THE EXTENDED D-TODA HIERARCHY

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Abstract. In a companion paper to this one, we proved that the Gromov–Witten theory of a Fano orbifold line of type D is governed by a system of Hirota Bilinear Equations. The goal of this paper is to prove that every solution to the Hirota Bilinear Equations determines a solution to a new integrable hierarchy of Lax equations. We suggest the name extended D-Toda hierarchy for this new system of Lax equations, because it should be viewed as the analogue of Carlet's extended bi-graded Toda hierarchy, which is known to govern the Gromov–Witten theory of Fano orbifold lines of type A.

Contents

1. Introduction

1.1. **Background and motivation.** Motivated by Gromov–Witten theory, Dubrovin and Zhang have proposed a general construction which associates an integrable hierarchy to every semi-simple Frobenius manifold. The definition however is very complicated and hence the study of these hierarchies is a very challenging problem. Our strategy is to concentrate on the cases when the Frobenius manifold corresponds to a semi-simple quantum cohomology of a complex orbifold X, whose coarse moduli space $|X|$ is a projective variety. We can further seperate these classes of Frobenius manifolds according to the dimension of X. Based on the examples worked out in the literature, one can speculate that in complex dimension 1, the corresponding integrable hierarchies can be understood in terms of the representation theory of generalized Kac–Moody Lie algebras.

Let us discuss the case when $\dim_{\mathbb{C}}(X) = 1$. The quantum cohomology of X is semi-simple if and only if the coarse moduli space of X is $\mathbb{P}^1$ (see [22]). Let us divide the orbifold lines into three groups depending on whether the orbifold Euler characteristic is > 0, $= 0$, or < 0. The orbifolds in these three groups will be called respectively Fano, elliptic, and hyperbolic orbifold lines. The Fano case is the easiest and nevertheless it is still unfinished. A Fano orbifold line has the form $\mathbb{P}^1_{a_1,a_2,a_3}$, that is, $\mathbb{P}^1$ with 3 orbifold points with isotropy groups of orders a_1, a_2, and a_3, such that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$. Tripples (a_1, a_2, a_3) satisfying the above inequality are in one-to-one correspondence with the Dynkin diagrams of type ADE. According to [19] the corresponding hierarchy must be an extension of a certain Kac–Wakimoto hierarchy. In the case A, the extension is known (see [3, 5, 20]) and it is called the Extended Bi-graded Toda Hierarchy. Our interest is in the case D. We divided the problem into two parts. In the first part, we find the extension of the corresponding Kac–Wakimoto hierarchy in the form of Hirota Bilinear Equations. The second part, which is the goal of this paper, is to describe the extension in terms of Lax equations.

Let us point out that although the Kac–Wakimoto hierarchies have been known for a while, it is still an open question to describe the flows of these hierarchies in terms of Lax equations. There are many cases in which the Lax equations are known – usually the answer is a reduction of some multicomponent KP hierarchy, but there is no general construction that works for all Kac–Wakimoto hierarchies. In particular, for the case D in our project, the Lax equations of the corresponding Kac–Wakimoto hierarchies were unkonwn, so we had to construct them. We believe that our methods can be generalized and that one should be able to construct the Lax equations of all Kac–Wakimoto hierarchies of type D. Finally, let us point out that the symbol of our Lax operator is very similar to the Landau–Ginzburg potential used in the construction of Frobenius structures on the orbit spaces of the extended Weyl groups in [9]. We can speculate that another possible generalization of our work is to construct the integrable hierarchies corresponding to the semi-simple Frobenius manifolds constructed in [9].

In the rest of the introduction we will focus on stating our results.

1.2. **Lax operators.** The idea of our construction is partially motivated by Shiota's approach to the 2-component BKP hierarchy (see [21]). Let $\mathcal{R}$ be the ring of formal power series in ϵ whose coefficients are differential polynomials in the set of $n+1$ variables $\Xi = \{a_1, \dots, a_{n-4}, \alpha, q_2, q_3, c_2, c_3\}$ and the functions $e^{\pm\alpha}$. Formally, $\mathcal{R}$ is defined by

$$\mathcal{R} := \mathbb{C}[\xi^i \,(i \geq 0,\ \xi \in \Xi), e^{\pm\alpha}][[\epsilon]],$$

where ξ^i is a formal variable. We identify $\xi^0 := \xi$ for $\xi \in \Xi$. Let us define the derivation ∂_x

$$\partial_x(P) = \sum_{i=0}^{\infty} \frac{\partial P}{\partial \xi^i}\, \xi^{i+1}.$$

Note that $\xi^i = \partial_x^i(\xi)$. The translation operator $\Lambda := e^{\epsilon\partial_x}$ acts naturally on the ring $\mathcal{R}$ and we put $P[m] := \Lambda^m(P)$.

Suppose that the ring $\mathcal{R}$ is equipped with two commuting derivations ∂_2 and ∂_3 both commuting with ∂_x. Given an operator series

$$A = \sum_{j_1,j_2,j_3\in\mathbb{Z}} f_{j_1j_2j_3}\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3} \tag{1}$$

where $f_{j_1,j_2,j_3} \in \mathcal{R}$ and the sum is possibly infinite, we define the truncations $A_{a,\leq k}$, $A_{a,<k}$, $A_{a,[k]}$, $A_{a,\geq k}$, and $A_{a,>k}$ by keeping only the terms in the sum (1) for which j_a is respectively $\leq k$, $< k$, $= k$, $\geq k$, and $> k$ and truncating the remaining ones, e.g.,

$$A_{1,\geq k} = \sum_{j_1\geq k}\sum_{j_2,j_3\in\mathbb{Z}} f_{j_1j_2j_3}\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3}.$$

Given another operator series $B = \sum_{l_1,l_2,l_3\in\mathbb{Z}} g_{l_1l_2l_3}\Lambda^{l_1}\partial_2^{l_2}\partial_3^{l_3}$, the operator composition of A and B, whenever it makes sense, will be denoted by AB or $A\cdot B$. If the sum (1) is finite and contains only terms for which $j_2, j_3 \geq 0$, then A is called a *differential-difference* operator. A differential-difference operator A acts naturally on the space of formal operator series of the type (1). We denote by $A(B)$ the operator series obtained by applying A to the coefficients of B, that is,

$$A(B) := \sum_{l_1,l_2,l_3\in\mathbb{Z}} A(g_{l_1l_2l_3})\Lambda^{l_1}\partial_2^{l_2}\partial_3^{l_3}.$$

Finally, let us introduce the adjoint operation #

$$A^\# = \sum_{j_1,j_2,j_3\in\mathbb{Z}} \Lambda^{-j_1}(-\partial_2)^{j_2}(-\partial_3)^{j_3} f_{j_1j_2j_3} \tag{2}$$

which obeys $(AB)^\# = B^\# A^\#$ for any two operator series A and B for which the composition AB makes sense.

Let us denote by $\mathcal{E} := \mathcal{R}[\Lambda^{\pm1}, \partial_2, \partial_3]$ the ring of differential-difference operators and define the following 4 operators in $\mathcal{E}$:

$$\begin{aligned}
\mathcal{L} &:= \Big(\sum_{i=1}^{n-3}(a_i\Lambda^i - \Lambda^{-i}a_i)\Big)(\Lambda-\Lambda^{-1}) + \frac{1}{2}\partial_2^2 + \frac{1}{2}\partial_3^2 + \frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1}) + \frac{1}{2}(c_2+c_3),\\
H_1 &:= \partial_2\partial_3 + q_1,\\
H_2 &:= (\Lambda-1)\partial_2 - q_2(\Lambda+1),\\
H_3 &:= (\Lambda+1)\partial_3 - q_3(\Lambda-1),
\end{aligned}$$

where $q_1 := -2(1+\Lambda)^{-1}(q_2q_3) \in \mathcal{R}$ and $a_{n-3} := \frac{1}{n-2}e^{(n-2)\alpha}$.

Let us introduce the following rings

$$\mathcal{E}_{(\pm)} = \mathcal{R}[\partial_2,\partial_3]((\Lambda^{\mp1})), \quad \mathcal{E}_{(2)} = \mathcal{R}[\Lambda,\Lambda^{-1},\partial_3]((\partial_2^{-1})), \quad \mathcal{E}_{(3)} = \mathcal{R}[\Lambda,\Lambda^{-1},\partial_2]((\partial_3^{-1}))$$

and

$$\mathcal{E}^0_{(\pm)} = \mathcal{R}((\Lambda^{\mp 1})), \quad \mathcal{E}^0_{(2)} = \mathcal{R}((\partial_2^{-1})), \quad \mathcal{E}^0_{(3)} = \mathcal{R}((\partial_3^{-1})).$$

Finally, let us denote by $\mathcal{A}H$ the left ideal in $\mathcal{A}$ generated by H_1, H_2, H_3, where $\mathcal{A}$ could be any of the rings $\mathcal{E}_{(\pm)}$, $\mathcal{E}_{(2)}$, or $\mathcal{E}_{(3)}$. Note that we always have a decomposition into sum of vector spaces

$$\mathcal{A} = \mathcal{A}^0 + \mathcal{A}H, \quad \forall\ \mathcal{A} \in \{\mathcal{E}_{(\pm)}, \mathcal{E}_{(2)}, \mathcal{E}_{(3)}\}. \tag{3}$$

We will work out a criteria for the derivations ∂_2 and ∂_3 that guarantees that (3) is a direct sum decomposition, that is, $\mathcal{A}^0 \cap \mathcal{A}H \neq \{0\}$. If (3) is a direct sum decomposition, then we denote by π_α ($\alpha = \pm, 2, 3$) the corresponding projection $\mathcal{E}_{(\alpha)} \to \mathcal{E}^0_{(\alpha)}$ and we have the following recursion formulas:

$$\begin{aligned}
\pi_\alpha(\Lambda^{j_1 \pm 1}\partial_2^{j_2}\partial_3^{j_3}) &= \Lambda^{\pm 1}\Big(\pi_\alpha(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3})\Big)\cdot \pi_\alpha(\Lambda^{\pm 1}),\\
\pi_\alpha(\Lambda^{j_1}\partial_2^{j_2+1}\partial_3^{j_3}) &= \partial_2\Big(\pi_\alpha(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3})\Big) + \pi_\alpha(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3})\cdot \pi_\alpha(\partial_2),\\
\pi_\alpha(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3+1}) &= \partial_3\Big(\pi_a(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3})\Big) + \pi_\alpha(\Lambda^{j_1}\partial_2^{j_2}\partial_3^{j_3})\cdot \pi_\alpha(\partial_3),
\end{aligned}$$

which reduce the computation of π_α to the following cases:

$$\begin{aligned}
&\pi_\pm(\partial_2) = \iota_{\Lambda^{\mp 1}} Q_2, \quad \pi_\pm(\partial_3) = \iota_{\Lambda^{\mp 1}} Q_3,\\
&\pi_2(\Lambda) = (\partial_2 - q_2)^{-1}\cdot(\partial_2 + q_2) = -1 + 2(\partial_2 - q_2)^{-1}\cdot\partial_2, \quad \pi_2(\partial_3) = -\partial_2^{-1}\cdot q_1,\\
&\pi_3(\Lambda) = -(\partial_3 - q_3)^{-1}\cdot(\partial_3 + q_3) = 1 - 2(\partial_3 - q_3)^{-1}\cdot\partial_3, \quad \pi_3(\partial_2) = -\partial_3^{-1}\cdot q_1,
\end{aligned}$$

where $Q_2 := (\Lambda - 1)^{-1} q_2 (\Lambda + 1)$, $Q_3 := (\Lambda + 1)^{-1} q_3 (\Lambda - 1)$ and ι_Λ (resp. $\iota_{\Lambda^{-1}}$) denotes the Laurent series expansion at $\Lambda = 0$ (resp. $\Lambda = \infty$).

Proposition 1. *a) The decomposition* (3) *is a direct sum of vector spaces if and only if the derivations ∂_2 and ∂_3 satisfy the following 0-curvature condition in $\mathcal{E}_{(\pm)}$*

$$\partial_2(Q_3) - \partial_3(Q_2) = [Q_2, Q_3].$$

b) There are unique derivations ∂_2 and ∂_3 such that (3) *is a direct sum of vector spaces and*

$$H_a\mathcal{L} \in \mathcal{E}H, \quad 2 \le a \le 3.$$

Remark 2. The 0-curvature condition can be justified as follows: Note that $\pi_\alpha(\partial_2\partial_3)$ ($\alpha = \pm, 2, 3$) can be computed using the recursion formulas from above in two different ways: reducing the power of ∂_2 or reducing the power of ∂_3. The two computation will agree if and only if

$$\partial_2\Big(\pi_\alpha(\partial_3)\Big) - \partial_3\Big(\pi_\alpha(\partial_2)\Big) + [\pi_\alpha(\partial_3), \pi_\alpha(\partial_2)] = 0,$$

which is equivalent to the 0-curvature condition. Similarly, the projection $\pi_\alpha(\Lambda\partial_b)$ ($\alpha=\pm,2,3,\ b=2,3$) can be computed in two different ways, which will agree if and only if

$$\partial_b\Big(\pi_\alpha(\Lambda)\Big)+\pi_\alpha(\Lambda)\cdot\pi_a(\partial_b)=\Lambda\Big(\pi_\alpha(\partial_b)\Big)\cdot\pi_\alpha(\Lambda).$$

This relation is also equivalent to the 0-curvature condition. □

1.3. **Dressing operators.** There are unique operators $L_1=b_{1,0}\Lambda+\sum_{i=1}^\infty b_{1,i}\Lambda^{1-i}$, $L_a=\partial_a+\sum_{i=1}^\infty b_{a,i}\partial_a^{1-i}$ ($a=2,3$) with coefficients in $\mathcal{R}$ such that

$$b_{1,0}=\exp\Big(\frac{(\Lambda-1)(n-2)}{\Lambda^{n-2}-1}(\alpha)\Big),\quad b_{2,1}=b_{3,1}=0$$

and

$$\pi_+(\mathcal{L})=\frac{1}{n-2}L_1^{n-2},\quad \pi_2(\mathcal{L})=\frac{1}{2}L_2^2,\quad \pi_3(\mathcal{L})=\frac{1}{2}L_3^2. \tag{4}$$

Let us construct 3 differential ring extensions $\mathcal{R}_i$ ($1\le i\le 3$) of $\mathcal{R}$. Put

$$\mathcal{R}_1:=\mathbb{C}[\xi^j(\xi\in\Xi,j\ge 0),e^{\pm\alpha},e^{\pm\phi},\psi_{1,1},\psi_{1,2},\dots][[\epsilon]],$$

and

$$\mathcal{R}_a=\mathbb{C}[\xi^j(\xi\in\Xi,j\ge 0),e^{\pm\alpha},\psi_{a,1},\psi_{a,2},\dots][[\epsilon]],\quad 2\le a\le 3.$$

The derivation $\epsilon\partial_x$ is extended uniquely to a derivation of $\mathcal{R}_1$ in such a way that $\epsilon\partial_x(\phi)=\alpha$ and $L_1=S_1\Lambda S_1^{-1}$, where

$$S_1=\psi_{1,0}+\sum_{i=1}^\infty\psi_{1,i}\Lambda^{-i},\quad \psi_{1,0}=e^{\frac{(n-2)\epsilon\partial_x}{1-\Lambda^{n-2}}(\phi)}.$$

More explicitly, by comparing the coefficients in front of Λ^{1-i} we get that the identity $L_1S_1=S_1\Lambda$ will be satisfied if we define

$$\epsilon\partial_x\Big(\frac{\psi_{1,i}}{\psi_{1,0}}\Big)=\frac{\epsilon\partial_x}{1-e^{\epsilon\partial_x}}\left(\sum_{s=1}^i b_{1,s}\frac{\psi_{1,i-s}[1-s]}{\psi_{1,0}}\right). \tag{5}$$

Similarly, there exists a unique extension of the derivation ∂_a ($a=2,3$) to a derivation of $\mathcal{R}_a$, such that, $L_aT_a=T_a\partial_a$, where

$$T_a=1+\psi_{a,1}\partial_a^{-1}+\psi_{a,2}\partial_a^{-2}+\cdots.$$

Such an extension exists and the derivation ∂_a ($a=2,3$) is uniquely determined. Indeed, substituting $L_a=\partial_a+\sum_{i=1}^\infty b_{a,i}\partial_a^{1-i}$ and the above expansion of T_a in $L_aT_a=T_a\partial_a$, comparing the coefficients in front of ∂_a^{-k} for $k\ge 1$, and using that $b_{a,1}=0$ we get

$$\partial_a(\psi_{a,k})+\sum_{i=2}^{k+1}\sum_{s=0}^{k+1-i}\binom{1-i}{s}b_{a,i}\partial_a^s(\psi_{a,k+1-i-s})=0,\quad k\ge 1.$$

This formula alows us to define recursively $\partial_a(\psi_{a,k})$ for all $k \geq 1$. We are going to prove that the derivations $\epsilon\partial_x$, ∂_2, and ∂_3 can be extended uniquely to pairwise commuting derivations of $\mathcal{R}_i$ ($1 \leq i \leq 3$), such that, the following conjugation relations hold: for the operator S_1

$$S_1 \Lambda S_1^{-1} = L_1, \quad S_1 \partial_a S_1^{-1} = \partial_a - Q_a \ (a = 2, 3), \tag{6}$$

for the operator T_2

$$T_2(\Lambda - 1)T_2^{-1} = (\partial_2 + q_2)^{-1} H_2, \quad T_2 \partial_2 T_2^{-1} = L_2, \quad T_2 \partial_3 T_2^{-1} = \partial_2^{-1} H_1, \tag{7}$$

and for the operator T_3

$$T_3(\Lambda + 1)T_3^{-1} = (\partial_3 + q_3)^{-1} H_3, \quad T_3 \partial_2 T_3^{-1} = \partial_3^{-1} H_1, \quad T_3 \partial_3 T_3^{-1} = L_3. \tag{8}$$

We need to modify slightly the definition of T_2 and T_3. It turns out that the following proposition holds:

Proposition 3. *There exists an operator $S_a \in 1 + \mathcal{R}_a[[\partial_a^{-1}]]$ ($a = 2, 3$), such that, $S_a^{-1} T_a$ commutes with the derivations $\epsilon\partial_x, \partial_2$, and ∂_3 and $S_a^{\#} = \partial_a S_a^{-1} \partial_a^{-1}$.*

Note that the conjugation formulas (7) and (8) remain valid if we replace T_a by S_a. The operators S_i ($1 \leq i \leq 3$) will be called *dressing operators*.

1.4. **Lax equations.** Let L_i ($1 \leq i \leq 3$) be the operator series defined by (4). Let us define

$$B_{1,k} := \sum_{m=0}^{\infty} \Big(\big(L_1^k \Lambda^{-2m-1} \big)_{1,\geq 0} + \big(\Lambda^{2m+1} (L_1^{\#})^k \big)_{1,<0} \Big) (\Lambda - \Lambda^{-1}), \quad k \geq 1,$$

and

$$B_{a,2l+1} := (L_2^{2l+1})_{2,\geq 0}, \quad a = 2, 3, \quad l \geq 0.$$

Using the dressing operators we define

$$\log L_1 := S_1 \epsilon \partial_x S_1^{-1} = \epsilon\partial_x - \ell_1, \quad \ell_1 \in \mathcal{R}[[\Lambda^{-1}]],$$

$$\log\Big((\partial_2 + q_2)^{-1} H_2 + 1 \Big) := S_2 \epsilon \partial_x S_2^{-1} = \epsilon\partial_x - \ell_2, \quad \ell_2 \in \mathcal{R}[[\partial_2^{-1}]]\partial_2^{-1},$$

and

$$\log\Big((\partial_3 + q_3)^{-1} H_3 - 1 \Big) := S_3 \epsilon \partial_x S_3^{-1} = \epsilon\partial_x - \ell_3, \quad \ell_3 \in \mathcal{R}[[\partial_3^{-1}]]\partial_3^{-1},$$

where $\ell_i := \epsilon\partial_x(S_i) \cdot S_i^{-1}$ and the fact that the coefficients of ℓ_i belong to $\mathcal{R}$ will be established later on. Put

$$A_{1,k}^+ := \frac{1}{(n-2)^k k!} L_1^{(n-2)k} (\epsilon\partial_x - \ell_1 - h_k),$$

$$A_{1,k}^- := \iota_\Lambda (\Lambda - \Lambda^{-1})^{-1} (A_{1,k}^+)^{\#} (\Lambda - \Lambda^{-1}),$$

and

$$A_{a,k} := \frac{1}{2^k k!} L_a^{2k}(\epsilon\partial_x - \ell_a), \quad a = 2,3,$$

where $k \geq 1$ and $h_k = \frac{1}{n-2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$. Let us define $B_{0,k} = B_{0,k,1} + B_{0,k,2} + B_{0,k,3}$, where

$$B_{0,k,1} = \sum_{m=0}^{+\infty} \left(\left(A_{1,k}^{+} \cdot \Lambda^{-2m-1} \right)_{1,\geq 0} + \left(A_{1,k}^{-} \cdot \Lambda^{2m+1} \right)_{1,<0} \right) \cdot (\Lambda - \Lambda^{-1}),$$

$$B_{0,k,2} = \left(A_{2,k} \right)_{2,>0} + \frac{1}{2}\left(A_{2,k} \right)_{2,[0]} \cdot (1 + \Lambda^{-1}),$$

$$B_{0,k,3} = \left(A_{3,k} \right)_{3,>0} + \frac{1}{2}\left(A_{3,k} \right)_{3,[0]} \cdot (1 - \Lambda^{-1}).$$

Our first result can be stated as follows.

Theorem 4. *Let (i,k) be an arbitrary pair of non-negative integers, such that $0 \leq i \leq 3$, $k \geq 1$, and $k = 2l+1$ is odd for $i = 2,3$.*

a) There exists a unique derivation $\partial_{i,k} : \mathcal{R} \to \mathcal{R}$ commuting with ∂_x such that

$$\partial_{i,k}(\mathcal{L}) - [B_{i,k}, \mathcal{L}] \in \mathcal{E}H,$$

$$\partial_{i,k}(H_a) - [B_{i,k}, H_a] \in \mathcal{E}H, \quad a = 2,3.$$

b) The set of derivations $\{\partial_{i,k}\}$ pairwise commute.

The integrable hierarchy defined by the infinite set of commuting derivations in Theorem 4 will be called the *Extended D-Toda Hierarchy*. Note that $\pi_+(\partial_{i,k}(M)) = \partial_{i,k}(M)$ for $M = \mathcal{L}, H_a (1 \leq a \leq 3)$. Therefore, the conditions in part a) of Theorem 4 yield the following formula:

$$\partial_{i,k}(M) = \pi_+[B_{i,k}^{+}, M], \quad M \in \{\mathcal{L}, H_1, H_2, H_3\}, \tag{9}$$

where $B_{i,k}^{+} := \pi_+(B_{i,k})$. We will prove also that formula (9) provides an equivalent formulation of the Extended D-Toda Hierarchy (see Lemma 10).

The Extended D-Toda hierarchy can be formulated also in terms of the dressing operators. In other words, the derivations $\partial_{i,k}$ can be extended to $\mathcal{R}_a$ $(1 \leq a \leq 3)$ and the extended derivations are pairwise commuting. The formulas for the derivatives of the dressing operators can be found in Section 5.1.

1.5. Hirota bilinear equations.

Let $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ be 4 sequences of formal variables of the following form:

$$\mathbf{t}_i = (t_{i,k})_{k\geq 1} \quad (i = 0,1), \quad \mathbf{t}_a = (t_{a,2k-1})_{k\geq 1}, \quad (a = 2,3).$$

The 2 variables $y_a := t_{a,1}$ $(2 \leq a \leq 3)$ will play a special role. Let $\mathcal{O}_\epsilon := \mathcal{O}(\mathbb{C})[[\epsilon]]$ be the ring of formal power series in ϵ whose coefficients are holomorphic functions on $\mathbb{C}$. Let x be the standard coordinate function on $\mathbb{C}$. The ring $\mathcal{O}_\epsilon[[\mathbf{t}]]$ is equipped with a translation operator

$$\Lambda : \mathcal{O}_\epsilon[[\mathbf{t}]] \to \mathcal{O}_\epsilon[[\mathbf{t}]], \quad \Lambda(f)(x,\mathbf{t}) := f(x+\epsilon,\mathbf{t})$$

and an infinite set of pairwise commuting differentiations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t_{i,k}}$. Note that a ring homomorphism

$$\phi : \mathcal{R} \to \mathcal{O}_\epsilon[[\mathbf{t}]], \quad \text{such that} \quad \phi\circ\partial_x = \frac{\partial}{\partial x}\circ\phi$$

is uniquely determined by a set of $n+1$ functions in $\mathcal{O}_\epsilon[[\mathbf{t}]]$

$$\phi(\xi) = \xi(x,\mathbf{t}) \quad \xi\in\Xi.$$

We say that the set of functions $\xi(x,\mathbf{t})$ ($\xi\in\Xi$) is a solution to the Extended D-Toda Hierarchy if the corresponding ring homomorphism satisfies

$$\phi\circ\partial_a = \tfrac{\partial}{\partial y_a}\circ\phi \ (a=2,3), \quad \phi\circ\partial_{i,k} = \tfrac{\partial}{\partial t_{i,k}}\circ\phi.$$

The second goal of our paper is to prove that solutions of the above type can be constructed from a system of Hirota Bilinear Equations (HBEs), which we describe now.

Suppose that $\tau(x,\mathbf{t})\in\mathcal{O}_\epsilon[[\mathbf{t}]]$ is an arbitary invertible formal power series, i.e., $\tau(x,0)\in\mathcal{O}(\mathbb{C})[[\epsilon]]$ is a formal power series in ϵ whose leading order term is a holomorphic function on $\mathbb{C}$ that has no zeros. Let $\mathcal{D}_\epsilon := \mathcal{D}(\mathbb{C})[[\epsilon]]$ be the ring of formal power series in ϵ whose coefficients belong to the ring $\mathcal{D}(\mathbb{C})$ of holomorphic differential operators on $\mathbb{C}$. We extend the anti-involution # to operator series of the form (1) with coefficients $f_{j_1,j_2,j_3}\in\mathcal{D}_\epsilon[[\mathbf{t}]]$, so that $(\partial_x)^\# = -\partial_x$. Let us introduce the formal power series

$$\Psi_1^+(x,\mathbf{t},z) = \psi_1^+(x,\mathbf{t},z)e^{\xi_1(\mathbf{t},z)}z^{x/\epsilon-\frac{1}{2}} \quad \text{and} \quad \Psi_1^-(x,\mathbf{t},z) = z^{-x/\epsilon-\frac{1}{2}}e^{-\xi_1(\mathbf{t},z)}\psi_1^-(x,\mathbf{t},z)$$

taking values in respectively $\mathcal{D}_\epsilon((z^{-1}))[[\mathbf{t}]]z^{x/\epsilon-\frac{1}{2}}$ and $z^{-x/\epsilon-\frac{1}{2}}\mathcal{D}_\epsilon((z^{-1}))[[\mathbf{t}]]$, where

$$\xi_1(\mathbf{t},z) = \sum_{k\geq 1}\left(t_{1,k}z^k + t_{0,k}\,(\epsilon\partial_x - h_k)\,\frac{z^{(n-2)k}}{(n-2)^k k!}\right),$$

$$\psi_1^\pm(x,\mathbf{t},z) = \frac{e^{\mp\sum_{k\geq 1}\frac{z^{-k}}{k}\partial_{t_{1,k}}}\,\tau(x\mp\epsilon,\mathbf{t})}{\tau(x,\mathbf{t})} = \sum_{k=0}^{\infty}\psi_{1,k}^\pm(x,\mathbf{t})z^{-k},$$

where $h_k = \frac{1}{n-2}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)$. Similarly, let us define for $a=2,3$ the formal series

$$\Psi_a^+(x,\mathbf{t},z) = \psi_a^+(x,\mathbf{t},z)e^{\xi_a(\mathbf{t},z)} \quad \text{and} \quad \Psi_a^-(x,\mathbf{t},z) = e^{-\xi_a(\mathbf{t},z)}\psi_a^-(x,\mathbf{t},z)$$

taking values in $\mathcal{D}_\epsilon((z^{-1}))[[\mathbf{t}]]$, where

$$\xi_a(\mathbf{t},z) = \sum_{l\geq 1}\left(t_{a,2l-1}z^{2l-1} + t_{0,l}\,\epsilon\partial_x\,\frac{z^{2l}}{2^l l!}\right),$$

$$\psi_a^{\pm}(x,\mathbf{t},z) = \frac{e^{\mp 2\sum_{l\geq 1}\frac{z^{-2l+1}}{2l-1}\partial_{t_{a,l}}}\tau(x,\mathbf{t})}{\tau(x,\mathbf{t})} = \sum_{k=0}^{\infty}\psi_{a,k}^{\pm}(x,\mathbf{t})z^{-k}.$$

The Hirota Bilinear Equations of the Extended D-Toda hierarchy are given by the following system of quadratic equations

$$\begin{aligned}
(10)\qquad & \mathrm{Res}_{z=0}\frac{z^{(n-2)k}}{(n-2)^k k!}\frac{dz}{z}\left(\Psi_1^+(x,\mathbf{t},z)\Psi_1^-(x+m\epsilon,\mathbf{t}',z)+(\Psi_1^+(x+m\epsilon,\mathbf{t}',z)\Psi_1^-(x,\mathbf{t},z))^{\#}\right)\\
= \;& \mathrm{Res}_{z=0}\frac{z^{2k}}{2^k k!}\frac{dz}{2z}\left(\Psi_2^+(x,\mathbf{t},z)\Psi_2^-(x+m\epsilon,\mathbf{t}',z)-(-1)^m\Psi_3^+(x,\mathbf{t},z)\Psi_3^-(x+m\epsilon,\mathbf{t}',z)\right),
\end{aligned}$$

where $k\geq 0$ and m are arbitary integers and $\Psi_i^{\pm}(x+m\epsilon,\mathbf{t}',z):=\Lambda^m(\Psi_i^{\pm}(x,\mathbf{t}',z))$. If the above equations are satisfied then we say that τ is a *tau-function* of the Extended D-Toda hierarchy, the formal series $\Psi_1^{\pm}$ and $\Psi_a:=\Psi_a^+$ $(a=2,3)$ will be called *wave functions*, and the operator series

$$S_1^+(x,\mathbf{t},\Lambda)=\sum_{j=0}^{\infty}\psi_{1,j}^+(x,\mathbf{t})\Lambda^{-j}$$

$$S_1^-(x,\mathbf{t},\Lambda)=\sum_{j=0}^{\infty}\Lambda^{-j}\psi_{1,j}^-(x,\mathbf{t})$$

$$S_a(x,\mathbf{t},\partial_a)=\sum_{j=0}^{\infty}\psi_{a,j}^+(x,\mathbf{t})\partial_a^{-j}\quad (a=2,3)$$

will be called *wave operators*. Let us introduce also the following auxiliarly Lax operators:

$$L_1^+(x,\mathbf{t},\Lambda):=S_1^+(x,\mathbf{t},\Lambda)\cdot\Lambda\cdot S_1^+(x,\mathbf{t},\Lambda)^{-1}=:u_{1,0}^+(x,\mathbf{t})\Lambda+\sum_{j=1}^{\infty}u_{1,j}^+(m,\mathbf{t})\Lambda^{1-j},$$

$$L_1^-(x,\mathbf{t},\Lambda):=S_1^-(x,\mathbf{t},\Lambda)^{\#}\cdot\Lambda^{-1}\cdot\left(S_1^-(x,\mathbf{t},\Lambda)^{\#}\right)^{-1}=:u_{1,0}^-(x,\mathbf{t})\Lambda^{-1}+\sum_{j=1}^{\infty}u_{1,j}^-(x,\mathbf{t})\Lambda^{j-1},$$

$$L_a(x,\mathbf{t},\partial_a):=S_a(x,\mathbf{t},\partial_a)\cdot\partial_a^{-1}\cdot S_a(x,\mathbf{t},\partial_a)^{-1}=:\partial_a+\sum_{j=1}^{\infty}u_{a,j}(x,\mathbf{t})\partial_a^{-j}\quad (a=2,3).$$

To avoid cumbersome notation we put $L_1(x,\mathbf{t},\Lambda):=L_1^+(x,\mathbf{t},\Lambda)$ and $u_{1,j}=u_{1,j}^+$.

Theorem 5. *If $\tau(x,\mathbf{t})$ is a tau-function of the Extended D-Toda hierarchy, then there is a uniquely determined solution of the Extended D-Toda hierarchy, such that,*

$$c_a(x,\mathbf{t})=\partial_{t_{a,1}}^2\log\tau(x,\mathbf{t}),\quad q_a(x,\mathbf{t})=\partial_{t_{a,1}}\log\frac{\tau(x,\mathbf{t})}{\tau(x+\epsilon,\mathbf{t})}\quad (a=2,3),$$

and the Lax operator

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{\partial_2^2}{2}+\frac{\partial_3^2}{2}+\left(\left(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\right)_{1,>0}-\left(\sum_{m=0}^{\infty}\Lambda^{2m+1}\tfrac{(L_1^{\#})^{n-2}}{n-2}\right)_{1,<0}\right)(\Lambda-\Lambda^{-1})+\\
&+\frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1})+\frac{1}{2}(c_2+c_3).
\end{aligned}$$

The proof of Theorem 5 will be given in Sections 6 and 7. In fact, Sections 6 and 7 can be read independently. Our strategy is to construct Lax operators and derive their evolution equations directly from the HBEs. In fact, the system of Lax equations (9) was discovered exactly in this way.

The inverse of Theorem 5 is also true, i.e., we have the following theorem:

Theorem 6. *Any solution to the Extended D-Toda hierarchy in $\mathcal{O}_\epsilon[[\mathbf{t}]]$ is obtained from a tau-function.*

The proof of Theorem 6 has two steps. The first one is to prove that for any solution of the Lax equations (9) the corresponding dressing operators satisfy the bilinear relations (10). Note that the wave function can be expressed only in terms of the dressing operators, so we can interpret (10) as a system of bilinear relations for the dressing operators. Using Taylor's series expansion, it is easy to check that (10) is equivalent to (52), so our proof of Proposition 17 (see Section 4.4) contains also the first step in the proof of the existence of a tau-function. The second step is to show the exsitence of the tau-function. The argument is essentially the same as in [18] (see also [5]) and we leave the details to the interested reader.

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2. Projections

The first goal of this section is to prove Proposition 1. The second goal is to establish several key properties of the rings $\mathcal{E}_{(\alpha)}$ ($\alpha \in \{\pm, 2, 3\}$) and the corresponding projections $\pi_\alpha : \mathcal{E}_{(\alpha)} \to \mathcal{E}^0_{(\alpha)}$. We prove two propositions which will be used in an essential way in the rest of the paper. While the second proposition (see Proposition 9) is straightforward to prove, the first one (see Proposition 8) will require some preparation. Namely, we will have to introduce rings of rational difference operators.

2.1. **Proof of Proposition 1.**

Proof. a) The neccessity of the condition is clear because

$$\partial_2(Q_3) - \partial_3(Q_2) - [Q_2, Q_3] = [\partial_2 - Q_2, \partial_3 - Q_3] \in \mathcal{E}^0_{(+)} \cap \mathcal{E}_{(+)} H.$$

Suppose that the 0-curvature condition is satisfied. Let us prove that (3) is a direct sum decomposition for $\mathcal{A} = \mathcal{E}_{(+)}$. The argument for $\mathcal{E}_{(-)}$ is identical. Note that H_1 already belongs to the ideal in $\mathcal{E}_{(+)}$ generated by H_2 and H_3. We have to prove that if $P := a(\partial_2 - Q_2) + b(\partial_3 - Q_3) \in \mathcal{E}^0_{(+)}$ then $P = 0$. Note that a and b must have the form

$$a = \sum_{i=0}^{m} a_i(\Lambda, \partial_3)\partial_2^i, \quad b = \sum_{j=0}^{m+1} b_j(\Lambda, \partial_3)\partial_2^j,$$

where the coefficients $a_i, b_j \in \mathcal{R}[\partial_3]((\Lambda^{-1}))$. We argue by induction on m. If $m = 0$ then by comparing the coefficients in front of ∂_2 we get $a_0 = -b_1(\partial_3 - Q_3)$. Using the 0-curvature condition we get $b_1 Q_2(\partial_3 - Q_3) \in \mathcal{E}^0_{(+)}$. This implies that $b_1 = 0$, otherwise we write $b_1 = \sum_{i=0}^{k} b_{1,i}\partial_3^i$ with $b_{1,k} \neq 0$. The coefficient in front of the highest power of ∂_3 is $b_{1,k}Q_2$ and it must vanish – contradiction because $\mathcal{E}^0_{(+)}$ does not have 0-divisors. If $m > 0$, then by comparing the coefficeints in front of ∂_2^{m+1} we get $a_m = -b_{m+1}(\partial_3 - Q_3)$. Using the 0-curvature condition we get

$$a_m \partial_2^m(\partial_2 - Q_2) + b_{m+1}\partial_2^{m+1}(\partial_3 - Q_3) = b_{m+1}\Big([Q_3, \partial_2^m](\partial_2 - Q_2) + \partial_2^m \cdot Q_2 \cdot (\partial_3 - Q_3)\Big).$$

Therefore we can write P as a linear combination of $\partial_2 - Q_2$ and $\partial_3 - Q_3$ whose coefficients are differential operators of ∂_2 of orders respectively $< m$ and $< m+1$. Recalling the inductive assumption we get $P = 0$.

