THE 2-COMPONENT BKP GRASSMANIAN AND SIMPLE SINGULARITIES OF TYPE D

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Abstract. It was proved in 2010 that the principal Kac–Wakimoto hierarchy of type D is a reduction of the 2-component BKP hierarchy. On the other hand, it is known that the total descendant potential of a singularity of type D is a tau-function of the principal Kac–Wakimoto hierarchy. We find explicitly the point in the Grassmanian of the 2-component BKP hierarchy (in the sense of Shiota) that corresponds to the total descendant potential. We also prove that the space of tau-functions of Gaussian type is parametrized by the base of the miniversal unfolding of the simple singularity of type D.

1. Introduction

The total descendant potential in singularity theory is defined through K. Saito's theory of primitive forms (see [20, 21, 13]) and Givental's higher genus reconstruction [9]. In the case of singularities of type A_{N-1} it was proved by Givental in [11] that the total descendant potential is a tau-function of the N-KdV hierarchy satisfying the string equation. Recalling the results of Kac and Schwartz (see [16]) such a tau-function is unique and it can be identified with a point in the big cell of Sato's Grassmanian (see [3, 22] for some background)

$$\mathrm{Gr}^{(0)} = \{U \subset \mathbb{C}(\!(z^{-1})\!) \mid \pi|_U : U \to \mathbb{C}[z] \text{ is an isomorphism}\},$$

where $\mathbb{C}(\!(z^{-1})\!)$ is the space of formal Laurent series near $z = \infty$ and $\pi : \mathbb{C}(\!(z^{-1})\!) \to \mathbb{C}[z]$ is the linear map that truncates the terms in the Laurent series that involve negative powers of z. Every point U in Sato's Grassmanian is determined uniquely by its *wave* or *Baker function*, which by definition is the unique formal function of the type

$$\Psi(x,z) = \Big(1 + w_1(x)z^{-1} + w_2(x)z^{-2} + \cdots\Big)e^{xz}, \quad w_i(x) \in \mathbb{C}[\![x]\!]$$

such that the Taylor's series expansion in x has coefficients that span the subspace U. Kac and Schwartz proved that the wave function corresponding to a solution of the N-KdV hierarchy satisfying the string equation can be identified uniquely with a certain solution to the generalized Airy equation.

The N-KdV hierarchy has a generalization to any simple Lie algebra of type ADE provided by the so called Kac–Wakimoto hierarchies (see [18]). Our main focus will be on the case D and our main goal is to obtain the analogue of Kac and Schwartz's result. The starting point of this paper is an observation due to Liu–Wu–Zhang [19] that the principal Kac–Wakimoto hierarchy of type D is a reduction of the 2-component BKP hierarchy. Furthermore, our work relies on the constructions of Shiota (see [23]) and ten Kroode–van de Leur (see [17]).

1.1. **The Grassmanian of the 2-component BKP hierarchy.** Let us recall the construction due to Shiota [23] of an inifinte Grassmanian that plays a key role in the study of the 2-component BKP hierarchy. Let

$$V := \mathbb{C}(\!(z_1^{-1})\!) \oplus \mathbb{C}(\!(z_2^{-1})\!),$$

where $z := (z_1, z_2)$ are formal variables and $(\ ,\ )$ be a symmetric non-degenerate bi-linear pairing defined by

$$(f(z), g(z)) := \mathrm{Res}_{z_1=0} \frac{dz_1}{z_1} f_1(z_1) g_1(-z_1) + \mathrm{Res}_{z_2=0} \frac{dz_2}{z_2} f_2(z_2) g_2(-z_2),$$

where $f(z) = (f_1(z_1), f_2(z_2)), g(z) = (g_1(z_1), g_2(z_2)) \in V$ and the residues are understood formally as the coefficients in front of dz_i/z_i, $i = 1, 2$. The vector space V has a direct sum decomposition $V = U_0 \oplus V_0$, where

$$U_0 := \mathbb{C}(e_1 + \mathbf{i}e_2) + \mathbb{C}[z_1]z_1\, e_1 + \mathbb{C}[z_2]z_2\, e_2,$$

and

$$V_0 := \mathbb{C}(e_1 - \mathbf{i}e_2) + \mathbb{C}[\![z_1^{-1}]\!]z_1^{-1}\, e_1 + \mathbb{C}[\![z_2^{-1}]\!]z_2^{-1}\, e_2,$$

where $e_1 := (1, 0)$, $e_2 := (0, 1)$, and $\mathbf{i} := \sqrt{-1}$. Let $\pi : V \to U_0$ be the projection along V_0. The big cell of the Grassmanian of the 2-component BKP hierarchy is by definition the set $\mathrm{Gr}_2^{I,(0)}$ of all linear subspaces $U \subset V$ such that

(1) The projection $\pi|_U : U \to U_0$ is an isomorphism.
(2) U is maximally isotropic.

Recall that a subspace U is called isotropic if $(u_1, u_2) = 0$ for all $u_1, u_2 \in U$. It is maximally isotropic if it is not contained in a larger isotropic subspace, i.e., if $u' \notin U$ then there exist $u \in U$, such that $(u, u') \neq 0$. Both U_0 and V_0 are maximal isotropic subspaces.

1.2. **The 2-component BKP hierarchy.** Let us recall the correspondence between the points of $\mathrm{Gr}_2^{I,(0)}$ and the tau-functions of the 2-component BKP hierarchy. Suppose that

$$\mathbf{t}^a := (t_m^a)_{m \in \mathbb{Z}_{\mathrm{odd}}^+}, \quad a = 1, 2$$

are two sequences of formal variables. We denote by $\mathbb{Z}_{\mathrm{odd}}$ the set of all odd integers and by $\mathbb{Z}_{\mathrm{odd}}^+$ the set of all positive odd integers. A formal power series

$$\tau(\mathbf{t}^1, \mathbf{t}^2) \in \mathbb{C}[\![\mathbf{t}^1, \mathbf{t}^2]\!]$$

is said to be a *tau-function* of the 2-component BKP if the following Hirota bilinear equations hold

$$\Omega_0(\tau\otimes\tau)=0,$$

where Ω_0 is the following bi-linear operator acting on $\mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]^{\otimes 2}$

$$\mathrm{Res}_{z_1=0}\frac{dz_1}{z_1}\Big(\Gamma(\mathbf{t}^1,z_1)\otimes\Gamma(\mathbf{t}^1,-z_1)\Big)-\mathrm{Res}_{z_2=0}\frac{dz_2}{z_2}\Big(\Gamma(\mathbf{t}^2,z_2)\otimes\Gamma(\mathbf{t}^2,-z_2)\Big),$$

where

$$\Gamma(\mathbf{t}^a,z_a):=\exp\Big(\sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}}t^a_m(z_a)^m\Big)\exp\Big(-\sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}}2\partial_{t^a_m}\frac{(z_a)^{-m}}{m}\Big)$$

are *vertex operators*. Suppose that τ is a tau-function of the 2-component BKP hierarchy. We will be interested only in tau-functions such that $\tau(0)\neq 0$, i.e., both $\log\tau$ and τ^{-1} exist in $\mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]$. The first two dynamical variables play a special role, so let us denote them by $x_1=t^1_1$ and $x_2=t^2_1$. We will refer to them as *spatial variables*. Then we define the *wave function* corresponding to the tau-function τ to be

$$\Psi(x,z):=\Psi^{(1)}(x,z_1)e_1+\mathbf{i}\,\Psi^{(2)}(x,z_2)e_2, \tag{1}$$

where $x=(x_1,x_2)$, $z=(z_1,z_2)$ and the components

$$\Psi^{(a)}(x,z_a):=\left.\frac{\Gamma(\mathbf{t}^a,z_a)\tau(\mathbf{t}^1,\mathbf{t}^2)}{\tau(\mathbf{t}^1,\mathbf{t}^2)}\right|_{t^b_m=0(b=1,2;m>1)},\quad a=1,2.$$

The wave function has the following three properties.

(W1) The components of the wave function are formal asymptotic series of the type

$$\Psi^{(a)}(x,z_a)=\Big(1+\sum_{k=1}^{\infty}W^{(a)}_k(x)(z_a)^{-k}\Big)\,e^{x_az_a},\quad a=1,2,$$

where $W^{(a)}_k(x)\in\mathbb{C}[\![x_1,x_2]\!]$.

(W2) There exists a formal function $q(x)\in\mathbb{C}[\![x_1,x_2]\!]$ such that

$$(\partial_1\partial_2+q(x))\Psi(x,z)=0,$$

where $\partial_a=\frac{\partial}{\partial x_a}$ $(a=1,2)$.

(W3) The wave function is isotropic

$$(\Psi(x',z),\Psi(x'',z))=0,$$

where x' and x'' are two copies of the spatial variables $x=(x_1,x_2)$.

It turns out that the wave function determines the corresponding tau-function uniquely up to a constant factor. Moreover, every function of the form (1) satisfying properties (W1)–(W3) is a wave function, i.e., it corresponds to a tau-function of the 2-component BKP hierarchy.

Suppose now that $\Psi(x,z)$ satisfies properties (W1)–(W3). According to Shiota (see [23], Lemma 9 and its Corollary) the coefficients of the Taylor's series expansion of $\Psi(x,z)$ at $x=(0,0)$ span a subspace that belongs to the Grassmanian $\mathrm{Gr}_2^{I,(0)}$. Moreover, this map establishes a one-to-one correspondance between the set of wave functions of the 2-component BKP hierarchy and the points of $\mathrm{Gr}_2^{I,(0)}$. For proofs of the statements in this section and for more details we refer to Section 3.1 in [23].

1.3. **Boson-fermion isomorphism.** Let us denote by $Cl(V)$ the Clifford algebra associated to the vector space V and the pairing $(\ ,\ )$, i.e., $Cl(V) = T(V)/I$ where

$$T(V) := \mathbb{C} \oplus \bigoplus_{n\geq 1} V^{\otimes n}$$

is the tensor algebra and $I \subset T(V)$ is the two-sided ideal generated by

$$v_1 \otimes v_2 + v_2 \otimes v_1 - (v_1, v_2), \quad v_1, v_2 \in V.$$

Following ten Kroode and van de Leur (see [17]) we construct the principal realization of the basic representation of type D via the spin representation of $Cl(V)$. Namely, let us introduce the fermionic Fock space

$$\mathcal{F} := Cl(V)/Cl(V)U_0,$$

where $U_0 \subset V$ is the maximal isotropic subspace introduced above. The image of $1 \in Cl(V)$ in $\mathcal{F}$ will be denoted by $|0\rangle$ and it will be called the *vacuum*. Note that $\mathcal{F}$ is an irreducible $Cl(V)$-module. We denote by $\phi_a(k)$ the linear operators induced by multiplication by $e_a(-z_a)^{-k}$. Recalling the definition of the pairing we get that these operators satisfy the following relations

$$\phi_a(k)\phi_b(\ell) + \phi_b(\ell)\phi_a(k) = (-1)^k\, \delta_{k,-\ell}\delta_{a,b}.$$

Linear operators satisfying such commutation relations are also known as *neutral free fermions*.

More explicitly, we can uniquely recover the $Cl(V)$-module structure on $\mathcal{F}$ by the relations

$$\begin{aligned} \phi_a(k)\,|0\rangle &= 0 \quad \text{for all } k<0, \\ (\phi_1(0) + \mathbf{i}\phi_2(0))\,|0\rangle &= 0, \\ \phi_1(0)^2 = \phi_2(0)^2 &= 1/2 \end{aligned}$$

and the fact that the following set of vectors

$$\phi_1(k_1^1)\cdots\phi_1(k_r^1)\phi_2(k_1^2)\cdots\phi_2(k_s^2)\,|0\rangle \tag{2}$$

where $k_1^1 > \cdots > k_r^1 > 0$ and $k_1^2 > \cdots > k_s^2 \geq 0$ is a linear basis of $\mathcal{F}$.

Let $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) be the subspace of $\mathcal{F}$ spanned by vectors (2) such that $r+s$ is even (resp. odd). We would like to equip both $\mathcal{F}_0$ and $\mathcal{F}_1$ with the structure of an irreducible highets weight module over a certain Heisenberg algebra. This is a standard construction. Namely, put

$$J_m^a := \sum_{j\in\mathbb{Z}} (-1)^j\ :\phi_a(-j-m)\phi_a(j):, \quad m \in \mathbb{Z}_{\mathrm{odd}}, \quad a = 1,2,$$

where the normal ordering is defined by

$$:ab: = ab - \langle ab\rangle, \quad a,b\in V,$$

where the vacuum expectation $\langle ab\rangle$ is the coefficient in front of the vacuum $|0\rangle$ when the vector $ab|0\rangle$ is written as a linear combination of the basis vectors (2). The operators satisfy Heisenberg commutation relations

$$[J_k^a, J_\ell^b] = 2k\delta_{k,-\ell}\delta_{a,b}.$$

Moreover, the fermions can be expressed in terms of the operators J_m^a as follows

$$\phi_a(z_a) := \sum_{k\in\mathbb{Z}} \phi_a(k)(z_a)^k = Q_a \exp\Big(\sum_{m\in\mathbb{Z}^+_{\rm odd}} J^a_{-m}\frac{(z_a)^m}{m}\Big)\exp\Big(-\sum_{m\in\mathbb{Z}^+_{\rm odd}} J^a_{m}\frac{(z_a)^{-m}}{m}\Big),$$

where $Q_a:\mathcal{F}\to\mathcal{F}$ are linear operators defined by the following relations

$$\begin{aligned} Q_a\,|0\rangle &= \phi_a(0)\,|0\rangle, \quad a=1,2,\\ Q_a\phi_a(k) &= \phi_a(k)Q_a, \quad a=1,2,\\ Q_a\phi_b(k) &= -\phi_b(k)Q_a, \quad \text{for } a\neq b. \end{aligned}$$

Finally, there is a unique isomorphism $\mathcal{F}_0\cong\mathbb{C}[\mathbf{t}^1,\mathbf{t}^2]$ such that the vacuum $|0\rangle\mapsto 1$ and

$$J^a_{-m}\mapsto m t^a_m, \quad J^a_m\mapsto 2\frac{\partial}{\partial t^a_m}, \quad 1\le a\le 2, \quad m\in\mathbb{Z}^+_{\rm odd}.$$

1.4. **The Kac–Wakimoto hierarchy and the 2-component BKP hierarchy.** To begin with note that under the boson-fermion isomorphism the bilinear operator Ω_0 of the Hirota bilinear equations satisfies the following relations

$$\Omega_0 = 4(Q_1\otimes Q_1)\sum_{a=1,2}\sum_{k\in\mathbb{Z}}(-1)^k\phi_a(k)\otimes\phi_a(-k),$$

where we used that

$$(Q_1\otimes Q_1)^2 = 1/4, \quad (Q_1\otimes Q_1)(Q_2\otimes Q_2) = -1/4.$$

Put $h_1 := h := 2N-2$ and $h_2 := 2$ ($N\ge 3$). We will be interested in the so-called (h_1,h_2)-reduction of the 2-component BKP. Let us introduce the bilinear operators Ω_m ($m\in\mathbb{Z}$) acting on $\mathbb{C}[\mathbf{t}^1,\mathbf{t}^2]^{\otimes 2}$ as follows:

$$\Omega_m := 4(Q_1\otimes Q_1)\sum_{a=1,2}\sum_{k\in\mathbb{Z}}(-1)^k\phi_a(-k)\otimes\phi_a(-k-m\,h_a).$$

The key observation is that if we set

$$H_{i,m} := \frac{1}{\sqrt{2}}J^1_m, \quad \text{for} \quad 1\le i\le N-1, \quad m\equiv 2i-1 (\mathrm{mod}\, h),$$

and

$$H_{n,m(N-1)} := \sqrt{\frac{N-1}{2}}J^2_m, \quad m\in\mathbb{Z}_{\rm odd},$$

then we get an isomorphism of the Heisenberg algebra spanned by J^a_m and the principal Heisenberg algebra of the affine Lie algebra $\widehat{so}_{2N}$. Therefore, we can identify $\mathcal{F}_0$ with the principal realization of the basic representation of $\widehat{so}_{2N}$. In particular, the Casimir of the Kac–Wakimoto hierarchy acts on $\mathcal{F}_0^{\otimes 2}$ via a certain bi-linear operator $\Omega_{\rm KW}$. Our first result can be stated as follows

Theorem 1. *The Casimir of the Kac–Wakimoto hierarchy satisfies the following relation*

$$-4\Omega_{\rm KW} = \frac{1}{2}\,\Omega_0^2 + \sum_{m=1}^{\infty} \Omega_{-m}\Omega_m.$$

Note that the action of the operators Ω_m $(m\in\mathbb{Z})$ extend to the completion $\mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]$. The above formula yields the following corollary.

