Abstracts

Symmetry Breaking Operators for Orthogonal Groups $O(n, 1)$

Toshiyuki Kobayashi

Given an irreducible representation π of a group G and a subgroup G', we may think of π as a representation of the subgroup G' (the *restriction* $\pi|_{G'}$). A typical example is the tensor product representation $\pi_1 \otimes \pi_2$ of two representations π_1 and π_2 of a group H, which is obtained by the restriction of the outer tensor product $\pi_1 \boxtimes \pi_2$ of the direct product group $G := H \times H$ to its subgroup $G' := \operatorname{diag}(H)$.

As *branching problems*, we wish to understand how the restriction $\pi|_{G'}$ behaves as a G'-module. For reductive groups, this is a difficult problem, partly because the restriction $\pi|_{G'}$ may not be well under control as a representation of G' even when G' is a maximal subgroup of G. Wild behavior such as infinite multiplicities may occur, for instance, already in the tensor product representation of $SL_3(\mathbb{R})$.

The author proposed in [3] to go on successively, further steps in the study of branching problems via the following three stages:

Stage A: Abstract feature of the restriction $\pi|_{G'}$

Stage B: Branching laws.

Stage C: Construction of symmetry breaking operators (Definition 0.1).

Here, *branching laws* in Stage B ask an explicit decomposition of the restriction into irreducible representations of the subgroup G' when π is a unitary representation, and also ask the *multiplicity* $m(\pi, \tau) := \dim \operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ for irreducible representations τ of G'. The latter makes sense even when π and τ are nonunitary. Stage C refines Stage B, by asking an explicit construction of SBOs when π and τ are realized geometrically.

Definition 0.1. An element in $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ is called a *symmetry breaking operator*, SBO for short.

Stage A includes a basic question whether spectrum is discrete or not, see [1]. Another fundamental question in Stage A is an estimate of multiplicities. In [2, 6], we discovered the following geometric creteria to control multiplicities:

Theorem 0.2 (geometric criteria for finite/bounded multiplicities)**.**

(1) *The dimension of* $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ *is finite for any irreducible representations* π *of* G *and any* τ *of* G' *iff* $G \times G'/\operatorname{diag}(G')$ *is real spherical.*

(2) *The dimension of* $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ *is uniformly bounded with respect to* π *and* τ *iff* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag}(G'_{\mathbb{C}})$ *is spherical.*

Here we recall

Definition 0.3. (1) A complex manifold $X_{\mathbb{C}}$ with holomorphic action of a complex reductive group $G_{\mathbb{C}}$ is *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

(2) A real manifold X with continuous action of a real reductive group G is *real spherical* if a minimal parabolic subgroup of G has an open orbit in X.

The latter terminology was introduced in [2] in search for a broader framework for global analysis on homogeneous spaces than the usual (*e.g.* group manifolds, symmetric spaces). That is, the function space $C^\infty(G/H)$ (or $L^2(G/H)$ *etc.*) should be under control by representation theory if

$$\dim \operatorname{Hom}_G(\pi, C^\infty(G/H)) < \infty \quad \text{for all } \pi \in \widehat{G}_{\mathrm{adm}}, \tag{1}$$

and hence we could expect to develop global analysis on G/H by using representation theory if (1) holds[2]. We discovered and proved that the geometric property "real spherical" characterizes exactly the representation-theoretic property (1):

Fact 0.4 ([2, 6])**.** *Let $X = G/H$ where $G \supset H$ are algebraic real reductive groups.*

(1) $\dim \operatorname{Hom}_G(\pi, C^\infty(X)) < \infty$ *($^\forall \pi \in \widehat{G}_{\mathrm{adm}}$) iff X is real spherical.*
(2) $\dim \operatorname{Hom}_G(\pi, C^\infty(X))$ *is uniformly bounded iff $X_{\mathbb{C}}$ is spherical.*

Theorem 0.2 follows from Fact 0.4.

The classification of the real spherical spaces of the form $(G \times G')/\mathrm{diag}(G')$ was accomplished in [5] when (G, G') is a reductive symmetric pair. This *a priori* estimate in Stage A singles out the settings which would be potentially promising for Stages B and C of branching problems. One of such settings arises from a different discipline, namely, from conformal geometry. The first complete solution to Stage C obtained [7] is related to this geometric setting as below.