Let us move to the case $\mathcal{A} = \mathcal{E}_{(2)}$. After a direct computation we get that the 0-curvature condition is equivalent to the following identities

$$\partial_2(q_3) = \partial_3(q_2) = \frac{e^{\epsilon\partial_x} - 1}{e^{\epsilon\partial_x} + 1}(q_2 q_3) \tag{11}$$

Using (11) we get the following relation:

$$\partial_2 \cdot H_3 + q_3 H_2 = (\Lambda + 1) \cdot H_1.$$

Therefore, the left ideal $\mathcal{E}_{(2)}H$ is generated by H_1 and H_2. Put

$$\widetilde{H}_1 := \partial_2^{-1} H_1 = \partial_3 + \partial_2^{-1} q_1$$

and

$$\widetilde{H}_2 := (\partial_2 + q_2)^{-1} H_2 = (1 + \partial_2^{-1} q_2)^{-1}(1 - \partial_2^{-1} q_2)\Lambda - 1.$$

We have to prove that if $P = a\widetilde{H}_1 + b\widetilde{H}_2 \in \mathcal{E}^0_{(2)}$ then $P = 0$. Similarly to the above case, since the coefficients in front of the highest power of ∂_3 in P vanish, we get that a and b must have the form $a = \sum_{i=0}^{m} a_i \partial_3^i$ and $b = \sum_{j=0}^{m+1} b_j \partial_3^j$, where $a_i, b_j \in \mathcal{R}[\Lambda^{\pm 1}]((\partial_2^{-1}))$. The coefficients in front of ∂_3^{m+1} in P must vanish $\Rightarrow a_m = -b_{m+1}\widetilde{H}_2$. We are going to prove that $[\widetilde{H}_1, \widetilde{H}_2] = 0$. Assuming this fact we get

$$a_m \partial_3^m \widetilde{H}_1 + b_{m+1}\partial_3^{m+1}\widetilde{H}_2 = b_{m+1}\Big([\partial_3^m, \widetilde{H}_2]\widetilde{H}_1 + \partial_3^m \cdot (-\partial_2^{-1} q_1) \cdot \widetilde{H}_2\Big).$$

Therefore we can complete the proof by induction on m. Let us prove that $[\widetilde{H}_1, \widetilde{H}_2] = 0$. We claim that the vanishing of this commutator follows from the identities

$$\partial_2(q_3) = \partial_3(q_2) = q_1 + q_2 q_3 = -q_2 q_3 - q_1[1], \tag{12}$$

where the last identity is just the definition of q_1, while the rest of the identities, being equivalent to (11), are consequence of the 0-curvature condition. It is convenient to put $M=(1+\partial_2^{-1}q_2)^{-1}(1-\partial_2^{-1}q_2)$. The vanishing of the commutator $[\widetilde{H}_1,\widetilde{H}_2]$ is equivalent to

$$\partial_3(M)=-\partial_2^{-1}q_1M+M\partial_2^{-1}q_1[1]. \tag{13}$$

We have

$$\partial_3(M)=-(1+\partial_2^{-1}q_2)^{-1}\partial_2^{-1}\Big(\partial_3(q_2)M+\partial_3(q_2)\Big).$$

Substituting the above formula in (13), collecting the terms that have M as the rightmost factor, and multiplying both sides of the equation from the left by $\partial_2(1+\partial_2^{-1}q_2)$, we get that (13) is equivalent to

$$(-\partial_3(q_2)+q_1+q_2\partial_2^{-1}q_1)M-\partial_3(q_2)=\partial_2(1+\partial_2^{-1}q_2)M\partial_2^{-1}q_1[1].$$

Using formula (12) we transform the above equation into

$$-q_2\partial_2^{-1}q_3(\partial_2-q_2)-\partial_3(q_2)=(\partial_2-q_2)\partial_2^{-1}q_1[1].$$

This however follows easily from (12).

b) Let us first prove the uniqueness. We will argue that the derivations ∂_2 and ∂_3 are uniquely determined from the 0-curvature condition and the 4 projection constraints

$$\pi_+(H_2\mathcal{L})=\pi_+(H_3\mathcal{L})=0,\quad \pi_2(H_2\mathcal{L})=0,\quad \pi_3(H_3\mathcal{L})=0.$$

We already know that the 0-curvature condition is equivalent to formulas (12), that is, $\partial_a(q_b)$ for $a\neq b$ is uniquely fixed. Let us prove that

$$\partial_a(q_a)=\frac{1}{2}\left(1-e^{\epsilon\partial_x}\right)(c_a),\quad a=2,3. \tag{14}$$

The proof of the two formulas is identical, so let us consider only the case $\partial_2(q_2)$. Note that $\pi_2(\mathcal{L})=\frac{\partial_2^2}{2}+c_2+O(\partial_2^{-1})$ and that

$$(\partial_2-q_2)^{-1}H_2=\Lambda-1-2\sum_{i=1}^{\infty}(\partial_2^{-1}q_2)^i.$$

Comparing the coefficients in front of ∂_2^0 in

$$\pi_2((\partial_2-q_2)^{-1}H_2\mathcal{L})=0$$

we get

$$(\Lambda-1)(c_2)+2\partial_2(q_2)=0.$$

The formula for $\partial_2(q_2)$ follows.

Note that $\pi_+(H_a\mathcal{L})=0$ is equivalent to

$$\partial_a(\mathcal{L})=\pi_+([Q_a,\mathcal{L}]). \tag{15}$$

The derivatives $\partial_2(a_i)$, $\partial_3(a_i)$ $(1 \le i \le n-4)$ and $\partial_a(c_b)$ $(2 \le a, b \le 3)$ can be determined by comparing the coefficients in front of Λ^i for $0 \le i \le n-3$. Indeed, let us consider only the case $\partial_2(a_i)$ and $\partial_2(c_b)$, because the other case is analogous. The operator $\mathcal{L}$ has the form

$$a_{n-3}\Lambda^{n-2} + a_{n-4}\Lambda^{n-3} + \sum_{k=2}^{n-4}(a_{k-1} - a_{k+1})\Lambda^k + \Big(-a_2 + \frac{1}{4}(c_2 - c_3)\Big)\Lambda +$$
$$\Big(-a_1 - a_1[-1] + \frac{1}{2}(c_2 + c_3)\Big)\Lambda^0 + \cdots,$$

where the dots stand for terms involving only negative powers of Λ. Let us split $[Q_2, \mathcal{L}]$ into sum of two commutators $[Q_2, \mathcal{L} - \frac{1}{2}(\partial_2^2 + \partial_3^2)]$ and

$$\frac{1}{2}[Q_2, \partial_2^2 + \partial_3^2] = -\frac{1}{2}(\partial_2^2 + \partial_3^2)(Q_2) - \partial_2(Q_2)\partial_2 - \partial_3(Q_2)\partial_3.$$

The first commutator is already in $\mathcal{E}^0_{(+)}$ and it is a Laurent series in Λ^{-1} whose coefficients are differential polynomials involving only ∂_x-derivatives. The projection π_+ of the second commutator is

$$-\frac{1}{2}(\partial_2^2 + \partial_3^2)(Q_2) - \partial_2(Q_2)Q_2 - \partial_3(Q_2)Q_3.$$

A straightforward computation, using formulas (11) and (14), shows that the above expression has leading order term of the type

$$\Big(\frac{1}{4}\partial_2(c_2 - c_2[-1]) - \frac{1}{2}\frac{1 - e^{-\epsilon\partial_x}}{1 + e^{\epsilon\partial_x}}\Big(\frac{e^{\epsilon\partial_x} - 1}{e^{\epsilon\partial_x} + 1}(q_2 q_3)\cdot q_3 + \frac{1}{2}(c_3 - c_3[1])q_2\Big) +$$
$$+\frac{1}{2}(c_2 - c_2[-1])q_2[-1] + \frac{1}{2}(c_3 - c_3[-1])q_3[-1]\Big)\Lambda^0 + O(\Lambda^{-1}).$$

Comparing the coefficients in front of Λ^k in (15) for $1 \le k \le n-2$ we get that $\partial_2(a_i)$ $(1 \le i \le n-3)$ and $\partial_2(c_2 - c_3)$ can be expressed as differential polynomials that involve only ∂_x-derivatives. Comparing the coefficients in front of Λ^0 in (15) we get that

$$\frac{1}{2}\partial_2(c_2 + c_3) - \frac{1}{4}\partial_2(c_2 - c_2[-1]) = \frac{1}{8}(3 + e^{-\epsilon\partial_x})\partial_2(c_2 + c_3) - \frac{1}{8}(1 - e^{-\epsilon\partial_x})\partial_2(c_2 - c_3)$$

can be expressed in terms of differential polynomials that involve only ∂_x-derivatives. The operator $3 + e^{-\epsilon\partial_x}$ is invertible, so the derivative $\partial_2(c_2 + c_3)$ is also a differential polynomial involving only ∂_x-derivatives. Finally, by comparing the coefficients in front of Λ^{n-2} we get

$$\partial_2(a_{n-3}) = a_{n-3}(1 - e^{(n-2)\epsilon\partial_x})(q_2[-1]) = a_{n-3}(q_2[-1] - q_2[n-3]).$$

Since $a_{n-3} = \frac{1}{n-2}e^{(n-2)\alpha}$ the above equation implies $\partial_2(\alpha) = \frac{1}{n-2}(q_2[-1] - q_2[n-3])$.

Let us define ∂_2 and ∂_3 as above, i.e.,

$$\partial_2(q_3) = \partial_3(q_2) = q_1 + q_2 q_3, \quad \partial_a(q_a) = \frac{1}{2}(c_a - c_a[1]),\ (a = 2, 3), \tag{16}$$

and

$$\Big(\partial_a(\mathcal{L}) - \pi_+([Q_a,\mathcal{L}])\Big)_{1,\geq 0} = 0.$$

We have to prove that $H_a\mathcal{L} \in \mathcal{E}H$ for $a = 2,3$. We will give the argument for $a = 2$ only, because the case $a = 3$ is analogous. Let us first prove that $\partial_2(\mathcal{L}) - \pi_+([Q_2,\mathcal{L}] = 0$. Using formulas (16) it is straightforward to verify that

$$\mathcal{L} = \mathcal{A} + \iota_{\Lambda^{\pm 1}}\mathcal{C} + \iota_{\Lambda^{\pm 1}}\mathcal{Q}, \tag{17}$$

where

$$\mathcal{A} = \sum_{i=1}^{3}(a_i\Lambda^i - \Lambda^{-i}a_i)(\Lambda - \Lambda^{-1}),$$

$$\mathcal{C} = \frac{1}{4}(\Lambda-1)^{-1}(\Lambda c_2 - c_2\Lambda^{-1})(\Lambda+1) - \frac{1}{4}(\Lambda+1)^{-1}(\Lambda c_3 - c_3\Lambda^{-1})(\Lambda-1) + \frac{1}{2}(Q_2^2 + Q_3^2),$$

and

$$\mathcal{Q} = \frac{1}{2}\Big((\partial_2 + Q_2)(\partial_2 - Q_2) + (\partial_3 + Q_3)(\partial_3 - Q_3)\Big).$$

Note that the above operators have the following symmetries

$$M^{\#} = (\Lambda - \Lambda^{-1})\cdot M\cdot(\Lambda - \Lambda^{-1})^{-1}, \quad M = \mathcal{A},\mathcal{C}, \tag{18}$$

and

$$\mathcal{Q}^{\#} - (\Lambda - \Lambda^{-1})\cdot\mathcal{Q}\cdot(\Lambda-\Lambda^{-1})^{-1} = (\Lambda - \Lambda^{-1})\cdot(\partial_2(Q_2) + \partial_3(Q_3))\cdot(\Lambda - \Lambda^{-1})^{-1}.$$

Using the 0-curvature condition we also have

$$\pi_+([Q_2,(\partial_a + Q_a)(\partial_a - Q_a)] = -Q_a\partial_2(Q_a) - \partial_2(Q_a)Q_a - \partial_2\partial_a(Q_a).$$

Therefore,

$$(\pi_+[Q_2,\mathcal{Q}])^{\#} - (\Lambda - \Lambda^{-1})[Q_2,\mathcal{Q}](\Lambda - \Lambda^{-1})^{-1} = (\Lambda - \Lambda^{-1})(\partial_2^2(Q_2) + \partial_2\partial_3(Q_3))(\Lambda - \Lambda^{-1})^{-1}.$$

Using the above symmetries it is strightforward to check that

$$\Big((\Lambda - \Lambda^{-1})\Big(\partial_2(\mathcal{L}) - \pi_+([Q_2,\mathcal{L}])\Big)\Big)^{\#} + (\Lambda - \Lambda^{-1})\Big(\partial_2(\mathcal{L}) - \pi_+([Q_2,\mathcal{L}])\Big) = 0.$$

By definition the operator $(\Lambda - \Lambda^{-1})\Big(\partial_2(\mathcal{L}) - \pi_+([Q_2,\mathcal{L}])\Big)$ has the form $\sum_{i=0}^{\infty} b_i\Lambda^{-i}$, so the above symmetry implies that $b_i = 0$ for all i.

Note that $\partial_2 - Q_2, \mathcal{Q} \in \mathcal{E}_{(+)}H$. Therefore,

$$[\partial_2 - Q_2, \mathcal{A} + \mathcal{C}] = \pi_+([\partial_2 - Q_2,\mathcal{L}]) = \partial_2(\mathcal{L}) - \pi_+([Q_2,\mathcal{L}]) = 0.$$

Since $H_2 = (\Lambda - 1)(\partial_2 - Q_2)$ the vanishing of the above commutator implies that

$$H_2\mathcal{L} = (\Lambda - 1)\mathcal{A}(\Lambda - 1)^{-1}H_2 + (\Lambda - 1)\mathcal{C}(\Lambda - 1)^{-1}H_2 + H_2\mathcal{Q}.$$

The first term on the RHS is already in $\mathcal{E}H$, so we need to verify that the remaining two terms add up to some element in $\mathcal{E}H$. A long but straightforward computation using the explicit formulas for $\mathcal{C}, \mathcal{Q}$, and H_a and formulas (16) gives that $(\Lambda-1)\mathcal{C}(\Lambda-1)^{-1}H_2+H_2\mathcal{Q}$ is given by the following formula

$$\Big(\frac{1}{2}(\partial_2^2+\partial_3^2)+\frac{1}{4}\Big(\Lambda(c_2-c_3)+(c_2-c_3)\Lambda^{-1}+c_2+c_3+c_2[1]+c_3[1]\Big)\Big)H_2+\partial_3(q_2)H_3.$$

Let us give the explicit formulas for the other case. We have

$$H_3\mathcal{L}=(\Lambda+1)\mathcal{A}(\Lambda+1)^{-1}H_3+(\Lambda+1)\mathcal{C}(\Lambda+1)^{-1}H_3+H_3\mathcal{Q}$$

and $(\Lambda+1)\mathcal{C}(\Lambda+1)^{-1}H_3+H_3\mathcal{Q}$ is given by

$$\Big(\frac{1}{2}(\partial_2^2+\partial_3^2)+\frac{1}{4}\Big(\Lambda(c_2-c_3)+(c_2-c_3)\Lambda^{-1}+c_2+c_3+c_2[1]+c_3[1]\Big)\Big)H_3+\partial_2(q_3)H_2. \quad \square$$

Let us point out that if the ring $\mathcal{R}$ is equipped with the derivations ∂_2 and ∂_3 defined by Proposition 1, b), then we have

$$H_1=\tfrac{1}{2}(\partial_2-q_2)H_3-\tfrac{1}{2}(\partial_3-q_3)H_2.$$

In particular, $H_1\mathcal{L}\in\mathcal{E}H$, that is, the condition in Proposition 1, b) holds also for $a=1$.

2.2. **Rational difference operators.** Let $\mathbb{C}(\Lambda)$ be the field of rational functions in Λ. If A is any $\mathbb{C}$-algebra, then we denote by $A(\Lambda):=A\otimes_{\mathbb{C}}\mathbb{C}(\Lambda)$. Using elementary fraction decomposition, we get that the A-module $A(\Lambda)$ can be decomposed into a direct sum of A-modules as follows:

$$A(\Lambda)=\bigoplus_{n\geq 0}A\otimes\Lambda^n\bigoplus\bigoplus_{a\in\mathbb{C},n\geq 1}A\otimes(\Lambda-a)^{-n}.$$

We apply this construction for $A:=\mathbb{C}[\xi^i \mid i\geq 0, \xi\in\Xi]$. The key observation is the following: if Λ commutes with the elements of A, then clearly $A(\Lambda)$ is a ring. This is not the case in our setting, but the commutator $[\Lambda,a]$ is proportional to ϵ, so the failiure of commutativity may be offset by allowing infinite power series in ϵ. This idea can be realized as follows: Put

$$\mathcal{E}_{\mathrm{rat}}:=\mathbb{C}[\xi^i\,(i\geq 0,\ \xi\in\Xi), e^{\pm\alpha}](\Lambda)[[\epsilon]].$$

We claim that $\mathcal{E}_{\mathrm{rat}}$ has a natural ring structure. It is sufficient to define $(\Lambda-a)^{-1}\cdot P$, where $P\in A$ is a differential polynomial. In order to justify the definition, note that we have the following formula:

$$(\Lambda-a)^{-1}P=\sum_{n=0}^{\infty}(-\mathrm{ad}_\Lambda)^n(P)\cdot(\Lambda-a)^{-n-1}, \tag{19}$$

where both sides make sense if we expand them in the powers of Λ^{-1}, that is, the above equality makes sense in the ring $A((\Lambda^{-1}))[[\epsilon]]$. For the proof, note that we have

$$(\Lambda-a)^{-1}P=P\cdot(\Lambda-a)^{-1}+(\Lambda-a)^{-1}(-\mathrm{ad}_\Lambda(P))\cdot(\Lambda-a)^{-1}. \tag{20}$$

Replacing in this formula P by $(-\mathrm{ad}_\Lambda(P))$, we get

$$(\Lambda-a)^{-1}(-\mathrm{ad}_\Lambda(P))=(-\mathrm{ad}_\Lambda(P))(\Lambda-a)^{-1}+(\Lambda-a)^{-1}(-\mathrm{ad}_\Lambda)^2(P)(\Lambda-a)^{-1}.$$

Therefore,

$$(\Lambda - a)^{-1}P = P(\Lambda - a)^{-1} + (-\mathrm{ad}_\Lambda)(P)(\Lambda - a)^{-2} + (\Lambda - a)^{-1}(-\mathrm{ad}_\Lambda)^2(P)(\Lambda - a)^{-2}.$$

Clearly, continuing this process by applying formula (20) with P replaced by $(-\mathrm{ad}_\Lambda)^m(P)$, we will get formula (19). On the other hand, if $P \in \mathcal{E}_{\mathrm{rat}}$, then $(-\mathrm{ad}_\Lambda)^m(P) \in \epsilon^m \mathcal{E}_{\mathrm{rat}}$. Therefore, the RHS of (19) defines an element of $\mathcal{E}_{\mathrm{rat}}$. In other words, the Laurent series expansion operation

$$\iota_{\Lambda^{-1}} : \mathcal{E}_{\mathrm{rat}} \to A((\Lambda^{-1}))[[\epsilon]]$$

provides an A-module embedding of $\mathcal{E}_{\mathrm{rat}}$ and the image is a subring of $A((\Lambda^{-1}))[[\epsilon]]$. In particular, the A-module $\mathcal{E}_{\mathrm{rat}}$ has a unique associative product, such that formula (19) holds. The elements of the ring $\mathcal{E}_{\mathrm{rat}}$ will be called *rational difference operators*.

For every $a \in \mathbb{C}$ let us denote by

$$\iota_{\Lambda - a} : \mathcal{E}_{\mathrm{rat}} \to A((\Lambda - a))[[\epsilon]]$$

the Laurent series expansion operations. Let us recall also the rings $\mathcal{E}^0_{(\pm)} = \mathcal{R}((\Lambda^{\mp 1}))$. Note that we have an embedding

$$\mathcal{R}((\Lambda^{\mp 1})) = A[[\epsilon]]((\Lambda^{\mp 1})) \subset A((\Lambda^{\mp 1}))[[\epsilon]],$$

which allows us to think of $\mathcal{R}((\Lambda^{\mp 1}))$ as the elements in $A((\Lambda^{\mp 1}))[[\epsilon]]$ that have a finite order pole at $\Lambda = \infty$ or 0. Finally, note that we can identify

$$\mathcal{R}[[\Lambda - a]] = A[[\epsilon]][[\Lambda - a]] = A[[\Lambda - a]][[\epsilon]]$$

with the elements of $A((\Lambda - a))[[\epsilon]]$ that are regular at $\Lambda = a$, that is, the series that do not involve negative powers of $\Lambda - a$. There is a unique associative ring structure on $\mathcal{R}[[\Lambda - a]]$, such that the multiplication satisfies

$$(\Lambda - a) \cdot P = P[1] \cdot (\Lambda - a) + a\Delta(P), \quad P \in \mathcal{R},$$

where $\Delta(P) := P[1] - P$ is the forward difference operator.

The following properties are straightforward to check:

(i) For every $a \in \mathbb{C}$, there exists a unique associative ring structure on $A((\Lambda - a))[[\epsilon]]$, such that formula (19) holds for $P \in A((\Lambda - a))[[\epsilon]]$.

(ii) The maps $\iota_{\Lambda^{\pm 1}}$ and $\iota_{\Lambda - a}$ are injective ring homomorphisms.

(iii) The ring $\mathcal{R}((\Lambda^{\mp 1}))$ is a subring of $A((\Lambda^{\mp 1}))[[\epsilon]]$.

(iv) The ring $\mathcal{R}[[\Lambda - a]] = A[[\Lambda - a]][[\epsilon]]$ is a subring of $A((\Lambda - a))[[\epsilon]]$.

2.3. **The kernel of the projections π_α.**

Lemma 7. *a) Suppose that $P=\sum_{i=0}^m a_i\partial_2^i+\sum_{i=0}^n b_i\partial_3^i$ is a differential operator with coefficients $a_i,b_i\in\mathcal{R}$. If $\pi_\pm(P)=0$ then $P=0$.*

b) Suppose that $P=\sum_{i=0}^m a_i\partial_2^i+\sum_{i=-n_1}^{n_2} b_i\Lambda^i$ is a differential operator with coefficients $a_i,b_i\in\mathcal{R}$. If $\pi_3(P)=0$ then $P=0$.

c) Suppose that $P=\sum_{i=0}^m a_i\partial_3^i+\sum_{i=-n_1}^{n_2} b_i\Lambda^i$ is a differential operator with coefficients $a_i,b_i\in\mathcal{R}$. If $\pi_2(P)=0$ then $P=0$.

Proof. Let us give the argument only for part a) for the case when $\pi_+(P)=0$. The remaining statements are proved in the same way.

Let us consider first the case when all $b_i=0$. We claim that $a_0=0$. Indeed, using the rlation $\partial_2=(\Lambda-1)^{-1}q_2(\Lambda+1)+(\Lambda-1)^{-1}H_2$, we get that $P=G_0+\sum_{i=1}^m G_iH_2^i$, where the coefficients $G_i\in\mathcal{E}_{\rm rat}$ are rational difference operators that have a finite order pole at $\Lambda=\infty$ and are regular at $\Lambda=a$ for all $a\neq 1$. By definition, $\iota_{\Lambda^{-1}}(G_0)=\pi_+(P)$, which is given to be 0. Therefore, $G_0=0$ and we get $P=GH_2$ for some $G\in\mathcal{E}_{\rm rat}[\partial_2]$. Since P is independent of Λ, we get $P=\iota_{\Lambda+1}(G)H_2$ in the ring $\mathcal{R}[\partial_2][[\Lambda+1]]$. Comparing the coefficients in front of $\partial_2^0(\Lambda+1)^0$ we get that $a_0=0$.

Let $m=\max\{i: a_i\neq 0\}$ be the order of the differential operator P. We argue by induction on m that $a_i=0$ for all i. Suppose that $\ker(\pi_+)$ does not contain differential operators of order $\leq m-1$. Note that the operator $\widetilde{P}:=(a_m\partial_3-\partial_3(a_m))\cdot P$ is also in the kernel of π_+. However,

$$\widetilde{P}=\sum_{i=1}^{m-1}(a_m\partial_3(a_i)-\partial_3(a_m)a_i)\partial_2^i+\sum_{i=1}^{m-1}a_ma_i\partial_2^{i-1}(-q_1+H_1)$$

and if we set $H_1=0$ in the above formula then we still have an operator whose projection π_+ is 0 and whose order is at most $m-1$. The inductive assumption implies that

$$\widetilde{P}=\sum_{i=1}^{m-1}(a_m\partial_3(a_i)-\partial_3(a_m)a_i)\partial_2^i+\sum_{i=1}^{m-1}a_ma_i\partial_2^{i-1}(-q_1)=0.$$

Comparing the coefficients in front of ∂_2^j for $j=0,1,\dots,m-2$ we get that $a_i=0$ for all $i=1,2,\dots,m-1$. This proves that $\widetilde{P}=0$ and since the ring $\mathcal{R}[\partial_2]$ does not have zero divisors we get $P=0$.

Similar argument works if $a_i=0$ for all i. Let $m:=\max\{i: a_i\neq 0\}$ and $n:=\max\{i: b_i\neq 0\}$. Then it remains only to consider the case $m>0$ and $n>0$. We argue by induction on n that such a case is impossible. Note that the operator $(b_n\partial_2-\partial_2(b_n))\cdot P$ has the form

$$\sum_{i=0}^{m}(b_n\partial_2-\partial_2(b_n))a_i\partial_2^i+\sum_{i=0}^{n-1}(b_n\partial_2(b_i)-\partial_2(b_n)b_i)\partial_3^i+b_nb_0\partial_2+\sum_{i=1}^{n}b_nb_i\partial_3^{i-1}(-q_1+H_1).$$

If we set $H_1=0$ in the above expression we will get an operator whose projection π_+ is also 0. The coefficient in front of ∂_2^{m+1} is $a_mb_n\neq 0$ while the highest possible power of ∂_3 is $n-1$ – contradiction with our inductive assumption. $\square$

Proposition 8. *The following equality of left ideals holds* $\mathcal{E}_{(\alpha)}H \cap \mathcal{E} = \mathcal{E}H$ *for all* $\alpha \in \{\pm, 2, 3\}$.

Proof. The arguments in all 4 cases are similar, so let us consider only the case $\alpha = +$. Suppose $P \in \mathcal{E}_{(+)}H \cap \mathcal{E}$. We may assume that P is polynomial in Λ. Indeed, if not then we can always find a positive integer n such that $\Lambda^n P \in \mathcal{R}[\partial_2, \partial_3, \Lambda]$. The operator $\Lambda^n P \in \mathcal{E}_{(+)}H \cap \mathcal{E}$, so if we knew that $\Lambda^n P \in \mathcal{E}H$, then $P \in \mathcal{E}H$.

Suppose that P is polynomial in Λ. We will reduce the proof to the case when $P = P_1 + P_2 + P_3$, where $P_1 \in \mathcal{R}[\Lambda]$, $P_a \in \mathcal{R}[\partial_a]$ $(a = 2, 3)$. Let us write

$$P = \sum_{i,j,k} p_{i,j}(\Lambda)\partial_2^i \partial_3^j, \quad p_{i,j}(\Lambda) \in \mathcal{R}[\Lambda],$$

where the sum in i and j is finite. Using that $\partial_2\partial_3 = H_1 - q_1$ we can write P as a sum of an element in $\mathcal{E}H$ and $P_2 + P_3$, where $P_a \in \mathcal{R}[\Lambda, \partial_a]$ $(a = 2, 3)$. Let us write P_2 (resp. P_3) as a sum of monomials of the type $(\Lambda - 1)^i \partial_2^j$ (resp. $(\Lambda + 1)^i \partial_3^j$). Using the relation $(\Lambda - 1)\partial_2 = H_2 + q_2(\Lambda + 1)$ (resp. $(\Lambda + 1)\partial_3 = H_3 + q_3(\Lambda - 1)$) we can reduce P_a to a sum of a polynomial in $\mathcal{R}[\Lambda]$ and a polynomial in $\mathcal{R}[\partial_a]$.

Suppose now that $P = P_1 + P_2 + P_3$, where $P_1 \in \mathcal{R}[\Lambda]\Lambda$, $P_a \in \mathcal{R}[\partial_a]$ $(a = 2, 3)$. We are given that $0 = \pi_+(P) = P_1 + \pi_+(P_2) + \pi_+(P_3)$. The projections $\pi_+(P_a)$ $(a = 2, 3)$ have only non-positive powers of Λ. Therefore, by comparing the coefficients in front of the postive powers of Λ, we get $P_1 = 0$. Therefore, $\pi_+(P_2 + P_3) = 0$. Recalling Lemma 7, a) we get $P_2 + P_3 = 0$. $\square$

2.4. **Residue formulas for the projections.**

Proposition 9. *a) If* $k > 0$ *then the following formulas hold*

$$\begin{aligned} \pi_\pm(\partial_2^k) &= \tfrac{1}{2}\iota_{\Lambda^{\mp 1}}\Big(\partial_2^k H_2^{-1}(\partial_2 + q_2)\Big)_{2,[0]}(\Lambda - \Lambda^{-1}), \\ \pi_3(\partial_2^k) &= \Big(\partial_2^k H_1^{-1}\, \partial_2\Big)_{2,[0]}\partial_3. \end{aligned}$$

b) If $k > 0$ *then the following formulas hold*

$$\begin{aligned} \pi_\pm(\partial_3^k) &= \tfrac{1}{2}\iota_{\Lambda^{\mp 1}}\Big(\partial_2^k H_3^{-1}(\partial_3 + q_3)\Big)_{3,[0]}(\Lambda - \Lambda^{-1}), \\ \pi_2(\partial_3^k) &= \Big(\partial_3^k H_1^{-1}\, \partial_3\Big)_{3,[0]}\partial_2. \end{aligned}$$

c) If $k > 0$ *then the following formulas hold*

$$\begin{aligned} \pi_2(\Lambda^{\pm k}) &= (-1)^k \pm 2\Big(\Lambda^{\pm k}\iota_{\Lambda^{\mp 1}}\Big(H_2^{-1}(\Lambda^{-1} + 1)^{-1}\Big)\Big)_{1,[0]} \cdot \partial_2, \\ \pi_3(\Lambda^{\pm k}) &= 1 \pm 2\Big(\Lambda^{\pm k}\iota_{\Lambda^{\mp 1}}\Big(H_3^{-1}(\Lambda^{-1} - 1)^{-1}\Big)\Big)_{1,[0]} \cdot \partial_3. \end{aligned}$$

Proof. Let us prove the first formula in a)

$$\pi_+(\partial_a^k) = \frac{1}{2}\,\iota_{\Lambda^{-1}}\Big(\partial_a^k H_2^{-1}(\partial_a + q_a)\Big)_{2,[0]} \cdot (\Lambda - \Lambda^{-1}). \tag{21}$$

The remaining formulas are proved with the same method. Put

$$H_2^{-1}\cdot(\partial_2+q_2)=(\Lambda-1)^{-1}+\sum_{j=1}^{+\infty}\partial_2^{-j}\cdot v_j. \tag{22}$$

Using the formula

$$f\cdot\partial_2^{-l}=\sum_{j=0}^{\infty}\binom{l+j-1}{j}\partial_2^{-l-j}\cdot\partial_2^{j}(f)$$

and comparing the coefficients on the right of ∂_2^{-l} in

$$(\partial_2+q_2)=((\Lambda-1)\partial_2-q_2(\Lambda+1))\Big((\Lambda-1)^{-1}+\sum_{j=1}^{+\infty}\partial_2^{-j}\cdot v_j\Big)$$

we get

$$v_1=2Q_2(\Lambda-\Lambda^{-1}),\ v_{l+1}=\sum_{j=0}^{l-1}\binom{l-1}{j}\partial_2^{j}(Q_2)\cdot v_{l-j},\quad l=1,2,3,\cdots, \tag{23}$$

where recall that $Q_2=(\Lambda-1)^{-1}q_2(\Lambda+1)$. Furthermore, note that

$$\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_{l+1}=Q_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l. \tag{24}$$

Indeed, the formula follows by comparing the pseudo-differential part, i.e., the terms involving only negative powers of ∂_2 in

$$\Big(\partial_2-Q_2\Big)\cdot\Big((\Lambda-1)^{-1}+\sum_{j=1}^{\infty}\partial_2^{-j}\cdot v_j\Big)=(\Lambda-1)^{-1}\cdot(\partial_2+q_2).$$

Recalling the definition of π_+ we get

$$\pi_+(\partial_2)=Q_2,\quad \pi_+(\partial_2^{l+1})=\partial_2(\pi_+(\pi_2^{l}))+\pi_+(\partial_2^{l})\cdot Q_2,\quad l=1,2,3,\cdots.$$

It is straightforward to check the relation (21) is correct for $l=1$. Let us denote by A_l the RHS of (21). We prove (21) by induction on k. Clearly, we need only to show that

$$A_{l+1}=\partial_2(A_l)+A_l\cdot Q_2. \tag{25}$$

Substituting the expansion (22) in A_l we get

$$A_l=\frac{1}{2}\sum_{j=1}^{l}\partial_2^{-j+l}(v_j)\cdot(\Lambda-\Lambda^{-1}). \tag{26}$$

After inserting the above formula for A_l in the recursion relation (25), we get that (25) is equivalent to

$$v_{l+1}=-\sum_{j=1}^{l}\partial_2^{-j+l}(v_j)\cdot Q_2^{\#}, \tag{27}$$

which according to formula (23) is equivalent to

$$-\sum_{j=0}^{l}\partial_2^{-j+l}(v_j)\cdot Q_2^{\#}=\sum_{j=0}^{l-1}\binom{l-1}{j}\partial_2^{j}(Q_2)\cdot v_{l-j}. \tag{28}$$

Furthermore, let us rewrite (28) as

$$-\operatorname{Res}_{\partial_2}\Big(\partial_2^{l}\cdot H_2^{-1}\cdot(\partial_2+q_2)\cdot\partial_2^{-1}\cdot Q_2^{\#}\Big)=\operatorname{Res}_{\partial_2}\Big(\partial_2^{l-1}\cdot Q_2\cdot H_2^{-1}\cdot(\partial_2+q_2)\Big),\quad l\geq 1, \tag{29}$$

which is equivalent to

$$-\Big(\partial_2\cdot H_2^{-1}\cdot(\partial_2+q_2)\cdot\partial_2^{-1}\cdot Q_2^{\#}\Big)_{2,<0}=\Big(Q_2\cdot H_2^{-1}\cdot(\partial_2+q_2)\Big)_{2,<0}. \tag{30}$$

Using the expansion (22) we transform the above relation into

$$\begin{aligned}
&-\partial_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot\partial_2^{-1}\cdot Q_2^{\#}=Q_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\\
\Leftrightarrow\quad&\sum_{l=1}^{\infty}\partial_2^{-l+1}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\cdot\partial_2^{-1}\cdot Q_2=Q_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\\
&\Big(\text{by } Q_2^{\#}=-(\Lambda-\Lambda^{-1})\cdot Q_2\cdot(\Lambda-\Lambda^{-1})^{-1}\Big)\\
\Leftrightarrow\quad&v_1\cdot(\Lambda-\Lambda^{-1})\cdot\partial_2^{-1}\cdot Q_2+Q_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\cdot\partial_2^{-1}\cdot Q_2\\
&=Q_2\cdot\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1}),\quad\Big(\text{by formula (24)}\Big)\\
\Leftrightarrow\quad&2\partial_2^{-1}\cdot Q_2+\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\cdot\partial_2^{-1}\cdot Q_2=\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\\
&\Big(\text{by } v_1=2Q_2\cdot(\Lambda-\Lambda^{-1})^{-1}\Big)\\
\Leftrightarrow\quad&2\partial_2^{-1}\cdot Q_2(1-\partial_2^{-1}\cdot Q_2)^{-1}=\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\\
&\Big(\text{by solving for }\sum_{l=1}^{\infty}\partial_2^{-l}\cdot v_l\cdot(\Lambda-\Lambda^{-1})\Big)\\
\Leftrightarrow\quad&2(1-\partial_2^{-1}\cdot Q_2)^{-1}-1+\Lambda^{-1}=H_2^{-1}\cdot(\partial_2+q_2)\cdot(\Lambda-\Lambda^{-1})\quad\Big(\text{by formula (22)}\Big)\\
\Leftrightarrow\quad&H_2=(\Lambda-1)\cdot\partial_2-q_2\cdot(\Lambda+1),\quad\Big(\text{by } 1-\partial_2^{-1}\cdot Q_2=\partial_2^{-1}\cdot(\Lambda-1)^{-1}\cdot H_2\Big),
\end{aligned}$$

which is just the definition of H_2. □

3. Existence of the derivations

In this section we are going to prove part a) of Theorem 4, except for the existence of the derivations of type $\partial_{0,k}$ ($k \geq 1$). The latter are called *extended flows*. Their construction, based on dressing operators, requires some additional work which we postpone until next section.

3.1. **Auxiliarly lemma.** The following simple lemma allows us to construct derivations of $\mathcal{R}$ commuting with ∂_x.

Lemma 10. *Suppose that $B \in \mathcal{E}$ is a differential-difference operator, s.t., the projections $B^{\pm} := \pi_{\pm}(B)$ satisfy the following conditions:*

(i) $(B^+)^{\#} = -(\Lambda - \Lambda^{-1})B^- \iota_\Lambda (\Lambda - \Lambda^{-1})^{-1}$.

(ii) $\pi_+[B^+, \mathcal{L}] = \pi_-[B^-, \mathcal{L}] \in \mathcal{E}$.

(iii) $\pi_+(H_a B^+) = \pi_-(H_a B^-) \in \mathcal{E}$.

