Fourier-Sato transform on hyperplane arrangements

Michael Finkelberg, Mikhail Kapranov, Vadim Schechtman

December 20, 2017

Contents

0 Introduction

A. Setup and goals, The theory of perverse sheaves can be said to provide an interpolation between homology and cohomology (or to mix them in a self-dual way). Since homology, sheaf-theoretically, can be understood as cohomology with compact support, interesting operations on perverse sheaves usually combine the functors of the types $f_!$ and f_* or, dually, the functors of the types $f^!$ and f^* in the classical formalism of Grothendieck.

An important context when this point of view can be pushed quite far, is that of perverse sheaves $\mathcal{F}$ on a complex affine space $\mathbb{C}^n$ smooth with respect to the stratification given by an arrangement $\mathcal{H}$ of hyperplanes with real equations [KS1]. Denoting by $i_\mathbb{R} : \mathbb{R}^n \hookrightarrow \mathbb{C}^n$ the embedding, we associate to such an $\mathcal{F}$ its *hyperbolic stalks*

$$E_A(\mathcal{F}) \;=\; R\Gamma(A, i_A^* i_\mathbb{R}^! \mathcal{F}).$$

Here $i_A : A \hookrightarrow \mathbb{R}^n$ is the embedding of a face (stratum) of the real arrangement. It is remarkable that the $E_A(\mathcal{F})$ reduce to single vector spaces, not complexes (while the ordinary stalks of $\mathcal{F}$ are of course complexes, $\mathcal{F}$ being a complex of sheaves). This type of phenomena was originally observed by T. Braden in the context of varieties with a $\mathbb{C}^*$-action [Br].

It was shown in [KS1] that the vector spaces $E_A(\mathcal{F})$ together with natural linear maps γ_{AB}, δ_{BA} ("generalization and specialization") connecting them, determine the perverse sheaf $\mathcal{F}$ uniquely. Moreover, the category $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ of perverse sheaves of the above type is equivalent to the category $\mathrm{Hyp}(\mathcal{H})$ formed by linear algebra data $(E_A, \gamma_{AB}, \delta_{BA})$ satisfying an explicit set of conditions. We call such linear algebra data *hyperbolic sheaves*, see §1D.

The goal of this paper is to develop the beginnings of a "hyperbolic calculus", describing the effect of several standard operations on perverse sheaves directly in terms of hyperbolic sheaves. These operations include forming vanishing cycles, specialization and Fourier-Sato transform. To illustrate the importance of such questions recall [BFS] that the the weight components of the highest weight modules (e.g. Verma, or their irreducible quotients) over quantized Kac-Moody algebras have interpretation as the spaces of vanishing cycles $\Phi_f(\mathcal{F})$ for appropriate $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ and f. In this case $\mathcal{H}$ is a so-called discriminantal arrangement, $\mathcal{F}$ is an extension of a 1-dimensional local system on the generic stratum, and f is a linear function. The monodromy of Fourier-Sato transforms of these sheaves is related to the action of Lusztig symmetries on the corresponding representations [FS].

B. Pattern of the results. To identify the effect of each operation on perverse sheaves above, we produce a new hyperbolic sheaf out of a given one. Our constructions and results fall into the following pattern.

(1) Each vector space of the new hyperbolic sheaf is identified with the 0th cohomology space of an otherwise acyclic complex formed by some of the vector spaces $E_A \otimes \mathrm{or}_A$ (here or_A is the orientation space), with the differential formed out of either the γ_{AB} or the δ_{BA}. So there are two versions of the answer: the γ-answer and the δ-answer, in each case.

(2) The complexes in (1) are subquotients of the two fundamental complexes (Proposition 1.12) calculating $R\Gamma_c(\mathbb{C}^n, \mathcal{F})$ and $R\Gamma(\mathbb{C}^n, \mathcal{F})$. These complexes are sums over all the faces A of the spaces $E_A \otimes \mathrm{or}_A$ and their differentials are formed out of the γ_{AB} and δ_{AB} respectively. The $R\Gamma_c(\mathbb{C}^n, \mathcal{F})$ and $R\Gamma(\mathbb{C}^n, \mathcal{F})$ typically have more than one nonzero cohomology, but the subquotients we take turn out to be acyclic outside degree 0.

(3) The choice of subquotient is obtained by taking not all but some summands $E_A \otimes \mathrm{or}_A$. The selection rule, depending on the problem, reflects the geometry of the problem in some rough ("tropical") way.

(4) In each case there is also a companion real statement, about complexes of sheaves on $\mathbb{R}^n$ constructible w.r.t. the stratification by the faces. This real statement is proved first, and the statement for perverse sheaves is deduced from it.

C. Structure of the paper. In §1 we recall the basics of the description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ by hyperbolic sheaves.

§2 is devoted to the calculation of the space of vanishing cycles $\Phi_f(\mathcal{F})$ in terms of hyperbolic sheaves. Here $f : \mathbb{C}^n \to \mathbb{C}$ is a linear function with real coefficients. The selection rule for subquotients of $R\Gamma_c(\mathbb{C}^n, \mathcal{F})$ and $R\Gamma(\mathbb{C}^n, \mathcal{F})$ consists in taking all faces $B \subset \mathbb{R}^n$ on which $f \geqslant 0$.

§3 describes the specialization of $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ along a $\mathbb{C}$-vector subspace $L_{\mathbb{C}} \subset \mathbb{C}^n$ with real equations. This is a perverse sheaf $\nu_L(\mathcal{F})$ on the normal bundle $T_L\mathbb{C}^n$ which is itself a vector space. In this case we have the real subspace $L_{\mathbb{R}}$, and the product arrangement $\nu_L(\mathcal{H})$ in $T_{L_{\mathbb{R}}}\mathbb{R}^n$. We further have the specialization at the level of faces which is a monotone map of posets

$$\nu : \{\text{faces of } \mathcal{H}\} \to \{\text{faces of } \nu_L(\mathcal{H})\}.$$

The selection rule for subquotients of $R\Gamma_c(\mathbb{C}^n, \mathcal{F})$ and $R\Gamma(\mathbb{C}^n, \mathcal{F})$ consists in taking all faces A with $\nu(A) = B$ being a fixed face B of $\nu_L(\mathcal{H})$. This produces complexes calculating the hyperbolic stalk of $\nu_L(\mathcal{F})$ at B.

We also give a description of the specialization for constructible sheaves of $\mathbb{R}^n$ as the direct image under an appropriate cellular map $q : \mathbb{R}^n \to T_{L_{\mathbb{R}}}\mathbb{R}^n$. This allows us to identify (in our particular case) different possible (and, in general, non-equivalent) definitions of the bispecialization functor [ST] [T] for a flag of subspaces $N \subset M \subset V$.

In §4 we give a similar description of the Fourier-Sato transform $\mathrm{FS}(\mathcal{F})$ which is a perverse sheaf on the dual space $(\mathbb{C}^n)^*$. It is smooth with respect to an appropriate arrangement $\mathcal{H}^\vee$. Each face $A^\vee$ on $\mathcal{H}^\vee$ gives a natural strictly convex cone $V(A^\vee) \subset \mathbb{R}^n$. The selection rule for subquotients of $R\Gamma_c(\mathbb{C}^n, \mathcal{F})$ and $R\Gamma(\mathbb{C}^n, \mathcal{F})$ consists in taking all faces $B \subset V(A^\vee)$ for a fixed $A^\vee$. This produces complexes calculating the hyperbolic stalk of $\mathrm{FS}(\mathcal{F})$ at $A^\vee$.

Combining the descriptions of the specialization and of the Fourier-Sato transform at the level of hyperbolic sheaves, one obtains a description of the microlocalization $\mu_L(\mathcal{F})$ along a linear subspace with real equations. The final §5 is dedicated to comparison, in our linear case, of several possible definitions of the second microlocalization of Kashiwara and Laurent, see [L] [ST] [T].

Acknowledgements. We would like to thank P. Schapira for remarks on a preliminary draft of the paper and for comminicating to us a proof of Theorem 5.7.

The research of M.F. was supported by the grant RSF-DFG 16-41-01013.

The research of M.K. was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

V. S. thanks Kavli IPMU for support of a visit during the preparation of this paper.

1 Real and complex data associated to perverse sheaves

A. The real setup. Let $V_{\mathbb{R}} = \mathbb{R}^n$ be a finite-dimensional vector space over $\mathbb{R}$ and $\mathcal{H}$ be a finite central arrangement of hyperplanes in $V_{\mathbb{R}}$. We denote by $\mathcal{S}_{\mathbb{R}} = \mathcal{S}_{\mathbb{R},\mathcal{H}}$ the poset of *faces* of $\mathcal{H}$, see, e.g., [KS1], §2A. Faces form a real stratification of $V_{\mathbb{R}}$ into (a disjoint union of) locally closed polyhedral cones. The order $\leqslant$ on $\mathcal{S}_{\mathbb{R}}$ is by inclusion of closures: $A \leqslant B$ means $A \subset \overline{B}$. For an integer $p \geqslant 0$ we use the notation $A <_p B$ to signify that $A \leqslant B$ and $\dim(B) = \dim(A) + p$, in particular $A <_0 B$ means $A = B$. We denote by $i_A : A \to V_{\mathbb{R}}$ the embedding of a face A.

Let $\mathbf{k}$ be a field and $\mathrm{Vect}_{\mathbf{k}}$ be the category of finite-dimensional $\mathbf{k}$-vector spaces. For any poset S we denote by $\mathrm{Rep}(S)$ the abelian category of *representations of S over* $\mathbf{k}$, i..e, of covariant functors from S (considered as a category) to $\mathrm{Vect}_{\mathbf{k}}$. By $D^b(\mathrm{Rep}(S))$ we denote the bounded derived category of $\mathrm{Rep}(S)$.

For a topological space X we denote by Sh_X the category of sheaves of $\mathbf{k}$-vector spaces on X and by $D^b(X)$ the derived category of Sh_X.

We denote by $\mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ the abelian category formed by sheaves of $\mathbf{k}$-vector spaces on $V_{\mathbb{R}}$ which are constructible with respect to the stratification $\mathcal{S}_{\mathbb{R}}$. Let also $D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ be the full subcategory in the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on $V_{\mathbb{R}}$ formed by complexes with all cohomology sheaves lying in $\mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$. For $\mathcal{G} \in D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ and a face A we denote

$$\mathcal{G}_A \;=\; R\Gamma(A, \mathcal{G}) \;:=\; R\Gamma(A, i_A^*\mathcal{G}) \;\in\; D^b(\mathrm{Vect}_{\mathbf{k}}) \tag{1.1}$$

the stalk of $\mathcal{G}$ at A. Thus $\mathcal{G}_A$ is a complex which is a single vector space, if $\mathcal{G}$ is a single sheaf. The following is well known.

Proposition 1.2. *(a) We have an equivalence of categories*

$$\mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}}) \longrightarrow \mathrm{Rep}(\mathcal{S}_{\mathbb{R}}), \quad \mathcal{G} \mapsto \big(\mathcal{G}_A, \gamma_{AB} : \mathcal{G}_A \to \mathcal{G}_B,\, A \leqslant B\big).$$

Here γ_{AB} is the generalization map.

(b) The natural functor $D^b(\mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})) \to D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ is an equivalence. In particular:

(c) We have an equivalence of categories $D^b(\mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})) \to D^b(\mathrm{Rep}(\mathcal{S}_{\mathbb{R}})$. $\square$

In view of (b), we can interpret the equivalence in (c) as sending a complex of sheaves $\mathcal{G}$ to the collection of complexes of vector spaces $\mathcal{G}_A$ defined by (1.1) and generalization maps (morphisms of complexes) γ_{AB} connecting them.

By a *cell* we mean a topological space B homeomorphic to $\mathbb{R}^d$ for some d. For a cell B we denote by $\mathrm{or}_B = H_c^{\dim(B)}(B, \mathbf{k})$ the 1-dimensional *orientation vector* $\mathbf{k}$*-space* of B. For two cells B, C we set $\mathrm{or}_{B/C} = \mathrm{or}_C \otimes \mathrm{or}_B^*$ and call it the *relative orientation space* of C and B.

In particular, any face $B \in \mathcal{S}_\mathbb{R}$ is a cell and so we have the space or_B. When B, C are two faces such that $B <_1 C$, we have a canonical "ncidence isomorphism"

$$\varepsilon_{BC} : \mathrm{or}_B \to \mathrm{or}_C \,.$$

It can be seen as a canonical trivialization of $\mathrm{or}_{C/B}$. If $B <_1 C_1, C_2 <_1 D$ is a square of codimension 1 inclusion of faces, then the diagram

$$\begin{tikzcd} \mathrm{or}_B \arrow[r, "\varepsilon_{B,C_1}"] \arrow[d, "\varepsilon_{B,C_2}"'] & \mathrm{or}_{C_1} \arrow[d, "\varepsilon_{C_1,D}"] \\ \mathrm{or}_{C_2} \arrow[r, "\varepsilon_{C_2,D}"'] & \mathrm{or}_D \end{tikzcd} \tag{1.3}$$

is anti-commutative.

Let $j_A : A \to V_\mathbb{R}$ be the embedding of a face A. If $A <_1 A'$ are two faces of $\mathcal{H}$, we have a canonical moprhism $\xi_{AA'} : j_{A!}\underline{\mathbf{k}}_A \longrightarrow j_{A'!}\underline{\mathbf{k}}_{A'}[1]$ in $D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$. Viewed as an element of $\mathrm{Ext}^1(j_{A!}\underline{\mathbf{k}}_A, j_{A'!}\underline{\mathbf{k}}_{A'})$, it represents the extension given by the subsheaf in $(j_{A'})_* \underline{\mathbf{k}}_{A'}$ formed by sections which vanish on all codimension 1 faces of A' except A. The moprhisms $\xi_{AA'}$ anticommute in squares of codimension 1 embeddings, just like the moprhisms $\varepsilon_{AA'}$ in (1.3).

Proposition 1.4. *For $\mathcal{G} \in D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$, the following are equivalent;*

(i) $\mathcal{G}$ corresponds to the data $(\mathcal{G}_A, \gamma_{AB})$.

(ii) We have a resolution of $\mathcal{G}$ (a complex over $D^b(Sh_V)$ with total object $\mathcal{G}$) of the form

$$\bigoplus_{\dim(A)=0} j_{A!}(\underline{\mathcal{G}_A}_A) \xrightarrow{\gamma\otimes\xi} \bigoplus_{\dim(A)=1} j_{A!}(\underline{\mathcal{G}_A}_A)[1] \xrightarrow{\gamma\otimes\xi} \bigoplus_{\dim(A)=2} j_{A!}(\underline{\mathcal{G}_A}_A)[2] \xrightarrow{\gamma\otimes\xi} \cdots,$$

the direct sums ranging over all faces of $\mathcal{H}$ of given dimension.

Proof: See, e.g., [KS1] Eq. (1.12). □

Corollary 1.5. *If $\mathcal{G} \in D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$ corresponds to $(\mathcal{G}_A, \gamma_{AB})$, then*

$$R\Gamma_c(V_\mathbb{R}, \mathcal{G}) \;\simeq\; \mathrm{Tot}\Big\{ \bigoplus_{\dim(A)=0} \mathcal{G}_A \otimes \mathrm{or}_A \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\dim(A)=0} \mathcal{G}_A \otimes \mathrm{or}_A \xrightarrow{\gamma\otimes\varepsilon} \cdots \Big\}$$

(the cohomology with compact supports is calculated by the cellular cochain complex).

Proof: This follows because $R\Gamma_c(V, j_{A!}\underline{\mathbf{k}}_A) = \mathrm{or}(A)[-\dim(A)]$ (cohomology of a cell with compact support). □

B. The complex setup. Let $V_\mathbb{C} = \mathbb{C}^n$ be the complexification of V, and $\mathcal{H}_\mathbb{C}$ the arrangement of hyperplanes in $V_\mathbb{C}$ formed by the $H_\mathbb{C}$, the complexifications of the hyperplanes $H \in \mathcal{H}$. By a *flat* of $\mathcal{H}_\mathbb{C}$ we will mean a subspace of the form $L = \bigcup_{H\in J} H_\mathbb{C}$ for a subset $J \subset \mathcal{H}$ (with $J = \varnothing$ or $J = \mathcal{H}$ allowed). Flats form a poset $\mathrm{Fl}(\mathcal{H}_\mathbb{C})$ ordered by inclusion. Becasue $\mathcal{H}$ is assumed central, $\mathrm{Fl}(\mathcal{H}_\mathbb{C})$ has 0 as the minimal element and $V_\mathbb{C}$ as the maximal element.

For a flat L we denote its *generic part* by

$$L^\circ \;=\; L \setminus \bigcup_{H\in\mathcal{H},\; H_\mathbb{C}\not\supset L} L \cap H_\mathbb{C}. \tag{1.6}$$

The subsets L° form a stratification of $V_\mathbb{C}$ which we denote by $\mathcal{S}_\mathbb{C} = \mathcal{S}_{\mathbb{C},\mathcal{H}}$. We view it as a poset, isomorphic to the poset of flats.

Note that faces can be defined as connected components of $L^\circ_\mathbb{R} = L^\circ \cap V_\mathbb{R}$ for strata L° of $\mathcal{S}_\mathbb{C}$. We therefore have the morphism of posets ("complexification")

$$c : \mathcal{S}_\mathbb{R} \longrightarrow \mathcal{S}_\mathbb{C}.$$

We denote by $D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$ the full subcategory in the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on $V_\mathbb{C}$ formed by complexes whose cohomology sheaves are constructible with respect to $\mathcal{S}_\mathbb{C}$. This category has a perfect duality given by passing from $\mathcal{F}$ to $\mathcal{F}^*$, the Verdier dual complex. Inside it, we have $\mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$ the abelian subcategory of perverse sheaves. We normalize the conditions of (middle) perversity so that $\underline{\mathbf{k}}_{V_\mathbb{C}}[n]$, the constant sheaf put in degree $(-n)$, is perverse. This normalization agrees with that of [BBD] and differs by shift from that of [KS1]. The abelian category $\mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$ is closed under Verdier duality.

