

# GROMOV–WITTEN INVARIANTS AND THE EXTENDED D-TODA HIERARCHY

JIPENG CHENG AND TODOR MILANOV

ABSTRACT. Fano orbifold lines are classified by the Dynkin diagrams of type  $A, D$ , and  $E$ . It is known that the corresponding total descendant potential is a tau-function of an appropriate Kac–Wakimoto hierarchy. It is also known that in the  $A$ -case the Kac–Wakimoto hierarchies admit an extension and that the total descendant potential is a tau-function of an extended Kac–Wakimoto hierarchy. The goal of this paper is to prove that in the  $D$ -case the total descendant potential is also a tau-function of an extended Kac–Wakimoto hierarchy.

## 1. INTRODUCTION

There are several ways to give a definition of an integrable hierarchy. In this paper we define an integrable hierarchy in the form of Hirota Bilinear Equations (HBEs). This approach to integrable hierarchies is closely related to representation theory of infinite dimensional Lie algebras and the outcome is an infinite system of quadratic equations involving the partial derivatives of a function  $\tau$ , called *tau-function*. The HBEs usually have a simple geometric interpretation, i.e., they provide the equations that define the orbit of the highest weight vector in some highest weight representation of a given infinite dimensional Lie algebra. On the other hand, Givental was able to reformulate the HBEs of the  $n$ -KdV hierarchy in terms of the Frobenius structure on the base space of a miniversal unfolding of the  $A_{n-1}$ -singularity (see [6]). All ingredients in Givental’s interpretation seem to be generalizable to any semi-simple Frobenius manifold. A natural question is whether HBEs can be constructed systematically for any semi-simple Frobenius manifold and whether such a class of HBEs, if it exists, has a representation-theoretic explanation. So far the answer is known to be positive for the Frobenius structures corresponding to simple singularities of type ADE (see [7]) and the quantum cohomology of  $\mathbb{P}_{a,b}^1$  (see [18]). In general, it might be too optimistic to expect that HBEs can be constructed for any orbifold  $X$  with semi-simple quantum cohomology. Especially if the dimension of  $X$  is  $> 1$ , then some new ideas are necessary.

On the other hand, if the target orbifold  $X$  has dimension 1, then  $X$  has semi-simple quantum cohomology if and only if  $X$  is an orbifold projective line  $\mathbb{P}_{a_1, \dots, a_k}^1$ . The problem of constructing HBEs seems to be approachable with the theory of generalized Kac–Moody Lie algebras. Let us separate the orbifold lines into three groups depending on whether the orbifold Euler characteristic is  $> 0$ ,  $= 0$ , or  $< 0$ . The easiest case is the case  $> 0$ . The corresponding orbifold lines are called *Fano orbifold lines* and they are classified by the Dynkin diagrams of type ADE. The problem of constructing HBEs for Fano orbifold lines was almost completely solved in [17]. Namely, Milanov–Shen–Tseng have

constructed a system of HBEs and identified them with an appropriate Kac–Wakimoto hierarchy, but the system of HBEs is not complete in a sense that certain dynamical variables were fixed. Therefore, the problem left is to construct an extension of the Kac–Wakimoto hierarchy.

The extension in the A-case is completely understood. Both the HBEs and the Lax equations of the corresponding integrable hierarchy are known (see [18] for more details and other references). In this paper, we would like to construct the extension of the Kac–Wakimoto hierarchy of type  $D$  and prove that the generating function of Gromov–Witten (GW) invariants of  $\mathbb{P}_{n-2,2,2}^1$  is a solution. The precise formulation of our result is given in Section 2. We also transform the HBEs in a form convenient for the applications to integrable hierarchies. Namely, in Section 3 we make a linear change of the dynamical variables and we work out the explicit form of our HBEs. In a subsequent paper we prove that the HBEs parametrize the solutions to an integrable hierarchy of Lax equations, which we propose to be called the *Extended D-Toda Hierarchy*. Our two papers solve the problem of finding an explicit description of the integrable hierarchy that governs the GW invariants of the Fano orbifold lines of type  $D$ , that is, the orbifold line  $\mathbb{P}_{n-2,2,2}^1$ .

**Acknowledgements.** The work of T.M. is partially supported by JSPS Grant-In-Aid (Kiban C) 17K05193 and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

## 2. HIROTA QUADRATIC EQUATIONS OF $\mathbb{P}_{n-2,2,2}^1$

Let us recall Givental’s construction of vertex operators in the settings of quantum cohomology of the orbifold  $\mathbb{P}_{n-2,2,2}^1$ .

**2.1. The orbifold line  $\mathbb{P}_{n-2,2,2}^1$ .** Put

$$G = \{t \in (\mathbb{C}^*)^3 \mid t_1^{n-2} = t_2^2 = t_3^2\}$$

then the orbifold  $\mathbb{P}_{n-2,2,2}^1$  is defined to be the translation groupoid  $[Z/G]$  where

$$Z := \{z_1^{n-2} + z_2^2 + z_3^2 = 0\} \subset (\mathbb{C} \setminus \{0\})^3.$$

The orbit space of  $\mathbb{P}_{n-2,2,2}^1$  is  $Z/G \cong \mathbb{P}^1$  where the isomorphism is induced from the map

$$Z \rightarrow \mathbb{P}^1, \quad (z_1, z_2, z_3) \mapsto [z_1^{n-2} : z_2^2].$$

By definition the inertia orbifold

$$I\mathbb{P}_{n-2,2,2}^1 = \bigsqcup_{g \in G} [Z^g/G], \quad Z^g := \{z \in Z \mid gz = z\}.$$

The connected components of the inertia orbifold are non-empty only if

$$g \in G_1 \cup G_2 \cup G_3,$$

where  $G_i$  is the subgroup of  $G$  consisting of  $t = (t_1, t_2, t_3)$  such that  $t_j = 1$  for all  $j \neq i$ . Note that  $G_i$  is a cyclic group of order  $a_i$ , where  $a_1 = n - 2$  and  $a_2 = a_3 = 2$ . The connected component for  $g = 1$  is  $\mathbb{P}_{n-2,2,2}^1$ , while for  $g \neq 1$  it is isomorphic to the orbifold point  $[\text{pt}/G_i]$  where  $i$  is such that  $g \in G_i$ . The connected components for  $g \neq 1$  are known as *twisted sectors*.

**2.2. Gromov–Witten invariants.** Let  $H := H_{\text{CR}}^*(\mathbb{P}_{n-2,2,2}^1; \mathbb{C})$  be the Chen–Ruan cohomology, i.e.,

$$H = \bigoplus_{g \in G_1 \cup G_2 \cup G_3} H^{*-l(g)}(Z^g/G, \mathbb{C}),$$

where  $*$  denotes complex degree (i.e. half of the standard degree) and the shift  $l(g)$  is defined for all finite order elements

$$g = (e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}, e^{2\pi i \alpha_3}) \in G, \quad \alpha_i \in \mathbb{Q} \cap [0, 1)$$

by

$$l(g) := \alpha_1 + \alpha_2 + \alpha_3.$$

Following [17] we fix a basis of  $H$  as follows

$$\phi_{0,0} = 1, \quad \phi_{0,1} = P,$$

and

$$\phi_{i,p} = 1 \in H^0(Z^{g_{i,p}}/G, \mathbb{C}), \quad 1 \leq i \leq 3, \quad 1 \leq p \leq a_i - 1,$$

where  $P \in H^2(\mathbb{P}^1, \mathbb{C})$  is the hyperplane class and  $g_{i,p} \in G_i$  is the element whose  $i$ th entry is  $e^{2\pi i p/a_i}$ . Let us define  $a_0 = 1$ , then the Chen–Ruan degree of the above basis is given by

$$\deg_{\text{CR}}(\phi_{i,p}) = p/a_i, \quad 0 \leq i \leq 3.$$

Recall that the descendant orbifold Gromov–Witten invariants of  $\mathbb{P}_{n-2,2,2}^1$  are defined as intersection numbers

$$\langle \phi_1 \psi_1^{k_1}, \dots, \phi_r \psi_r^{k_r} \rangle_{g,r,d} := \int_{[\overline{\mathcal{M}}_{g,r}(\mathbb{P}_{n-2,2,2}^1, d)]^{\text{virt}}} \text{ev}^*(\phi_1 \otimes \dots \otimes \phi_r) \psi_1^{k_1} \dots \psi_r^{k_r},$$

where  $\phi_i \in H$ ,  $g, r, d \in \mathbb{Z}_{\geq 0}$ , and the integral is interpreted as cap product with the virtual fundamental cycle on the moduli space  $\overline{\mathcal{M}}_{g,r}(\mathbb{P}_{n-2,2,2}^1, d)$  of orbifold stable maps. For more details we refer to [17] and the references therein. The following generating function

$$\mathcal{D}(\hbar, \mathbf{t}) = \exp \left( \sum_{g,r,d=0}^{\infty} \hbar^{g-1} \frac{Q^d}{r!} \langle \mathbf{t}(\psi_1), \dots, \mathbf{t}(\psi_r) \rangle_{g,n,d} \right)$$

is called the *total descendant potential*. Here  $Q \in \mathbb{C}^*$  is a complex parameter called the *Novikov variable*,  $\hbar, t_0, t_1, \dots \in H$  are formal vector variables, and  $\mathbf{t}(z) = \sum_{k=0}^{\infty} t_k z^k$ . The components of the vector variable  $t_k$  with respect to the basis  $\phi_{i,p}$  of  $H$  from above will be denoted by  $t_{i,p,k}$ .