Then there exists a unique derivation ∂ of $\mathcal{R}$ commuting with ∂_x, such that one of the following two equivalent conditions is satisfied

$$\partial(\mathcal{L}) = \pi_+([B^+, \mathcal{L}]), \tag{31}$$

$$\partial(H_a) = -\pi_+(H_a B^+), \quad a = 2, 3, \tag{32}$$

or

$$\partial(\mathcal{L}) - [B, \mathcal{L}] \in \mathcal{E}H,$$

$$\partial(H_a) - [B, H_a] \in \mathcal{E}H, \quad a = 2, 3.$$

Moreover $\partial\partial_a(q_a) = \partial_a \partial(q_a)$ for $a = 2, 3$.

Proof. We claim that it is sufficient to prove that there exists a unique derivation satisfying (31) and (32). Indeed, if we knew this, then let us write $B^+ = B + \sum_{b=2,3} P_b H_b$, where $P_b \in \mathcal{E}_{(+)}$. We have

$$[B^+, \mathcal{L}] = [B, \mathcal{L}] + \sum_{b=2,3} ([P_b, \mathcal{L}]H_b + P_b[\mathcal{L}, H_b]), \tag{33}$$

$$[B^+, H_a] = [B, H_a] + \sum_{b=2,3} ([P_b, H_a]H_b + P_b[\mathcal{L}, H_b]). \tag{34}$$

Since $H_b \mathcal{L} \in \mathcal{E}H$, we get that $\pi_+([B^+, \mathcal{L}]) = \pi_+([B, \mathcal{L}])$. Therefore, $\partial(\mathcal{L}) - [B, \mathcal{L}] \in \mathcal{E}_{(+)}H \cap \mathcal{E} = \mathcal{E}H$. Similarly, $\partial(H_a) - [B, H_a] \in \mathcal{E}_{(+)}H \cap \mathcal{E} = \mathcal{E}H$. This proves that there exists a derivation satisfying the scond set of equations. Conversely, suppose that there exists a derivation ∂ satisfying the scond set of equations. Then the relations (33) and (34) prove that the 1st set of equations (31) and (32) are also satisfied and hence ∂ must be the unique derivation satisfying the firs set of equations. Our claim is proved.

It remains to prove that there exists a unique derivation satisfying (31) and (32). Note that ∂ is uniquely defined by the equations

$$\begin{aligned}\Big(\partial_{1,k}(\mathcal{L}) - \pi_+([B^+_{1,k},\mathcal{L}])\Big)_{1,\geq 0} &= 0,\\ \Big(\partial_{i,k}(H_a) + \pi_+(H_a B^+_{1,k})\Big)_{1,[0]} &= 0, \quad a=2,3\end{aligned}$$

and the requirement that it commutes with ∂_x. Indeed, the first equation determines $\partial(a_i), \partial(c_2)$, and $\partial(c_3)$, while the remaining two equations determine $\partial(q_a)$ ($a = 2,3$). We would like to prove that

$$\begin{aligned}\partial(\mathcal{L}) &= \pi_+([B^+,\mathcal{L}]),\\ \partial(H_a) &= -\pi_+(H_a B^+), \quad a=2,3.\end{aligned}$$

First we will prove the second equation. Let us consider only the case $a = 2$. The argument for $a = 3$ is similar. Put

$$\mathcal{A} := (\Lambda^{-1}+1)(\partial(H_2)+\pi_+(H_2B^+)) = (\Lambda^{-1}+1)\partial(H_2) + (\Lambda-\Lambda^{-1})\Big(\partial_2(B^+)+[B^+,Q_2]\Big).$$

Using $H_2^{\#} = (\Lambda^{-1}+1)H_2(\Lambda+1)^{-1}$ and conditions (i) and (iii) we get that $\mathcal{A}^{\#} = \mathcal{A}$. Recalling again condition (iii) we write

$$\partial(H_2)+\pi_+(H_2B^+) = a_1\Lambda + a_0 + \sum_{i=1}^{m} a_{-i}\Lambda^{-i}.$$

Then

$$\mathcal{A} = a_1\Lambda + (a_0 + a_1[-1]) + \sum_{i=1}^{m+1}(a_{-i}+a_{-i+1}[-1])\Lambda^{-i},$$

where $a_{-m-1} := 0$. Since $\mathcal{A}^{\#} = \mathcal{A}$ we get the following $m+1$ equations: $a_{-i}+a_{-i+1}[-1] = 0$ for $2 \leq i \leq m+1$ and $a_1 = a_{-1}[1]+a_0$. The first m equations imply that $a_{-i} = 0$ for all $1 \leq i \leq m$. Furthermore, $a_0 = 0$ by the definition of $\partial(H_2)$. Finally, we get $a_1 = 0$ from the last equation.

Let us prove the first equation. We will use the following identity

$$(\partial(\mathcal{L})-\pi_+[B^+,\mathcal{L}])^{\#} - (\Lambda-\Lambda^{-1})\,(\partial(\mathcal{L})-\pi_+[B^+,\mathcal{L}])\,(\Lambda-\Lambda^{-1})^{-1} = \sum_{a=2,3}(\Lambda-\Lambda^{-1})\,(\partial\partial_a(Q_a)-\partial_a\partial(Q_a))\,(\Lambda-\Lambda^{-1})^{-1}, \tag{35}$$

where on both sides we could take either the expansion ι_Λ or the expansion $\iota_{\Lambda^{-1}}$ and the identity holds in both cases. Let us prove (35). The main difficulty is to compute

$$\Big(\pi_+([B^+,\mathcal{L}])\Big)^{\#} = [B^+,M]^{\#} - \sum_{a=2,3}\Big(\partial_a(B^+)Q_a + \tfrac{1}{2}\partial_a^2(B^+)\Big)^{\#}, \tag{36}$$

where $M := \mathcal{L} - \frac{1}{2}(\partial_2^2+\partial_3^2)$. It is straightforward to check that

$$\mathcal{L}^{\#} = (\Lambda-\Lambda^{-1})\,(\mathcal{L}+\partial_2(Q_2)+\partial_3(Q_3))\,(\Lambda-\Lambda^{-1})^{-1}, \tag{37}$$

where on the RHS we take either the expansion in the powers of Λ^{-1} or the expansion in Λ. Recalling conditions (i) and (ii), after a short computation, we get that (36) transforms into

$$(\Lambda-\Lambda^{-1})\left(\pi_+[B^+,\mathcal{L}]+\sum_{a=2,3}\left(\partial_a^2(B^-)+[\partial_a(B^-),Q_a]+[B^-,\partial_a(Q_a)]\right)\right)(\Lambda-\Lambda^{-1})^{-1}.$$

On the other hand,

$$\partial_a\partial(Q_a)=-\partial_a\partial(\partial_a-Q_a)=\partial_a\pi_-((\partial_a-Q_a)B^-)=\partial_a^2(B^-)+\partial_a[B^-,Q_a].$$

Therefore,

$$\left(\pi_+([B^+,\mathcal{L}])\right)^\#=(\Lambda-\Lambda^{-1})\,(\pi_+[B^+,\mathcal{L}]+\partial_a\partial(Q_a))\,(\Lambda-\Lambda^{-1})^{-1}.$$

The above identity and (37) yields (35).

Put $A:=(\Lambda-\Lambda^{-1})(\partial(\mathcal{L})-\pi_+[B^+,\mathcal{L}])$. Then (35) yields

$$-A^\#-A=(\Delta_2+\Delta_3)\Lambda+(\Delta_2+\Delta_3-\Delta_2[-1]-\Delta_3[-1])+\Lambda^{-1}(\Delta_2+\Delta_3), \tag{38}$$

where $\Delta_a:=\partial\partial_a(q_a)-\partial_a\partial(q_a)$. The definition of ∂ and condition (ii) imply that $\partial(\mathcal{L})-\pi_+[B^+,\mathcal{L}]=\sum_{i\geq 1}a_i\Lambda^{-i}$ has only negative powers of Λ and that only finitely many a_i's are not zero. Suppose that there exists at least one $i\geq 1$ such that $a_i\neq 0$. Let us denote by m the largest such i. Comparing the coefficients in front Λ^{-m-1} in (38) we get $a_m=0$ – contradiction. Therefore $A=0$, i.e., $\partial(\mathcal{L})-\pi_+[B^+,\mathcal{L}]=0$ which is exactly what we wanted to prove. Note that as a biproduct of our argument we get also that $\partial\partial_a(q_a)=\partial_a\partial(q_a)$. □

3.2. **The derivation $\partial_{1,k}$.** Let us first prove the existence of the Lax operator L_1.

Lemma 11. *There exists a unique operator $L_1=b_{1,0}\Lambda+\sum_{i=1}^\infty b_{1,i}\Lambda^{1-i}$ with coefficients $b_i\in\mathcal{R}$, such that,*

$$b_{1,0}=\exp\Big(\frac{(\Lambda-1)(n-2)}{\Lambda^{n-2}-1}(\alpha)\Big),\qquad \pi_+(\mathcal{L})=\frac{1}{n-2}\,L_1^{n-2}.$$

Proof. In order to avoid cumbersome notation let us write b_i for $b_{1,i}$. Let us first check that the coefficients in front of Λ^{n-2} in $\pi_+(\mathcal{L})=\frac{1}{n-2}L_1^{n-2}$ match, i.e., the following identity must hold:

$$a_{n-3}=\frac{1}{n-2}b_0b_0[1]\cdots b_0[n-3].$$

On the other hand, $a_{n-3}=\frac{1}{n-2}e^{(n-2)\alpha}$ and

$$\frac{(\Lambda-1)(n-2)}{\Lambda^{n-2}-1}(1+\Lambda+\cdots+\Lambda^{n-3})(\alpha)=(n-2)\alpha.$$

Our claim is proved.

Comparing the coefficients in front of the powers of Λ^{n-2-i} for $i\geq 1$ we get a relation of the following form

$$\Big(\sum_{s=0}^{n-3}(b_0\cdots b_0[s-1])(b_0[s+1-i]\cdots b_0[n-3-i])\Lambda^s\Big)(b_i)=\cdots,$$

where on the RHS we have an expression involving $b_0,\ldots,b_{i-1}$ and the coefficients of $\pi_+(\mathcal{L})$. The operator on the LHS that is applied to b_i belongs to $\mathcal{R}[\partial_x][[\epsilon]]$, i.e., it is a formal power series in ϵ whose coefficients are differential operators in ∂_x with coefficients in $\mathcal{R}$. The leading order term is the constant $n-2$ because $b_0 = 1 + O(\epsilon)$ and $\Lambda = 1 + O(\epsilon)$, so the operator is invertible as an element in $\mathcal{R}[\partial_x][[\epsilon]]$. Therefore we can solve for b_i, get a recursion that uniquely determines L_1, and using a simple induction on i prove that $b_i \in \mathcal{R}$. $\square$

Let us recall the difference operators $B_{1,k}$ ($k \geq 1$) defined in Section 1.4. We will check that $B^+ := B_{1,k}$ and $B^- := B_{1,k}$ satisfy conditions (i)–(iii) in Lemma 10. Note that condition (i) is immediate from the definition of $B^+_{1,k}$. We have to verify only (ii) and (iii).

To begin with, note that the decomposition (17) and the symmetry (18) implies that

$$\mathcal{L} = \tfrac{1}{n-2}\, L_1^{n-2} + \iota_{\Lambda^{-1}}\mathcal{Q} = \tfrac{1}{n-2}\, (L_1^-)^{n-2} + \iota_\Lambda \mathcal{Q}, \tag{39}$$

where

$$\mathcal{Q} = \tfrac{1}{2}\Big((\partial_2 + Q_2)(\partial_2 - Q_2) + (\partial_3 + Q_3)(\partial_3 - Q_3)\Big)$$

and

$$L_1^- := \iota_\Lambda (\Lambda - \Lambda^{-1})^{-1}\, L_1^{\#}\, (\Lambda - \Lambda^{-1}). \tag{40}$$

In particular, $\pi_-(\mathcal{L}) = \frac{1}{n-2}(L_1^-)^{n-2}$.

We have by definition that

$$\begin{aligned}
\pi_+(H_2 B_{1,k}) &= (\Lambda - 1)\big(\partial_2(B_{1,k}) + [B_{1,k}, \iota_{\Lambda^{-1}} Q_2]\big)\\
\pi_-(H_2 B_{1,k}) &= (\Lambda - 1)\big(\partial_2(B_{1,k}) + [B_{1,k}, \iota_{\Lambda} Q_2]\big).
\end{aligned}$$

Subtracting the second equation from the first one we get

$$\pi_+(H_2 B_{1,k}) - \pi_-(H_2 B_{1,k}) = (\Lambda - 1)\, B_{1,k} \left(\sum_{m\in\mathbb{Z}} \Lambda^m\right) H_2 = 0$$

where we used that $B_{1,k}$ is divisible from the right by $(\Lambda - \Lambda^{-1})$. We claim that $[\partial_2 - Q_2, L_1] = 0$. Indeed, using (39) we get that $[\partial_2 - Q_2, L_1^{n-2}] = \pi_+[\partial_2 - Q_2, L_1^{n-2}] = 0$. Suppose that that $[\partial_2 - Q_2, L_1] = a_m \Lambda^m + O(\Lambda^{m-1})$ with $a_m \neq 0$, then the vanishing of the coefficients of the highest power of Λ in $[\partial_2 - Q_2, L_1^{n-2}]$ yields

$$\sum_{i=1}^{n-2} b_{1,0} b_{1,0}[1] \cdots b_{1,0}[i-1] a_m[i] b_{1,0}[m+i] \cdots b_{1,0}[m+n-3] = 0.$$

Since $b_{1,0} = 1 + O(\epsilon)$ the above equation implies that $a_m = 0$ – contradiction. In particular, our claim implies that $\pi_+(H_2 L_1^k) = 0$. Therefore, $\pi_+(H_2 B_{1,k}) = \pi_+(H_2(B_{1,k} - L_1^k))$ is a Laurent series in Λ^{-1} that involves only powers of Λ that are ≤ 1. This completes the proof of condition (iii).

Let us prove that condition (ii) holds. We have

$$\pi_+[B_{1,k},\mathcal{L}] = [B_{1,k},M] + \tfrac{1}{2}\pi_+[B_{1,k},\partial_2^2+\partial_3^2],$$

where $M := \mathcal{L} - \frac{1}{2}(\partial_2^2+\partial_3^2)$. Similarly,

$$\pi_-[B_{1,k},\mathcal{L}] = [B_{1,k},M] + \tfrac{1}{2}\pi_-[B_{1,k},\partial_2^2+\partial_3^2].$$

It remains only to notice that

$$\tfrac{1}{2}\pi_+[B_{1,k},\partial_a^2] = -\partial_a(B_{1,k})\iota_{\Lambda^{-1}}Q_a - \tfrac{1}{2}\partial_a^2(B_{1,k}) = \tfrac{1}{2}\pi_-[B_{1,k},\partial_a^2],$$

where we used that $B_{1,k}$ is divisible from the right by $\Lambda-\Lambda^{-1}$, so $\partial_a(B_{1,k})Q_a \in \mathcal{R}[\Lambda,\Lambda^{-1}]$ is a difference operator. Finally, note that $\pi_+[B_{1,k},\mathcal{L}] = \pi_+[B_{1,k}-L_1^k,\mathcal{L}] \in \Lambda^{n-2}\mathcal{R}[\Lambda^{-1}]$ because $B_{1,k}-L_1^k$ does not involve terms with positive powers of Λ.

3.3. **The derivation $\partial_{a,2l+1}$.** Let us prove the existence of the operators L_a.

Lemma 12. *There exists a unique pseudo-differential operator $L_a = \partial_a + \sum_{i=1}^\infty b_{a,i}\partial_a^{1-i}$ ($a = 2,3$) with coefficients in $\mathcal{R}$, such that,*

$$b_{a,1} = 0, \quad \pi_a(\mathcal{L}) = \frac{1}{2}L_a^2.$$

Proof. The proof is straightforward computation by comparing the coefficients in front of ∂_a^2, ∂_a, and ∂_a^{1-i} ($i \geq 1$). The first identity is trivially satisfied. The second one is equivalent to $b_{a,1} = 0$, and the remaining ones, give a recursion for solving $b_{1,i+1}$ in terms of $b_{1,1}, b_{1,2}, \ldots, b_{1,i}$. □

Let us recall the differential operators $B_{a,2l+1}$ from Section 1.4. Put

$$B^{\pm}_{a,2l+1} := \frac{1}{2}\iota_{\Lambda^{\mp1}}\Big(L_a^{2l+1}H_a^{-1}(\partial_a+q_a)\Big)_{a,[0]}(\Lambda-\Lambda^{-1}), \quad a = 2,3,$$

where H_a^{-1} is the inverse of H_a in the ring $\mathcal{R}((\Lambda))((\partial_a^{-1}))$. Recalling Proposition 9, we get $B^{\pm}_{a,2l+1} = \pi_{\pm}(B_{a,2l+1})$.

Lemma 13. *Suppose that $a \in \{2,3\}$. Then the following properties are satisfied.*

a) The operators $\partial_a^{-1}H_1$, $(\partial_a+q_a)^{-1}H_a$, and L_a pairwise commute.

b) $L_a^{\#} = -\partial_a \cdot L_a \cdot \partial_a^{-1}$ and $(L_a^{2l+1})_{a,[0]} = 0$.

c) $\pi_+(H_bB^+_{a,2l+1}) = \pi_-(H_bB^-_{a,2l+1})$ ($b = 1,2,3$).

d) $\pi_+([B^+_{a,2l+1},\mathcal{L}]) = \pi_-([B^-_{a,2l+1},\mathcal{L}])$.

Proof. a) Let us argue in the case when $a = 2$. The case $a = 3$ is similar. We have already proved that $\widetilde{H}_1 = \partial_2^{-1}H_1 = \partial_3 + \partial_2^{-1}q_1$ and

$$\widetilde{H}_2 := (\partial_2+q_2)^{-1}H_2 = (\partial_2+q_2)^{-1}(\partial_2-q_2)\Lambda - 1$$

commute (see Proposition 1). Let us prove that $[\widetilde{H}_2, L_2] = 0$. The argument for the remaining identity $[\widetilde{H}_1, L_2] = 0$ is similar. We have

$$[\widetilde{H}_2, L_2^2] = \Big((\partial_2 + q_2)^{-1}(\partial_2 - q_2)(L_2[1])^2 - L_2^2(\partial_2 + q_2)^{-1}(\partial_2 - q_2)\Big)\Lambda.$$

By definition $L_2^2 = 2\pi_2(\mathcal{L})$ and we know that $\widetilde{H}_2\mathcal{L} \in \mathcal{E}_{(2)}H$. Therefore, $\pi_2([\widetilde{H}_2, L_2^2]) = 0$. On the other hand, if $\pi_2(A\Lambda) = 0$ for some $A \in \mathcal{E}_{(2)}^0$, then $A = 0$. Indeed, by definition

$$\pi_2(A\Lambda) = A(\partial_2 - q_2)^{-1}(\partial_2 + q_2)$$

and the operator $(\partial_2 - q_2)^{-1}(\partial_2 + q_2)$ is invertible. This proves that $[\widetilde{H}_2, L_2^2] = 0$. Suppose that $[\widetilde{H}_2, L_2] = a_m(\Lambda)\partial_2^m + O(\partial_2^{m-1})$ with $a_m \neq 0$. The coefficients in front of ∂_2^{2m+1} in $[\widetilde{H}_2, L_2^2] = [\widetilde{H}_2, L_2]L_2 + L_2[\widetilde{H}_2, L_2]$ is $2a_m(\Lambda)$ and it must be 0 – contradiction.

b) It is enough to check that $(L_2^2)^\# = \partial_2 \cdot (L_2^2) \cdot \partial_2^{-1}$. Indeed, if this was known then $X := -\partial_2^{-1} \cdot L_2^\# \cdot \partial_2$ is a pseudo-differential operator of the form $\partial_2 + O(\partial_2^{-1})$ solving the equation $\pi_2(\mathcal{L}) = \frac{X^2}{2}$. However, we know that such a pseudo-differential operator is unique and by definition it is L_2, i.e., $-\partial_2^{-1} \cdot L_2^\# \cdot \partial_2 = L_2$, which is the first formula that we have to prove. For the second formula $(L_2^{2l+1})_{2,[0]} = 0$ we argue as follows. We have $(L_2^{2l+1})_{2,[0]} = \mathrm{Res}_{\partial_2}\left(L_2^{2l+1}\partial_2^{-1}\right)$. Let us recall the following formula $\left(\mathrm{Res}_{\partial_2}(P)\right)^\# = -\mathrm{Res}_{\partial_2}(P^\#)$ for every pseudodifferential operator $P \in \mathcal{R}(\!(\partial_2^{-1})\!)$. In the case $P = L_2^{2l+1}\partial_2^{-1}$, we have $P^\# = -P$, so $\mathrm{Res}_{\partial_2}(P) = 0$.

Let us prove that $(L_2^2)^\# = \partial_2 \cdot (L_2^2) \cdot \partial_2^{-1}$. By definition $L_2^2 = 2\pi_2(\mathcal{L})$. Let us write the operator $\mathcal{L} = A + \frac{1}{2}\partial_2^2 + \frac{1}{2}\partial_3^2$. Recalling Proposition 9, c) we have

$$\pi_2(A) = 2\Big(A_{1,>0}\,\iota_{\Lambda^{-1}}\left(H_2^{-1}(\Lambda^{-1}+1)^{-1}\right)\Big)_{1,[0]}\partial_2 - 2\Big(A_{1,<0}\,\iota_{\Lambda}\left(H_2^{-1}(\Lambda^{-1}+1)^{-1}\right)\Big)_{1,[0]}\partial_2 + A|_{\Lambda=-1},$$

where for $A = \sum_i \alpha_i\Lambda^i$ we set $A|_{\Lambda=-1} := \sum_i(-1)^i\alpha_i$. It is straightforward to check that $A|_{\Lambda=-1} = c_3$. On the other hand, by definition $\frac{1}{n-2}L_1^{n-2} = \pi_+(\mathcal{L})$ and the π_+ projection of ∂_2^2 and ∂_3^2 involves only non-positive powers of Λ, while $\pi_+(A) = A$. Therefore, $A_{1,>0} = \frac{1}{n-2}(L_1^{n-2})_{1,>0}$. Furthermore,

$$\iota_{\Lambda^{-1}}\left(H_2^{-1}(\Lambda^{-1}+1)^{-1}\right) = \iota_{\Lambda^{-1}}\left((\partial_2 - Q_2)^{-1}(\Lambda - \Lambda^{-1})^{-1}\right)$$

involves only negative powers of Λ. Therefore, replacing $A_{1,>0}$ by $\frac{1}{n-2}L_1^{n-2}$ does not change the contribution to the projection $\pi_2(\mathcal{L})$. Similarly, replacing $A_{1,<0}$ by $\frac{1}{n-2}(L_1^-)^{n-2}$ does not change the contribution to the projection. We got the following formula for $\pi_2(A)$

$$2\Big(\tfrac{1}{n-2}L_1^{n-2}\iota_{\Lambda^{-1}}\left((\partial_2 - Q_2)^{-1}(\Lambda - \Lambda^{-1})^{-1}\right)\Big)_{1,[0]}\partial_2 - 2\Big(\tfrac{1}{n-2}(L_1^-)^{n-2}\iota_{\Lambda}\left((\partial_2 - Q_2)^{-1}(\Lambda - \Lambda^{-1})^{-1}\right)\Big)_{1,[0]}\partial_2 + c_3.$$

Using $L_1^\# = (\Lambda - \Lambda^{-1})L_1^-(\Lambda - \Lambda^{-1})^{-1}$, $Q_2^\# = -(\Lambda - \Lambda^{-1})Q_2(\Lambda - \Lambda^{-1})^{-1}$, and that L_1 (resp. L_1^-) commutes with $\partial_2 - \iota_{\Lambda^{-1}}(Q_2)$ (resp. $\partial_2 - \iota_\Lambda(Q_2)$) we get that

$$(\pi_2(A))^\# = \partial_2 \cdot \pi_2(A) \cdot \partial_2^{-1} - \partial_2(c_3) \cdot \partial_2^{-1}.$$

A direct computation (or using Proposition 9, b)) yields

$$\tfrac{1}{2}\Big(\pi_2(\partial_3^2)\Big)^{\#} = \tfrac{1}{2}\,\partial_2\cdot\pi_2(\partial_3^2)\cdot\partial_2^{-1} + \partial_3(q_1)\cdot\partial_2^{-1}.$$

Therefore, we get the following relation

$$(\pi_2(\mathcal{L}))^{\#} - \partial_2\cdot\pi_2(\mathcal{L}))\cdot\partial_2^{-1} = (\partial_3(q_1) - \partial_2(c_3))\cdot\partial_2^{-1}.$$

We claim that the RHS is 0. This follows from part a). Namely, the operators $\frac{1}{2}L_3^2 = \pi_3(\mathcal{L}) = \frac{\partial_3^2}{2} + c_3 + O(\partial_3^{-1})$ and $\partial_3^{-1}H_1 = \partial_2 + \partial_3^{-1}q_1$ commute, i.e.,

$$\Big[\tfrac{\partial_3^2}{2} + c_3 + O(\partial_3^{-1}), \partial_2 + \partial_3^{-1}q_1\Big] = 0.$$

Comapring the coefficients in front of ∂_3^0 we get $\partial_2(c_3) = \partial_3(q_1)$.

c) We will prove a stronger statement. Namely, we claim that for each pair (a,b) there are $\alpha_{a,b}$ and $\beta_{a,b}\in\mathcal{R}$ such that $H_bB_{a,2l+1} - \alpha_{a,b}\Lambda - \beta_{a,b}\in\mathcal{E}H$. Suppose that $a=2$, the case $a=3$ is similar. Let us prove the statement for $b=2$. Recalling Proposition 9 we get

$$\pi_+(H_2B_{2,2l+1}) = \tfrac{1}{2}\iota_{\Lambda^{-1}}\Big(H_2(L_2^{2l+1})_{2,\geq 0}\widetilde{H}_2^{-1}\Big)(\Lambda - \Lambda^{-1}).$$

Using $(H_2L_2^{2l+1})_{2,\geq 0} = H_2(L_2^{2l+1})_{2,\geq 0} + (\Lambda - 1)\mathrm{res}_{\partial_2}(L_2^{2l+1})$ and the fact that $\widetilde{H}_2^{-1} = (\Lambda-1)^{-1} + O(\partial_2^{-1})$ has only non-positive powers of ∂_2 (see formula (22)) we get

$$\pi_+(H_2B_{2,2l+1}) = \tfrac{1}{2}\iota_{\Lambda^{-1}}\Big(H_2L_2^{2l+1})\widetilde{H}_2^{-1}\Big)(\Lambda - \Lambda^{-1}) - \tfrac{1}{2}(\Lambda-1)\mathrm{res}_{\partial_2}(L_2^{2l+1})(\Lambda^{-1}+1).$$

According to part a) L_2 and $\widetilde{H}_2$ commute, $H_2\widetilde{H}_2^{-1} = \partial_2 + q_2$, and by part b) $(L_2^{2l+1})_{2,[0]} = 0$. Therefore

$$\pi_+(H_2B_{2,2l+1}) = \tfrac{r}{2}(\Lambda - \Lambda^{-1}) - (\Lambda - 1)\tfrac{r}{2}(\Lambda^{-1}+1) = \tfrac{1}{2}(r - r[1])(\Lambda+1),$$

where $r = \mathrm{Res}_{\partial_2}(L_2^{2l+1})$. From here, recalling Proposition 8, we get $H_2B_{2,2l+1} - \frac{1}{2}(r - r[1])(\Lambda+1)\in\mathcal{E}_{(+)}H\cap\mathcal{E} = \mathcal{E}H$. In particular, we get the following formulas

$$\alpha_{2,2} = \beta_{2,2} = \tfrac{1}{2}(r - r[1]).$$

Let us move to the case $b=1$. Recalling Proposition 9 we get

$$\pi_3(H_1B_{2,2l+1}) = \Big(H_1B_{2,2l+1}\widetilde{H}_1^{-1}\Big)_{2,[0]}\partial_3.$$

On the other hand, using that $(H_1L_2^{2l+1})_{2,\geq 0} = H_1(L_2^{2l+1})_{2,\geq 0} + \partial_3\cdot\mathrm{res}_{\partial_2}(L_2^{2l+1})$ and that $\widetilde{H}_1^{-1} = \partial_3^{-1} + O(\partial_2^{-1})$ has only non-positive powers of ∂_2 we get

$$\pi_3(H_1B_{2,2l+1}) = \Big(H_1L_2^{2l+1}\widetilde{H}_1^{-1}\Big)_{2,[0]}\partial_3 - \partial_3\cdot r,$$

where $r = \mathrm{Res}_{\partial_2}(L_2^{2l+1})$. According to part a), the operators L_2 and $\widetilde{H}_1$ commute. Since $H_1\widetilde{H}_1^{-1} = \partial_2$ we get

$$\pi_3(H_1B_{2,2l+1}) = r\cdot\partial_3 - \partial_3\cdot r = -\partial_3(r).$$

Recalling Proposition 8 we get $H_1 B_{2,2l+1} + \partial_3(r) \in \mathcal{E}_{(3)} H \cap \mathcal{E} = \mathcal{E}H$. Let us point out that

$$\alpha_{2,1} = 0, \quad \beta_{2,1} = -\partial_3(r).$$

Finally, for the case $b = 3$, let us recall the identity

$$(\Lambda - 1)H_1 = q_2 H_3 + \partial_3 \cdot H_2.$$

Multiplying both sides by $B_{2,2l+1}$, taking the projection π_+, and using that

$$\pi_+(H_1 B_{2,2l+1}) = -\partial_3(r)\pi_+(H_2 B_{2,2l+1}) = \tfrac{1}{2}(r - r[1])(\Lambda + 1),$$

we get the following identity

$$-(\Lambda - 1)\partial_3(r) = q_2 \pi_+(H_3 B_{2,2l+1}) + \tfrac{1}{2}\partial_3(r - r[1])(\Lambda + 1) + \tfrac{1}{2}(r - r[1])q_2(\Lambda - 1),$$

which yields

$$q_2 \pi_+(H_3 B_{2,2l+1}) = -\tfrac{1}{2}\Big(\partial_3(r + r[1]) + q_2(r - r[1])\Big)(\Lambda - 1).$$

Comparing the coefficients in front of Λ we get that $\pi_+(H_3 B_{2,2l+1}) = \alpha_{2,3}\Lambda + \beta_{2,3}$ for some $\alpha_{2,3}, \beta_{2,3} \in \mathcal{R}$ satisfying

$$q_2 \beta_{2,3} = -q_2 \alpha_{2,3} = \tfrac{1}{2}\Big(\partial_3(r + r[1]) + q_2(r - r[1])\Big).$$

It remains only to use that $H_3 B_{2,2l+1} - \alpha_{2,3}(\Lambda - 1) \in \mathcal{E}_{(+)} H \cap \mathcal{E} = \mathcal{E}H$.

d) Let us give the argument for $a = 2$. The case $a = 3$ is similar. Since $B^{\pm}_{2,2l+1} = \pi_{\pm}((L_2^{2l+1})_{2,\geq 0})$ and $H_i \mathcal{L} \in \mathcal{E}H$, we have $\pi_{\pm}[B^{\pm}_{2,2l+1}, \mathcal{L}] = \pi_{\pm}[(L_2^{2l+1})_{2,\geq 0}, \mathcal{L}]$. It is sufficient to prove that the π_+-projection of $[(L_2^{2l+1})_{2,\geq 0}, \mathcal{L}]$ is a difference operator, i.e., it belongs to the ring $\mathcal{R}[\Lambda, \Lambda^{-1}]$. Indeed, if we knew this fact, then

$$[(L_2^{2l+1})_{2,\geq 0}, \mathcal{L}] - \pi_+[(L_2^{2l+1})_{2,\geq 0}), \mathcal{L}] \in \mathcal{E} \cap \mathcal{E}_{(+)} H = \mathcal{E}H,$$

where we used Proposition 8. The above statement implies that $\pi_+[(L_2^{2l+1})_{2,\geq 0}), \mathcal{L}] = \pi_-[(L_2^{2l+1})_{2,\geq 0}), \mathcal{L}]$, which would complete the proof.

Note that

$$[(L_2^{2l+1})_{2,\geq 0}, \mathcal{L}] = [L_2^{2l+1}, \mathcal{L}]_{2,\geq 0} + \partial_2(\mathrm{res}_{\partial_2}(L_2^{2l+1})).$$

Therefore, it is enough to prove that $\pi_+[L_2^{2l+1}, \mathcal{L}]_{2,\geq 0}$ is a difference operator. Let us decompose the operator

$$\mathcal{L} = \tfrac{1}{2}L_2^2 + A\widetilde{H}_2 + \partial_3 \cdot \widetilde{H}_1 - \tfrac{1}{2}\widetilde{H}_1^2, \tag{41}$$

where $\widetilde{H}_2 = (\partial_2 + q_2)^{-1} H_2$ and $\widetilde{H}_1 = \partial_2^{-1} H_1$. The last two terms in (41) coincide with $\frac{1}{2}(\partial_3^2 - \pi_2(\partial_3^2))$. The terms of $\mathcal{L}$ that involve only Λ decompose by the following recursive rules:

$$\Lambda^i = (\partial_2 - q_2[i-1])^{-1}\Lambda^{i-1} H_2 + (\partial_2 - q_2[i-1])^{-1}(\partial_2 + q_2[i-1])\Lambda^{i-1}$$

and

$$\Lambda^{-i} = -(\partial_2 + q_2[-i])^{-1}\Lambda^{-i}H_2 + (\partial_2 + q_2[-i])^{-1}(\partial_2 - q_2[-i])\Lambda^{-i+1},$$

where $i > 0$ is an integer. Therefore, the coefficient $A \in \mathcal{R}[\Lambda, \Lambda^{-1}]((\partial_2^{-1}))$. According to part a) the operators $L_2, \widetilde{H_2}$, and $\widetilde{H}_1$ pairwise commute. Therefore,

$$[L_2^{2l+1}, \mathcal{L}]_{2,\geq 0} = \Big([L_2^{2l+1}, A]\widetilde{H}_2\Big)_{2,\geq 0} - \Big(\partial_3(L_2^{2l+1}\partial_2^{-1})H_1\Big)_{2,\geq 0}.$$

Note that

$$\Big(\partial_3(L_2^{2l+1}\partial_2^{-1})H_1\Big)_{2,\geq 0} - \Big(\partial_3(L_2^{2l+1}\partial_2^{-1})\Big)_{2,\geq 0} H_1 = \mathrm{Res}_{\partial_2}(\partial_3(L_2^{2l+1}\partial_2^{-1}))\cdot\partial_3 = 0,$$

where for the last equality we used that $(L_2^{2l+1})_{2,[0]} = 0$ (see part b)). We get

$$\pi_+\Big([L_2^{2l+1}, \mathcal{L}]_{2,\geq 0}\Big) = \Big([L_2^{2l+1}, A]\widetilde{H}_2\Big)_{2,[0]} + \pi_+\Big(\Big([L_2^{2l+1}, A]\widetilde{H}_2\Big)_{2,>0}\Big).$$

The first term on the RHS is a difference operator, while for the second one we recall Proposition 9 part a) and we get

$$\pi_+\Big(\Big([L_2^{2l+1}, A]\widetilde{H}_2\Big)_{2,>0}\Big) = \tfrac{1}{2}\Big([L_2^{2l+1}, A] - ([L_2^{2l+1}, A]\widetilde{H}_2)_{2,[0]}\widetilde{H}_2^{-1}\Big)_{2,[0]}(\Lambda - \Lambda^{-1}).$$

This is clearly a difference operator. □

We claim that the operators $B^{\pm} := B^{\pm}_{a,2l+1}$ satisfy the conditions (i)–(iii) of Lemma 10. Conditions (ii) and (iii) follow from respectively part d) and c) of Lemma 13. We need only to verify that condition (i) is satisfied, that is,

$$(B^{+}_{a,2l+1})^{\#} = -(\Lambda - \Lambda^{-1})\, B^{-}_{a,2l+1}\, \iota_\Lambda(\Lambda - \Lambda^{-1})^{-1}. \tag{42}$$

Again let us give the argument only for $a = 2$. The case $a = 3$ is similar. Recall that the residue of a pseudo-differential operator $P = \sum_j p_j\partial_2^j$ is defined by $\mathrm{res}_{\partial_2}(P) := p_{-1}$. We have

$$B^{\pm}_{2,2l+1} = \tfrac{1}{2}\iota_{\Lambda^{\mp 1}}\,\mathrm{Res}_{\partial_2}\Big(L_2^{2l+1}\widetilde{H}_2^{-1}\partial_2^{-1}\Big)(\Lambda - \Lambda^{-1}).$$

Using the following simple formula

$$\Big(\mathrm{Res}_{\partial_2} P(\Lambda, \partial_2)\Big)^{\#} = -\mathrm{Res}_{\partial_2} P^{\#}(\Lambda, \partial_2) \tag{43}$$

we get

$$(B^{+}_{2,2l+1})^{\#} = \tfrac{1}{2}(\Lambda - \Lambda^{-1})\,\mathrm{Res}_{\partial_2}\Big((-\partial_2)^{-1}(\widetilde{H}_2^{\#})^{-1}(L_2^{\#})^{2l+1}\Big).$$

Recalling Lemma 13, a) and b) and

$$\widetilde{H}_2^{\#} = -\partial_2\cdot\widetilde{H}_2\cdot(\widetilde{H}_2 + 1)^{-1}\cdot\partial_2^{-1}$$

we get

$$(B^{+}_{2,2l+1})^{\#} = -\tfrac{1}{2}(\Lambda - \Lambda^{-1})\,\mathrm{Res}_{\partial_2}\Big(L_2^{2l+1}\widetilde{H}_2^{-1}\partial_2^{-1} + L_2^{2l+1}\partial_2^{-1}\Big).$$

In order to prove (42) we need only to recall that according to Lemma 13, b) $\mathrm{Res}_{\partial_2}\Big(L_2^{2l+1}\partial_2^{-1}\Big) = 0$.