C. Real data: stalks and hyperbolic stalks. Let $i_{\mathbb{R}}; V_{\mathbb{R}} \to V_{\mathbb{C}}$ be the embedding. It induces exact functors of triangulated categories

$$i_{\mathbb{R}}^*, i_{\mathbb{R}}^! : D^b(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}}) \longrightarrow D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}}).$$

To every complex $\mathcal{F} \in D^b(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ and every face $A \in \mathcal{S}_{\mathbb{R}}$ we can associate therefore two complexes of vector spaces, which we call the *stalk* and the *hyperbolic stalk* of $\mathcal{F}$ at A:

$$\mathcal{F}_A \;=\; (i_{\mathbb{R}}^*\mathcal{F})_A \;=\; R\Gamma(A, i_A^* i_{\mathbb{R}}^* \mathcal{F}), \quad E_A(\mathcal{F}) \;=\; (i_{\mathbb{R}}^!\mathcal{F})_A \;=\; R\Gamma(A, i_A^* i_{\mathbb{R}}^! \mathcal{F}).$$

For any pair of faces $A \leqslant B$ we have the generalization maps (morphisms of complexes) for $i_{\mathbb{R}}^*\mathcal{F}$ and $i_{\mathbb{R}}^!\mathcal{F}$:

$$\digamma_{AB} : \mathcal{F}_A \longrightarrow \mathcal{F}_B, \quad \gamma_{AB} : E_A(\mathcal{F}) \longrightarrow E_B(\mathcal{F}). \tag{1.7}$$

By the Duality Theorem, see [KS1] Prop. 4.6 or [BFS] Pt. I, Thm. 3.9, we have natural isomorphisms

$$E_A(\mathcal{F}^*) \;\simeq\; E_A(\mathcal{F})^*. \tag{1.8}$$

which imply the following.

Proposition 1.9. *(a) We have a canonical identification* $E_A(\mathcal{F}) \;\simeq\; R\Gamma(A, i_A^! i_{\mathbb{R}}^* \mathcal{F})$.

(b) The hyperbolic stalk $E_A(\mathcal{F})$ *is identified with the complex*

$$\mathcal{F}_{\geqslant A} \;:=\; \mathrm{Tot}\left\{ \mathcal{F}_A \xrightarrow{\digamma\otimes\varepsilon} \bigoplus_{B >_1 A} \mathcal{F}_B \otimes \mathrm{or}_{B/A} \xrightarrow{\digamma\otimes\varepsilon} \bigoplus_{B >_2 A} \mathcal{F}_B \otimes \mathrm{or}_{B/A} \xrightarrow{\gamma\otimes\varepsilon} \cdots \right\}$$

with the differential $\digamma \otimes \varepsilon$ *having matrix elements* $\digamma_{BC} \otimes \varepsilon_{BC}$, $B <_1 C$.

□

For a dual statement, expressing ordinary stalks through hyperbolic stalks, see Corollary 1.14.

Proof: Part (a) follows from (1.8) and the fact that Verdier duality interchanges i^* and $i^!$. Part (b) follows by interpreting $i_A^! i_{\mathbb{R}}^* \mathcal{F}$ as $\underline{R\Gamma}_A(i_{\mathbb{R}}^*\mathcal{F})$, the complex of sheaves formed by (derived) global sections with support in A. The stalk of this complex at any $a \in A$ can be seen as

$$R\Gamma_{\{a\}}(D, i_{\mathbb{R}}^*\mathcal{F}) \;=\; R\Gamma_c(D, i_{\mathbb{R}}^*\mathcal{F}),$$

where $D \subset V_{\mathbb{R}}$ is a small transverse open ball (of complementary dimension) to A centered at a. The situation is similar to that of Corollary 1.5 (with a ball instead of a vector space) and the same argument gives the result. $\square$

It was proved in [KS1] Prop. 4.9(a) that for $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ the complex $i^!_{\mathbb{R}}(\mathcal{F})$ is exact in degrees $\neq 0$, and so the functor

$$(1.10) \quad \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}}) \to \mathrm{Sh}(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}}), \quad \mathcal{F} \mapsto \mathcal{E}(\mathcal{F}) := \underline{H}^0(i^!_{\mathbb{R}}\mathcal{F}) = \underline{\mathbb{H}}^0_{V_{\mathbb{R}}}(\mathcal{F})$$

is an exact functor of abelian categories. In particular, each $E_A(\mathcal{F})$ reduces to a single vector space. Further, (1.8) allows us to define maps of vector spaces

$$\delta_{BA} = \delta^{\mathcal{F}}_{BA} : E_B(\mathcal{F}) \longrightarrow E_A(\mathcal{F}), \ A \leqslant B, \quad \delta^{\mathcal{F}}_{BA} := (\gamma^{\mathcal{F}*}_{AB})^*.$$

which form an anti-representation of $\mathcal{S}_{\mathbb{R}}$, i.e., a contravariant functor $(\mathcal{S}_{\mathbb{R}}, \leqslant) \to \mathrm{Vect}_{\mathbf{k}}$. This leads to the following concept.

D. Hyperbolic sheaves. By a *hyperbolic sheaf* on $\mathcal{H}$ we will mean a datum

$$\mathcal{Q} = \big(E_A, \gamma_{AB} : E_A \to E_B, \, \delta_{BA} : E_B \to E_A, A \leqslant B\big)$$

where $E_A, A \in \mathcal{S}_{\mathbb{R}}$, are finite-dimensional $\mathbf{k}$-vector spaces, (γ_{AB}) form a representation of $\mathcal{S}_{\mathbb{R}}$, and (δ_{BA}) form an anti-representation so that the following additional conditions hold:

(i) For each $B \leqslant A$, $\delta_{AB}\gamma_{BA} = \mathrm{Id}_{E_B}$. This allows us to define for arbitrary $A, B \in \mathcal{S}_{\mathbb{R}}$, the "flopping operator"

$$\phi_{AB} := \gamma_{CB}\delta_{AC} : \ E_A \longrightarrow E_B.$$

Here $C \in \mathcal{S}_{\mathbb{R}}$ is any face such that $C \leqslant A, B$, and the definition does not depend on the choice of C.

(ii) Let us call a triple of faces (A, B, C) *collinear* if there exist points $x \in A, y \in B, z \in C$ lying on the same straight line, with $y \in [x, z]$. Then for any such collinear triple we must have

$$\phi_{AC} = \phi_{BC}\,\phi_{AB}.$$

(iii) Let A, B be two faces. Let us say that they are *neighbors* if they have the same dimension d, and there exists a face $C \leqslant A, C \leqslant B$, with $\dim C = d - 1$ (a *wall* separating A and B). Such a wall is unique if it exists. For any such pair of neighbors we require that ϕ_{AB} is an isomorphism.

We denote by $\mathrm{Hyp}(\mathcal{H})$ the abelian category formed by hyperbolic sheaves on $\mathcal{H}$. This category has a perfect duality

$$\mathcal{Q} = (E_A, \gamma_{AB}, \delta_{BA}) \ \mapsto \ \mathcal{Q}^* = (E_A^*, \delta_{BA}^*, \gamma_{AB}^*).$$

The main result of [KS1] can be formulated as follows.

Theorem 1.11. *The functor*

$$\mathcal{F} \mapsto \mathcal{Q}(\mathcal{F}) \ = \ \big(E_A(\mathcal{F}), \gamma_{AB} : E_A(\mathcal{F}) \to E_B(\mathcal{F}),\ \delta_{BA} : E_B(\mathcal{F}) \to E_A(\mathcal{F}), A \leqslant B\big)$$

defines an equivalence $\mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}}) \to \mathrm{Hyp}(\mathcal{H})$. *This equivalence commutes with duality:* $\mathcal{Q}(\mathcal{F}^*) \simeq \mathcal{Q}(\mathcal{F})^*$. □

The goal of this paper is to describe various features of perverse sheaves explicitly, in terms of the linear algebra data given by the associated hyperbolic sheaves.

Let us first note the following.

Proposition 1.12. *If* $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ *corresponds to a hyperbolic sheaf* $\mathcal{Q}(E_A, \gamma_{AB}, \delta_{BA})$, *then*

$$R\Gamma_c(V_{\mathbb{C}}, \mathcal{F}) \ \simeq \ \Big\{ \bigoplus_{\dim(A)=0} E_A \otimes \mathrm{or}_A \xrightarrow{\gamma \otimes \varepsilon} \bigoplus_{\dim(A)=1} E_A \otimes \mathrm{or}_A \xrightarrow{\gamma \otimes \varepsilon} \cdots \Big\},$$

$$R\Gamma(V_{\mathbb{C}}, \mathcal{F}) \ \simeq \ \Big\{ \bigoplus_{\mathrm{codim}(A)=0} E_A \otimes \mathrm{or}_A \xrightarrow{\delta \otimes \varepsilon} \bigoplus_{\mathrm{codim}(A)=1} E_A \otimes \mathrm{or}_A \xrightarrow{\delta \otimes \varepsilon} \cdots \Big\}.$$

Proof: The first quasi-isomorphism follows from Corollary 1.5 and the lemma below. The second quasi-isomorphism follows from the first one by applying the Verdier duality. □

Lemma 1.13. *For any* $\mathcal{F} \in D^b(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ *we have*

$$R\Gamma_c(V_{\mathbb{C}}, \mathcal{F}) \ \simeq \ R\Gamma_c(V_{\mathbb{R}}, i_{\mathbb{R}}^! \mathcal{F}).$$

Proof of the lemma: Let $i_{0,\mathbb{C}} : \{0\} \to V_{\mathbb{C}}$ and $i_{0,\mathbb{R}} : \{0\} \to V_{\mathbb{R}}$ be the embedings of the origin. Any $\mathcal{F} \in D^b(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ is $\mathbb{R}_+$-*conic*, i..e, each cohomology sheaf of $\mathcal{F}$ is locally constant on each orbit of the scaling action of $\mathbb{R}_{>0}$ on $V_{\mathbb{C}}$. This implies that

$$R\Gamma_c(V_{\mathbb{C}}, \mathcal{F}) \simeq R\Gamma_{\{0\}}(V_{\mathbb{C}}, \mathcal{F}) = R\Gamma(V_{\mathbb{C}}, i_{0,\mathbb{C}}^! \mathcal{F}).$$

Similarly, $i_{\mathbb{R}}^! \mathcal{F}$ is $\mathbb{R}_+$-conic on $V_{\mathbb{R}}$ and

$$R\Gamma_c(V_{\mathbb{R}}, i_{\mathbb{R}}^! \mathcal{F}) \simeq R\Gamma_{\{0\}}(V_{\mathbb{R}}, i_{\mathbb{R}}^! \mathcal{F}) = R\Gamma(V_{\mathbb{C}}, i_{0,\mathbb{R}}^! i_{\mathbb{R}}^! \mathcal{F}),$$

which is the same as the above because $i_{\mathbb{R}} i_{0,\mathbb{R}} = i_{0,\mathbb{C}}$. □

We can now complement Proposition 1.9 by a "Koszul dual" statement.

Corollary 1.14. *For $\mathcal{F} \in \operatorname{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ the ordinary stalk $\mathcal{F}_A$, $A \in \mathcal{S}_{\mathbb{R}}$ is expressed through hyperbolic stalks as follows:*

$$\mathcal{F}_A \simeq \left\{ \bigoplus_{\substack{B \geqslant A \\ \operatorname{codim}(B)=0}} E_B \otimes \operatorname{or}_{B/A} \xrightarrow{\delta \otimes \varepsilon} \bigoplus_{\substack{B \geqslant A \\ \operatorname{codim}(B)=1}} E_B \otimes \operatorname{or}_{B/A} \xrightarrow{\delta \otimes \varepsilon} \cdots \right\}.$$

That is, the complex in question is exact everywhere except the leftmost term where the cohomology (kernel) is identified with $\mathcal{F}_A$.

Proof: For $A = 0$ this is the second identification of Proposition 1.12, since $\mathcal{F}_0 = R\Gamma(U, \mathcal{F})$ for a small convex open $U \ni 0$, and this complex is independent of U, so is the same for $U = V_{\mathbb{C}}$.

For an arbitrary A the statement reduces to the above by considering the quotient arrangement $\mathcal{H}/L_{\mathbb{R}}$ in $V_{\mathbb{R}}/L_{\mathbb{R}}$, where $L_{\mathbb{R}}$ is the $\mathbb{R}$-linear span of A. Faces of $\mathcal{H}/L_{\mathbb{R}}$ are in bijection with faces B of $\mathcal{H}$ such that $B \geqslant A$.

The arrangement $\mathcal{H}/L_{\mathbb{R}}$ represents the transversal slice M to A; the restriction $\mathcal{F}|_{M_{\mathbb{C}}}$ to the complexified transversal slice is, by [KS1] Prop. 5.3, represented by the hyperbolic sheaf $\mathcal{Q}^{\geqslant A}$ formed by $E_B, B \geqslant A$, so the calculation of

$$\mathcal{F}_A = R\Gamma(M_{\mathbb{C}}, \mathcal{F}|_{M_{\mathbb{C}}}) = (\mathcal{F}|_{M_{\mathbb{C}}})_0$$

reduces to the above case.

2 Vanishing cycles in terms of hyperbolic sheaves

The standard microlocal approach to study of perverse sheaves on any stratification is in terms of the local systems of vanishing cyclies on the generic parts of conormal bundles to the strata, see [MV] [KS2]. Our first result provides an explicit description of the fibers of these local systems for perverse sheaves from $\mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$.

A. Background on vanishing cycles. We recall that for any (polynomial) function $f : V_\mathbb{C} \to \mathbb{C}$ and any perverse sheaf $\mathcal{F}$ on $V_\mathbb{C}$ we have a perverse sheaf $\Phi_f(\mathcal{F})$ on $V_\mathbb{C}$ supported on the hypersurface $\{f = 0\}$ and known as the *perverse sheaf of vanishing cycles*, see [Be][De]. We will use the following real analytic interpretation of this perverse sheaf [KS2]. This interpretation reflects the intuitive meaning of the term "vanishing cycles".

Proposition 2.1. *We have an isomorphism in the derived category of sheaves on $V_\mathbb{C}$:*

$$\Phi_f(\mathcal{F}) \;\simeq\; \underline{R\Gamma}_{\{\Re(f)\geqslant 0\}}(\mathcal{F}).$$

That is, the complex $\underline{R\Gamma}_{\{\Re(f)\geqslant 0\}}(\mathcal{F})$ which is, a priori, supported on the closed set $\{\Re(f) \geqslant 0\}$, is in fact supported on the subset $\{f = 0\}$ and is identified with $\Phi_f(\mathcal{F})$. $\square$

We will be interested in the case when f is linear. More precisely, let $L^\circ \in \mathcal{S}_\mathbb{C}$ be a stratum, i.e., the generic part of a flat L, as in (1.6). The conormal bundle to L° is

$$T^*_{L^\circ} V_\mathbb{C} \;=\; L^\circ \times (V_\mathbb{C}/L)^* \;\subset\; V_\mathbb{C} \times V_\mathbb{C}^* \;=\; T^* V_\mathbb{C}.$$

A hyperplane $\Pi \subset V_\mathbb{C}$ is said to be *transversal to $\mathcal{S}_\mathbb{C}$ at L* if $L \subset \Pi$, and $L' \in \mathrm{Fl}(\mathcal{H}_\mathbb{C})$ with $L' \subset \Pi$ implies $L' \subset L$. Let us call a *polarization at L* a linear function $f : V_\mathbb{C} \to \mathbb{C}$ such that $\Pi := \mathrm{Ker}\, f$ is transversal to $\mathcal{S}_\mathbb{C}$ at L. Polarizations of L form an open subset $\mathrm{Pol}(L) \subset (V_\mathbb{C}/L)^*$, and we define the generic part of the conormal bundle to L° as

$$(T^*_{L^\circ} V_\mathbb{C})^\circ \;=\; L^\circ \times \mathrm{Pol}(L).$$

Proposition 2.2. *Let $\mathcal{F} \in \mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$. If $L \in \mathrm{Fl}(\mathcal{H}_\mathbb{C})$ and $f \in \mathrm{Pol}(L)$, then $\Phi_f(\mathcal{F})$ is supported on L. In particular, being perverse, it reduces to a local system in degree $(-\dim(L))$ on L°.*

Proof: Let $x \in \{f = 0\} \subset V_{\mathbb{C}}$ and suppose $x \notin L$. Since $f \in \mathrm{Pol}(L)$, the hyperplane $\Pi = \{f = 0\}$ cannot contain any flats L' which are not contained in L. So x is not contained in any flat other than $V_{\mathbb{C}}$ itself, which means that near x the perverse sheaf $\mathcal{F}$ is reduced to a local system in degree $(-n)$, and so $\Phi_f(\mathcal{F})_x = 0$. $\square$

We now describe the stalks of the local system $\Phi_f(\mathcal{F})$ at the maximal faces of $L_{\mathbb{R}}$.

B. The complex result.

Theorem 2.3. *Let $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ and $\mathcal{Q} = (E_A, \gamma_{AB}, \delta_{BA})$ be the corresponding hyperbolic sheaf as in Theorem 1.11. Suppose further that $f \in \mathrm{Pol}(L)$ is real, i.e., takes $V_{\mathbb{R}}$ to $\mathbb{R}$. Let A be a connected component of $L^{\circ}_{\mathbb{R}}$, so A is a face of $\mathcal{H}$. Consider the complex*

$$E^{\bullet}_{f,A} = \left\{ E_A \overset{\gamma\otimes\varepsilon}{\to} \bigoplus_{B>_1 A,\ f|_B \geqslant 0} E_B \otimes \mathrm{or}_{B/A} \overset{\gamma\otimes\varepsilon}{\to} \bigoplus_{B>_2 A,\ f|_B\geqslant 0} E_B \otimes \mathrm{or}_{B/A} \overset{\gamma\otimes\varepsilon}{\to} \cdots \right\}$$

with the differential $\gamma \otimes \varepsilon$ having matrix elements $\gamma_{BC} \otimes \varepsilon_{BC}$, $B >_1 C$. Then $E^{\bullet}_{f,A}$ is exact outside of the leftmost term, and its leftmost cohomology is identified with the vector space $\Phi_f(\mathcal{F})_a[-\dim(L)]$ for any $a \in A$.

The theorem implies that the shifted space of vanishing cycles is identified with the subspace

$$E_{f,A} = H^0(E^{\bullet}_{f,A}) = \bigcap_{B>_1 A,\ f|_B \geqslant 0} \mathrm{Ker}(\gamma_{AB}) \subset E_A.$$

It also implies the following.

