \mathbf{P}_1$ given by the embedding

$$\mathcal{O}_{\mathbf{P}_1} \subset \mathcal{O}_{\mathbf{P}_1} \oplus \mathcal{O}_{\mathbf{P}_1}(-1) \oplus \mathcal{O}_{\mathbf{P}_1}(-1)$$

into the first factor. The curve e is a $(-1, -1)$ -curve which does not intersect with H_1 . The contributions of the relative stable pairs on \mathbf{P}_1 with curve class $m[e]$ is identified with the stable pair invariants on \tilde{X} with curve class $m[e_1]$ ⁸. Hence we obtain the desired result for $k = 1$. The case of $k > 1$ is similarly discussed. \square

Proposition 3.8. *For any $d \in \mathbb{Z}_{>0}$, the degeneration formula (37) implies that*

$$\sum_{\substack{\beta \in H_2(X_t) \\ L_t \cdot \beta = d}} \text{PT}(X_t)_\beta = \sum_{\substack{\beta' \in H_2(X_0) \\ L_0 \cdot \beta' = d}} \text{PT}(Y/D)_{f^! \beta'}.$$

Proof. By the agreement of the virtual dimensions, a non-zero term of the RHS of (37) satisfies

$$c_1(Y) \cdot \beta_1 = -D_1 \cdot \beta_1 = \deg C_\eta \geq 0.$$

⁸This follows by a parallel argument as (4.4) in the proof of [HL12, Theorem 4.2].

We also have the following compatibility condition

$$D_1 \cdot \beta_1 = H_2 \cdot \beta_2 = |\eta| \geq 0.$$

The above inequalities imply that $D_1 \cdot \beta_1 = |\eta| = 0$ and $H_2 \cdot \beta_2 = 0$. The first equality implies that β_1 is written as $f^! \beta'$ for some $\beta' \in H_2(X_0)$. The second equality implies that $\beta_2 = 0$, since H_2 is an ample divisor in \mathbf{P}_2 . By rearranging the sum, we obtain the desired formula for $k = 1$. The case of $k > 1$ is similarly discussed. \square

3.5. Relation between $\text{PT}(X)$ and $\text{PT}(\tilde{X})$. Now we choose k' generic fibers $S_1, \dots, S_{k'}$ of $X \rightarrow C$. By our assumption S_i is a $K3$ surface. Let $X_i = S_i \times \mathbb{P}^1$. Then X_i is a smooth $K3$ -fibration over \mathbb{P}^1 . We identify the surface S_i with the divisor $S_i \times \{0\}$ in X_i .

Lemma 3.9. *For any curve class β contained in fibers of $X_i \rightarrow \mathbb{P}^1$, we have $P_{n,\beta}(X_i/S_i) = 0$.*

Proof. By (21) in [MPT10] the cup product map, $\cup \beta : H^1(T_{S_i}) \rightarrow H^2(\mathcal{O}_{S_i})$ is surjective, and hence by the proof of [KT14, Theorem 2.7] the obstruction theory of $\mathcal{P}_n(X_i/S_i, \beta)$ contains a trivial factor which implies the vanishing of the invariants⁹. \square

We set S to be the disjoint union of S_i for $1 \leq i \leq k'$.

Lemma 3.10. $P_{n,\beta}(X/S) = P_{n,\beta}(X)$.

Proof. Use the degeneration formula for the degeneration of X into

$$X \bigcup_{S_1, \dots, S_r} \coprod_i X_i.$$

The vanishing of $P_{n,\beta}(X_i/S_i)$ from Lemma 3.9 proves the claim. \square

We use Lemma 3.10 to relate the PT invariants of X to X_t . To achieve this we use the degeneration of X_t obtained by degenerating its base \tilde{C} to two copies of C by attaching two copies of X along the generic fibers $S_1, \dots, S_{k'}$. The degeneration formula then implies that

$$(38) \quad \sum_{L_t \cdot \beta = d} \text{PT}(X_t)_\beta = \sum_{d_1 + d_2 = d} \sum_{L \cdot \beta_1 = d_1} \text{PT}(X/S)_{\beta_1} \sum_{L \cdot \beta_2 = d_2} \text{PT}(X/S)_{\beta_2}$$

for each $d \in \mathbb{Z}_{>0}$. Let $\text{PT}^h(\tilde{X})$ be the generating series defined by

$$(39) \quad \begin{aligned} \text{PT}^h(\tilde{X}) &:= \sum_{h_* \beta = 0} P_{n,\beta}(\tilde{X}) q^n t^\beta \\ &= \prod_{1 \leq i \leq k, n \geq 1} (1 - (-q)^n t^{[e_i]})^n. \end{aligned}$$

Here the second identity follows from the computation of stable pair invariants on a $(-1, -1)$ -curve [NN11]. Let $\tilde{\epsilon}: \tilde{X} \rightarrow X$ be the natural morphism given in the diagram (34). Combined with the results in the previous subsection, we obtain the following result:

⁹The proof of [KT14, Theorem 2.7] is given for absolute geometries but a parallel argument applies to the relative geometry here.

Theorem 3.11. *We have the formula*

$$(40) \quad \tilde{\epsilon}_* \frac{\mathrm{PT}(\tilde{X})}{\mathrm{PT}^h(\tilde{X})} = \mathrm{PT}(X)^2.$$

Here $\tilde{\epsilon}_*$ is the variable change $t^\beta \mapsto t^{\tilde{\epsilon}_*\beta}$.

Proof. For each $d \in \mathbb{Z}_{>0}$, Proposition 3.8, Lemma 3.10 and (38) imply

$$\sum_{\substack{\beta \in H_2(X_0) \\ L \cdot \epsilon_* \beta = d}} \mathrm{PT}(Y/D)_{f^! \beta} = \sum_{\substack{\beta_1, \beta_2 \in H_2(X) \\ L \cdot \beta_1 + L \cdot \beta_2 = d}} \mathrm{PT}(X)_{\beta_1} \cdot \mathrm{PT}(X)_{\beta_2}.$$

Since the above formula holds for any L , we have

$$(41) \quad \tilde{\epsilon}_* \sum_{\beta \in H_2(X_0)} \mathrm{PT}(Y/D)_{f^! \beta} t^{h^! \beta} = \mathrm{PT}(X)^2.$$

On the other hand, the LHS of (41) coincides with the LHS of (40) by Proposition 3.7. Therefore we obtain the result. \square

Remark 3.12. *The variable change $t^\beta \mapsto t^{\tilde{\epsilon}_*\beta}$ only makes sense after taking the quotient series in the LHS of (40). Otherwise there are infinite numbers of contributions to the coefficients.*

Remark 3.13. *The LHS of (40) is known to be independent of a choice of a small resolution [HL12], [Tod13b], [Cal]. The formula (40) is consistent with this fact.*

In an irreducible curve class case, we have the following corollary:

Corollary 3.14. *For an irreducible curve class $\beta \in H_2(X, \mathbb{Z})^\pi$, we have the following formula:*

$$(42) \quad \sum_{\tilde{\epsilon}_* \tilde{\beta} = \beta} \tilde{\epsilon}_* \frac{\sum_{h_* e = 0} \mathrm{PT}(\tilde{X})_{\tilde{\beta} + e}}{\mathrm{PT}^h(\tilde{X})} = 2 \mathrm{PT}(X)_\beta.$$

Here $\tilde{\beta} \in H_2(\tilde{X}, \mathbb{Z})$ are irreducible curve classes, giving lifts of β .