**2.3. The calibration operator.** If  $t \in H$  then we define

$$\langle \phi_1 \psi_1^{k_1}, \dots, \phi_r \psi_r^{k_r} \rangle_{g,r}(t) = \sum_{\ell, d=0}^{\infty} \frac{Q^d}{\ell!} \langle \phi_1 \psi_1^{k_1}, \dots, \phi_r \psi_r^{k_r}, t, \dots, t \rangle_{g, r+\ell, d}.$$

Using the divisor equation and degree reasons one can prove that the above correlator is polynomial in  $t$  and  $Qe^{t_{0,1}}$ . In particular, it defines an analytic function on  $H$ .

The quantum cup product  $\bullet_{t,Q} ((t, Q) \in H \times \mathbb{C}^*)$  is a family of multiplications in  $H$  defined by the identity

$$(\phi_1 \bullet_{t,Q} \phi_2, \phi_3) := \langle \phi_1, \phi_2, \phi_3 \rangle_{0,3}(t)$$

for all  $\phi_1, \phi_2, \phi_3 \in H$ , where

$$(\phi_1, \phi_2) := \langle \phi_1, \phi_2, 1 \rangle_{0,3,0}$$

is the orbifold Poincare pairing. The specialization of the quantum cup product to  $t = Q = 0$  is known as the Chen–Ruan cup product  $\cup_{\text{CR}} := \bullet_{0,0}$ . Let us point out also that the orbifold Poincare pairing is given explicitly by the following formulas

$$(\phi_{i,p}, \phi_{j,q}) = \frac{1}{a_i} \delta_{i,j} \delta_{p+q, a_i}, \quad 0 \leq i, j \leq 3.$$

The multiplication and the Poincare pairing define a conformal Frobenius structure on  $H$  of conformal dimension 1 with Euler vector field

$$E = \sum_{i,p} (1 - p/a_i) t_{i,p} \frac{\partial}{\partial t_{i,p}} + \frac{1}{n-2} \frac{\partial}{\partial t_{0,1}}.$$

Note that  $\chi_{\text{orb}}(\mathbb{P}_{n-2,2,2}^1) = 1/(n-2)$  is the orbifold Euler characteristic. In particular, the connection  $\nabla$  on the vector bundle  $TH \times \mathbb{C}^* \rightarrow H \times \mathbb{C}^*$  defined by

$$\begin{aligned} \nabla_{\partial/\partial t_{i,p}} &= \frac{\partial}{\partial t_{i,p}} - z^{-1} \phi_{i,p} \bullet_{t,Q} \\ \nabla_{\partial/\partial z} &= \frac{\partial}{\partial z} + z^{-2} E \bullet_{t,Q} - z^{-1} \theta \end{aligned}$$

is flat, where

$$\theta : H \rightarrow H, \quad \theta(\phi_{i,p}) = \left( \frac{1}{2} - \frac{p}{a_i} \right) \phi_{i,p}$$

is the so called *Hodge grading operator*. We refer to [2] for further details on Frobenius structures.

The Dubrovin's connection has a solution of the form  $S(t, Q, z)z^\theta z^{-\rho}$ , where

$$\rho = \frac{1}{n-2} P \cup_{\text{CR}} = c_1(T\mathbb{P}_{n-2,2,2}^1) \cup_{\text{CR}}.$$

is a nilpotent operator and

$$S(t, Q, z) = 1 + S_1(t, Q)z^{-1} + S_2(t, Q)z^{-2} + \dots, \quad S_k(t, Q) \in \text{End}(H)$$

is an operator series defined by

$$(S(t, Q, z)a, b) = (a, b) + \sum_{k=0}^{\infty} z^{-k-1} \langle a \psi_1^k, b \rangle_{0,2}(t).$$

**2.4. Periods.** Let us recall the second structure connection

$$\begin{aligned} \nabla_{\partial_{i,p}}^{(n)} &= \partial_{i,p} + (\lambda - E_{\bullet})^{-1}(\phi_{i,p} \bullet)(\theta - n - 1/2) \\ \nabla_{\partial_{\lambda}}^{(n)} &= \partial_{\lambda} - (\lambda - E_{\bullet_t})^{-1}(\theta - n - 1/2), \end{aligned}$$

where  $n \in \mathbb{C}$  is a complex parameter and  $\partial_{i,p} := \partial/\partial t_{i,p}$ . This is a connection on the trivial bundle

$$(H \times \mathbb{C})' \times H \rightarrow (H \times \mathbb{C})',$$

where

$$(H \times \mathbb{C})' = \{(t, \lambda) \mid \det(\lambda - E_{\bullet}) \neq 0\}.$$

The hypersurface  $\det(\lambda - E_{\bullet_t}) = 0$  in  $H \times \mathbb{C}$  is called the *discriminant*.

Let us fix a reference point  $(t^{\circ}, \lambda^{\circ}) \in (H \times \mathbb{C})'$ , such that  $\lambda^{\circ}$  is a sufficiently large real number. Suppose that  $m \in \mathbb{Z}$  is an integer, such that  $m < \alpha$  for all eigenvalues  $\alpha$  of  $\theta + \frac{1}{2}$ . It is easy to check that the following functions provide a fundamental solution to the 2nd structure connection  $\nabla^{(m)}$

$$I^{(m)}(t, \lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t, Q) \tilde{I}^{(m+k)}(\lambda),$$

where

$$\tilde{I}^{(m)}(\lambda) = e^{-\rho \partial_{\lambda} \partial_m} \left( \frac{\lambda^{\theta - m - \frac{1}{2}}}{\Gamma(\theta - m + \frac{1}{2})} \right).$$

Note that the following relation is satisfied:

$$(1) \quad \partial_{\lambda} I^{(m)}(t, \lambda) = I^{(m+1)}(t, \lambda)$$

for all  $m$ , such that  $m + 1 < \alpha$  for all eigenvalues  $\alpha$  of  $\theta + \frac{1}{2}$ . Therefore, we can uniquely extend the definition of  $I^{(m)}(t, \lambda)$  for all  $m \in \mathbb{Z}$  in such a way that (1) is satisfied for all  $m \in \mathbb{Z}$ . The 2nd structure connection has a Fuchsian singularity at infinity, therefore the series  $I^{(m)}(t, \lambda)$  is convergent for all  $(t, \lambda)$  sufficiently close to  $(t^{\circ}, \lambda^{\circ})$ . Using the differential equations we extend  $I^{(m)}$  to a multi-valued analytic function on  $(H \times \mathbb{C})'$ . We define the following multi-valued functions taking values in  $H$ :

$$I_a^{(m)}(t, \lambda) := I^{(m)}(t, \lambda) a, \quad a \in H, \quad m \in \mathbb{Z}.$$

The functions  $I_a^{(m)}$  and  $\tilde{I}_a^{(m)}$  will be called respectively *periods* and *calibrated periods*.

Using analytic continuation we get a representation

$$(2) \quad \pi_1((H \times \mathbb{C})', (t^{\circ}, \lambda^{\circ})) \rightarrow \text{GL}(H)$$

called the *monodromy representation* of the Frobenius manifold. The image  $W$  of the monodromy representation is called the *monodromy group*.