4. Existence of the extended flows

4.1. Dressing operators. Let us recall the notation from Section 1.3.

Proposition 14. *a) The derivations ∂_2 and ∂_3 can be extended uniquely to derivations of $\mathcal{R}_1$ such that $\epsilon\partial_x$, ∂_2, and ∂_3 pairwise commute and $\partial_a(S_1) = Q_a S_1$, $a = 2,3$.*

b) The coefficients of the operator series $\ell_1 := \epsilon\partial_x(S_1)\cdot S_1^{-1}$ belong to $\mathcal{R}$.

Proof. Let us extend the derivations ∂_a $(a = 2,3)$ to derivations of $\mathcal{R}_1$ via the given formula $\partial_a(S_1) = Q_a \cdot S_1$. We need only to prove that $\epsilon\partial_x$, ∂_2, and ∂_3 pairwise commute as derivations of $\mathcal{R}_1$. The commutativity of ∂_2 and ∂_3 is a direct consequence of the 0-curvature condition. Let us prove that ∂_2 and $\epsilon\partial_x$ commute. The argument for ∂_3 and $\epsilon\partial_x$ is similar. We have to prove that $\epsilon\partial_x\partial_2(S_1) = \partial_2\epsilon\partial_x(S_1)$.

The first step in our argument is to prove that the derivation ∂_2 commutes with the translation operator Λ^{n-2}. Using that

$$\frac{1}{n-2}L_1^{n-2} = \pi_+(\mathcal{L}) = \mathcal{L} - \frac{1}{2}\Big((\partial_2+Q_2)(\partial_2-Q_2) + (\partial_3+Q_3)(\partial_3-Q_3)\Big)$$

and

$$\partial_2(\mathcal{L}) = [Q_2, \mathcal{L} - \tfrac{1}{2}(\partial_2^2+\partial_3^2)] - \sum_{a=2,3}\left(\frac{1}{2}\partial_a^2(Q_2) + \partial_a(Q_2)Q_a\right)$$

we get $\partial_2(L_1^{n-2}) = [Q_2, L_1^{n-2}]$. Let us substitute $L_1^{n-2} = \sum_{i=0}^\infty b_{n-2,i}\Lambda^{n-2-i}$ and $S_1 = \sum_{j=0}^\infty \psi_{1,j}\Lambda^{-j}$ in $L_1^{n-2}S_1 = S_1\Lambda^{n-2}$ and differentiate the resulting identity with respect to ∂_2. We get

$$\partial_2(L_1^{n-2})S_1 + \sum_{k=0}^\infty\sum_{i=0}^k b_{m,i}\partial_2(\psi_{1,k-i}[n-2-i])\Lambda^{n-2-k} = \partial_2(S_1)\Lambda^{n-2}.$$

Since $\partial_2(L_1^{n-2}) = [Q_2, L_1^{n-2}]$, $\partial_2(S_1) = Q_2S_1$, and $L_1^{n-2}S_1 = S_1\Lambda^{n-2}$, the above identity yields

$$\sum_{i=0}^k b_{m,i}\partial_2(\psi_{1,k-i}[n-2-i]) = \sum_{i=0}^k b_{m,i}\partial_2(\psi_{1,k-i})[n-2-i].$$

A simple induction on k yields $\partial_2\Lambda^{n-2}(\psi_{1,k}) = \Lambda^{n-2}\,\partial_2(\psi_{1,k})$ for all $k \ge 0$.

Now we are in position to prove that $(\epsilon\partial_x\circ\partial_2)(\psi_{1,i}) = (\partial_2\circ\epsilon\partial_x)(\psi_{1,i})$ for all $i \ge 0$. For $i = 0$ the equality is straightforward to check. In particular, it is sufficient to prove that $(\epsilon\partial_x\circ\partial_2)(\psi_{1,i}/\psi_{1,0}) = (\partial_2\circ\epsilon\partial_x)(\psi_{1,i}/\psi_{1,0})$. We argue by induction on i. Formula (5) can be written as $(1-\Lambda)(\psi_{1,i}/\psi_{1,0}) = f_k$, where $f_k \in \mathcal{R}_1$ is a formal power series in ϵ whose coeffcients are differential polynomials in $\phi, \psi_{1,1}, \dots, \psi_{1,i-1}$ with coefficients in $\mathcal{R}$. Using the iductive assumption we get $\partial_2(\epsilon\partial_x)^l(f_k) = (\epsilon\partial_x)^l\partial_2(f_k)$ for all $l \ge 0$. Therefore

$$\partial_2\,(1-\Lambda^{n-2})\Big(\frac{\psi_{1,i}}{\psi_{1,0}}\Big) = \partial_2\,\frac{1-\Lambda^{n-2}}{1-\Lambda}(f_k) = \frac{1-\Lambda^{n-2}}{1-\Lambda}\,\partial_2(f_k).$$

Let us take the equality between the first and the last term in the above formula. After exchanging the order of ∂_2 and $1-\Lambda^{n-2}$ on the LHS and applying to both sides the operator $\frac{\epsilon\partial_x}{1-\Lambda^{n-2}}$ we get

$$\epsilon\partial_x\,\partial_2\Big(\frac{\psi_{1,i}}{\psi_{1,0}}\Big)=\frac{\epsilon\partial_x}{1-\Lambda}\,\partial_2(f_k)=\partial_2\,\frac{\epsilon\partial_x}{1-\Lambda}(f_k)=\partial_2\,\epsilon\partial_x\Big(\frac{\psi_{1,i}}{\psi_{1,0}}\Big),$$

where for the second equality we used the inductive assumption and for the thrid equality we recall formula (5).

b) The idea of the proof is the same as the proof of Theorem 2.1 in [4]. Let us first prove the following general statement about pseudo-difference operators: let

$$\widetilde{\mathcal{R}}=\mathbb{C}[\partial_x^m\widetilde{\psi}_j\,|\,m\geq 0,\ j\geq 1][[\epsilon]]$$

be the ring of formal power series in ϵ whose coefficients are differential polynomials on the infinite number of formal variables $\widetilde{\psi}_1,\widetilde{\psi}_2,\ldots$. The translation operator $\Lambda=e^{\epsilon\partial_x}$ acts naturally on $\widetilde{\mathcal{R}}$, so we can define the ring of pseudo-difference operators $\widetilde{\mathcal{R}}((\Lambda^{-1}))$. Let us define the following pseudo-difference operators:

$$\widetilde{S}:=1+\sum_{j\geq 1}\widetilde{\psi}_j\Lambda^{-j},\quad \widetilde{L}:=\widetilde{S}\Lambda\widetilde{S}^{-1},\quad \widetilde{\ell}:=\epsilon\partial_x(\widetilde{S})\widetilde{S}^{-1}.$$

We claim that the coefficients of $\widetilde{\ell}$ are differential polynomials in the coefficients of $\widetilde{L}$, that is, the coefficients of $\widetilde{\ell}$ belong to the differential subring of $\widetilde{\mathcal{R}}$ generated by the coefficients of $\widetilde{L}$.

Let us prove the above claim. The idea is to compare the coefficient in front of Λ^0 in the following equations:

$$\epsilon\partial_x(\widetilde{L}^m)=[\widetilde{\ell},\widetilde{L}^m],\quad m\geq 1.$$

Let us write $\widetilde{\ell}=\sum_{i=1}^{\infty}a_i\Lambda^{-i}$ and $\widetilde{L}^m=\Lambda^m+\sum_{j=1}^{\infty}u_{m,j}\Lambda^{m-j}$, then we get

$$\epsilon\partial_x(u_{m,m})=(1-e^{m\epsilon\partial_x})(a_m)+\sum_{i=1}^{m-1}(1-e^{i\epsilon\partial_x})(a_iu_{m,m-i}[-i]). \tag{44}$$

In order to complete the proof it is sufficient to prove that

$$a_m=\frac{\epsilon\partial_x}{1-e^{m\epsilon\partial_x}}(u_{m,m})-\sum_{i=1}^{m-1}\frac{1-e^{i\epsilon\partial_x}}{1-e^{m\epsilon\partial_x}}(a_iu_{m,m-i}[-i]). \tag{45}$$

The recursion (44) implies that the derivatives of the LHS and the RHS of (45) with respect to x are the same. We have to prove only that the integration constant is 0. Let us turn $\widetilde{\mathcal{R}}$ into a graded ring by assigning degree i to $\partial_x^m\widetilde{\psi}_i$ for $i\geq 1$, $m\geq 0$. We leave it as an exercise to check that the ring of constants of $\widetilde{R}$, that is, the set of elements $A\in\widetilde{R}$, such that, $\partial_x(A)=0$, is $\mathbb{C}[[\epsilon]]$. In other words, the ring of constants coincides with the subring of homogeneous elements of degree 0. On the other hand, note that the coefficients a_i and $u_{m,i}$ are homogeneous of degree i. The difference of the LHS and the RHS in (45) is homogeneous of degree $m \geq 1$ and its derivative is 0, so it must be identically 0.

Now the proof of part b) can be completed as follows: Put $\widetilde{S} = \psi_{1,0}^{-1} S_1$. Note that the coefficients of the pseudo-difference operator $\widetilde{L} := \widetilde{S}\Lambda\widetilde{S}^{-1} = \psi_{1,0}^{-1} L_1 \psi_{1,0}$ belong to $\mathcal{R}$, because $e^{\phi[i]-\phi} \in \mathcal{R}$ for all $i \in \mathbb{Z}$. Recalling the claim from above, we get that the coefficients of $\widetilde{\ell} = \epsilon\partial_x(\widetilde{S})\widetilde{S}^{-1}$ also belong to $\mathcal{R}$. Finally, the coefficients of the operator

$$\ell_1 = \epsilon\partial_x(S_1)\cdot S_1^{-1} = \frac{\epsilon\partial_x(\psi_{1,0})}{\psi_{1,0}} + \psi_{1,0}\cdot\widetilde{\ell}\cdot\psi_{1,0}^{-1} = \frac{(n-2)\epsilon\partial_x}{1-\Lambda^{n-2}}(\alpha) + \psi_{1,0}\cdot\widetilde{\ell}\cdot\psi_{1,0}^{-1}$$

belong to $\mathcal{R}$, which is what we have to prove. □

Proposition 15. *Suppose that $a = 2, b = 3$ or $a = 3, b = 2$.*

a) The translation operator $\Lambda = e^{\epsilon\partial_x}$ and ∂_b can be extended uniquely to respectively an automorphism and a derivation of the ring $\mathcal{R}_a$ in such a way that Λ, ∂_2, and ∂_3 pairwise commute, and

$$T_a[1] = (\partial_a - q_a)^{-1}(\partial_a + q_a)T_a, \quad \partial_b(T_a) = -\partial_a^{-1}\cdot q_1\cdot T_a.$$

b) The derivation $\epsilon\partial_x$ can be extended uniquely to a derivation on $\mathcal{R}_a$, such that $\epsilon\partial_x, \partial_2$, and ∂_3 pairwise commute and $\Lambda = e^{\epsilon\partial_x}$. The coefficients of the operator $\ell_a := \epsilon\partial_x(S_a)\cdot S_a^{-1}$ belong to $\mathcal{R}$.

Proof. a) Let us give the argument for the case $a = 2$ and $b = 3$. The other case, $a = 3$ and $b = 2$ is similar. By definition,

$$\partial_2(T_2) = (\partial_2 - L_2)\cdot T_2.$$

Let us extend the translation operator Λ and the derivation ∂_3 to $\mathcal{R}_2$ by the given formulas, that is,

$$T_2[1] = (\partial_2 - q_2)^{-1}(\partial_2 + q_2)T_2,$$
$$\partial_3(T_2) = -\partial_2^{-1}\cdot q_1\cdot T_2.$$

The commutativity of the extended translation operator and the extended derivation ∂_3 is equivalent to

$$\partial_3\Big((\partial_2 - q_2)^{-1}(\partial_2 + q_2)T_2\Big) = -\partial_2^{-1}\cdot q_1[1]\cdot T_2[1]$$

Substituting the above formulas, after a short computation, we get that the above identity is equivalent to the fact that $\widetilde{H}_1 := \partial_3 + \partial_2^{-1}q_1$ and $\widetilde{H}_2 := (\partial_2 + q_2)^{-1}(\partial_2 - q_2)\Lambda$ commute. This however, follows from part a) of Lemma 13. Similarly, the commutativity of Λ and ∂_2 is equivalent to

$$\partial_2\Big((\partial_2 - q_2)^{-1}(\partial_2 + q_2)T_2\Big) = (\partial_2 - L_2[1])\cdot T_2[1].$$

The above identity is equivalent to the fact that L_2 and $\widetilde{H}_2$ commute, which again follows from part a) of Lemma 13. Finally, the commutativity of ∂_2 and ∂_3 is equivalent to the commutativity of L_2 and $\widetilde{H}_1$.

b) Let us define pseudo-differential operators $\widetilde{\ell}_2^{(m)} \in \mathcal{R}[[\partial_2^{-1}]]\partial_2^{-1}$ ($m \geq 1$) by the following two conditions:

(i) $\tilde{\ell}_2^{(m+1)} = \tilde{\ell}_2 \cdot \tilde{\ell}_2^{(m)} + \epsilon\partial_x \tilde{\ell}_2^{(m)}$ with $\tilde{\ell}_2^{(0)} = 1$ and $\tilde{\ell}_2^{(1)} = \tilde{\ell}_2$,

(ii) $\sum_{m=0}^{+\infty} \tilde{\ell}_2^{(m)}/m! = (\partial_2 + q_2)^{-1}(\partial_2 - q_2)$.

Let us check that conditions (i) and (ii) uniquely determine the sequence $\widetilde{\ell}_2^{(m)}$. Condition (i) defines $\widetilde{\ell}_2^{(m)}$ in terms of $\widetilde{\ell}_2$, so we need only to check that $\widetilde{\ell}_2 = \sum_{i\geq 1} a_i \partial_2^{-i}$ is uniquely determined by condition (ii). Note that $\widetilde{\ell}_2^{(m)} = (\epsilon\partial_x)^{m-1}(\ell_2) + \cdots$, where the dots stand for a differential polynomial in $\widetilde{\ell}_2$ that involves at least quadratic terms in $\widetilde{\ell}_2$. In particular, the ceofficient in front of ∂_2^{-i} in $\sum_{m\geq 1} \widetilde{\ell}_2^{(m)}/m!$ has the form

$$\frac{e^{\epsilon\partial_x} - 1}{\epsilon\partial_x}(a_i) + \text{terms involving } a_1, \ldots, a_{i-1}$$

Since the operator $\frac{e^{\epsilon\partial_x}-1}{\epsilon\partial_x}$ is invertible, the identity in condition (ii) is equivalent to a system of recursions, which uniquely define the coefficients a_i ($i \geq 1$).

Let us extend the derivation $\epsilon\partial_x$ to a derivation on $\mathcal{R}_2$ by the following formula: $\epsilon\partial_x(T_2^{-1}) = T_2^{-1} \cdot \tilde{\ell}_2$. Using the fact that ∂_2 commutes with $\epsilon\partial_x$ on $\mathcal{R}$ and condition (i), we get $(\epsilon\partial_x)^m(T_2^{-1}) = T_2^{-1}\tilde{\ell}_2^{(m)}$ for all $m \geq 0$. Recalling condition (ii), we get that the extension of the translation operator to $\mathcal{R}_2$ coincides with $e^{\epsilon\partial_x}$. It remains only to prove that $[\partial_2, \epsilon\partial_x] = 0$ on $\mathcal{R}_2$.

Put $c_m = T_2 \cdot [\partial_2, (\epsilon\partial_x)^m](T_2)$. According to part a), we have $[\partial_2, \Lambda] = 0$ on $\mathcal{R}_2 \Rightarrow \sum_{m=1}^{+\infty} c_m/m! = 0$. By using $[\partial_2, \epsilon\partial_x] = 0$ on $\mathcal{R}$, we get the following recursion relations for the commutators c_m: $c_{m+1} = \tilde{\ell}_2 c_m + c_1 \tilde{\ell}_2^{(m)} + \epsilon\partial_x c_m$. Using this recursion relation, by induction on m, we get the following formula:

$$c_m = \sum_{k=0}^{m-1}\sum_{i=0}^{k}\binom{k}{i}\tilde{\ell}_2^{(i)}(\epsilon\partial_x)^{k-i}(c_1 \cdot \tilde{\ell}_2^{(m-1-k)}) = (\epsilon\partial_x)^{m-1}(c_1) + \text{at least quadratic terms.} \tag{46}$$

Then assume $c_1 = \sum_{i=1}^{\infty} c_{1,i}\partial_2^{-i}$. By comparing the coefficient of ∂_2^{-i} in $\sum_{m=1}^{+\infty} c_m/m! = 0$, we get

$$\frac{e^{\epsilon\partial_x} - 1}{\epsilon\partial_x}(c_{1,i}) + \text{terms involving } c_{1,1}, \ldots, c_{1,i-1} = 0.$$

Since the operator $\frac{e^{\epsilon\partial_x}-1}{\epsilon\partial_x}$ is invertible, by induction on i, we get $a_i = 0$ for all $i \geq 1$, that is, $c_1 = 0$.

Finally, note that $\ell_2 = -\widetilde{\ell}_2$, so the coefficients of ℓ_2 belong to $\mathcal{R}$ as claimed. □

Corollary 16. *The operator*

$$T_a^{\#}\,\partial_a\,T_a\,\partial_a^{-1} \quad \in \quad \mathcal{R}_a[[\partial_a^{-1}]], \quad a = 2, 3,$$

commutes with $\epsilon\partial_x, \partial_2$, and ∂_3.

Proof. The commutativity $[P, \epsilon\partial_x] = 0$ is equivalent to $P[1] = P$. Indeed, let us assume that $P[1] = P$ (the other implication is obvious). Expanding $P = \sum_{k=0}^{\infty} P_k \epsilon^k$ and comparing the coefficients in front of ϵ^1 in $P[1] = P$ we get $\partial_x(P_0) = 0$ and hence $P_0[1] = P_0$. Clearly, we can continue inductively and prove that $\partial_x(P_k) = 0$ and $P_k[1] = P_k$ for all k.

Let us check that

$$T_a^\#[1]\,\partial_a\,T_a[1]\,\partial_a^{-1} = T_a^\#\,\partial_a\,T_a\,\partial_a^{-1}. \tag{47}$$

Using that

$$T_a[1] = (\partial_a - q_a)^{-1}(\partial_a + q_a)T_a$$

and

$$T_a^\#[1] = \Lambda T_a^\# \Lambda^{-1} = (T_a[1])^\# = T_a^\#(\partial_a - q_a)(\partial_a + q_a)^{-1},$$

we get that the LHS of (47) is equal to

$$T_a^\#(\partial_a - q_a)(\partial_a + q_a)^{-1}\partial_a(\partial_a - q_a)^{-1}(\partial_a + q_a)T_a\partial_a^{-1}.$$

This expression coincides with the RHS of (47), because

$$\partial_a(\partial_a - q_a)^{-1}(\partial_a + q_a) = (1 - q_a\partial_a^{-1})^{-1}(1 + q_a\partial_a^{-1})\partial_a$$

and $(\partial_a - q_a)(\partial_a + q_a)^{-1} = (1 - q_a\partial_a^{-1})(1 + q_a\partial_a^{-1})^{-1}$.

The commutativity with ∂_b is proved in a similar way, using the explicit formula $\partial_b(T_a) = -\partial_a^{-1}q_1 T_a$. Finally, to prove the commutativity with ∂_a, let us conjugate the identity $L_a T_a = T_a\partial_a$, we get $T_a^\# L_a^\# = -\partial_a T_a^\#$. On the other hand, according to part b) of Lemma 13, we have $L_a^\# = -\partial_a L_a \partial_a^{-1} = -\partial_a T_a \partial_a T_a^{-1}\partial_a^{-1}$. Therefore,

$$T_a^\#\partial_a T_a \partial_a T_a^{-1}\partial_a^{-1} = \partial_a T_a^\# \quad \Rightarrow \quad [T_a^\#\partial_a T_a \partial_a^{-1}, \partial_a] = 0. \quad \square$$

Using Corollary 16, we can give a proof of Proposition 3 as follows: Put $A := T_a^\#\partial_a T_a\partial_a^{-1} \in 1 + \mathcal{R}_a[[\partial_a^{-1}]]$. Note that $A^\# = \partial_a^{-1}A\partial_a = A$, where for the second identity we used that A commutes with ∂_2. Therefore, $A = 1 + \sum_{i=1}^\infty a_i\partial_2^{-2i}$. Recalling Corollary 16 we also have that the derivatives with respect to ∂_x, ∂_2 and ∂_3 of a_i are 0. There exists a unique operator $B = 1 + \sum_{j=1}^\infty b_j\partial_2^{-2j}$ such that $B^2 = A$. Indeed, comparing the coefficients in front of ∂_2^{-2i} we get a system of recursion relations for b_i of the form $2b_i + \cdots = a_i$, where the dots stand for terms involving $b_1, \ldots, b_{i-1}$. Clearly, the operator B commutes with the 3 derivations and $B^\# = B$. Therefore the operator $S_a := T_a B^{-1}$ has the required properties. $\square$

4.2. **Extension of the coefficient ring $\mathcal{R}$.** It will be convenient to extended the coefficient ring $\mathcal{R}$ to $\hat{\mathcal{R}} := \mathcal{R}[\partial_x]$, that is, let us define

$$\hat{\mathcal{E}} = \hat{R}[\Lambda^{\pm 1}, \partial_2, \partial_3],$$

$$\hat{\mathcal{E}}_{(\pm)} = \hat{\mathcal{R}}[\partial_2, \partial_3]((\Lambda^{\mp 1})), \quad \hat{\mathcal{E}}_{(2)} = \hat{\mathcal{R}}[\Lambda, \Lambda^{-1}, \partial_3]((\partial_2^{-1})), \quad \hat{\mathcal{E}}_{(3)} = \hat{\mathcal{R}}[\Lambda, \Lambda^{-1}, \partial_2]((\partial_3^{-1}))$$

and

$$\hat{\mathcal{E}}^0_{(\pm)} = \hat{\mathcal{R}}((\Lambda^{\mp 1})), \quad \hat{\mathcal{E}}^0_{(2)} = \hat{\mathcal{R}}((\partial_2^{-1})), \quad \hat{\mathcal{E}}^0_{(3)} = \hat{\mathcal{R}}((\partial_3^{-1})).$$

We still have a direct sum decomposition $\hat{\mathcal{A}} = \hat{\mathcal{A}}^0 \oplus \hat{\mathcal{A}}H$ for all $\mathcal{A} \in \{\mathcal{E}_{(\pm)}, \mathcal{E}_{(2)}, \mathcal{E}_{(3)}\}$, where $\hat{\mathcal{A}}H$ is the left ideal in $\hat{\mathcal{A}}$ generated by H_1, H_2, and H_3. Therefore, we can define the projections $\pi_\alpha : \hat{\mathcal{A}}_\alpha \to \hat{\mathcal{A}}^0_\alpha$. Moreover, the natural generalization of both Proposition 8 and Proposition 9 still hold.

4.3. **The extended flows.** We have the following symmetry

$$(A_{a,k})^\# = -\partial_a A_{a,k} \partial_a^{-1}, \quad a = 2, 3. \tag{48}$$

Indeed, since $S_a^\# = \partial_a S_a^{-1} \partial_a^{-1}$ and by definition,

$$A_{a,k} = S_a \tfrac{\partial_a^{2k}}{2^k k!}(\epsilon \partial_x) S_a^{-1},$$

the identity (48) is straightforward to prove. Let us define

$$B^+_{0,l} = (B^+_{0,l,1} + B^+_{0,l,2} + B^+_{0,l,3})(\Lambda - \Lambda^{-1}),$$

where

$$B^+_{0,l,1} := \sum_{m=0}^{\infty} \left(\left(\frac{L_1^{(n-2)l}}{(n-2)^l l!} (\log L_1 - h_l) \Lambda^{-2m-1} \right)_{1,\geq 0} + \left(\Lambda^{2m+1} \left(\frac{L_1^{(n-2)l}}{(n-2)^l l!} (\log L_1 - h_l) \right)^\# \right)_{1,<0} \right),$$

$$B^+_{0,l,2} := \frac{1}{2}\, \iota_{\Lambda^{-1}} \left(\frac{L_2^{2l}}{2^l l!} H_2^{-1} (\partial_2 + q_2) \log\Big((\partial_2 + q_2)^{-1} H_2 + 1 \Big) \right)_{2,[0]},$$

and

$$B^+_{0,l,3} := \frac{1}{2}\, \iota_{\Lambda^{-1}} \left(\frac{L_3^{2l}}{2^l l!} H_3^{-1} (\partial_3 + q_3) \log\Big((\partial_3 + q_3)^{-1} H_3 - 1 \Big) \right)_{3,[0]}.$$

Note that $B_{0,k,1} = B^+_{0,k,1}(\Lambda - \Lambda^{-1})$. Recalling parts a) and b) of Proposition 9 we get that $B^+_{0,k} = \pi_+(B_{0,k})$.

We claim that the operators $B^+_{0,k}$ and $B^-_{0,k} = \pi_-(B_{0,k})$ satisfy conditions (i)–(iii) in Lemma 10 and hence the extended flows $\partial_{0,k}$ exist. We divide the argument into two parts. First, we show that conditions (i)–(iii) of Lemma 10 hold provided that the coefficient ring $\mathcal{R}$ is replaced with $\hat{\mathcal{R}}$. Second, we will prove that the coefficients of $B^\pm_{0,k}$ belong to $\mathcal{R}$, that is, the terms that involve the differential operator $\epsilon \partial_x$ cancel out.

Let us check condition (i). The identity

$$B^\#_{0,k,1} = -(\Lambda - \Lambda^{-1}) B_{0,k,1} (\Lambda - \Lambda^{-1})^{-1}$$

is obvious. We have

$$\pi_+(B_{0,k,2}) = \frac{1}{2} \iota_\Lambda \operatorname{Res}_{\partial_2} \Big(A_{2,k} \widetilde{H}_2^{-1} \partial_2^{-1} \Big) (\Lambda - \Lambda^{-1}).$$

Note that $\widetilde{H}_2 = S_2(\Lambda - 1)S_2^{-1}$ and $A_{2,k}$ commute, and that the conjugate of $A_{2,k}$ can be computed by formula (48). Therefore, the same argument used in the proof of formula (42) yields

$$(\pi_+(B_{0,k,2}))^{\#} = -(\Lambda - \Lambda^{-1})\pi_-(B_{0,k,2})\iota_\Lambda(\Lambda - \Lambda^{-1})^{-1}.$$

Similarly,

$$(\pi_+(B_{0,k,3}))^{\#} = -(\Lambda - \Lambda^{-1})\pi_-(B_{0,k,3})\iota_\Lambda(\Lambda - \Lambda^{-1})^{-1}.$$

Let us check condition (iii). It is sufficient to prove that each $B_{0,k,i}$ $(i = 1,2,3)$ satisfies condition (iii) with $B^+ = B^- = B_{0,k,i}$ and $\mathcal{R}$ replaced by $\hat{\mathcal{R}}$. For $i = 1$ the statement is obvious, because $B_{0,k,1}$ is a difference operator divisible by $\Lambda - \Lambda^{-1}$. For $i = 2$ we will use the same argument as in the proof of part c) of Lemma 13 to prove that $\pi_+(H_a B_{0,k,2}) \in \hat{\mathcal{R}}[\Lambda^{\pm 1}]$ for $a = 1,2,3$. This would imply that $H_a B_{0,k,2} - \pi_+(H_a B_{0,k,2}) \in \hat{\mathcal{E}}^0_{(+)}H \cap \hat{\mathcal{E}} = \hat{\mathcal{E}}H$, that is,

$$H_a B_{0,k,2} \in \hat{\mathcal{R}}[\Lambda^{\pm 1}] \oplus \hat{\mathcal{E}}H, \tag{49}$$

so the projections π_+ and π_- of $H_a B_{0,k,2}$ coincide with the projection on the first factor of the above direct sum (49). Let us see how the argument from the proof of part c) of Lemma 13 is modified in order to prove that $\pi_+(H_a B_{0,k,2}) \in \hat{\mathcal{R}}[\Lambda^{\pm 1}]$ for $a = 2$. The modification for the other two cases $a = 1$ and $a = 3$ is analogous. To begin with, we have

$$\frac{1}{2}\pi_+(H_2(A_{2,k})_{2,[0]}(\Lambda^{-1}+1)) = \frac{1}{2}(\Lambda - 1)\Big(\partial_2(A_{2,k})_{2,[0]}(\Lambda^{-1}+1) + [(A_{2,k})_{2,[0]}(\Lambda^{-1}+1), Q_2]\Big). \tag{50}$$

Note that

$$H_2(A_{2,k})_{2,>0} = (H_2 A_{2,k})_{2,\geq 0} - H_2 \cdot (A_{2,k})_{2,[0]}$$

Recalling part a) of Proposition 9 and the expansion $\widetilde{H}_2^{-1} = (\Lambda - 1)^{-1} + 2Q_2(\Lambda - \Lambda^{-1})^{-1}\partial_2^{-1} + \cdots$, we get that

$$\begin{aligned}\pi_+\left((A_{2,k})_{2,>0}\right) = {} & \frac{1}{2}\iota_{\Lambda^{-1}}\Big(H_2 A_{2,k}\widetilde{H}_2^{-1}\Big)_{2,0}(\Lambda - \Lambda^{-1}) + \\ & + \frac{1}{2}\Big(q_2(\Lambda+1)(A_{2,k})_{2,[0]} - \partial_2(A_{2,k})_{2,[0]}\Big)(\Lambda^{-1}+1) - (\Lambda - 1)(A_{2,k})_{2,[0]}Q_2.\end{aligned}$$

Adding up the above formula and (50), we get that

$$\pi_+(H_2 B_{0,k,2}) = \frac{1}{2}\iota_{\Lambda^{-1}}\Big(H_2 A_{2,k}\widetilde{H}_2^{-1}\Big)_{2,0}(\Lambda - \Lambda^{-1}) + \frac{1}{2}(\Lambda - 1)(A_{2,k})_{2,[0]}(\Lambda^{-1} - 1)Q_2.$$

The second term on the RHS is a difference operator, because $(\Lambda^{-1} - 1)Q_2 = -q_2[-1](\Lambda^{-1}+1)$. The first term is also a difference operator, because $A_{2,k}$ and $\widetilde{H}_2$ commute and $H_2\widetilde{H}_2^{-1} = \partial_2 + q_2$.

The proof that $B_{0,k,3}$ satisfies condition (iii) is completely analogous. Let us move to the last step, i.e., we will prove that $B_{0,k,i}$ $(i = 1,2,3)$ satisfy condition (ii) with $B^+ = B^- = B_{0,k,i}$ and $\mathcal{R}$ replaced by $\hat{\mathcal{R}}$. Again, this fact is obvious for $i = 1$ and the arguments for $i = 2$ and $i = 3$ are identical, so let us give the details only for $i=2$. We follow the same idea is in the proof of part d of Lemma 13. Since $[B_{0,k,2},\mathcal{L}]\in\hat{\mathcal{E}}$, it is sufficient to prove that the projection $\pi_+([B_{0,k,2},\mathcal{L}])\in\hat{\mathcal{R}}[\Lambda,\Lambda^{-1}]$. We have

$$[B_{0,k,2},\mathcal{L}]=([A_{2,k},\mathcal{L}])_{2,\geq 0}+\frac{1}{2}[(A_{2,k})_{2,[0]}(\Lambda^{-1}-1),\mathcal{L}]+\partial_2(\mathrm{res}_{\partial_2}(A_{k,2})). \tag{51}$$

The π_+-projections of the 3rd term is difference operators, so we have to check that the projection of the 1st and the 2nd terms add up to a difference operator. Let us decompose $\mathcal{L}$ as in (41). Since $A_{2,k}$-commutes with $\widetilde{H}_2$ and $\widetilde{H}_1$ (see the conjugation formulas for S_2), we get

$$[A_{2,k},\mathcal{L}]_{2,\geq 0}=([A_{2,k},A]\widetilde{H}_2)_{2,\geq 0}-\Big(\partial_3(A_{2,k}\partial_2^{-1})H_1\Big)_{2,\geq 0}.$$

Note that

$$\pi_+(([A_{2,k},A]\widetilde{H}_2)_{2,>0})=\frac{1}{2}\iota_{\Lambda^{-1}}\Big(([A_{2,k},A]\widetilde{H}_2)_{2,>0}\widetilde{H}_2^{-1}\Big)_{2,[0]}(\Lambda-\Lambda^{-1})$$

is a difference operator, because in the above formula we can replace $([A_{2,k},A]\widetilde{H}_2)_{2,>0}$ by $([A_{2,k},A]\widetilde{H}_2-([A_{2,k},A]\widetilde{H}_2)_{2,[0]}$, where only the second term could have a contribution which is not a difference operator. However, this contribution involves only the 0-th order term of $\widetilde{H}_2^{-1}(\Lambda-\Lambda^{-1})$ which is $\Lambda^{-1}+1$, that is, a difference operator. Clearly, $\pi_+(([A_{2,k},A]\widetilde{H}_2)_{2,[0]})$ is a difference operator. Note that

$$\pi_+\Big(\partial_3(A_{2,k}\partial_2^{-1})H_1\Big)_{2,\geq 0}=\partial_3(A_{2,k})_{2,[0]}Q_3.$$

Recalling formula (51), we get that up to terms that are difference operators, the projection $\pi_+([B_{0,k,2},\mathcal{L}])$ coincides with

$$-\partial_3(A_{2,k})_{2,[0]}Q_3+\frac{1}{4}\pi_+([(A_{2,k})_{2,[0]},\partial_3^2](\Lambda^{-1}-1))=-\frac{1}{2}\partial_3(A_{2,k})_{2,[0]}(\Lambda^{-1}+1)Q_3.$$

However, $(\Lambda^{-1}+1)Q_3=q_3[-1](1-\Lambda^{-1})$ is a difference operator, so $\pi_+([B_{0,k,2},\mathcal{L}])$ is also a difference operator.

Let us prove that the coefficients of $B^{\pm}_{0,k}$ belong to $\mathcal{R}$. We will consider only the case $B^+_{0,k}$. The other case is similar. Since $\pi_+([A_{1,k},\mathcal{L}])=\pi_+(H_aA^+_{1,k})=0$, it will be sufficient to prove that the coefficients of the operator $B_{0,k}-A^+_{1,k}\in\hat{\mathcal{E}}_{(+)}$ belong to $\mathcal{R}$. The coefficients of the operator $B_{0,k}-A^+_{1,k}$ are at most 1st order differential operators in $\epsilon\partial_x$. The vanishing of the coefficient in front of $\epsilon\partial_x$ is equivalent to the following identity

$$\sum_{m=0}^{\infty}\Big(\frac{L_1^{(n-2)k}}{(n-2)^k}\Lambda^{-2m-1}+\Lambda^{2m+1}\frac{(L_1^{\#})^{(n-2)k}}{(n-2)^k}\Big)_{1,<0}(\Lambda-\Lambda^{-1})=\frac{1}{2}\iota_{\Lambda^{-1}}\Big(\Big(\frac{L_2^{2k}}{2^k}\widetilde{H}_2^{-1}\Big)_{2,[0]}+\Big(\frac{L_3^{2k}}{2^k}\widetilde{H}_3^{-1}\Big)_{3,[0]}\Big)(\Lambda-\Lambda^{-1}).$$

Proposition 17. *If $k\geq 0$ is an integer, then the following identity holds*

$$\sum_{m=0}^{\infty}\Big(\frac{L_1^{(n-2)k}}{(n-2)^k}\Lambda^{-2m-1}+\Lambda^{2m+1}\frac{(L_1^{\#})^{(n-2)k}}{(n-2)^k}\Big)=\frac{1}{2}\sum_{m\in\mathbb{Z}}\Big(\Big(\frac{L_2^{2k}}{2^k}S_2\Lambda^mS_2^{-1}\Big)_{2,[0]}-\Big(\frac{L_3^{2k}}{2^k}S_3(-\Lambda)^mS_3^{-1}\Big)_{3,[0]}\Big).$$

The above proposition yields the identity that we want to prove. Indeed, since

$$\sum_{m<0} S_2 \Lambda^m S_2^{-1} = \iota_{\Lambda^{-1}} S_2 (\Lambda - 1)^{-1} S_2^{-1} = \iota_{\Lambda^{-1}} \widetilde{H}_2^{-1}$$

and

$$-\sum_{m<0} S_3 (-\Lambda)^m S_3^{-1} = \iota_{\Lambda^{-1}} S_3 (\Lambda + 1)^{-1} S_3^{-1} = \iota_{\Lambda^{-1}} \widetilde{H}_3^{-1},$$

we just have to remove the terms that involve non-negative powers of Λ and multiply both sides by $(\Lambda - \Lambda^{-1})$.