Corollary 2.4. *Consider the complex*

$$\check{E}^{\bullet}_{f,A} = \left\{ \cdots \overset{\delta\otimes\varepsilon}{\to} \bigoplus_{B>_2 A,\ f|_B\geqslant 0} E_B \otimes \mathrm{or}_{B/A} \overset{\delta\otimes\varepsilon}{\to} \bigoplus_{B>_2 A,\ f|_B\geqslant 0} E_B \otimes \mathrm{or}_{B/A} \overset{\delta\otimes\varepsilon}{\to} E_A \right\}$$

with the differential $\delta \otimes \varepsilon$ having matrix elements $\delta_{CB} \otimes \varepsilon_{CB}$, $B >_1 C$. Then $E^{\bullet}_{f,A}$ is exact outside of the rightmost term, and its righttmost cohomology is identified with the vector space $\Phi_f(\mathcal{F})_a[-\dim(L)]$ for any $a \in A$. In other words,

$$E_{f,A} \simeq \mathrm{Coker}\left(\sum \delta_{BA} : \bigoplus_{B>_1 A,\ f|_B \geqslant 0} E_B \longrightarrow E_A \right).$$

Proof of the corollary: The vanishing cycle functor commutes with Verdier duality. Therefore the vector spaces $\Phi_f(\mathcal{F})_a[-\dim(L)]$ and $\Phi_f(\mathcal{F}^*)_a[-\dim(L)]$ are canonically dual to each other. On the other hand, the hyperbolic sheaf corresponding to $\mathcal{F}^*$ is, by Theorem 1.11, identified with $\mathcal{Q}^* = (E_A^*, \delta_{BA}^*, \gamma_{AB}^*)$. Our statement follows by combining this with Theorem 2.3 for $\mathcal{F}$ and $\mathcal{F}^*$. $\square$

Remark 2.5. Theorem 2.3 and Corollary 2.4 can be interpreted as follows. The same graded space $E_{f,A}^\bullet$ possesses two differentials going in the opposite directions: one induced by the maps γ, and the other one induced by the maps δ. It is natural therefore to form the "Laplacian" $\Delta = \delta\gamma + \gamma\delta$ out of them.

In the examples we have calculated, $\Delta : E_{f,A}^i \to E_{f,A}^i$ is an isomorphism for $i > 0$. This of course implies the acyclicity statements above. One may wonder if this stronger property (Laplacian being an isomorphism for $i > 0$) holds more generally.

C. The real analog. Before proving Theorem 2.3, we establish its real counterpart.

Let $\mathcal{G} \in D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$ and let $(\mathcal{G}_A, \gamma_{AB})$ be the complex of representations of $\mathcal{S}_\mathbb{R}$ corresponding to $\mathcal{G}$ by Proposition 1.2. That is, $\mathcal{G}_A$ is the ordinary stalk of $\mathcal{G}$ at A, and γ_{AB} is the generalization map.

Given a nonzero $f \in V_\mathbb{R}^*$, we have the real hyperplane $\Pi = \{f = 0\} \subset V_\mathbb{R}$. The arrangement $\mathcal{H}$ cuts out an arrangement $\mathcal{H} \cap \Pi$ in Π. We denote by $\mathcal{S}_{\mathbb{R},\Pi}$ the stratification of Π into cells of $\mathcal{H} \cap \Pi$. We then have the real version of the vanishing cycle sheaf. It is the complex of sheaves

$$\underline{R\Gamma}_{f \geqslant 0}(\mathcal{G}) \in D^b(\Pi, \mathcal{S}_{\mathbb{R},\Pi}).$$

Proposition 2.6. *(a) Let C' be a cell of $\mathcal{H} \cap \Pi$ and C be the unique cell of $\mathcal{H}$ such that $C' = C \cap \Pi$. The stalk of $\underline{R\Gamma}_{f \geqslant 0}(\mathcal{G})$ at C' is quasi-isomorphic to the total complex of the double complex*

$$\left\{ \mathcal{G}_C \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{D >_1 C,\ f|_D \geqslant 0} \mathcal{G}_D \otimes \mathrm{or}_{D/C} \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{D >_2 C,\ f|_D \geqslant 0} \mathcal{G}_D \otimes \mathrm{or}_{D/C} \xrightarrow{\gamma\otimes\varepsilon} \cdots \right\}$$

(b) Let $C_1' \leqslant C_2'$ be an inclusion of cells of $\mathcal{H} \cap \Pi$. The generalization map

$$\gamma_{C_1', C_2'} : \underline{R\Gamma}_{f \geqslant 0}(\mathcal{G})_{C_1} \longrightarrow \underline{R\Gamma}_{f \geqslant 0}(\mathcal{G})_{C_2}$$

is given by the maps $\gamma_{DD'}$ for $\mathcal{G}$ which induce a moprhism of complexes in (a).

Proof: Let $x \in C'$ and U be a small open ball centered at x. By definition,

$$\underline{R\Gamma}_{f\geqslant 0}(\mathcal{G})_{C'} \;=\; R\Gamma(U, U \cap \{f < 0\}; \mathcal{G})$$

The relative cellular cochain complex representing this, is precisely the complex in (a). Part (b) also follows immediately. □

D. Proof of Theorem 2.3. Let f be as in the theorem. Considering f as a complex functional on $V_{\mathbb{C}}$, we have the complex hyperplane $\Pi_{\mathbb{C}} = \{f = 0\} \subset V_{\mathbb{C}}^*$ and the perverse sheaf $\Phi_f(\mathcal{F})$ on $\Pi_{\mathbb{C}}$. By Proposition 2.1 we can express the hyperbolic stalk of $\Phi_f(\mathcal{F})$ at a cell $C' \in \mathcal{S}_{\mathbb{R},\Pi}$ is as

$$E_{C'}(\Phi_f(\mathcal{F})) \;=\; (\underline{R\Gamma}_{\Pi_{\mathbb{R}}}\, \underline{R\Gamma}_{\Re(f)\geqslant 0}(\mathcal{F}))_{C'} \;=\; (\underline{R\Gamma}_{f\geqslant 0}\, \underline{R\Gamma}_{V_{\mathbb{R}}}(\mathcal{F}))_{C'}.$$

Now, the complex (actually a sheaf) $\mathcal{G} = \underline{R\Gamma}_{V_{\mathbb{R}}}(\mathcal{F})$ on $V_{\mathbb{R}}$ is given by the stalks E_B and generalization maps γ_{BC} from the hyperbolic sheaf $\mathcal{Q}$. So applying Proposition 2.6 to this $\mathcal{G}$ and to the cell $C' = A$ as in the formulation of theorem, we get the statement. □

Remark 2.7. It is worth noticing the following contrast between Proposition 2.6 and Theorem 2.3. If $\mathcal{G}$ is an arbitrary sheaf (not a complex) on $V_{\mathbb{R}}$, then Proposition 2.6 gives, in general, a complex with several nontrivial cohomology spaces, because $\underline{R\Gamma}_{f\geqslant 0}(\mathcal{G})$ need not reduce to a single sheaf. However, in the case when $\mathcal{G}$ has the form $\mathcal{G} = \underline{R\Gamma}_{V_{\mathbb{R}}}(\mathcal{F})$ for a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$, this complex is, by Theorem 2.3, quasi-isomorphic to a single vector space in degree 0.

A more immediate instance of such special behavior of the sheaves $\underline{R\Gamma}_{V_{\mathbb{R}}}(\mathcal{F})$ can be seen from the property (i) of hyperbolic sheaves in §1D: the condition $\delta_{AB}\gamma_{BA} = \mathrm{Id}$ implies that each γ_{BA} is surjective.

3 Specialization and hyperbolic sheaves

A. Generalities on specialization. We recall the necessary material from [KS2] §4.1-4.2. Let X be a C^∞-manifold, $M \subset X$ a locally closed submanifold and $T_M X$ the normal bundle to M in X. Any subset $S \subset X$ gives rise to its *normal cone* with center M, which is a closed subset $C_M S \subset T_M X$ depending only on the closure $\overline{S}$. We will need the following example.

Example 3.1. Let X be a finite-dimensional $\mathbb{R}$-vector space and $M \subset X$ is an $\mathbb{R}$-vector subspace. Then $T_M X = M \times (X/M)$. If S is also an $\mathbb{R}$-vector subspace, then, with respect to the above identification,

$$C_M(S) \;=\; (M \cap S) \times \big(M/(M \cap S)\big).$$

For any complex of sheaves $\mathcal{G} \in D^b(\mathrm{Sh}_X)$ we have its *specialization* at M which is an $\mathbb{R}_{>0}$-conic complex of sheaves $\nu_M(\mathcal{G}) \in D^b(T_M X)$. We will later recall its definition in the case we need.

B. The case of sheaves on arrangements. We will study this construction in two related cases, related to the data of a real arrangement $(V_\mathbb{R}, \mathcal{H})$.

Complex case: $X = V_\mathbb{C}$, $M = L_\mathbb{C}$ a complex flat of $\mathcal{H}$ and $\mathcal{G} = \mathcal{F} \in \mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$ a perverse sheaf smooth with respect to $\mathcal{S}_\mathbb{C}$.

Real case: $X = V_\mathbb{R}$, $M = L_\mathbb{R}$ is a real flat and $\mathcal{G} \in D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$ is any complex smooth with respect to the cell decomposition $\mathcal{S}_\mathbb{R}$.

In each of these cases the normal bundle is itself a vector space:

$$T_{L_\mathbb{C}} V_\mathbb{C} \;=\; L_\mathbb{C} \times (V_\mathbb{C}/L_\mathbb{C}), \quad T_{L_\mathbb{R}} V_\mathbb{R} \;=\; L_\mathbb{R} \times (V_\mathbb{R}/L_\mathbb{R}). \tag{3.2}$$

The subspace $L_\mathbb{R}$ carries the *induced arrangement* $\mathcal{H} \cap L_\mathbb{R}$ formed by the hyperplanes $H \cap L_\mathbb{R}$ for $H \in \mathcal{H}$, $H \not\supset L_\mathbb{R}$. The quotient space $V_\mathbb{R}/L_\mathbb{R}$ carries the *quotient arrangement* $\mathcal{H}/L_\mathbb{R}$ formed by the hyperplanes $H/L_\mathbb{R}$ for $H \in \mathcal{H}$, $H \supset L_\mathbb{R}$. We equip $T_{L_\mathbb{R}} V_\mathbb{R}$ with the *product arrangement*

$$\nu_L \mathcal{H} \;:=\; (\mathcal{H} \cap L_\mathbb{R}) \oplus (\mathcal{H}/L_\mathbb{R}) \;=\;$$
$$\big\{(H \cap V_\mathbb{R}) \times V_\mathbb{R}/L_\mathbb{R},\; H \not\supset V_\mathbb{R}\big\} \;\cup\; \big\{L_\mathbb{R} \times (V_\mathbb{R}/L_\mathbb{R}),\; H \supset V_\mathbb{R}\big\}.$$

We have a surjective map $\mathcal{H} \to \nu_L(\mathcal{H})$ between (the sets of hyperplanes of) the two arrangements. Two hyperplanes H, H' of $\mathcal{H}$ can give the same hyperplane of $\nu_L(\mathcal{H})$, if $H \cap L_\mathbb{R} = H' \cap L_\mathbb{R}$ is the same hyperplane in $L_\mathbb{R}$.

We denote by

$$\mathcal{S}^\nu_\mathbb{R} = \mathcal{S}_{1,\mathbb{R}} \times \mathcal{S}_{2,\mathbb{R}}, \quad \mathcal{S}^\nu_\mathbb{C} = \mathcal{S}_{1,\mathbb{C}} \times \mathcal{S}_{2,\mathbb{C}} \tag{3.3}$$

the stratification of $T_{L_\mathbb{R}} V_\mathbb{R}$ by the faces of $\nu_L(\mathcal{H})$, and the stratification of $T_{L_\mathbb{C}} V_\mathbb{C}$ by the generic parts of the complex flats of $\nu_L(\mathcal{H})$. Here $\mathcal{S}_{1,\mathbb{R}}$ is the stratification of $L_\mathbb{R}$ by the faces of $\mathcal{H} \cap L$, while $\mathcal{S}_{2,\mathbb{R}}$ is the stratification of $V_\mathbb{R}/L_\mathbb{R}$ by the faces of $\mathcal{H}/L$, and similarly for $\mathcal{S}_{i,\mathbb{C}}$.

Proposition 3.4. *(a) If $\mathcal{F} \in D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$, then $\nu_{L_\mathbb{C}} \mathcal{F} \in D^b(T_{L_\mathbb{C}} V_\mathbb{C}, \mathcal{S}^\nu_\mathbb{C})$.*

(b) If $\mathcal{G} \in D^b(V_\mathbb{R}, \mathcal{S}_\mathbb{R})$, then $\nu_{L_\mathbb{R}} \mathcal{G} \in D^b(T_{L_\mathbb{R}} V_\mathbb{R}, \mathcal{S}^\nu_\mathbb{R})$.

Proof: We treat only the real case (b), the complex case (a) being identical. In the proof we simply write V for the ambient vector space $V_\mathbb{R}$, as well as L for a real flat and so on. We denote by $\mathrm{SS}(\mathcal{G}) \subset T^*V$ the microsupport of the complex $\mathcal{G}$, and similarly for complexes of sheaves on other spaces, see [KS2] Ch. VI. The statement that $\mathcal{G} \in D^b(V, \mathcal{S})$, resp. that $\nu_L(\mathcal{G}) \in D^b(T_L V, \mathcal{S}_\nu)$, is equivalent to

$$\mathrm{SS}(\mathcal{G}) \subset \bigcup_{P \in \mathrm{Fl}(\mathcal{H})} T^*_P V, \quad \text{resp.} \quad \mathrm{SS}(\nu_L(\mathcal{G})) \subset \bigcup_{Q \in \mathrm{Fl}(\nu_L(\mathcal{H}))} T^*_Q(L \times (V/L)).$$

So we deduce the second inclusion from the first. By Theorem 6.4.1 of [KS2], for any manifold X, a submanifold M and a complex of sheaves $\mathcal{G}$ on X we have

$$\mathrm{SS}(\nu_M(\mathcal{G})) \subset C_{T^*_M X}(\mathrm{SS}(\mathcal{G})) \subset T_{T^*_M X} T^* X \overset{(!)}{\simeq} T^*(T_M X).$$

Here $C_{T^*_M X}(\mathrm{SS}(\mathcal{G}))$ is the normal cone to $\mathrm{SS}(\mathcal{G}) \subset T^*X$, and the identification (!) looks, in our concrete case, as follows.

We have $T^*V = V \times V^*$, and $T^*_L V = L \times L^\perp$. Therefore

$$\begin{aligned} T_{T^*_L V} T^* V &= T_{L \times L^\perp}(V \times V^*) = (L \times L^\perp) \times \big((V/L) \times L^*\big), \\ T^*(T_L V) &= T^*\big(L \times (V/L)\big) = \big(L \times (V/L)\big) \times (L^* \times L^\perp), \end{aligned}$$

and (!) identifies factors number 1,2,3,4 of the first product with factors number 1,4,2,3 of the second one.

With this understanding, we need to prove that for any flat P of $\mathcal{H}$ the normal cone $C_{T^*_L V}(T^*_P V)$ is contained in the union of $T^*_Q\big(L \times (V/L)\big)$ over flats Q of the product arrangement in $L \times (V/L)$. In fact, it is contained in a single $T^*_Q\big(L \times (V/L)\big)$, where Q is the product flat $(P \cap L) \times \big(P/(P \cap L)\big)$, as follows from Example 3.1. This finishes the proof of Proposition 3.4.

C. Specialization of faces as a continuous map. Given a face A of $\mathcal{H}$, the intersection $\overline{A} \cap L_{\mathbb{R}}$ is the closure of a unique face of the arrangement $\mathcal{H} \cap L_{\mathbb{R}}$ which we denote by $\nu'_L(A)$. Further, the image of A in $V_{\mathbb{R}}/L_{\mathbb{R}}$ is a face of the quotient arrangement $\mathcal{H}/L_{\mathbb{R}}$ which we denote by $\nu''_L(A)$. The pair $\nu_L(A) = (\nu'_L(A), \nu''_L(A))$ is then a face of the product arrangement $\nu_L(\mathcal{H})$ which we call the *specialization* of A.

Proposition 3.5. *The closure of $\nu_L(A)$ is identified with the normal cone $C_{L_{\mathbb{R}}}(A)$. Thus $\nu_L(A)$ is the interior (complement of the boundary) of $C_{L_{\mathbb{R}}}(A)$.*

Proof: This is similar to Example 3.1. □

Example 3.6. The concept of specialization is illustrated in Fig. 1, where $\mathcal{H}$ consists of 5 lines in the plane, $L_{\mathbb{R}}$ is the horizontal line, and $\mathcal{H}/L_{\mathbb{R}}$ is the coordinate arrangement of two lines in $\mathbb{R}^2$. The three open sectors (colored red) on top, together with the open half-lines bounding them, specialize to the upward half-line (also colored red) in $\mathbb{R}^2$. The open sector (colored blue) with one side being the positive part of $L_{\mathbb{R}}$, specializes to the first quadrant in $\mathbb{R}^2$ (also colored blue).

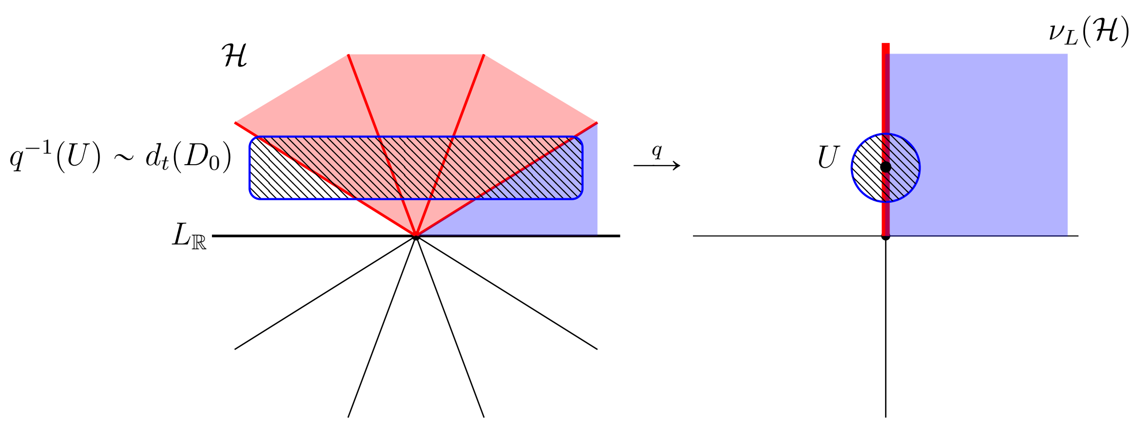

Figure 1: Specialization of faces.

The following is obvious.