3.6. Conjectural relation to perverse (non-commutative) stable pair theory. In this subsection, we introduce perverse (or non-commutative) stable pair theory, and propose a conjectural relationship between the quotient series $\mathrm{PT}(\tilde{X})/\mathrm{PT}^h(\tilde{X})$ in Theorem 3.11 and the generating series of the perverse (non-commutative) stable pair invariants. We first recall the heart of the perverse t-structure on $D^b \mathrm{Coh}(\tilde{X})$ associated to the small resolution $h: \tilde{X} \rightarrow X_0$, introduced by Bridgeland [Bri02]. Let $\mathcal{C} \subset D^b \mathrm{Coh}(\tilde{X})$ be the full subcategory defined by

$$\mathcal{C} = \{E \in D^b \mathrm{Coh}(\tilde{X}) : \mathbf{R}h_* E = 0\}.$$

By [Bri02], the standard t-structure on $D^b \mathrm{Coh}(\tilde{X})$ induces a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on \mathcal{C} . We define $\mathrm{Per}(\tilde{X}/X_0)$ to be

$$\mathrm{Per}(\tilde{X}/X_0) = \left\{ E \in D^b \mathrm{Coh}(\tilde{X}) : \begin{array}{l} \mathbf{R}h_* E \in \mathrm{Coh}(X_0), \\ \mathrm{Hom}(\mathcal{C}^{<0}, E) = \mathrm{Hom}(E, \mathcal{C}^{>0}) = 0 \end{array} \right\}.$$

The category $\text{Per}(\tilde{X}/X_0)$ is the heart of a bounded t-structure on $D^b \text{Coh}(\tilde{X})$, hence it is an abelian category. Note that $\mathcal{O}_{\tilde{X}} \in \text{Per}(\tilde{X}/X_0)$ by definition. Let $\text{Per}_{\leq 1}(\tilde{X}/X_0)$, $\text{Per}_0(\tilde{X}/X_0)$ be the subcategories of $\text{Per}(\tilde{X}/X_0)$, defined by

$$\begin{aligned}\text{Per}_{\leq 1}(\tilde{X}/X_0) &= \{E \in \text{Per}(\tilde{X}/X_0) : \dim h(\text{Supp}(E)) \leq 1\}, \\ \text{Per}_0(\tilde{X}/X_0) &= \{E \in \text{Per}(\tilde{X}/X_0) : \dim h(\text{Supp}(E)) = 0\}.\end{aligned}$$

It is easy to see that $\text{Per}_{\leq 1}(\tilde{X}/X_0)$ and $\text{Per}_0(\tilde{X}/X_0)$ are closed under quotients and subobjects in $\text{Per}(\tilde{X}/X_0)$. In particular, they are abelian subcategories of $\text{Per}(\tilde{X}/X_0)$.

Definition 3.15. A perverse stable pair for $h: \tilde{X} \rightarrow X_0$ consists of data (F, s)

$$(43) \quad F \in \text{Per}_{\leq 1}(\tilde{X}/X_0), \quad s: \mathcal{O}_{\tilde{X}} \rightarrow F$$

satisfying the following conditions:

- $\text{Hom}(\text{Per}_0(\tilde{X}/X_0), F) = 0$.
- The cokernel of s in $\text{Per}(\tilde{X}/X_0)$ is an object in $\text{Per}_0(\tilde{X}/X_0)$.

The above definition coincides with the usual definition of stable pairs if $h: \tilde{X} \rightarrow X_0$ is an isomorphism (i.e. there is no nodal fiber for $\pi: X \rightarrow C$). Also the above perverse stable pairs are related to the non-commutative version of stable pairs as follows. Suppose that \tilde{X} were a projective variety. Then, Van den Bergh [dB04] shows that there is a vector bundle \mathcal{E}_0 on \tilde{X} such that $\mathcal{E} = \mathcal{O}_{\tilde{X}} \oplus \mathcal{E}_0$ gives a derived equivalence

$$\Phi = \mathbf{R}h_* \mathbf{R}\mathcal{H}om(\mathcal{E}, *): D^b \text{Coh}(\tilde{X}) \xrightarrow{\sim} D^b \text{Coh}(\mathcal{A}_{X_0})$$

where $\mathcal{A}_{X_0} = h_* \mathcal{E}nd(\mathcal{E})$ is the sheaf of non-commutative algebras on X_0 . The equivalence Φ restricts to an equivalence between $\text{Per}(\tilde{X}/X_0)$ and $\text{Coh}(\mathcal{A}_{X_0})$. Let $\mathcal{U} = \Phi(\mathcal{O}_{\tilde{X}}) \in \text{Coh}(\mathcal{A}_{X_0})$ be the local projective generator of \mathcal{A}_{X_0} . Let

$$\text{Coh}_{\leq 1}(\mathcal{A}_{X_0}), \quad \text{Coh}_0(\mathcal{A}_{X_0})$$

be the subcategories of $\text{Coh}(\mathcal{A}_{X_0})$, consisting of $E \in \text{Coh}(\mathcal{A}_{X_0})$ whose support as \mathcal{O}_{X_0} -module has dimension ≤ 1 , 0 respectively. It is easy to see that

$$\begin{aligned}\Phi(\text{Per}_{\leq 1}(\tilde{X}/X_0)) &= \text{Coh}_{\leq 1}(\mathcal{A}_{X_0}), \\ \Phi(\text{Per}_0(\tilde{X}/X_0)) &= \text{Coh}_0(\mathcal{A}_{X_0}).\end{aligned}$$

Under the equivalence Φ , the data (43) is equivalent to the data (F', s')

$$(44) \quad F' \in \text{Coh}_{\leq 1}(\mathcal{A}_{X_0}), \quad s': \mathcal{U} \rightarrow F'$$

such that $\text{Hom}(\text{Coh}_0(\mathcal{A}_{X_0}), F') = 0$ and the cokernel of s in $\text{Coh}(\mathcal{A}_{X_0})$ is an object in $\text{Coh}_0(\mathcal{A}_{X_0})$.

The pair (44) is a non-commutative analogue of stable pairs. However unfortunately, our 3-fold \tilde{X} may not be projective, and the sheaf of non-commutative algebras \mathcal{A}_{X_0} only exists at formal neighborhoods at each point

in X_0 . Because of the absence of the global sheaf of non-commutative algebras \mathcal{A}_{X_0} , we formulate our non-commutative stable pairs as a morphism in the perverse heart as in (43).