4.4. **Proof of Proposition 17.** We are going to prove a more general statement. Namely, put

$$\begin{aligned}
M_{+,k}^{r,s} &:= \sum_{m=0}^{\infty} S_1 \cdot \mu_{1,k} \cdot (\partial_2^r \partial_2^s (S_1^{-1})) \cdot \Lambda^{-2m-1}, \\
M_{-,k}^{r,s} &:= \sum_{m=0}^{\infty} \Lambda^{2m+1} \cdot \Big((\partial_2^r \partial_3^s (S_1)) \cdot \mu_{1,k} \cdot S_1^{-1} \Big)^{\#}, \\
M_{2,k}^{r,s} &:= \Big(S_2 \cdot \mu_{2,k} \cdot \partial_3^s (S_2^{-1}) \cdot (-\partial_2)^r \Big)_{2,[0]}, \\
M_{3,k}^{r,s} &:= \Big(S_3 \cdot \mu_{3,k} \cdot \partial_2^r (S_3^{-1}) \cdot (-\partial_3)^s \Big)_{3,[0]},
\end{aligned}$$

where

$$\mu_{1,k} := \tfrac{\Lambda^{(n-2)k}}{(n-2)^k}, \quad \mu_{2,k} := \sum_{m \in \mathbb{Z}} \tfrac{\partial_2^{2k}}{2^k} \Lambda^m, \quad \mu_{3,k} := -\sum_{m \in \mathbb{Z}} \tfrac{\partial_3^{2k}}{2^k} (-\Lambda)^m.$$

We are going to prove the following identity

$$M_{+,k}^{r,s} + M_{-,k}^{r,s} = \frac{1}{2} \Big(M_{2,k}^{r,s} + M_{3,k}^{r,s} \Big). \tag{52}$$

The identity stated in the proposition is the case when $r = s = 0$. Let us introduce the following notation: If $P(\Lambda, \partial_2, \partial_3) \in \mathcal{E}$ is a differential-difference operator, then we denote by $\vec{P}$ the operator, acting on $\mathcal{E}_{(\pm)}$, such that Λ acts by multiplication, while ∂_2 and ∂_3 act by derivations, e.g.,

$$\vec{H}_2(M) = (\Lambda - 1) \cdot \partial_2(M) - q_2(\Lambda + 1) \cdot M = H_2 \cdot M - (\Lambda - 1) \cdot M \cdot \partial_2.$$

Lemma 18. *Suppose that $i \in \{+, -, 2, 3\}$ and $k, r, s \geq 0$ are arbitrary integers.*

a) The following formulas hold:

$$\vec{H}_2(M_{i,k}^{r,s}) = (\Lambda - 1) M_{i,k}^{r+1,s}, \quad \vec{H}_3(M_{i,k}^{r,s}) = (\Lambda + 1) M_{i,k}^{r,s+1}.$$

b) The following formulas hold:

$$\vec{\mathcal{L}}(M_{i,k}^{r,s}) = M_{i,k+1}^{r,s} + \partial_2(M_{i,k}^{r+1,s}) + \partial_3(M_{i,k}^{r,s+1}) - \frac{1}{2} M_{i,k}^{r+2,s} - \frac{1}{2} M_{i,k}^{r,s+2}.$$

Proof. a) Let us prove the formulas for $\vec{H}_2(M^{r,s}_{+,k})$ and $\vec{H}_2(M^{r,s}_{3,k})$. The argument in all other cases is similar. We have

$$\vec{H}_2(M^{r,s}_{+,k}) = H_2 \cdot M^{r,s}_{+,k} - (\Lambda - 1)M^{r,s}_{+,k} \cdot \partial_2.$$

On the other hand, $H_2 S_1 = (\Lambda - 1)S_1\partial_2$. Therefore, the above identity is transformed into

$$\vec{H}_2(M^{r,s}_{+,k}) = \sum_{m=0}^{\infty} (\Lambda - 1)S_1\mu_{1,k}(\partial_2 \cdot (\partial_2^r \partial_3^s (S_1^{-1})) - (\partial_2^r \partial_3^s (S_1^{-1})) \cdot \partial_2) = (\Lambda - 1)M^{r+1,s}_{+,k}.$$

We have

$$\vec{H}_2(M^{r,s}_{3,k}) = \Big(H_2 S_3 \mu_{3,k} \partial_2^r (S_2^{-1})(-\partial_3)^s - (\Lambda - 1)S_3\mu_{3,k}\partial_2^r(S_2^{-1})(-\partial_3)^s\partial_2\Big)_{3,[0]}.$$

Recall that $H_2 = (\Lambda - 1)\partial_3^{-1} H_1 - \partial_3^{-1} q_2 H_3$. Recalling also the conjugation formulas for S_3, we get $H_3 S_3 = (\partial_3 + q_3)S_3(\Lambda + 1)$ and $\partial_3^{-1} H_1 S_3 = S_3\partial_2$. Finally, note that $(\Lambda + 1)\mu_{3,k} = 0$. Putting these facts together we get

$$\vec{H}_2(M^{r,s}_{3,k}) = (\Lambda - 1)\Big(S_3 \cdot \mu_{3,k} \cdot \partial_2 \cdot \partial_2^r(S_2^{-1}) \cdot (-\partial_3)^s - S_3 \cdot \mu_{3,k} \cdot \partial_2^r(S_2^{-1}) \cdot (-\partial_3)^s \cdot \partial_2\Big)_{3,[0]}.$$

The RHS of the above formula coincides with $M^{r+1,s}_{2,k}$.

b) Let us prove the formula for $\vec{\mathcal{L}}(M^{r,s}_{2,k})$. The remaining ones are proved with the same technique. The action of $\vec{\mathcal{L}}$ can be computed as follows:

$$\vec{\mathcal{L}}(M) = \Big(\mathcal{L} - \tfrac{\partial_2^2}{2} - \tfrac{\partial_3^2}{2}\Big) \cdot M + \tfrac{\partial_2^2(M)}{2} + \tfrac{\partial_3^2(M)}{2} = \mathcal{L} \cdot M - \sum_{a=2,3} \Big(\partial_a \cdot M \cdot \partial_a - M \cdot \tfrac{\partial_a^2}{2}\Big) \tag{53}$$

Let us apply formula (53) to $M := S_2 \cdot \mu_{2,k} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r$. The operator $\mathcal{L}$ has the following decomposition (see formula (41)):

$$\mathcal{L} = \tfrac{L_2^2}{2} + A\widetilde{H}_2 + \partial_3 \cdot \widetilde{H}_1 - \tfrac{\widetilde{H}_1^2}{2}.$$

Recalling the conjugation formulas for S_2, we get

$$\tfrac{L_2^2}{2} \cdot M = S_2 \cdot \mu_{2,k+1} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r,$$

$\widetilde{H}_2 \cdot M = 0$, because $\widetilde{H}_2 S_2 = (\partial_2 + q_2)S_2(\Lambda - 1)$ and $(\Lambda - 1)\mu_{2,k} = 0$, and finally,

$$\Big(\partial_3 \cdot \widetilde{H}_1 - \tfrac{\widetilde{H}_1^2}{2}\Big) \cdot M = \Big(\partial_3 \cdot S_2 \cdot \partial_3 - S_2 \cdot \tfrac{\partial_3^2}{2}\Big) \cdot \mu_{2,k} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r$$

Substituting these formulas in (53), we get

$$\vec{\mathcal{L}}(M) = S_2 \cdot \mu_{2,k+1} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r + \Big(\partial_3 \cdot S_2 \cdot \partial_3 - S_2 \cdot \tfrac{\partial_3^2}{2}\Big) \cdot \mu_{2,k} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r - \sum_{a=2,3} \Big(\partial_a \cdot M \cdot \partial_a - M \cdot \tfrac{\partial_a^2}{2}\Big).$$

Substituting also the formula for M, after a short computation we get the following formula:

$$\vec{\mathcal{L}}(M) = S_2 \cdot \mu_{2,k+1} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^r$$
$$+ \partial_3\Big(S_2 \cdot \mu_{2,k} \cdot \partial_3^{s+1}(S_2^{-1})(-\partial_2)^r\Big) + \partial_2\Big(S_2 \cdot \mu_{2,k} \cdot \partial_3^s(S_2^{-1})(-\partial_2)^{r+1}\Big) +$$

$$-\frac{1}{2}\Big(S_2\cdot\mu_{2,k}\cdot\partial_3^{s+2}(S_2^{-1})(-\partial_2)^r\Big)-\frac{1}{2}\Big(S_2\cdot\mu_{2,k}\cdot\partial_3^{s}(S_2^{-1})(-\partial_2)^{r+2}\Big).$$

Since $\vec{\mathcal{L}}(M^{r,s}_{2,k})=(\vec{\mathcal{L}}(M))_{2,[0]}$, we see that the 1st line of the RHS of the above formula contributes $M^{r,s}_{2,k+1}$, the 2nd line contributes $\partial_3(M^{r,s+1}_{2,k})+\partial_2(M^{r+1,s}_{2,k})$ and the 3rd line contributes $-\frac{1}{2}M^{r,s+2}_{2,k}-\frac{1}{2}M^{r+2,s}_{2,k}$. This is exactly the formula that we wanted to prove. □

We prove (52) by induction on the lexicographical order of the tripple (k,r,s). If $k=r=s=0$, then the identity is obvious. Let

$$P_{k,r,s}:=M^{r,s}_{+,k}+M^{r,s}_{-,k}-\frac{1}{2}\Big(M^{r,s}_{2,k}+M^{r,s}_{3,k}.$$

Note that the operator

$$P_k:=\sum_{r,s=0}^{\infty}P_{k,r,s}(-1)^r\partial_2^{-r-1}(-1)^s\partial_3^{-s-1}$$

coincides with

$$\sum_{m=0}^{\infty}\Big(S_1\mu_{1,k}\partial_2^{-1}\partial_3^{-1}S_1^{-1}\Lambda^{-2m-1}+\Lambda^{2m+1}\Big(S_1\mu_{1,k}\partial_2^{-1}\partial_3^{-1}S_1^{-1}\Big)^{\#}\Big)+$$
$$-\frac{1}{2}\Big(S_2\mu_{2,k}\partial_3^{-1}S_2^{-1}\partial_2^{-1}\Big)_{2,<0}-\frac{1}{2}\Big(S_3\mu_{3,k}\partial_2^{-1}S_3^{-1}\partial_3^{-1}\Big)_{3,<0}.$$

This formula implies that $P_k^{\#}=P_k$. Let us prove that $P_0=0$. If $k=0$, then let us check that (52) holds if $r=0$ or $s=0$. In terms of the operator P_0, this statement is equivalent to $\operatorname{res}_{\partial_2}(P_0)=\operatorname{res}_{\partial_3}(P_0)=0$. We have

$$\operatorname{res}_{\partial_2}(P_0)_{1,<0}=S_1\partial_3^{-1}S_1^{-1}(\Lambda-\Lambda^{-1})^{-1}-\frac{1}{2}\Big(\partial_3^{-1}(\Lambda-1)^{-1}+S_3(\Lambda+1)^{-1}S_3^{-1}\partial_3^{-1}\Big),$$

where the rational expressions in Λ should be expanded in the powers of Λ^{-1}. Recalling the conjugation formulas for S_1 and S_3, we get

$$(\partial_3-Q_3)^{-1}(\Lambda-\Lambda^{-1})^{-1}-\frac{1}{2}\Big(\partial_3^{-1}(\Lambda-1)^{-1}+H_3^{-1}(\partial_3+q_3)\partial_3^{-1}\Big)=0.$$

Since $P_0^{\#}=P_0$, the residue satisfies $(\operatorname{res}_{\partial_2}(P_0))^{\#}=-\operatorname{res}_{\partial_2}(P_0)$. Therefore, $\operatorname{res}_{\partial_2}(P_0)=0$ as claimed. The vanishing of the other residue is proved in a similar way. Suppose now that (r_0,s_0) is a lexicographically minimal pair such that $P_{0,r_0,s_0}\neq 0$. Then $r_0>0$ and $s_0>0$ by what we have just proved. Recalling part a) of Lemma 18, we get

$$(\Lambda-1)P_{0,r_0,s_0}=\vec{H}_2(P_{0,r_0-1,s_0})=0$$

and

$$(\Lambda+1)P_{0,r_0,s_0}=\vec{H}_3(P_{0,r_0,s_0-1})=0.$$

The first identity implies that $P_{0,r_0,s_0}=(\sum_{m\in\mathbb{Z}}\Lambda^m)a$ for some $a\in\mathcal{R}$, while the second one implies that $P_{0,r_0,s_0}=(\sum_{m\in\mathbb{Z}}(-\Lambda)^m)b$ for some $b\in\mathcal{R}$. Comparing the coefficients in front of Λ^0 and Λ^1, we get respectively that $a = b$ and $a = -b$. This is possible only if $a = b = 0$ – contradiction. Therefore, $P_0 = 0$, that is (52) holds for $k = 0$ and for all $r,s \in \mathbb{Z}_{\geq 0}$. Recalling part b) of Lemma 18, we get that if (52) holds for some k and for all $r,s \in \mathbb{Z}_{\geq 0}$, then it must hold for $k+1$ and for all $r,s \in \mathbb{Z}_{\geq 0}$. This completes the inductive step, so formula (52) is proved.

5. Commutativity of flows

The main goal in this section is to prove part b) of Theorem 4.

5.1. **Evolution of the dressing operators.**

Proposition 19. *The projections of $\mathcal{L}$ satisfy the following differential equations:*

$$\partial_{i,k}\pi_b(\mathcal{L}) = \pi_b([B_{i,k}, \pi_b(\mathcal{L})]) = [\pi_b(B_{i,k}), \pi_b(\mathcal{L})],$$

where $i = 0,1,2,3$, $b = \pm,2,3$ and k is odd when $i = 2,3$.

Proof. First, let us compute the derivatives of the operator H_1. Using the relation $H_1 = \frac{1}{2}(\partial_2 - q_2)\cdot H_3 - \frac{1}{2}(\partial_3 - q_3)\cdot H_2$, we get $\partial_{i,k}H_1 \in \frac{1}{2}(\partial_2 - q_2)\cdot\partial_{i,k}H_3 - \frac{1}{2}(\partial_3 - q_3)\cdot\partial_{i,k}H_2 + \mathcal{E}H = -H_1B_{i,k} + \mathcal{E}H$. Therefore

$$\partial_{i,k}H_1 + H_1B_{i,k} \in \mathcal{E}H.$$

The identity $\pi_b([B_{i,k}, \pi_b(\mathcal{L})]) = [\pi_b(B_{i,k}), \pi_b(\mathcal{L})]$ follows from the definition of the projection π_b and part b) of Proposition 1.

By definition, the projection $\pi_b(\mathcal{L}) = \mathcal{L} + \sum_{i=1}^{3} A_iH_i$, where $A_i \in \mathcal{E}_{(b)}$. Recalling Theorem 4, we get

$$\begin{aligned}
\partial_{i,k}\pi_b(\mathcal{L}) &\in [B_{i,k}, \mathcal{L}] - \sum_{i=1}^{3} A_iH_iB_{i,k} + \mathcal{E}_{(b)}H \\
&= [B_{i,k}, \pi_b(\mathcal{L})] - [B_{i,k}, \sum_{i=1}^{3} A_iH_i] - \sum_{i=1}^{3} A_iH_iB_{i,k} + \mathcal{E}_{(b)}H \\
&= [\pi_b(B_{i,k}), \pi_b(\mathcal{L})] + \mathcal{E}_{(b)}H.
\end{aligned}$$

Therefore, $\partial_{i,k}\pi_b(\mathcal{L}) - [\pi_b(B_{i,k}), \pi_b(\mathcal{L})] \in \mathcal{E}^0_{(b)} \cap \mathcal{E}_{(b)}H = \{0\}$. □

Furthermore, recalling the definitions (4), we get the following corollary:

Corollary 20. *The operators $L_1^{\pm}$ and L_a ($a = 2,3$) satisfy the following differential equations:*

$$\begin{aligned}
\partial_{i,k}L_1^{\pm} &= \pi_{\pm}([B_{i,k}, L_1^{\pm}]) = [\pi_{\pm}(B_{i,k}), L_1^{\pm}], \\
\partial_{i,k}L_a &= \pi_a([B_{i,k}, L_a]) = [\pi_a(B_{i,k}), L_a],
\end{aligned}$$

where $L_1^+ := L_1$ and L_1^- is defined by formula (40). □

We would like to extend the derivations $\partial_{i,k}$ of $\mathcal{R}$ to derivations of $\mathcal{R}_i$ ($i = 1,2,3$), that is, the rings that contain the coefficients of the dressing operators. Recall that the Lax operators can be expressed in terms of the dressing operators as follows: $S_1 \cdot \Lambda \cdot S_1^{-1} = L_1^+$ and $S_a \cdot \partial_a \cdot S_a^{-1} = L_a$ ($a = 2,3$). Comparing these formulas with the formulas for the derivatives of the Lax operators (see Corollary 20), we get that a natural choice for the extension is given by the following formulas:

a) The derivation $\partial_{1,k}$ ($k \geq 1$)

$$\begin{aligned} \partial_{1,k} S_1 &= (-(L_1^+)^k + B_{1,k}) \cdot S_1, \\ \partial_{1,k} S_2 &= \pi_2(B_{1,k}) \cdot S_2, \\ \partial_{1,k} S_3 &= \pi_3(B_{1,k}) \cdot S_3. \end{aligned}$$

b) The derivation $\partial_{a,2l+1}$ ($a = 2,3$, $l \geq 0$)

$$\begin{aligned} \partial_{a,2l+1} S_1 &= \pi_+(B_{a,2l+1}) \cdot S_1, \\ \partial_{a,2l+1} S_a &= (-L_a^{2l+1} + B_{a,2l+1}) \cdot S_a, \\ \partial_{a,2l+1} S_b &= \pi_b(B_{a,2l+1}) \cdot S_b, \end{aligned}$$

where $b = \{2,3\} \setminus \{a\}$.

c) The derivation $\partial_{0,k}$ ($k \geq 1$)

$$\begin{aligned} \partial_{0,k} S_1 &= (-A_{1,k}^+ + \pi_+(B_{0,k})) \cdot S_1, \\ \partial_{0,k} S_2 &= (-A_{2,k} + \pi_2(B_{0,k})) \cdot S_2, \\ \partial_{0,k} S_3 &= (-A_{3,k} + \pi_3(B_{0,k})) \cdot S_3. \end{aligned}$$

Recall that the rings $\mathcal{R}_i$ ($i = 1,2,3$) are equipped with the derivations $\epsilon\partial_x, \partial_2, \partial_3$ and the action of the translation operator $\Lambda = e^{\epsilon\partial_x}$.

Proposition 21. *The following commutators in $\mathcal{R}_i$ ($i = 1,2,3$) vanish: $[\partial_{j,k}, \epsilon\partial_x] = [\partial_{j,k}, \partial_2] = [\partial_{j,k}, \partial_3]$, that is, the extended derivation $\partial_{j,k}$ commutes with $\epsilon\partial_x, \partial_2$, and ∂_3.*

Proof. The vanishing of the commutators $[\partial_{j,k}, \partial_x] = [\partial_{j,k}, \partial_2] = [\partial_{j,k}, \partial_3] = 0$ in $\mathcal{R}_1$ can be proved in the same way as in Proposition 14, a). We only prove the vanishing of the commutators in $\mathcal{R}_2$. The proof of the vanishing in $\mathcal{R}_3$ is similar.

Let us prove that $[\partial_{j,k}, \partial_2] = 0$ in $\mathcal{R}_2$. Put $L_2 = \sum_{i=0}^{+\infty} b_{2,i} \partial_2^{1-i}$ and $S_2 = \sum_{j=0}^{+\infty} \psi_{2,j} \partial_2^{-j}$, where $b_{2,0} = 1$, $b_{2,1} = 0$ and $\psi_{2,0} = 1$. Applying $\partial_{j,k}$ to the relation $L_2 S_2 = S_2 \partial_2$, we get

$$\partial_{j,k}(L_2) S_2 + \sum_{l=0}^{+\infty} \sum_{i=0}^{l} \sum_{p=0}^{l-i} b_{2,i} \binom{1-i}{l-i-p} \partial_{j,k}\left(\psi_{2,p}^{(l-i-p)}\right) \partial_2^{1-l} = \partial_{j,k}(S_2) \cdot \partial_2.$$

Here $\psi_{2,p}^{(s)} = \partial_2^s(\psi_{2,p})$. Since $\partial_{j,k}L_2 = [\pi_2(\tilde{B}_{j,k}), L_2)]$ and $\partial_{j,k}(S_2) = \pi_2(\tilde{B}_{j,k})S_2$, where

$$\tilde{B}_{j,k} = \begin{cases} B_{j,k}, & j \neq 2 \text{ and } j \neq 0; \\ -L_2^k + B_{j,k}, & j = 2 \text{ and } j \neq 0; \\ -A_{2,k} + B_{0,k}, & j = 0, \end{cases}$$

we get

$$\sum_{i=0}^{l}\sum_{p=0}^{l-i} b_{2,i}\binom{1-i}{l-i-p} \cdot \partial_{j,k}\left(\psi_{2,p}^{(l-i-p)}\right) = \sum_{i=0}^{l}\sum_{p=0}^{l-i} b_{2,i}\binom{1-i}{l-i-p}\left(\partial_{j,k}\psi_{2,p}\right)^{(l-i-p)}. \tag{54}$$

Formula (54) for $l = 2$ yields $\partial_{j,k}(\psi_{2,1}^{(1)}) = (\partial_{j,k}\psi_{2,1})^{(1)}$. Arguing by induction on l we get $\partial_{j,k}\partial_2(\psi_{2,l}) = \partial_2\partial_{j,k}(\psi_{2,l})$. Thus $[\partial_{j,k}, \partial_2] = 0$ in $\mathcal{R}_2$.

Let us prove that $[\partial_{j,k}, \Lambda] = 0$ and $[\partial_{j,k}, \partial_3] = 0$ in $\mathcal{R}_2$. For brevity, put $A_2 = (\partial_2 - q_2)^{-1} \cdot (\partial_2 + q_2)$ and $A_2 = \Lambda + A \cdot H_2$, where $A = -(\partial_2 - q_2)^{-1} \in \mathcal{E}_{(2)}$. Recalling Proposition 15 and using that $[\partial_{j,k}, \partial_2] = [\Lambda, \partial_2] = 0$, we get

$$[\partial_{j,k}, \Lambda](S_2) = \left(\partial_{j,k}(A_2) + A_2 \cdot \pi_2(\tilde{B}_{j,k}) - \pi_2(\tilde{B}_{j,k})[1] \cdot A_2\right) \cdot S_2$$

Therefore, we only need to prove that $\partial_{j,k}(A_2) + A_2 \cdot \pi_2(\tilde{B}_{j,k}) - \pi_2(\tilde{B}_{j,k})[1] \cdot A_2 = 0$. In fact, since $\partial_{j,k}H_2 = -H_2B_{j,k} + \mathcal{E}H$, we have

$$\begin{aligned} \partial_{j,k}(A_2) &+ A_2 \cdot \pi_2(\tilde{B}_{j,k}) - \pi_2(\tilde{B}_{j,k})[1] \cdot A_2 \\ &= A \cdot \partial_{j,k}H_2 + \Lambda \cdot \pi_2(\tilde{B}_{j,k}) + A \cdot H_2 \cdot \pi_2(\tilde{B}_{j,k}) \\ &\quad - \pi_2(\tilde{B}_{j,k})[1] \cdot \Lambda + \mathcal{E}_{(2)}H \in \mathcal{E}_{(2)}H \cap \mathcal{E}_{(2)}^0 = \{0\}. \end{aligned}$$

The vanishing of $[\partial_{j,k}, \partial_3]$ in $\mathcal{R}_2$ is proved similarly. Finally, let us prove that $[\partial_{j,k}, \epsilon\partial_x] = 0$ in $\mathcal{R}_2$. Put $c_m = S_2 \cdot [\partial_{j,k}, (\epsilon\partial_x)^m](S_2)$. Since $[\partial_{j,k}, \Lambda] = 0$ in $\mathcal{R}_2$, we have $\sum_{m=1}^{+\infty} c_m/m! = 0$. Using that $[\partial_{j,k}, \epsilon\partial_x] = 0$ in $\mathcal{R}$, we get the following recursion relations: $c_{m+1} = \tilde{\ell}_2 c_m + c_1\tilde{\ell}_2^{(m)} + \epsilon\partial_x c_m$. The rest of the proof is the same as the proof of $[\partial_2, \epsilon\partial_x] = 0$ in $\mathcal{R}_2$ (see Proposition 15). $\square$

Remark 22. Proposition 21 is very important, because it implies that the Leibniz rule holds for pseudo-differential-difference operators with coefficients in $\mathcal{R}_i$. In particular, the following formula holds:

$$\partial_{j,k}\left(S_i \cdot \Lambda^{m_1}\partial_2^{m_2}\partial_3^{m_3} \cdot S_i^{-1}\right) = \partial_{j,k}(S_i) \cdot \Lambda^{m_1}\partial_2^{m_2}\partial_3^{m_3} \cdot S_i^{-1} + S_i \cdot \Lambda^{m_1}\partial_2^{m_2}\partial_3^{m_3} \cdot \partial_{j,k}(S_i^{-1}). \quad \square$$

We will use quite frequently the following dressing formula: $L_1^- = (S_1^-)^{\#}\Lambda^{-1}((S_1^-)^{\#})^{-1}$, where $S_1^- := (\Lambda - \Lambda^{-1})S_1^{-1}\iota_{\Lambda^{-1}}(\Lambda - \Lambda^{-1})^{-1}$. Using the formulas for the derivatives of S_1, we get that the following formulas for the derivatives of the dressing operator S_1^-:

$$\partial_{1,k}(S_1^-)^{\#} = ((L_1^-)^k + B_{1,k}) \cdot (S_1^-)^{\#},$$

$$\partial_{a,2l+1}(S_1^-)^{\#} = \pi_-(B_{a,2l+1}) \cdot (S_1^-)^{\#},$$

$$\partial_{0,k}(S_1^-)^{\#} = (A_{1,k}^- + \pi_-(B_{0,k}))\cdot(S_1^-)^{\#},$$

where $k \geq 1$ and $a = 2,3$. The operator series $A_{1,l}^{\pm}$ and $A_{a,l}$ (see Section 1.4) can be expressed in terms of the dressing operators, which gives immediately the following corollary:

Corollary 23. *The operator series $A_{1,l}^{\pm}$ and $A_{a,l}$ satisfy the following differential equations:*

$$\partial_{i,k}A_{1,l}^{\pm} = \pi_{\pm}([B_{i,k}, A_{1,l}^{\pm}]) = [\pi_{\pm}(B_{i,k}), A_{1,l}^{\pm}],$$
$$\partial_{i,k}A_{a,l} = \pi_a([B_{i,k}, A_{a,l}]) = [\pi_a(B_{i,k}), A_{a,l}]. \quad \square$$

In the next two Lemmas we derive formulas for the operator series $B_{1,k}$ and $B_{0,k}$, which will be needed for the proof of Theorem 4, b).

Lemma 24. *The following formula holds:*

$$B_{1,k} = (L_1^+)^k_{1,\geq 1} - (L_1^-)^k_{1,\leq -1} - \left((L_1^+)^k_{1,\geq 1} - (L_1^-)^k_{1,\leq -1}\right)\Big|_{\Lambda=1}$$

Proof. Recall the definition of $B_{1,k}$ from Section 1.4. Note that

$$(B_{1,k})_{1,\geq 1} = \left(\left((L_1^+)^k_{1,\geq 1}\sum_{m=0}^{+\infty}\Lambda^{-2m-1}\right)_{1,\geq 0}\cdot(\Lambda - \Lambda^{-1})\right)_{1,\geq 1}$$

Let us rewrite the term $\left((L_1^+)^k_{1,\geq 1}\sum_{m=0}^{+\infty}\Lambda^{-2m-1}\right)_{1,\geq 0}$ in the form $(\quad) - (\quad)_{1,<0}$. Since $\left(A_{1,<0}\cdot(\Lambda - \Lambda^{-1})\right)_{1,\geq 1} =$ 0 for any operator A, we get $(B_{1,k})_{1,\geq 1} = (L_1^+)^k_{1,\geq 1}$. Similarly, using the conjugation relation

$$\left((L_1^+)^k\sum_{m=0}^{+\infty}\Lambda^{-2m-1}\right)^{\#} = \left((L_1^-)^k\sum_{m=0}^{+\infty}\Lambda^{2m+1}\right),$$

we get $(B_{1,k})_{1,\leq -1} = -(L_1^-)^k_{1,\leq -1}$. Finally, since $B_{1,k}\big|_{\Lambda=1} = 0$, we get $\left((L_1^+)^k_{1,\geq 1} - (L_1^-)^k_{1,\leq -1}\right)\Big|_{\Lambda=1} + (B_{1,k})_{1,[0]} =$ 0. Solving for $(B_{1,k})_{1,[0]}$, we get that $B_{1,k} = (B_{1,k})_{1,\geq 1} + (B_{1,k})_{1,[0]} + (B_{1,k})_{1,\leq -1}$ coincides with the RHS of the formula that we wanted to prove. $\square$

Lemma 25. *a) The following formula holds:*

$$B_{0,k} = (A_{1,k}^+)_{1,\geq 1} - (A_{1,k}^-)_{1,\leq -1} + (A_{2,k})_{2,\geq 1} + (A_{3,k})_{3,\geq 1} + (B_{0,k})_{[0]}, \tag{55}$$

where

$$(B_{0,k})_{[0]} = -\left((A_{1,k}^+)_{1,\geq 1} - (A_{1,k}^-)_{1,\leq -1}\right)\Big|_{\Lambda=1} + (A_{2,k})_{2,[0]}$$
$$= -\left((A_{1,k}^+)_{1,\geq 1} - (A_{1,k}^-)_{1,\leq -1}\right)\Big|_{\Lambda=-1} + (A_{3,k})_{3,[0]}.$$

b) The following formula holds:

$$B_{0,k} = B_{k,\Lambda} + (A_{2,k})_{2,\geq 0} + (A_{3,k})_{3,\geq 1} = B_{k,-\Lambda} + (A_{2,k})_{2,\geq 1} + (A_{3,k})_{3,\geq 0},$$

where

$$B_{k,\Lambda} =(A^+_{1,k})_{1,\geq 1} - (A^-_{1,k})_{1,\leq -1} - \Big((A^+_{1,k})_{1,\geq 1} - (A^-_{1,k})_{1,\leq -1}\Big)\Big|_{\Lambda=1}$$

$$B_{k,-\Lambda} =(A^+_{1,k})_{1,\geq 1} - (A^-_{1,k})_{1,\leq -1} - \Big((A^+_{1,k})_{1,\geq 1} - (A^-_{1,k})_{1,\leq -1}\Big)\Big|_{\Lambda=-1}.$$

c) The projections of $B_{0,k}$ are given by the following formulas:

$$\pi_2(B_{0,k})_{2,\geq 0} = (A_{2,k})_{2,\geq 0}, \quad \pi_3\left(B_{0,k}\big|_{\Lambda\to -\Lambda}\right)_{3,\geq 0} = (A_{3,k})_{3,\geq 0}.$$

Proof. a) Recalling the definition of $B_{0,k}$, we get $(B_{0,k})_{a,\geq 1} = (A_{a,k})_{a,\geq 1}$ for $a = 2, 3$. Using the identity from Proposition 17 we get $(B_{0,k})_{1,\geq 1} = (A^+_{1,k})_{1,\geq 1}$ and $(B_{0,k})_{1,\leq -1} = -(A^-_{1,k})_{1,\leq -1}$, where both identities are proved in the same ways as the corresponding identities for $B_{1,k}$ were proved in Lemma 24. Note that

$$B_{0,k}|_{\Lambda=1} = (A_{2,k})_{2,[0]}, \quad B_{0,k}|_{\Lambda=-1} = (A_{3,k})_{2,[0]}.$$

The formula for $(B_{0,k})_{[0]}$ follows easily.

b) This is just a reformulation of a).

c) Follows again from a) and the following relations:

$$\pi_2(\Lambda^j) =1 + O(\partial_2^{-1}) + \mathcal{E}_{(2)}H_2,$$

$$\pi_3(\Lambda^j) =(-1)^j + O(\partial_3^{-1}) + \mathcal{E}_{(3)}H_3. \quad \square$$

Next proposition is the main result of this section

Proposition 26. *The following Zakhalov-Shabat equations hold*

(56) $$\partial_{a,k}B_{b,l} - \partial_{b,l}B_{a,k} + [B_{b,l}, B_{a,k}] \in \mathcal{A}H,$$

where $a, b \in \{0, 1, 2, 3\}$, k (resp. l) is odd when $a = 2, 3$ (resp. $b = 2, 3$), $\mathcal{A} = \mathcal{E}$ if $ab \neq 0$, and $\mathcal{A} = \hat{\mathcal{E}}$ if $ab = 0$.

5.2. **Proof of the case $a = 1$ and $b = 1$.** In this case, we have to prove that

(57) $$\partial_{1,k}B_{1,l} - \partial_{1,l}B_{1,k} + [B_{1,l}, B_{1,k}] = 0.$$

Let us substitute the formulas for $B_{1,k}$ and $B_{1,l}$ from Lemma 24 in the projection (LHS of (57))$_{1,\geq 1}$ = $\left[B_{1,k}, (L_1^+)^l\right]_{1,\geq 1} - \left[B_{1,l}, (L_1^+)^k\right]_{1,\geq 1} + [B_{1,l}, B_{1,k}]_{1,\geq 1}$. The result can be written as the sum of the following three parts:

- $a_{11} = \left[\left((L_1^+)^k\right)_{1,\geq 1}, (L_1^+)^l\right]_{1,\geq 1} - \left[\left((L_1^+)^l\right)_{1,\geq 1}, (L_1^+)^k\right]_{1,\geq 1} + \left[\left((L_1^+)^l\right)_{1,\geq 1}, \left((L_1^+)^k\right)_{1,\geq 1}\right],$
- $b_{11} = -\left[\left((L_1^+)^k\right)_{1,\leq -1}, (L_1^+)^l\right]_{1,\geq 1} + \left[\left((L_1^+)^l\right)_{1,\leq -1}, (L_1^+)^k\right]_{1,\geq 1}$
  $- \left[\left((L_1^+)^l\right)_{1,\leq -1}, \left((L_1^+)^k\right)_{1,\geq 1}\right]_{1,\geq 1} - \left[\left((L_1^+)^l\right)_{1,\geq 1}, \left((L_1^+)^k\right)_{1,\leq -1}\right]_{1,\geq 1},$
- $c_{11} = \left[(B_{1,k})_{1,[0]}, (L_1^+)^l\right]_{1,\geq 1} - \left[(B_{1,l})_{1,[0]}, (L_1^+)^k\right]_{1,\geq 1} +$

$$+\Big[(B_{1,l})_{1,[0]},\Big((L_1^+)^k\Big)_{1,\geq 1}\Big]-\Big[\Big((L_1^+)^l\Big)_{1,\geq 1},(B_{1,k})_{1,[0]}\Big].$$

Note that $b_{11}=c_{11}=0$ and

$$\begin{aligned} a_{11} &= \Big[\Big((L_1^+)^k\Big)_{1,\geq 1},\Big((L_1^+)^l\Big)_{1,\leq 0}\Big]_{1,\geq 1}-\Big[\Big((L_1^+)^l\Big)_{1,\geq 1},(L_1^+)^k\Big]_{1,\geq 1} \\ &= \Big[(L_1^+)^k,\Big((L_1^+)^l\Big)_{1,\leq 0}\Big]_{1,\geq 1}-\Big[\Big((L_1^+)^l\Big)_{1,\geq 1},(L_1^+)^k\Big]_{1,\geq 1}=\Big[(L_1^+)^k,(L_1^+)^l\Big]_{1,\geq 1}=0, \end{aligned}$$

where we have used that $[A_{1,\geq 1},B_{1,\geq 1}]=[A_{1,\geq 1},B_{1,\geq 1}]_{1,\geq 1}$ and $[A_{1,\leq 0},B_{1,\leq 0}]_{1,\geq 1}=0$ for any two operators A and B. Therefore $\Big(\text{LHS of (57)}\Big)_{1,\geq 1}=0$. Similarly, one can show $\Big(\text{LHS of (57)}\Big)_{1,\leq -1}=0$. Finally, for the zero-order term of (57), using Lemma 24, we get

$$(\text{LHS of (57)})_{1,[0]}=-\Big((\text{LHS of (57)})_{1,\geq 1}+(\text{LHS of (57)})_{1,\leq -1}\Big)(1)=0,$$

where we have used the relation $[B_{1,l},B_{1,k}]_{1,[0]}=-([B_{1,l},B_{1,k}]_{1,\geq 1}+[B_{1,l},B_{1,k}]_{1,\leq -1})(1)$, which is obtained from $[B_{1,l},B_{1,k}](1)=0$. Summarize the the above results, we get (57).

5.3. **Proof of the case** $a=1$ **and** $b=2,3$. Suppose that $a=1$ and $b=2$. The case when $b=3$ is similar. We have to prove the following relation:

$$\partial_{1,k}B_{2,l}-\partial_{2,l}B_{1,k}+[B_{2,l},B_{1,k}]\in\mathcal{E}H. \tag{58}$$

According to Corollary 20 and Lemma 24, we have

$$\begin{aligned} &\partial_{2,l}\Big(B_{1,k}\Big)_{1,\geq 1}=\pi_+([B_{2,l},B_{1,k}])_{1,\geq 1}, \\ &\partial_{2,l}\Big(B_{1,k}\Big)_{1,\leq -1}=\pi_-([B_{2,l},B_{1,k}])_{1,\leq -1}, \\ &\partial_{1,k}B_{2,l}=\pi_2([B_{1,k},B_{2,l}])_{2,\geq 1}, \end{aligned}$$

where we used that $\pi_+(\partial_2^k\Lambda^{-l})\in(\mathcal{E}^0_{(+)})_{1,\leq 0}$, $\pi_-(\partial_2^k\Lambda^l)\in(\mathcal{E}^0_{(-)})_{1,\geq 0}$, and $\pi_2(\partial_2^{-k}\Lambda^p)\in(\mathcal{E}^0_{(2)})_{1,\leq 0}$ for $k,l\geq 0,p\in\mathbb{Z}$. Moreover, using the relation $\Lambda\partial_2=\partial_2+q_2\Lambda+q_2+H_2$, we can remove the terms involving $\Lambda\partial_2$ in $[B_{2,l},B_{1,k}]$ and obtain

$$[B_{2,l},B_{1,k}]=h+\pi_+([B_{2,l},B_{1,k}])_{1,\geq 1}+\pi_-([B_{2,l},B_{1,k}])_{1,\leq -1}-\pi_2([B_{2,l},B_{1,k}])_{2,\geq 1}+\mathcal{E}H_2,$$

where h is some function. Therefore, we only need to prove that $\partial_{2,l}(B_{1,k})_{1,[0]}=h$. In fact, note that $[B_{2,l},B_{1,k}]|_{\Lambda\to-\Lambda}(1)=H_2|_{\Lambda\to-\Lambda}(1)=0$. Therefore, $h=-\Big(\pi_+([B_{2,l},B_{1,k}])_{1,\geq 1}+\pi_-([B_{2,l},B_{1,k}])_{1,\leq -1}\Big)(1)$. Recalling Lemma 24, we get $\partial_{2,l}(B_{1,k})_{1,[0]}=h$.