Using the differential equations of the 2nd structure connection it is easy to prove that the pairing

$$(a|b) := (I_a^{(0)}(t, \lambda), (\lambda - E \bullet) I_b^{(0)}(t, \lambda))$$

is independent of  $t$  and  $\lambda$ . This pairing is known as the *intersection pairing*. The monodromy group  $W$  is generated by a set of reflections

$$w_a(x) = x - (a|x)a, \quad a \in \mathfrak{R},$$

where the set  $\mathfrak{R}$  is defined as follows. Under the semi-simplicity assumption, we may choose a generic reference point, such that the Frobenius multiplication  $\bullet_{t^\circ, Q}$  is semi-simple and the operator  $E \bullet_{t^\circ, Q}$  has  $N := n + 1$  pairwise different eigenvalues  $u_i^\circ$ . Let  $\mathfrak{R}$  be the set of all  $a \in H$  such that  $(a|a) = 2$  and there exists a simple loop in  $\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$  based at  $\lambda^\circ$  such that monodromy transformation along it transforms  $a$  into  $-a$ . Here simple loop means a loop that starts at  $\lambda^\circ$ , approaches one of the punctures  $u_i^\circ$  along a path  $\gamma$  that ends at a point sufficiently close to  $u_i^\circ$ , goes around  $u_i^\circ$ , and finally returns back to  $\lambda^\circ$  along  $\gamma$ .

**2.5. The integral lattice of Iritani.** Let  $X$  be a complex orbifold groupoid whose orbit space  $|X|$  is a projective variety. Using the  $K$ -ring  $K^0(X)$  of topological orbifold vector bundles on  $X$  and a certain  $\Gamma$ -modification of the Chern character map, Iritani has introduced an integral lattice in the Chen-Ruan cohomology group  $H_{\text{CR}}(X; \mathbb{C})$ . If  $X$  has semi-simple quantum cohomology, then it is expected that  $X$  has a LG mirror model and that Iritani's lattice coincide with the image of the Milnor lattice via an appropriate period map. Let us recall Iritani's construction in a form suitable for our purposes.

Let  $IX$  be the *inertia orbifold* of  $X$ , that is, as a groupoid the points of  $IX$  are

$$(IX)_0 = \{(x, g) \mid x \in X_0, g \in \text{Aut}(x)\}$$

while the arrows from  $(x', g')$  to  $(x'', g'')$  consists of all arrows  $g \in X_1$  from  $x'$  to  $x''$ , such that,  $g'' \circ g = g \circ g'$ . It is known that  $IX$  is an orbifold consisting of several connected components  $X_v$ ,  $v \in T := \pi_0(|IX|)$ .

Following Iritani [8], we define a linear map

$$\Psi : K^0(X) \rightarrow H^*(IX; \mathbb{C}) = \bigoplus_{v \in T} H^*(X_v; \mathbb{C})$$

via

$$(3) \quad \Psi(E) = (2\pi)^{(1-\dim X)/2} \left( \widehat{\Gamma}(X) e^{-\sum_{i=1}^r P_i \log Q_i} \right) \cup (2\pi\sqrt{-1})^{\deg} \text{inv}^* \widetilde{\text{ch}}(E),$$

where  $\{P_i\}_{i=1}^r \subset H^2(|X|, \mathbb{Z})$  is an ample basis and  $Q_i$  are the corresponding Novikov variables. Let us recall the rest of the notation. The linear operator

$$\deg : H^*(IX; \mathbb{C}) \rightarrow H^*(IX; \mathbb{C})$$

is defined by  $\deg(\phi) = i\phi$  if  $\phi \in H^{2i}(IX; \mathbb{C})$ . The involution  $\text{inv} : IX \rightarrow IX$  inverts all arrows while on the points it acts as  $(x, g) \mapsto (x, g^{-1})$ . If  $E$  is an orbifold vector bundle, then we have an eigenbasis decomposition

$$\text{pr}^*(E) = \bigoplus_{v \in T} E_v = \bigoplus_{v \in T} \bigoplus_{0 \leq f < 1} E_{v,f},$$

where  $\text{pr} : IX \rightarrow X$  is the forgetful map  $(x, g) \mapsto x$  and  $E_{v,f}$  is the subbundle of  $E_v := \text{pr}^*(E)|_{X_v}$  whose fiber over a point  $(x, g) \in (IX)_0$  is the eigenspace of  $g$  corresponding to the eigenvalue  $e^{2\pi\sqrt{-1}f}$ . Let us denote by  $\delta_{v,f,i}$  ( $1 \leq i \leq l_{v,f} := \text{rk}(E_{v,f})$ ) the Chern roots of  $E_{v,f}$ , then the Chern character and the  $\Gamma$ -class of  $E$  are defined by

$$\widetilde{\text{ch}}(E) = \sum_{v \in T} \sum_{0 \leq f < 1} e^{2\pi\sqrt{-1}f} \sum_{i=1}^{l_{v,f}} e^{\delta_{v,f,i}}$$

$$\widehat{\Gamma}(E) = \sum_{v \in T} \prod_{0 \leq f < 1} \prod_{i=1}^{l_{v,f}} \Gamma(1 - f + \delta_{v,f,i}),$$

where the value of the  $\Gamma$ -function  $\Gamma(1 - f + y)$  at  $y = \delta_{v,f,i}$  is obtained by first expanding in Taylor's series at  $y = 0$  and then formally substituting  $y = \delta_{v,f,i}$ . By definition  $\widehat{\Gamma}(X) := \widehat{\Gamma}(TX)$ . The cup product in (3) is the usual topological cup product on  $|IX|$ .

**Remark 1.** There are two differences between the maps  $\Psi$  defined respectively by (3) and formula (37) in [17]. First, the normalization factor in [17] is  $(2\pi)^{-\dim X/2}$  instead of  $(2\pi)^{(1-\dim X)/2}$  and second, the map  $\Psi$  in [17] does not depend on the Novikov variables.  $\square$

**Lemma 2.** *The intersection pairing is independent of the Novikov variable  $Q$  and the following formula holds:*

$$(a|b) = \langle a, b \rangle + \langle b, a \rangle, \quad \forall a, b \in H,$$

where

$$\langle a, b \rangle := \frac{1}{2\pi} \left( a, e^{\pi\sqrt{-1}\theta} e^{\pi\sqrt{-1}\rho} b \right).$$

The proof of Lemma 2 will be given in Section 4.1. Using the Kawasaki–Riemann–Roch formula, one can prove that the map  $\Psi$  intertwines the pairing  $\langle \cdot, \cdot \rangle$  defined in Lemma 2 and the Euler pairing on  $K^0(X)$ , that is,

$$(4) \quad \chi(E_1^\vee \otimes E_2) = \langle \Psi(E_1), \Psi(E_2) \rangle.$$

We refer to [8], Proposition 2.10, (iii) for the details of this computation.

**Remark 3.** Formula (4) is used only to conclude that its RHS is an integer, which in our settings could also be checked directly, because we will compute the RHS of (4) explicitly.

According to Lemma 2, we have

$$(5) \quad (\Psi(a)|\Psi(b)) = \chi(a^\vee \otimes b) + \chi(a \otimes b^\vee), \quad a, b \in K^0(X).$$

From now on we will use  $\Psi$  to identify  $H$  and  $K^0(X) \otimes \mathbb{C}$ . In particular, the above formula allows us to say that the intersection pairing is the symmetrization of the Euler pairing.