For $\beta \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}$ and $n \in \mathbb{Z}$, let

$$(45) \quad \mathcal{P}_n^{\text{per}}(\tilde{X}, \beta)$$

be the moduli space of perverse stable pairs $(\mathcal{O}_{\tilde{X}} \rightarrow F)$ with $[F] = \beta$ and $\chi(F) = n$. We expect that (see Remark 3.16) the moduli space (45) exists as a proper algebraic space, with a perfect obstruction theory with virtual dimension zero given by

$$(46) \quad \mathcal{E}_{\text{per}}^{\bullet} := \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{P}}}(\mathbb{I}_{\text{per}}^{\bullet}, \mathbb{I}_{\text{per}}^{\bullet} \otimes \omega_{\pi})_0[2] \rightarrow \mathbb{L}_{\mathcal{P}_n^{\text{per}}(\tilde{X}, \beta)}^{\bullet}.$$

Here $\mathbb{I}_{\text{per}}^{\bullet}$ is the total complex associated to the universal morphism

$$s: \mathcal{O}_{\tilde{X} \times \mathcal{P}_n^{\text{per}}(\tilde{X}, \beta)} \rightarrow \mathbb{F}_{\text{per}}.$$

Remark 3.16. *Here we assumed that (45) is a proper algebraic space with perfect obstruction theory (46). We have not checked all the details but we believe that they should all follow from the similar arguments in the existing literatures. First the argument in [Tod13b, Section 6.4, Step 2] should show that (45) is an algebraic space of finite type. The properness should follow from the arguments in [Lo13]. Also the existence of the perfect obstruction theory (46) should follow from the argument in [HT10] by noting that the complexes associated to perverse stable pairs in Definition 3.15 are simple objects of the derived category as they are stable with respect to some weak stability condition (see Remark 3.18).*

By taking the integration, we obtain the invariant

$$P_{n, \beta}^{\text{per}} = \int_{[\mathcal{P}_n^{\text{per}}(\tilde{X}, \beta)]^{\text{vir}}} 1.$$

We define the generating series $\text{PT}^{\text{per}}(\tilde{X})_{\beta}$ and $\text{PT}^{\text{per}}(\tilde{X})$ to be

$$\text{PT}^{\text{per}}(\tilde{X})_{\beta} = \sum_{n \in \mathbb{Z}} P_{n, \beta}^{\text{per}} q^n, \quad \text{PT}^{\text{per}}(\tilde{X}) = \sum_{\beta \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}}} \text{PT}(\tilde{X})_{\beta} t^{\beta}.$$

Conjecture 3.17. *We have the equality*

$$(47) \quad \frac{\text{PT}(\tilde{X})}{\text{PT}^h(\tilde{X})} = \text{PT}^{\text{per}}(\tilde{X}).$$

Remark 3.18. *The conjectural equality (47) is motivated by the third author's work [Tod13b]. Indeed the equality (47) holds true for the Euler characteristic version, i.e. replace all the invariants by the topological Euler characteristic of the moduli spaces. This follows immediately from the proof of [Tod13b, Theorem 5.8], and the fact that $\sigma_{\xi(0)}$ -semistable objects in the notation of [Tod13b, Theorem 5.8] consist of total complexes associated to perverse stable pairs in Definition 3.15. (The latter fact can be easily proved using the same argument of [Tod13b, Proposition 5.5]). Moreover, if our 3-fold \tilde{X} were projective Calabi-Yau 3-fold, then the equality (47) follows from the results of [Tod13b], [Cal]. However, since our 3-fold \tilde{X} is no longer*

Calabi-Yau, we need a new technique of wall-crossing to show the inequality (47).

Remark 3.19. *If Conjecture 3.17 is true, then Theorem 3.11 implies that*

$$(48) \quad \tilde{\epsilon}_* \mathrm{PT}^{\mathrm{per}}(\tilde{X}) = \mathrm{PT}(X)^2.$$

Hence we can reduce the computation of $\mathrm{PT}(X)$ for a nodal K3 fibration $X \rightarrow C$ to the computation of $\mathrm{PT}^{\mathrm{per}}(\tilde{X})$ for a smooth K3 fibration $\tilde{X} \rightarrow \tilde{C}$.

3.7. Computation of perverse stable pair invariants. Finally in this section, we aim at computing perverse stable pair invariants. We prove the following lemma:

Lemma 3.20. *Let $(\mathcal{O}_{\tilde{X}} \rightarrow F)$ be a perverse stable pair for $\tilde{X} \rightarrow X_0$ such that $h_*[F]$ is an irreducible curve class in X_0 . Then we have*

- (1) $\mathrm{Hom}(F, F) = \mathbb{C}$.
- (2) $\mathrm{Supp}(F) \subset S$ where $i : S \hookrightarrow \tilde{X}$ is a K3 fiber and $\mathrm{Supp}(F)$ is the scheme theoretic support of the complex F .

Proof. (1) It is enough to show that any non-zero morphism $u : F \rightarrow F$ in $\mathrm{Per}(\tilde{X}/X_0)$ is an isomorphism. Let us take a factorization in $\mathrm{Per}(\tilde{X}/X_0)$

$$F \twoheadrightarrow \mathrm{Im} u \hookrightarrow F.$$

Let $U = \tilde{X} \setminus \mathrm{Ex}(h)$. Suppose first that $u|_U = 0$. Then $\mathrm{im} u$ is a non-zero object in $\mathrm{Per}_0(\tilde{X}/X_0)$, which contradicts to Definition 3.15. Hence we have $u|_U \neq 0$. Since the object $F|_U$ is a pure one dimensional sheaf whose scheme theoretic support is irreducible, the morphism $u|_U$ is an injective morphism in $\mathrm{Coh}(U)$. This implies that, if $\mathrm{Ker}(u) \in \mathrm{Per}(\tilde{X}/X_0)$ is non-zero, then it is an object in $\mathrm{Per}_0(\tilde{X}/X_0)$. This contradicts to Definition 3.15, hence $\mathrm{Ker}(u) = 0$.

Now we have proved that u is injective in $\mathrm{Per}(\tilde{X}/X_0)$. It remains to show that $\mathrm{Cok}(u) \in \mathrm{Per}(\tilde{X}/X_0)$ vanishes. Since $\mathbf{R}h_*$ takes $\mathrm{Per}(\tilde{X}/X_0)$ to $\mathrm{Coh}(X_0)$, we have the exact sequence in $\mathrm{Coh}(X_0)$

$$0 \rightarrow \mathbf{R}h_* F \rightarrow \mathbf{R}h_* F \rightarrow \mathbf{R}h_* \mathrm{Cok}(u) \rightarrow 0.$$

The above sequence implies that $\mathbf{R}h_* \mathrm{Cok}(u) = 0$, hence $\mathrm{Cok}(u) \in \mathcal{C}$. It follows that, by [dB04, Proposition 3.5.8], the object $\mathrm{Cok}(u)$ is isomorphic to the direct sum of the sheaves $\mathcal{O}_{e_i}(-1)$. Therefore by taking the cohomology long exact sequence associated to $0 \rightarrow F \rightarrow F \rightarrow \mathrm{Cok}(u) \rightarrow 0$, and noting that $\mathcal{H}^i(F) = 0$ for $i \neq 0, -1$ (because $F \in \mathrm{Per}(\tilde{X}/X_0)$), we obtain the exact sequence of sheaves

$$0 \rightarrow \mathcal{H}^0(F) \rightarrow \mathcal{H}^0(F) \rightarrow \mathrm{Cok}(u) \rightarrow 0.$$

Therefore we have $\mathrm{Cok}(u) = 0$.