5.4. **Proof of the case** $a,b = 2$ **or** 3**.** Firstly, in the case $a = b = 2$ or 3, one only needs to check $\partial_{a,k}B_{a,l} - \partial_{a,l}B_{a,k} + [B_{a,l}, B_{a,k}] = 0$. The proof is very standard just like the KP case [6].

Then in the case $a = 2$ and $b = 3$, the goal is to show

$$\partial_{2,k}B_{3,l} - \partial_{3,l}B_{2,k} + [B_{3,l}, B_{2,k}] \in \mathcal{E}H \tag{59}$$

According to Corollary 20 and the relations $\partial_a^{k+1}\partial_b^{-l} \in (\mathcal{E}^0_{(b)})_{b,<0} + \mathcal{E}_{(b)}H$, $k,l \geq 0$, $a \neq b$, we get $\partial_{2,k}B_{3,l} = (\partial_{2,k}L_3^l)_{3,\geq 1} = [\pi_3(B_{2,k}), L_3^l]_{3,\geq 1} = \pi_3([B_{2,k}, L_3^l])_{3,\geq 1} = \pi_3([B_{2,k}, B_{3,l}])_{3,\geq 1}$. Similarly, $\partial_{3,l}B_{2,k} = \pi_2([B_{3,l}, B_{2,k}])_{2,\geq 1}$. Furthermore, let us rewrite $[B_{3,l}, B_{2,k}] = \sum_{i,j\geq 1} a_{ij}\partial_2^i\partial_3^j$ into $h + \pi_2([B_{3,l}, B_{2,k}])_{2,\geq 1} + \pi_3([B_{3,l}, B_{2,k}])_{3,\geq 1} + \mathcal{E}H$ by using the relation $\partial_2\partial_3 = H_1 - q_1$. We only need to prove that $h = 0$. On the other hand, recalling part b) of Proposition 1 and part b) of Lemma 13, we get $h = (\pi_2([B_{2,k}, B_{3,l}]))_{2,[0]} = \left(\pi_2([L_2^k, B_{3,l}])\right)_{2,[0]} = [L_2^k, \pi_2(B_{3,l})]_{2,[0]} = -(\partial_{3,l}L_2^k)_{2,[0]} = 0$.

5.5. **Proof of the case** $a = 0$ **and** $b = 2,3$**.** Suppose that $a = 0$ and $b = 2$. The case when $b = 3$ is similar. We have to prove that

$$\partial_{0,k}B_{2,l} - \partial_{2,l}B_{0,k} + [B_{2,l}, B_{0,k}] \in \hat{\mathcal{E}}H. \tag{60}$$

According to Corollary 23 and Lemma 25, $\pi_+(\text{LHS of } (60))_{1,\geq 1} = -\left(\partial_{2,l}A^+_{1,k}\right)_{1,\geq 1} + \pi_+([B_{2,l}, B_{0,k}])_{1,\geq 1} = -\pi_+\left([B_{2,l}, A^+_{1,k}]\right)_{1,\geq 1} + \pi_+\left([B_{2,l}, (A^+_{1,k})_{1,\geq 1}]\right)_{1,\geq 1} = 0$. Here $\pi_+([B_{2,\geq 1}, A_{1,\leq 0}])_{1,\geq 1} = \pi_+([B_{2,\geq 1}, A_{b,\geq 0}])_{1,\geq 1} = 0$ for any operators A and B. Similarly, $\pi_-(\text{LHS of } (60))_{1,\leq -1} = 0$.

Using Lemma 25, we get that $\pi_2(\text{LHS of } (60))_{2,\geq 0}$ can be written as a sum of the following three parts:

$$\begin{aligned}
a_{02} &= [(A_{2,k})_{2,\geq 0}, L_2^l]_{2,\geq 0} - [B_{2,l}, A_{2,k}]_{2,\geq 0} + [B_{2,l}, (A_{2,k})_{2,\geq 0}]_{2,\geq 0} = [A_{2,k}, L_2^l]_{2,\geq 0} = 0,\\
b_{02} &= \pi_2([B_{k,\Lambda}, L_2^l])_{2,\geq 0} + \pi_2([B_{2,l}, B_{k,\Lambda}])_{2,\geq 0} = 0,\\
c_{02} &= \pi_2([(A_{3,k})_{3,>0}, L_2^l])_{2,\geq 0} + \pi_2([B_{2,l}, (A_{3,k})_{3,>0}])_{2,\geq 0} = 0.
\end{aligned}$$

Therefore $\pi_2\Big(\text{LHS of } (60)\Big)_{2,\geq 0} = 0$. As for the projection of π_3, one can obtain $\pi_3(\text{LHS of } (60))_{3,\geq 1} = -\partial_{2,l}(A_{3,k})_{3,\geq 1} + \pi_3([B_{2,l}, B_{0,k}])_{3,\geq 1} = -\pi_3([B_{2,l}, A_{3,k}])_{3,\geq 1} + \pi_3([B_{2,l}, (A_{3,k})_{3,\geq 1}])_{3,\geq 1} = 0$.

Note that by the relation $\Lambda\partial_2 = \partial_2 + q_2\cdot\Lambda + q_2 + H_2$ and $\partial_2\partial_3 = -q_1 + H_1$, one can obtain $[B_{2,l}, B_{0,k}] = h + \pi_+([B_{2,l}, B_{0,k}])_{1,\geq 1} + \pi_-([B_{2,l}, B_{0,k}])_{1,\leq -1} + + \pi_2([B_{2,l}, B_{0,k}])_{2,\geq 1} + \pi_3([B_{2,l}, B_{0,k}])_{3,\geq 1} + \hat{\mathcal{E}}H$. Thus by the same method as the one in Lemma 25, $\pi_2([B_{2,l}, B_{0,k}])_{2,[0]} = h + \pi_+([B_{2,l}, B_{0,k}])_{1,\geq 1}(1) + \pi_-([B_{2,l}, B_{0,k}])_{1,\leq -1}(1)$. Therefore

$$\begin{aligned}
[B_{2,l}, B_{0,k}] &= \pi_+([B_{2,l}, B_{0,k}])_{1,\geq 1} + \pi_-([B_{2,l}, B_{0,k}])_{1,\leq -1}\\
&\quad -\Big(\pi_+([B_{2,l}, B_{0,k}])_{1,\geq 1} + \pi_-([B_{2,l}, B_{0,k}])_{1,\leq -1}\Big)(1)\\
&\quad +\pi_2([B_{2,l}, B_{0,k}])_{2,\geq 0} + \pi_3([B_{2,l}, B_{0,k}])_{3,\geq 1} + \hat{\mathcal{E}}H.
\end{aligned}$$

Finally, it remains only to check that $-\partial_{2,l}(B_{k,\Lambda})_{1,[0]} - \big(\pi_+([B_{2,l},B_{0,k}])_{1,\geq 1} + \pi_-([B_{2,l},B_{0,k}])_{1,\leq -1}\big)(1) = 0$, which is correct thanks to Lemma 25.

5.6. **Proof of the case** $a = 0$ **and** $b = 1$**.** The task is to prove

$$\partial_{0,k}B_{1,l} - \partial_{1,l}B_{0,k} + [B_{1,l},B_{0,k}] \in \hat{\mathcal{E}}H. \tag{61}$$

By Corollary 23, Lemmas 24 and 25, one can rewrite $\pi_+(\text{LHS of (61)})_{1,\geq 1}$ as the sum of the following terms:

$$\begin{aligned}
a_{01} &= [(A^+_{1,k})_{1,\geq 1},(L^+_1)^l]_{1,\geq 1} - [((L^+_1)^l)_{1,\geq 1},A^+_{1,k}]_{1,\geq 1} + [((L^+_1)^l)_{1,\geq 1},(A^+_{1,k})_{1,\geq 1}] \\
&= [A^+_{1,k},(L^+_1)^l]_{1,\geq 1} = 0, \\
b_{01} &= -[(A^-_{1,k})_{1,\leq -1},(L^+_1)^l]_{1,\geq 1} + [((L^-_1)^l)_{1,\leq -1},A^+_{1,k}]_{1,\geq 1} \\
&\quad -[((L^-_1)^l)_{1,\leq -1},(A^+_{1,k})_{1,\geq 1}]_{1,\geq 1} - [((L^+_1)^l)_{1,\geq 1},(A^-_{1,k})_{1,\leq -1}] = 0, \\
c_{01} &= [(B_{k,\Lambda})_{1,[0]},(L^+_1)^l]_{1,\geq 1} - [(B_{1,l})_{1,[0]},A^+_{1,k}]_{1,\geq 1} \\
&\quad +[(B_{1,l})_{1,[0]},(A^+_{1,k})_{1,\geq 1}]_{1,\geq 1} + [((L^+_1)^l)_{1,\geq 1},(B_{k,\Lambda})_{1,[0]}] = 0, \\
d_{01} &= \pi_+([(A_{2,k})_{2,\geq 0},(L^+_1)^l])_{1,\geq 1} + \pi_+([((L^+_1)^l)_{1,\geq 1},(A_{2,k})_{2,\geq 0}])_{1,\geq 1} = 0, \\
e_{01} &= \pi_+([(A_{3,k})_{3,\geq 1},(L^+_1)^l])_{1,\geq 1} + \pi_+([((L^+_1)^l)_{1,\geq 1},(A_{3,k})_{3,\geq 1}])_{1,\geq 1} = 0.
\end{aligned}$$

So $\pi_+\Big(\text{LHS of (61)}\Big)_{1,\geq 1} = 0$. And similarly $\pi_-\Big(\text{LHS of (61)}\Big)_{1,\leq -1} = 0$. In order to discuss π_2 and π_3, one has to know the structure of $[B_{1,l},B_{0,k}]$. In fact, similar to the case of $a = 0$ and $b = 2$,

$$\begin{aligned}
[B_{1,l},B_{0,k}] &= \pi_+([B_{1,l},B_{0,k}])_{1,\geq 1} + \pi_-([B_{1,l},B_{0,k}])_{1,\leq -1} \\
&\quad -\Big(\pi_+([B_{1,l},B_{0,k}])_{1,\geq 1} + \pi_-([B_{1,l},B_{0,k}])_{1,\leq -1}\Big)(1) \\
&\quad +\pi_2([B_{1,l},B_{0,k}])_{2,\geq 0} + \pi_3([B_{1,l},B_{0,k}])_{3,\geq 1} + \hat{\mathcal{E}}H.
\end{aligned}$$

Thus $\pi_2(\text{LHS of (61)})_{2,\geq 0} = -\pi_2([B_{1,l},A_{2,k}])_{2,\geq 0} + \pi_2([B_{1,l},(A_{2,k})_{2,\geq 0}])_{2,\geq 0} = 0$. Similarly, $\pi_3(\text{LHS of (61)})_{3,\geq 1} = 0$.

Finally, $\partial_{0,k}(B_{1,l})_{1,[0]} - \partial_{1,l}(B_{k,\Lambda})_{1,[0]}$-$\big(\pi_+([B_{1,l},B_{0,k}])_{1,\geq 1} + \pi_-([B_{1,l},B_{0,k}])_{1,\leq -1}\big)(1) = 0$ can be proved by Lemmas 24 and 25. Summarize the above results completes the proof of (61).

5.7. **Proof of the case** $a = 0$ **and** $b = 0$**.** We have to prove that

$$\partial_{0,k}B_{0,l} - \partial_{0,l}B_{0,k} + [B_{0,l},B_{0,k}] \in \hat{\mathcal{E}}H. \tag{62}$$

Using Corollary 23 and Lemma 25, we can write $\pi_+(\text{LHS of (62)})_{1,\geq 1}$ as the sum of the following terms:

$$\begin{aligned}
a_{001} &= [(A^+_{1,k})_{1,\geq 1},A^+_{1,l}]_{1,\geq 1} - [(A^+_{1,k})_{1,\geq 1},A^+_{1,l}]_{1,\geq 1} + [(A^+_{1,l})_{1,\geq 1},(A^+_{1,k})_{1,\geq 1}] \\
&= [A^+_{1,k},A^+_{1,l}]_{1,\geq 1} = 0,
\end{aligned}$$

$$
\begin{aligned}
b_{001} &= -[(A^{-}_{1,k})_{1,\leq -1}, A^{+}_{1,l}]_{1,\geq 1} + [(A^{-}_{1,l})_{1,\leq -1}, A^{+}_{1,k}]_{1,\geq 1} \\
&\quad -[(A^{-}_{1,l})_{1,\leq -1}, (A^{+}_{1,k})_{1,\geq 1}]_{1,\geq 1} - [(A^{+}_{1,l})_{1,\geq 1}, (A^{-}_{1,k})_{1,\leq -1}] = 0, \\
c_{001} &= [(B_{k,\Lambda})_{1,[0]}, A^{+}_{1,l}]_{1,\geq 1} - [(B_{l,\Lambda})_{1,[0]}, A^{+}_{1,k}]_{1,\geq 1} \\
&\quad +[(B_{l,\Lambda})_{1,[0]}, (A^{+}_{1,k})_{1,\geq 1}]_{1,\geq 1} + [(A^{+}_{1,l})_{1,\geq 1}, (B_{k,\Lambda})_{1,[0]}] = 0, \\
d_{001} &= \pi_{+}([(A_{2,k})_{2,\geq 0}, A^{+}_{1,l}])_{1,\geq 1} - \pi_{+}([(A_{2,l})_{2,\geq 0}, A^{+}_{1,k}])_{1,\geq 1} \\
&\quad +\pi_{+}([(A^{+}_{1,l})_{1,\geq 1}, (A_{2,k})_{2,\geq 0}])_{1,\geq 1} + \pi_{+}([(A_{2,l})_{2,\geq 0}, (A^{+}_{1,k})_{1,\geq 1}])_{1,\geq 1} = 0, \\
e_{001} &= \pi_{+}([(A_{3,k})_{3,\geq 1}, A^{+}_{1,l}])_{1,\geq 1} - \pi_{+}([(A_{3,l})_{3,\geq 1}, A^{+}_{1,k}])_{1,\geq 1} \\
&\quad +\pi_{+}([(A^{+}_{1,l})_{1,\geq 1}, (A_{3,k})_{3,\geq 1}])_{1,\geq 1} + \pi_{+}([(A_{3,l})_{3,\geq 1}, (A^{+}_{1,k})_{1,\geq 1}])_{1,\geq 1} = 0.
\end{aligned}
$$

Similarly, $\pi_{-}(\text{LHS of (62)})_{1,\leq -1} = 0$. Next, we have $\pi_2(\text{LHS of (62)})_{2,\geq 0} = a_{002} + b_{002} + c_{002}$, where

$$
\begin{aligned}
a_{002} &= [(A_{2,k})_{2,\geq 0}, A_{2,l}]_{2,\geq 0} - [(A_{2,l})_{2,\geq 0}, A_{2,k}]_{2,\geq 0} + [(A_{2,l})_{2,\geq 0}, (A_{2,k})_{2,\geq 0}]_{2,\geq 0} \\
&= [A_{2,k}, A_{2,l}]_{2,\geq 0} = 0, \\
b_{002} &= \pi_2([B_{k,\Lambda}, A_{2,l}])_{2,\geq 0} - \pi_2([B_{l,\Lambda}, A_{2,k}])_{2,\geq 0} \\
&\quad +\pi_2([(A_{2,l})_{2,\geq 0}, B_{k,\Lambda}])_{2,\geq 0} + \pi_2([B_{l,\Lambda}, (A_{2,k})_{2,\geq 0}])_{2,\geq 0} = 0, \\
c_{002} &= \pi_2([(A_{3,k})_{3,>0}, A_{2,l}])_{2,\geq 0} - \pi_2([(A_{3,l})_{3,>0}, A_{2,k}])_{2,\geq 0} \\
&\quad +\pi_2([(A_{2,l})_{2,\geq 0}, (A_{3,k})_{3,>0}])_{2,\geq 0} + \pi_2([(A_{3,l})_{3,>0}, (A_{2,k})_{2,\geq 0}])_{2,\geq 0} = 0.
\end{aligned}
$$

Similarly, we have $\pi_3(\text{LHS of (62)})_{3,\geq 1} = 0$. Just as in the case of $a = 0$ and $b = 2$, we have

$$
\begin{aligned}
[B_{0,l}, B_{0,k}] &= \pi_{+}([B_{0,l}, B_{0,k}])_{1,\geq 1} + \pi_{-}([B_{0,l}, B_{0,k}])_{1,\leq -1} \\
&\quad -\big(\pi_{+}([B_{0,l}, B_{0,k}])_{1,\geq 1} + \pi_{-}([B_{0,l}, B_{0,k}])_{1,\leq -1}\big)(1) \\
&\quad +\pi_2([B_{0,l}, B_{0,k}])_{2,\geq 0} + \pi_3([B_{0,l}, B_{0,k}])_{3,\geq 1} + \hat{\mathcal{E}}H.
\end{aligned}
$$

Finally, $\partial_{0,k}(B_{l,\Lambda})_{1,[0]} - \partial_{0,l}(B_{k,\Lambda})_{1,[0]} - \big(\pi_{+}([B_{0,l}, B_{0,k}])_{1,\geq 1} + \pi_{-}([B_{0,l}, B_{0,k}])_{1,\leq -1}\big)(1) = 0$ can be obtained by Lemma 25 and the above results, which completes the proof of (62). This is the last possible case, so the proof of Proposition 26 is completed.

5.8. **Commutativity of $\partial_{a,k}$ and $\partial_{b,l}$.** Proposition 26, part a) of Theorem 4, and Proposition 8, yield the following corollary:

Corollary 27. *Under projections π_c,*

$$
\partial_{a,k}\pi_c(B_{b,l}) - \partial_{b,l}\pi_c(B_{a,k}) + [\pi_c(B_{b,l}), \pi_c(B_{a,k})] = 0. \tag{63}
$$

Here $a, b = 0, 1, 2, 3$, $c = \pm, 2, 3$ and k (resp. l) is odd when $a = 2, 3$ (resp. $b = 2, 3$).

Proof. Put $\pi_c(B_{b,l}) = B_{b,l} + \sum_{j=1}^3 A_{cbj}H_j$, $\pi_c(B_{a,k}) = B_{a,k} + \sum_{j=1}^3 A_{caj}H_j$ and $\partial_{a,k}B_{b,l} - \partial_{b,l}B_{a,k} + [B_{b,l}, B_{a,k}] = \sum_{j=1}^3 A_{cj}H_j$ with $A_{cbj}, A_{caj}, A_{cj} \in \mathcal{A}_{(c)}$. Then by using part a) of Theorem 4,

$$
\begin{aligned}
&\partial_{a,k}\pi_c(B_{b,l}) - \partial_{b,l}\pi_c(B_{a,k}) + [\pi_c(B_{b,l}), \pi_c(B_{a,k})] \\
\in&\partial_{a,k}B_{b,l} + \sum_{j=1}^{3} A_{cbj}\cdot\partial_{a,k}H_j - \partial_{b,l}B_{a,k} - \sum_{j=1}^{3} A_{caj}\cdot\partial_{b,l}H_j + [B_{b,l} + \sum_{j=1}^{3} A_{cbj}H_j, B_{a,k} + \sum_{j=1}^{3} A_{caj}H_j] + \mathcal{A}_{(c)}H \\
=&\partial_{a,k}B_{b,l} - \partial_{b,l}B_{a,k} + [B_{b,l}, B_{a,k}] - \sum_{j=1}^{3} A_{cbj}\cdot H_j\cdot B_{a,k} + \sum_{j=1}^{3} A_{caj}\cdot H_j\cdot B_{b,l} \\
&+ \sum_{j=1}^{3} A_{cbj}\cdot H_j\cdot B_{a,k} - \sum_{j=1}^{3} A_{caj}\cdot H_j\cdot B_{b,l} + \mathcal{A}_{(c)}H = \mathcal{A}_{(c)}H.
\end{aligned}
$$

Therefore $\partial_{a,k}\pi_c(B_{b,l}) - \partial_{b,l}\pi_c(B_{a,k}) + [\pi_c(B_{b,l}), \pi_c(B_{a,k})] \in \mathcal{A}^0_{(c)} \cap \mathcal{A}_{(c)}H = \{0\}$. □

Now we can prove part b) of Theorem 4, that is, $[\partial_{a,k}, \partial_{b,l}] = 0$ in $\mathcal{R}$. We have to prove that $[\partial_{a,k}, \partial_{b,l}]\mathcal{L} = 0$ and $[\partial_{a,k}, \partial_{b,l}]H_c = 0$. Note that

$$
\begin{aligned}
\partial_{a,k}\mathcal{L} &= [B_{a,k}, \mathcal{L}] + \sum_{j=1}^{3} A_{akj}H_j, \quad \partial_{b,l}\mathcal{L} = [B_{b,l}, \mathcal{L}] + \sum_{j=1}^{3} A_{blj}H_j \\
\partial_{a,k}H_c &= -H_cB_{a,k} + \sum_{j=1}^{3} A'_{akj}H_j, \quad \partial_{b,l}H_c = -H_cB_{b,l} + \sum_{j=1}^{3} A'_{blj}H_j,
\end{aligned}
$$

for some $A_{akj}, A_{blj}, A'_{akj}, A'_{blj} \in \mathcal{A}$. The second derivatives take the form

$$
\begin{aligned}
\partial_{a,k}(\partial_{b,l}\mathcal{L}) &\in [\partial_{a,k}B_{b,l}, \mathcal{L}] + [B_{b,l}, [B_{a,k}, \mathcal{L}]] - \sum_{j=1}^{3}\Big(A_{akj}H_jB_{b,l} + A_{blj}H_jB_{a,k}\Big) + \mathcal{A}H, \\
\partial_{a,k}(\partial_{b,l}H_c) &\in H_c(B_{a,k}B_{b,l} - \partial_{a,k}B_{b,l}) - \sum_{j=1}^{3}(A'_{akj}H_jB_{b,l} + A'_{blj}H_jB_{a,k}) + \mathcal{A}H.
\end{aligned}
$$

Therefore, according to Proposition 26 we have

$$
\begin{aligned}
[\partial_{a,k}, \partial_{b,l}]\mathcal{L} &\in \big[\partial_{a,k}B_{b,l} - \partial_{b,l}B_{a,k} + [B_{b,l}, B_{a,k}], \mathcal{L}\big] + \mathcal{A}H = \mathcal{A}H, \\
[\partial_{a,k}, \partial_{b,l}]H_c &\in -H_c\cdot\Big(\partial_{a,k}B_{b,l} - \partial_{b,l}B_{a,k} + [B_{b,l}, B_{a,k}]\Big) + \mathcal{A}H = \mathcal{A}H.
\end{aligned}
$$

Furthermore, we have $[\partial_{a,k}, \partial_{b,l}]\mathcal{L} \in \mathcal{A}^0_{(+)} \cap \mathcal{A}H = \{0\}$ and $[\partial_{a,k}, \partial_{b,l}]H_c \in \mathcal{A}^0_{(+)} \cap \mathcal{A}H = \{0\}$, because $[\partial_{a,k}, \partial_{b,l}]\mathcal{L} \in \mathcal{A}^0_{(+)}$ and $[\partial_{a,k}, \partial_{b,l}]H_c \in \mathcal{A}^0_{(+)}$.

We claim that $[\partial_{a,k}, \partial_{b,l}]L_c = 0$. Let us prove only the case $[\partial_{a,k}, \partial_{b,l}]L_1 = 0$. The remaining two cases are essentially the same. In fact, $[\partial_{a,k}, \partial_{b,l}]L_1 = [\partial_{a,k}\pi_+(B_{b,l}) - \partial_{b,l}\pi_+(B_{a,k}) + [\pi_+(B_{b,l}), \pi_+(B_{a,k})], L_1]$, which is zero according to Corollary 27.

Let us recall the settings from Section 1.5. Our goal is to construct Lax operators $\mathcal{L}$ and H_a ($1 \leq a \leq 3$) with coefficients in $\mathcal{O}_\epsilon[[\mathbf{t}]]$ and establish some of their properties, which will be needed for the proof of Theorem 5.

6.1. **Wave operators.** The first step is to transform the Hirota Bilinear Equations into an equivalent system of quadratic equations involving operator series. Let us introduce the wave operators

$$W_1^+(x,\mathbf{t},\Lambda) = S_1^+(x,\mathbf{t},\Lambda)e^{\xi_1(\mathbf{t},\Lambda)}, \quad S_1^+(x,\mathbf{t},\Lambda) = \sum_{j=0}^\infty \psi_{1,j}^+(x,\mathbf{t})\Lambda^{-j}$$

$$W_1^-(x,\mathbf{t},\Lambda) = e^{-\xi_1(\mathbf{t},\Lambda)}S_1^-(x,\mathbf{t},\Lambda), \quad S_1^-(x,\mathbf{t},\Lambda) = \sum_{j=0}^\infty \Lambda^{-j}\psi_{1,j}^-(x,\mathbf{t})$$

$$\xi_1(\mathbf{t},\Lambda) = \sum_{k=1}^\infty \left(t_{1,k}\,\Lambda^k + t_{0,k}\,(\epsilon\partial_x - h_k)\frac{\Lambda^{(n-2)k}}{(n-2)^k k!}\right)$$

and

$$W_a(x,\mathbf{t},\partial_a) = S_a(x,\mathbf{t},\partial_a)e^{\overline{\xi}_a(\mathbf{t},\partial_a)}, \quad S_a(x,\mathbf{t},\partial_a) = \sum_{j=0}^\infty \psi_{a,j}^+(x,\mathbf{t})\partial_a^{-j}$$

$$\overline{\xi}_a(\mathbf{t},\partial_a) = \sum_{k=1}^\infty \left(t_{a,2k+1}\,\partial_a^{2k+1} + t_{0,k}\,\epsilon\partial_x\,\frac{\partial_a^{2k}}{2^k k!}\right).$$

Note that in the definition of $\overline{\xi}_a$ compared to the definition of ξ_a ($a = 2,3$) the terms involving $y_a = t_{a,1}$ are missing. The operator series $W_1^\pm$ and W_a ($a = 2,3$) take values respectively in $\mathcal{D}_\epsilon((\Lambda^{-1}))[[\mathbf{t}]]$ and $\mathcal{D}_\epsilon((\partial_a^{-1}))[[\mathbf{t}]]$.

Given a pseudo-differential operator $P(x,\mathbf{t},\partial_a) = \sum_j p_j(x,\mathbf{t})\partial_a^{-j} \in \mathcal{D}_\epsilon((\partial_a^{-1}))[[\mathbf{t}]]$ then we define its action on $(y_a - y_a')^0$ by the following rule

$$\partial_a^{-j}(y_a - y_a')^0 = \begin{cases} \frac{1}{j!}(y_a - y_a')^j & \text{, if } j \geq 0, \\ 0 & \text{, if } j < 0, \end{cases}$$

In other words, $P(x,\mathbf{t},\partial_a)(y_a - y_a')^0 = \sum_{j\geq 0} p_j(x,\mathbf{t})\frac{(y_a-y_a')^j}{j!}$.

Proposition 28. *The Hirota Bilinear Equations are equivalent to the following system of quadratic equations for the operator series $W_1^\pm$ and W_a ($a = 2,3$):*

$$W_1^+(x,\mathbf{t},\Lambda)\frac{\Lambda^{(n-2)k-1}}{(n-2)^k k!}W_1^-(x,\mathbf{t}',\Lambda) + \left(W_1^+(x,\mathbf{t}',\Lambda)\frac{\Lambda^{(n-2)k-1}}{(n-2)^k k!}W_1^-(x,\mathbf{t},\Lambda)\right)^\#$$

$$= \sum_{m\in\mathbb{Z}} \frac{1}{2}\left(\left(W_2(x,\mathbf{t},\partial_2)\frac{\partial_2^{2k-1}}{2^k k!}W_2(x+m\epsilon,\mathbf{t}',\partial_2)^\#\partial_2\right)\Big|_{y_2'=y_2}(y_2-y_2')^0\right.$$

$$-(-1)^m\Big(W_3(x,\mathbf{t},\partial_3)\frac{\partial_3^{2k-1}}{2^k k!}W_3(x+m\epsilon,\mathbf{t}',\partial_3)^{\#}\partial_3\Big)\Big|_{y_3'=y_3}(y_3-y_3')^0\Big)\Lambda^m,$$

where $k\geq 0$ is an arbitrary non-negative integer.

The proof of Proposition 28 is based on the following two lemmas.

Lemma 29. *Let $A(x,\Lambda),B(x,\Lambda)$ be two operator series in $\mathcal{D}_\epsilon((\Lambda^{-1}))$. Then*

$$A(x,\Lambda)\cdot B(x,\Lambda)^{\#}=\sum_{j\in\mathbb{Z}}\mathrm{Res}_{z=0}\frac{dz}{z}\Big(A(x,\Lambda)(z^{\pm x/\epsilon})\Big(B(x+j,\Lambda)(z^{\mp(x/\epsilon+j)})\Big)^{\#}\Big)\Lambda^j.$$

Proof. Similar formula can be found in [1]. Put $A=\sum_k A_k(x)\Lambda^k$ and $B=\sum_l B_l(x)\Lambda^l$, then the RHS of the identity takes the form

$$\sum_{k,l,j}\mathrm{Res}_{z=0}\frac{dz}{z}\Big(A_k(x)z^{\pm(x/\epsilon+k)}\Big(B_l(x+j\epsilon)z^{\mp(x/\epsilon+l+j)}\Big)^{\#}\Lambda^j\Big)=\sum_{l,j}A_{l+j}(x)\Lambda^j B_l(x)=A(x,\Lambda)\cdot B(x,\Lambda)^{\#},$$

where we used that $\Lambda^j f(x)=f(x+j\epsilon)\Lambda^j$. $\square$

Lemma 30. *Let $A(y,\partial)=\sum_i a_i(y)\partial^i$ and $B(y,\partial)=\sum_j b_j(y)\partial^j$, where $\partial:=\frac{\partial}{\partial y}$, be two pseudo-differential operators. Then we have*

$$\Big(A(y,\partial)(B(y,\partial))^{\#}\partial\Big)(y-y')^0=\mathrm{Res}_{z=0}\,dz\Big(A(y,\partial)(e^{yz})B(y',\partial')(e^{-y'z})\Big),$$

where $\partial'=\frac{\partial}{\partial y'}$ and for $P(y,\partial)=\sum_k p_k(y)\partial^{-k}$ we define $P(y,\partial)(y-y')^0=\sum_{k\geq 0}\frac{1}{k!}p_k(y)\,(y-y')^k$.

Proof. Let us recall the following formula

$$\mathrm{Res}_{z=0}\,dz\Big(A(e^{yz})\,B(e^{-yz})\Big)=\mathrm{Res}_\partial\,AB^{\#},$$

where $\mathrm{Res}_\partial P(y,\partial)$ denotes the coefficient in front of ∂^{-1} in $P(y,\partial)$. The proof is straightforward (see also [6]). Using the Taylor's expansion formula at $y'=y$ we get

$$\begin{aligned}
&\mathrm{Res}_{z=0}\,dz\Big(A(y,\partial)(e^{yz})\,B(y',\partial'))(e^{-y'z})\Big)\\
=\;&\mathrm{Res}_{z=0}\,dz\Big(A(y,\partial)(e^{yz})\sum_{n=0}^{\infty}\frac{(y'-y)^n}{n!}\partial_y^n B(y,\partial)(e^{-yz})\Big)\\
=\;&\sum_{n=0}^{\infty}\frac{(y'-y)^n}{n!}\,\mathrm{Res}_\partial\,A(y,\partial)B^{\#}(y,\partial)(-1)^n\partial_y^n\\
=\;&\sum_{n=0}^{\infty}\frac{(y-y')^n}{n!}\,\mathrm{Res}_\partial\,A(y,\partial)B^{\#}(y,\partial)\partial^n\\
=\;&\sum_{n=0}^{\infty}\partial^{-n}\,(y-y')^0\,\mathrm{Res}_\partial\,A(y,\partial)B^{\#}(y,\partial)\partial^n.
\end{aligned}$$

Put $A(y,\partial)B^{\#}(y,\partial)=\sum_{i\in\mathbb{Z}}c_i\partial^i$ and notice that

$$\operatorname{Res}_\partial A(y,\partial)B^{\#}(y,\partial)\partial^n=\operatorname{Res}_\partial\sum_{i\in\mathbb{Z}}c_i\partial^{i+n}=c_{-n-1}.$$

We get

$$\begin{aligned}&\operatorname{Res}_{z=0}\frac{dz}{z}\Big(A(y,\partial)(e^{yz})B(y',\partial')(e^{-y'z})\Big)\\ =\;&\sum_{n=0}^{\infty}c_{-n-1}\partial^{-n}(y-y')^0=\sum_{k\in\mathbb{Z}}c_k\partial^{k+1}(y-y')^0\\ =\;&\Big(A(y,\partial)B^{\#}(y,\partial)\partial\Big)(y-y')^0.\quad\square\end{aligned}$$

Proof of Proposition 28. Note that

$$\Psi_1^+(x,\mathbf{t},z)=W_1^+(x,\mathbf{t},\Lambda)(z^{x/\epsilon-\frac{1}{2}})\quad\text{and}\quad\Psi_1^-(x,\mathbf{t},z)^{\#}=W_1^-(x,\mathbf{t},\Lambda)^{\#}(z^{-x/\epsilon-\frac{1}{2}}).$$

Using Lemma 29 we get

$$\sum_{m\in\mathbb{Z}}\operatorname{Res}_{z=0}\frac{z^{(n-2)k}}{(n-2)^kk!}\frac{dz}{z}\Big((\Psi_1^+(x,\mathbf{t},z)\Psi_1^-(x+m\epsilon,\mathbf{t}',z)\Big)\Lambda^m=W_1^+(x,\mathbf{t},\Lambda)\frac{\Lambda^{(n-2)k-1}}{(n-2)^kk!}W_1^-(x,\mathbf{t}',\Lambda)$$

and

$$\sum_{m\in\mathbb{Z}}\operatorname{Res}_{z=0}\frac{z^{(n-2)k}}{(n-2)^kk!}\frac{dz}{z}\Big(\Psi_1^+(x+m\epsilon,\mathbf{t}',z)\Psi_1^-(x,\mathbf{t},z)\Big)^{\#}\Lambda^m=\Big(W_1^+(x,\mathbf{t}',\Lambda)\frac{\Lambda^{(n-2)k-1}}{(n-2)^kk!}W_1^-(x,\mathbf{t},\Lambda)\Big)^{\#}.$$

Similarly, for $a=2,3$, using that

$$\Psi_a^+(x,\mathbf{t},z)=W_a(x,\mathbf{t},\partial_a)e^{y_az}\quad\text{and}\quad\Psi_a^-(x,\mathbf{t},z)=W_a(x,\mathbf{t},\partial_a)e^{-y_az}$$

and Lemma 30 we get

$$\begin{aligned}&\operatorname{Res}_{z=0}\frac{z^{2k}}{2^kk!}\frac{dz}{2z}\left(\Psi_a^+(x,\mathbf{t},z)\Psi_a^-(x+m\epsilon,\mathbf{t}',z)\right)=\\ &\left(W_a(x,\mathbf{t},\partial_a)\frac{\partial_a^{2k-1}}{2^kk!}W_a(x+m\epsilon,\mathbf{t}',\partial_a)^{\#}\partial_a\right)\Big|_{y_a=y_a'}(y_a-y_a')^0.\quad\square\end{aligned}$$

Proposition 31. *a) The operators $S_1^{\pm}$ satisfy the following relation*

$$S_1^+(x,\mathbf{t},\Lambda)\Lambda^{-1}S_1^-(x,\mathbf{t},\Lambda)=\sum_{m=0}^{\infty}\Lambda^{-2m-1}.$$

b) The operators $S_a(x,\mathbf{t},\partial_a)$ ($a=2,3$) satisfy the following relations

$$S_a(x,\mathbf{t},\partial_a)\partial_a^{-1}S_a(x,\mathbf{t},\partial_a)^{\#}=\partial_a^{-1}.$$

Proof. a) Let us substitute in the identity of Proposition 28 $k=0$, $\mathbf{t}'=\mathbf{t}$ and compare the coefficients in front of the negative powers of Λ. The identity that we want to prove follows.

b) Let us make the same substitution as in a) except for $t'_{a,1}=y'_a$, i.e., we keep $y'_a\neq y_a$. Comparing the coefficients in front of Λ^0 yields the identity that we want to prove. $\square$

For brevity put $B = \mathcal{O}_\epsilon[[\mathbf{t}]]$ and let us denote by $\mathcal{E} = B[\Lambda, \Lambda^{-1}, \partial_2, \partial_3]$ the ring of differential difference operators. Note that the ring $\mathcal{E}$ acts on $\Psi := (\Psi_1^+, (\Psi_1^-)^\#, \Psi_2, \Psi_3)$ by acting on each component.