Let us specialize again to  $X = \mathbb{P}_{n-2,2,2}^1$ . According to Milanov–Shen–Tseng (see Theorem 12 and Proposition 13 in [17]) the set of reflection vectors

$$\mathfrak{R} = \{\Psi(a) \mid a \in K^0(X) \text{ such that } (a|a) = 2\}.$$

**2.6. Calibrated periods.** The  $K$ -ring of  $\mathbb{P}_{n-2,2,2}^1$  can be identified with

$$K := \mathbb{Z}[L_1, L_2, L_3] / \langle L_i^{a_i} - L_j^{a_j}, (L_i - 1)(L_j - 1) \mid 1 \leq i \neq j \leq 3 \rangle.$$

The generators  $L_i$  correspond to the orbifold line bundles  $[(Z \times \mathbb{C})/G]$ , where  $G$  acts on the fiber  $\mathbb{C}$  via the character

$$G \rightarrow \mathbb{C}^*, \quad g = (g_1, g_2, g_3) \mapsto g_i.$$

The orbifold tangent bundle  $T\mathbb{P}_{n-2,2,2}^1$  can be identified with  $[Q/G]$  where  $Q$  is the  $G$ -equivariant bundle on  $Z$  defined as a quotient

$$0 \rightarrow Z \times \mathfrak{g} \rightarrow TZ \rightarrow Q \rightarrow 0,$$

where  $\mathfrak{g}$  is the Lie algebra of  $G$  and the map  $Z \times \mathfrak{g} \rightarrow TZ$  is defined by

$$(z, \xi) \mapsto \left. \frac{d}{d\epsilon} \left( e^{\epsilon \xi} z \right) \right|_{\epsilon=0} \in T_z Z.$$

Note that in the  $K$ -ring we have

$$[TZ/G] = L_1 + L_2 + L_3 - L,$$

where  $L := L_i^{a_i}$ . Therefore

$$T\mathbb{P}_{n-2,2,2}^1 = L_1 + L_2 + L_3 - L - 1 = L_1 L_2 L_3 L^{-1}.$$

Let us introduce the following notation for the components of the Chern character map:

$$\widetilde{\text{ch}}(E) = \text{rk}(E)\phi_{0,0} + \deg(E)\phi_{0,1} + \sum_{j=1}^3 \sum_{p=1}^{a_j-1} \chi_{j,p}(E)\phi_{j,p}.$$

The first component  $\text{rk}(E)$  is the rank of  $E$ , the second one  $\deg(E) := \int_{[\mathbb{P}_{n-2,2,2}^1]} c_1(E)$  is the degree of  $E$ , and the remaining ones  $\chi_{j,p} : K \rightarrow \mathbb{C}$  are some *ring* homomorphisms. The rank is straightforward to compute, while for the remaining components we have the following explicit formulas:

$$\deg(L_i) = 1/a_i, \quad \chi_{j,p}(L_i) := e^{-\frac{2\pi i}{a_j} p \delta_{j,i}}.$$



Recalling the definition, we get that Iritani's map  $\Psi$  has the following explicit form (see also [17], formula (39)):

$$\Psi(E) = \text{rk}(E) \left( 1 - \frac{\gamma}{n-2} P \right) + 2\pi i \deg(E) P + \sum_{j=1}^3 \sum_{p=1}^{a_j-1} \Gamma(1 - p/a_j) \chi_{j,p}(E) \phi_{j,p},$$

where  $\gamma = -\Gamma'(1) + (n-2)\log Q$ .

Using formulas (4) and (5), we get (see Section 4.2 for similar computations)

$$(L_i^m | L_j^n) = 0 \quad \text{for all} \quad i \neq j, \quad 1 \leq m \leq a_i - 1, \quad 1 \leq n \leq a_j - 1$$

and

$$(L_i^m | L_i^n) = \begin{cases} 2 & \text{if } m \equiv n \pmod{a_i}, \\ 1 & \text{otherwise.} \end{cases}$$

The pairing  $(|)$  is degenerate with kernel spanned by  $(L-1)$ . Using the above formulas or the formulas in Section 4.2, we get that the following vectors project to an orthonormal basis of  $K/\mathbb{Z}(L-1)$ :

$$\begin{aligned} \epsilon_i^1 &= L_1^i + \frac{1}{2}(L_2 + L_3) - 1 \quad (1 \leq i \leq n-2), \\ \epsilon_1^2 &= \frac{1}{2}(L_2 + L_3) - 1, \\ \epsilon_1^3 &= \frac{1}{2}(L_2 - L_3). \end{aligned}$$

**Proposition 4.** *a) The calibrated period corresponding to an arbitrary  $\alpha \in K$  is given by the following formula*

$$\begin{aligned} \widetilde{I}_\alpha^{(-\ell-1)}(\lambda) &= \text{rk}(\alpha) \frac{\lambda^{\ell+1}}{(\ell+1)!} \phi_{0,0} + \frac{\lambda^\ell}{\ell!} \left( \frac{1}{n-2} \text{rk}(\alpha) (\log \lambda - C_\ell) + 2\pi i \deg(\alpha) \right) \phi_{0,1} + \\ &\quad \sum_{j=1}^3 \sum_{p=1}^{a_j-1} \frac{\lambda^{\ell+1-p/a_j}}{\left( \ell + 1 - \frac{p}{a_j} \right) \cdots \left( 1 - \frac{p}{a_j} \right)} \chi_{j,p}(\alpha) \phi_{j,p}, \end{aligned}$$

where  $C_0 := (n-2)\log Q$  and  $C_{\ell+1} = C_\ell + \frac{1}{\ell+1}$  for  $\ell \geq 0$ .

b) Let  $\sigma : K \rightarrow K$  be the classical monodromy operator defined by: the analytic continuation in anti-clockwise direction of  $\widetilde{I}_\alpha^{(m)}(\lambda)$  is  $\widetilde{I}_{\sigma(\alpha)}^{(m)}(\lambda)$ . Then  $\sigma(\alpha) = \alpha T \mathbb{P}_{n-2,2,2}^1$ .

c) We have

$$\begin{aligned} \sigma(\epsilon_i^1) &= \begin{cases} \epsilon_{i+1}^1, & 1 \leq i \leq n-3 \\ \epsilon_1^1 + L - 1, & i = n-2, \end{cases} \\ \sigma(\epsilon_1^2) &= -\epsilon_1^2 + L - 1, \\ \sigma(\epsilon_1^3) &= -\epsilon_1^3. \end{aligned}$$

The proof of Proposition 4 is a straightforward computation using formula (13), so it will be omitted. Finally, let us point out that the set of reflection vectors is given explicitly by

$$(6) \quad \mathfrak{R} = \{\pm(\epsilon_i^a \pm \epsilon_j^b) + m(L-1)\},$$

where  $(a, i) \neq (b, j)$  and  $m \in \mathbb{Z}$  take all possible values.

**2.7. Vertex operators.** Let us change the variables in the total descendant potential

$$q_k := t_k - \delta_{k,1} \mathbf{1}, \quad k \in \mathbb{Z}_{\geq 0},$$

where  $\mathbf{1} := \phi_{0,0}$  is the unit. The components of the formal vector variable  $q_k$  with respect to the basis  $\{\phi_{i,p}\} \subset H$  will be denoted by  $q_{i,p,k}$ . This substitution identifies the total descendant potential with a vector in the ring of formal power series with a shifted origin  $\mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \dots]]$ , where  $\mathbb{C}_\hbar = \mathbb{C}((\hbar^{1/2}))$ .

Following Givental [5] we introduce the symplectic vector space  $\mathcal{H} := H((z^{-1}))$  with symplectic form

$$\Omega(f, g) = \text{Res}_{z=0}(f(-z), g(z))dz.$$

The vector space  $\mathcal{H} \oplus \mathbb{C}$  has a natural structure of a Heisenberg Lie algebra

$$[a, b] := \Omega(a, b), \quad a, b \in \mathcal{H}.$$

Let  $\{\phi^{i,p}\} \subset H$  be the basis dual to  $\{\phi_{i,p}\}$  with respect to the orbifold Poincare pairing. The formulas

$$(\phi_{i,p} z^k)^\frown := -\sqrt{\hbar} \partial / \partial q_{i,p,k}, \quad (\phi^{i,p} (-z)^{-k-1})^\frown := q_{i,p,k} / \sqrt{\hbar},$$

define a representation of the Heisenberg Lie algebra on  $\mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \dots]]$ .

Given  $\alpha \in K$  put

$$\mathbf{f}_\alpha(t, \lambda, z) = \sum_{m \in \mathbb{Z}} I_\alpha^{(m)}(t, \lambda) (-z)^m.$$

and

$$\widetilde{\mathbf{f}}_\alpha(\lambda, z) = \sum_{m \in \mathbb{Z}} \widetilde{I}_\alpha^{(m)}(\lambda) (-z)^m.$$

We will be interested in the vertex operators

$$\Gamma^\alpha(t, \lambda) := e^{\mathbf{f}_\alpha^-(t, \lambda, z)} e^{\mathbf{f}_\alpha^+(t, \lambda, z)}$$

and

$$\widetilde{\Gamma}^\alpha(\lambda) := e^{\widetilde{\mathbf{f}}_\alpha^-(\lambda, z)} e^{\widetilde{\mathbf{f}}_\alpha^+(\lambda, z)},$$

where the superscript + (resp. -) denotes the z-series obtained by truncating all terms that contain negative (resp. non-negative) powers of z.