Proposition 3.7. *The correspondence $A \mapsto \nu_L(A)$ defines a surjective monotone map $\nu_L : \mathcal{S}_{\mathbb{R}} \to \mathcal{S}^{\nu}_{\mathbb{R}}$ between the posets of faces of $\mathcal{H}$ and $\nu_L(\mathcal{H})$ such that* $\dim \nu_L(A) \leqslant \dim A$. □

We now form the "geometric realization" of the morphism of posets ν_L to construct a continuous map $q : V_{\mathbb{R}} \to L_{\mathbb{R}} \times (V_{\mathbb{R}}/L_{\mathbb{R}})$ from $V_{\mathbb{R}}$ to the normal bundle. That is, choose a point x_A in each face A of $\mathcal{H}$. Then we have the barycentric subdivision of V into based simplicial convex cones

$$C_{A_1,\cdots,A_p} = \mathbb{R}_{>0} \cdot x_{A_1} + \ \cdots \ + \mathbb{R}_{>0} \cdot x_{A_p}$$

corresponding to all increasing chains $A_1 < \cdots < A_p$ in $\mathcal{S}_{\mathbb{R}}$. In particular each A is the union of the $C(A_1, \cdots, A_p)$ with $A_p = A$. Similarly, choose a point y_B in each face B of $\nu_L(\mathcal{H})$. Then we have the barycentric subdivision of $L \times (V/L)$ into similarly defined based simplicial convex cones $C(B_1, \cdots, B_p)$ for all chains $B_1 < \cdots < B_p$ in $\mathcal{S}^{\nu}_{\mathbb{R}}$. For each chain $A_1 < \cdots < A_p$ we define

$$p_{A_1,\cdots,A_p} : C(A_1, \cdots, A_p) \longrightarrow C(\nu_L(A_1), \cdots, \nu_L(A_p))$$

to be the unique $\mathbb{R}$-linear map taking x_{A_i} to $y_{\nu_L(A_i)}$.

Proposition 3.8. *q is a continuous, proper, piecewise linear surjective map. Further, each face A of $\mathcal{H}$ is mapped by q to $\nu_L(A)$ in a surjective, piecewise-linear way.*

Proof: Clear from construction. □

D. The real result. In this subsection we deal only with the real situation so we write V for $V_{\mathbb{R}}$ etc. Let $\mathcal{G} \in D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ be a constructible complex.

Theorem 3.9. *The specialization $\nu_L(\mathcal{G})$ is identifed with the topological direct image $Rq_*\mathcal{G}$ where q is the map from Proposition 3.8.*

Proof: We first recall the definition ([KS2] §4.1-2) of $\nu_L(\mathcal{G})$ in terms of the *normal deformation* $\widetilde{V}_L$ which, in our linear case, reduces to a single chart.

Choose a linear complement L' to L in V so $V = L \oplus L'$. Then L' is identified with V/L and $T_L V$ is also identified with $L \oplus L'$, i.e., with V. We write a general vector of V as $v = (l, l')$ with $l \in L$ and $l' \in L'$. Then we define the commutative diagram with Cartesian squares:

(3.10)
$$\begin{tikzcd}
T_L V = V \times \{0\} \arrow[r, "s"] \arrow[d] & \widetilde{V}_L := V \times \mathbb{R} \arrow[d, "\tau"'] \arrow[rr, bend left, "p"] & \Omega \arrow[l, "j"'] \arrow[r, "\widetilde{p}"] \arrow[d, "\widetilde{\tau}"] & V \\
0 \arrow[r] & \mathbb{R} & \mathbb{R}_{>0}. \arrow[l] &
\end{tikzcd}$$

where

$$p : (l, l', t) = (l, t \cdot l'), \quad \tau(l, l', t) = t, \quad l \in L, \, l' \in L', \, t \in \mathbb{R}.$$

The space Ω is defined as $\tau^{-1}(\mathbb{R}_{>0}) = V \times \mathbb{R}_{>0}$, and $\widetilde{p}$ is the restriction of p to Ω.

After that the specialization is defined by

$$\nu_L(\mathcal{G}) \;=\; s^* Rj_* \widetilde{p}^*(\mathcal{G}) \;\in\; D^b(\mathrm{Sh}_{T_M X}).$$

Let now $\xi = (l, l')$ be a point of $L \oplus L' = T_L V = \tau^{-1}(0)$. By definition, the stalk of $\nu_L(\mathcal{G})$ at ξ is

$$\nu_L(\mathcal{G})_\xi \;=\; R\Gamma(D \cap \Omega, p^* \mathcal{G})$$

where $D \subset V \times \mathbb{R}$ is a small $(n+1)$-dimensional open ball around $(\xi, 0) = (l, l', 0)$. Now, $\Omega = V \times \mathbb{R}_{>0}$. For each $t > 0$ consider the slice $D_t = D \cap (V \times \{t\})$. The restriction of p to D_t is the dilation $d_t : (l, l') \mapsto (l, t \cdot l')$ in the direction of L'.

Since D is a ball, the intersections $D_t \cap \Omega$ are nonempty for t lying in an open interval of the form $(0, \varepsilon)$ for some $\varepsilon > 0$ (the radius of D). For such t we have that $D_t \cap \Omega = D_t$ is the slice over t. Since D is a small ball, these nonempty slices together with the complexes $d_t^* \mathcal{G}$ form a topologically trivial family over $(0, \varepsilon)$. This means that we can replace the cohomology of $D \cap \Omega$ (the union of all slices $D_t, t \in (0, \varepsilon)$) by the cohomology of any single slice, i.e.,

$$\nu_L(\mathcal{G})_\xi \;\simeq\; R\Gamma(D_t, d_t^* \mathcal{G})$$

for any suffuciently small $t > 0$. We can further replace D_t for such t with 0th slice $D_0 = D \cap (V \times \{0\})$. This slice is just a small n-dimensional open ball in $L \oplus L' = V$ around (l, l'). This gives

$$\nu_L(\mathcal{G})_\xi \;\simeq\; R\Gamma(D_0, d_t^* \mathcal{G}) \;=\; R\Gamma(d_t(D_0), \mathcal{G}), \quad 0 < t \ll 1.$$

When $t \to 0$, the open sets $d_t(D_0)$ become more and more flattened. We compare them with open sets of the form $q^{-1}(U)$ where U is a small ball in $T_L V = L \oplus L'$ around $d_t(\xi) = (l, t \cdot l')$. More precisely, we notice that $d_t(D_0)$ and $q^{-1}(U)$ become homotopy equivalent relatively to the stratification by the faces, see Fig. 1. This means that we have identifications (the last one expressing the conic nature of $Rq_*(\mathcal{G})$:

$$\nu_L(\mathcal{G})_\xi \;\simeq\; R\Gamma(q^{-1}(U), \mathcal{G}) \;=\; Rq_*(\mathcal{G})_{d_t(\xi)} \;\simeq\; Rq_*(\mathcal{G})_\xi.$$

This identifies the stalks. The same considerations show that the generalization maps between the stalks match as well. The theorem is proved. $\square$

Assume now that $\mathcal{G}$ is given by a complex of representations $G = (\mathcal{G}_A, \gamma_{AA'})$ of $\mathcal{S}_{\mathbb{R}}$. So the complexes $\mathcal{G}_A$ are the stalks of $\mathcal{G}$ and the $\gamma_{AA'}$ are the generalization maps. For any face $B \in \mathcal{S}^{\nu}_{\mathbb{R}}$ of $\nu_L(\mathcal{H})$ define a complex

(3.11)

$$\mathcal{G}_{L,B} \;=\; \mathrm{Tot}\left\{ \bigoplus_{\substack{\nu_L(A)=B\\ \dim(A)=\dim(B)}} \mathcal{G}_A \otimes \mathrm{or}_{A/B} \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\substack{\nu_L(A)=B\\ \dim(A)=\dim(B)+1}} \mathcal{G}_A \otimes \mathrm{or}_{A/B} \xrightarrow{\gamma\otimes\varepsilon} \cdots \right\}.$$

Let $B <_d B'$ be two faces of $\nu_L(\mathcal{H})$. We define a morphism of complexes

$$\gamma^L_{B,B'} : \mathcal{G}_{L,B} \to \mathcal{G}_{L,B'}$$

as follows. Let $A \in \mathcal{S}_{\mathbb{R}}$ be such that $\nu_L(A) = B$ and $\dim(A) = \dim(B) + p$, so that $\mathcal{G}_A \otimes \mathrm{or}_{A/B}$ is a summand in the pth term of $\mathcal{G}_{L,B}$. Similarly let $A' \in \mathcal{S}_{\mathbb{R}}$ be such that $\nu_L(A') = B'$ and $\dim(A') = \dim(B') + p$, so that $\mathcal{G}'_A \otimes \mathrm{or}_{A'/B'}$ is a summand in the pth term of $\mathcal{G}_{L,B'}$. If $A \leqslant A'$, then $A <_d A'$ and the identification of the quotient spaces

$$\mathrm{Lin}_{\mathbb{R}}(A')/\,\mathrm{Lin}_{\mathbb{R}}(A) \xrightarrow{\simeq} \mathrm{Lin}_{\mathbb{R}}(B')/\,\mathrm{Lin}_{\mathbb{R}}(B)$$

gives, passing to the determinants and transposing, an isomorphism

$$\sigma^*_{AA'} : \mathrm{or}_{A/B} \longrightarrow \mathrm{or}_{A'/B'}\,.$$

We define the matrix element

$$(\gamma^L_{B,B'})^{A'}_A : \mathcal{G}_A \otimes \mathrm{or}_{A/B} \longrightarrow \mathcal{G}_{A'} \otimes \mathrm{or}_{A'/B}$$

to be equal to $\gamma_{AA'} \otimes \sigma^*_{AA'}$ if $A < A'$ and to 0 otherwise.

Corollary 3.12. *Each $\gamma^L_{BB'}$ is indeed a morphism of complexes, and the data $(\mathcal{G}_{L,B}, \gamma^L_{BB'})$ is a complex of representations of $\mathcal{S}_{\nu,\mathbb{R}}$, the poset of faces of the arrangement $\nu_L(\mathcal{H})$. This complex of representations describes the constructible complex $\nu_L(\mathcal{G})$.*

Proof: Choose any point $b \in B$. Since q is a proper map, the stalk of $Rq_*(\mathcal{G})$ at b is identified with $R\Gamma(q^{-1}(b), \mathcal{G})$. Now $\mathcal{G}_{L,B}$ is nothing but the cellular cochain complex calculating $R\Gamma(q^{-1}(b), \mathcal{G})$. We similarly identify the generalization maps. $\square$

Remark 3.13. At the formal algebraic level, the property that $\gamma^L_{BB'}$ is indeed a morphism of complexes, simply reflects the fact that the differential in $R\Gamma(V,\mathcal{G})$, the cellular cochain complex, satisfies $d^2=0$. More precisely, we have an identification (isomorphism, not just a quasi-isomorpism) of cellular cochain complexes

$$R\Gamma(V,\mathcal{G}) \;\simeq\; R\Gamma(L\times(V/L), Rq_*(\mathcal{G})) \;\;\simeq\; R\Gamma(L\times(V/L),\nu_L(\mathcal{G})).$$

The RHS of this identification represents the same complex in a "block" form, with blocks (stalks of $\nu_L(\mathcal{G})$) parametrized by faces B of $\nu_L(\mathcal{H})$. The fact that the maps $\gamma^L_{BB'}$ between the blocks are morphisms of complexes is implied by the fact that the total differential squares to 0.

E. Bispecialization. We first consider the general situation studied in [ST] [T]. Let $N\subset M\subset X$ be a flag of C^∞ submanifolds in a C^∞ manifold X. In the normal bundle T_NX we have the submanifold (subbundle) T_NM. In the normal bundle T_MX we have the submanifold N, emdedded into M (the zero section of T_MX). It turns out that the normal bundles of these new submanifolds are identified.

Proposition 3.14. *We have identifications*

$$T_{T_NM}(T_NX) \overset{(1)}{\simeq} T_NM\oplus(T_MX)|_N \overset{(2)}{\simeq} T_N(T_MX).$$

Proof: The statement is a part of Prop. 2.1 of [T]. For convenience of the reader we give a sketch of the proof. The identification (1) is a particular case of the well known fact which generalizes, to vector bundles, the identification (3.2) for vector spaces: If $L\subset V$ is a C^∞ vector subbundle in a C^∞ vector bundle over a C^∞-manifold B, then $T_LV\simeq L\oplus(V/L)$. To see (2), we recognize, inside $T_N(T_MX)$ two subbundles: first, T_NM (the normal bundle to N inside the zero section of T_MX), and, second $(T_MX)|_N$ (the restriction to N of the normal bundle). Inspection in local coordinates shows that these two subbundles form a direct sum decomposition. □

In this context Schapira and Takeuchi [ST] [T] defined a functor

$$\nu_{NM}: D^b(X)\longrightarrow D^b(T_N\oplus(T_XM)|_N)$$

called *bispecialization*. It is defined, similarly to the usual specialization, through the *binormal deformation* $\widetilde{X}_{NM}$, recalled below. On the other hand, we can iterate the specialization functors, getting a diagram of functors between derived categories of sheaves on the manifolds in question:

$$
\begin{tikzcd}
D^b(X) \arrow[r, "\nu_N"] \arrow[d, "\nu_M"'] \arrow[dr, "\nu_{NM}"] & D^b(T_N X) \arrow[d, "\nu_{T_N M}"] \\
D^b(T_M X) \arrow[r, "\nu_N"'] & D^b(T_N M \oplus (T_M X)|_N).
\end{tikzcd} \tag{3.15}
$$

This diagram is not (2-)commutative, i.e., the two composite functors (iterated specializations) are not isomorphic.

Example 3.16. Let $X = \mathbb{R}^2$ with coordinates x, y, let M be the line $y = 0$ and N be the origin $(0, 0)$. Let $P \subset X$ be the parabola $y = x^2$ and $\mathcal{G} = \underline{\mathbf{k}}_P$ be the constant sheaf on P. We identify all three manifolds $T_N X$, $T_M X$ and $T_N M \oplus (T_M X)|_N$ back with $\mathbb{R}^2$ with the same coordinates. Then $\nu_N(\mathcal{G})$ is the constant sheaf on the horizontal line $y = 0$ (the tangent line to P), and $\nu_{T_N M}(\nu_N(\mathcal{G}))$ is again the constant sheaf on the line $y = 0$. On the other hand, $\nu_M(\mathcal{G})$ is supported on the vertical half-line $x = 0, y \geqslant 0$ (since P is contained in the upper half plane $y \geqslant 0$ and does not meet M except for $x = 0$). So $\nu_N(\nu_M(\mathcal{G}))$ will be again supported on this half-line.

Nevertheless, in the linear case all three possible functors are identified.

Theorem 3.17. *Let $X = V$ be an $\mathbb{R}$-vector space and $N \subset M \subset V$ be a flag of $\mathbb{R}$-linear subspaces. Let $\mathcal{H}$ be an arrangement of hyperplanes in V and $\mathcal{S}_{\mathbb{R}}$ the corresponding stratification by faces. Then for $\mathcal{G} \in D^b(V, \mathcal{S}_{\mathbb{R}})$ we have canonical quasi-isomorphisms*

$$\nu_N(\nu_M(\mathcal{G})) \;\simeq\; \nu_{T_N M}(\nu_N(\mathcal{G})) \;\simeq\; \nu_{NM}(\mathcal{F}).$$

In other words, the diagram (3.15) *becomes 2-commutative if the top left corner is replaced by $D^b(V, \mathcal{S}_{\mathbb{R}})$.*

Proof: Enlarging $\mathcal{H}$ if necessary, we can assume that N and M are flats of $\mathcal{H}$. The space $T_N M \oplus (T_M X)|_N$ is identified with vector space $V'' = N \oplus (M/N) \oplus (V/M)$ which carries the *triple product arrangement*

$$\nu_{NM}(\mathcal{H}) := (\mathcal{H} \cap N) \oplus ((\mathcal{H} \cap M)/N) \oplus (\mathcal{H}/M).$$

Denote by $\mathcal{S}_{\mathbb{R}}^{\nu_{NM}}$ the stratification given by the faces of this arrangement. Also denote $\mathcal{S}_{\mathbb{R}}^{\nu_N}$ and $\mathcal{S}_{\mathbb{R}}^{\nu_M}$ the stratifications given by the faces of the arrangements $\nu_N(\mathcal{H})$ and $\nu_M(\mathcal{H})$. Now notice that specialization of faces gives a *commutative* diagram of morphisms of posets which we then use to construct a commutative diagram of proper piecewise linear maps:

$$\begin{tikzcd} \mathcal{S}_{\mathbb{R}} \arrow[r, "\nu_N"] \arrow[d, "\nu_M"'] & \mathcal{S}_{\mathbb{R}}^{\nu_N} \arrow[d, "\nu_{T_N M}"] \\ \mathcal{S}_{\mathbb{R}}^{\nu_M} \arrow[r, "\nu'_N"'] & \mathcal{S}_{\mathbb{R}}^{\nu_{NM}} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "q_N"] \arrow[d, "q_M"'] & T_N V \arrow[d, "q_{T_N M}"] \\ T_M V \arrow[r, "q'_N"] & T_N M \oplus (T_M V)|_N . \end{tikzcd}$$

The direct images in this second diagram correspond, by Theorem 3.9, to the specialization functors on the outer edges of the diagram (3.15). This shows that the outer rim of (3.15) is 2-commutative.

We now show that the composite functor given by the outer rim of (3.15), is isomorphic to ν_{NM}. (This will also give another proof of the commutativity of the outer rim.) For this we recall the explicit form of the binormal deformation diagram, see [T] Eq. (2.20). We choose a complement L' to N in M and a complement L'' to M in V, thus identifying V, as well as $T_N M \oplus (T_M V)|_N$, with $N \oplus L' \oplus L''$. So we write elements of either of this spaces as (n, l', l''). Then the "bi"-analog of the diagram (3.10) has the form

$$\begin{tikzcd} T_N M \oplus (T_M V)|_N = V \times \{(0,0)\} \arrow[r, "s"] \arrow[d] & \widetilde{V}_{NM} = V \times \mathbb{R}^2 \arrow[d, "\tau"'] \arrow[rr, bend left, "p"] & \Omega \arrow[l, "j"'] \arrow[r, "\widetilde{p}"] \arrow[d, "\widetilde{\tau}"] & V \\ 0 \arrow[r] & \mathbb{R}^2 & \mathbb{R}^2_{>0}, \arrow[l] & \end{tikzcd}$$

with

$$p((n, l', l''), (t', t'')) \;=\; (n,\, t'l',\, t't''l''), \quad \tau((n, l', l''), (t', t'')) = (t', t''),$$

so the restriction of p to $V \times \{(t', t'')\}$ is the map

$$p_{(t',t'')} : (n, l', l'') \mapsto (n,\, t'l',\, t't''l'').$$

The bispecialization is defined as $\nu_{NM}(\mathcal{G}) = s^* R j_* \widetilde{p}^* \mathcal{G}$ with respect to this diagram, so its stalk at (n, l', l'') is $R\Gamma(D \cap \Omega, p^*\mathcal{G})$ where D is a small open $(n+2)$-dimensional ball around $((n, l', l''), (0, 0))$. We slice D into n-dimensional balls $D_{(t',t'')} = D \cap \tau^{-1}(t', t'')$.