Corollary 1. *A formal power series $\tau\in\mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]$ is a solution to the Kac–Wakimoto hierarchy $\Omega_{\rm KW}(\tau\otimes\tau)=0$ if and only if one of the following two equivalent conditions are satisfied*

a) $\Omega_m(\tau\otimes\tau)=0$ for all $m\geq 0$.

b) τ is a tau-function of the 2-component BKP hierarchy and the corresponding point $U\in {\rm Gr}_2^{I,(0)}$ has the following symmetry

$$(z_1^h, z_2^2)\, U\subset U,$$

where (z_1^h,z_2^2) acts on V by component-wise multiplication

$$(z_1^h, z_2^2)\,(f_1(z_1), f_2(z_2)) := (z_1^h f_1(z_1), z_2^2 f_2(z_2)).$$

Remark 1. The result in part a) of Corollary 1 is stated without proof in [19]. The authors only made a comment that the proof follows from the fact that the Casimir of the Kac–Wakimoto hierarchy has the form $\Omega_0^*\Omega_0$. This statement is conceptually true, but nevertheless a small correction is needed. The goal of Theorem 1 is to clarify the remark of Liu–Wu–Zhang.

Remark 2. Recall that the N-KdV hierarchy is a reduction of the KP hierarchy. It is well known that it can be identified also with the principal Kac–Wakimoto hierarchy of type A_{N-1}. The precise relation between the Casimirs of the Kac–Wakimoto and the KP hierarchies should be given by a formula similar to the one in Theorem 1. Quite surprisingly such a formula seems to be missing in the literature, or at least we could not find it.

1.5. **Virasoro Constraints and the dilaton equation.** According to Frenkel–Givental–Milanov (see [7, 12]) the total descendant potential of the simple singularity of type D_N is a tau-function of the principal Kac–Wakimoto hierarchy of type D_N. Recalling Corollay 1 we get that the total descendant potential is a tau-function of the (h_1,h_2)-reduction of the 2-component BKP with $h_1=2N-2$ and $h_2=2$. On the other hand, the total descendant potential is known to satisfy Virasoro constraints and the *dilaton equation* (see [10]). In the case at hands the Virasoro constraints and the dilaton equation can be stated as follows. The Virasoro operators are given by

$$L_k = -\frac{\mathbf{i}}{2}\, J^1_{1+(1+k)h} + D_k, \quad k \in \mathbb{Z}, \tag{3}$$

where

$$D_k = \delta_{k,0}\,\frac{N(h+1)}{24h} + \sum_{a=1,2} \frac{1}{4h_a} \sum_{m\in\mathbb{Z}_{\mathrm{odd}}} :J^a_m J^a_{-m+kh_a}: \ . \tag{4}$$

Let us point out that the operator L_k is obtained from D_k via the so-called *dilaton shift* $J^1_{-2N+1} \mapsto J^1_{-2N+1} - \mathbf{i}\,h$.

The total descendent potential depends on an extra parameter $\hbar$. Namely

$$\mathcal{D}(\hbar,\mathbf{t}) = \exp\Big(\sum_{g=0}^{\infty} F^{(g)}(\mathbf{t})\hbar^{g-1}\Big). \tag{5}$$

In order to identify $\mathcal{D}$ with a tau-function of the Kac–Wakimoto hierarchy we put $\hbar=1$. The dependence on $\hbar$ can be recovered via the so called dilaton equation

$$\Big(-\frac{\mathbf{i}h}{h+1}\partial_{t^1_{1+h}} + \sum_{a=1,2}\sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} t^a_m\partial_{t^a_m} + \frac{N}{24} + 2\hbar\partial_\hbar\Big)\mathcal{D} = 0. \tag{6}$$

Note that the first term is obtained from the second one via the dilaton shift. We will say that a formal power series $\tau(\mathbf{t})$ satisfies the dilaton constraint if the solution to the linear PDE (6) with initial condition $\mathcal{D}(1,\mathbf{t}) = \tau(\mathbf{t})$ is a formal power series $\mathcal{D}(\hbar,\mathbf{t})$ that has the form (5).

The Virasoro symmetries can be expressed in terms of the Grassmanian $\mathrm{Gr}_2^{I,(0)}$ as follows. Let us introduce the following differential operators acting component-wise on V

$$\ell_k(z) = (\ell_k^{(1)}(z_1), \ell_k^{(2)}(z_2)), \quad k\in\mathbb{Z}, \tag{7}$$

where

$$\ell_k^{(a)}(z_a) := -\mathbf{i}(z_a)^{1+(1+k)h_a}\delta_{1,a} + \frac{k}{2}(z_a)^{kh_a} + \frac{1}{h_a}(z_a)^{1+kh_a}\frac{\partial}{\partial z_a}, \quad a=1,2.$$

We will prove that if a tau-function of the Kac–Wakimoto hierarchy satisfies the Virasoro constraints $L_k\tau = 0$ for all $k\geq -1$ then the corresponding plane $U\in \mathrm{Gr}_2^{I,(0)}$ satisfies

$$\ell_k(z)U \subset U, \quad k\geq -1.$$

Note that the constraint for $k=-1$, also known as the *string equation*, and the symmetry $(z_1^h, z_2^2)U\subset U$ imply the rest of the Virasoro constraints. The main result of this paper can be stated as follows.

Theorem 2. *If a tau-function of the Kac–Wakimoto hierarchy satisfies the Virasoro and the dilaton constraints, then the corresponding wave function $\Psi(x,z)$ satisfies the following system of PDEs*

$$\begin{aligned}(\partial_1^h+\partial_2^2-\mathbf{i}\,x_1)\Psi(x,z) &= (z_1^h,z_2^2)\Psi(x,z),\\ \partial_1\Psi(x,z) &= \mathbf{i}\ell_{-1}(z)\Psi(x,z),\\ \partial_1\partial_2\Psi(x,z) &= \frac{\mathbf{i}}{2}x_2\Psi(x,z),\end{aligned}$$

where $\partial_1:=\partial/\partial x_1$, $\partial_2:=\partial/\partial x_2$.

Corollary 2. *The total descendant potential is the unique tau-function of the Kac–Wakimoto hierarchy satisfying the string and the dilaton constraints.*

Remark 3. In the case of singularities of type A the Virasoro constraints uniquely determine the tau-function. We do not know whether the additional dilaton constraint imposed in Corollary 2 is necessary.

1.6. **Tau-functions of Gaussian type.** The problem of classifying tau-functions whose logorithm is a quadratic form in the dynamical variables was proposed by Givental in the settings of the N-KdV hierarchy (see [11]). We solve this problem in the case of the principal Kac–Wakimoto hierarchy of type D. The answer is quite suggestive. Namely, one can speculate that there is a general theory of Hirota bilinear equations in which the tau-functions of Gaussian type provide an embedding of the theory of semi-simple Frobenius manifolds into the theory of integrable systems. This expectation is in some sense compatible with the general framework developped by Dubrovin and Zhang in [5].

Suppose that

$$\tau(\mathbf{t}^1,\mathbf{t}^2)=\exp\Big(\frac{1}{2}\sum_{a,b=1}^{2}\sum_{k,\ell\in\mathbb{Z}^+_{\rm odd}}W^{ab}_{k\ell}t^a_kt^b_\ell\Big). \tag{8}$$

Under what conditions τ is a tau-function of the Kac–Wakimoto hierarchy? The answer is given in terms of the following system of equations

$$\begin{aligned}x^{2(N-1)}+\sum_{i=1}^{N-1}t_ix^{2(N-1-i)}+y^2 &= \lambda,\\ xy+t_N &= 0,\end{aligned}$$

where $t=(t_1,\dots,t_N)\in\mathbb{C}^N$ are fixed parameters. We will prove that the set of all coefficients $W^{ab}_{k\ell}$ is uniquely determined from the subset of coefficients with $k=1$. The coefficients $W^{ab}_{1\ell}$ are determined uniquely from the above system of algebraic equations as follows. Put $\lambda=z_1^h$. Then the system admits a formal solution of the form

$$x=z_1-2\sum_\ell W^{11}_{1\ell}(t)z_1^{-\ell}/\ell,\quad y=-2\sum_\ell W^{21}_{1\ell}(t)z_1^{-\ell}/\ell,$$

where both sums are over all $\ell \in \mathbb{Z}^+_{\mathrm{odd}}$ and $W^{11}_{1\ell}(t), W^{21}_{1\ell}(t) \in \mathbb{C}[t]$. Similarly, put $\lambda = z_2^2$, then the system admits a formal solution of the form

$$x = -2\sum_\ell W^{12}_{1\ell}(t) z_2^{-\ell}/\ell, \quad y = z_2 - 2\sum_\ell W^{22}_{1\ell}(t) z_2^{-\ell}/\ell.$$

Theorem 3. *The map $t \mapsto W^{ab}_{k\ell}(t)$ defined above establishes a one-to-one correspondence between $\mathbb{C}^N$ and the set of tau-functions of Gaussian type.*

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2. The principal Kac–Wakimoto hierarchy of type D_N

The main goal of this section is to prove Theorem 1. We will use the language of twisted representations of lattice vertex algebras, because it seems to be the most appropriate one. We have verified the formula in Theorem 1 also directly but the computation is very long. For some quick introduction to twisted vertex algebra modules with an extensive list of references for more systematic study we refer to [2].

2.1. **The Euclidean lattice vertex algebra.** Let $\mathfrak{h} := \mathbb{C}^N$ and $\{v_i\}_{i=1}^N \subset \mathfrak{h}$ be the standard basis. Put

$$\mathbb{Z}^N := \bigoplus_{i=1}^N \mathbb{Z}\, v_i, \quad (v_i|v_j) = \delta_{i,j}$$

for the standard Euclidean lattice. Let

$$\epsilon : \mathbb{Z}^N \times \mathbb{Z}^N \to \{\pm 1\},$$

be the bi-multiplicative function defined by

$$\epsilon(v_i, v_j) = \begin{cases} -1 & \text{if } i \leq j\ , \\ 1 & \text{if } i > j. \end{cases}$$

Recall the *twisted group* algebra $\mathbb{C}_\epsilon[\mathbb{Z}^N]$ with basis e^a ($a \in \mathbb{Z}^N$) and multiplication

$$e^a\, e^b := \epsilon(a,b) e^{a+b}.$$

As a vector space the lattice vertex algebra is defined by

$$V_{\mathbb{Z}^N} := \mathrm{Sym}(\mathfrak{h}[s^{-1}]s^{-1}) \otimes \mathbb{C}_\epsilon[\mathbb{Z}^N],$$

where $\mathrm{Sym}(A)$ is the symmetric algebra of a vector space A. Slightly abusing the notation we will write e^a for $1 \otimes e^a$ and a for $(as^{-1}) \otimes e^0$. The vector $\mathbf{1} := 1 \otimes e^0$ is called the *vacuum*.

The structure of a vertex algebra is given by a bi-linear map

$$Y(\cdot,z): V_{\mathbb{Z}^N}\otimes V_{\mathbb{Z}^N}\to V_{\mathbb{Z}^N}(\!(z)\!),$$

which is also known as the *state-field correspondence*. Let us recall the construction of Y. Let $\widehat{\mathfrak{h}}=\mathfrak{h}[s,s^{-1}]\oplus\mathbb{C}\,K$ be the Heisenberg Lie algebra whose commutator is defined by

$$[as^m,bs^n]=m\delta_{m,-n}(a|b)\,K.$$

There is a unique way to turn $V_{\mathbb{Z}^N}$ into a $\widehat{\mathfrak{h}}$-module such that as^m for $m<0$ acts as multipliction by $(as^m)\otimes 1$, the central element K acts by 1, and for $m\geq 0$

$$as^m(1\otimes e^b)=\delta_{m,0}(a|b)\,1\otimes e^b,\quad a\in\mathfrak{h},\quad b\in\mathbb{Z}^N.$$

The linear operator representing $as^n\in\widehat{\mathfrak{h}}$ is denoted by $a_{(n)}$. Put

$$Y(a,z):=\sum_{n\in\mathbb{Z}}a_{(n)}z^{-n-1},\quad a\in\mathfrak{h}$$

and

$$Y(e^b,z):=e^bz^{b_{(0)}}e^{\sum_{j>0}b_{(-j)}\frac{z^j}{j}}e^{\sum_{j<0}b_{(-j)}\frac{z^j}{j}},\quad b\in\mathbb{Z}^N.$$

The definition of the state-field correspondence can be extended uniquely so that the following formula holds

$$Y(a_{(n)}b,z)=\frac{1}{k!}\,\partial_w^k\Big((w-z)^{n+1+k}Y(a,w)Y(b,z)\Big)\Big|_{w=z}\tag{9}$$

for every $a,b\in V_{\mathbb{Z}^N}$, where we choose $k\gg 0$ so big that

$$(w-z)^{n+1+k}[Y(a,w),Y(b,z)]=0,\tag{10}$$

and denote by $a_{(n)}$ the Fourier modes $Y(a,z)=:\sum_n a_{(n)}z^{-n-1}$. This formula defines $Y(a_{(n)}b,z)$ assuming that we already know $Y(a,z)$ and $Y(b,z)$. Clearly this is a recursive procedure that defines the state-field correspondence in terms of $Y(a,z)$ $(a\in\mathfrak{h})$ and $Y(e^a,z)$ $(a\in\mathbb{Z}^N)$. For more details we refer to [2], Proposition 3.2.