Let us denote by  $\mathcal{A}$  the algebra of differential operators in one variable

$$A(x, \partial_x) = \sum_{i=0}^d A_i(x) \partial_x^i, \quad A_i(x) \in \mathbb{C}[x].$$

The algebra is equipped with an anti-involution

$$\# : \mathcal{A} \rightarrow \mathcal{A}, \quad A \mapsto A^\# := \sum_{i=0}^d (-\partial_x)^i \circ A_i(x),$$

where  $\circ$  means composition (not action!). Let us introduce the following vertex operator with coefficients in  $\mathcal{A}$ :

$$\tilde{\Gamma}(\lambda) = \exp\left(-\sum_{\ell>0} \frac{\lambda^\ell}{\ell!} \partial_x q_{0,0,\ell}\right) \exp\left(x \partial_{0,0,0}\right),$$

where  $\partial_{i,p,k} := \partial/\partial q_{i,p,k}$ .

Now we can introduce the main object of our investigation. That is a system of quadratic equations, which will be called the Hirota Quadratic Equations (HQEs) of  $\mathbb{P}_{n-2,2,2}^1$ . Put

$$\mathcal{E} := \{\pm \epsilon_i^1 \ (1 \leq i \leq n-2), \pm \epsilon_1^2, \pm \epsilon_1^3\},$$

$$\tilde{b}_{\pm \epsilon_i^1}(\lambda) = \frac{Q}{n-2} \lambda^{-1/(n-2)} \eta_1^{-i}, \quad \eta_1 := e^{2\pi i/(n-2)},$$

and

$$\tilde{b}_{\pm \epsilon_1^2}(\lambda) = -\frac{1}{4}, \quad \tilde{b}_{\pm \epsilon_1^3}(\lambda) = \frac{1}{4}.$$

The coefficients  $\tilde{b}_\epsilon$  can be defined also in terms of the phase factors corresponding to the composition of the vertex operators  $\tilde{\Gamma}^\epsilon(\lambda) \tilde{\Gamma}^{-\epsilon}(\mu)$  (see Section 4.3 and Lemma 14 for more details).

We say that a function (or formal power series in  $\mathbf{q} = (q_{i,p,k})$ )  $\tau(\hbar, \mathbf{q})$  satisfies the HQEs of  $\mathbb{P}_{n-2,2,2}^1$  if for every integer  $m \in \mathbb{Z}$  the 1-form

$$(7) \quad \frac{d\lambda}{\lambda} \left( \tilde{\Gamma}^\#(\lambda) \otimes \tilde{\Gamma}(\lambda) \right) \left( \sum_{\epsilon \in \mathcal{E}} \tilde{b}_\epsilon(\lambda) \tilde{\Gamma}^\epsilon(\lambda) \otimes \tilde{\Gamma}^{-\epsilon}(\lambda) \right) (\tau \otimes \tau)$$

computed at  $q'_{0,0,0} - q''_{0,0,0} = m\hbar^{1/2}$  is regular in  $\lambda$ . Note that the entire expression makes sense as an element in

$$d\lambda \mathcal{A}((\lambda^{-1}))((\hbar^{1/2}))[[\mathbf{q}' + \mathbf{1}, \mathbf{q}'' + \mathbf{1}]].$$

The regularity requirement means that the coefficients in front of the monomials involving  $\mathbf{q}'$ ,  $\mathbf{q}''$ , and  $\hbar^{1/2}$  are in fact polynomial in  $\lambda$ .

**Theorem 5.** *The total descendant potential of  $\mathbb{P}_{n-2,2,2}^1$  satisfies the HQEs of  $\mathbb{P}_{n-2,2,2}^1$ .*

Let us clarify the relation of our HQEs (7) to the Kac–Wakimoto hierarchies. To begin with, note that the set of reflection vectors (6), modulo the rank 1 lattice  $\mathbb{Z}(L-1)$ , form a root system of type  $D$  with respect to the intersection pairing. It is an easy exercise, using the formulas in Proposition 4 c), to check that the automorphism  $\sigma$  induces an element of the Weyl group given by the composition of the reflections corresponding to the non-branching nodes of the Dynkin diagram (see also [17], Proposition 17). If we specialize  $q_{0,0,\ell} \otimes 1 = 1 \otimes q_{0,0,\ell}$  for all  $\ell > 0$ , then (7) becomes a system of HBEs which is equivalent to the HBEs of the Kac–Wakimoto hierarchy of type  $D$  corresponding to the conjugacy class of  $\sigma$ . Let us point out that here we get a realization of the Kac–Wakimoto hierarchy corresponding to a fermionic realization of the basic representation (see [11]).

**Remark 6.** Our interpretation of (7) leads to a system of quadratic equations for the Taylor’s coefficients of the tau-function  $\tau$ . On the other hand, in the theory of integrable systems, there is a different interpretation that leads to a system of PDEs, which is usually called HBEs (e.g. see [9], Section 14.11). We refer to (7) as HQEs or HBEs depending on whether we would like to think of (7) as a system of algebraic equations or a system of PDEs.

### 3. THE EXTENDED D-TODA HIERARCHY

In this section, by making an explicit linear change of the variables  $q_{i,p,k}$ , we will transform the HQEs (7) into form convenient for the applications to integrable hierarchies. In a companion paper to this one, we prove that our system of HBEs parametrizes the solutions to an integrable hierarchy, which we suggest to be called the *Extended D-Toda Hierarchy*.

**3.1. Change of variables.** Let  $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$  be 4 sequences of formal variables, where

$$\mathbf{t}_0 = (t_{0,\ell})_{\ell \geq 1}, \quad \mathbf{t}_1 = (t_{1,\ell})_{\ell \geq 1}, \quad \mathbf{t}_2 = (t_{2,2\ell+1})_{\ell \geq 0}, \quad \mathbf{t}_3 = (t_{3,2\ell+1})_{\ell \geq 0}.$$

Let us introduce the vertex operators

$$\begin{aligned} \Gamma_1^\pm(z) &:= \exp\left(\pm \sum_{\ell=1}^{\infty} t_{1,\ell} z^\ell\right) \exp\left(\mp \sum_{\ell=1}^{\infty} \frac{z^{-\ell}}{\ell} \frac{\partial}{\partial t_{1,\ell}}\right), \\ \Gamma_a(z) &:= \exp\left(\sum_{\ell=0}^{\infty} t_{2,2\ell+1} z^{2\ell+1}\right) \exp\left(-2 \sum_{\ell=0}^{\infty} \frac{z^{-2\ell-1}}{2\ell+1} \frac{\partial}{\partial t_{a,2\ell+1}}\right), \quad a = 2, 3. \end{aligned}$$

Using Proposition 4 we get the following formulas

$$\begin{aligned} \widetilde{\Gamma}_i^{\epsilon_1}(\lambda) &= \exp\left(\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \left(\frac{1}{n-2}(\log \lambda - C_\ell) + 2\pi i \left(\frac{i}{n-2} + \frac{1}{2}\right)\right) q_{0,0,\ell} / \sqrt{\hbar}\right) e^{-\sqrt{\hbar} \partial_{0,0,0}} \\ &\exp\left(\sum_{\ell=0}^{\infty} \left(\frac{\lambda^{\ell+1}}{(\ell+1)!} q_{0,1,\ell} / \sqrt{\hbar} + \sum_{j=1}^{n-3} \frac{\eta_1^{ij} \lambda^{\ell+\frac{j}{n-2}}}{\left(\ell + \frac{j}{n-2}\right) \cdots \left(0 + \frac{j}{n-2}\right) (n-2)} q_{1,j,\ell} / \sqrt{\hbar}\right)\right) \end{aligned}$$

$$\exp\left(-\sum_{\ell=0}^{\infty}\left(\frac{\ell!}{n-2}\lambda^{-1-\ell}\sqrt{\hbar}\partial_{0,1,\ell}+\sum_{j=1}^{n-3}\prod_{k=0}^{\ell-1}\left(k+\frac{j}{n-2}\right)\lambda^{-\frac{j}{n-2}-\ell}\eta_1^{-ij}\sqrt{\hbar}\partial_{1,j,\ell}\right)\right).$$

Let us make the following substitutions:  $\lambda = \frac{z_1^{n-2}}{n-2}$ ,

$$(8) \quad t_{1,\ell(n-2)} = \hbar^{-1/2} \frac{q_{0,1,\ell-1}}{(n-2)^\ell \ell!}, \quad \ell > 0,$$

$$(9) \quad t_{1,\ell(n-2)+i} = \hbar^{-1/2} (n-2)^{-\frac{i}{n-2}} \frac{q_{1,i,\ell}}{i(i+n-2)\cdots(i+(n-2)\ell)}, \quad 1 \leq i \leq n-3, \quad \ell \geq 0.$$