Proposition 32. *The annihilator* $\mathrm{Ann}(\Psi) = \{P \in \mathcal{E} \mid P(\Psi) = 0\}$ *is a left ideal in* $\mathcal{E}$ *generated by*

$$H_1 = \partial_2 \cdot \partial_3 + q_1, \quad H_2 = (\Lambda - 1) \cdot \partial_2 - q_2 \cdot (\Lambda + 1), \quad H_3 = (\Lambda + 1) \cdot \partial_3 - q_3 \cdot (\Lambda - 1),$$

where $q_1 = 2\partial_2\partial_3(\log \tau)$, $q_a = \partial_a(\log f)$ $(a = 2, 3)$, *where* $f := \frac{\tau(x,\mathbf{t})}{\tau(x+\epsilon,\mathbf{t})}$.

Proof. It is enough to prove that $H_i \in \mathrm{Ann}(\Psi)$. Indeed, suppose that this is proved. Let $P \in \mathrm{Ann}(\Psi)$ be arbitrary. The operator P can be decomposed as a sum of an operator in the left ideal $\mathcal{E}(H_1, H_2, H_3)$ generated by H_i and $Q = P_1 + P_2 + P_3$, where $P_1 \in B[\Lambda^{\pm 1}]$, $P_2 \in B[\partial_2]$, and $P_3 \in B[\partial_3]$. If $P_2 \neq 0$ then let $a(x,\mathbf{t})\partial_2^m$ be its highest order term. We get that $Q(\Psi_2) = a(x,\mathbf{t})z^m(1 + O(z^{-1}))e^{\xi_2(\mathbf{t},z)}$ can not be zero – contradiction. Similar argument shows that $P_3 = 0$, and finally $0 = P(\Psi_1^+) = (P_1 \cdot W_1^+)(z^{x/\epsilon - 1/2})$ implies $P_1 \cdot W_1^+ = 0 \Rightarrow P_1 = 0$.

Let us prove that $H_1(\Psi) = 0$. The argument in the remaining two cases is similar. Let us apply the differential operator $\partial_{t'_{2,1}}\partial_{t'_{3,1}}$ to the identiy in Proposition 28 with $k = 0$, substitute $\mathbf{t}' = \mathbf{t}$, and compare the coefficients in front of the negative powers of Λ. We get

$$S_1^+(x,\mathbf{t},\Lambda)\Lambda^{-1}\partial_2\partial_3 S_1^-(x,\mathbf{t},\Lambda) = \frac{1}{2}\sum_{m<0}\Lambda^m\Big(\partial_3\psi_{2,1}^+(x,\mathbf{t}) - (-1)^m\partial_2\psi_{3,1}^+(x,\mathbf{t})\Big).$$

Note that $\partial_3\psi_{2,1}^+(x,\mathbf{t}) = \partial_2\psi_{3,1}^+(x,\mathbf{t}) = -q_1$. Recalling Proposition 31, a) we get $\partial_2\partial_3 S_1^- = -S_1^- q_1$ and $\partial_2\partial_3 S_1^+ = -q_1 S_1^+$. These two relations are equivalent to $H_1(\Psi_1^+) = 0$ and $H_1((\Psi_1^-)^\#) = 0$. In order to prove that $H_1(\Psi_2) = 0$, we proceed as above except that we leave $t'_{2,1}$ arbitrary, set the remaining components of $\mathbf{t}'$ and $\mathbf{t}$ to be equal, and compare the coefficients in front of Λ^0. We get $(S_2 \cdot \partial_2^{-1} \cdot (\partial_3(S_2))^\# \cdot \partial_2^2)(y_2 - y_2')^0 = q_1(y_2')$, where we suppressed the dependance of $q_1(x,\mathbf{t})$ on x and on the remaining components of $\mathbf{t}$. Taking the Taylor's expansion of $-q_1(y_2')$ at $y_2' = y_2$ and comparing the coefficients in front of the powers of $y_2' - y_2$ we get

$$S_2 \cdot \partial_2^{-1} \cdot (\partial_3(S_2))^\# \cdot \partial_2^2 = \sum_{j=0}^{\infty}(-1)^j(\partial_2^j q_1(x,\mathbf{t}))\partial_2^{-j} = \partial_2^{-1} \cdot q_1(x,\mathbf{t}) \cdot \partial_2.$$

Recalling Proposition 31, b) we get $(\partial_3 S_2)^\#\partial_2 = S_2^\# q_1 \Rightarrow \partial_2 \cdot (\partial_3 S_2(x,\mathbf{t},\partial_2)) = -q_1 S_2(x,\mathbf{t},\partial_2)$. This identity is equivalent to $H_1(\Psi_2) = 0$. Finally, the argument for $H_1(\Psi_3) = 0$ is completely analogus. $\square$

To avoid cumbersome notation let us put $S_1 := S_1^+$. Proposition 32 is equivalent to a set of relations involving the operators H_i $(1 \le i \le 3)$ and the wave operators S_j $(1 \le j \le 3)$. These relations take the following form.

Corollary 33. *a) The operator* H_1 *and the wave operators satisfy the following relations:*

$$H_1 \cdot S_1 = \partial_2 \cdot S_1 \cdot \partial_3 + \partial_3 \cdot S_1 \cdot \partial_2 - S_1 \cdot \partial_2 \cdot \partial_3, \tag{64}$$

$$H_1 \cdot S_2 = \partial_2 \cdot S_2 \cdot \partial_3,$$

$$H_1 \cdot S_3 = \partial_3 \cdot S_3 \cdot \partial_2.$$

b) The operator H_2 and the wave operators satisfy the following relations:

$$
\begin{aligned}
&H_2 \cdot S_1 = (\Lambda - 1) \cdot S_1 \cdot \partial_2, \\
&H_2 \cdot S_2 = (\partial_2 + q_2) \cdot S_2 \cdot (\Lambda - 1), \\
&H_2 \cdot S_3 = (\Lambda \cdot S_3 + S_3 \cdot \Lambda) \cdot \partial_2 - (\partial_2 + q_2) \cdot S_3 \cdot (\Lambda + 1).
\end{aligned} \tag{65}
$$

c) The operator H_3 and the wave operators satisfy the following relations:

$$
\begin{aligned}
&H_3 \cdot S_1 = (\Lambda + 1) \cdot S_1 \cdot \partial_3, \\
&H_3 \cdot S_2 = (\Lambda \cdot S_2 + S_2 \cdot \Lambda) \cdot \partial_3 - (\partial_3 + q_3) \cdot S_2 \cdot (\Lambda - 1), \\
&H_3 \cdot S_3 = (\partial_3 + q_3) \cdot S_3 \cdot (\Lambda + 1). \quad \square
\end{aligned} \tag{66}
$$

A straightforward computation yields that the complicated looking relations (64), (65), and (66) can be replaced equvalently with the following simple relation

$$q_1 + q_1[1] + 2q_2q_3 = 0, \tag{67}$$

where the functions q_1, q_2, and q_3 are defined in Proposition 32 and for a pseudo-differential-difference operator P we put $P[m] := \Lambda^m(P) = \Lambda^m \cdot P \cdot \Lambda^{-m}$ for its translation by m, i.e., substituting $x \mapsto x + m\epsilon$.

6.2. **The Lax operator.** Following Shiota's construction in [21] we introduce the following rings of pseudo-differential-difference operators:

$$\mathcal{E}_{(\pm)} = B[\partial_2, \partial_3]((\Lambda^{\mp 1})), \quad \mathcal{E}_{(2)} = B[\Lambda, \Lambda^{-1}, \partial_3]((\partial_2^{-1})), \quad \mathcal{E}_{(3)} = B[\Lambda, \Lambda^{-1}, \partial_2]((\partial_3^{-1})),$$

$$\mathcal{E}^0_{(\pm)} = B((\Lambda^{\mp 1})), \quad \mathcal{E}^0_{(2)} = B((\partial_2^{-1})), \quad \mathcal{E}^0_{(3)} = B((\partial_3^{-1})),$$

and the quotient rings $\mathcal{A}'' = \mathcal{A}'/\mathcal{A}H$, for $\mathcal{A} = \mathcal{E}, \mathcal{E}_{(\pm)}, \mathcal{E}_{(2)}$ or $\mathcal{E}_{(3)}$, where $\mathcal{A}H = \mathcal{A}H_1 + \mathcal{A}H_2 + \mathcal{A}H_3$ is the left ideal of $\mathcal{A}$ generated by H_i $(1 \le i \le 3)$ and

$$\mathcal{A}' = \{P \in \mathcal{A} \mid H_i P \in \mathcal{A}H \ \forall\ i = 1, 2, 3\}.$$

Let us introduce the following auxiliarly Lax operators

$$
\begin{aligned}
L_1^+(x, \mathbf{t}, \Lambda) &:= S_1^+(x, \mathbf{t}, \Lambda) \cdot \Lambda \cdot S_1^+(x, \mathbf{t}, \Lambda)^{-1} =: u^+_{1,0}(x, \mathbf{t})\Lambda + \sum_{j=1}^{\infty} u^+_{1,j}(m, \mathbf{t})\Lambda^{1-j}, \\
L_1^-(x, \mathbf{t}, \Lambda) &:= S_1^-(x, \mathbf{t}, \Lambda)^{\#} \cdot \Lambda^{-1} \cdot \left(S_1^-(x, \mathbf{t}, \Lambda)^{\#}\right)^{-1} =: u^-_{1,0}(x, \mathbf{t})\Lambda^{-1} + \sum_{j=1}^{\infty} u^-_{1,j}(x, \mathbf{t})\Lambda^{j-1}, \\
L_a(x, \mathbf{t}, \partial_a) &:= S_a(x, \mathbf{t}, \partial_a) \cdot \partial_a^{-1} \cdot S_a(x, \mathbf{t}, \partial_a)^{-1} =: \partial_a + \sum_{j=1}^{\infty} u_{a,j}(x, \mathbf{t})\partial_a^{-j} \quad (a = 2, 3).
\end{aligned}
$$

To avoid cumbersome notation we put $L_1(x,\mathbf{t},\Lambda)=L_1^+(x,\mathbf{t},\Lambda)$ and $u_{1,j}=u_{1,j}^+$. Recalling Proposition 31 we get the following relations

$$L_1^+(x,\mathbf{t},\Lambda)^{\#}=(\Lambda^{-1}-\Lambda)\cdot L_1^-(x,\mathbf{t},\Lambda)\cdot\ \iota_\Lambda\left(\Lambda^{-1}-\Lambda\right)^{-1},$$
$$L_a(x,\mathbf{t},\partial_a)^{\#}=-\partial_a\cdot L_a(x,\mathbf{t},\partial_a)\cdot\partial_a^{-1}\quad(a=2,3),$$

where ι_Λ denotes the operation that takes the Laurent series expansion at $\Lambda=0$. Note that the coefficient of L_a in front of ∂_a^0 must be $u_{a,0}=0$.

Lemma 34. *The following formula holds*

$$c_a(x,\mathbf{t}):=\left(L_a(x,\mathbf{t},\partial_a)^2/2\right)_{a,[0]}=2\partial_a^2(\log\tau(x,\mathbf{t}))\quad(a=2,3).$$

Proof. By definition $S_a=1+\psi_{a,1}^+\partial_a^{-1}+\psi_{a,2}^+\partial_a^{-2}+\cdots$ with $\psi_{a,1}^+=-2\partial_a(\log\tau)$. Put $L_a^2=\partial_a^2+2c_a+\cdots$. By definition $L_aS_a=S_a\partial_a^2$. Comparing the coefficients in front of ∂_a^0 we get

$$\psi_{a,2}^++2\partial_a(\psi_{a,1}^+)+2c_a=\psi_{a,2}^+.$$

Solving for c_a we get $c_a=-\partial_a(\psi_{a,1}^+)=2\partial_a^2(\log\tau)$. □

Let us define the following differential-difference operator

$$\begin{aligned}\mathcal{L}\ =\ &\frac{\partial_2^2}{2}+\frac{\partial_3^2}{2}+\left(\left(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\right)_{1,>0}-\left(\sum_{m=0}^{\infty}\Lambda^{2m+1}\tfrac{(L_1^{\#})^{n-2}}{n-2}\right)_{1,<0}\right)(\Lambda-\Lambda^{-1})+\\&+\frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1})+\frac{1}{2}(c_2+c_3),\end{aligned}$$

where c_a $(a=2,3)$ are the same as in Lemma 34. Note that in both sums only finitely many terms contribute and that the term involving $(\ )_{1,<0}$ is conjugated via the anti-involution # to the term involving $(\ )_{1,>0}$.

Proposition 35. *Let us introduce the following two rational difference operators*

$$Q_2(x,\mathbf{t},\Lambda):=(\Lambda-1)^{-1}\cdot q_2(x,\mathbf{t})\cdot(\Lambda+1)\quad\text{and}\quad Q_3(x,\mathbf{t},\Lambda):=(\Lambda+1)^{-1}\cdot q_3(x,\mathbf{t})\cdot(\Lambda-1)$$

Then the following formulas hold

$$\mathcal{L}=\tfrac{(L_1^{\pm})^{n-2}}{n-2}+\tfrac{1}{2}\iota_{\Lambda^{\mp1}}\Big((\partial_2+Q_2)(\partial_2-Q_2)+(\partial_3+Q_3)(\partial_3-Q_3)\Big).$$

Proof. Let us recall the Hirota quadratic equations from Proposition 28 with $k=1$ and $\mathbf{t}'=\mathbf{t}$. We get

$$\begin{aligned}&S_1\left(\tfrac{\Lambda^{n-2}}{n-2}\right)S_1^{-1}\sum_{m=0}^{\infty}\Lambda^{-2m-1}+\left(S_1\left(\tfrac{\Lambda^{n-2}}{n-2}\right)S_1^{-1}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\right)^{\#}=\\&\frac{1}{2}\sum_{m\in\mathbb{Z}}\left(\left(\tfrac{L_2^2}{2}S_2\Lambda^mS_2^{-1}\right)_{2,[0]}-\left(\tfrac{L_3^2}{2}S_3(-\Lambda)^mS_3^{-1}\right)_{3,[0]}\right)\end{aligned}$$

Using that

$$\sum_{m\in\mathbb{Z}} \Lambda^m = \iota_{\Lambda^{-1}}(\Lambda-1)^{-1} - \iota_\Lambda(\Lambda-1)^{-1} \quad \text{and} \quad \sum_{m\in\mathbb{Z}} (-\Lambda)^m = -\iota_{\Lambda^{-1}}(\Lambda+1)^{-1} + \iota_\Lambda(\Lambda+1)^{-1}$$

we write the above identity as

$$S_1\Big(\tfrac{\Lambda^{n-2}}{n-2}\Big)S_1^{-1}\sum_{m=0}^{\infty}\Lambda^{-2m-1} + \Big(S_1\Big(\tfrac{\Lambda^{n-2}}{n-2}\Big)S_1^{-1}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\Big)^{\#} = \tag{68}$$
$$\frac{1}{2}(\iota_{\Lambda^{-1}}-\iota_\Lambda)\Big(\Big(\tfrac{L_2^2}{2}S_2(\Lambda-1)^{-1}S_2^{-1}\Big)_{2,[0]} + \Big(\tfrac{L_3^2}{2}S_3(\Lambda+1)^{-1}S_3^{-1}\Big)_{3,[0]}\Big).$$

Recalling the second identity from Corollary 33, b) we get

$$S_2(\Lambda-1)S_2^{-1} = (\partial_2+q_2)^{-1}H_2.$$

We will need the coefficients v_1 and v_2 from the expansion

$$S_2(\Lambda-1)^{-1}S_2^{-1} = (\Lambda-1)^{-1} + v_1\partial_2^{-1} + v_2\partial_2^{-2} + \cdots,$$

where the coefficients v_i on the RHS should be interpreted as rational difference operators. Since $H_2 = (\Lambda-1)\partial_2 - q_2(\Lambda+1)$ the above formula implies that

$$((\Lambda-1)\partial_2 - q_2(\Lambda+1)\cdot((\Lambda-1)^{-1} + v_1\partial_2^{-1} + v_2\partial_2^{-2} + \cdots) = \partial_2 + q_2.$$

Comparing the coefficients in front of ∂_2^0 and ∂_2^{-1} yields

$$v_1 = 2Q_2(\Lambda-\Lambda^{-1})^{-1}, \quad (v_2 + \partial_2(v_1)) = 2Q_2^2(\Lambda-\Lambda^{-1})^{-1}. \tag{69}$$

Similar argument using the 3rd identity in Corollary 33, c) implies that the coefficients w_1 and w_2 in the expansion

$$S_3(\Lambda+1)^{-1}S_3^{-1} = (\Lambda+1)^{-1} + w_1\partial_3^{-1} + w_2\partial_3^{-2} + \cdots$$

are gven by

$$w_1 = 2Q_3(\Lambda-\Lambda^{-1})^{-1}, \quad (w_2 + \partial_2(w_1)) = 2Q_3^2(\Lambda-\Lambda^{-1})^{-1}. \tag{70}$$

Formulas (69) and (70) imply that

$$\Big(\tfrac{L_2^2}{2}S_2(\Lambda-1)^{-1}S_2^{-1}\Big)_{2,[0]} = c_2(\Lambda-1)^{-1} + \Big(\partial_2(Q_2) + Q_2^2\Big)(\Lambda-\Lambda^{-1})^{-1}$$

and

$$\Big(\tfrac{L_3^2}{2}S_3(\Lambda+1)^{-1}S_3^{-1}\Big)_{3,[0]} = c_3(\Lambda+1)^{-1} + \Big(\partial_3(Q_3) + Q_3^2\Big)(\Lambda-\Lambda^{-1})^{-1}.$$

The identity (68) takes the form

$$\Bigg(\Big(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\Big) + \Big(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\Big)^{\#}\Bigg)(\Lambda-\Lambda^{-1}) = \tag{71}$$

$$(72)\qquad \frac{1}{2}(\iota_{\Lambda^{-1}}-\iota_\Lambda)\Big(c_2(1+\Lambda^{-1})+c_3(1-\Lambda^{-1})+\Big(\partial_2(Q_2)+Q_2^2+\partial_3(Q_3)+Q_3^2\Big)\Big).$$

Let us proof the formula in the proposition for L_1^+. The other case is similar. By definition we have

$$\begin{aligned}\mathcal{L}-\tfrac{L_1^{n-2}}{n-2} &= \frac{\partial_2^2}{2}+\frac{\partial_3^2}{2}-\left(\left(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\right)_{1,<0}+\left(\sum_{m=0}^{\infty}\Lambda^{2m+1}\tfrac{(L_1^{\#})^{n-2}}{n-2}\right)_{1,<0}\right)(\Lambda-\Lambda^{-1})+\\ &\quad +c_1(\Lambda-\Lambda^{-1})+\frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1})+\frac{1}{2}(c_2+c_3),\end{aligned}$$

where

$$c_1:=-\left(\tfrac{L_1^{n-2}}{n-2}\sum_{m=0}^{\infty}\Lambda^{-2m-1}\right)_{1,[0]}=-\frac{1}{4}(c_2-c_3),$$

where the second equality in the above formula is proved by comparing the coefficients in front of Λ^0 in (68). The identity (71) allows us to transform the above formula into

$$\begin{aligned}\mathcal{L}-\tfrac{L_1^{n-2}}{n-2} &= \frac{\partial_2^2}{2}+\frac{\partial_3^2}{2}-\frac{1}{2}\iota_{\Lambda^{-1}}\Big(\partial_2(Q_2)+Q_2^2+\partial_3(Q_3)+Q_3^2\Big)-\frac{1}{2}\Big(c_2(1+\Lambda^{-1})+c_3(1-\Lambda^{-1})\Big)+\\ &\quad +c_1(\Lambda-\Lambda^{-1})+\frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1})+\frac{1}{2}(c_2+c_3).\end{aligned}$$

It remains only to check that the terms that involve c_1, c_2, and c_3 add up to 0 and to use that

$$(\partial_a+Q_a)(\partial_a-Q_a)=\partial_a^2-\partial_a(Q_a)-Q_a^2\quad (a=2,3).\quad\square$$

Proposition 36. *The wave functions are eigenfunctions of $\mathcal{L}$*

$$\mathcal{L}(\Psi_1^+)=\tfrac{z^{n-2}}{n-2}\Psi_1^+,\quad \mathcal{L}((\Psi_1^-)^{\#})=\tfrac{z^{n-2}}{n-2}(\Psi_1^-)^{\#},\quad \mathcal{L}(\Psi_a)=\tfrac{z^2}{2}\Psi_a\quad (a=2,3).$$

Proof. The first identity is a direct consequence from Propositon 35. Indeed, using that $\Psi_1^+(x,\mathbf{t},z)=S_1(x,\mathbf{t},\Lambda)\Big(e^{\xi_1(\mathbf{t},z)}z^{x/\epsilon-\frac{1}{2}}\Big)$ and that the expansions at $\Lambda=0$ of $(\partial_2-Q_2)=(\Lambda-1)^{-1}H_2$ and $\partial_3-Q_3=(\Lambda+1)^{-1}H_3$ anihilate Ψ_1^+ we get

$$\mathcal{L}(\Psi_1^+)=\frac{1}{n-2}(L_1^{n-2}\cdot S_1)\Big(e^{\xi_1(\mathbf{t},z)}z^{x/\epsilon-\frac{1}{2}}\Big)=\frac{1}{n-2}(S_1\cdot\Lambda^{n-2})\Big(e^{\xi_1(\mathbf{t},z)}z^{x/\epsilon-\frac{1}{2}}\Big)=\frac{z^{n-2}}{n-2}\Psi_1^+.$$

The formula for $\mathcal{L}((\Psi_1^-)^{\#})$ is proved in a similar way.

Let us prove the formula for $\mathcal{L}(\Psi_2)$. The computation for $\mathcal{L}(\Psi_3)$ is similar. Let us apply the operator $\mathcal{L}(\Lambda M^{-1},\partial_2,\partial_3)$ to the set of Hirota bilinear equations (HBEs) (10), where M acts on the set of HBEs via the shift $m\mapsto m+1$. Note that

$$\mathcal{L}(\Lambda M^{-1},\partial_2,\partial_3)\big(F(x,\mathbf{t},z)F(x+m\epsilon,\mathbf{t}',z)\big)=\big(\mathcal{L}(\Lambda,\partial_2,\partial_3)(F(x,\mathbf{t},z))\big)\,F(x+m\epsilon,\mathbf{t}',z),$$

and

$$\mathcal{L}(\Lambda M^{-1},\partial_2,\partial_3)\big((-1)^mF(x,\mathbf{t},z)F(x+m\epsilon,\mathbf{t}',z)\big)=\big(\mathcal{L}(-\Lambda,\partial_2,\partial_3)(F(x,\mathbf{t},z))\big)\,F(x+m\epsilon,\mathbf{t}',z).$$

In other words the action of $\mathcal{L}(\Lambda M^{-1},\partial_2,\partial_3)$ on the set of HBEs is equivalent to acting by $\mathcal{L}(\Lambda,\partial_2,\partial_3)$ (or $\mathcal{L}^*:=\mathcal{L}(-\Lambda,\partial_2,\partial_3)$ whenever the sign factor $(-1)^m$ is present) only on the first factors of the bilinear identities. The LHS of (10) with $k=0$ is transformed into the LHS of (10) with $k=1$, because Ψ_1^+ and $(\Psi_1^-)^\#$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $\frac{z^{n-2}}{n-2}$. Therefore using also (10) with $k=1$ we get that the residue

$$\mathrm{Res}_{z=0}\frac{dz}{2z}\Big(\Big(\mathcal{L}-\tfrac{z^2}{2}\Big)(\Psi_2^+(x,\mathbf{t},z))\,\Psi_2^-(x+m\epsilon,\mathbf{t}',z)-(-1)^m\Big(\mathcal{L}^*-\tfrac{z^2}{2}\Big)(\Psi_3^+(x,\mathbf{t},z))\,\Psi_3^-(x+m\epsilon,\mathbf{t}',z)\Big)$$

is 0. Let us substitute now $m=0$ and $\mathbf{t}'=\mathbf{t}$, except for $t'_{2,1}$, i.e., we keep $y'_2=t'_{2,1}$ and $y_2=t_{2,1}$ arbitrary. Note that

$$\Big(\mathcal{L}-\tfrac{z^2}{2}\Big)(\Psi_2^+)=\Big(\mathcal{L}(S_2^+)+\partial_2(S_2^+)\cdot\partial_2\Big)(e^{\xi_2(\mathbf{t},z)})=\sum_{j=1}^\infty\widetilde{\psi}^+_{2,j}(x,\mathbf{t})z^{-j}\,e^{\xi_2(\mathbf{t},z)},$$

where we used that $\mathcal{L}(1)+\partial_2(\psi^+_{2,1})=c_2+\partial_2(\psi^+_{2,1})=0$. Similarly,

$$\Big(\mathcal{L}^*-\tfrac{z^2}{2}\Big)(\Psi_3^+)=\Big(\mathcal{L}^*(S_3^+)+\partial_3(S_3^+)\cdot\partial_3\Big)(e^{\xi_3(\mathbf{t},z)})=\sum_{j=1}^\infty\widetilde{\psi}^+_{3,j}(x,\mathbf{t})z^{-j}\,e^{\xi_3(\mathbf{t},z)},$$

where we used that $\mathcal{L}^*(1)+\partial_3(\psi^+_{3,1})=c_3+\partial_3(\psi^+_{3,1})=0$. Therefore the vanishing of the above residue yields the following identity

$$\sum_{j=1}^\infty\mathrm{Res}_{z=0}\frac{dz}{2z}\Big(\widetilde{\psi}^+_{2,j}\,z^{-j}\,e^{(y_2-y_2')z}\,\psi_2^-(x,\mathbf{t}',z)-\widetilde{\psi}^+_{3,j}\,z^{-j}\,\psi_3^-(x,\mathbf{t}',z)\Big)=0$$

The terms involving ψ_3 do not contribute. Expanding the exponential in the powers of y_2-y_2' we get a sequence of vanishing residues. Since $\psi_2^-=1+O(z^{-1})$, a simple induction on j shows that $\widetilde{\psi}^+_{2,j}=0$ for all $j\geq 1$. This is exactly what we had to prove. □

Let us point out that in the above proof we have established also the identities $\mathcal{L}(S_2^+)=-\partial_2(S_2^+)\cdot\partial_2$ and $\mathcal{L}^*(S_3^+)=-\partial_3(S_3^+)\cdot\partial_3$.

Proposition 37. *The differential-difference operator $\mathcal{L}$ satisfies the following relations*

a) $\mathcal{L}-\frac{(L_1^\pm)^{n-2}}{n-2}\in\mathcal{E}_{(\pm)}H$.
b) $\mathcal{L}-\frac{L_2^2}{2}\in\mathcal{E}_{(2)}H$.
c) $\mathcal{L}-\frac{L_3^2}{2}\in\mathcal{E}_{(3)}H$.

Proof. Part a) is a direct consequence of Proposition 35. For part b). First, note that $\mathcal{L}(S_2)=-\partial_2(S_2)\cdot\partial_2$ implies

$$\Big(\mathcal{L}-\frac{\partial_2^2}{2}\Big)(S_2)=\frac{1}{2}(S_2\,\partial_2^2-\partial_2^2\,S_2)=\frac{1}{2}(L_2^2)_{2,\leq 0}\,S_2,$$

where we used that $(L_2^2)_{2,>0}=\partial_2^2$. Using the above identity we get

$$\mathcal{L}-\frac{1}{2}L_2^2=\Big(\Big(\mathcal{L}-\frac{1}{2}\partial_2^2\Big)\cdot S_2-\Big(\mathcal{L}-\frac{1}{2}\partial_2^2\Big)(S_2)\Big)\cdot S_2^{-1}. \tag{73}$$

The operator $\mathcal{L}-\frac{1}{2}\partial_2^2$ is a sum of terms of the form $a(x,\mathbf{t})\Lambda^i$ and $\partial_3^2/2$. We claim that each of these terms contributes to (73) a term that belongs to $\mathcal{E}_{(2)}H$. Indeed, we have

$$\Big(a\Lambda^i\cdot S_2 - a\,\Lambda^i(S_2)\Big)\cdot S_2^{-1} = aS_2[i](\Lambda^i-1)S_2^{-1}$$

and

$$\Big(\partial_3^2\cdot S_2-\partial_3^2(S_2)\Big)S_2^{-1} = (\partial_3(S_2)+S_2\,\partial_3)\cdot\partial_3\cdot S_2^{-1}$$

On the other hand, recalling Corollary 33 we get

$$(\Lambda-1)S_2^{-1} = S_2^{-1}(\partial_2+q_2)^{-1}H_2 \quad\in\quad \mathcal{E}_{(2)}H$$

and

$$\partial_3\cdot S_2^{-1} = S_2^{-1}\cdot\partial_2^{-1}\cdot H_1 \quad\in\quad \mathcal{E}_{(2)}H.$$

This completes the proof of part b). The proof of part c) is similar, except for the following point. It is convenient to prove that $\mathcal{L}^*-\frac{L_3^2}{2}\in\mathcal{E}_{(3)}H^*$ where * is the involution $\Lambda\mapsto-\Lambda$ and $\mathcal{E}_{(3)}H^*$ is the left ideal of $\mathcal{E}_{(3)}$ generated by H_i^* $(1\le i\le 3)$. The argument goes in the same way as in part b). The only point worth mentioning is the identity

$$(\Lambda-1)S_3^{-1} = -\Big((\Lambda+1)S_3^{-1}\Big)^* = -S_3^{-1}(\partial_3+q_3)^{-1}H_3^* \quad\in\quad \mathcal{E}_{(3)}H^*. \quad\square$$

Corollary 38. *The operator $\mathcal{L}$ has the following properties.*

a) $\mathcal{L}\in\mathcal{E}'_{(\pm)}$ and $\mathcal{L}=\frac{(L_1^{\pm})^{n-2}}{n-2}$ in the quotient ring $\mathcal{E}''_{(\pm)}$.

b) $\mathcal{L}\in\mathcal{E}'_{(2)}$ and $\mathcal{L}=\frac{L_2^2}{2}$ in the quotient ring $\mathcal{E}''_{(2)}$.

c) $\mathcal{L}\in\mathcal{E}'_{(3)}$ and $\mathcal{L}=\frac{L_3^2}{2}$ in the quotient ring $\mathcal{E}''_{(3)}$.

Proof. We need only to check that $L_1^{\pm}\in\mathcal{E}'_{(\pm)}$ and that $L_a\in\mathcal{E}'_{(a)}$ $(a=2,3)$. This however is a straightforward computation using Corollary 33. $\square$

7. The evolution of $\mathcal{L}$, H_1, H_2, and H_3

Note that the operator $\mathcal{L}$ has the form

$$\mathcal{L}=\Big(\sum_{i=1}^{n-3}(a_i\Lambda^i-\Lambda^{-i}a_i)\Big)(\Lambda-\Lambda^{-1})+\frac{1}{2}\partial_2^2+\frac{1}{2}\partial_3^2+\frac{1}{4}(c_2-c_3)(\Lambda+\Lambda^{-1})+\frac{1}{2}(c_2+c_3)$$

and that the coefficient $a_{n-3}=\frac{1}{n-2}e^{(n-2)\alpha}$ for some $\alpha\in\mathcal{O}_\epsilon[[\mathbf{t}]]$. Let us denote by $\mathcal{M}$ the set of operators $\mathcal{L}$, H_2, and H_3. They depend on the $n+1$ functions

$$\Xi := \{a_i(x,\mathbf{t})\,(1\le i\le n-4),\quad \alpha(x,\mathbf{t}),\quad q_j(x,\mathbf{t})\,(j=2,3),\quad c_k(x,\mathbf{t})\,(k=2,3)\}, \tag{74}$$

which will be viewed as coordinates on $\mathcal{M}$. Let us point out that the operator $H_1=\partial_2\partial_3+q_1$ is uniquely determined from H_2 and H_3, because the relation $q_1+q_1[1]+2q_2q_3=0$ can be solved uniquely for $q_1\in\mathcal{O}_\epsilon[[\mathbf{t}]]$ in terms of q_2 and q_3.

We would like to prove that the operators $\mathcal{L}$, H_1, H_2, and H_3 satisfy the Lax equations (9) of the Extended D-Toda hierarchy. In other words, the Extended D-Toda hierarchy defines an infinite sequence of commuting flows on the manifold $\mathcal{M}$. Let us give an outline of our argument. Put

$$\mathcal{R} := \mathbb{C}[\partial_x^i \partial_2^j \partial_3^k(\xi)\,(\xi \in \Xi,\, i, j, k \in \mathbb{Z}_{\geq 0}), e^{\pm\alpha}][[\epsilon]]$$

for the ring of formal power series in ϵ whose coefficients are polynomials in the partial derivatives $\partial_x^i \partial_2^j \partial_3^k(\xi)$ where ξ is one of the functions in (74). We refer to the elements of $\mathcal{R}$ as differential polynomials. For brevity, put $\partial_{i,k} = \frac{\partial}{\partial t_{i,k}}$. First, we are going to prove that $P_{i,k,\xi} := \partial_{i,k}(\xi) \in \mathcal{R}$ for all $\xi \in \Xi$. Then we will argue that $P_{a,\xi} := \partial_a(\xi)$ for $a = 2, 3$ can be expressed in terms of the partial derivatives of (74) with respect to x only. In other words the ring $\mathcal{R}$ is in fact equal to

$$\mathbb{C}[\partial_x^i(\xi)\,(\,\xi \in \Xi,\, i \in \mathbb{Z}_{\geq 0}), e^{\pm\alpha}][[\epsilon]].$$

7.1. **Projections.**

Lemma 39. *Let $\mathcal{A}$ be one of the rings $\mathcal{E}_{(\pm)}$, $\mathcal{E}_{(2)}$, or $\mathcal{E}_{(3)}$. Then we have an orthogonal decomposition*

$$\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A} H.$$

Proof. Let us give the argument for the case $\mathcal{A} = \mathcal{E}_{(+)}$. The remaining cases are similar. If $P \in \mathcal{E}_{(+)}$ then it can be written as

$$\sum_{j=j_0}^{\infty} \Lambda^{-j} p_j(\partial_2, \partial_3),$$

where the coefficients p_j are differential operators in ∂_2 and ∂_3. Since $H_2 = (\Lambda - 1)\partial_2 - q_2(\Lambda + 1)$ we have

$$\partial_2 = \iota_{\Lambda^{-1}} \Big((\Lambda - 1)^{-1} q_2 (\Lambda + 1) + (\Lambda - 1)^{-1} H_2 \Big).$$

Similarly

$$\partial_3 = \iota_{\Lambda^{-1}} \Big((\Lambda + 1)^{-1} q_3 (\Lambda - 1) + (\Lambda + 1)^{-1} H_3 \Big).$$

Using these identities and that $[\partial_a, (\Lambda \pm 1)^{-1} H_b] \in B[[\Lambda^{-1}]]$ we get that each differential operator $p_j(\partial_2, \partial_3)$ can be decomposed as $p_j^{(0)} + p_j^{(2)}(\Lambda - 1)^{-1} H_2 + p_j^{(3)}(\Lambda + 1)^{-1} H_3$, where $p_j^{(0)} \in B[[\Lambda^{-1}]]$ and $p_j^{(2)}$, $p_j^{(3)} \in B[\partial_2, \partial_3][[\Lambda^{-1}]]$. This proves that $\mathcal{E}_{(+)} = \mathcal{E}_{(+)}^0 + \mathcal{E}_{(+)} H$. The sum must be direct because if $P(x, \mathbf{t}, \Lambda) \in \mathcal{E}_{(+)}^0$ anihilates the wave function $\Psi_1(x, \mathbf{t}, z)$, then $P(x, \mathbf{t}, \Lambda) S_1(x, \mathbf{t}, \Lambda) = 0$. However, the ring $\mathcal{E}_{(+)}^0$ does not have zero divisors and $S_1 \neq 0$, so $P = 0$. □

Let us denot by $\pi_\alpha : \mathcal{E}_{(\alpha)} \to \mathcal{E}_{(\alpha)}^0$ ($\alpha = \pm, 2, 3$) the projection defined via the orthogonal decomposition from Lemma 39. Note that if we have an operator $P = \sum_{j,k,l} p_{j,k,l}(x, \mathbf{t}) \Lambda^{-j} \partial_2^k \partial_3^l \in \mathcal{E}_{(\alpha)}$, then the projection $\pi_\alpha(P)$ is a pseudo-difference (if $\alpha = \pm$) or a pseudo-differential (if $\alpha = 2, 3$) operator whose coefficients are polynomials on the shifted derivatives $\partial_2^b \partial_3^c(p_{j,k,l}[a])$, where $b, c \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$.

7.2. **The $t_{1,k}$-flows.** Let us define

$$B_{1,k}^{+} := \left(-\left(L_1^k \sum_{m=0}^{\infty} \Lambda^{-2m-1} \right)_{1,<0} + \left(\sum_{m=0}^{\infty} \Lambda^{2m+1} \, (L_1^{\#})^k \right)_{1,<0} \right) (\Lambda - \Lambda^{-1}).$$

Note that $B_{1,k}^{+} \in B((\Lambda^{-1})) = -L_1^k + B_{1,k}$ is a pseudo-difference operator. We claim that the coefficients of $B_{1,k}^{+}$ are differential polynomials in $\mathcal{R}$. Indeed, it is sufficient to prove that the coefficients of L_1 belong to $\mathcal{R}$. Recalling Corollary 38 we get $\pi_+(\mathcal{L}) = \frac{L_1^{n-2}}{n-2}$. The coefficients of $\pi_+(\mathcal{L})$ belong to $\mathcal{R}$. The rest of the proof is the same as the proof of Lemma 11.