The main property of the above definition is the so-called Borcherd's identity for the modes

$$\begin{aligned}&\sum_{j=0}^{\infty}(-1)^j\binom{n}{j}\Big(a_{(m+n-j)}(b_{(k+j)}c)-(-1)^n\,b_{(k+n-j)}(a_{(m+j)}c)\Big)\\&\qquad=\sum_{j=0}^{\infty}\binom{m}{j}(a_{(n+j)}b)_{(k+m-j)}c,\end{aligned}\tag{11}$$

where $a,b,c\in V_{\mathbb{Z}^N}$. Observe that the above sums are finite, because $a_{(n)}b=0$ for sufficiently large n. We will need also the following commutator formula

$$[Y(a,w),Y(b,z)]=\sum_{n=0}^{\infty}\frac{1}{n!}Y(a_{(n)}b,w)\partial_w^n\delta(z,w),$$

where $\delta(z,w)=\sum_{m\in\mathbb{Z}} z^m w^{-m-1}$ is the formal delta-function. The above formula is equivalent to the Borcherd's identity (11) with $n=0$. For more details on lattice vertex algebras and for the proofs of the statements in this section we refer to [15].

2.2. **The Frenkel-Kac construction.** Let us recall the Frenkel–Kac construction (see [8]) of the affine Kac–Moody Lie algebra of type D_N. Recall that the root system of type D_N consists of the following vectors

$$\Delta=\{\pm(v_i\pm v_j)\mid 1\leq i\neq j\leq N\}.$$

The lattice vertex algebra $V_{\mathbb{Z}^N}$ has a *conformal vector*

$$\nu := \frac{1}{2}\sum_{i=1}^{N} v_{i(-1)}v_{i(-1)}\mathbf{1}. \tag{12}$$

The affine Kac–Moody Lie algebra of type D_N can be constructed by the following simple formula

$$\widehat{so}_{2N}(\mathbb{C})\cong\bigoplus_{\alpha\in\Delta}\bigoplus_{n\in\mathbb{Z}}\mathbb{C}e^{\alpha}_{(n)}\oplus\bigoplus_{i=1}^{N}\bigoplus_{n\in\mathbb{Z}}\mathbb{C}v_{i(n)}\oplus\mathbb{C}\nu_{(1)}\oplus\mathbb{C}\,\mathbf{1}_{(-1)}\ ,$$

where the RHS is equipped with a Lie bracket given by the commutator:

$$[x_{(m)},y_{(n)}]:=x_{(m)}y_{(n)}-y_{(n)}x_{(m)},\quad x,y\in V_{\mathbb{Z}^N},\quad m,n\in\mathbb{Z}.$$

We leave it to the reader as an exercise to use the Borcherd's identity (11) with $n=0$ to check that the above formula indeed defines a Lie algebra isomorphic to $\widehat{so}_{2N}(\mathbb{C})$.

2.3. **Casimirs of $\widehat{so}_{2N}(\mathbb{C})$.** The tensor product $V_{\mathbb{Z}}\otimes V_{\mathbb{Z}}$ also has the stucture of a vertex operator algebra with state-field correspondence defined by

$$Y(a\otimes b,z):=Y(a,z)\otimes Y(b,z).$$

Let us define the following two vectors in $V_{\mathbb{Z}^N}^{\otimes 2}$:

$$\omega_{\mathrm{KW}}=-\sum_{\alpha\in\Delta}e^{\alpha}\otimes e^{-\alpha}+\sum_{i=1}^{N}v_i\otimes v_i-\nu\otimes 1-1\otimes\nu$$

and

$$\omega_{\mathrm{BKP}}=\sum_{i=1}^{N}(e^{v_i}\otimes e^{-v_i}+e^{-v_i}\otimes e^{v_i}).$$

Lemma 4. *The following relation holds*

$$\omega_{\mathrm{KW}}=-\frac{1}{2}(\omega_{\mathrm{BKP}})_{(-1)}\omega_{\mathrm{BKP}}.$$

Proof. By definition

$$(\omega_{\rm BKP})_{(-1)}\omega_{\rm BKP} = \operatorname{Res}_{z=0}\frac{dz}{z}\Big(Y(\omega_{\rm BKP},z)\omega_{\rm BKP}\Big).$$

For $\lambda,\mu\in\{+1,-1\}$ and $1\leq i,j\leq N$ we have

$$\frac{dz}{z}\Big((Y(e^{\lambda v_i},z)\otimes Y(e^{-\lambda v_i},z))e^{\mu v_j}\otimes e^{-\mu v_j}\Big) = \frac{dz}{z}\, z^{2\lambda\mu\delta_{i,j}}\times$$
$$(e^{\sum_{n>0}\lambda v_{i(-n)}\frac{z^n}{n}}e^{\lambda v_i+\mu v_j})\otimes(e^{-\sum_{n>0}\lambda v_{i(-n)}\frac{z^n}{n}}e^{-\lambda v_i-\mu v_j}).$$

The residue of the above 1-form is not 0 only if $i\neq j$ or $i=j$ and $\mu=-\lambda$. In the former case the residue is $e^\alpha\otimes e^{-\alpha}$ with $\alpha=\lambda v_i+\mu v_j\in\Delta$, while in the latter it is

$$\frac{\lambda}{2}(v_{i(-2)}\otimes 1-1\otimes v_{i(-2)})+\frac{1}{2}(v_{i(-1)}^2\otimes 1+1\otimes v_{i(-1)}^2)-v_{i(-1)}\otimes v_{i(-1)}.$$

Summing over all $i,j\in\{1,2,\ldots,N\}$ and over all $\lambda,\mu\in\{+1,-1\}$ we get

$$\operatorname{Res}_{z=0}\frac{dz}{z}\Big(Y(\omega_{\rm BKP},z)\omega_{\rm BKP}\Big) = -2\omega_{\rm KW}.$$

2.4. **The Coxeter transformation.** Let $\alpha_i=v_i-v_{i+1}$ $(1\leq i\leq N-1)$ and $\alpha_N=v_{N-1}+v_N$ be a set of simple roots. Let us fix a Coxeter transformation

$$\sigma := r_{\alpha_1}\cdots r_{\alpha_N},$$

where $r_{\alpha_i}(x)=x-(\alpha_i|x)\alpha_i$ are the simple reflections. The action of σ on the standard basis is represented by the diagram

$$v_1\mapsto v_2\mapsto\cdots\mapsto v_{N-1}\mapsto -v_1,\quad v_N\mapsto -v_N.$$

Let us choose an eigenbasis $\{H_i\}_{i=1}^N$ of σ such that

$$\sigma(H_i)=\eta^{m_i}H_i,\quad (H_i|H_j)=h\delta_{i,j^*},$$

where $\eta:=e^{2\pi\mathbf{i}/h}$ (recall that $h=2N-2$ is the Coxeter number), $m_i:=2i-1$ $(1\leq i\leq N-1)$ and $m_N=N-1$ are the so-called *exponents*, and * is an involution defined by

$$j^*:=\begin{cases} n-j & \text{if } 1\leq j\leq N-1,\\ j & \text{if } j=N.\end{cases}$$

To be more specific, let us define H_i as the solutions to the following linear system of equations (with unknowns H_i):

$$v_i=\frac{\sqrt{2}}{h}\Big(\eta^{m_1 i}H_1+\cdots+\eta^{m_{N-1}i}H_{N-1}\Big),\quad 1\leq i\leq N-1,$$
$$v_N=\frac{1}{\sqrt{h}}H_N.$$

Following Bakalov–Kac (see [1], Proposition 4.1) we extend the Coxeter transformation σ to a vertex algebra automorphism of $V_{\mathbb{Z}^N}$. The action of σ on $\operatorname{Sym}(\mathfrak{h}[s^{-1}]s^{-1})$ is induced from the action of σ on $\mathfrak{h}$. While the definition of the action of σ on the twisted group algebra involves the choice of a function

$$\zeta : \mathbb{Z}^N \to \{+1, -1\},$$

such that

$$\epsilon(\sigma(a), \sigma(b))\epsilon(a, b)^{-1} = \zeta(a + b)\zeta(a)^{-1}\zeta(b)^{-1}, \quad a, b \in \mathbb{Z}^N.$$

The definition

$$\sigma(e^a) = \zeta(a)e^{\sigma(a)}, \quad a \in \mathbb{Z}^N$$

extends uniquely to an automorphism of the twisted group algebra $\mathbb{C}_\epsilon[\mathbb{Z}^N]$. The linear map

$$\sigma : V_{\mathbb{Z}^N} \to V_{\mathbb{Z}^N}, \quad \sigma(x \otimes e^a) := \sigma(x) \otimes \sigma(e^a),$$

is an automorphism of vertex algebras, i.e., $\sigma(x_{(n)}y) = \sigma(x)_{(n)}\sigma(y)$ for all $x, y \in V_{\mathbb{Z}^N}$ and for all $n \in \mathbb{Z}$.

Let us define

$$\pi_n(v) := \frac{1}{h} \sum_{j=1}^{h} \eta^{jn} \sigma^j(v), \quad v \in V_{\mathbb{Z}^N}.$$

Note that π_n is the projection of v onto the eigensubspace of σ corresponding to eigenvalue $e^{-2\pi \mathbf{i} n/h}$. After a small modification of the Frenkel–Kac construction we define the following Lie algebra:

$$\widehat{so}_{2N}(\mathbb{C}, \sigma) := \bigoplus_{\alpha \in \Delta/\sigma} \bigoplus_{n \in \mathbb{Z}} \mathbb{C}(e^\alpha)^\sigma_{(n)} \oplus \bigoplus_{i \in \{1,N\}} \bigoplus_{n \in \mathbb{Z}} \mathbb{C}(v_i)^\sigma_{(n)} \oplus \mathbb{C}\nu_{(1)} \oplus \mathbb{C}\, \mathbf{1}_{(-1)} \ ,$$

where for $x \in V_{\mathbb{Z}^N}$ we put $x^\sigma_{(n)} := (\pi_n(x))_{(n)}$ and Δ/σ is the set of orbits of σ in Δ. Note that if x and y belong to the same orbit of σ, i.e., $x = \sigma^\ell(y)$ for some $\ell \in \mathbb{Z}$, then the complex lines $\mathbb{C}\, x^\sigma_{(n)}$ and $\mathbb{C}\, y^\sigma_{(n)}$ coincide. Using the Borcherd's identities one can check that $\widehat{so}_{2N}(\mathbb{C}, \sigma)$ is isomorphic to the *twisted affine Lie algebra* of type D_N corresponding to the automorphism σ. We refer to [14], Section 8.2 for some background on twisted affine Lie algebras.

2.5. **Twisted representations.** Suppose that M is a vector space and that

$$Y^M(\cdot, \lambda) : V_{\mathbb{Z}^N} \otimes M \to M(\!(\lambda^{1/h})\!)$$

is a linear map that defines a σ-twisted representation of $V_{\mathbb{Z}^N}$ on M (see [1], Definition 3.1). If $a \in V_{\mathbb{Z}^N}$ then we denote by $a^M_{(m)}$ $(m \in \frac{1}{h}\mathbb{Z})$ the modes of the twisted field

$$Y^M(a, \lambda) = \sum_{m \in \frac{1}{h}\mathbb{Z}^N} a^M_{(m)}\, \lambda^{-m-1}.$$

Let us recall the following three properties (see [2]) of a twisted representation:

(i) σ-invariance: $Y^M(\sigma(v),\lambda)$ coincides with the analytic continuation in counter clock-wise direction around $\lambda=0$ of $Y^M(v,\lambda)$.

(ii) Locality: for every $a,b\in V_{\mathbb{Z}^N}$ there exists $n_{ab}\geq 0$ such that

$$(\lambda_1-\lambda_2)^{n_{ab}}[Y^M(a,\lambda_1),Y^M(b,\lambda_2)]=0.$$

(iii) Product formula: (9) remains true if we replace Y with Y^M, i.e.,

$$Y^M(a_{(n)}b,\lambda)=\frac{1}{k!}\,\partial_\mu^k\Big((\mu-\lambda)^{n+1+k}Y^M(a,\mu)Y^M(b,\lambda)\Big)\Big|_{\mu=\lambda}, \tag{13}$$

for all $a,b\in V_{\mathbb{Z}^N}$ and $n\in\mathbb{Z}$.

In fact the above properties and the vacuum axiom $Y_\sigma(\mathbf{1},\lambda)=1$ can be used as a definition of a twisted representation. In particular, the locality and the product formula imply the Borcherd's identity for the twisted modes

$$\begin{aligned}\sum_{j=0}^\infty(-1)^j\binom{n}{j}\Big(a^M_{(m+n-j)}(b^M_{(k+j)}c)-(-1)^n\,b^M_{(k+n-j)}(a^M_{(m+j)}c)\Big)\\ =\sum_{j=0}^\infty\binom{m}{j}(a_{(n+j)}b)^M_{(k+m-j)}c\,,\end{aligned} \tag{14}$$

for all $a\in V_{\mathbb{Z}^N}$, s.t., $\sigma(a)=e^{-2\pi\mathbf{i}m}a$, $b\in V_{\mathbb{Z}^N}$, $c\in M$, $m,k\in\frac{1}{h}\mathbb{Z}$, and $n\in\mathbb{Z}$.