(2) By part (1), F is necessarily supported (set theoretically) on only one fiber S . Tensoring the natural short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}}(-S) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow i_* \mathcal{O}_S \rightarrow 0$ by F , and noting that $F \otimes \mathcal{O}_{\tilde{X}}(-S) \cong F$, we get the exact triangle $F \rightarrow F \rightarrow i_* \mathbf{L}i^* F$. Since $\mathbf{L}i^* F \not\cong 0$, the first map in the exact triangle cannot be an isomorphism, and hence it must be zero map by part (1) of the lemma. The exact triangle then implies that $i_* \mathbf{L}i^* F \cong F \oplus F[1]$,

and so in particular $\text{Supp}(F) = \text{Supp}(\mathbf{L}i^*F)$. Part (2) now follows from [Huy06, Lemma 3.29]. \square

For the smooth K3 fibration $\tilde{\pi} : \tilde{X} \rightarrow \tilde{C}$, let us take a curve class

$$(49) \quad \bar{\beta} = \tilde{\beta} + me_i \in H_2(\tilde{X}, \mathbb{Z})^{\tilde{\pi}},$$

for some irreducible curve class $\tilde{\beta} \in H_2(\tilde{X}, \mathbb{Z})$ and $m \in \mathbb{Z}_{\geq 0}$. Since $\tilde{\beta}$ is an irreducible class Lemma 3.20 applies for the perverse stable pairs $(\mathcal{O}_{\tilde{X}} \rightarrow F)$ such that $[F] = \bar{\beta}$. Motivated by the result of this lemma we assume the following conjecture:¹⁰

Conjecture 3.21. *Let $(\mathcal{O}_{\tilde{X}} \rightarrow F)$ be a perverse stable pair for $\tilde{X} \rightarrow X_0$ such that $[F] = \bar{\beta}$. Using the notation of Lemma 3.20 Part (2), there exists a complex G on S such that $F = i_*G$.*

Assuming Conjecture 3.21, there is a natural morphism

$$\rho : \mathcal{P}_n^{\text{per}}(\tilde{X}, \bar{\beta}) \rightarrow \tilde{C}.$$

Also similarly to Definition 2.6, we can define the type I and type II components $\mathcal{P}_I^{\text{per}}, \mathcal{P}_{II}^{\text{per}}$ of $\mathcal{P}_n^{\text{per}}(\tilde{X}, \bar{\beta})$. For a point $p \in \tilde{C}$, suppose that $S := \tilde{\pi}^{-1}(p) \xrightarrow{i} \tilde{X}$ and γ is a curve class on S with $i_*\gamma = \bar{\beta}$, then we denote the fiber $\rho^{-1}(p)$ of ρ restricted to a type I or an isolated type II component by $\mathcal{P}_n^{\text{per}}(S, h)$ where $h := \gamma^2/2 + 1$. Note that by the choice of $\bar{\beta}$ one can find a smooth deformation S' of S in which γ becomes irreducible, and hence a perverse stable pair on S' in this class is the same as a usual stable pair. From this and Proposition 3.22, one can see that $\mathcal{P}_n^{\text{per}}(S, h)$ is deformation equivalent to $\mathcal{P}_n(\text{K3}, h)$. We have the analogs of Propositions 2.4, 2.8, 2.10 and Corollary 2.15 that can be summarized as follows:

Proposition 3.22. *The type I components and isolated type II components of the moduli space of perverse stable pairs $\mathcal{P}^{\text{per}}(\tilde{X}, \bar{\beta})$ are smooth.*

Proof. We first prove the proposition for isolated type II components. The proof is along the lines of the proof of [PT10, Proposition C.2]. We show that the obstruction space vanishes for any given perverse stable pair $I_{\text{per}}^\bullet = (\mathcal{O}_{\tilde{X}} \xrightarrow{s} F)$ with $F = i_*G$ is supported on a K3 fiber $i : S \hookrightarrow \tilde{X}$. We will slightly abuse the notation and use the same symbol I_{per}^\bullet for the perverse stable pair on S and its pushforward on \tilde{X} . As in the proof of [ibid], the obstruction space is the kernel of v in the following natural exact sequence:

$$\begin{aligned} \text{Hom}(I_{\text{per}}^\bullet, G) &\rightarrow \text{Ext}^1(G, G) \xrightarrow{u} \text{Ext}^1(\mathcal{O}, G) \\ &\rightarrow \text{Ext}^1(I_{\text{per}}^\bullet, G) \xrightarrow{v} \text{Ext}^2(G, G)_0 \rightarrow 0. \end{aligned}$$

Lemma 3.20, and Serre duality implies that $\text{Ext}^2(G, G)_0 = 0$, so to prove the claim we need to show that the map u is surjective. Let $S_0 := h(S)$, $p : S_0 \rightarrow \text{Spec } \mathbb{C}$ be the structure morphism, and

$$G_0 := \mathbf{R}h_*G \in \text{Coh}(S_0), \quad D_0 := \text{Supp}(G_0).$$

¹⁰This is an analog of Lemma 2.5.

We can write $\mathbf{R}\Gamma(G) = \mathbf{R}p_*G_0$ and hence

$$\mathrm{Ext}^1(\mathcal{O}, G) \cong H^1(G_0).$$

On the other hand, applying $\mathbf{R}h_*$ to $\mathbf{R}\mathcal{H}om(G, G) \xrightarrow{\mathrm{tr}} \mathcal{O}_S \xrightarrow{s} G$, we get the chain of maps

$$\mathbf{R}h_*\mathbf{R}\mathcal{H}om(G, G) \rightarrow \mathbf{R}\mathcal{H}om(G_0, G_0) \xrightarrow{\mathrm{tr}} \mathcal{O}_{S_0} \xrightarrow{s_0} G_0.$$

Clearly, s_0 has a zero dimensional cokernel, and the trace map

$$\mathcal{H}om(G_0, G_0) \rightarrow \mathcal{O}_{D_0}$$

is surjective with at most a zero dimensional kernel, and therefore $s_0 \circ \mathrm{tr}$ induces the surjection $H^1(\mathcal{H}om(G_0, G_0)) \rightarrow H^1(G_0)$. By the local to global spectral sequence

$$H^1(\mathcal{H}^0(\mathbf{R}h_*\mathbf{R}\mathcal{H}om(G, G))) \subseteq \mathrm{Ext}^1(G, G),$$

and hence u is surjective as desired.