Then the vertex operator

$$\widetilde{\Gamma}^{\pm\epsilon_i^1}(\lambda) = e^{\pm\sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\left(\frac{1}{n-2}(\log\lambda-C_\ell)+2\pi i\left(\frac{i}{n-2}+\frac{1}{2}\right)\right)q_{0,0,\ell}/\sqrt{\hbar}} e^{\mp\sqrt{\hbar}\partial_{0,0,0}} \Gamma_1^\pm(\eta_1^i z_1).$$

and the 1-form

$$\widetilde{b}_{\epsilon_i^1}(\lambda) \frac{d\lambda}{\lambda} = -C \eta_1^{-i} \frac{dz_1}{z_1^2}, \quad C := -Q(n-2)^{\frac{1}{n-2}}.$$

A straightforward computation yields

$$\widetilde{\Gamma}^\# \widetilde{\Gamma}^{\pm\epsilon_i^1} \tau = \left( \Gamma_1^\pm(\eta_1^i z_1) \tau(x \mp \sqrt{\hbar}, \mathbf{q}) \right) e^{\sum_{\ell>0} \frac{\lambda^\ell}{\ell!} (\sqrt{\hbar}\partial_x \mp h_\ell) q_{0,0,\ell}/\sqrt{\hbar}} \left( \frac{\eta_1^i z_1}{C} \right)^{\pm(q_{0,0,0}+x)/\sqrt{\hbar}}$$

and

$$\widetilde{\Gamma} \widetilde{\Gamma}^{\pm\epsilon_i^1} \tau = \left( \frac{\eta_1^i z_1}{C} \right)^{\pm(q_{0,0,0}+x)/\sqrt{\hbar}} e^{\sum_{\ell>0} \frac{\lambda^\ell}{\ell!} (-\sqrt{\hbar}\partial_x \mp h_\ell) q_{0,0,\ell}/\sqrt{\hbar}} \left( \Gamma_1^\pm(\eta_1^i z_1) \tau(x \mp \sqrt{\hbar}, \mathbf{q}) \right),$$

where  $\tau(x, \mathbf{q}) := e^{x\partial_{0,0,0}} \tau(\mathbf{q})$  and  $h_\ell = \frac{1}{n-2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{\ell}\right)$  for  $\ell > 0$ . The residue in the  $\lambda$ -plane

$$\text{Res}_{\lambda=\infty} \frac{\lambda^r}{r!} \sum_{i=1}^{n-2} \widetilde{b}_{\epsilon_i^1}(\lambda) \frac{d\lambda}{\lambda} \left( \widetilde{\Gamma}^\# \otimes \widetilde{\Gamma} \right) \left( \widetilde{\Gamma}^{\epsilon_i^1}(\lambda) \otimes \widetilde{\Gamma}^{-\epsilon_i^1}(\lambda) + \widetilde{\Gamma}^{-\epsilon_i^1}(\lambda) \otimes \widetilde{\Gamma}^{\epsilon_i^1}(\lambda) \right) (\tau \otimes \tau)$$

turns into the following residue in the  $z_1$ -plane

$$\begin{aligned} & -\text{Res}_{z_1=\infty} \frac{z_1^{(n-2)r}}{(n-2)^r r!} \frac{dz_1}{z_1} \left( (z_1/C)^{m-1} \times \right. \\ & \left( \Gamma_1^+(z_1) \tau(x - \sqrt{\hbar}, \mathbf{q}') \right) e^{\sum_{\ell>0} \frac{z_1^{(n-2)\ell}}{(n-2)^\ell \ell!} (\sqrt{\hbar}\partial_x - h_\ell) (q'_{0,0,\ell} - q''_{0,0,\ell})/\sqrt{\hbar}} \left( \Gamma_1^-(z_1) \tau(x + \sqrt{\hbar}, \mathbf{q}'') \right) + \\ & (z_1/C)^{-m-1} \times \\ & \left. \left( \Gamma_1^-(z_1) \tau(x + \sqrt{\hbar}, \mathbf{q}') \right) e^{\sum_{\ell>0} \frac{z_1^{(n-2)\ell}}{(n-2)^\ell \ell!} (\sqrt{\hbar}\partial_x + h_\ell) (q'_{0,0,\ell} - q''_{0,0,\ell})/\sqrt{\hbar}} \left( \Gamma_1^+(z_1) \tau(x - \sqrt{\hbar}, \mathbf{q}'') \right) \right). \end{aligned}$$

Similarly,

$$\widetilde{\Gamma}^{\epsilon_i^2}(\lambda) = \exp\left(-\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)\cdots\left(1+\frac{1}{2}\right)} (q_{2,1,\ell} + q_{3,1,\ell})/\sqrt{\hbar}\right) e^{\pi i \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} q_{0,0,\ell}/\sqrt{\hbar}}$$

$$\exp\left(\sum_{\ell=0}^{\infty}\left(0+\frac{1}{2}\right)\cdots\left(\ell-1+\frac{1}{2}\right)\lambda^{-\ell-\frac{1}{2}}\sqrt{\hbar}(\partial_{2,1,\ell}+\partial_{3,1,\ell})\right).$$

Let us make the substitutions:  $\lambda = z_2^2/2$  and

$$(10) \quad t_{2,2\ell+1} = \hbar^{-1/2} \frac{q_{2,1,\ell} + q_{3,1,\ell}}{\sqrt{2}(2\ell+1)!!}.$$

Then, using also (11), we get

$$\widetilde{\Gamma}^{\pm\epsilon_1^2}(\lambda) = e^{\pm\pi i \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} q_{0,0,\ell}/\sqrt{\hbar}} \Gamma_2(\mp z_2)$$

and

$$\widetilde{b}_{\epsilon_1^2}(\lambda) \frac{d\lambda}{\lambda} = -\frac{dz_2}{2z_2}.$$

We have

$$\widetilde{\Gamma}^{\#} \widetilde{\Gamma}^{\pm\epsilon_1^2} \tau = \Gamma_2(\mp z_2) \tau(x, \mathbf{q}) e^{\pm \sum_{\ell>0} \frac{\lambda^\ell}{\ell!} \partial_x q_{0,0,\ell}} (-1)^{\pm(q_{0,0,0}+x)/\sqrt{\hbar}}$$

and

$$\widetilde{\Gamma} \widetilde{\Gamma}^{\pm\epsilon_1^2} \tau = (-1)^{\pm(q_{0,0,0}+x)/\sqrt{\hbar}} e^{\mp \sum_{\ell>0} \frac{\lambda^\ell}{\ell!} \partial_x q_{0,0,\ell}} \Gamma_2(\mp z_2) \tau(x, \mathbf{q}).$$

The residue in the  $\lambda$ -plane

$$\text{Res}_{\lambda=\infty} \frac{\lambda^r}{r!} \widetilde{b}_{\epsilon_1^2}(\lambda) \frac{d\lambda}{\lambda} \left( \widetilde{\Gamma}^{\#} \otimes \widetilde{\Gamma} \right) \left( \widetilde{\Gamma}^{\epsilon_1^2}(\lambda) \otimes \widetilde{\Gamma}^{-\epsilon_1^2}(\lambda) + \widetilde{\Gamma}^{-\epsilon_1^2}(\lambda) \otimes \widetilde{\Gamma}^{\epsilon_1^2}(\lambda) \right) (\tau \otimes \tau)$$

turns into the following residue in the  $z_2$ -plane

$$-\text{Res}_{z_2=\infty} \frac{dz_2}{2z_2} \frac{z_2^{2r}}{(2r)!!} (-1)^m \left( \Gamma_2(z_2) \tau(x, \mathbf{q}') \right) e^{\sum_{\ell>0} \frac{z_2^{2\ell}}{(2\ell)!!} \partial_x (q'_{0,0,\ell} - q''_{0,0,\ell})} \left( \Gamma_2(-z_2) \tau(x, \mathbf{q}'') \right).$$

Finally

$$\begin{aligned} \widetilde{\Gamma}^{\epsilon_1^3}(\lambda) &= \exp\left(-\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)\cdots\left(1+\frac{1}{2}\right)} (q_{2,1,\ell} - q_{3,1,\ell})/\sqrt{\hbar}\right) \\ &\quad \exp\left(\sum_{\ell=0}^{\infty} \left(0+\frac{1}{2}\right)\cdots\left(\ell-1+\frac{1}{2}\right) \lambda^{-\ell-\frac{1}{2}} \sqrt{\hbar}(\partial_{2,1,\ell} - \partial_{3,1,\ell})\right). \end{aligned}$$