Using the two Borcherd's identities (11) and (14) it is straightforward to check that the maps

$$(e^\alpha)^\sigma_{(n)}\mapsto(e^\alpha)^M_{(n/h)},\quad(\alpha\in\Delta,\quad n\in\mathbb{Z}),\quad \nu_{(1)}\mapsto h\nu^M_{(1)},\quad \mathbf{1}_{(-1)}\mapsto h^{-1}\mathrm{Id}_M$$

define a representation of $\widehat{so}_{2N}(\mathbb{C},\sigma)$ on M. Let us observe also that the standard Euclidean pairing on $\mathfrak{h}$ can be extended uniquely to an invariant bi-linear form on $\widehat{so}_{2N}(\mathbb{C})$, such that

$$(e^\alpha_{(m)}|e^\beta_{(n)})=-\delta_{\alpha,-\beta}\delta_{m,-n},\quad (a_{(m)}|b_{(n)})=(a|b)\delta_{m,-n},\quad(\nu_{(1)}|\mathbf{1}_{(-1)})=-1,$$

where $a,b\in\mathfrak{h}$, $\alpha,\beta\in\Delta$, $m,n\in\mathbb{Z}$, and all other pairings vanish. The twisted affine Lie algebra $\widehat{so}_{2N}(\mathbb{C},\sigma)$ is a Lie subsalgebra of $\widehat{so}_{2N}(\mathbb{C})$. The induced bi-linear form is still non-degenerate and invariant. Therefore we can construct a Casimir for $\widehat{so}_{2N}(\mathbb{C},\sigma)$ via $\sum_a x_a\otimes x^a$, where $\{x_a\}$ and $\{x^a\}$ is a pair of dual bases of $\widehat{so}_{2N}(\mathbb{C},\sigma)$. Let us fix a basis of the following type

$$(e^{\alpha_i})^\sigma_{(n)},\quad (v_1)^\sigma_{(m)},\quad (v_N)^\sigma_{(m(N-1))},\quad \nu_{(1)},\quad \mathbf{1}_{(-1)},$$

where $n\in\mathbb{Z}$, $m\in\mathbb{Z}_{\rm odd}$, and the roots $\alpha_i\in\Delta$ $(1\leq i\leq N)$ are such that the corresponding orbits of the Coxeter transformation are pairwise disjoint. Using the formulas

$$\begin{aligned}((e^{\alpha_i})^\sigma_{(m)}|(e^{-\alpha_j})^\sigma_{(n)})&=-\delta_{m,-n}\delta_{i,j}/h\\ ((v_i)^\sigma_{(m)}|(v_i)^\sigma_{(n)})&=\begin{cases}\frac{1}{N-1}\,\delta_{m,-n} & \text{if } 1\leq i\leq N-1,\\ \delta_{m,-n} & \text{if } i=N,\end{cases}\end{aligned}$$

it is straightforward to construct a dual basis:

$$-h(e^{-\alpha_i})^\sigma_{(-n)} \quad (N-1)(v_1)^\sigma_{(-m)} \quad (v_N)^\sigma_{(-m(N-1))}, \quad -\mathbf{1}_{(-1)}, \quad -\nu_{(1)}.$$

The Casimir takes the form

$$-\sum_{\alpha\in\Delta}\sum_{n\in\mathbb{Z}}(e^\alpha)^\sigma_{(n)}\otimes(e^{-\alpha})^\sigma_{(-n)} - \nu_{(1)}\otimes\mathbf{1}_{(-1)} - \mathbf{1}_{(-1)}\otimes\nu_{(1)} + \\ + \sum_{i=1}^N\sum_{n\in\mathbb{Z}}(v_i)^\sigma_{(n)}\otimes(v_i)^\sigma_{(-n)},$$

where we used that the expressions $(e^\alpha)^\sigma_{(n)}\otimes(e^{-\alpha})^\sigma_{(-n)}$ and $(v_i)^\sigma_{(n)}\otimes(v_i)^\sigma_{(-n)}$ are invariant under the Coxeter transformations $\alpha\mapsto\sigma\alpha$ and $v_i\mapsto\sigma v_i$, respectively. Note that the action of the Casimir on $M^{\otimes 2}$ is given by $\mathrm{Res}_{\lambda=0}\left(Y^M(\omega_{\mathrm{KW}},\lambda)\lambda d\lambda\right)$.

2.6. **Twisted modules over the Euclidean lattice vertex algebra.** The σ-twisted modules over a lattice vertex algebra are classified by Bakalov and Kac in [1]. Let us recall their result in the case of $V_{\mathbb{Z}^N}$ when σ is the Coxeter transformation.

The σ-twisted module structure is determined uniquely by the so called *Heisenberg pair* $(\widehat{\mathfrak{h}}_\sigma, G_\sigma)$, which is defined as follows. The first member of the pair is the σ-twisted Heisenberg algebra

$$\widehat{\mathfrak{h}}_\sigma = \mathbb{C}\,K \oplus \bigoplus_{i=1}^N\bigoplus_{n\in\mathbb{Z}}\mathbb{C}\,H_i s^{-n-m_i/h},$$

where the notation is the same as in Section 2.4. The Lie bracket is given by

$$[h's^m, h''s^n] := m\delta_{m,-n}(h'|h'')\,K, \quad h',h''\in\mathfrak{h}, \quad m,n\in\frac{1}{h}\mathbb{Z}.$$

Let $G=\mathbb{C}^*\times\mathbb{Z}^N$ be the set whose elements will be written as cU_α, $c\in\mathbb{C}^*$, $\alpha\in\mathbb{Z}^N$. The following multiplication turns G into a group

$$U_\alpha U_\beta := \epsilon(\alpha,\beta)B(\alpha,\beta)^{-1}U_{\alpha+\beta},$$

where

$$B(\alpha,\beta) := h^{-(\alpha|\beta)}\prod_{k=1}^{h-1}(1-\eta^k)^{(\sigma^k(\alpha)|\beta)}.$$

Put

$$N_\sigma := \{\zeta(\alpha)^{-1}U_{\sigma\alpha}^{-1}U_\alpha\,(-1)^{|\alpha|^2} \mid \alpha\in\mathbb{Z}^N\},$$

where $|\alpha|^2 := (\alpha|\alpha)$. Using the commutator formula

$$U_\alpha U_\beta U_\alpha^{-1}U_\beta^{-1} = \exp\ 2\pi\mathbf{i}\Big(\frac{1}{2}|\alpha|^2|\beta|^2 + ((1-\sigma)^{-1}\alpha|\beta)\Big)$$

it is easy to check that N_σ is a subroup of the center $Z(G)$. The second member of the Heisenberg pair is the quotient group $G_\sigma := G/N_\sigma$. Let us denote by $\overline{U}_\alpha \in G_\sigma$ the image of U_α under the quotient map. We will be interested in representations of G_σ that are $\mathbb{C}^*$-invariant, i.e., the element $c\overline{U}_0 \in G_\sigma$ $(c \in \mathbb{C}^*)$ acts by multiplication by the scalar c.

Lemma 5. *a) The following relations hold*

$$\overline{U}_{v_1}^2 = (-1)^N \zeta(v_1)\cdots\zeta(v_{N-1})\,\frac{1}{2h}, \quad \overline{U}_{v_N}^2 = \frac{1}{4}\,\zeta(v_N)\,, \quad \overline{U}_{v_1}\overline{U}_{v_N} = -\overline{U}_{v_N}\overline{U}_{v_1}.$$

b) The elements

$$X := \Big((-1)^{N-1}\zeta(v_1)\cdots\zeta(v_{N-1})\,\frac{1}{2h}\Big)^{-1/2}\overline{U}_{v_1} \quad Y := \frac{1}{2}\,(-\zeta(v_N))^{-1/2}\overline{U}_{v_N}$$

generate a subgroup of G_σ isomorphic to the quaternion group

$$Q_8 := \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\},$$

where the multiplication is given by the standard quaternion relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}$.

c) The $\mathbb{C}^*$-invariant representations of G_σ are given by $\mathbb{H}^n$ $(n \geq 1)$, where $\mathbb{H}$ is the quaternion algebra and we fix an identification $\mathbb{H} \cong \mathbb{C}^2$, such that the natural left action of Q_8 on $\mathbb{H}$ becomes a 2-dimensional representation.

Proof. a) Using that $\sigma^{N-1}(v_1) = -v_1$ we get that

$$(-1)^{N-1}\zeta(v_1)\cdots\zeta(v_{N-1})U_{v_1}^{-1}U_{-v_1} \in N_\sigma.$$

On the other hand by definition

$$U_{-v_1}U_{v_1} = \frac{\epsilon(-v_1, v_1)}{B(-v_1, v_1)}\,U_0 = -\frac{1}{2h},$$

so the relation for $\overline{U}_{v_1}$ follows. The proof of the second relation is similar, while the last one follows directly from the definitions.

b) By part a) we have $X^2 = -1$, $Y^2 = -1$ and $XY = -YX$. We need just to prove that there are no further relations between X and Y. This is equivalent to proving that elements of the type cU_{v_1}, cU_{v_N}, or $cU_{v_1}U_{v_N}$ do not belong to N_σ. On the other hand, every element of N_σ has the form $c_\alpha U_{(1-\sigma)\alpha}$ for some $\alpha \in \mathbb{Z}^N$ and $c_\alpha \in \mathbb{C}^*$. Our claim follows from the fact that

$$(1-\sigma)^{-1}(v_1) = \frac{1}{2}(v_1 + \cdots + v_{N-1}), \quad (1-\sigma)^{-1}(v_N) = \frac{1}{2}v_N.$$

c) Since the action of the Coxeter transformation on v_1 and v_N produces two orbits that contain $\pm v_i$ for all i, we get that the group G_σ is generated by $\mathbb{C}^*$, X, and Y. The $\mathbb{C}^*$-invariance implies that every representation is uniquely determined by its restriction to the quaternion subgroup $Q_8 = \langle X, Y\rangle$. Moreover, the $\mathbb{C}^*$-invariance implies that $-1 \in Q_8$ acts by -1 in the representation. The representations of the quaternion group Q_8 are completely reducible and it is known that the there are 5 irreducible ones. Their dimensions are $1, 1, 1, 1$, and 2 and only the 2-dimensional one has the property that -1 acts by -1. $\square$

The result of Bakalov and Kac (see [1], Proposition 4.2) can be stated as follows. Suppose that M is a σ-twisted $V_{\mathbb{Z}^N}$-module. Note that due to *sigma*-invariance the modes $(H_i)^M_{(-m/h)}$ are not zero only if $m \equiv m_i (\mathrm{mod}\ h)$. The twisted Borcherd's identity implies that

$$K \mapsto 1, \quad H_i s^{-n-m_i/h} \mapsto (H_i)^M_{(-n-m_i/h)} \quad (1 \leq i \leq N, n \in \mathbb{Z})$$

is a representation of $\widehat{\mathfrak{h}}_\sigma$. The axioms of a twisted module imply that

$$Y^M(e^\alpha, \lambda) = U^M_\alpha\, \lambda^{-|\alpha|^2/2} : \exp\Big(\sum_{n \in \frac{1}{h}\mathbb{Z} - \{0\}} \alpha^M_{(-n)} \frac{\lambda^n}{n} \Big) :$$

where U^M_α are certain linear operators that commute with the representation of $\widehat{\mathfrak{h}}_\sigma$. Note that the σ-invariance imply that $U^M_{\sigma\alpha} = \zeta(\alpha)^{-1} U^M_\alpha (-1)^{|\alpha|^2}$, which is the relation imposed on the elements of the group G. Moreover, the assignment

$$cU_\alpha \mapsto cU^M_\alpha, \quad c \in \mathbb{C}^*, \quad \alpha \in \mathbb{Z}^N,$$

defines a representation of G_σ on M such that $U^M_0 = \mathrm{Id}_M$. The map just described establishes an equivalence between the category of σ-twisted $V_{\mathbb{Z}^N}$-modules and the category of $(\widehat{\mathfrak{h}}_\sigma, G_\sigma)$-modules in which $K \in \widehat{\mathfrak{h}}_\sigma$ is represented by the identity operator and the representation of G_σ is $\mathbb{C}^*$-invariant.

Lemma 6. *The fermionic Fock space $\mathcal{F}$ has a structure of a σ-twisted $V_{\mathbb{Z}^N}$-module such that*

$$Y^{\mathcal{F}}(e^{\pm v_i}, \lambda) = C^\pm_i\, \lambda^{-1/2} \phi_1(\pm \lambda^{1/h} \eta^i), \quad 1 \leq i \leq N-1$$
$$Y^{\mathcal{F}}(e^{\pm v_N}, \lambda) = C^\pm_N\, \lambda^{-1/2} \phi_2(\pm \lambda^{1/2}),$$

where $C^\pm_i$ $(1 \leq i \leq N)$ are some constants satisfying

$$C^+_i\, C^-_i = -1/h \quad (1 \leq i \leq N-1), \quad C^+_N\, C^-_N = -1/2.$$

Proof. Recalling the commutation relations for J^a_m we get that the map

$$H_i s^{-m/h} \mapsto \frac{1}{\sqrt{2}} J^1_{-m} \quad (1 \leq i \leq N-1) \quad H_N s^{-m(N-1)/h} \mapsto \sqrt{\frac{N-1}{2}} J^2_{-m}$$

defines a representation of the twisted Heisenberg algebra $\widehat{\mathfrak{h}}_\sigma$ in which $K \mapsto 1$. Since the operators Q_1 and Q_2 satisfy the relations $Q_1^2 = Q_2^2 = 1/2$ and $Q_1 Q_2 = -Q_2 Q_1$, we can choose constants C^+_1 and C^+_N such that the map

$$X \mapsto \Big((-1)^{N-1} \zeta(v_1) \cdots \zeta(v_{N-1})\, \frac{1}{2h} \Big)^{-1/2} C^+_1 Q_1,$$
$$Y \mapsto \frac{1}{2} \Big(-\zeta(v_N) \Big)^{-1/2} C^+_N Q_2$$

defines a representation of the quaternion group Q_8 and hence a $\mathbb{C}^*$-invariant representation of G_σ on the fermionic Fock space $\mathcal{F}$. Therefore, we can equip $\mathcal{F}$ with the structure of a σ-twisted $V_{\mathbb{Z}^N}$-module, such that $Y^{\mathcal{F}}(e^{\pm v_i}, \lambda)$ has the form stated in the lemma.

Let us porve the relations between the constants. Note that $U^{\mathcal{F}}_{v_1} = C_1^+ Q_1$ and $U^{\mathcal{F}}_{-v_1} = C_1^- Q_1$, so

$$-\frac{1}{2h} = U^{\mathcal{F}}_{v_1} U^{\mathcal{F}}_{-v_1} = C_1^+ C_1^- \,\frac{1}{2}.$$

The proof of the remaining relations is similar. □

2.7. **Proof of Theorem 1.** The Casimir Ω_{KW} of the Kac–Wakimoto hierarchy is given by the residue of the 1-form $Y^{\mathcal{F}}(\omega_{\mathrm{KW}}, \lambda)\lambda d\lambda$. Recall Lemma 4 and the product formula (13). After a direct computation using Lemma 6 we get that

$$Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda) = \sum_{m\in\mathbb{Z}} \widetilde{\Omega}_m \lambda^{-m-1},$$

where

$$\widetilde{\Omega}_m = -\sum_{a=1}^{2}\sum_{k\in\mathbb{Z}} (-1)^k \phi_a(k)\otimes \phi_a(-k-mh_a).$$

Moreover another straightforward computation using the identity

$$[Y^{\mathcal{F}}(a,\lambda_1), Y^{\mathcal{F}}(b,\lambda_2)] = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\mathcal{F}}(a_{(n)}b, \lambda_2)\partial_{\lambda_2}^n \delta(\lambda_1,\lambda_2),$$

where $a, b \in V_{\mathbb{Z}^N}$ are σ-invariant and $\delta(\lambda_1,\lambda_2) := \sum_{n\in\mathbb{Z}} \lambda_1^n\lambda_2^{-n-1}$ is the formal delta function, yields the following commutation relations

$$[\widetilde{\Omega}_m, \widetilde{\Omega}_n] = 2Nm\delta_{m,-n}.$$

Let us define normal ordering

$$:\widetilde{\Omega}_m\widetilde{\Omega}_n := \begin{cases} \widetilde{\Omega}_m\widetilde{\Omega}_n & \text{if } n\geq 0,\\ \widetilde{\Omega}_n\widetilde{\Omega}_m & \text{if } n<0.\end{cases}$$

Then we have

$$Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda_1)Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda) =: Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda_1)Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda): + \frac{2N}{(\lambda_1-\lambda_2)^2}.$$

Recalling the product formula (with $k=2$) we get

$$-\frac{1}{2}Y^{\mathcal{F}}((\omega_{\mathrm{BKP}})_{(-1)}\omega_{\mathrm{BKP}}, \lambda) = -\frac{1}{2} : Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda)Y^{\mathcal{F}}(\omega_{\mathrm{BKP}}, \lambda): \;.$$

We get the following formula for the Casimir

$$\Omega_{\mathrm{KW}} = -\frac{1}{2}\,\widetilde{\Omega}_0^2 - \sum_{m=1}^{\infty} \widetilde{\Omega}_{-m}\widetilde{\Omega}_m.$$

Finally, it remains only to recall that by definition

$$\widetilde{\Omega}_m = -(Q_1\otimes Q_1)\,\Omega_m$$

and that $(Q_1\otimes Q_1)^2 = \frac{1}{4}$. □

3. Virasoro symmetries

The goal in this section is to prove Corollary 1 and Theorem 2. We will need to work out the commutation relations between the fermions $\phi_a(k)$ and the Virasoro operators D_k (see (4)). This can be done directly (see for example [17]). For the sake of completeness we would like to use the twisted module structure on $\mathcal{F}$ and derive all commutation relations from the Borcherd's identities.