The proof of proposition for the type I components essentially follows the same analysis as above but this time carried out relative to the base curve C . \square

Proposition 3.23. *Suppose that any component of $\mathcal{P}_{\mathrm{II}}^{\mathrm{per}}$ is isolated. Then we have*

$$\begin{aligned} [\mathcal{P}_{\mathrm{II}}^{\mathrm{per}}, E_{\mathrm{per}}^\bullet]^{\mathrm{vir}} &= [\mathcal{P}_{\mathrm{II}}^{\mathrm{per}}] \cap c_{\mathrm{top}}(\Omega_{\mathcal{P}_{\mathrm{II}}^{\mathrm{per}}}), \\ [\mathcal{P}_{\mathrm{I}}^{\mathrm{per}}, E_{\mathrm{per}}^\bullet]^{\mathrm{vir}} &= [\mathcal{P}_{\mathrm{I}}^{\mathrm{per}}] \cap (c_1(\rho^*\tilde{\mathcal{K}}^\vee) \cup c_{\mathrm{top}}(\Omega_{\mathcal{P}_{\mathrm{I}}^{\mathrm{per}}/\tilde{C}})), \end{aligned}$$

where $\tilde{\mathcal{K}} = \tilde{\pi}_*\omega_{\tilde{X}/\tilde{C}}$.

Proof. The same as the proofs of Proposition 2.4 and Corollary 2.15 in which the stable pair I^\bullet is replaced with the perverse stable pair $I_{\mathrm{per}}^\bullet = (\mathcal{O}_{\tilde{X}} \xrightarrow{s} F)$. The surjectivity $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \rightarrow \mathrm{Ext}^1(\mathcal{O}_{\tilde{X}}, F) = H^1(F)$ which is one of the important requirements can be obtained as the proof of surjectivity of the map u in Proposition 3.22. \square

Now we provide an analog of Theorem 2.16 for perverse stable pair invariants:

Theorem 3.24. *For the smooth K3 fibration $\tilde{\pi} : \tilde{X} \rightarrow \tilde{C}$, suppose that $\bar{\beta}$ is given as in (49), and assume Conjecture 3.21. Then we have*

$$\mathrm{PT}^{\mathrm{per}}(\tilde{X})_{\bar{\beta}} = \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n-1} \chi(\mathcal{P}_n^{\mathrm{per}}(S, h)) \cdot NL_{h, \bar{\beta}}^{\tilde{\pi}} q^n,$$

where S is a K3 fiber of \tilde{X} . \square

Remark 3.25. *By the discussion before Proposition 3.22, we can see that the Euler characteristics $\chi(\mathcal{P}_n^{\mathrm{per}}(S, h)) = \chi(\mathcal{P}_n(\mathbf{K3}, h))$ in Theorem 3.24 can be read off from Kawai-Yoshioka's formula (4).*

Remark 3.26. *If the conjectural relation (48) holds, then by Theorem 3.24, it enables us to compute stable pair invariants for the nodal K3 fibration $X \rightarrow C$ with irreducible curve classes.*

4. WALL-CROSSING FORMULA FOR $K3$ FIBRATIONS

In this section, we study the stable pair invariants on $K3$ fibrations from a different approach. We first relate the stable pair invariants on a $K3$ fibration (with possibly singular fibers) to the generalized DT invariants counting torsion sheaves supported on fibers of the $K3$ fibration. This is achieved by an argument similar to [Tod12b], where a similar problem was studied for the trivial $K3$ fibration $S \times \mathbb{C} \rightarrow \mathbb{C}$ for a $K3$ surface S . The key ingredient was the wall-crossing formula of DT type invariants in the derived category [JS12], [KS]. The latter invariants are calculated in [GS13] for nodal $K3$ fibrations, and we apply its result to calculate the stable pair invariants on nodal $K3$ fibrations with irreducible curve classes. For a non-irreducible curve classes, the two approaches give different descriptions of the stable pair invariants on nodal $K3$ fibrations. This gives a non-trivial relationship between the generalized DT invariants on nodal $K3$ fibrations and the stable pair invariants on smooth $K3$ fibrations, which may have some applications to the study of the generalized DT invariants.

4.1. Stability conditions for $K3$ fibrations. Let X be a smooth projective Calabi-Yau 3-fold over \mathbb{C} , i.e.

$$K_X = 0, \quad H^1(X, \mathcal{O}_X) = 0.$$

We assume that there is a morphism

$$\pi: X \rightarrow \mathbb{P}^1$$

such that every scheme-theoretic fiber is an integral scheme. Note that a generic fiber of π is a smooth algebraic $K3$ surface. An example is given in Example 3.1. Let $\text{Coh}_\pi(X)$ be the subcategory of $E \in \text{Coh}(X)$ such that $\text{Supp}(E)$ is contained in the fibers of π . We consider its bounded derived category

$$\mathcal{D}_0 := D^b \text{Coh}_\pi(X).$$

Note that for any object $E \in \mathcal{D}_0$, we have

$$\begin{aligned} \text{ch}(E)\sqrt{\text{td}_X} &= (0, r[F], \beta, n) \\ (50) \quad &\in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X). \end{aligned}$$

Here $[F]$ is a fiber class of π , and $r, n \in \mathbb{Z}$. Under the Poincaré duality, the element β is regarded as an element in $H_2(X, \mathbb{Z})^\pi$. By setting

$$\Gamma_0 := \mathbb{Z} \oplus H_2(X, \mathbb{Z})^\pi \oplus \mathbb{Z}$$

the expression (50) gives a group homomorphism

$$(51) \quad \text{cl}_0: K(\mathcal{D}_0) \rightarrow \Gamma_0, \quad E \mapsto (r, \beta, n).$$

Let ω be an ample divisor on X . Using the map (51), we define the slope function μ_ω on $\text{Coh}_\pi(X)$ by

$$\mu_\omega(E) = \int_X \omega \cdot \beta / r \in \mathbb{Q} \cup \{\infty\}.$$

Here $\text{cl}_0(E) = (r, \beta, n)$ and $\mu_\omega(E) = \infty$ if $r = 0$, i.e. E is one or zero dimensional sheaf. The slope function μ_ω defines the μ_ω -stability on $\text{Coh}_\pi(X)$ in an obvious way. We define the following full subcategories in $\text{Coh}_\pi(X)$:

$$\begin{aligned}\mathcal{T}_\omega &:= \langle F : \mu_\omega\text{-semistable with } \mu_\omega(F) > 0 \rangle_{\text{ex}} \\ \mathcal{F}_\omega &:= \langle F : \mu_\omega\text{-semistable with } \mu_\omega(F) \leq 0 \rangle_{\text{ex}}.\end{aligned}$$