Lemma 3.18. *For sufficiently small $\varepsilon > 0$, the slices $D_{(t',t'')}$ together with the restrictions $p^*\mathcal{G}|_{D_{(t',t'')}} = p^*_{(t',t'')}\mathcal{G}$, form a topologically trivial family over the product of open intervals $(0,\varepsilon) \times (0,\varepsilon)$.*

Proof of the lemma: For $u', u'' > 0$ let

$$d_{(u',u'')} : V \to V, \quad (n, l', l'') \mapsto (n,\, u'l',\, u''l'')$$

be the bi-dilation in the last two variables. Then $p_{(t',t'')} = d_{c(t',t'')}$, where $c : \mathbb{R}^2 \to \mathbb{R}^2$ is the map

$$(t', t'') \mapsto (u', u'') = (t',\, t't'').$$

Now, c maps the open square $(0,\varepsilon)^2$ homeomorphically onto the open triangular wedge ∇_ε of slope ε, see Fig. 2.

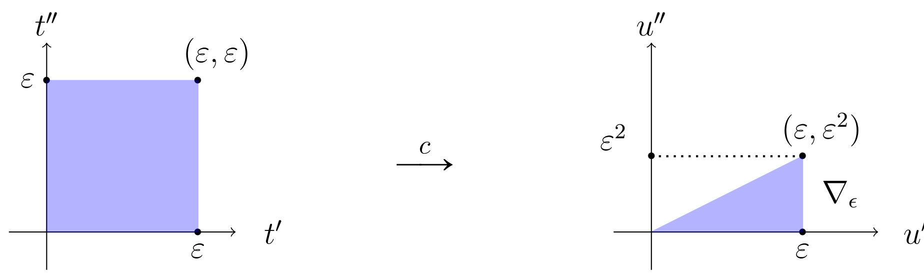

Figure 2: The wedge ∇_ϵ.

For small $t', t'' > 0$ we can identify the slices $D_{(t',t'')}$ with $D_{(0,0)}$ (alternatively, we could have taken D to be the product of balls in V and in $\mathbb{R}^2$ so that the slices would not change at all).

We recall that $\mathcal{G}$ is smooth with respect to a hyperplane arrangement $\mathcal{H}$ (so the slopes of the hyperplanes are fixed). On the other hand, the slope of the wedge ∇_ϵ is shrinking as $\varepsilon \to 0$. Therefore, for sufficiently small ε we will have that for all $(u', u'') \in \nabla_\epsilon$ the topological structure of $d^*_{(u',u'')}\mathcal{G}$ on $D_{(0,0)}$ will stabilize. This proves the lemma. $\square$

The lemma implies that the stalk of $\nu_{NM}(\mathcal{G})$ at (n, l', l'') can be written as

$$R\Gamma(D_{0,0}, p^*_{(t',t'')}\mathcal{G}) \;=\; R\Gamma(p_{(t',t'')}(D_{0,0}), \mathcal{G})$$

for any sufficiently small positive t', t''.

It remains to similarly analyze the two outer composite functors (iterated specializations) in (3.15) and to find that they correspond to the choice of $0 < t' \ll t'' \ll 1$, resp. $0 < t'' \ll t' \ll 1$. Because of the topological triviality of the family over all $(t', t'') \in (0, \varepsilon) \times (0, \varepsilon)$, all three results are the same. $\square$

F. The complex result. We now consider the complex situation: that of a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ and the corresponding hyperbolic sheaf $\mathcal{Q} = (E_A, \gamma_{AA'}, \delta_{A'A})$. Let $\mathcal{Q}^\nu = (E_B^\nu, \gamma_{BB'}^\nu, \delta_{B'B}^\nu)$ be the hyperbolic sheaf corresponding to $\nu_{L_{\mathbb{C}}}(\mathcal{F}) \in \mathrm{Perv}(T_{L_{\mathbb{C}}} V_{\mathbb{C}}, \mathcal{S}_{\nu,\mathbb{C}})$. Here B, B' are faces of the product arrangement $\nu_L(\mathcal{H})$.

Theorem 3.19. *(a) The hyperbolic stalk E_B^ν is identified as*

$$E_B^\nu \simeq \left\{ \bigoplus_{\substack{\nu_L(A)=B \\ \dim(A)=\dim(B)}} E_A \otimes \mathrm{or}_{A/B} \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\substack{\nu_L(A)=B \\ \dim(A)=\dim(B)+1}} E_A \otimes \mathrm{or}_{A/B} \xrightarrow{\gamma\otimes\varepsilon} \cdots \right\}$$

That is, the complex in the RHS is exact everywhere except the leftmost term, where the kernel is identified with E_B^ν.

(a') We also have an identification

$$E_B^\nu \simeq \left\{ \cdots \xrightarrow{\delta\otimes\varepsilon} \bigoplus_{\substack{\nu_L(A)=B \\ \dim(A)=\dim(B)+1}} E_A \otimes \mathrm{or}_{A/B} \xrightarrow{\delta\otimes\varepsilon} \bigoplus_{\substack{\nu_L(A)=B \\ \dim(A)=\dim(B)+1}} E_A \otimes \mathrm{or}_{A/B} \right\}.$$

That is, the complex in the RHS is exact everywhere except the rightmost term, where the cokernel is identified with E_B^ν.

(b) The maps $\gamma_{BB'}^\nu$ are induced by the maps $\gamma_{AA'}$ which induce morphisms of complexes in (a), similarly to Corollary 3.12.

(b') The maps $\delta_{B'B}^\nu$ are induced by the map $\delta_{A'A}$ which induce morphisms of complexes in (a').

Proof: We first prove parts (a) and (b). Let $i_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{C}}$ and $i_{\mathbb{R},\nu} : T_{L_{\mathbb{R}}} V_{\mathbb{R}} \to T_{L_{\mathbb{C}}} V_{\mathbb{C}}$ be the embeddings of the real parts. Put

$$\mathcal{G} = i_{\mathbb{R}}^! \mathcal{F}, \quad \mathcal{G}_\nu = i_{\mathbb{R},\nu}^! \, \nu_{L_{\mathbb{C}}}(\mathcal{F}).$$

These are ordinary sheaves (not just complexes) on $V_{\mathbb{R}}$ and $T_{L_{\mathbb{R}}} V_{\mathbb{R}}$, smooth with respect to $\mathcal{S}_{\mathbb{R}}$ and $\mathcal{S}_{\mathbb{R}}^\nu$ respectively. Their stalks are given by the E_A and

E^ν_B and their generalization maps are given by the $\gamma_{AA'}$ and $\gamma^\nu_{BB'}$ respectively. Note that we have a canonical morphism

$$\nu_{L_\mathbb{R}}(\mathcal{G}) \;=\; \nu_{L_\mathbb{R}}(i^!_\mathbb{R}\mathcal{F}) \xrightarrow{\beta} i^!_{\mathbb{R},\nu}\nu_{L_\mathbb{C}}(\mathcal{F}) \;=\; \mathcal{G}_\nu,$$

see [KS2] Prop. 4.2.5. So our statements will follow from Corollary 3.12 if we establish the following.

Proposition 3.20. *For any $\mathcal{F} \in D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$, the morphism $\beta : \nu_{L_\mathbb{R}}(i^!_\mathbb{R}\mathcal{F}) \to i^!_{\mathbb{R},\nu}\nu_{L_\mathbb{C}}(\mathcal{F})$ is a quasi-isomorphism.*

Proof of Proposition 3.20: Since $\nu_{L_\mathbb{R}}$ and $\nu_{L_\mathbb{C}}$ commute with Verdier duality, it is enough to show that for any $\mathcal{F} \in D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$, the dual morphism $\alpha : i^*_{\mathbb{R},\nu}\nu_{L_\mathbb{C}}(\mathcal{F}) \to \nu_{L_\mathbb{R}}(i^*_\mathbb{R}\mathcal{F})$ is a quasi-isomorphism. Such a morphism is defined for any $\mathcal{F} \in D^b(\mathrm{Sh}_{V_\mathbb{C}})$ whatsoever, see [KS2] Prop. 4.2.5. So we show that it is a quasi-isomorphism for a more general class of complexes. Namely, $V_\mathbb{C}$ has the *product stratification* $\mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}$ formed by the cells of the form $A' + iA'' \subset V_\mathbb{C} = V_\mathbb{R} + iV_\mathbb{R}$, where A' and A'' are arbitrary faces of $\mathcal{H}$ and $i = \sqrt{-1}$. This stratification refines $\mathcal{S}_\mathbb{C}$, so $D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{C}) \subset D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R})$. Therefore it suffices to prove:

Lemma 3.21. *For any $\mathcal{F} \in D^b(V_\mathbb{C}, \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R})$, the morphism $\alpha : i^*_{\mathbb{R},\nu}\nu_{L_\mathbb{C}}(\mathcal{F}) \to \nu_{L_\mathbb{R}}(i^*_\mathbb{R}\mathcal{F})$ is a quasi-isomorphism.*

Proof of Lemma 3.21: The stratification on $V_\mathbb{R}$ induced by $i_\mathbb{R}$ from $\mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}$, is $\mathcal{S}_\mathbb{R}$. This means that the specializations maps of the posets of of faces are compatible, and therefore we have a commutative diagram

$$\begin{tikzcd} V_\mathbb{R} \arrow[r, "q_\mathbb{R}"] \arrow[d, "i_\mathbb{R}"'] & L_\mathbb{R} \times (V_\mathbb{R}/L_\mathbb{R}) \arrow[d, "i_{\mathbb{R},\nu}"] \\ V_\mathbb{C} \arrow[r, "q_\mathbb{C}"'] & L_\mathbb{C} \times (V_\mathbb{C}/L_\mathbb{C}), \end{tikzcd}$$

where $q_\mathbb{R}$ and $q_\mathbb{C}$ are the proper maps constructed in Proposition 3.8. So our statement follows from Theorem 3.9 by proper base change. $\square$

This finishes the proof of Proposition 3.20 and of parts (a) and (b) of Theorem 3.19.

Now, parts (a') and (b') of Theorem 3.19 follow from (a) and (b) because $\nu_{L_\mathbb{C}}$ commutes with Verdier duality whose effect on hyperbolic sheaves exchanges γ and δ, see Theorem 1.11. Theorem 3.19 is proved.

4 Fourier transform and hyperbolic sheaves

A. Generalities on the Fourier-Sato transform. Let W be a finite-dimensional $\mathbb{R}$-vector space and W^* the dual space. We denote by $D^b_{\mathrm{con}}(E) \subset D^b(E)$ the full subcategory formed by complexes $\mathcal{G}$ which are *conic*, i.e., such that each sheaf $\underline{H}^j(\mathcal{G})$ is locally constant on any orbit of the scaling action of $\mathbb{R}_{>0}$ on W.

Set

$$P \;=\; \big\{(x,f)\in W\times W^* \,\big|\, f(x)\geqslant 0\big\} \xhookrightarrow{i_P} W\times W^*$$

and denote by p_1, p_2 the projections of P to W and W^* respectively. The *Fourier-Sato transform* is an equivalence of categories

$$\mathrm{FS}: D^b_{\mathrm{con}}(W) \longrightarrow D^b_{\mathrm{con}}(W^*), \quad \mathrm{FS}(\mathcal{G}) \;=\; Rp_{2!}(p_1^*\mathcal{G}),$$

see [KS2] Def. 3.7.8. The base change theorem implies at once the following.

Proposition 4.1. *Let $f\in W^*$. The stalk of* $\mathrm{FS}(\mathcal{G})$ *at f is found as*

$$\mathrm{FS}(\mathcal{G})_f \;\simeq\; R\Gamma_c(P_f,\mathcal{G}),$$

where $P_f = p_2^{-1}(f) = \{x\in W | f(x)\geqslant 0\}$. (Thus P_f is a closed half-space for $f\neq 0$ and $P_f = W$ for $f=0$.)

□

B. The dual arrangement. We specialize the above to the two situations related to an arrangement of hyperplanes $\mathcal{H}$ in $V_\mathbb{R}$. We denote $n=\dim_\mathbb{R} V_\mathbb{R}$.

(1) $W = V_\mathbb{R}$ and $\mathcal{G}\in D^b(V_\mathbb{R},\mathcal{S}_\mathbb{R})$. In this case we would like to find the stalks of $\mathrm{FS}(\mathcal{G})$.

(2) $W = V_\mathbb{C}$ and $\mathcal{G}\in \mathrm{Perv}(V_\mathbb{C},\mathcal{S}_\mathbb{C})$. We identify $W^* = \mathrm{Hom}_\mathbb{C}(V,\mathbb{C})$ with the real dual $\mathrm{Hom}_\mathbb{R}(V_\mathbb{C},\mathbb{R})$ by means of the form

$$(x,f)\mapsto \Re(f(x)), \quad x\in V_\mathbb{C}, f\in V_\mathbb{C}^*.$$

In this case it is known, see [KS2] Ch. X, that $\mathrm{FS}(\mathcal{G})[-n]$ is a perverse sheaf on $V_\mathbb{C}^*$ with respect to some stratification. We would like to relate this stratification to an arrangement of hyperplanes and to find the hyperbolic stalks of $\mathrm{FS}(\mathcal{G})$.

This leads to the following definition.

Definition 4.2. The *dual arrangement* $\mathcal{H}^\vee$ of hyperplanes in $V_\mathbb{R}^*$ consists of orthogonals $l^\perp$ where l is a 1-dimensional flat of $\mathcal{H}$. We denote by $\mathcal{S}_\mathbb{R}^\vee$ the stratification of $V_\mathbb{R}^*$ into faces of $\mathcal{H}^\vee$ and by $\mathcal{S}_\mathbb{C}^\vee$ the stratification of $V_\mathbb{C}^*$ into generic parts of the complex flats of $\mathcal{H}^\vee$.

Proposition 4.3. *We have an inclusion $\mathcal{H} \subset \mathcal{H}^{\vee\vee}$ (as sets of hyperplanes in $V_\mathbb{R}$).*

Proof: 1-dimensional flats of $\mathcal{H}^\vee$ are the orthogonals $M^\perp$, where M runs over hyperplanes in V which are sums of 1-dimensional flats of $\mathcal{H}$. Such M are therefore, precisely the hyperplanes of $\mathcal{H}^{\vee\vee}$. Now the statement means that each hyperplane $H \in \mathcal{H}$ can be obtained as a sum of 1-dimensional flats of $\mathcal{H}$. This is indeed the case, since we have assumed from the outset that $\mathcal{H}$ is central, i.e., the intersection of all $H \in \mathcal{H}$ is 0. $\square$

Examples 4.4. (a) Call an arrangement $\mathcal{H}$ *reflexive*, if $\mathcal{H}^{\vee\vee} = \mathcal{H}$. A sufficient condition for this is that the set of flats of $\mathcal{H}$ is closed not only under intersections but also under sums, i.e., it forms a lattice. This follows from the proof of Proposition 4.3. Examples of reflexive arrangements include any arrangement with $\dim(V) \leqslant 2$, as well as any direct sum of such arrangements.

(b) In general, forming the union of the arrangements

$$\mathcal{H} \subset \mathcal{H}^{\vee\vee} \subset \mathcal{H}^{\vee\vee\vee\vee} \subset \cdots$$

amounts to closing $\mathcal{H}$ under the operations of sum and intersection, i.e., to forming the lattice of subspaces generated by $\mathcal{H}$ and taking all $(n-1)$-dimensional elements of it. Such a lattice (and therefore the above union) is typically infinite. For instance, for $n = 3$ we start with a finite set of lines in $\mathbb{R}P^2$, form all their intersection points, then draw new lines through these points and so on.

(c) Let $V_\mathbb{R} = \mathbb{R}^n$ with coordinates $x_1, \cdots, x_n$. Take $\mathcal{H}$ to be the arrangement of the following hyperplanes:

$$\{x_i = 0\},\ i = 1, \cdots, n, \quad \{x_i = x_{i+1}\},\ i = 1, \cdots, n-1.$$

There are $\binom{n+1}{2}$ one-dimensional flats of $\mathcal{H}$, they have the form

$$L_{[i,j]} = \big\{x \,\big|\, x_i = x_{i+1} = \cdots = x_j;\ x_k = 0, k \notin [i,j]\big\}, \quad 1 \leqslant i \leqslant j \leqslant n.$$

On the other hand, consider $\mathbb{R}^{n+1}$ with coordinates $y_0, \cdots, y_n$ and let $W_{\mathbb{R}} = \mathbb{R}^{n+1}/\mathbb{R}\cdot(1,\cdots,1)$. Thus $W_{\mathbb{R}} = \mathfrak{h}^*$ is the space of weights for the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{R})$. We have an isomorphism $V_{\mathbb{R}} \to W_{\mathbb{R}}^*$ which takes the ith basis vector $e_i \in V_{\mathbb{R}}$, $i = 1, \cdots, n$, to the functional $y \mapsto y_{i-1} - y_i$ (simple co-root). This isomorphism takes $L_{[i,j]}$ to the co-root hyperplane $\{y_{i-1} = y_j\}$. Therefore the dual arrangement $\mathcal{H}^\vee$ is the co-root arrangement in $\mathfrak{h}^*$.

Next, flats of $\mathcal{H}^\vee$ are in bijection with equivalence relations R on the set $\{0, 1, \cdots, n\}$. The flat corresponding to R has the form

$$M_R \;=\; \big\{ y \,\big|\, y_i = y_j \text{ whenever } i \equiv_R j. \big\}.$$

It is one-dimensional if and only if R has only 2 equivalence classes, both non-empty. Thus there are $2^{n-1} - 1$ one-dimensional flats of $\mathcal{H}^\vee$ and so the double dual arrangement $\mathcal{H}^{\vee\vee}$ consists of $2^{n-1} - 1$ hyperplanes and is much bigger than $\mathcal{H}$.

Proposition 4.5. *(a) If $\mathcal{G} \in D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$, then $\mathrm{FS}(\mathcal{G}) \in D^b(V_{\mathbb{R}}^*, \mathcal{S}_{\mathbb{R}}^\vee)$.*

(b) If $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$, then $\mathrm{FS}(\mathcal{F})[-n] \in \mathrm{Perv}(V_{\mathbb{C}}^*, \mathcal{S}_{\mathbb{C}}^\vee)$.