3.1. **Commutation relations.**

Recall the σ-twisted $V_{\mathbb{Z}^N}$-module structure on $\mathcal{F}$ from the previous section (see Lemma 6). By definition we have

$$Y^{\mathcal{F}}(H_N,\lambda)=\sqrt{\frac{N-1}{2}}\sum_{m\in\mathbb{Z}_{\mathrm{odd}}}J^2_{-m}\lambda^{m/2-1}$$

and

$$Y^{\mathcal{F}}(H_i,\lambda)=\frac{1}{\sqrt{2}}\sum_{m}J^1_{-m}\lambda^{m/h-1},\quad 1\leq i\leq N-1,$$

where the sum is over all $m\in\mathbb{Z}$ such that $m\equiv m_i(\mathrm{mod}\ h)$. Therefore, the twisted fields representing the standard basis v_i are given by

$$Y^{\mathcal{F}}(v_i,\lambda)=\frac{1}{h\lambda}\sum_{m\in\mathbb{Z}_{\mathrm{odd}}}J^1_{-m}(\lambda^{1/h}\eta^i)^m,\quad 1\leq i\leq N-1,$$

and

$$Y^{\mathcal{F}}(v_N,\lambda)=\frac{1}{2\lambda}\sum_{m\in\mathbb{Z}_{\mathrm{odd}}}J^2_{-m}(\lambda^{1/2})^m.$$

Lemma 7. *The conformal vector* (12) *is represented by the twisted field*

$$Y^{\mathcal{F}}(\nu,\lambda)=\sum_{k\in\mathbb{Z}}D_k\lambda^{-k-2},$$

where D_k are the operators (4)*.*

Proof. This is a straightforward computation using the product formula (13). Let us point out the main steps leaving some of the details to the reader. We have

$$\begin{aligned}&Y^{\mathcal{F}}(v_i,\lambda_1)Y^{\mathcal{F}}(v_i,\lambda_2)=\\&:Y^{\mathcal{F}}(v_i,\lambda_1)Y(v_i,\lambda_2):+\frac{\lambda_1^{1/h-1}\lambda_2^{1/h-1}}{(\lambda_1^{1/h}-\lambda_2^{1/h})^2h^2}+\frac{\lambda_1^{1/h-1}\lambda_2^{1/h-1}}{(\lambda_1^{1/h}+\lambda_2^{1/h})^2h^2},\end{aligned}$$

where $1\leq i\leq N-1$ and the normal ordering means that the currents J^1_m with $m>0$ should be applied first. The formula for $i=N$ is the same except that we have to replace everywhere h with 2.

Note that we have the following Taylor's series expansions at $\lambda_1=\lambda_2$:

$$(\lambda_1-\lambda_2)^2\,\frac{\lambda_1^{1/h-1}\lambda_2^{1/h-1}}{(\lambda_1^{1/h}-\lambda_2^{1/h})^2h^2}=1+\frac{h^2-1}{12h^2\lambda_2^2}\,(\lambda_1-\lambda_2)^2+\cdots$$

and

$$(\lambda_1-\lambda_2)^2\,\frac{\lambda_1^{1/h-1}\lambda_2^{1/h-1}}{(\lambda_1^{1/h}+\lambda_2^{1/h})^2h^2}=\frac{1}{4h^2\lambda_2^2}\,(\lambda_1-\lambda_2)^2+\cdots\ .$$

Recalling the product formula (13) we get

$$Y^{\mathcal{F}}(v_{i(-1)}v_i,\lambda)=:Y^{\mathcal{F}}(v_i,\lambda)Y^{\mathcal{F}}(v_i,\lambda):\ +\ \frac{h^2+2}{12h^2}\,\lambda^{-2}$$

for all $1\leq i\leq N-1$. For $i=N$ the formula remains the same except that we have to replace h with 2. Recalling the formulas for $Y^{\mathcal{F}}(v_i,\lambda)$ we get

$$\frac{1}{2}\sum_{i=1}^{N-1}:Y^{\mathcal{F}}(v_i,\lambda)Y^{\mathcal{F}}(v_i,\lambda):=\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}_{\rm odd}}\frac{1}{4h}:J^1_mJ^1_{-m-kh}:\lambda^{-k-2},$$

and

$$\frac{1}{2}:Y^{\mathcal{F}}(v_N,\lambda)Y^{\mathcal{F}}(v_N,\lambda):=\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}_{\rm odd}}\frac{1}{8}:J^2_mJ^2_{-m-2k}:\lambda^{-k-2}.$$

Finally it remains only to sum up the extra contributions to the coefficient in front of λ^{-2}:

$$\frac{h^2+2}{24h^2}\,(N-1)+\frac{2^2+2}{24\cdot 2^2}=\frac{(h+1)N}{24h}.\quad\Box$$

The twisted Borcherd's identity (14) with $n=0$ imply the following formula

$$[Y^{\mathcal{F}}(\nu,\lambda_1),Y^{\mathcal{F}}(a,\lambda_2)]=\sum_{m=0}^{\infty}\frac{1}{m!}Y^{\mathcal{F}}(\nu_{(m)}a,\lambda_2)\,\partial^m_{\lambda_2}\delta(\lambda_1,\lambda_2),$$

where $a\in V_{\mathbb{Z}^N}$ and we used that $\sigma(\nu)=\nu$.

Lemma 8. *The following formulas hold:*

$$[D_k,\phi_a(l)]=\Big(\frac{l}{h_a}-\frac{k}{2}\Big)\phi_a(l-kh_a)$$

and

$$[D_k,\phi_a(z_a)]=\Big(\frac{1}{h_a}z_a^{1+kh_a}\partial_{z_a}+\frac{k}{2}\,z_a^{kh_a}\Big)\phi_a(z_a),\qquad a=1,2.$$

Proof. The first formula is equivalent to the second one as one can see immediately by comparing the coefficients in front of z_a^l. Let us prove the second formula. We will consider only the case $a=1$. The argument for $a=2$ is similar.

Since

$$\nu_{(0)}e^{v_1}=v_{1(-1)}e^{v_1},\quad \nu_{(1)}e^{v_1}=\frac{1}{2}e^{v_1},\quad \nu_{(m)}e^{v_1}=0\ (m>1),$$

we get

$$[Y^{\mathcal{F}}(\nu,\lambda_1),Y^{\mathcal{F}}(e^{v_1},\lambda_2)]=Y^{\mathcal{F}}(v_{1(-1)}e^{v_1},\lambda_2)\delta(\lambda_1,\lambda_2)+\frac{1}{2}Y^{\mathcal{F}}(e^{v_1},\lambda_2)\partial_{\lambda_2}\delta(\lambda_1,\lambda_2).$$

After a straightforward computation we get

$$Y^{\mathcal{F}}(v_1,\lambda_1)Y^{\mathcal{F}}(e^{v_1},\lambda_2) =$$
$$:Y^{\mathcal{F}}(v_1,\lambda_1)Y^{\mathcal{F}}(e^{v_1},\lambda_2): +\Big(\frac{\lambda_1^{1/h-1}}{(\lambda_1^{1/h}-\lambda_2^{1/h})h}-\frac{\lambda_1^{1/h-1}}{(\lambda_1^{1/h}+\lambda_2^{1/h})h}\Big)\, Y^{\mathcal{F}}(e^{v_1},\lambda_2).$$

Using the Taylor's series expansion

$$(\lambda_1-\lambda_2)\Big(\frac{\lambda_1^{1/h-1}}{(\lambda_1^{1/h}-\lambda_2^{1/h})h}-\frac{\lambda_1^{1/h-1}}{(\lambda_1^{1/h}+\lambda_2^{1/h})h}\Big) = 1-\frac{1}{2\lambda_2}\,(\lambda_1-\lambda_2)+\cdots$$

and the product formula (13) we get

$$Y^{\mathcal{F}}(v_{1(-1)}e^{v_1},\lambda) = \partial_\lambda\, Y^{\mathcal{F}}(e^{v_1},\lambda),$$

where we used that

$$:Y^{\mathcal{F}}(v_1,\lambda)Y^{\mathcal{F}}(e^{v_1},\lambda): = \Big(\partial_\lambda+\frac{1}{2\lambda}\Big)\, Y^{\mathcal{F}}(e^{v_1},\lambda).$$

The formula for the commutator that we want to compute takes the form

$$[Y^{\mathcal{F}}(\nu,\lambda_1), Y^{\mathcal{F}}(e^{v_1},\lambda_2)] = \Big(\delta(\lambda_1,\lambda_2)\partial_{\lambda_2}+\frac{1}{2}\partial_{\lambda_2}\delta(\lambda_1,\lambda_2)\Big)\, Y^{\mathcal{F}}(e^{v_1},\lambda_2).$$

Comparing the coefficients in front of λ_1^{-k-2} we get

$$[D_k, Y^{\mathcal{F}}(e^{v_1},\lambda_2)] = \Big(\lambda_2^{k+1}\partial_{\lambda_2}+\frac{(k+1)}{2}\,\lambda_2^{k}\Big)\, Y^{\mathcal{F}}(e^{v_1},\lambda_2).$$

It remains only to recall that $Y^{\mathcal{F}}(e^{v_1},\lambda_2) = C_1^+\lambda_2^{-1/2}\phi_1(\lambda_2^{1/h}\eta)$ and to substitute $z_1 = \lambda_2^{1/h}\eta$. $\square$

3.2. **Proof of Corollary 1.** Let us prove part a) of Corollary 1. We have to prove that if $\Omega_{\mathrm{KW}}(\tau\otimes\tau)=0$ then $\Omega_m(\tau\otimes\tau)=0$ for all $m\geq 0$. Following ten Kroode–van de Leur (see [17]) we equip the Fock space $\mathcal{F}$ with a positive definite Hermitian form H, such that

(1) $H(|0\rangle,|0\rangle)=1$
(2) $H(\phi_a(k)v_1,v_2) = (-1)^k H(v_1,\phi_a(-k)v_2)$ for all $k\in\mathbb{Z}$, $a=1,2$, and $v_1,v_2\in\mathcal{F}$.

In particual $\mathcal{F}^{\otimes 2}$ also has an induced positive definite Hermitian form. Observe that if $\Omega_{\mathrm{KW}}(v)=0$ for some $v\in\mathcal{F}^{\otimes 2}$, then since the Hermitian adjoint of the operator Ω_{-m} is Ω_m we get

$$-4H(\Omega_{\mathrm{KW}}(v),v) = \frac{1}{2}H(\Omega_0(v),\Omega_0(v)) + \sum_{m=1}^{\infty} H(\Omega_m(v),\Omega_m(v)).$$

The positive definitness of H implies that $\Omega_m(v)=0$ for all $m\geq 0$.

We can not apply the above argument directly, because τ is a formal power series. It belongs to the completion $\widehat{\mathcal{F}_0} = \mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]$ and the Hermitian form does not extend to $\widehat{\mathcal{F}}_0$. On the other hand the grading operator

$$D_0 - \frac{(h+1)N}{24h} = \sum_{m\in\mathbb{Z}^+_{\rm odd}} \frac{m}{h} t^1_m \partial_{t^1_m} + \frac{m}{2} t^2_m \partial_{t^2_m}$$

commutes with $\Omega_{\rm KW}$ (see Lemma 8) and $\widehat{\mathcal{F}}_0$ decomposes as an infinite product of finite dimensional eigensubspaces. Therefore if we decompose $\tau\otimes\tau = \sum_{n\geq 0} v_n$ where each $v_n \in \mathcal{F}_0$ is homogeneous of degree n, then $\Omega_{\rm KW}(v_n)=0$ for all n. Recalling the argument from above $\Omega_m(v_n)=0$ for all $m\geq 0$, i.e., $\Omega_m(\tau\otimes\tau)=0$.

Let us prove part b). Suppose that τ is a tau-function of the Kac–Wakimoto hierarchy. By part a) $\Omega_m(\tau\otimes\tau)=0$ for all $m\geq 0$. In particular τ is a tau-function of the 2-component BKP. Let us define the wave function

$$\Psi(\mathbf{t},z) = \Psi^{(1)}(\mathbf{t},z_1)e_1 + \mathbf{i}\Psi^{(2)}(\mathbf{t},z_2)e_2,$$

where

$$\Psi^{(a)}(\mathbf{t},z_a) = \Gamma(\mathbf{t}^a,z_a)\tau/\tau, \quad a=1,2.$$

The subspace $U\in \mathrm{Gr}_2^{I,(0)}$ is spanned by the coefficients of the Taylor's series expansion of $\Psi(\mathbf{t},z)$ at $\mathbf{t}=0$. Therefore we need to prove that $(z_1^h,z_2^2)\Psi(\mathbf{t},z)\in U$.

By definition Ω_m is the following bi-linear operator acting on $\mathbb{C}[\![\mathbf{t}^1,\mathbf{t}^2]\!]^{\otimes 2}$

$$\mathrm{Res}_{z_1=0}\frac{dz_1}{z_1} z_1^{mh}\Big(\Gamma(\mathbf{t}^1,z_1)\otimes\Gamma(\mathbf{t}^1,-z_1)\Big) - \mathrm{Res}_{z_2=0}\frac{dz_2}{z_2} z_2^{2m}\Big(\Gamma(\mathbf{t}^2,z_2)\otimes\Gamma(\mathbf{t}^2,-z_2)\Big).$$

The equation $\Omega_m(\tau\otimes\tau)=0$ is equivalent to

$$((z_1^{mh},z_2^{2m})\Psi(\mathbf{t}',z),\Psi(\mathbf{t}'',z)=0.$$

Therefore $(z_1^{mh},z_2^{2m})\Psi(\mathbf{t}',z)$ is orthogonal to every vector in U. Since U is maximally isotropic, we must have $(z_1^{mh},z_2^{2m})\Psi(\mathbf{t}',z)\in U$. Note that the argument is invertible, i.e., the condition that $(z_1^h,z_2^2)U\subset U$ implies that the tau-function τ satisfies all bi-linear equations $\Omega_m(\tau\otimes\tau)=0$, so by Theorem 1 τ is a tau-function of the Kac–Wakimoto hierarchy.