Here $\langle * \rangle_{\text{ex}}$ is the extension closure of $*$, i.e. the smallest extension closed subcategory which contains $*$. The above pair $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ determines a torsion pair (cf. [HRS96]) on $\text{Coh}_\pi(X)$. We have the associated tilting

$$\mathcal{B}_\omega := \langle \mathcal{F}_\omega, \mathcal{T}_\omega[-1] \rangle_{\text{ex}} \subset D^b \text{Coh}_\pi(X).$$

The subcategory \mathcal{B}_ω is the heart of a bounded t-structure of $D^b \text{Coh}_\pi(X)$, hence in particular it is an abelian category. Let $Z_{\omega,0}: \Gamma_0 \rightarrow \mathbb{C}$ be the group homomorphism defined by

$$\begin{aligned}(52) \quad Z_{\omega,0}(v) &:= \int_X e^{-i\omega} v \\ &= n - \frac{r}{2}(\omega|_{X_p})^2 - (\omega \cdot \beta)\sqrt{-1}.\end{aligned}$$

Here we have written $v = (r, \beta, n)$, and $X_p = \pi^{-1}(p)$ for a closed point $p \in C$. Below we assume the familiarity of Bridgeland stability conditions [Bri07] on triangulated categories.

Lemma 4.1. *The pairs*

$$\sigma_{t\omega,0} := (Z_{t\omega,0}, \mathcal{B}_\omega), \quad t > 0$$

determine Bridgeland stability conditions on \mathcal{D}_0 .

Proof. The proof is almost the same as in the $K3$ surface case in [Bri08, Proposition 7.1]. However, we need to take a little care since there may be possible singular fibers of π . For non-zero $E \in \mathcal{B}_\omega$, let us show the following property

$$(53) \quad Z_{t\omega,0}(E) \in \{r \exp(i\pi\phi) : r > 0, \phi \in (0, 1]\}.$$

By our construction of \mathcal{B}_ω , we have $\text{Im } Z_{t\omega,0}(E) \geq 0$. If $\text{Im } Z_{t\omega,0}(E) = 0$, then $\mathcal{H}^1(E)$ is zero dimensional and $\mathcal{H}^0(E)$ is a μ_ω -semistable sheaf with $\mu_\omega(\mathcal{H}^0(E)) = 0$. Therefore by (52), it is enough to show that for any μ_ω -stable sheaf $F \in \text{Coh}_\pi(X)$ with $\text{cl}_0(F) = (r, \beta, n)$ and $\omega \cdot \beta = 0$, we have $n \leq 0$. Since F is μ_ω -stable, it is an \mathcal{O}_{X_p} -module for some $p \in \mathbb{P}^1$. If X_p is smooth, then the inequality $n \leq 0$ this is a consequence of Bogomolov-Gieseker inequality. In general, since X_p is an integral scheme, the structure sheaf \mathcal{O}_{X_p} is a μ_ω -stable sheaf with $\mu_\omega(\mathcal{O}_{X_p}) = 0$. By the μ_ω -stability of F , if $r \geq 2$, we have $\text{Hom}(\mathcal{O}_{X_p}, F) = \text{Hom}(F, \mathcal{O}_{X_p}) = 0$. By the Serre duality and the Riemann-Roch theorem, we obtain

$$(54) \quad \chi(F) = r + n \leq 0.$$

If $r = 1$, we may assume that $\text{Hom}(F, \mathcal{O}_{X_p}) = 0$. Indeed if otherwise, then F is a ideal sheaf of \mathcal{O}_{X_p} , hence the inequality $n \leq 0$. Then we also have $\text{Hom}(\mathcal{O}_{X_p}, F) = 0$. Indeed if there is a non-zero morphism $\mathcal{O}_{X_p} \rightarrow F$, then it must be injective and the cokernel must be a zero dimensional sheaf. But as $\text{Ext}_{X_p}^1(\mathcal{O}_x, \mathcal{O}_{X_p}) = 0$ for any $x \in X_p$, the sheaf F must contain a zero

dimensional sheaf, which contradicts to the μ_ω -stability of F . Therefore $\text{Hom}(F, \mathcal{O}_{X_p}) = \text{Hom}(\mathcal{O}_{X_p}, F) = 0$, and the Riemann-Roch computation (54) shows $n \leq 0$.

We finally check the Harder-Narasimhan property. By following the arguments as in [Bri08, Proposition 7.1], we finally arrive at the exact sequences of sheaves for $i \geq 1$

$$0 \rightarrow Q \rightarrow \mathcal{H}^{-1}(E_i) \rightarrow \mathcal{H}^0(L_i) \rightarrow 0.$$

Here we have used the same notation of [Bri08, Proposition 7.1]. The sheaf Q is a pure two dimensional sheaf and $\mathcal{H}^0(L_i)$ is a zero dimensional sheaf. A difference from [Bri08, Proposition 7.1] is that our sheaf Q is no longer a torsion free sheaf. However by pushing forward the above sequence by the generic projection $\text{Supp}(\mathcal{Q}) \rightarrow \mathbb{P}^2$, we arrive at the same situation in [Bri08, Proposition 7.1]. Therefore the Harder-Narasimhan property also holds. \square

4.2. Weak stability conditions on \mathcal{D} . Let \mathcal{D} be the triangulated subcategory of $D^b \text{Coh}(X)$ defined by

$$\mathcal{D} := \langle \pi^* \text{Pic}(\mathbb{P}^1), \text{Coh}_\pi(X) \rangle_{\text{tr}} \subset D^b \text{Coh}(X).$$

Here $\langle * \rangle_{\text{tr}}$ is the triangulated closure, i.e. the smallest triangulated subcategory which contains $*$. We construct weak stability conditions on \mathcal{D} in the sense of [Tod10a], following the same argument of [Tod12b, Subsection 3.3]. We refer to [Tod10a, Section 2] for details on the space of weak stability conditions on triangulated categories. Let \mathcal{A}_ω be the subcategory of \mathcal{D} defined by

$$\mathcal{A}_\omega := \langle \pi^* \text{Pic}(\mathbb{P}^1), \mathcal{B}_\omega \rangle_{\text{ex}} \subset \mathcal{D}.$$

One can check that, using the same argument of [Tod12b, Proposition 2.9], the subcategory \mathcal{A}_ω is the heart of a bounded t-structure on \mathcal{D} . In particular it is an abelian category. We define the group homomorphism

$$\text{cl}: K(\mathcal{D}) \rightarrow \Gamma := \mathbb{Z} \oplus \Gamma_0$$

by $\text{cl}|_{K(\mathcal{D}_0)} = (0, \text{cl}_0)$ and

$$\text{cl}(\pi^* \mathcal{L}) = (1, \deg \mathcal{L}, 0, 0)$$

for $\mathcal{L} \in \text{Pic}(\mathbb{P}^1)$. We take a filtration Γ_\bullet of Γ by

$$0 = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 := \Gamma$$

where the second inclusion is given by $v \mapsto (0, v)$. We define the element

$$Z_\omega \in \prod_{i=0}^1 \text{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$

to be the following:

$$\begin{aligned} Z_{\omega,1}(R) &:= R\sqrt{-1}, \quad R \in \Gamma_1/\Gamma_0 = \mathbb{Z} \\ Z_{\omega,0}(v) &:= \int_X e^{-i\omega} v, \quad v \in \Gamma_0. \end{aligned}$$

Lemma 4.2. *The pairs*

$$\sigma_{t\omega} := (Z_{t\omega}, \mathcal{A}_\omega), \quad t \in \mathbb{R}_{>0}$$

determine weak stability conditions on \mathcal{D} with respect to the filtration Γ_\bullet on Γ .