Proof: As in the proof of Proposition 3.4, the real and complex case are completely parallel, so we treat the real case, dropping the subscript $\mathbb{R}$. The microsupport of $\mathcal{G}$ is contained in the union of the $T_L^*V = L \times L^\perp$ over $L \in \mathrm{Fl}(\mathcal{H})$. Now, the effect of FS on microsupports is via the identification ("Legendre transform")

$$T^*V \;=\; V \times V^* \longrightarrow V^* \times V \;=\; T^*V^*.$$

This identification takes T_L^*V to $T_{L^\perp}^*V^*$. This means that $\mathrm{FS}(\mathcal{G})$ is smooth with respect to the stratification $\mathcal{S}^*$ formed by the generic parts

$$L_\circ^\perp \;=\; L^\perp \setminus \bigcup_{L_1^\perp \not\supset L^\perp} L_1^\perp, \quad L \in \mathrm{Fl}(\mathcal{H}).$$

Now, $\mathcal{S}^\vee$ refines $\mathcal{S}^*$, so $\mathrm{FS}(\mathcal{G})$ is smooth with respect to $\mathcal{S}^\vee$. $\square$

C. Big and small dual cones. Let $A^\vee \in \mathcal{S}_{\mathbb{R}}^\vee$ be a face. Its *big dual cone* is defined as

$$U(A^\vee) \;=\; \big\{ x \in V \,\big|\, f(x) \geqslant 0, \; \forall f \in A^\vee \big\} \;\subset\; V_{\mathbb{R}}. \tag{4.6}$$

It is a closed polyhedral cone in V with nonempty interior, the union of the closures of (in general, several) chambers of $\mathcal{H}$.

The *small dual cone* of $A^\vee$ is defined as

$$V(A^\vee) \;=\; \bigcap_{B^\vee \geqslant A^\vee \text{ chamber}} U(B^\vee). \tag{4.7}$$

It is a strictly convex (not containing $\mathbb{R}$-linear subspaces) closed polyhedral cone in $V_{\mathbb{R}}$. Note that $U(A^\vee) = V(A^\vee)$ if $A^\vee$ is a chamber but $U(A^\vee)$ can be strictly larger than $V(A^\vee)$ in general. For example, if $A^\vee$ is a half-line (1-dimensional face) of $\mathcal{H}^\vee$, then $U(A^\vee)$ is a closed half-space in $V_{\mathbb{R}}$, while V_A is strictly convex, cf. Fig. 3.

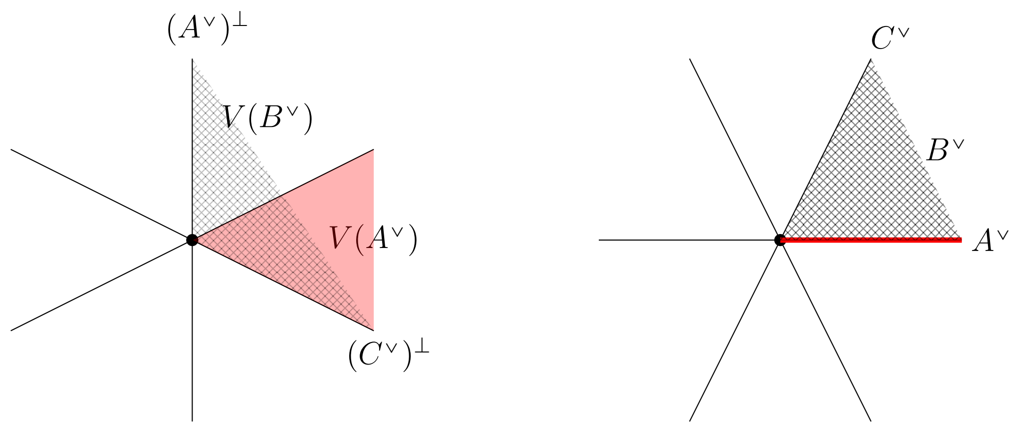

Figure 3: Small dual cones.

The next statement is clear from the definitions.

Proposition 4.8. *If $A_1^\vee \leqslant A_2^\vee$, then $U(A_1^\vee) \supset U(A_2^\vee)$ and $V(A_1^\vee) \subset V(A_2^\vee)$.* □

Proposition 4.9. *Let $f \in A^\vee$ be arbitrary. Then:*

(a) $U(A^\vee)$ is the union of all faces B of $\mathcal{H}$ such that $f|_B \geqslant 0$ (non-strict inequality).

(b) $V(A^\vee)$ is the union of 0 and all the faces B of $\mathcal{H}$ such that $f|_B > 0$ (strict inequality everywhere).

Proof: (a) Since $A^\vee$ is a face of $\mathcal{H}^\vee$, for each $f \in A^\vee$ the pattern of signs (positive, negative or zero) of f on faces of $\mathcal{H}$ is the same. So the requirement that $f|_B \geqslant 0$ for each $f \in A^\vee$ (appearing in the definition of $U(A^\vee)$) is equivalent to the requirement that $f|_B \geqslant 0$ for any partiular choice of $f \in A^\vee$ (appearing in the statement of the proposition).

(b) Let V' be the union of the faces in question. If $B \neq 0$ is a face of $\mathcal{H}$ such that $B \subset V'$, i.e., that $f|_B > 0$, then $g|_B > 0$ for any $g \in A^\vee$, by definition of the dual arrangement. This means that for any $B^\vee \geqslant A^\vee$ and any $g \in B^\vee$ sufficiently close to $A^\vee$ we still have $g|_B > 0$. This, further implies (again, by the definition of the dual arrangement) that for any $B^\vee \geqslant A^\vee$ and any $g \in B^\vee$ whatsoever we still have $g|_B > 0$. This means that $B \subset V(B^\vee)$ for any $B^\vee \geqslant A^\vee$, in other words, that $B \subset V(A^\vee)$. We proved that $V' \subset V(A^\vee)$.

Conversely, suppose $B \subset V(A^\vee)$. For any chamber $B^\vee \subset A^\vee$ and any $g \in B^\vee$ the restriction $g|_B$ cannot vanish, since that would mean that g is not inside a chamber of a dual arrangement. Therefore $g|_B > 0$ everywhere. Now, if $f \in A^\vee$ and $A^\vee$ is not a chamber, then looking at g varying in a small transverse ball to $A^\vee$ near f in $V_\mathbb{R}^*$, we see that all such $g|_B$ must be positive and therefore $f|_B$ must be positive. In other words, we proved that $V(A^\vee) \subset V'$.

□

Corollary 4.10. *We have*

$$U(A^\vee) \;=\; \bigcup_{B^\vee \geqslant A^\vee} U(B^\vee) \;=\; \bigcup_{B^\vee \geqslant A^\vee} V(B^\vee).$$

□

We now analyze the nature of the covering of $U(A^\vee)$ by the $U(B^\vee)$, $B^\vee \geqslant A^\vee$. All $B^\vee \geqslant A^\vee$ are in bijection with faces of the *quotient arrangement* $\mathcal{H}^\vee/A^\vee$ in the quotient space $V_\mathbb{R}^*/\mathrm{Lin}_\mathbb{R}(A^\vee)$, cf. [KS1] §2B. We denote by $B^\vee/A^\vee$ the face of $\mathcal{H}^\vee/A^\vee$ corresponding to $B^\vee \geqslant A^\vee$.

Proposition 4.11. *Let $A^\vee \in \mathcal{S}_\mathbb{R}^\vee$ and $B \in \mathcal{S}_\mathbb{R}$. Then:*

(a) There is a closed convex polyhedral cone $K(A^\vee, B) \subset V_\mathbb{R}^*/\mathrm{Lin}_\mathbb{R}(A^\vee)$, a union of faces of $\mathcal{H}^\vee/A^\vee$, which has the following property:

For $B^\vee \geqslant A^\vee$ we have $B \subset U(B^\vee)$ if and only $B^\vee/A^\vee \subset K(A^\vee, B)$.

(b) The cone $K(A^\vee, B)$ coincides with the whole $V_\mathbb{R}^*/\mathrm{Lin}_\mathbb{R}(A^\vee)$ if and only if $B \subset V(A^\vee)$.

Proof: (a) Let $U(B) \subset V_{\mathbb{R}}^*$ be the dual cone to B, i.e., the set of $f \in V_{\mathbb{R}}^*$ such that $f|_B \geqslant 0$. It is a convex, closed polyhedral cone in $V_{\mathbb{R}}^*$ which is a union of faces of $\mathcal{H}^\vee$. In fact, the condition $B^\vee \subset U(B)$ is equivalent to $B \subset U(B^\vee)$, both meaning that $(b^\vee, b) \geqslant 0$ for each $b^\vee \in B^\vee$ and $b \in B$.

Let also $(V_{\mathbb{R}}^*)^{\geqslant A^\vee} \subset V_{\mathbb{R}}^*$ be the union of all faces $B^\vee$ of $\mathcal{H}^\vee$ such that $B^\vee \geqslant A^\vee$. It is a convex, open polyhedral cone in $V_{\mathbb{R}}^*$. The intersection $U(B) \cap (V_{\mathbb{R}}^*)^{\geqslant A^\vee}$ is then a convex polyhedral cone which is closed in $(V_{\mathbb{R}}^*)^{\geqslant A^\vee}$. Since this cone is a union of faces $B^\vee \geqslant A^\vee$, it projects to a convex closed polyhedral cone in $V_{\mathbb{R}}^*/\mathrm{Lin}_{\mathbb{R}}(A^\vee)$ which we denote $K(A^\vee, B)$. By construction, $K(A^\vee, B)$ satisfies the required property.

(b) This is a reformulation of the formula (4.7) defining $V(A^\vee)$. $\square$

Note an appealing numerical corollary of Proposition 4.11. For any subset $Z \subset V$ we denote by $\mathbf{1}_Z : V \to \mathbb{R}$ its characteristic function, equal to 1 on Z and to 0 elsewhere.

Corollary 4.12. *(inclusion - exclusion formulas) We have the identities*

$$(a) \qquad \mathbf{1}_{U(A^\vee)} = \sum_{B^\vee \supset A^\vee} (-1)^{\dim(B^\vee) - \dim(A^\vee)} \mathbf{1}_{V(A^\vee)},$$

$$(b) \qquad \mathbf{1}_{V(A^\vee)} = \sum_{B^\vee \supset A^\vee} (-1)^{\dim(B^\vee) - \dim(A^\vee)} \mathbf{1}_{U(A^\vee)}.$$

Identities of this general nature (representing the characteristic function of a convex polytope as an alternating sum of characteristic functions of simplices or cones) are familiar in the theory of convex polytopes [V] [FL] and the theory of automorphic forms, see, e.g., [Ar], §11.

We note the similarity of the identities (a) and (b) with Proposition 1.9(b) and Corollary 1.14 relating the usual stalks and hyperbolic stalks of a perverse sheaf. In fact, we will use a "categorified" version of these identities to relate the usual and hyperbolic stalks of the Fourier-Sato transform.

Proof of Corollary 4.12 : (b) Write the RHS of the proposed identity as $\sum_B c_B \mathbf{1}_B$ with B running over faces of $\mathcal{H}$. Part (a) of Proposition 4.11 implies that

$$c_B = \sum_{\substack{B^\vee \geqslant A^\vee \\ B^\vee/A^\vee \subset K(A^\vee, B)}} (-1)^{\dim(B^\vee) - \dim(A^\vee)} = (-1)^{n - \dim(A^\vee)} \chi\big(H_c^\bullet(K(A^\vee, B), \mathbf{k}\big)$$

is the signed (calculated from the top) Euler characteristic of the cohomology with compact support of the cone $K(A^\vee, B)$. This signed Euler characteristic is equal to 0 unless $K(A^\vee, B)$ is the entire vector space, in which case it is 1. By Part (b) of Proposition 4.11 this happens precisely when $B \subset V(A^\vee)$, so the identify is proved.

(a) is a formal consequence of (b) in virtue of the identity

$$\sum_{B^\vee \geqslant A^\vee} (-1)^{\dim(B^\vee) - \dim(A^\vee)} \;=\; 1$$

(the Euler characteristic of the link of $A^\vee$). $\square$

D. The real result.

Theorem 4.13. *Let $\mathcal{G} \in D^b(V_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ be represented by a complex $(\mathcal{G}_A, \gamma_{AB})$ of representations of $(\mathcal{S}_{\mathbb{R}}, \leqslant)$.*

(a) The stalk $\mathrm{FS}(\mathcal{G})_{A^\vee}$ of $\mathrm{FS}(\mathcal{G})$ at a face $A^\vee \in \mathcal{S}_{\mathbb{R}}^\vee$ is identified with the complex

$$\mathcal{U}_{A^\vee} \;:=\; \mathrm{Tot}\left\{ \bigoplus_{\substack{\dim(B)=0,\\ B \subset U(A^\vee)}} \mathcal{G}_B \otimes \mathrm{or}(B) \xrightarrow{\gamma \otimes \varepsilon} \bigoplus_{\substack{\dim(B)=1,\\ B \subset U(A^\vee)}} \mathcal{G}_B \otimes \mathrm{or}(B) \xrightarrow{\gamma \otimes \varepsilon} \cdots \right\}.$$

(b) Let $A_1^\vee \leqslant A_2^\vee$ be two faces of $\mathcal{H}^\vee$. Then the inclusion $U(A_1^\vee) \supset U(A_2^\vee)$ (Proposition 4.8) exhibits $\mathcal{U}(A_2^\vee)$ as a quotient complex of $\mathcal{U}(A_1^\vee)$, and the the generalization map $\gamma_{A_1^\vee, A_2^\vee} : \mathrm{FS}(\mathcal{G})_{A_1^\vee} \to \mathrm{FS}(\mathcal{G})_{A_2^\vee}$ of $\mathrm{FS}(\mathcal{G})$ is identified with the quotient map $\mathcal{U}_{A_1^\vee} \to \mathcal{U}_{A_2^\vee}$.

Proof: (a) Let $f \in A^\vee$. By Proposition 4.1 we have

$$\mathrm{FS}(\mathcal{G})_{A^\vee} \;\simeq\; \mathrm{FS}(\mathcal{G})_f \;\simeq\; R\Gamma_c(P_f, \mathcal{G}).$$

We now use the resolution of $\mathcal{G}$ given by Proposition 1.4(ii). The pth term of this resolution is the direct sum of $j_{B!}\, \underline{\mathcal{G}_B}_B[p]$ where B runs over p-dimensional faces of $\mathcal{H}$.

Lemma 4.14. *Let B be a face of $\mathcal{H}$ and E be any $\mathbf{k}$-vector space. We have natural quasi-isomorphisms*

$$R\Gamma_c(P_f, j_{B!}\underline{E}_B)[\dim B] \simeq \begin{cases} E \otimes \mathrm{or}(B), & \text{if } B \subset P_f, \\ 0, & \text{if } B \not\subset P_f. \end{cases}$$

Proof of the lemma: The case $B \subset P_f$ follows from the canonical identification $R\Gamma_c(B, \mathbf{k}) \simeq \mathrm{or}(B)[-\dim(B)]$ (compactly supported cohomology of a cell with constant coefficients). Suppose $B \not\subset P_f$. If B does not meet P_f at all, then the statement is obvious. If B does meet P_f, then the intersection $B \cap P_f$ is homeomorphic to a closed half-space in a Euclidean space, i.e., to a Cartesian product of several open intervals $(0,1)$ and one half-open interval $[0,1)$. So our statement follows from the fact that $H_c^\bullet([0,1), \mathbf{k}) = 0$. $\square$

Applying this lemma to the resolution of $\mathcal{G}$ given by Proposition 1.4(ii), we obtain a complex representing $R\Gamma_c(P_f, \mathcal{G})$ whose pth term is the sum of the $\mathcal{G}_B \otimes \mathrm{or}_B$ for B running over p-dimensional faces $B \subset P_f$ and the differential is formed by the maps $\gamma \otimes \varepsilon$. By Proposition 4.10 (a), the conditon $B \subset P_f$ is equivalent to $B \subset U(A^\vee)$. This proves part (a) of Theorem 4.13.

We now prove part (b). Let $f_1 \in A_1^\vee$ and $f_2 \in A_2^\vee$ be a small deformation of f_1. As in the proof of (a), we can write our generalization map as

$$\gamma_{A_1^\vee, A_2^\vee} : R\Gamma_c(P_{f_1}, \mathcal{G}) \longrightarrow R\Gamma_c(P_{f_2}, \mathcal{G}).$$

As before, consider first the case $\mathcal{G} = j_{B!}\underline{E}_B$ for some face B and some $\mathbf{k}$-vector space E. In this case we find that $\gamma_{A_1^\vee, A_2^\vee}$ is equal to the identity map, if B is contained in P_{f_2} (and therefore in P_{f_1}), and it is equal to 0 otherwise (since the target is the zero vector space). That is, claim (b) obviously holds in this case. The case of general $\mathcal{G}$ is now obtained from this by considering the resolution of $\mathcal{G}$ given by Proposition 1.4(ii). Theorem 4.13 is proved.

E. The complex result. Let $\mathcal{F} \in \mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$ correspond to a hyperbolic sheaf $\mathcal{Q} = (E_A, \gamma_{AB}, \delta_{BA})$. By Proposition 4.5, $\mathrm{FS}(\mathcal{F})[-n]$ lies in $\mathrm{Perv}(V_\mathbb{C}^*, \mathcal{S}_\mathbb{C}^\vee)$ and so is described by a hyperbolic sheaf which we denote $\mathcal{Q}^\vee = (E_{A^\vee}^\vee, \gamma_{A^\vee, 'A^\vee}, \delta_{'A^\vee, A^\vee})$. Here $A^\vee \leqslant {}'A^\vee$ are faces of the arrangement $\mathcal{H}^\vee$.

It turns out that the hyperbolic stalks $E_{A^\vee}^\vee$ are governed by the small dual cones $V(A^\vee)$.

Theorem 4.15. *(a) The space $E_{A^\vee}^\vee$ is quasi-isomorphic to the complex*

$$\mathcal{V}_{A^\vee} = \left\{ \bigoplus_{\substack{\dim(B)=0, \\ B \subset V(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma \otimes \varepsilon} \bigoplus_{\substack{\dim(B)=1, \\ B \subset V(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma \otimes \varepsilon} \cdots \right\}.$$

In other words, $\mathcal{V}_{A^\vee}$ is exact everywhere except the leftmost term, where the cohomology (kernel) is identified with $E_{A^\vee}^\vee$.

(a') The space $E^\vee_{A^\vee}$ is also quasi-isomorphic to the complex

$$\mathcal{V}^\dagger_{A^\vee} = \left\{ \cdots \xrightarrow{\delta\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=1,\\ B\subset V(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\delta\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=0,\\ B\subset V(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \right\}.$$

In other words, $\mathcal{V}^\dagger_{A^\vee}$ is exact everywhere except the rightmost term, where the cohomology (cokernel) is identified with $E^\vee_{A^\vee}$.