3.3. **Virasoro constraints.** Suppose now that τ is a tau-function of the Kac–Wakimoto hierarchy satisfying the Virasoro constraints $L_k\tau=0$ for all $k\geq -1$, where L_k are the operators (3). Recall the differential operators $\ell_k(z)$ (see formula (7)).

Lemma 9. *Let $U\in\mathrm{Gr}_2^{I,(0)}$ be the point corresponding to a tau-function τ of the Kac–Wakimoto hierarchy. Then the condition $L_k(\tau)=0$ implies $\ell_k(z)U\subset U$.*

Proof. Note that

$$[J^1_{1+(1+k)h},\Gamma(\mathbf{t}^a,z_a)] = 2\delta_{1,a} z_1^{1+(1+k)h}\Gamma(\mathbf{t}^a,z_a).$$

Recalling the commutation relations in Lemma 8 and the boson-fermion isomorphism $\phi_a(z_a) = Q_a\Gamma(\mathbf{t}^a, z_a)$ we get

$$[L_k, \Gamma(\mathbf{t}^a, z_a)] = \ell_k^{(a)}(z_a)\, \Gamma(\mathbf{t}^a, z_a).$$

If $\Psi(\mathbf{t}, z)$ is the wave function then

$$\ell_k(z)\Psi(\mathbf{t}, z) = \tau^{-1}(\mathbf{t})\, \ell_k(z)\, (\Gamma(\mathbf{t}^1, z_1)e_1 + \mathbf{i}\Gamma(\mathbf{t}^2, z_2)e_2)\, \tau(\mathbf{t}).$$

Since

$$\begin{aligned}&\ell_k(z)\, (\Gamma(\mathbf{t}^1, z_1)e_1 + \mathbf{i}\Gamma(\mathbf{t}^2, z_2)e_2) = \\ &L_k \circ (\Gamma(\mathbf{t}^1, z_1)e_1 + \mathbf{i}\Gamma(\mathbf{t}^2, z_2)e_2) - (\Gamma(\mathbf{t}^1, z_1)e_1 + \mathbf{i}\Gamma(\mathbf{t}^2, z_2)e_2) \circ L_k\end{aligned}$$

and L_k anihilates τ we get

$$\ell_k(z)\Psi(\mathbf{t}, z) = (\tau^{-1} \circ L_k \circ \tau)\Psi(\mathbf{t}, z). \tag{15}$$

The RHS of the above formula belongs to U, because $\Psi(\mathbf{t}, z)$ does and $\tau^{-1} \circ L_k \circ \tau$ is a differential operator in the dynamical variables $\mathbf{t}$. □

Lemma 10. *If $p \geq 0$ is an integer, then the following formulas hold:*

$$(\mathbf{i}\ell_{-1}^{(1)}(z_1))^p\Psi^{(1)}(x, z_1) = z_1^p\Psi^{(1)}(x, z_1) + \Big(\frac{\mathbf{i}p}{h}\, x_1 z_1^{p-h} + O(z_1^{p-h-1})\Big)e^{x_1z_1}$$

and

$$(\mathbf{i}\ell_{-1}^{(2)}(z_2))^p\Psi^{(2)}(x, z_2) = O(z_2^{-p})e^{x_2z_2}.$$

Proof. The second formula is obvious. Let us prove the first one. We argue by induction on p. For $p = 0$ the identity is obvious. Suppose the formula is true for p. We will prove it for $p + 1$. To begin with note that

$$\ell_{-1}^{(1)}(z_1) = z_1 - \frac{\mathbf{i}}{2}z_1^{-h} + \frac{\mathbf{i}}{h}z_1^{-h}(z_1\partial_{z_1}).$$

We have

$$\begin{aligned}&\Big(z_1 - \frac{\mathbf{i}}{2}z_1^{-h} + \frac{\mathbf{i}}{h}z_1^{-h}(z_1\partial_{z_1})\Big)z_1^p\Psi^{(1)}(x, z_1) = \\ &z_1^p\Big(z_1\, \Psi^{(1)}(x, z_1) + \Big(\frac{\mathbf{i}}{h}x_1 z_1^{1-h} + O(z_1^{-h})\Big)e^{x_1z_1}\Big) + O(z_1^{p-h})e^{x_1z_1} = \\ &z_1^{p+1}\Psi^{(1)}(x, z_1) + \Big(\frac{\mathbf{i}}{h}x_1 z_1^{p+1-h} + O(z_1^{p-h})\Big)e^{x_1z_1},\end{aligned}$$

where we used that $\Psi^{(1)}(x, z_1) = (1 + O(z_1^{-1}))e^{x_1z_1}$. We also have

$$\begin{aligned}&\Big(z_1 - \frac{\mathbf{i}}{2}z_1^{-h} + \frac{\mathbf{i}}{h}z_1^{-h}(z_1\partial_{z_1})\Big)\Big(\frac{\mathbf{i}p}{h}x_1 z_1^{p-h}\, e^{x_1z_1}\Big) = \\ &\Big(\frac{\mathbf{i}p}{h}x_1 z_1^{p+1-h} + O(z_1^{p-2h+1})\Big)\, e^{x_1z_1}.\end{aligned}$$

Note that $p-2h+1 \leq p-h$, so combining the above two formulas completes the inductive step. □

Let us prove the first equation in Theorem 2. We will need the following simple lemma.

Lemma 11. *Suppose that $U \in \mathrm{Gr}_2^{I,(0)}$ and that the series*

$$w(x,z) = (u_1(x,z_1)e^{x_1z_1}, u_2(x,z_2)e^{x_2z_2}) = \sum_{m,n=0}^{\infty} w_{m,n}(z)x_1^m x_2^n$$

belongs to the intersection

$$U \cap (e^{x_1z_1}, e^{x_2z_2}) \cdot V_0,$$

where the multiplication $\cdot$ is the componentwise multiplication. Then $w = 0$.

Proof. Using Taylor's formula we get

$$w_{m,n} = \frac{1}{m!n!}((\partial_1 + z_1)^m \partial_2^n u_1, \partial_1^m(\partial_2 + z_2)^n u_2)\Big|_{x_1=x_2=0}.$$

Using the binomial formula we get

$$w_{m,n} = \sum_{i=0}^{m} u_{1,i,n}(z_1)\frac{z_1^{m-i}}{(m-i)!}e_1 + \sum_{j=0}^{n} u_{2,m,j}(z_2)\frac{z_2^{n-j}}{(n-j)!}e_2,$$

where $u_{a,m,n}(z_a)$ is the coefficient in front of $x_1^m x_2^n$ of $u_a(x,z_a)$. We argue by induction on $m+n$ that $u_{a,m,n} = 0$ for all $a = 1,2$ and $m,n \geq 0$. If $m = n = 0$ the definition of the Grassmanian implies that $(u_{1,0,0}, u_{2,0,0}) = w_{0,0} \in U \cap V_0 = \{0\}$. Suppose $u_{a,m,n} = 0$ for $m+n < k$. If $m+n = k$ then $u_{1,i,n} = 0$ and $u_{2,m,j} = 0$ for $i < m$ and $j < n$, respectively. Hence

$$w_{m,n} = (u_{1,m,n}, u_{2,m,n}) \in U \cap V_0 = \{0\},$$

where we used that $(u_1(x,z_1), u_2(x,z_2)) \in V_0$. □

Lemma 12. *If $\tau(\mathbf{t})$ satisfies the Virasoro and the dilaton constraints then the restriction of τ to $t_m^a = 0$ for all $a = 1,2$ and $m > 1$ is given by $e^{c - \mathbf{i}x_1x_2^2/8}$, where c is some constant independent of x_1 and x_2.*

Proof. The restriction of the string equation to $t_m^a = 0$ for $a = 1,2$, $m > 1$ yields

$$-\mathbf{i}\partial_1\tau(x_1,x_2) + \frac{1}{8}x_2^2\tau(x_1,x_2) = 0.$$

Therefore $\tau(x) = e^{f(x_2) - \mathbf{i}x_1x_2^2/8}$ for some function $f(x_2) \in \mathbb{C}[\![x_2]\!]$. We have to check that f is a constant. Let us write $\tau(\mathbf{t}) = \mathcal{D}(1,\mathbf{t})$ where $\mathcal{D}(\hbar,\mathbf{t})$ is a solution of the dilaton equation (6). The differential operator defining the dilaton equation commutes with

$$L_0 = -\mathbf{i}\partial_{t_{1+h}^1} + \sum_{m\in\mathbb{Z}_{\mathrm{odd}}^+}(\frac{m}{h}t_m^1\partial_{t_m^1} + \frac{m}{2}t_m^2\partial_{t_m^2}) + \frac{N(h+1)}{24h}.$$

Therefore $L_0\mathcal{D} = 0$. Subtracting the dilaton equation from $\frac{h}{h+1}L_0\mathcal{D} = 0$ we get

$$\Big(-2\hbar\partial_\hbar + \frac{1}{h+1}\sum_{m\in\mathbb{Z}_{\mathrm{odd}}^+}((m-1-h)t_m^1\partial_{t_m^1} + ((N-1)m-1-h)t_m^2\partial_{t_m^2}\Big)\mathcal{D}(\hbar,\mathbf{t}) = 0.$$

This differential equation imposes a non-trivial constraint on the coefficients of $\mathcal{D}$. Namely, let us write

$$\mathcal{D}(\hbar, \mathbf{t}) = \exp\Big(\sum_{g,\mu,\nu} C^{(g)}(\mu,\nu) t^1_{m_1} \cdots t^1_{m_r} t^2_{n_1} \cdots t^2_{n_s}\, \hbar^{g-1} \Big)$$

where the sum is over all $g \geq 0$, $\mu = (m_1, m_2, \dots, m_r)$, and $\nu = (n_1, n_2, \dots, n_s)$. Then if $C^{(g)}(\mu,\nu) \neq 0$ we must have

$$\frac{1}{h+1}\Big(\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} (N-1) n_j \Big) = 2g-2+r+s.$$

In order to prove that $f(x_2)$ is a constant independent of x_2 we just need to check that we can not have monomials with $r=0$ and $n_j = 1$ for $1 \leq j \leq s$. Suppose that such a monomial contributes to the tau-function. Then

$$\frac{s(N-1)}{2N-1} = 2g-2+s.$$

Since $N-1$ and $2N-1$ are relatively prime, we get $s = \ell(2N-1)$ for some integer $\ell \geq 1$. The above equation takes the form $\ell(N-1) = 2g-2+\ell(2N-1)$, i.e., $2g-2+\ell N = 0$. This equation does not have solutions for $g \geq 0$ and $N \geq 4$. Hence $\log \tau(x)$ does not have non-trivial monomials in x_2. $\square$

For future references let us state an important corollary of the proof of Lemma 12. We got that if some monomial $t^1_{m_1} \cdots t^1_{m_r} t^2_{n_1} \cdots t^2_{n_s}$ contributes to the tau-function, then the number $\sum_{i=1}^r m_i + \sum_{j=1}^s (N-1) n_j$ is an integer divisible by $h+1$. In particular, we get that the total descendant potential is invariant under the rescaling

$$t^1_m \mapsto \omega^m\, t^1_m, \quad t^2_m \mapsto \omega^{m(N-1)} t^2_m\ , \tag{16}$$

where $\omega = e^{2\pi \mathbf{i}/(h+1)}$.

3.4. **Proof of Theorem 2.** The third equation is a direct consequence of Lemma 12. Indeed, by property (W2) of wave functions we know that there exists $q(x) \in \mathbb{C}[\![x_1, x_2]\!]$ such that $(\partial_1 \partial_2 + q(x))\Psi(x,z) = 0$, where recall that $x_1 := t^1_1$, $x_2 := t^2_1$, and $\partial_a = \partial/\partial x_a$.. By definition

$$\Psi^{(1)}(x, z_1) = \Big(1 - 2\frac{\partial_1 \tau}{\tau} z_1^{-1} + \cdots \Big) e^{x_1 z_1}.$$

Comparing the coeffiecients in front of z_1^0 in $(\partial_1\partial_2 + q(x))\Psi^{(1)}(x,z_1) = 0$ we get

$$q(x) = 2\partial_1 \partial_2 \log \tau = -\frac{\mathbf{i}}{2}\, x_2.$$

Let us prove the second equation. Note that the operator L_{-1} is given explicitly by

$$L_{-1} = -\mathbf{i}\partial_{t_1^1} + \sum_{a=1,2} \sum_{2i+2j=h_a+2} \frac{(2i-1)(2j-1)}{4h_a}\, t^a_{2i-1} t^a_{2j-1} + \sum_{a=1,2} \frac{1}{h_a} \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} (m+h_a) t^a_{m+h_a} \partial_{t^a_m}.$$

By using Lemma 12 and restricting the equation (15) with $k=-1$ to $t^a_m=0$ for $a=1,2$ and $m>1$, we get the second equation in Theorem 2:

$$\ell_{-1}(z)\Psi(x,z) = -\mathbf{i}\partial_1\Psi(x,z). \tag{17}$$

It remains to prove the first equation. Recalling Lemma 10 with $p=h$ we get

$$(\partial_1^h + \partial_2^2 - \mathbf{i}x_1 - z_1^h)\Psi^{(1)}(x,z_1) = O(z_1^{-1})e^{x_1z_1}.$$

By definition

$$\Psi^{(2)}(x,z_2) = \Big(1 - 2\frac{\partial_2\tau}{\tau} z_2^{-1} + \cdots\Big)e^{x_2z_2} = \Big(1 + \frac{\mathbf{i}}{2}x_1x_2z_2^{-1} + \cdots\Big)e^{x_2z_2},$$

where in the second equality we used Lemma 12. It is straightforward to check that

$$(\partial_1^h + \partial_2^2 - \mathbf{i}x_1 - z_2^2)\Psi^{(2)}(x,z_2) = O(z_2^{-1})e^{x_2z_2}.$$

Hence

$$(\partial_1^h + \partial_2^2 - \mathbf{i}x_1 - (z_1^h, z_2^2))\Psi(x,z) \in U \cap (e^{x_1z_1}, e^{x_2z_2})V_0.$$

It remains only to recall Lemma 11. □

3.5. **The wave function.** Our goal is to prove Corollary 2. We are going to prove that the system of differntial equations in Theorem 2 uniquely determines the wave function $\Psi(x,z)$.