Proof. Using Lemma 4.1 instead of [Tod12b, Lemma 3.3], the same argument of [Tod12b, Lemma 3.4] is applied without any major modification. \square

4.3. Donaldson-Thomas type invariants. Let M be a \mathbb{C} -scheme. In the paper [Beh09], Behrend associated a canonical constructible function ν on it, called *Behrend function*. The ν -weighted Euler characteristic

$$(55) \quad \int_M \nu \, d\chi := \sum_{m \in \mathbb{Z}} m \cdot \chi(\nu^{-1}(m))$$

is called the DT-type invariants. This is due to the fact by Behrend [Beh09] that, if M is proper and admits a symmetric perfect obstruction theory, we have the identity

$$\int_{[M]^{\text{vir}}} 1 = \int_M \nu \, d\chi.$$

On the other hand, the naive Euler characteristic $\chi(M)$ is called the Euler characteristic version of the invariant (55).

Since we assumed that X is a smooth projective Calabi-Yau 3-fold, the perfect obstruction theory in Theorem 2.1 is symmetric, hence we have

$$P_{n,\beta} = \int_{\mathcal{P}_n(X,\beta)} \nu \, d\chi.$$

We can similarly define DT type invariants on X using the Behrend function. For $(r, \beta, n) \in \Gamma_0$, let $\mathcal{M}_\omega(r, \beta, n)$ be the moduli stack of ω -Gieseker stable sheaves $E \in \text{Coh}_\pi(X)$ with $\text{cl}_0(E) = (r, \beta, n)$. If any closed point of $\mathcal{M}_\omega(r, \beta, n)$ is stable, then it is a \mathbb{C}^* -gerbe over a projective scheme, hence the following integration makes sense:

$$(56) \quad J(r, \beta, n) = \int_{\mathcal{M}_\omega(r, \beta, n)} \nu \, d\chi.$$

In general, there may be a closed point of $\mathcal{M}_\omega(r, \beta, n)$ corresponding to a strictly semistable sheaf. In such a case, the invariant (56) is defined in [JS12], [KS], as the generalized DT invariant, and takes its value in \mathbb{Q} . It is defined as the weighted Euler number of the ‘logarithm’ of $\mathcal{M}_\omega(r, \beta, n)$ in the motivic Hall algebra of $\text{Coh}_\pi(X)$. An Euler characteristic version of the invariant (56) for the trivial $K3$ fibration $S \times \mathbb{C} \rightarrow \mathbb{C}$ for a $K3$ surface S is available in [Tod12b, Definition 4.23].

Lemma 4.3. *The invariant (56) is independent of a choice of ω .*

Proof. Since the Euler pairing on \mathcal{D}_0 is trivial, the same argument of [Tod12b, Lemma 4.16] is applied. \square

4.4. Wall-crossing formula. The following is the main result in this section:

Theorem 4.4. *We have the following formula*

$$(57) \quad \text{PT}(X) = \prod_{r \geq 0, \beta > 0, n \geq 0} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^n t^\beta \right)^{n+2r} \cdot \prod_{r > 0, \beta > 0, n > 0} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^{-n} t^\beta \right)^{n+2r}.$$

Proof. The Euler characteristic version of the above result is proved when X is a trivial fibration in [Tod12b]. The proof proceeds along with the same argument of [Tod12b] without any major modification, using the derived category version of [JS12, Theorem 5.18] proved in [Tod, Theorem 2.8] instead of [Joy08, Theorem 6.28]. We only give an outline of the proof.

Step 1. *First wall-crossing.*

By the result of [Bri11], [Tod] (also see [Tod10b, Theorem 1.3], [Tod12a, Theorem 3.11]), we have the following formula

$$(58) \quad \text{PT}(X) = \prod_{\beta > 0, n > 0} \exp \left((-1)^{n-1} J(0, \beta, n) q^n t^\beta \right)^n \left(\sum_{\beta, n} L(\beta, n) q^n t^\beta \right).$$

The above formula is obtained by applying the wall-crossing formula of DT type invariants in the category of perverse coherent sheaves on $D^b \text{Coh}(X)$, which are certain two term complexes of coherent sheaves. The invariant $L(\beta, n)$ counts perverse coherent sheaves $E \in D^b \text{Coh}(X)$, which are semistable with respect to a certain self dual weak stability condition on the perverse heart. (It was denoted by $\mu_{i\omega}$ -limit stability in [Tod10b, Section 3], and $Z_{\omega, 1/2}$ -stability in [Tod12a, Section 5].) The object E satisfies the numerical condition

$$\text{ch}(E) = (1, 0, -\beta, -n) \in H^*(X, \mathbb{Q}).$$

The precise definition of the invariant $L(\beta, n)$ is available in [Tod12a, Definition 5.5], where it is denoted by $L_{n, \beta}$. When $X = S \times \mathbb{P}^1$ for a $K3$ surface S , its Euler characteristic version is available in [Tod12b, Subsection 4.6].

Step 2. *Generating series of DT type invariants.*

For $(r, \beta, n) \in \Gamma_0$, let $\mathcal{M}_{t\omega}(r, \beta, n)$ be the moduli stack of $\sigma_{t\omega}$ -semistable objects $E \in \mathcal{A}_\omega$ satisfying

$$\text{cl}(E) = (1, -r, -\beta, -n) \in \Gamma.$$

Similarly to [Tod12b, Lemma 4.13], the above moduli stack is realized as a constructible subset of an Artin stack \mathcal{M} locally of finite type. Similarly to the previous subsection and [Tod12b, Definition 4.10], it defines the (generalized) DT type invariant

$$\text{DT}_{t\omega}(r, \beta, n) := \int_{\mathcal{M}_{t\omega}(r, \beta, n)} \nu \, d\chi \in \mathbb{Q}.$$

We form the following generating series

$$\mathrm{DT}_{t\omega}(X) := \sum_{(r,\beta,n) \in \Gamma_0} \mathrm{DT}_{t\omega}(r, \beta, n) q^n t^\beta s^r.$$

We consider the behavior of the above generating series for $t \gg 0$ and $0 < t \ll 1$. We have the following proposition:

Proposition 4.5. *We have the following formulas*

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathrm{DT}_{t\omega}(X) &= \sum_{r,\beta,n} L(\beta, n) q^n t^\beta s^r \\ \lim_{t \rightarrow +0} \mathrm{DT}_{t\omega}(X) &= \lim_{t \rightarrow +0} \sum_{r,\beta} \mathrm{DT}_{t\omega}(r, \beta, 0) t^\beta s^r. \end{aligned}$$

Proof. The same proof of [Tod12b, Proposition 4.16] is applied without any major modification. \square

Step 3. *Wall-crossing in \mathcal{A}_ω .*

We investigate the difference of the generating series $\mathrm{DT}_{t\omega}(X)$ for $t = t_0 + 0$ and $t = t_0 - 0$ for a fixed $t_0 \in \mathbb{R}_{>0}$. We have the following result:

Proposition 4.6. *We have the following formula*

$$\begin{aligned} \lim_{t \rightarrow t_0+0} \mathrm{DT}_{t\omega}(X) &= \lim_{t \rightarrow t_0-0} \mathrm{DT}_{t\omega}(X) \\ &\cdot \prod_{\substack{\beta > 0, \\ n = \frac{1}{2} r t_0^2 \omega|_{X_p}^2}} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^n t^\beta s^r \right)^{\epsilon(r)(n+2r)}. \end{aligned}$$

Here $\epsilon(r) = 1$ if $r > 0$, $\epsilon(r) = -1$ if $r < 0$ and $\epsilon(r) = 0$ if $r = 0$.

Proof. The same proof of [Tod12b, Theorem 5.1] is applied by using the results of [JS12], [Tod] instead of [Joy08]. We just give an outline of the proof. For $v \in \Gamma_0$, let $\mathcal{M}_{t\omega,0}(v)$ be the moduli stack of $\sigma_{t\omega,0}$ -semistable objects $E \in \mathcal{B}_\omega$ satisfying $\mathrm{cl}_0(E) = v$. Similarly to the invariant $J(v)$, we have the generalized DT type invariant

$$J_{t\omega}(v) := \int_{\mathcal{M}_{t\omega,0}(v)} \nu \, d\chi \in \mathbb{Q}$$

which counts $\sigma_{t\omega,0}$ -semistable objects $E \in \mathcal{B}_\omega$ with $\mathrm{cl}_0(E) = v$. The wall-crossing formula [JS12] describes the difference of the two limiting series $\mathrm{DT}_{t\omega \pm 0}(X)$ in terms of the invariants $J_{t\omega}(v)$, where $v \in \Gamma_0$ satisfies $Z_{t\omega,0}(v) \in \mathbb{R}_{>0}\sqrt{-1}$. If we write $v = (r, \beta, n)$, the latter condition implies that

$$n = \frac{1}{2} r t_0^2 \omega|_{X_p}^2.$$

On the other hand, the proof of [Tod12b, Corollary 4.27] shows that the invariant $J_{t\omega}(v)$ is independent of t , and coincides with $J(v)$. Now by applying the wall-crossing formula in [JS12], and computing the relevant combinatorial coefficients as in [Tod10b], [Tod10a], we obtain the result. \square

By combining the above result with Proposition 4.5, we have the following corollary (cf. [Tod12b, Corollary 5.2]):

Corollary 4.7. *We have the following formula:*

$$\begin{aligned} & \sum_{(r,\beta,n) \in \Gamma_0} L(\beta, n) q^n t^\beta s^r \\ &= \prod_{\beta > 0, rn > 0} \exp \left((-1)^{n-1} J(r, \beta, r+n) q^n t^\beta s^r \right)^{\epsilon(r)(n+2r)} \cdot \widehat{\text{DT}}(X) \end{aligned}$$

where $\widehat{\text{DT}}(X)$ is defined to be

$$\widehat{\text{DT}}(X) := \lim_{t \rightarrow +0} \sum_{(r,\beta,n) \in \Gamma_0} \text{DT}_{t\omega}(r, \beta, 0) t^\beta s^r$$

Step 4. *Further wall-crossing.*

The final step is to decompose the series $\widehat{\text{DT}}(X)$ into a product which involves the invariants $J(v)$. Following [Tod12b, Subsection 3.6], let

$$\mathcal{A}_\omega(1/2) \subset \mathcal{A}_\omega$$

be the subcategory given as the extension closure of $Z_{t\omega}$ -semistable objects $E \in \mathcal{A}_\omega$ for $0 < t \ll 1$, satisfying

$$\lim_{t \rightarrow +0} \arg Z_{t\omega}(E) = \frac{\pi}{2}.$$

Note that any object $E \in \mathcal{A}_\omega$ which contributes to the invariant $\text{DT}_{t\omega}(r, \beta, 0)$ is an object in $\mathcal{A}_\omega(1/2)$. Similarly to [Tod12b, Subsection 3.7], we are able to construct a one parameter family of weak stability conditions

$$(\widehat{Z}_{\omega,\theta}, \mathcal{A}_\omega(1/2)), \quad \theta \in (0, 1)$$

on $\mathcal{A}_\omega(1/2)$. Moreover by [Tod12b, Proposition 3.17], an object $E \in \mathcal{A}_\omega(1/2)$ contributes to the invariant $\text{DT}_{t\omega}(r, \beta, 0)$ if and only if it is $\widehat{Z}_{\omega,1/2}$ -semistable, and an object $E \in \mathcal{A}_\omega(1/2)$ with $\text{rank}(E) = 1$ is $\widehat{Z}_{t\omega,\theta}$ -semistable for $0 < \theta \ll 1$ if and only if it is an object in $\pi^* \text{Pic}(\mathbb{P}^1)$. Applying the wall-crossing formula, we obtain

$$\widehat{\text{DT}}(X) = \prod_{r > 0, \beta > 0} \exp \left(-J(r, \beta, r) t^\beta s^r \right)^{2r} \cdot \sum_{r \in \mathbb{Z}} s^r.$$

The above formula is obtained by the same argument of [Tod12b, Proposition 5.3] without any major modification. Combined with (58), Corollary 4.7, and taking the s^0 -term, we obtain the desired identity (57). \square

4.5. Irreducible curve class case. When the curve class β is irreducible, we have the following corollary of Theorem 4.4:

Corollary 4.8. *Suppose that $\beta \in H_2(X, \mathbb{Z})^\pi$ is irreducible. Then for $n \geq 0$, we have*

$$\begin{aligned} P_{n,\beta} &= \sum_{r \geq 0} (-1)^{n-1} (n+2r) J(r, \beta, r+n) \\ (59) \quad P_{-n,\beta} &= \sum_{r > 0} (-1)^{n-1} (n+2r) J(r, \beta, r+n). \end{aligned}$$

Remark 4.9. *Note that when the data (r, β, n) is chosen so that there does not exist any strictly semistable sheaf, then the invariants $J(r, \beta, n)$ appearing on the right hand side of (59) coincide with the invariants computed in [GS13, Section 2].*

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