Let us make the substitutions:  $\lambda = z_3^2/2$  and

$$(11) \quad t_{3,2\ell+1} = \hbar^{-1/2} \frac{q_{2,1,\ell} - q_{3,1,\ell}}{\sqrt{2}(2\ell+1)!!}.$$

Then, using also (10), we get  $\widetilde{\Gamma}^{\pm\epsilon_1^3}(\lambda) = \Gamma_3(\mp z_3)$  and

$$\widetilde{b}_{\epsilon_1^3}(\lambda) \frac{d\lambda}{\lambda} = \frac{dz_3}{2z_3}.$$

We have

$$\widetilde{\Gamma}^{\#} \widetilde{\Gamma}^{\pm\epsilon_1^3} \tau = \Gamma_3(\mp z_3) \tau(x, \mathbf{q}) e^{\pm \sum_{\ell>0} \frac{\lambda^\ell}{\ell!} \partial_x q_{0,0,\ell}}$$

and

$$\widetilde{\Gamma}^{\pm\epsilon_1^3} \tau = e^{\mp \sum_{\ell>0} \frac{\lambda^\ell}{\ell!} \partial_x q_{0,0,\ell}} \Gamma_3(\mp z_3) \tau(x, \mathbf{q}).$$

The residue in the  $\lambda$ -plane

$$\text{Res}_{\lambda=\infty} \frac{\lambda^r}{r!} \widetilde{b}_{\epsilon_1^3}(\lambda) \frac{d\lambda}{\lambda} \left( \widetilde{\Gamma}^\# \otimes \widetilde{\Gamma} \right) \left( \widetilde{\Gamma}^{\epsilon_1^3}(\lambda) \otimes \widetilde{\Gamma}^{-\epsilon_1^3}(\lambda) + \widetilde{\Gamma}^{-\epsilon_1^3}(\lambda) \otimes \widetilde{\Gamma}^{\epsilon_1^3}(\lambda) \right) (\tau \otimes \tau)$$

turns into the following residue in the  $z_3$ -plane

$$\text{Res}_{z_3=\infty} \frac{dz_3}{2z_3} \frac{z_3^{2r}}{(2r)!!} \left( \Gamma_3(z_3) \tau(x, \mathbf{q}') \right) e^{\sum_{\ell>0} \frac{z_3^{2\ell}}{(2\ell)!!} \partial_x (q'_{0,0,\ell} - q''_{0,0,\ell})} \left( \Gamma_3(-z_3) \tau(x, \mathbf{q}'') \right).$$

**3.2. Wave functions.** Let  $\mathcal{O}_\epsilon(\mathbb{C}) := \mathcal{O}(\mathbb{C})[[\epsilon]]$  denote the ring of formal power series in  $\epsilon$  whose coefficients are holomorphic functions on  $\mathbb{C}$ . Let us denote by  $x$  the standard coordinate function on  $\mathbb{C}$ . We will be interested in formal functions of the form  $\tau(x, \mathbf{t}) = e^{\mathcal{F}(\epsilon, x, \mathbf{t})\epsilon^{-1}}$ , where  $\mathcal{F}(\epsilon, x, \mathbf{t})$  is a formal power series in  $\mathbf{t}$  with coefficients in  $\mathcal{O}_\epsilon(\mathbb{C})$  satisfying the following condition: the coefficient in front of  $\epsilon^0$  in  $\mathcal{F}(\epsilon, x, \mathbf{t})$  is at most linear in  $\mathbf{t}$ . For example, if  $\mathcal{D}(\hbar, \mathbf{q})$  is the total descendent potential of  $\mathbb{P}_{n-2,2,2}^1$ , then the substitution  $(\hbar, \mathbf{q}) \mapsto (\epsilon, x, \mathbf{t})$  defined by

$$\sqrt{\hbar} := \epsilon, \quad q_{0,0,0} := x, \quad q_{0,0,k} := t_{0,k}\epsilon - \delta_{k,1} \quad (k > 0)$$

and the linear change of variables (8)–(9), (10), and (11), identify the total descendent potential with a formal function of the above type. In order to prove this, we need only to recall the dimension formula for the virtual fundamental cycle, that is, if the Gromov–Witten invariant

$$\langle \phi_{i_1, p_1} \psi^{k_1}, \dots, \phi_{i_r, p_r} \psi^{k_r} \rangle_{g, r, d}$$

is not 0, then

$$\sum_{s=1}^r \left( k_s + \frac{p_s}{a_{is}} \right) = 2g - 2 + r + \frac{d}{n-2}.$$

Let us point out that, using the above formula, we can prove also that the coefficient in front of each monomial in  $\mathbf{t}$  and  $\epsilon$  in  $\log \mathcal{D}(\hbar, \mathbf{q})$  is a polynomial in  $x$  and  $Q$ .

Let us introduce the following functions:

$$\Psi_1^+(x, \mathbf{t}, z) := \psi_1^+(x, \mathbf{t}, z) e^{\xi_1(\mathbf{t}, z)} (-z/C)^{\frac{x}{\epsilon} - \frac{1}{2}}, \quad \Psi_1^-(x, \mathbf{t}, z) := (-z/C)^{-\frac{x}{\epsilon} - \frac{1}{2}} e^{-\xi_1(\mathbf{t}, z)} \psi_1^-(x, \mathbf{t}, z),$$

where

$$\xi_1(\mathbf{t}, z) := \sum_{k=1}^{\infty} \left( t_{1,k} z^k + t_{0,k} \frac{z^{(n-2)k}}{(n-2)^k k!} (\epsilon \partial_x - h_k) \right),$$

$$\psi_1^\pm(x, \mathbf{t}, z) := \frac{e^{\mp \sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial_{t_{1,k}} \tau(x \mp \epsilon, \mathbf{t})}}{\tau(x, \mathbf{t})} =: \sum_{k=0}^{\infty} \psi_{1,k}^\pm(x, \mathbf{t}) z^{-k},$$

and

$$\Psi_a^+(x, \mathbf{t}, z) := \psi_a^+(x, \mathbf{t}, z) e^{\xi_a(\mathbf{t}, z)}, \quad \Psi_a^-(x, \mathbf{t}, z) := e^{-\xi_a(\mathbf{t}, z)} \psi_a^-(x, \mathbf{t}, z) \quad (a = 2, 3),$$

where

$$\xi_a(\mathbf{t}, z) := \sum_{k=1}^{\infty} \left( t_{a,2k-1} z^{2k-1} + t_{0,k} \frac{z^{2k}}{2^k k!} \epsilon \partial_x \right),$$

$$\psi_a^{\pm}(x, \mathbf{t}, z) := \frac{e^{\mp 2 \sum_{k=0}^{\infty} \frac{z^{-2k-1}}{2^{k+1}} \partial_{t_{a,2k+1}} \tau(x, \mathbf{t})}}{\tau(x, \mathbf{t})} =: \sum_{k=0}^{\infty} \psi_{a,k}^{\pm}(x, \mathbf{t}) z^{-k}.$$

The functions  $\Psi_i^{\pm}(x, \mathbf{t}, z)$  ( $1 \leq i \leq 3$ ) will be called *wave functions* of the Extended D-Toda Hierarchy if they satisfy the following bilinear equations:

$$\text{Res}_{z=0} \frac{z^{(n-2)r}}{(n-2)^r r!} \frac{dz}{z} \left( \Psi_1^+(x, \mathbf{t}', z) \Psi_1^-(x + m\epsilon, \mathbf{t}'', z) + (\Psi_1^+(x + m\epsilon, \mathbf{t}'', z) \Psi_1^-(x, \mathbf{t}', z))^{\#} \right) =$$

$$\text{Res}_{z=0} \frac{z^{2r}}{2^r r!} \frac{dz}{2z} \left( \Psi_2^+(x, \mathbf{t}', z) \Psi_2^-(x + m\epsilon, \mathbf{t}'', z) - (-1)^m \Psi_3^+(x, \mathbf{t}', z) \Psi_3^-(x + m\epsilon, \mathbf{t}'', z) \right),$$

where  $r \geq 0$  and  $m$  are arbitrary integers.