Using the method of the characteristics it is straightforward to check that the components of a wave function solving the linear PDEs

$$\partial_1\Psi^{(a)}(x,z_a) = \mathbf{i}\ell^{(a)}_{-1}(z_a)\Psi(x,z_a), \quad a=1,2,$$

must have the following form

$$\Psi^{(1)}(x,z_1) = e^{\frac{\mathbf{i}h}{h+1}(z_1^{h+1} - (\mathbf{i}x_1+z_1^h)^{\frac{h+1}{h}})} \frac{z_1^{h/2}}{(\mathbf{i}x_1+z_1^h)^{1/2}} \Big(1 + \sum_{k=1}^\infty \frac{\psi_k^{(1)}(x_2)}{(\mathbf{i}x_1+z_1^h)^{k/h}}\Big)$$

and

$$\Psi^{(2)}(x,z_2) = e^{x_2(\mathbf{i}x_1+z_2^2)^{1/2}} \frac{z_2}{(\mathbf{i}x_1+z_2^2)^{1/2}} \Big(1 + \sum_{k=1}^\infty \frac{\psi_k^{(2)}(x_2)}{(\mathbf{i}x_1+z_2^2)^{k/2}}\Big),$$

where $\psi_k^{(a)}(x_2) \in \mathbb{C}[\![x_2]\!]$ are to be determined from the remaining two equations.

Let us prove that the 3rd equation uniquely determines $\Psi(x,z)$ from its restriction to $x_2=0$. Conjugating the differential operator $\partial_1\partial_2 - \frac{\mathbf{i}}{2}x_2$ with the exponential factor of $\Psi^{(1)}$, we get that the series

$$\sum_{k=0}^{\infty} \psi_k^{(1)}(x_2)(\mathbf{i}x_1+z_1^h)^{-k/h-1/2}, \quad \psi_0^{(1)}(x_2):=1,$$

is anihilated by the differential operator

$$(\partial_1+(\mathbf{i}x_1+z_1^h)^{1/h})\partial_2-\frac{\mathbf{i}}{2}x_2.$$

Comparing the coefficients in front of $(\mathbf{i}x_1+z_1^h)^{-k/h-1/2}$, we get the following recursion relation

$$\mathbf{i}\Big(-\frac{k}{h}+\frac{1}{2}\Big)\partial_2\psi_{k-h}^{(1)}(x_2)+\partial_2\psi_{k+1}^{(1)}(x_2)-\frac{\mathbf{i}}{2}x_2\,\psi_k^{(1)}(x_2)=0. \tag{18}$$

Assuming that we have determined $\psi_k^{(1)}(0)$ for all k, then the relation (18) allows us to reconstruct recursively $\psi_k^{(1)}(x_2)$ for all k. Similarly the series

$$\sum_{k=0}^{\infty} \psi_k^{(2)}(x_2)(\mathbf{i}x_1+z_2^2)^{-k/2-1/2}, \quad \psi_0^{(2)}(x_2):=1,$$

is anihilated by the differential operator

$$\Big(\partial_1+\frac{\mathbf{i}}{2}x_2(\mathbf{i}x_1+z_2^2)^{-1/2}\Big)\Big(\partial_2+(\mathbf{i}x_1+z_2^2)^{1/2}\Big)-\frac{\mathbf{i}}{2}x_2=$$
$$\partial_1\partial_2+\partial_1\circ(\mathbf{i}x_1+z_2^2)^{1/2}+\frac{\mathbf{i}}{2}x_2(\mathbf{i}x_1+z_2^2)^{-1/2}\partial_2.$$

Comparing the coefficients in front of $(\mathbf{i}x_1+z_2^2)^{-k/2-1/2}$ we get the following recursion relation

$$(-k+1)\partial_2\psi_{k-2}^{(2)}(x_2)+(x_2\partial_2-k+1)\psi_{k-1}^{(2)}(x_2)=0. \tag{19}$$

Writing each $\psi_k^{(2)}(x_2)=\sum_{a=0}^{\infty}\psi_{k,a}^{(2)}x_2^a$ and comparing the coefficients in front of x_2^a we get

$$(-k+1)(a+1)\psi_{k-2,a+1}^{(2)}+(a-k+1)\psi_{k-1,a}^{(2)}=0.$$

Using these relations we get that $\psi_{k,a}^{(2)}=0$ for all $a>k+1$ and that

$$\psi_{k,a}^{(2)}=\frac{(a-k-2)\cdots(a-k-a-1)}{(k+1)\cdots(k+a)a!}\,\psi_{k+a,0}^{(2)}.$$

Therefore the set of functions $\psi_k^{(2)}(x_2)$ $(k\geq 0)$ is uniquely determined from its restriction to $x_2=0$ as claimed.

It remains to prove that the restriction $\Psi(x_1,0,z)$ is determined from the 1st equation. To be more precise, differentiating the third equation in Theorem 2 with respect to x_2 and restricting to $x_2=0$ we get

$$\Big(\partial_1\partial_2^2\Psi\Big)\Big|_{x_2=0}=\frac{\mathbf{i}}{2}\,\Psi|_{x_2=0}.$$

Differentiating the first equation in Theorem 2 in x_1, restricting to $x_2 = 0$, and using the above relation we get the following scalar differential equation for $\Psi|_{x_2=0}$

$$\Big(\partial_1^{h+1} - (\mathbf{i}x_1 + z_1^h, \mathbf{i}x_1 + z_2^2))\partial_1 - \frac{\mathbf{i}}{2}\Big)\Psi|_{x_2=0} = 0.$$

Conjugating with the exponential factors of the wave function, we get that the series

$$\sum_{k=0}^{\infty} \psi_k^{(1)}(0)(\mathbf{i}x_1 + z_1^h)^{-k/h-1/2}, \quad \psi_0^{(1)}(0) := 1$$

and

$$\sum_{k=0}^{\infty} \psi_k^{(2)}(0)(\mathbf{i}x_1 + z_2^2)^{-k/2-1/2}, \quad \psi_0^{(2)}(0) := 1,$$

are anihilated by the differential operators

$$(\partial_1 + (\mathbf{i}x_1 + z_1^h)^{1/h})^{h+1} - (\mathbf{i}x_1 + z_1^h)(\partial_1 + (\mathbf{i}x_1 + z_1^h)^{1/h}) - \frac{\mathbf{i}}{2} \tag{20}$$

and

$$\partial_1^{h+1} - (\mathbf{i}x_1 + z_2^2)\partial_1 - \frac{\mathbf{i}}{2}, \tag{21}$$

respectively. We claim that comparing the coefficients in front of $(\mathbf{i}x_1 + z_1^h)^{-k/h-1/2}$ and $(\mathbf{i}x_1 + z_2^2)^{-k/2-1/2}$ yields a recursion that uniquely determines $\psi_k^{(a)}(0)$ for all $a = 1, 2$ and $k > 0$.

Let us change the variables in the differential operator (20) via $y_1 = (\mathbf{i}x_1 + z_1^h)^{1/h}$. Since

$$\partial_1 + (\mathbf{i}x_1 + z_1^h)^{1/h} = y_1^{-h}(y_1^{h+1} + \mathfrak{D}_1),$$

where $\mathfrak{D}_1 := \frac{\mathbf{i}}{h} y_1 \partial_{y_1}$ we get that the differential operator (20) takes the form

$$y_1^{-h(h+1)}(y_1^{h+1} + \mathfrak{D}_1 - \mathbf{i}\cdot h)\cdots(y_1^{h+1} + \mathfrak{D}_1 - \mathbf{i}\cdot 0) - \mathfrak{D}_1 - y_1^{h+1} - \frac{\mathbf{i}}{2},$$

where we used that $\mathfrak{D}_1 y_1^m = y_1^m(\mathfrak{D}_1 + \mathbf{i}m/h)$. The above operator can be expanded as a Laurent polynomial in y_1^{h+1} with coefficients (written on the left of the powers of y_1) that are polynomials in $\mathfrak{D}_1$. Note that the coefficient in front of y_1^{h+1} is 0, while the free term is

$$\sum_{k=0}^{h}(\mathfrak{D}_1 - \mathbf{i}k) - \mathfrak{D}_1 - \frac{\mathbf{i}}{2} = h\Big(\mathfrak{D}_1 + \frac{\mathbf{i}}{2}\Big).$$

Therefore the differential operator (20) has the form

$$h\Big(\mathfrak{D}_1 + \frac{\mathbf{i}}{2}\Big) + \sum_{s=1}^{h} y_1^{-s(h+1)}\, P_s(\mathfrak{D}_1),$$

where $P_s(\mathfrak{D}_1) \in \mathbb{C}[\mathfrak{D}_1]$ are some polynomials. The above operator anihilates the series $\sum_{k\geq 0} \psi_k^{(1)}(0) y_1^{-k-N+1}$. Comparing the coefficients in front of y_1^{-k-N+1}, we get the recursion relation

$$-\mathbf{i}k\psi_k^{(1)}(0) + \sum_{s=1}^{h} \psi^{(1)}_{k-s(h+1)}(0)\, P_s(-\mathbf{i}(k/h+1/2)) = 0$$

valid for all $k \in \mathbb{Z}$ provided that we define $\psi_\ell^{(1)}(0) := 0$ for $\ell < 0$. By definition $\psi_0^{(1)}(0) = 1$, so the above recursion determines uniquely $\psi_k^{(1)}(0)$ for all $k \geq 0$.

The argument for the differential operator (21) is similar. After the change $y_2 = (\mathbf{i}x_1 + z_2^2)^{1/2}$, the differential operator (21) takes the form

$$y_2^{-2(h+1)}(\mathfrak{D}_2 - \mathbf{i}\cdot h)\cdots(\mathfrak{D}_2 - \mathbf{i}\cdot 0) - \mathfrak{D}_2 - \frac{\mathbf{i}}{2},$$

where $\mathfrak{D}_2 = \frac{\mathbf{i}}{2} y_2 \partial_{y_2}$. This operator anihilates the series $\sum_{k\geq 0} \psi_k^{(2)}(0) y_2^{-k-1}$. Comparing the coefficients in front of y_2^{-k-1}, we get the following recursion

$$\frac{\mathbf{i}k}{2}\psi_k^{(2)}(0) + (-1)^{N-1}\mathbf{i}\Big(\prod_{s=0}^{h}(s-(k-1)/2)\Big)\psi^{(2)}_{k-2h-2}(0) = 0.$$

Note that for $0 \leq k \leq 2h+1$ the second term is 0, because $\psi_\ell^{(2)}(0)$ is by definition 0 for $\ell < 0$. Therefore $\psi_k^{(2)}(0) = 0$ for all $1 \leq k \leq 2h+1$. By definition $\psi_0^{(2)}(0) = 1$, so the above recursion uniquely determines $\psi_k^{(2)}(0)$ for all $k > 0$. This completes the proof of Corollary 2.

4. Tau functions of Gaussian type

The goal in this section is to prove Theorem 3.

4.1. **Time evolution.** Suppose that $\Psi(x,z)$ is a wave function of the 2-component BKP hierarchy. We would like to determine under what conditions the logarithm of the corresponding tau-function is a quadratic form.

By definition the components of the wave function have the form

$$\Psi^{(a)}(x,z_a) = \Big(1 + \sum_{k=1}^{\infty} w_k^{(a)}(x) z_a^{-k}\Big) e^{x_a z_a}.$$

The 2-component BKP hierarchy can be formulated as a deformation of the wave function of the form

$$\Psi^{(a)}(\mathbf{t},z) = \Big(1 + \sum_{k=1}^{\infty} w_k^{(a)}(\mathbf{t}) z_a^{-k}\Big) e^{\sum_{k\in\mathbb{Z}^+_{\rm odd}} t_k^a z_a^k},$$

where as usual we identify $t_1^1 = x_1$ and $t_1^2 = x_2$. The pseudodifferential operators

$$S^{(a)}(\mathbf{t}, \partial_a) = 1 + \sum_{k=1}^{\infty} w_k^{(a)}(\mathbf{t})\partial_a^{-k}, \quad \partial_a = \partial/\partial x_a \quad (a = 1, 2)$$

and

$$\mathcal{L}_a(\mathbf{t}, \partial_a) := S^{(a)}(\mathbf{t}, \partial_a)\partial_a S^{(a)}(\mathbf{t}, \partial_a)^{-1} \quad (a = 1, 2)$$

are called *dressing operators* and *Lax operators*, respectively. The dressing operators are called *wave operators* if the following system of differential equations is satisfied

$$\partial_{t_k^a} \Psi(\mathbf{t}, z) = (\mathcal{L}_a(\mathbf{t}, \partial_a)^k)_+ \Psi(\mathbf{t}, z), \quad a = 1, 2, \quad k \in \mathbb{Z}_{\mathrm{odd}}^+,$$

where the $+$ index of a pseudodifferential operator means truncating the non-differential operator part. The above system of differential equations uniquely determines the wave operators from their restriction to $t_1^1 = x_1, t_1^2 = x_2$, and $t_m^a = 0$ for $m > 1$. In particular, the deformation $\Psi(\mathbf{t}, z)$ of the wave function is uniquely determined and it is still refered to as a wave function. It is a very non-trivial result that for every wave function $\Psi(\mathbf{t}, z)$ there exists a formal series $\tau(\mathbf{t})$, called *tau function*, such that

$$\Psi^{(a)}(\mathbf{t}, z_a) = \frac{\Gamma(\mathbf{t}^a, z_a)\tau(\mathbf{t})}{\tau(\mathbf{t})}.$$

Such a formal series is unique up to a constant factor and it satisfies the Hirota bilinear equations of the 2-component BKP hierarchy. It is straightforward to prove that $\log \tau$ has at most quadratic terms in $\mathbf{t}$ if an only if the non-deformed Lax operators $\mathcal{L}_a(x, \partial_a) = \mathcal{L}_a(0, \partial_a)$ are constant, i.e., independent of x.

4.2. **Properties.** Suppose that $\tau(\mathbf{t})$ is a tau-function of the Kac–Wakimoto hierarchy that has the form (8). Let us assume also that the coefficients $W_{k\ell}^{ab}$ are symmetric: $W_{k\ell}^{ab} = W_{\ell k}^{ba}$. The components of the corresponding wave function have the form

$$\Psi^{(1)}(\mathbf{t}, z_1) = \Psi^{(1)}(0, z_1) \times \exp\Big(\sum_{k \in \mathbb{Z}_{\mathrm{odd}}^+} t_k^1 \Big(z_1^k - 2 \sum_{\ell \in \mathbb{Z}_{\mathrm{odd}}^+} W_{k\ell}^{11} \frac{z_1^{-\ell}}{\ell} \Big) + t_k^2 \Big(-2 \sum_{\ell \in \mathbb{Z}_{\mathrm{odd}}^+} W_{k\ell}^{21} \frac{z_1^{-\ell}}{\ell} \Big) \Big)$$

and

$$\Psi^{(2)}(\mathbf{t}, z_2) = \Psi^{(2)}(0, z_2) \times \exp\Big(\sum_{k \in \mathbb{Z}_{\mathrm{odd}}^+} t_k^1 \Big(-2 \sum_{\ell \in \mathbb{Z}_{\mathrm{odd}}^+} W_{k\ell}^{12} \frac{z_2^{-\ell}}{\ell} \Big) + t_k^2 \Big(z_2^k - 2 \sum_{\ell \in \mathbb{Z}_{\mathrm{odd}}^+} W_{k\ell}^{22} \frac{z_2^{-\ell}}{\ell} \Big) \Big),$$

where

$$\Psi^{(a)}(0,z_a) = \exp\Big(2\sum_{k,\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{aa}_{k\ell}\frac{z_a^{-k-\ell}}{k\ell}\Big), \quad a=1,2.$$

The first observation is that the subspace $U\in \mathrm{Gr}_2^{I,(0)}$ corresponding to the wave function is a $\mathbb{C}[X_1,X_2]$-module, where $X_a = X_a(z)\in V$ $(a=1,2)$ are defined by $\partial_{t_1^a}\Psi = X_a(z)\Psi$, i.e.,

$$X_1 = \Big(z_1 - 2\sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{11}_{1\ell} z_1^{-\ell}/\ell\ ,\ -2\sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{12}_{1\ell} z_2^{-\ell}/\ell\Big) \tag{22}$$

and

$$X_2 = \Big(-2\sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{21}_{1\ell} z_1^{-\ell}/\ell\ ,\ z_2 - 2\sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{22}_{1\ell} z_2^{-\ell}/\ell\Big), \tag{23}$$

where the action of X_a on the wave function is via componentwise multiplication. Note also that $U = \mathbb{C}[X_1,X_2]\Psi(0,z)$.