Given a Riemannian manifold (X, g), we write $G = \mathrm{Conf}(X, g)$ for the group of conformal diffeomorphisms of X. Then there is a natural family of representations π_λ of G on $C^\infty(X)$ for $\lambda \in \mathbb{C}$ given by

$$(\pi_\lambda(h)f)(x) = \Omega(h^{-1}, x)^\lambda f(h^{-1} \cdot x) \quad \text{for } h \in G, x \in X.$$

We can extend this to a family of representations on the space $\mathcal{E}^i(X)$ of differential i-forms, to be denoted by $\pi_\lambda^{(i)}$.

If Y is a submanifold of X, then there is a natural morphism

$$G' := \{h \in G : h \cdot Y \subset Y\} \to \mathrm{Conf}(Y, g|_Y).$$

Then we may compare two families of representations of the group G':

- the restriction $\pi_\lambda^{(i)}|_{G'}$ acting on $\mathcal{E}^{(i)}(X)$,
- the representation ${\pi'}_\nu^{(j)}$ acting on $\mathcal{E}^{(j)}(Y)$.

A conformally covariant SBO on differential forms is a linear map $\mathcal{E}^i(X) \to \mathcal{E}^j(Y)$ that intertwines $\pi_\lambda^{(i)}|_{G'}$ and ${\pi'}_\nu^{(j)}$. Here is a basic question arising from conformal geometry:

Question 0.5. *Let X be a Riemannian manifold X, and Y a hypersurface. Construct and classify conformally covariant SBOs from $\mathcal{E}^i(X)$ to $\mathcal{E}^j(Y)$.*

We are interested in "natural operators" D that persist for all pairs (X, Y). The larger $\mathrm{Conf}(X; Y)$ is, the more constrains are on D, and hence, we first focus on the model space with largest symmetries which is given by $(X, Y) = (S^n, S^{n-1})$. In this case the pair (G, G') of conformal groups is locally isomorphic to $(O(n+1, 1), O(n, 1))$. It then turns out that the criterion in Theorem 0.2 (2) for Stage A is fulfilled. Then Question 0.5 is regarded as Stages B and C of branching problems. Recently, we have solved completely Question 0.5 in the model space:

- Continuous SBOs for $i = j = 0$ were constructed and classified in [7].
- Differential SBOs for general i and j were constructed and classified in [4].
- The final classification is announced in [8].

Here is a flavor of the complete classification:

Theorem 0.6. *If* $\mathrm{Hom}_{G'}(\pi_\lambda^{(i)}|_{G'}, {\pi'}_\nu^{(j)}) \neq \{0\}$ *for some* λ*,* $\nu \in \mathbb{C}$*, then* $j \in \{i-2, i-1, i, i+1\}$ *or* $i + j \in \{n-2, n-1, n, n+1\}$*.*

In the talk, I gave briefly the methods of the complete solution [4, 7, 8], some of which are also applicable in a more general setting that Theorem 0.2 (an *a priori* estimate for Stage A) suggests.

Finally, some applications of these results include:

• an evidence of a conjecture of Gross and Prasad for $O(n, 1)$, see [8];

• periods of irreducible unitary representations with nonzero cohomologies;

• a construction of discrete spectrum of the branching laws of complementary series [7, Chap. 15].

References

[1] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math. **117** (1994), 181–205; Part II, Ann. of Math. (2) **147** (1998), 709–729; Part III, Invent. Math. **131** (1998), 229–256.

[2] T. Kobayashi, Introduction to harmonic analysis on real spherical homogeneous spaces, Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano (F. Sato, ed.), 1995, 22–41.

[3] T. Kobayashi, A program for branching problems in the representation theory of real reductive groups, In: Representations of Lie Groups, In Honor of David A. Vogan on his 60th Birthday, Progress in Mathematics **312**, pp. 277–322. Birkhäuser, 2015.

[4] T. Kobayashi, T. Kubo, and M. Pevzner, Conformal Symmetry Breaking Differential Operators for Differential Forms on Spheres, Lecture Notes in Mathematics, vol. **2170**, iv + 192 pages, 2016. ISBN: 978-981-10-2657-7.

[5] T. Kobayashi, T. Matsuki, Classification of finite-multiplicity symmetric pairs, Transform. Groups **19** (2014), 457–493, Special issue in honor of Dynkin for his 90th birthday.

[6] T. Kobayashi and T. Oshima, Finite multiplicity theorems for induction and restriction, Adv. Math. **248** (2013), 921–944.

[7] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, Mem. Amer. Math. Soc., vol. **238**, No. 1126, (2015), v+112 pages, ISBN: 978-1-4704-1922-6.

[8] T.Kobayashi, B.Speh, Symmetry breaking for orthogonal groups and a conjecture by B. Gross and D. Prasad, submitted, arXiv:1702.00263.