Lemma 40. *The operator series S_1 satisfies the differential equations*

$$\partial_{1,k} S_1 = B_{1,k}^{+} S_1, \quad k \geq 1.$$

Proof. Let us differentiate the HBEs in Proposition 28 with respect to $\partial_{1,k}$, substitute $\mathbf{t}' = \mathbf{t}$, and compare the coefficients in front of the negative powers of Λ. We get

$$\partial_{1,k} S_1^{+} \Lambda^{-1} S_1^{-} + \left(S_1^{+} \Lambda^{k-1} S_1^{-} - (S_1^{-})^{\#} \Lambda^{1-k} (S_1^{+})^{\#} \right)_{1,<0} = 0.$$

It remains only to recall Proposition 31, a). □

Proposition 41. *We have*

$$\begin{aligned} \partial_{1,k} \mathcal{L} &= \pi_+([B_{1,k}^{+}, \mathcal{L}]) \\ \partial_{1,k} H_i &= -\pi_+(H_i B_{1,k}^{+}), \quad (1 \leq i \leq 3). \end{aligned}$$

Proof. Recalling Corollary 33 we have $H_2 = (\Lambda - 1) S_1 \cdot \partial_2 \cdot S_1^{-1}$, $H_3 = (\Lambda + 1) S_1 \cdot \partial_3 \cdot S_1^{-1}$, and

$$H_1 = \partial_2 (\Lambda + 1)^{-1} H_3 + \partial_3 (\Lambda - 1)^{-1} H_2 - (\Lambda - 1)^{-1} H_2 (\Lambda + 1)^{-1} H_3.$$

The differential equations for H_i follow from the above formulas and Lemma 40.

It remain only to prove the differential equation for $\mathcal{L}$. Using that

$$\mathcal{L} = \frac{1}{n-2} S_1 \Lambda^{n-2} S_1^{-1} + A_2 H_2 + A_3 H_3$$

for some $A_2, A_3 \in \mathcal{E}_{(+)}$ we get that modulo $\ker(\pi_+)$ the derivative $\partial_{1,k} \mathcal{L}$ is given by

$$\frac{1}{n-2} [B_{1,k}, S_1 \Lambda^{n-2} S_1^{-1}] - A_2 H_2 B_{1,k} - A_3 H_3 B_{1,k},$$

where we used Lemma 40 and the differential equations for H_i which we have established already. The above expression coincides with $[B_{1,k}, \mathcal{L}]$ modulo terms in $\ker(\pi_+)$. □

7.3. **The $t_{a,2l+1}$-flows.** Let us introduce the pseudo-difference operators

$$B^+_{2,k} = \frac{1}{2}\,\iota_{\Lambda^{-1}}\left(S_2\partial_2^k(\Lambda-1)^{-1}S_2^{-1}\right)_{2,[0]}(\Lambda-\Lambda^{-1})$$

and

$$B^+_{3,k} = \frac{1}{2}\,\iota_{\Lambda^{-1}}\left(S_3\partial_3^k(\Lambda+1)^{-1}S_3^{-1}\right)_{3,[0]}(\Lambda-\Lambda^{-1})$$

We claim that $B^+_{2,k}$ and $B^+_{3,k}$ are formal Laurent series in Λ^{-1} whose coefficients belong to $\mathcal{R}$. Let us prove this for $B^+_{2,k}$. The argument for $B^+_{3,k}$ is similar. Using Corollary 33, b) we have

$$B^+_{2,k} = L_2^k H_2^{-1}(\partial_2+q_2)(\Lambda-\Lambda^{-1}) = L_2^k\left((1+\partial_2^{-1}q_2)^{-1}(1-\partial_2^{-1}q_2)\Lambda-1\right)^{-1}(\Lambda-\Lambda^{-1}).$$

It is enough to prove that the coefficients of the pseudo-differential operator L_2 belong to $\mathcal{R}$. On the other hand, $\pi_2(\mathcal{L}) = \frac{L_2^2}{2}$. Solving for L_2 and using that the coefficients of $\pi_2(\mathcal{L})$ belong to $\mathcal{R}$, we get that L_2 has coefficients in $\mathcal{R}$ (see the proof of Lemma 12 for more details).

Lemma 42. *We have*

$$\partial_{a,2l+1}(S_1) = B^+_{a,2l+1}\,S_1, \quad l\geq 0, \quad a=2,3.$$

Proof. Again we give the argument only for $a=2$the first differential equation when $a=2$, because the argument for the second one is identical. Let us differentiate the HBEs in Proposition 28 corresponding to $k=0$ with respect to $\partial_{2,2l+1}$, set $\mathbf{t}'=\mathbf{t}$, and compare the coefficients in front of the negative powers of Λ. If $l>0$ then we get

$$\partial_{2,2l+1}(S_1)S_1^{-1}\iota_{\Lambda^{-1}}(\Lambda-\Lambda^{-1})^{-1} = \frac{1}{2}\sum_{m=1}^{\infty}(S_2\partial_2^{2l+1}\Lambda^{-m}S_2^{-1})_{2,[0]}.$$

This is equivalent to what we have to prove. If $l=0$ we get

$$\partial_2(S_1)S_1^{-1}\iota_{\Lambda^{-1}}(\Lambda-\Lambda^{-1})^{-1} = \frac{1}{2}\sum_{m=1}^{\infty}(\partial_2\cdot S_2\Lambda^{-m}S_2^{-1})_{2,[0]}.$$

It remains only to use that $L_2 = S_2\partial_2 S_2^{-1} = \partial_2 + O(\partial_2^{-1})$, i.e., the coefficients in front of ∂_2^0 of L_2 is 0. □

The same argument that we used to prove Proposition 41 yields the following Proposition.

Proposition 43. *We have*

$$\begin{aligned}
\partial_{a,2l+1}\mathcal{L} &= \pi_+([B^+_{a,2l+1},\mathcal{L}]),\\
\partial_{a,2l+1}H_i &= -\pi_+(H_iB^+_{a,2l+1}), \quad (1\leq i\leq 3),
\end{aligned}$$

where $a=2,3$.

7.4. **The derivations ∂_2 and ∂_3.** Let ξ be one of the coordinate functions (74). We would like to prove that the derivative $\partial_a(\xi)$ $(a=2,3)$ is a formal power series in ϵ whose coefficients are differential polynomials involving only the derivative ∂_x. To begin with let us point out that by comparing the formulas for q_a and c_a $(a=2,3)$ in terms of the tau-function it follows that

$$\partial_2(q_3)=\partial_3(q_2)=\frac{e^{\epsilon\partial_x}-1}{e^{\epsilon\partial_x}+1}(q_2q_3) \tag{75}$$

and that

$$\partial_a(q_a)=\frac{1}{2}\left(1-e^{\epsilon\partial_x}\right)(c_a),\quad a=2,3. \tag{76}$$

Let us check that the derivatives $\partial_2(a_i)$ $(1\le i\le n-3)$ and $\partial_2(c_a)$ $(a=2,3)$ can be expressed in terms of ∂_x-differential polynomials. The argument for $\partial_3(a_i)$ and $\partial_3(c_a)$ is identical. Let us recall the differential equation

$$\partial_2(\mathcal{L})=\pi_+([B^+_{2,1},\mathcal{L}]). \tag{77}$$

We have $B^+_{2,1}=Q_2$ and the operator $\mathcal{L}$ has the form

$$a_{n-3}\Lambda^{n-2}+a_{n-4}\Lambda^{n-3}+\sum_{k=2}^{n-4}(a_{k-1}-a_{k+1})\Lambda^k+\Big(-a_2+\frac{1}{4}(c_2-c_3)\Big)\Lambda+$$
$$\Big(-a_1-a_1[-1]+\frac{1}{2}(c_2+c_3)\Big)\Lambda^0+\cdots,$$

where the dots stand for terms involving only negative powers of Λ. Let us split $[Q_2,\mathcal{L}]$ into sum of two commutators $[Q_2,\mathcal{L}-\frac{1}{2}(\partial_2^2+\partial_3^2)]$ and

$$\frac{1}{2}[Q_2,\partial_2^2+\partial_3^2]=-\frac{1}{2}(\partial_2^2+\partial_3^2)(Q_2)-\partial_2(Q_2)\partial_2-\partial_3(Q_2)\partial_3.$$

The first commutator is already in $\mathcal{E}^0_{(+)}$ and it is a Laurent series in Λ^{-1} whose coefficients are differential polynomials involving only ∂_x-derivatives. The projection π_+ of the second commutator is

$$-\frac{1}{2}(\partial_2^2+\partial_3^2)(Q_2)-\partial_2(Q_2)Q_2-\partial_3(Q_2)Q_3.$$

A straightforward computation, using formulas (75) and (76), shows that the above expression has leading order term of the type

$$\Big(\frac{1}{4}\partial_2(c_2-c_2[-1])-\frac{1}{2}\frac{1-e^{-\epsilon\partial_x}}{1+e^{\epsilon\partial_x}}\Big(\frac{e^{\epsilon\partial_x}-1}{e^{\epsilon\partial_x}+1}(q_2q_3)\cdot q_3+\frac{1}{2}(c_3-c_3[1])q_2\Big)+$$
$$+\frac{1}{2}(c_2-c_2[-1])q_2[-1]+\frac{1}{2}(c_3-c_3[-1])q_3[-1]\Big)\Lambda^0+O(\Lambda^{-1}).$$

Comparing the coefficients in front of Λ^k in (77) for $1\le k\le n-2$ we get that $\partial_2(a_i)$ $(1\le i\le n-3)$ and $\partial_2(c_2-c_3)$ can be expressed as differential polynomials that involve only ∂_x-derivatives. Comparing the coefficients in front of Λ^0 in (77) we get that

$$\frac{1}{2}\partial_2(c_2+c_3)-\frac{1}{4}\partial_2(c_2-c_2[-1]) = \frac{1}{8}(3+e^{-\epsilon\partial_x})\partial_2(c_2+c_3)-\frac{1}{8}(1-e^{-\epsilon\partial_x})\partial_2(c_2-c_3)$$

can be expressed in terms of differential polynomials that involve only ∂_x-derivatives. The operator $3+e^{-\epsilon\partial_x}$ is invertible, so the derivative $\partial_2(c_2+c_3)$ is also a differential polynomial involving only ∂_x-derivatives.

7.5. **The extended flows.** Let us define the following operator series

$$B^+_{0,l} := (B^+_{0,l,1}+B^+_{0,l,2}+B^+_{0,l,3})(\Lambda-\Lambda^{-1}),$$

where

$$B^+_{0,l,1} := -\left(S_1\left(\frac{\Lambda^{(n-2)l}}{(n-2)^l l!}(\epsilon\partial_x-h_l)\right)S_1^{-1}\sum_{m=0}^{\infty}\Lambda^{-2m-1}-\sum_{m=0}^{\infty}\Lambda^{2m+1}\left(S_1\left(\frac{\Lambda^{(n-2)l}}{(n-2)^l l!}(\epsilon\partial_x-h_l)\right)S_1^{-1}\right)^{\#}\right)_{1,<0},$$

$$B^+_{0,l,2} := \frac{1}{2}\iota_{\Lambda^{-1}}\left(S_2\,\frac{\partial_2^{2l}}{2^l l!}\,\frac{\epsilon\partial_x}{\Lambda-1}\,S_2^{-1}\right)_{2,[0]},$$

and

$$B^+_{0,l,3} := \frac{1}{2}\iota_{\Lambda^{-1}}\left(S_3\,\frac{\partial_3^{2l}}{2^l l!}\,\frac{\epsilon\partial_x}{\Lambda+1}\,S_3^{-1}\right)_{3,[0]}.$$

Note that $B^+_{0,l} = -A^+_{1,k}+\pi_+(B_{0,l})$. Next we will prove that $B^+_{0,l}$ are Laurent series in Λ^{-1} whose coefficients belong to $\mathcal{R}$. Note that the coefficients are apriori differential operators in ∂_x of order 1. However, recalling Proposition 28 with $k=l$ and $\mathbf{t}'=\mathbf{t}$ we see that the coefficients in front of ∂_x is 0, i.e., the coefficients of $B^+_{0,l}$ viewed as a Laurent series in Λ^{-1} are scalar functions. Since we already know that the coefficients of $L_1 = S_1\Lambda S_1^{-1}$ and $L_a = S_a\partial_a S_a^{-1}$ $(a=2,3)$ are in $\mathcal{R}$ it is enough to prove the following Lemma.

Lemma 44. *The coefficients of the operator series*

$$\ell_1 := \epsilon\partial_x(S_1)\,S_1^{-1},\quad \ell_a := \epsilon\partial_x(S_a)\,S_a^{-1}\ (a=2,3)$$

are in $\mathcal{R}$.

The proof of Lemma 44 is the same as the proof of Proposition 14, b) and Proposition 15, b).

Lemma 45. *The wave operator* S_1 *satisfies the differential equations*

$$\partial_{0,l}(S_1) = B^+_{0,l}\,S_1,\quad \ell\geq 1.$$

Proof. Let us differentiate the HBEs corresponding to $k=0$ in Proposition 28 with respect to $t_{0,l}$, put $\mathbf{t}'=\mathbf{t}$, and compare the coefficients in front of the negative powers of Λ. We get exactly the identity stated in the lemma. □

Just like before, this lemma alows us to prove the following proposition.

Proposition 46. *We have*

$$\begin{aligned}\partial_{0,l}\mathcal{L} &= \pi_+([B^+_{0,l},\mathcal{L}]),\\ \partial_{0,l}H_i &= -\pi_+(H_iB^+_{0,l}),\quad (l\geq 1).\end{aligned}$$

Let us point out that $B^+_{0,0}=\ell_1$. Therefore, if we put $\partial_{0,0}=\epsilon\partial_x$, then the equations in Proposition 46 will hold for $l=0$ as well.

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8. Appendix

In this section, the examples of the projections and flows equations are given.

8.1. Example of projections.

• The projection π_+ of ∂_2^k,

$$\pi_+(\partial_2) = q_2[-1] + \sum_{l=1}^{+\infty}(q_2[-l-1] + q_2[-l])\Lambda^{-l},$$

$$\pi_+(\partial_2^2) = \Big(\partial_2(q_2) + q_2^2\Big)[-1] + \Big(\partial_2(q_2[-1] + q_2) + (q_2[-1] + q_2)^2\Big)[-1]\Lambda^{-1}$$
$$+ \Big(\partial_2(q_2[-3] + q_2[-2]) + (q_2[-3] + q_2[-2])(2q_2[-1] + q_2[-2] + q_2[-3])\Big)\Lambda^{-2} + O(\Lambda^{-3}),$$

$$\pi_+(\partial_2^3) = \Big(\partial_2^2(q_2) + 3\partial_2(q_2)\cdot q_2 + q_2^3\Big)[-1] + O(\Lambda^{-1}),$$

$$\pi_+(\partial_2^4) = \Big(\partial_2^3(q_2) + 4\partial_2^2(q_2)\cdot q_2 + 3(\partial_2 q_2)^2 + 6\partial_2(q_2)q_2^2 + q_2^4\Big)[-1] + O(\Lambda^{-1}).$$

• The projection π_+ of ∂_3^k,

$$\pi_+(\partial_3) = q_3[-1] + \sum_{k=1}^{+\infty}(-1)^k(q_3[-k-1] + q_3[-k])\Lambda^{-k},$$

$$\pi_+(\partial_3^2) = \Big(\partial_3(q_3) + q_3^2\Big)[-1] - \Big(\partial_3(q_3[-1] + q_3) + (q_3[-1] + q_3)^2\Big)[-1]\Lambda^{-1}$$
$$+ \Big(\partial_2(q_3[-3] + q_3[-2]) + (q_3[-3] + q_3[-2])(2q_3[-1] + q_3[-2] + q_3[-3])\Big)\Lambda^{-2} + O(\Lambda^{-2}),$$

$$\pi_+(\partial_3^3) = \Big(\partial_3^2(q_3) + 3\partial_3(q_3)\cdot q_3 + q_3^3\Big)[-1] + O(\Lambda^{-1}),$$

$$\pi_+(\partial_3^4) = \Big(\partial_3^3(q_3) + 4\partial_3^2(q_3)\cdot q_3 + 3(\partial_3 q_3)^2 + 6\partial_3(q_3)q_3^2 + q_3^4\Big)[-1] + O(\Lambda^{-1}).$$

• The projection π_2 of $\Lambda^{\pm k}$,

$$\pi_2(\Lambda) = 1 + 2q_2\partial_2^{-1} + 2(q_2^2 - \partial_2(q_2))\partial_2^{-2} + O(\partial_2^{-3}),$$

$$\pi_2(\Lambda^{-1}) = 1 - 2q_2[-1]\partial_2^{-1} + 2(q_2^2 - \partial_2(q_2))[-1]\partial_2^{-2} + O(\partial_2^{-3}),$$

$$\pi_2(\Lambda^2) = 1 + 2(q_2 + q_2[1])\partial_2^{-1} + 2\Big((q_2 + q_2[1])^2 - \partial_2(q_2 + q_2[1])\Big)\partial_2^{-2} + O(\partial_2^{-3}),$$

$$\pi_2(\Lambda^{-2}) = 1 - 2(q_2 + q_2[-1])[-1]\cdot\partial_2^{-1} + 2\Big((q_2 + q_2[-1])^2 - \partial_2(q_2 + q_2[-1])\Big)[-1]\cdot\partial_2^{-2} + O(\partial_2^{-3}).$$

- The projection π_2 of ∂_3^k,

$$\pi_2(\partial_3) = -\partial_2^{-1} q_1,$$
$$\pi_2(\partial_3^2) = -\partial_3(q_1)\partial_2^{-1} + (\partial_2\partial_3(q_1) + q_1^2)\partial_2^{-2} + O(\partial_2^{-2}).$$

- The projection π_3 of $\Lambda^{\pm k}$,

$$\pi_3(\Lambda) = -1 - 2q_3\partial_3^{-1} + 2(\partial_3(q_3) - q_3^2)\partial_3^{-2} + O(\partial_3^{-3}),$$
$$\pi_3(\Lambda^{-1}) = -1 + 2q_3[-1]\partial_3^{-1} - 2(q_3^2 + \partial_3(q_3))[-1]\partial_2^{-2} + O(\partial_3^{-3}),$$
$$\pi_3(\Lambda^2) = 1 + 2(q_3 + q_3[1])\partial_3^{-1} + 2\Big((q_3 + q_3[1])^2 - \partial_3(q_3 + q_3[1])\Big)\partial_3^{-2} + O(\partial_3^{-3}),$$
$$\pi_2(\Lambda^{-2}) = 1 - 2(q_3 + q_3[-1])[-1]\cdot\partial_3^{-1} + 2\Big((q_3 + q_3[-1])^2 + \partial_3(q_3 + q_3[-1])\Big)[-1]\cdot\partial_3^{-2} + O(\partial_3^{-3}).$$

- The projection π_3 of ∂_2^k,

$$\pi_3(\partial_2) = -\partial_3^{-1} q_1,$$
$$\pi_3(\partial_2^2) = -\partial_2(q_1)\partial_3^{-1} + (\partial_2\partial_3(q_1) + q_1^2)\partial_3^{-2} + O(\partial_3^{-2}).$$

8.2. **Examples of Lax operator.** Take $n = 4$ as an example. The Lax operator will be

$$\mathcal{L} = a\Lambda^2 + \frac{1}{4}(c_2 - c_3)\Lambda + \frac{1}{2}(c_2 + c_3) - a - a[-1] + \frac{1}{4}(c_2 - c_3)\Lambda^{-1} + a[-1]\Lambda^{-2} + \frac{1}{2}\partial_2^2 + \frac{1}{2}\partial_3^2,$$

where $a = \frac{1}{2}e^{\frac{\alpha}{2}}$. Then we will obtain

$$\pi_+(\mathcal{L}) = a\Lambda^2 + \frac{1}{4}(c_2 - c_3)\Lambda + v_{1,0} + v_{1,1}\Lambda^{-1} + v_{1,2}\Lambda^{-2} + O(\Lambda^{-3}),$$
$$\pi_i(\mathcal{L}) = \frac{1}{2}\partial_i^2 + c_i + v_{i,1}\partial_i^{-1} + v_{i,2}\partial_i^{-2} + O(\partial_i^{-3}),$$

where

$$\begin{aligned}
v_{1,0} =& \frac{1}{4}(c_2 + c_3) + \frac{1}{4}(c_2 + c_3)[1] - a - a[-1] + \frac{1}{2}\Big(q_2^2 + q_3^2\Big)[-1],\\
v_{1,1} =& \frac{1}{4}(c_2 - c_3)[-2] + \frac{1}{2}\sum_{l=2}^{3}(-1)^l(q_l[-2] + q_l[-1])^2,\\
v_{1,2} =& a[-1] + \frac{1}{4}(c_2 + c_3)[-3] - \frac{1}{4}(c_2 + c_3)[-1]\\
&+ \frac{1}{2}\sum_{l=2}^{3}(q_l[-3] + q_l[-2])(2q_l[-1] + q_l[-2] + q_l[-3]),\\
v_{i,1} =& 2a(q_i + q_i[1]) + \frac{(-1)^i}{2}(c_2 - c_3)(q_i + q_i[-1]) - 2a[-1](q_i[-1] + q_i[-2]),\\
v_{i,2} =& 2a((q_i + q_i[1])^2 - \partial_i(q_i + q_i[1])) + \frac{(-1)^i}{2}(c_2 - c_3)(q_i^2 - q_i[-1]^2\\
&- \partial_i(q_i - q_i[-1])) + 2a[-1]((q_i[-1] + q_i[-2])^2 - \partial_i(q_i[-1] + q_i[-2])).
\end{aligned}$$

The expressions of $B_{i,k}$ are listed as follows

$$
\begin{aligned}
B_{0,1} &= \mathcal{L}\cdot\epsilon\partial_x - \frac{1}{2}(a_{2,1}\partial_2 + a_{3,1}\partial_3) + \sum_{l=-2}^{2} b_{0,l}\Lambda^l \\
B_{1,1} &= e^{\beta}(\Lambda-\Lambda^{-1}), \quad B_{2,1} = \partial_2, \quad B_{3,1} = \partial_3 \\
B_{1,2} &= \left(2a\Lambda + \frac{1}{2}(c_2-c_3) - 2a[-1]\Lambda^{-1}\right)\cdot(\Lambda-\Lambda^{-1}) \\
B_{2,3} &= \partial_2^3 + 3c_2\partial_2, \quad B_{3,3} = \partial_3^2 + 3c_3\partial_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\beta &= \frac{1}{1+e^{\epsilon\partial_x}}\left(\frac{\alpha}{2}\right), \quad b_{0,2} = -a\left(\frac{1}{2} + a_{1,0}[2]\right), \\
b_{0,1} &= -\frac{1}{4}(c_2-c_3)(\frac{1}{2} + a_{1,0}[1]) - aa_{1,1}[2], \\
b_{0,0} &= -(b_{0,2}+b_{0,-2}) + \epsilon\partial_x a[-1] - a_{1,0}[1]\cdot a[-1] - a_{2,1} - \frac{1}{2}a_{2,2}, \\
b_{0,-1} &= -b_{0,1} + \frac{1}{4}\epsilon\partial_x(c_2-c_3), \\
b_{0,-2} &= \left(\frac{1}{2} + a_{1,0}[2]\right)\cdot a[-1] + \epsilon\partial_x a[-1].
\end{aligned}
$$

Here $a_{k,l}$ comes from $\ell_i = \epsilon\partial_x(S_i)\cdot S_i^{-1}$ with $i = 1,2,3$,

$$
\begin{aligned}
&\ell_1 = a_{1,0} + a_{1,1}\Lambda^{-1} + O(\Lambda^{-2}) \\
&\ell_2 = a_{2,1}\partial_2^{-1} + a_{2,2}\partial_2^{-2} + O(\partial_2^{-3}), \quad \ell_3 = a_{3,1}\partial_3^{-1} + a_{3,2}\partial_3^{-2} + O(\partial_3^{-3}),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{1,0} &= \frac{\epsilon\partial_x}{1-e^{\epsilon\partial_x}}(\beta), \quad a_{1,1} = \frac{1}{2}e^{\beta}\frac{\epsilon\partial_x}{1-e^{2\epsilon\partial_x}}\left((c_2-c_3)e^{-\beta}\right), \quad a_{i,1} = \frac{2\epsilon\partial_x}{e^{\epsilon\partial_x}-1}(q_i), \\
a_{i,2} &= \frac{2\epsilon\partial_x}{e^{\epsilon\partial_x}-1}\left(-\partial_i(q_i) - q_i^2 + \sum_{m=0}^{+\infty}\sum_{i=1}^{m+1}\sum_{j=0}^{i-1}\binom{i-1}{j}(\epsilon\partial_x)^j(a_{i,1})(\epsilon\partial_x)^{m-j}(a_{i,1})\right), \quad i = 2,3.
\end{aligned}
$$

8.3. **Examples of flows of $t_{i,1}$ for $i = 1,2,3$.** Flows of $t_{1,1}$ are

$$
\begin{aligned}
\partial_{1,1}(a) &= \frac{1}{4}\left(e^{\beta}\cdot(c_2-c_3)[1] - e^{\beta[1]}\cdot(c_2-c_3)\right), \\
\partial_{1,1}(q_2) &= e^{\beta[1]}(q_2+q_2[1]) - e^{\beta}(q_2+q_2[-1]), \\
\partial_{1,1}(q_3) &= e^{\beta}(q_3+q_3[-1]) - e^{\beta[1]}(q_3+q_3[1]), \\
\partial_{1,1}(c_2) &= c_2[1] - c_2[-1], \quad \partial_{1,1}(c_3) = c_3[-1] - c[1].
\end{aligned}
$$

Flows of $t_{2,1}$ are

$$
\partial_2(a) = a(q_2[-1] - q_2[1]), \quad \partial_2(q_2) = \frac{1}{2}(c_2 - c_2[1]), \quad \partial_2(q_3) = \frac{e^{\epsilon\partial_x}-1}{e^{\epsilon\partial_x}+1}(q_2q_3),
$$

$$
\begin{aligned}
\partial_2(c_2) =&\frac{1}{1+e^{-\epsilon\partial_x}}\Big(-(c_2-c_3)q_2+(c_2-c_3)[-1]\cdot q_2[-2]-4a(q_2+q_2[1])\\
&+4a[-1](q_2[-2]-q_2)+4a[-2](q_2[-2]+q_2[-3])-q_2[-1](c_2[-1]-c_2)\\
&-2q_3[-1]\frac{1-e^{-\epsilon\partial_x}}{1+e^{\epsilon\partial_x}}(q_2q_3)\Big),\\
\partial_2(c_3) =&\frac{1}{1+e^{-\epsilon\partial_x}}\Big(q_2[-1](c_3-c_3[-1])-2q_3[-1]\frac{1-e^{-\epsilon\partial_x}}{1+e^{\epsilon\partial_x}}(q_2q_3)\Big).
\end{aligned}
$$

And flows of $t_{3,1}$ are given by

$$
\begin{aligned}
\partial_3(a) =&a(q_3[-1]-q_3[1]),\quad \partial_3(q_3)=\frac{1}{2}(c_3-c_3[1]),\quad \partial_3(q_3)=\frac{e^{\epsilon\partial_x}-1}{e^{\epsilon\partial_x}+1}(q_2q_3),\\
\partial_3(c_2) =&\frac{1}{1+e^{-\epsilon\partial_x}}\Big(q_3[-1](c_2-c_2[-1])-2q_2[-1]\frac{1-e^{-\epsilon\partial_x}}{1+e^{\epsilon\partial_x}}(q_2q_3)\Big),\\
\partial_3(c_3) =&\frac{1}{1+e^{-\epsilon\partial_x}}\Big((c_2-c_3)q_3-(c_2-c_3)[-1]\cdot q_3[-2]-4a(q_3+q_3[1])\\
&+4a[-1](q_3[-2]-q_3)+4a[-2](q_3[-2]+q_3[-3])-q_3[-1](c_3[-1]-c_3)\\
&-2q_2[-1]\frac{1-e^{-\epsilon\partial_x}}{1+e^{\epsilon\partial_x}}(q_2q_3)\Big).
\end{aligned}
$$

8.4. **Examples of flows of $t_{1,2}$, $t_{2,3}$ and $t_{3,2}$.** Flows of $t_{1,2}$ are

$$
\begin{aligned}
\partial_{1,2}(a) =&\frac{1}{2}a\Big((c_2+c_3)[2]+(c_2+c_3)[1]-(c_2+c_3)-(c_2+c_3)[-1]\\
&+2q_2[1]^2+2q_3[1]^2-2q_2[-1]^2-2q_3[-1]^2-8a[1]+8a[-1]\Big),\\
\partial_{1,2}(q_2) =&2a[1](q_2[1]+q_2[2])+2a(q_2[-1]-q_2[1])-2a[-1](q_2[-1]+q_2[-2])\\
&-\frac{1}{2}(c_2-c_3)(q_2[-1]+q_2)+\frac{1}{2}(c_2-c_3)[1]\cdot(q_2+q_2[1]),\\
\partial_{1,2}(q_3) =&2a[1](q_3[1]+q_3[2])+2a(q_3[-1]-q_3[1])-2a[-1](q_3[-1]+q_3[-2])\\
&+\frac{1}{2}(c_2-c_3)(q_3[-1]+q_3)-\frac{1}{2}(c_2-c_3)[1]\cdot(q_3+q_3[1]),\\
\partial_{1,2}(c_2) =&-2a\Big(2(q_2+q_2[1])^2+c_2-c_2[2]\Big)+2a[-1]\Big(2(q_2[-1]+q_2[-2])^2+c_2-c_2[-2]\Big)\\
&-\frac{1}{2}(c_2-c_3)\Big(2q_2[-1]^2-2q_2^2+c_2[-1]-c_2[1]\Big),\\
\partial_{1,2}(c_3) =&2a\Big(2(q_3+q_3[1])^2+c_3[2]-c_3\Big)-2a[-1]\Big(2(q_3[-1]+q_3[-2])^2+c_3[-2]-c_3\Big)\\
&+\frac{1}{2}(c_2-c_3)\Big(2q_3[-1]^2-2q_3^2+c_3[-1]-c_3[1]\Big).
\end{aligned}
$$

The flows of $t_{2,3}$ are given by

$$
\begin{aligned}
\partial_{2,3}(a) =&a\Big(\partial_2^2(q_2[-1]-q_2[1])+3\partial_2(q_2[-1])\cdot q_2[-1]-3\partial_2(q_2[1])\cdot q_2[1]\\
&+q_2[-1]^3-q_2[1]^3+3c_2\cdot q_2[-1]-3c_2[2]\cdot q_2[1]\Big),
\end{aligned}
$$

$$
\begin{aligned}
\partial_{2,3}(c_2) =&\partial_2^3(c_2)+3\partial_2^2(v_{2,1})+3\partial_2(v_{2,2})+3c_2\partial_2(c_2),\\
\partial_{2,3}(c_3) =&\partial_2^2\partial_3(q_1)+3\partial_3(c_2q_1),\\
\partial_{2,3}(q_2) =&\partial_2^3(q_2)+3q_2\cdot\partial_2^2(q_2)+3\partial_2(q_2)\cdot q_2^2+3\partial_2(c_2[1]q_2)+3(\partial_2(q_2))^2\\
\partial_{2,3}(q_3) =&-(3c_2[1]+q_2^2)\cdot(q_2q_3+q_1[1])+3\partial_3(c_2[1])q_2-\partial_2(q_2)\cdot(3q_2q_3+q_1[1])\\
&-2\partial_2(q_1[1])\cdot q_2-\partial_2^2(q_1[1])-q_3\cdot\partial_2^2(q_2).
\end{aligned}
$$

The flows of $t_{3,3}$ are given by

$$
\begin{aligned}
\partial_{3,3}(a) =&a\Big(\partial_3^2(q_3[-1]-q_3[1])+3\partial_3(q_3[-1])\cdot q_3[-1]-3\partial_3(q_3[1])\cdot q_3[1]\\
&+q_3[-1]^3-q_3[1]^3+3c_3\cdot q_3[-1]-3c_3[2]\cdot q_3[1]\Big),\\
\partial_{3,3}(c_2) =&\partial_2\Big(\partial_3^2(q_1)+3c_3q_1\Big),\\
\partial_{3,3}(c_3) =&\partial_3^3(c_3)+3\partial_3^2(v_{3,1})+3\partial_3(v_{3,2})+3c_3\partial_3(c_3),\\
\partial_{3,3}(q_2) =&-(3c_3[1]+q_3^2)\cdot(q_2q_3+q_1[1])+3\partial_2(c_3[1])q_3-\partial_3(q_3)\cdot(3q_2q_3+q_1[1])\\
&-2\partial_3(q_1[1])\cdot q_3-\partial_3^2(q_1[1])-q_2\cdot\partial_3^2(q_3),\\
\partial_{3,3}(q_3) =&\partial_3^3(q_3)+3q_3\cdot\partial_3^2(q_3)+3\partial_3(q_3)\cdot q_3^2+3\partial_3(c_3[1]q_3)+3(\partial_3(q_3))^2.
\end{aligned}
$$

8.5. **Examples of flows of $t_{0,1}$.** Flows of $t_{0,1}$ are

$$
\begin{aligned}
\partial_{0,1}(a) =&a\cdot\epsilon\partial_xv_{1,0}[2]+\frac{1}{16}(c_2-c_3)\cdot\epsilon\partial_x(c_2-c_3)[1]+v_{1,0}\cdot\epsilon\partial_x(a)+b_{0,2}(v_{1,0}[2]-v_{1,0})\\
&+\frac{1}{4}\Big(b_{0,1}\cdot(c_2-c_3)[1]-(c_2-c_3)\cdot b_{1,0}[1]\Big)+a\Big(b_{0,0}-b_{0,0}[2]-\frac{1}{2}(a_{2,1}\cdot q_2[-1]\\
&+a_{3,1}\cdot q_3[-1])+\frac{1}{2}(a_{2,1}[2]q_2[1]+a_{3,1}[2]\cdot q_3[1])\Big),\\
\partial_{0,1}(c_2) =&\frac{1}{2}\epsilon\partial_x\Big(\partial_2^2(c_2)+c_2^2+2\partial_2(v_{2,1})+v_{2,2}\Big)-\frac{1}{2}a_{2,1}\partial_2(c_2)-\frac{1}{2}\partial_2^2(b_{0,0,2})-\partial_2(b_{0,1,2}),\\
\partial_{0,1}(c_3) =&\frac{1}{2}\epsilon\partial_x\Big(\partial_3^2(c_3)+c_2^2+2\partial_3(v_{3,1})+v_{3,2}\Big)-\frac{1}{2}a_{3,1}\partial_3(c_3)-\frac{1}{2}\partial_3^2(b_{0,0,3})-\partial_3(b_{0,1,3}),\\
\partial_{0,1}(q_2) =&\partial_2\Big(b_{0,0}[1]-b_{0,1}-\frac{1}{2}a_{2,1}[1]q_2-\frac{1}{2}a_{3,1}[1]q_3\Big)+q_2\Big(b_{0,2}[1]-b_{0,2}+b_{0,1}[1]-2b_{0,1}\Big)\\
&+q_2[1]\Big(b_{0,2}[1]-b_{0,2}+b_{0,1}[1]\Big)-\epsilon\partial_xA_2,\\
\partial_{0,1}(q_3) =&\partial_3\Big(b_{0,0}[1]+b_{0,1}-\frac{1}{2}a_{2,1}[1]q_2-\frac{1}{2}a_{3,1}[1]q_3\Big)+q_2\Big(b_{0,2}[1]+b_{0,2}+b_{0,1}[1]+2b_{0,1}\Big)\\
&+q_2[1]\Big(b_{0,2}[1]+b_{0,2}+b_{0,1}[1]\Big)-\epsilon\partial_xA_3.
\end{aligned}
$$

Here

$$
b_{0,0,2} =\epsilon\partial_x\Big(a[-1]+\frac{1}{4}(c_2-c_3)\Big)-a_{1,0}[1]\cdot a[-1]-a_{2,1}-\frac{1}{2}a_{2,2},
$$

$$
\begin{aligned}
b_{0,0,3} =& \epsilon\partial_x\Big(a[-1]-\frac{1}{4}(c_2-c_3)\Big)-a_{1,0}[1]\cdot a[-1]-a_{2,1}-\frac{1}{2}a_{2,2},\\
b_{0,1,2} =& 2b_{0,2}\Big(q_2+q_2[1]\Big)+2b_{0,1}q_2+2b_{0,-1}q_2[-1]-2b_{0,-2}\Big(q_2[-1]+q_2[-2]\Big)+\frac{1}{2}a_{3,1}q_1,\\
b_{0,1,3} =& 2b_{0,2}\Big(q_3+q_3[1]\Big)-2b_{0,1}q_3-2b_{0,-1}q_3[-1]-2b_{0,-2}\Big(q_3[-1]+q_3[-2]\Big)+\frac{1}{2}a_{2,1}q_1,\\
A_2 =& \partial_2\Big(\frac{1}{2}(c_2+c_3)[1]-a-a[1]-\frac{1}{4}(c_2-c_3)\Big)+q_2\Big(a[1]-a+\frac{1}{4}(c_2-c_3)[1]-\frac{1}{2}(c_2-c_3)\\
&-\frac{1}{2}(\partial_2(q_2)+\partial_3(q_3)+q_2^2+q_3^2)\Big)+q_2[1]\cdot\Big(a[1]-a+\frac{1}{4}(c_2-c_3)[1]\Big),\\
A_3 =& \partial_3\Big(\frac{1}{2}(c_2+c_3)[1]-a-a[1]-\frac{1}{4}(c_2-c_3)\Big)+q_3\Big(a[1]-a-\frac{1}{4}(c_2-c_3)[1]+\frac{1}{2}(c_2-c_3)\\
&+\frac{1}{2}(\partial_2(q_2)+\partial_3(q_3)+q_2^2+q_3^2)\Big)+q_3[1]\cdot\Big(a[1]-a-\frac{1}{4}(c_2-c_3)[1]\Big).
\end{aligned}
$$

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