(b) Let $A_1^\vee \leqslant A_2^\vee$ be two faces of $\mathcal{H}^\vee$. Then the embedding $V(A_1^\vee) \subset V(A_2^\vee)$ realizes $\mathcal{V}_{A_1^\vee}$ as a quotient complex of $\mathcal{V}_{A_2^\vee}$, and the map $\delta_{A_2^\vee, A_1^\vee} : E^\vee_{A_2^\vee} \to E^\vee_{A_1^\vee}$ is identified with the quotient map $\mathcal{V}_{A_2^\vee} \to \mathcal{V}_{A_1^\vee}$.

(b') In the situation of (b), the embedding $V(A_1^\vee) \subset V(A_2^\vee)$ realizes $\mathcal{V}^\dagger_{A_1^\vee}$ as a subcomplex of $\mathcal{V}^\dagger_{A_2^\vee}$, and the map $\gamma_{A_1^\vee, A_2^\vee}$ is identified with the embedding $\mathcal{V}^\dagger_{A_1^\vee} \to \mathcal{V}^\dagger_{A_2^\vee}$.

Remarks 4.16. (a) Note that for $A^\vee = 0$, the cone $V(A^\vee)$ is equal to $\{0\}$, therefore $E^\vee_0$ is identified with E_0.

(b) Let $A^\vee \neq 0$. Then, by Proposition 4.9(b) one can re-write the complex $\mathcal{V}_{A^\vee}$ as

$$E_0 \otimes \mathrm{or}_V \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=1\\ f|_B>0}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=2\\ f|_B>0}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma\otimes\varepsilon} \cdots,$$

where $f \in A^\vee$ is an arbitrary element. Similarly for $\mathcal{V}^\dagger_{A^\vee}$.

The proof of Theorem 4.15 is based on the following preliminary result which shows that the big dual cones $U(A^\vee)$ govern the ordinary stalks, not hyperbolic stalks of $\mathrm{FS}(\mathcal{F})$,

Proposition 4.17. *(a) If $A^\vee \in \mathcal{S}_\mathbb{R}$ is any face, then the ordinary stalk $\mathrm{FS}(\mathcal{F})_{A^\vee}$ is quasi-isomorphic to the complex*

$$\mathrm{FS}(\mathcal{F})_{A^\vee} \simeq \left\{ \bigoplus_{\substack{\dim(B)=0,\\ B\subset U(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=1,\\ B\subset U(A^\vee)}} E_B \otimes \mathrm{or}_{V/B} \xrightarrow{\gamma\otimes\varepsilon} \cdots \right\}.$$

(b) The generalization maps for the $\mathrm{FS}(\mathcal{F})_{A^\vee}$ are induced by the projections of the complexes in (a), similarly to Theorem 4.13(b).

Proof of Proposition 4.17: Our statement will follow from Theorem 4.13, if we establish the following.

Proposition 4.18. *For any $\mathcal{F} \in \mathrm{Perv}(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ we have an identification*

$$\mathrm{FS}(i_{\mathbb{R}}^! \mathcal{F}) \ \simeq \ i_{\mathbb{R}}^* \, \mathrm{FS}(\mathcal{F}),$$

where $i_{\mathbb{R}}$ on the right means the embedding $V_{\mathbb{R}}^* \to V_{\mathbb{C}}^*$.

Proof of Proposition 4.18: We first recall the behavior of the Fourier-Sato transform with respect to an arbitrary $\mathbb{R}$-linear map $\phi : W_1 \to W_2$ of $\mathbb{R}$-vector spaces. Denoting ${}^t\phi : W_2^* \to W_1^*$ the transposed map, we have, for any conic complex $\mathcal{G}$ on W_2:

$$\mathrm{FS}(\phi^! \mathcal{G}) \ \simeq \ R({}^t\phi)_* \, \mathrm{FS}(\mathcal{G})$$

see [KS2] Prop. 3.7.14.

We specialize this to $\phi = i_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{C}}$ and $\mathcal{G} = \mathcal{F}$. In this case

$${}^t\phi \ = \ \Re : V_{\mathbb{C}}^* \longrightarrow V_{\mathbb{R}}^*$$

is the real part map. So after replacing V^* by V and $\mathrm{FS}(\mathcal{F})$ by $\mathcal{F}$, Proposition 4.18 reduces to the following.

Lemma 4.19. *For any $\mathcal{F} \in D^b(V_{\mathbb{C}}, \mathcal{S}_{\mathbb{C}})$ we have an identification $i_{\mathbb{R}}^* \mathcal{F} \simeq R\,\Re_*(\mathcal{F})$, where $\Re : V_{\mathbb{C}} \to V_{\mathbb{R}}$ is the real part map for V.*

Proof of Lemma 4.19: We consider $\Re : V_{\mathbb{C}} \to V_{\mathbb{R}}$ as a real vector bundle over $V_{\mathbb{R}}$. The complex $\mathcal{F}$, being constructible with respect to the complexification of a real hyperplane arrangement, is conic with respect to this vector bundle structure. Therefore the stalk at $x \in V_{\mathbb{R}}$ of $i_{\mathbb{R}}^* \mathcal{F}$ which is $R\Gamma(U, \mathcal{F})$ for a small open $U \subset V_{\mathbb{C}}$ containing x, is equal to $R\Gamma(\Re^{-1}(U \cap V_{\mathbb{R}}), \mathcal{F})$ which is the stalk of $R\,\Re_*(\mathcal{F})$ at x. $\square$

This finishes the proof of Propositions 4.18 and 4.17.

Proof of Theorem 4.15: We prove (a') and (b'). Parts (a) and (b) follow by Verdier duality.

We denote $\mathcal{K} = \mathrm{FS}(\mathcal{F})$, and let $\mathcal{L} = \mathcal{K}^* = \mathrm{FS}(\mathcal{F}^*)$ be the Verdier dual perverse sheaf. By definition, $E_{A^\vee}^\vee$ is the stalk at $A^\vee$ of

$$i_{\mathbb{R}}^! \mathcal{K} \ \simeq \ (i_{\mathbb{R}}^* \mathcal{L})^* \ \simeq \ \big(i_{\mathbb{R}}^* \mathrm{FS}(\mathcal{F}^*)\big)^*.$$

First, we recall that $\mathcal{F}^*$ is represented by the hyperbolic sheaf $(E_A^*, \delta_{BA}^*, \gamma_{AB}^*)$. Applying Proposition 4.17 to $\mathcal{F}^*$ we write the stalk of $i_{\mathbb{R}}^*\mathcal{L}$ at $A^\vee$ as

$$\mathcal{L}_{A^\vee} \simeq \left\{ \bigoplus_{\substack{\dim(B)=0\\ B\subset U(A^\vee)}} E_B^* \otimes \mathrm{or}_{V/B} \xrightarrow{\delta^*\otimes\varepsilon} \bigoplus_{\substack{\dim(B)=1\\ B\subset U(A^\vee)}} E_B^* \otimes \mathrm{or}_{V/B} \xrightarrow{\delta^*\otimes\varepsilon} \cdots \right\}. \tag{4.20}$$

Further, for $A_1^\vee \leqslant A_2^\vee$ we have $U(A_1^\vee) \supset U(A_2^\vee)$ and Proposition 4.17 implies that the generalization map $\digamma_{A_1^\vee, A_2^\vee} : \mathcal{L}_{A_1^\vee} \to \mathcal{L}_{A_2^\vee}$ is given by the projection of the corresponding complexes in (4.20).

We now recall the following general procedure on finding the stalks and generalization maps of the Verdier dual complex. See, e.g., [KS1] Prop. 1.11. We formulate it here for complexes on $V_{\mathbb{R}}^*$ constructible with respect to $\mathcal{S}_{\mathbb{R}}^\vee$.

Lemma 4.21. *Let $\mathcal{M} \in D^b(V_{\mathbb{R}}^*, \mathcal{S}_{\mathbb{R}}^\vee)$ correspond to a complex $(\mathcal{M}_{A^\vee}, \digamma_{A_1^\vee, A_2^\vee})$ of representations of $\mathcal{S}_{\mathbb{R}}^\vee$. Then:*

(a) The stalk of $\mathcal{M}^*$ at $A^\vee$ is identified with the complex

$$\mathcal{D}_{A^\vee} = \mathrm{Tot}\left\{ \cdots \xrightarrow{\digamma^*\otimes\varepsilon^*} \bigoplus_{C^\vee >_1 A^\vee} (\mathcal{M}_{C^\vee})^* \otimes \mathrm{or}_{C^\vee} \xrightarrow{\digamma^*\otimes\varepsilon^*} (\mathcal{M}_{A^\vee})^* \otimes \mathrm{or}_{A^\vee} \right\},$$

with the horizontal grading associating to the summand $(\mathcal{M}_{A^\vee})^* \otimes \mathrm{or}_{A^\vee}$ degree $-\dim(A^\vee)$. The horizontal differential $\digamma^* \otimes \varepsilon^*$ has, as the matrix element corresponding to $C_2^\vee >_1 C_1^\vee \geqslant A^\vee$, the tensor product of the dual maps to $\digamma_{C_1^\vee, C_2^\vee}$ and to $\varepsilon_{C_1^\vee, C_2^\vee}$.

(b) For two faces $A_1^\vee \leqslant A_2^\vee$ the generalization map $(\mathcal{M}^*)_{A_1^\vee} \to (\mathcal{M}^*)_{A_2^\vee}$ of $\mathcal{M}^*$, is identified with the projection of the complexes $\mathcal{D}_{A_1^\vee} \to \mathcal{D}_{A_2^\vee}$.

Applying part (a) of the lemma to $\mathcal{M} = \mathcal{L}$ and substituting, instead of each $\mathcal{L}_{C^\vee}$, its expansion (4.20), we identify (quasi-isomorphically) $E_{A^\vee}^\vee$ with the total complex of the following double complex. We denote this total complex $\mathcal{E}_{A^\vee}$.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{emb}\otimes\varepsilon"] & \bigoplus_{C^\vee >_1 A^\vee} \bigoplus_{\substack{\dim(B)=0 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{C^\vee} \arrow[r, "\mathrm{emb}\otimes\varepsilon"] & \bigoplus_{\substack{\dim(B)=0 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{A^\vee} \\
\cdots \arrow[r, "\mathrm{emb}\otimes\varepsilon"] & \bigoplus_{C^\vee >_1 A^\vee} \bigoplus_{\substack{\dim(B)=1 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{C^\vee} \arrow[r, "\mathrm{emb}\otimes\varepsilon"] \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"'] & \bigoplus_{\substack{\dim(B)=1 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{A^\vee} \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"'] \\
\cdots \arrow[r, "\mathrm{emb}\otimes\varepsilon"] & \bigoplus_{C^\vee >_1 A^\vee} \bigoplus_{\substack{\dim(B)=2 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{C^\vee} \arrow[r, "\mathrm{emb}\otimes\varepsilon"] \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"'] & \bigoplus_{\substack{\dim(B)=2 \\ B\subset U(C^\vee)}} E_B\otimes \mathrm{or}_{V/B}\otimes \mathrm{or}_{A^\vee} \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"'] \\
 & \vdots \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"'] & \vdots \arrow[u, "\delta\otimes\varepsilon\otimes\mathrm{Id}"']
\end{tikzcd}$$

Here the vertical differentials are dual to those in $\mathcal{L}_{C^\vee}$, i.e., given by the δ maps. Matrix elements of the horizontal differential are dual to the $\digamma$ maps for $\mathcal{L}$, and those $\digamma$ maps are given by the projections. So each matrix element in question is in fact the product of an *embedding* of δ-complexes and the ε map of orientation torsors.

For two faces $A_1^\vee \leqslant A_2^\vee$ the generalization map $\gamma_{A_1^\vee, A_2^\vee} : E^\vee_{A_1^\vee} \to E^\vee_{A_2^\vee}$ is identified, by part (b) of Lemma 4.21, with the projection $\mathcal{E}_{A_1^\vee} \to \mathcal{E}_{A_2^\vee}$.

We now compare $\mathcal{E}_{A^\vee}$ with the complex $\mathcal{V}^\dagger_{A^\vee}$ from the formulation of Theorem 4.15(a'). Let B be a face of $\mathcal{H}$. The summand corresponding to B in $\mathcal{V}^\dagger_{A^\vee}$, is either $E_B\otimes\mathrm{or}_{V/B}$ or 0 depending on whether $B\subset V(A^\vee)$ or not. On the other hand, $\mathcal{E}_{A^\vee}$ has many summands associated to B, they are labelled by $C^\vee > A^\vee$ such that $B\subset U(C^\vee)$. By Proposition 4.11, such $C^\vee$ are in bijection with faces of the closed polyhedral cone $K(A^\vee, B)$. So in the double complex above the summand $E_B\otimes\mathrm{or}_{V/B}$ is multiplied by a combinatorial complex which is easily found to calculate the cohomology with compact support $H^\bullet_c(K(A^\vee,B),\mathbf{k})$. If $B\not\subset V(A^\vee)$, then, by the same Proposition 4.11, $K(A^\vee,B)$ is a proper closed cone with nonempty interior in $V^*_\mathbb{R}/\mathrm{Lin}_\mathbb{R}(A^\vee)$ and so its cohomology with compact support vanishes entirely. If $B\subset V(A^\vee)$, then $K(A^\vee,B) = V^*_\mathbb{R}/\mathrm{Lin}_\mathbb{R}(A^\vee)$ so it has the top cohomology with compact support identified with $\mathrm{or}_{V/A^\vee}$, so the part of $\mathcal{E}_{A^\vee}$ corresponding to B is quasi-isomorphic to $E_B\otimes\mathrm{or}_{V/B}$. Moreover, we see that these quasi-isomorphisms combine into a quasi-isomorphism between $\mathcal{E}_{A^\vee}$ and $\mathcal{V}^\dagger_{A^\vee}$. This shows part (a') of Theorem 4.17. Part (b') follows by noticing that the projection $\mathcal{E}_{A_1^\vee} \to \mathcal{E}_{A_2^\vee}$ corresponds, under our quasi-isomorphism, to the embedding $\mathcal{V}^\dagger_{A_1^\vee} \to \mathcal{V}^\dagger_{A_2^\vee}$. Theorem 4.17 is proved.

5 Applications to second microlocalization

A. Microlocalization. If $M \subset X$ is a C^∞ submanifold of a C^∞ manifold, as in §§3A, then for any $\mathcal{G} \in D^b(X)$ the *microlocalization* of $\mathcal{G}$ along M is defined as

$$\mu_M(\mathcal{G}) = \mathrm{FS}_M(\nu_M(\mathcal{G})) \in D^b(T^*_M X),$$

see [KS2] Ch. 4. Here FS_M is the relative Fourier-Sato transform on the vector bundle $T_M X \to M$.

If $X = V$ is a real vector space with an arrangement $\mathcal{H}$, if $\mathcal{G} \in D^b(V, \mathcal{S}_\mathbb{R})$ and M is a vector subspace, then our descriptions of the Fourier-Sato transform and the specialization functors can be combined to obtain a combinatorial description of $\mu_M(\mathcal{G})$. We leave this to the reader, establishing instead some compatibility properties of various approaches to "second microlocalization" of Kashiwara and Laurent, see [L] and references therein. For convenience we give a brief general introduction.

B. Iterated microlocalization.

Lemma 5.1. *Let (W, ω) be a symplectic $\mathbb{R}$-vector space, and $L_1, L_2 \subset W$ are Lagrangian vector subspaces. Then the restriction of ω gives an identification*

$$\left(\frac{L_1}{L_1 \cap L_2}\right)^* \simeq \left(\frac{L_2}{L_1 \cap L_2}\right).$$

Proof: Consider the restriction of ω to the subspace $L_1 + L_2$. Its kernel on this subspace is

$$(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp = L_1 \cap L_2.$$

Therefore the restriction of ω makes

$$\frac{L_1 + L_2}{L_1 \cap L_2} = \frac{L_1}{L_1 \cap L_2} \oplus \frac{L_2}{L_1 \cap L_2}$$

into a symplectic vector space decomposed into the direct sum of two Lagrangian subspaces, So these Lagrangan subspaces become dual to each other. □

Let now (S, ω) be a C^∞ symplectic manifold and $\Lambda_1, \Lambda_2 \subset S$ be two (smooth) Lagrangian submanifolds. We say that Λ_1 and Λ_2 *intersect cleanly* (in the symplectic sense), if, locally near each $x \in \Lambda_1 \cap \Lambda_2$, there is a symplectomorphism of a neighborhood of x in S to a neighborhood of 0 in a symplectic vector space W, sending Λ_i to linear Lagrangian subspaces L_i as above. This implies that $\Lambda_1 \cap \Lambda_2$ is smooth.