Lemma 13. *a) Put*

$$t_N := 2\partial_1\partial_2 \log\tau = 2W^{12}_{11} = 2W^{21}_{11},$$

then $X_1X_2 + t_N = 0$.

b) There are parameters t_i $(1\le i\le N-1)$, such that

$$X_1^h + \sum_{i=1}^{N-1} t_i X_1^{h-2i} + X_2^2 = (z_1^h, z_2^2).$$

c) There are uniquely defined polynomials

$$p_k^{(a)}(t,x)\in\mathbb{C}[t,x],\quad k\in\mathbb{Z}^+_{\mathrm{odd}},\quad a=1,2,$$

where $t=(t_1,t_2,\dots,t_N)$ such that with respect to x they are monic of degree k and

$$p_k^{(a)}(t,X_a(z)) = z_a^k e_a + O(z_1^{-1})e_1 + O(z_2^{-1})e_2.$$

Proof. a) Recall property $(W2)$ of the wave function (see Section 1.2). There exists function $q(x_1,x_2)$ such that $(\partial_1\partial_2 + q)\Psi = 0$. Comparing the coefficients in front of z_1^0 we get that $q = 2\partial_1\partial_2\log\tau = t_N$. By definition $\partial_1\partial_2\Psi = X_1X_2\Psi$, so the relation $X_1X_2 + t_N = 0$ follows.

b) Note that

$$X_1^{2m} = z_1^{2m}e_1 + (O(z_1^{2m-2}), O(z_2^{-2m}))$$

and

$$X_2^2 = z_2^2 e_2 + (O(z_1^{-2}), O(z_2^0)).$$

Therefore we can choose the coefficients $t_1,\dots,t_{N-1}$ in such a way that

$$F := X_1^h + \sum_{i=1}^{N-1} t_i X_1^{h-2i} + X_2^2 - (z_1^h, z_2^2)$$

satisfies $F(e_1+\mathbf{i}e_2)\in V_0$. This however implies that $F\Psi(0,z)\in U\cap V_0=\{0\}$, i.e., $F=0$.

c) Using part a) and b) we can express all $W^{ab}_{1\ell}$, $(a,b=1,2,\ \ell\in\mathbb{Z}^+_{\rm odd})$ as polynomials in t. It follows that if $k\in\mathbb{Z}^+_{\rm odd}$, then

$$X_a(z)^k=z_a^ke_a+O(z_a^{k-2})e_a+O(z_2^{-k})e_b,$$

where b is such that $\{a,b\}=\{1,2\}$, the RHS is a Laurent series in z_1 and z_2 that involves only odd powers of z_1 and z_2 and its coefficients are polynomials in t. Clearly we can remove the lower order terms on the RHS that involve non-negative powers of z_a by adding a linear combination of $X_a(z)^{k-2},\dots,X_a(z)$. □

We claim that the coefficients $W^{ab}_{k\ell}$ are polynomials in the parameters t. We already argued that part a) and b) of Lemma 13 implies that $W^{ab}_{1\ell}$ are polynomials in t. Let us prove that $W^{1b}_{k\ell}$ is polynomial in t. The argument for $W^{2b}_{k\ell}$ is similar. Put

$$A=\Big(z_1^k-2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}}W^{11}_{k\ell}\frac{z_1^{-\ell}}{\ell}\Big)e_1+\Big(-2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}}W^{12}_{k\ell}\frac{z_2^{-\ell}}{\ell}\Big)e_2.$$

By definition $\partial_{t^1_k}\Psi=A\Psi$, so $A\Psi(0,z)\in U$. On the other hand

$$(A-p^{(1)}_k(t,X_1))\Psi(0,z)\in U\cap V_0=\{0\}.$$

Hence $A=p^{(1)}_k(t,X_1)$.

Lemma 14. *The components of the wave function $\Psi(0,z)$ are given by the following formulas*

$$\Psi^{(a)}(0,z_a)^2=\frac{z_a\partial_{z_a}X_{aa}}{X_{aa}},\quad a=1,2,$$

where we define X_{ab} by $X_a=X_{a1}e_1+X_{a2}e_2$.

Proof. Since U is an isotropic subspace the vectors $X_1^m\Psi(0,z)$ must be isotropic for all integers $m\geq 0$. This imples that

$$\mathrm{Res}_{z_1=0}\,X_{11}^{2m}\Psi^{(1)}(0,z_1)^2\frac{dz_1}{z_1}=\delta_{m,0}.$$

Note that series $\Psi^{(1)}(0,z_1)^2=1+E_1z_1^{-2}+E_2z_1^{-4}+\cdots$ is uniquely determined in terms of X_1 from the above residue relations: the relation for $m=1$ determines E_1, the relation for $m=2$ determines E_2, etc.. On the other hand

$$\mathrm{Res}_{z_1=0}\,X_{11}^{2m}\frac{dX_{11}}{X_{11}}=-\,\mathrm{Res}_{X_{11}=\infty}\,X_{11}^{2m}\frac{dX_{11}}{X_{11}}=\delta_{m,0}.$$

This relation implies the formula for $a=1$. The proof of the formula for $a=2$ is similar. □

4.3. **Proof of Theorem 3.** We have constructed a map from the set of tau-functions of the Kac–Wakimoto hierarchy of Gaussian type to $\mathbb{C}^N$. The map is given by the parameters $t = (t_1, \ldots, t_N)$ from Lemma 13. Moreover, we proved that the coefficients $W^{ab}_{k\ell}$ can be expressed as polynomials in t. This proves that the map is injective. It remains only to prove that it is surjective.

Given parameters $t = (t_1, \ldots, t_N)$ we can uniquely define $W^{ab}_{1\ell}$ such that X_1 and X_2 given by formulas (22) and (23) respectively are solutions to the polynomial equations in parts a) and b) of Lemma 13. Furthermore, we define the remaining coefficients $W^{ab}_{k\ell}$ via the relations

$$\Big(z_1^k - 2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}} W^{11}_{k\ell}\frac{z_1^{-\ell}}{\ell}\Big)e_1 + \Big(-2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}} W^{12}_{k\ell}\frac{z_2^{-\ell}}{\ell}\Big)e_2 := p^{(1)}_k(t, X_1)$$

and

$$\Big(-2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}} W^{21}_{k\ell}\frac{z_1^{-\ell}}{\ell}\Big)e_1 + \Big(z_2^k - 2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}} W^{22}_{k\ell}\frac{z_2^{-\ell}}{\ell}\Big)e_2 := p^{(2)}_k(t, X_2),$$

where $p^{(a)}_k(t, x)$ are the polynomials from Lemma 13, part c). We claim that

$$\Psi^{(a)}(\mathbf{t}, z_a) = \Big(\frac{z_a\partial_{z_a}X_{aa}}{X_{aa}}\Big)^{1/2}\ \exp\Big(\sum_{k\in\mathbb{Z}^+_{\rm odd}} t^1_k p^{(1)}_k(t, X_{1a}) + t^2_k p^{(2)}_k(t, X_{2a})\Big)$$

are the components of a wave function. By definition

$$\partial_a\Psi^{(a)} = X_{aa}(z_a)\Psi^{(a)} = X_{aa}(\mathcal{L}_a)\Psi^{(a)}.$$

Hence the Lax operators can be found by solving the equations

$$\partial_a = \mathcal{L}_a - 2\sum_{\ell\in\mathbb{Z}^+_{\rm odd}} W^{aa}_{1\ell}\frac{\mathcal{L}_a^{-\ell}}{\ell}\quad (a = 1, 2)$$

for $\mathcal{L}_a$ in terms of ∂_a. In particular, the Lax operators are independent of all dynamical variables $\mathbf{t}$. Note that by definition

$$p^{(a)}_k(t, X_{aa}(\mathcal{L}_a)) = \mathcal{L}_a^k + O(\mathcal{L}_a^{-1}).$$

Taking the differential operator part of both sides and recalling $X_{aa}(\mathcal{L}_a) = \partial_a$ we get that

$$(\mathcal{L}_a(0, \partial_a)^k)_+ = p^{(a)}_k(t, \partial_a).$$

On the other hand, by definition

$$\partial_a\Psi^{(b)}(\mathbf{t}, z_b) = X_{ab}(z_b)\Psi^{(b)}(\mathbf{t}, z_b).$$

Therefore

$$\begin{aligned}\partial_{t^a_k}\Psi^{(b)}(\mathbf{t}, z_b) = p^{(a)}_k(t, X_{ab})\Psi^{(b)}(\mathbf{t}, z_b) = p^{(a)}_k(t, \partial_a)\Psi^{(b)}(\mathbf{t}, z_b) =\\ = (\mathcal{L}_a(0, \partial_a)^k)_+\Psi^{(b)}(\mathbf{t}, z_b).\end{aligned}$$

This proves that $\Psi(\mathbf{t},z)$ satisfies the differential equations of the 2-component BKP hierarchy. We need only to varify that $\Psi(x,z)$ is a wave-function, i.e., it satisfies the axioms (W1)–(W3). Axioms (W1) and (W2) hold by definition. Only (W3) requires an argument. Recalling the definition of $\Psi(x,z)$, we get that the pairing

$$(\Psi(x',z),\Psi(x'',z))$$

is a difference of two residues

$$\operatorname{Res}_{z_1=0}\frac{dX_{11}}{X_{11}}e^{(x_1'-x_1'')X_{11}+(x_2'-x_2'')X_{21}} \tag{24}$$

and

$$\operatorname{Res}_{z_2=0}\frac{dX_{22}}{X_{22}}e^{(x_1'-x_1'')X_{12}+(x_2'-x_2'')X_{22}}. \tag{25}$$

Note that $X_1X_2=-t_N$ implies that $X_{11}X_{21}=-t_N$ and $X_{12}X_{22}=-t_N$. Since X_{11} is a coordinate near $z_1=\infty$, the first residue can be written as

$$-\operatorname{Res}_{X_{11}=\infty}\frac{dX_{11}}{X_{11}}e^{(x_1'-x_1'')X_{11}+(x_2'-x_2'')(-t_N/X_{11})},$$

where the 1-form should be expanded as a power series in $x_1'-x_1''$ and $x_2'-x_2''$. The residue is interpreted first formally as the negative of the coefficient in front of dX_{11}/X_{11}. Note however that the coefficients of the formal power series are Laurent polynomials in X_{11}, so each residue can be interpreted analytically. Similarly the second residue (25) equals

$$-\operatorname{Res}_{X_{22}=\infty}\frac{dX_{22}}{X_{22}}e^{(x_1'-x_1'')(-t_N/X_{22})+(x_2'-x_2'')X_{22}}.$$

Changing the variables $X_{22}\mapsto -t_N/X_{11}$ yields

$$\operatorname{Res}_{X_{11}=0}\frac{dX_{11}}{X_{11}}e^{(x_1'-x_1'')X_{11}+(x_2'-x_2'')(-t_N/X_{11})},$$

so we need to prove that

$$(\operatorname{Res}_{X_{11}=0}+\operatorname{Res}_{X_{11}=\infty})\frac{dX_{11}}{X_{11}}e^{(x_1'-x_1'')X_{11}+(x_2'-x_2'')(-t_N/X_{11})}=0.$$

This however follows from the residue theorem for $\mathbb{P}^1$, because the residues involved are residues of Laurent polynomials in X_{11}, so the only poles are at $X_{11}=0$ and $X_{11}=\infty$. This completes the proof that $\Psi(\mathbf{t},z)$ is a wave function of the 2-component BKP hierarchy. The corresponding point $U\in \mathrm{Gr}_2^{I,(0)}$ is invariant under multiplication by X_1 and X_2 and since by definition X_1 and X_2 satisfy the relation in part b) of Lemma 13 we get that U is invariant under multiploication by (z_1^h,z_2^2). Hence $\Psi(\mathbf{t},z)$ is a wave function of the Kac–Wakimoto hierarchy. Furthermore, since the Lax operators are constant the tau-function corresponding to $\Psi(\mathbf{t},z)$ must have the form

$$\widetilde{\tau}(\mathbf{t})=\exp\Big(\sum_{a=1,2}\sum_{k\in\mathbb{Z}_{\rm odd}^+}\widetilde{W}_k^a t_k^a+\frac{1}{2}\sum_{a,b=1,2}\sum_{k,\ell\in\mathbb{Z}_{\rm odd}^+}\widetilde{W}_{k\ell}^{ab}t_k^at_\ell^b\Big).$$

We need only to prove that $\widetilde{W}^a_k = 0$ and $\widetilde{W}^{ab}_{k\ell} = W^{ab}_{k\ell}$. Writing $\Psi(\mathbf{t}, z)$ in terms of the tau-function $\widetilde{\tau}$ we get that $\log \Psi^{(a)}(\mathbf{t}, z_a)$ can be written as a power series in $\mathbf{t}$ that involves at most linear terms. Recalling the definition of $\Psi(\mathbf{t}, z)$ and comparing the free terms, we get

$$-2 \sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}} \widetilde{W}^a_k \frac{z_a^{-k}}{k} + 2 \sum_{k,\ell\in\mathbb{Z}^+_{\mathrm{odd}}} \widetilde{W}^{aa}_{k\ell} \frac{z_a^{-k-\ell}}{k\ell} = \frac{1}{2} \log \frac{z_a \partial_{z_a} X_{aa}}{X_{aa}}, \quad a = 1, 2.$$

The RHS involves only even powers of z_a^{-1}. Therefore all coefficients $\widetilde{W}^a_k = 0$. Comparing the coefficients in front of t^a_k in $\log \Psi^{(b)}(\mathbf{t}, z_b)$ we get

$$\sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} \widetilde{W}^{ab}_{k\ell} \frac{z_b^{-\ell}}{\ell} = \sum_{\ell\in\mathbb{Z}^+_{\mathrm{odd}}} W^{ab}_{k\ell} \frac{z_b^{-\ell}}{\ell}, \quad a, b = 1, 2, \quad k \in \mathbb{Z}^+_{\mathrm{odd}}.$$

This implies that $\widetilde{W}^{ab}_{k\ell} = W^{ab}_{k\ell}$. □

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