Corollary 5.2. *If Λ_1, Λ_2 intersect cleanly, then the restriction of ω gives an identification*

$$T^*_{\Lambda_1 \cap \Lambda_2}\Lambda_1 \ \simeq\ T_{\Lambda_1 \cap \Lambda_2}\Lambda_2. \quad \square$$

Now let X be a C^∞ manifolds and $M, N \subset X$ be two smooth submanifolds. We assume that they intersect cleanly in the sense that they can locally be brought by a diffeomorphism to two vector subspaces in a vector space. Take $S = T^*X$ has two Lagrangian submanifolds $\Lambda_1 = T^*_M X$, $\Lambda_2 = T^*_N X$ which intersect cleanly in the symplectic sense. Given a complex of sheaves $\mathcal{G} \in D^b(X)$, we have microlocalizations

$$\mu_M(\mathcal{G}) \ \in\ D^b(\Lambda_1), \quad \mu_N(\mathcal{G}) \in D^b(\Lambda_2)$$

and we can specialize and microlocalize further, getting two complexes of sheaves

$$\mu_{\Lambda_1 \cap \Lambda_2}\mu_M(\mathcal{G}) \ \in\ D^b(T^*_{\Lambda_1 \cap \Lambda_2}\Lambda_1), \quad \nu_{\Lambda_1 \cap \Lambda_2}\mu_N(\mathcal{G}) \ \in\ D^b(T_{\Lambda_1 \cap \Lambda_2}\Lambda_2)$$

on two spaces which are identified by Corollary 5.2, so we can consider them as living on the same space. One can then formulate

Second Microlocalization Problem 5.3. Under which conditions on M, N and $\mathcal{G}$ can we guarantee that

$$\mu_{\Lambda_1 \cap \Lambda_2}\, \mu_M(\mathcal{G}) \ \simeq\ \nu_{\Lambda_1 \cap \Lambda_2}\, \mu_N(\mathcal{G}) \quad ?$$

C. Bi-microlocalization. Let us restrict to the case $N \subset M$. In this case we have

Proposition 5.4. *We have identifications*

$$T^*_{\Lambda_1 \cap \Lambda_2}\Lambda_1 \ =\ T_{\Lambda_1 \cap \Lambda_2}\Lambda_2 \ \simeq\ T^*_N M \oplus (T^*_M X)|_N.$$

Proof: Obviously, $\Lambda_1 \cap \Lambda_2$ projects, under $T^*X \to X$, to N. Looking at the fibers of this projection, we find that $\Lambda_1 \cap \Lambda_2 = (T^*_M X)|_N$. Looking at the Cartesian square

$$\begin{tikzcd}
\Lambda_1 \cap \Lambda_2 \arrow[r] \arrow[d, "\rho"'] & T^*_M X = \Lambda_1 \arrow[d, "\pi"] \\
N \arrow[r] & M
\end{tikzcd}$$

with π being a smooth fibration (projection of a vector bundle), we find that

$$T^*_{\Lambda_1 \cap \Lambda_2}\Lambda_1 \;\simeq\; \rho^* T^*_N M \;\simeq\; T^*_N M \oplus (T^*_M X)|_N. \quad \square$$

We already considered the situation of a flag $N \subset M \subset X$ in discussing bi-specialization $\nu_{NM}(\mathcal{G})$ in §3E. Further, in this context Schapira and Takeuchi [ST] [T] have defined the *bimicrolocalization*

$$\mu_{NM}(\mathcal{G}) \;=\; \mathrm{FS}_N(\nu_{NM}(\mathcal{G})) \;\in\; D^b(T^*_N M \oplus (T^*_M X)|_N).$$

Here FS_N is the relative Fourier-Sato transform on the vector bundle $T_N M \oplus (T_M X)|_N \to N$. So we have the following specialization-microlocalization diagram:

(5.5)

$$\begin{tikzcd}
D^b(X) \arrow[r, "\mu_M"] \arrow[d, "\mu_N"'] \arrow[dr, "\mu_{NM}"] & D^b(T^*_M X = \Lambda_1) \arrow[d, "\mu_{\Lambda_1 \cap \Lambda_2}"] \\
D^b(T^*_N X = \Lambda_2) \arrow[r, "\nu_{\Lambda_1 \cap \Lambda_2}"'] & D^b\big((T^*_N X \oplus (T^*_M X)|_N)) = T^*_{\Lambda_1 \cap \Lambda_2}\Lambda_1 = T_{\Lambda_1 \cap \Lambda_2}\Lambda_2\big)
\end{tikzcd}$$

which gives three possible "second microlocaliizations".

D. Comparisons in the linear case.

Theorem 5.6. *Let $X = V$ be an $\mathbb{R}$-vector space, $N \subset M \subset V$ be vector subspaces and $\mathcal{H}$ an arrangement of hyperplanes in V with the corresponding face stratification $\mathcal{S}_\mathbb{R}$. Then the diagram* (5.5) *is canonically 2-commutative if we replace $D^b(V)$ with $D^b(V, \mathcal{S}_\mathbb{R})$.*

In the complex situation, when $V = V_\mathbb{C}$ is a $\mathbb{C}$-vector space, $N \subset M \subset V_\mathbb{C}$ are $\mathbb{C}$-subspaces and $D^b(V, \mathcal{S}_\mathbb{R})$ is replaced by $\mathrm{Perv}(V_\mathbb{C}, \mathcal{S}_\mathbb{C})$, the commutativity of the outer square of (5.5) was proved in [FS] using the $\mathcal{D}$-module techiques.

We will deduce Theorem 5.6 from the following result.

Theorem 5.7 (P. Schapira)**.** *Let B be a C^∞-manifold and V be a smooth $\mathbb{R}$-vector bundle on B. Let $M \subset V$ be a vector subbundle. Then, the Fourier-Sato transforms on V and $T_M(V) = M \oplus (V/M)$ are compatible with specializations. In other words, the following diagram of functors is canonically 2-commutative:*

$$\begin{tikzcd}
D^b_{\mathrm{con}}(V) \arrow[r, "\mathrm{FS}_V"] \arrow[d, "\nu_M"'] & D^b(V^*) \arrow[d, "\nu_{M^\perp}"] \\
D^b(M \oplus (V/M)) \arrow[r, "P_{12} \circ \mathrm{FS}_{M \oplus (V/M)}"'] & D^b(M^\perp \oplus M^*).
\end{tikzcd}$$

Here P_{12} is the permutation of the two direct summands in $M^* \oplus M^\perp$.

The notation $\oplus$ here and below means direct sum of vector bundles, i.e., fiber product over B.

We note that the diagram in Theorem 5.7 can be seen as a particular case of the outer rim of the diagram (5.5) for the case when $X = V$, when $M \subset V$ is our subbundle and $N = B$ is the zero section of V. In other words, Theorem 5.7 can be seen as a parametrized version (with arbitrary base B instead of $B = \mathrm{pt}$) of a particular case of Theorem 5.6 corresponding to $N = 0$.

E. Proof of Theorem 5.7. The following proof is an adaptation of the argument communicated to us by P. Schapira.

We consider three pairs

$$M^\perp \subset V^*, \ \ M \oplus M^\perp \subset V \oplus V^*, \ \ M \subset V,$$

and the corresponding normal deformations which are related by the natural projections:

$$\widetilde{V}^*_{M^\perp} \longleftarrow \widetilde{V \oplus V^*}_{M \oplus M^\perp} \longrightarrow \widetilde{V}_M. \tag{5.8}$$

Each of the three normal deformations fits into its own diagram of the form (3.10) whose spaces and maps will be decorated by the subscripts $M^\perp$, $M \oplus M^\perp$ and M. In particular, the projections of the three spaces in (5.8) to the line $\mathbb{R}$ will be denoted $\tau_{M^\perp}, \tau_{M \times M^\perp}$ and τ_M. These projections commute with the maps in (5.8). The coordinate in $\mathbb{R}$ will be denoted t.

Now, the Fourier-Sato transform on any vector bundle W is defined using the region

$$P = P_W = \left\{(x, f) \in W \oplus W^* \middle| \, (f(x) \geqslant 0\right\},$$

cf. §4A. We apply this to $W = V$ and $W = M \oplus (V/M)$ and denote the corresponding regions

$$P_V \subset V \oplus V^*, \quad P_{M\oplus(V/M)} \subset M \oplus (V/M) \oplus M^* \oplus (V/M)^*.$$

We want to lift P_V into a region $\overline{\mathcal{P}} \subset \widetilde{V \oplus V^*}_{M\oplus M^\perp}$ which specializes, for $t > 0$, to P_V and for $t = 0$, to $P_{M\oplus(V/M)}$.

For this we consider the region $\Omega_{M\oplus M^\perp} \subset \widetilde{V \oplus V^*}_{M\oplus M^\perp}$, defined as the preimage $\tau^{-1}_{M\oplus M^\perp}(\mathbb{R}_{>0})$, cf. (3.10). It is identified with $V \oplus V^* \times \mathbb{R}_{>0}$. Let $\mathcal{P} \subset \Omega$ be the image of $P_V \times \mathbb{R}_{>0}$.

Proposition 5.9. *The closure $\overline{\mathcal{P}}$ of $\mathcal{P}$ in $\widetilde{V \oplus V^*}_{M\oplus M^\perp}$ is the union of $\mathcal{P}$ and $P_{M\oplus(V/M)} \subset \tau^{-1}_{M\oplus M^\perp}(0)$.*

Proof: The statement is local in B. So we can assume that there exists a complement M' to M and to write $V = M \oplus M'$. We then identify, as in (3.10),

$$\widetilde{V \oplus V^*}_{M\oplus M^\perp} = M \oplus M' \oplus M^* \oplus M'^* \times \mathbb{R}$$

and the projection $p_{M\oplus M^\perp} : \widetilde{V \oplus V^*}_{M\oplus M^\perp} \to V \oplus V^*$ can be written as

$$\begin{aligned} p_{M\oplus M^\perp} : M \oplus M' \oplus M^* \oplus M'^* \times \mathbb{R} &\longrightarrow M \oplus M' \oplus M^* \oplus M'^*, \\ \big(m, m', \phi, \phi', t\big) &\mapsto \big(m, tm', t\phi, \phi'\big). \end{aligned} \tag{5.10}$$

Recall that the identification $\Omega_{M\oplus M^\perp} \to V \oplus V^* \times \mathbb{R}_{>0}$ is given by the map $(p_{M\oplus M^\perp}, \tau_{M\oplus M^\perp})$, the second component being projection to t. It follows from (5.10) that for any $t > 0$ the image, under $p_{M\oplus M^\perp}$, of $P_V \times \{t\}$ is P_V. Therefore the inverse of $p_{M\oplus M^\perp}, \tau_{M\oplus M^\perp})$ identifies $P_V \times \mathbb{R}_{\geqslant 0}$ with $P_V \times \mathbb{R}_{\geqslant 0}$, where for $t = 0$ our choice of complement has identified P_V with $P_{M\oplus M'} = P_{M\oplus(V/M)}$. □

We now consider the following diagram:

$$
\begin{tikzcd}
V^* & P_V \arrow[l, "p_{2,V}"'] \arrow[r, "p_{1,V}"] & V \\
\Omega_{M^\perp} \arrow[u, "\widetilde{p}_{M^\perp}"] \arrow[d, "j_{M^\perp}"'] & \mathcal{P} \arrow[l, "\rho_2"'] \arrow[r, "\rho_1"] \arrow[u] \arrow[d] & \Omega_M \arrow[u, "\widetilde{p}_M"'] \arrow[d, "j_M"] \\
\widetilde{V}^*_{M^\perp} & \widetilde{\mathcal{P}} \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] & \widetilde{V}_M \\
M^\perp \oplus M^* \arrow[u, "s_{M^\perp}"] & P_{M\oplus(V/M)} \arrow[l, "p_{2,M\oplus(V/M)}"] \arrow[r, "p_{1,M\oplus(V/M)}"'] \arrow[u] & M \oplus (V/M) \arrow[u, "s_M"']
\end{tikzcd}
$$

Given $\mathcal{G} \in D^b_{\mathrm{con}}(V)$, we have that

$$
\begin{aligned}
\nu_{M^\perp}\mathrm{FS}_V(\mathcal{G}) &= s^*_{M^\perp} R(j_{M^\perp})_* \, \widetilde{p}^*_{M^\perp} \, (p_{2,V})_! \, p^{-1}_{1,V}(\mathcal{G}), \\
\mathrm{FS}_{M\oplus(V/M)}\nu_M(\mathcal{G}) &= (p_{2,M\oplus(V/M)})_! \; p^*_{1,M\oplus(V/M)} \, s^*_M \, R(j_M)_* \, \widetilde{p}^*_M(\mathcal{G})
\end{aligned}
$$

are given by moving along the two boundary paths of this diagram from the northeast to the southwest corner. We identify these functors using the base change theorem for the Cartesian squares forming this diagram.

F. Proof of Theorem 5.6. We write the diagram (5.5) in our case as follows:

$$
\begin{tikzcd}
D^b(V, \mathcal{S}_\mathbb{R}) \arrow[r, "\mu_M"] \arrow[d, "\mu_N"'] \arrow[dr, "\mu_{NM}"] & D^b_{\mathrm{con}}(M \times (V/M)^*) \arrow[d, "\mu_{N\times(V/M)^*}"] \\
D^b_{\mathrm{con}}(N \times (V/N)^*) \arrow[r, "\nu_{N\times(V/M)^*}"'] & D^b_{\mathrm{bicon}}(N \times (M/N)^* \times (V/M)^*).
\end{tikzcd}
\tag{5.11}
$$

Here and below the subscript "con" means complexes which are $\mathbb{R}_{>0}$-conic with to the second argument, and "bico" means cmplexes which are $(\mathbb{R}_{>0})^2$-biconic with respect to the second and third arguments.

We recall that μ_{NM} is the composition

$$
D^b(V, \mathcal{S}_\mathbb{R}) \xrightarrow{\nu_{NM}} D^b_{\mathrm{bicon}}(N \times (M/N) \times (V/M)) \xrightarrow{\mathrm{FS}_{(M/N)\times(V/M)}} D^b_{\mathrm{bicon}}(N \times (M/N)^* \times (V/M)^*).
$$

We now prove the 2-commutativity of each of the two triangles in (5.11).

Upper triangle. We write each μ as the composition of the corresponding FS and ν and apply Theorem 3.17 to decompose ν_{NM} as the composition of two specializations. After this we represent the two paths in the triangle as the two boundary paths in the following diagram:

$$\begin{tikzcd}
D^b(V,\mathcal{S}_{\mathbb{R}}) \arrow[d, "\nu_M"'] \arrow[dr, "\mu_M"] & \\
D^b_{\mathrm{con}}(M\times(V/M)) \arrow[r, "\mathrm{FS}_{V/M}"] \arrow[d, "\nu_N"'] & D^b_{\mathrm{con}}(M\times(V/M)^*) \arrow[d, "\nu_N"] \\
D^b_{\mathrm{bicon}}(N\times(M/N)\times(V/M)) \arrow[r, "\mathrm{FS}_{V/M}"] \arrow[dr, "\mathrm{FS}_{(M/N)\times(V/M)}"'] & D^b_{\mathrm{bicon}}(N\times(M/N)\times(V/M)^*)) \arrow[d, "\mathrm{FS}_{M/N}"] \\
& D^b_{\mathrm{bicon}}(N\times(M/N)^*\times(V/M)^*).
\end{tikzcd}$$

In this diagram, the top triangle commutes by definition of μ_M and the commutativity of the bottom triangle expresses the fact that the Fourier-Sato transform of biconic sheaves on the direct sum of vector bundles can be done in stages, cf. [KS2] Prop. 3.7.15. The commutativity of the middle square follows because specialization along N and the Fourier-Sato transform along V/M operate in different factors so they are independent of each other and can be permuted.

Lower triangle. As before, by unravelling the definitions of various μ and applying Theorem 3.17, we represent the two paths in the triangle as the two boundary paths in the following diagram:

$$\begin{tikzcd}
& D^b(V,\mathcal{S}_{\mathbb{R}}) \arrow[dl, "\mu_N"'] \arrow[d, "\nu_N"] \\
D^b_{\mathrm{con}}(N\times(V/N)^*) \arrow[d, "\nu_{N\times(V/M)^*}"'] & D^b_{\mathrm{con}}(N\times(V/N)) \arrow[l, "\mathrm{FS}_{V/N}"'] \arrow[d, "\nu_{N\times(M/N)}"] \\
D^b_{\mathrm{bicon}}(N\times(M/N)\times(V/M)) & D^b_{\mathrm{bicon}}(N\times(M/N)\times(V/M)) \arrow[l, "\mathrm{FS}_{(M/N)\times(V/M)}"]
\end{tikzcd}$$

The commutativity of the top triangle in this diagram is the definition of μ_N. The commutativity of the lower square is an instance of Theorem 5.7 for the trivial vector bundle over $B = N$ with fiber V/N and the trivial subbundle with fiber M/N. Theorem 5.6 is proved.

References

[Ar] J. Arthur. An introduction to the trace formula. In: "Harmonic Analysis, Trace Formula and Schimura Varieties" (J. Arthur, D. Ellwood, R. Kottwitz Eds.) *Clayt Math. Proceedings* **4** p. 3-263, Amer. Math. Soc. 2005.

[Be] A. Beilinson. How to glue perverse sheaves. In: K-theory, arithmetic and geometry (Moscow, 1984), *Lecture Notes in Math.* **1289**, Springer-Verlag, 1987, 42-51.

[BBD] A. Beilinson, J. Bernstein, P. Deligne. Faisceaux pervers. *Astérisque* **100**.

[BFS] R. Bezrukavnikov, M. Finkelberg, V. Schechtman. Factorizable sheaves and quantum groups. *Lecture Notes in Math.* **1691**, Springer-Verlag, 1998.

[Br] T. Braden, Hyperbolic localization of intersection cohomology, *Transform. Groups* , **8** (2003), 209-216.

[De] P. Deligne, Le formalisme des cycles évanescents. SGA 7, Exp. 13, 14 *Lecture Notes in Math.* **340**, Springer-Verlag 1973.

[DG] V. Drinfeld, D. Gaitsgory, On a theorem of Braden. *Transform. Groups* **19** (2014) 313-358.

[FL] T. Finis, E. Lapid. On the spectral side of the Arthur's trace formula - combinatorial setup. *Ann. Math.* **174** (2011) 197-223.

[FS] M. Finkelberg, V. Schechtman. Microlocal approach to Lusztig's symmetries. *arXiv:*1401.5885.

[HP] N. Honda, L. Prelli. Multi-specialization and multi-asymptotic expansions. *Adv. Math.* **232** (2013) 432-498.

[HPY] N. Honda, L. Prelli, S. Yamazaki. Multi-microlocalization and microsupport. Bull. Soc. Math. France, **144** (2016) 569-611.

[KS1] M. Kapranov, V. Schechtman. Perverse sheaves over real hyperplane arrangements. *Ann. Math.* **183** (2016) 619-679.

[KS2] M. Kashiwara, P. Schapira, Sheaves on Manifolds. Springer-Verlag, 1990.

[KS3] M. Kashiwara, P. Schapira, Persistent homology and microlocal sheaf theory. arXiv: 1705.00955.

[L] Y. Laurent. Théorie de la Deuxieme Microlocalization dans le Domaine Complexe. *Progress in Math.* **53**, Birkhäuser, Boston, 1985.

[MV] R. D. McPherson, K. Vilonen. Elementary construction of perverse sheaves. *Invent. Math.* **84** (1986), 403-435.

[ST] P. Schapira, K. Takeuchi. Déformation binormale et bispecialization. *C. R. Acad. Sci.* **319** (1994) 707-712.

[T] K. Takeuchi. Binormal deformation and bimicrolocalization. *Publ.RIMS Kyoto Univ.* **32** (1996) 277-322.

[V] A. N. Varchenko, Combinatorics and topology of the arrangement of affine hyperplanes in the real space. *Funct. Anal. Appl.* **21** (1987) 11-22.

M.F.: Department of Mathematics, National Research University Higher School of Economics, Russian Federation, 6 Usacheva st, Moscow 119048; Skolkovo Institute of Science and Technology, `fnklberg@gmail.com`

M.K.: Kavli IPMU, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583 Japan, `mikhail.kapranov@ipmu.jp`

V.S.: Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France, `schechtman@math.ups-tlse.fr`