Suppose that  $\mathcal{D}(\hbar, \mathbf{q}) \in \mathbb{C}_{\hbar}[[q_0, q_1 + 1, q_2, \dots]]$  is a formal power series, such that, under the substitution  $(\hbar, \mathbf{q}) \mapsto (\epsilon, x, \mathbf{t})$  from above,  $\mathcal{D}(\hbar, \mathbf{q})$  becomes a formal function in  $\epsilon, x$ , and  $\mathbf{t}$  for which the definition of  $\Psi_i^{\pm}(x, \mathbf{t}, z)$  ( $1 \leq i \leq 3$ ) makes sense. Then one can check that the corresponding functions  $\Psi_i^{\pm}(x, \mathbf{t}, z)$  ( $1 \leq i \leq 3$ ) are wave functions of the Extended D-Toda Hierarchy if and only if  $\mathcal{D}(\hbar, \mathbf{t})$  satisfies the HQEs (7).

#### 4. PHASE FACTORS

The proof of Theorem 5 follows the method developed by Givental in [6]. It relies on Givental's higher genus reconstruction formula (see [4]) proved by Teleman [19]. There is a part of Givental's argument in [6] which was hard to generalize. The difficulty however was offset in [14] in the settings of singularity theory and after a small modification in the current settings as well (see also [17]). The goal of this section is to prove the results that we need in order to make Givental's argument from [6] work.

**4.1. Proof of Lemma 2.** The calibration  $S(t, Q, z)$  satisfies the so-called *divisor equation*

$$\partial_{t_{0,1}} S(t, Q, z) = Q \partial_Q S(t, Q, z) + S(t, Q, z) z^{-1} P \cup,$$

where  $P \cup$  is the linear operator defined by topological cup product by the cohomology class  $\phi_{0,1} = P$ . In order to avoid cumbersome notation, let us agree to denote by  $P$  the linear operator  $P \cup$  and by  $\phi_{0,1}$  the cohomology class  $P$ . Using the commutation relation  $\theta P = P(\theta - 1)$  and (1), we get

$$\widetilde{I}^{(m)}(\lambda) P = P \partial_{\lambda} \widetilde{I}^{(m)}(\lambda).$$



The above formula implies that

$$I^{(m)}(t, \lambda) e^{-P \log Q} a = S(t, Q, -\partial_\lambda^{-1}) e^{-P \log Q} \partial_\lambda \tilde{I}^{(m)}(\lambda) a.$$

On the other hand, according to the divisor equation, we have  $Q \partial_Q (S(t, Q, z) e^{P \log Q/z}) = z^{-1} P \bullet S(t, Q, z) e^{P \log Q/z}$ . Therefore,

$$Q \partial_Q (I^{(m)}(t, \lambda) e^{-P \log Q} a) = -P \bullet I^{(m+1)}(t, \lambda) e^{-P \log Q} a.$$

The above differential equation and  $(\lambda - E \bullet) I^{(m+1)} = (\theta - m - \frac{1}{2}) I^{(m)}$  imply that the pairing

$$(e^{-P \log Q} a | e^{-P \log Q} b) = (I^{(0)}(t, \lambda) e^{-P \log Q} a, (\lambda - E \bullet) I^{(0)}(t, \lambda) e^{-P \log Q} b)$$

is independent of  $Q$ . On the other hand the operators  $S_k(t, Q)$  vanish at  $t = Q = 0$ . Therefore, the only contribution to the intersection pairing is given by

$$(12) \quad (e^{-P \log Q} a | e^{-P \log Q} b) = (\tilde{I}^{(0)}(\lambda) a, (\lambda - \rho) \tilde{I}^{(0)}(\lambda) b).$$

On the other hand, since  $\rho$  is a multiplication by a top degree class and  $\theta(P) = -\frac{1}{2} P$ , we have  $(\theta + \frac{1}{2}) \rho = 0$ . This relation allows us to get the following simple formula for the calibrated periods (for  $m < 0$ )

$$(13) \quad \tilde{I}^{(m)}(\lambda) = \frac{\lambda^{\theta-m-\frac{1}{2}}}{\Gamma(\theta-m+\frac{1}{2})} + \frac{\lambda^{-m-1}}{\Gamma(-m)} (\log \lambda - \psi(-m)) \rho,$$

where  $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$  is the di-gamma function. From here, since  $\tilde{I}^{(0)} = \partial_\lambda \tilde{I}^{(-1)}$ , we get

$$\tilde{I}^{(0)}(\lambda) = \frac{\lambda^{\theta-\frac{1}{2}}}{\Gamma(\theta+\frac{1}{2})} + \frac{1}{\lambda} \rho.$$

Substituting the above formula in (12), after some straightforward computation, we get

$$(e^{-P \log Q} a | e^{-P \log Q} b) = \frac{1}{2\pi} \left( (e^{\pi\sqrt{-1}\theta} + e^{-\pi\sqrt{-1}\theta} + 2\pi\rho) a, b \right).$$

In order to complete the proof of Lemma 2, it remains only to notice the following two facts: the RHS of the above formula is invariant under the action  $(a, b) \mapsto (e^{P \log Q} a, e^{P \log Q} b)$  and

$$e^{\pi\sqrt{-1}\theta} + e^{-\pi\sqrt{-1}\theta} + 2\pi\rho = e^{\pi\sqrt{-1}\theta} e^{\pi\sqrt{-1}\rho} + e^{\pi\sqrt{-1}\rho} e^{-\pi\sqrt{-1}\theta},$$

where we used the relations  $\rho^2 = 0$  and  $\theta\rho = -\frac{1}{2}\rho$ .

**4.2. Euler pairing.** Let us denote the imaginary unit by  $\mathbf{i} := \sqrt{-1}$ . In what follows, we will have to use quite frequently the following formulas for the Euler pairing:

$$\langle a, b \rangle := \chi(a^\vee \otimes b), \quad a, b \in K^0(\mathbb{P}_{n-2,2,2}^1).$$

If  $1 \leq i, j \leq n-3$ , then

$$\langle \epsilon_i^1, \epsilon_j^1 \rangle = \begin{cases} \frac{1}{2} & \text{if } i \leq j, \\ -\frac{1}{2} & \text{if } i > j. \end{cases}$$

If  $1 \leq i \leq n-3$ , then

$$\langle \epsilon_i^1, \epsilon_1^2 \rangle = \frac{1}{2}, \quad \langle \epsilon_1^2, \epsilon_i^1 \rangle = -\frac{1}{2}, \quad \langle \epsilon_i^1, \epsilon_1^3 \rangle = \langle \epsilon_1^3, \epsilon_i^1 \rangle = 0.$$

Finally

$$\langle \epsilon_1^2, \epsilon_1^2 \rangle = \langle \epsilon_1^3, \epsilon_1^3 \rangle = \frac{1}{2}, \quad \langle \epsilon_1^2, \epsilon_1^3 \rangle = \langle \epsilon_1^3, \epsilon_1^2 \rangle = 0.$$

Let us sketch the proof of the first formula. The computations in the remaining ones are similar and much shorter. Note that

$$\Psi(\epsilon_i^1) = 1 + \frac{\pi \mathbf{i}(2i + n - 2) - \gamma}{n - 2} P + \sum_{p=1}^{n-3} \Gamma\left(1 - \frac{p}{n-2}\right) \eta_1^{-pi} \phi_{1,p},$$

and

$$e^{\pi i \theta} e^{\pi i \rho} \Psi(\epsilon_j^1) = \mathbf{i} + \frac{\pi(2j + n - 1) + \mathbf{i}\gamma}{n - 2} P + \sum_{p=1}^{n-3} \Gamma\left(1 - \frac{p}{n-2}\right) \eta_1^{-p(j+1/2)} \mathbf{i} \phi_{1,p}$$

where  $\eta_1 = e^{2\pi \mathbf{i}/(n-2)}$ . Recalling formula (4), we get

$$\langle \epsilon_i^1, \epsilon_j^1 \rangle = \frac{1}{n-2} \left( j - i + \frac{1}{2} - \sum_{p=1}^{n-3} \frac{\eta_1^{(i-j)p}}{\eta_1^p - 1} \right).$$

Suppose that  $i - j > 0$  then the sum

$$\sum_{p=1}^{n-3} \frac{\eta_1^{(i-j)p}}{\eta_1^p - 1} = \sum_{p=1}^{n-3} \left( \frac{1}{\eta_1^p - 1} + 1 + \eta_1^p + \dots + \eta_1^{p(i-j-1)} \right) = j - i + \frac{1}{2} + \frac{n-2}{2}.$$

If  $j \geq i$  then since the above sum is invariant under  $p \mapsto -p$  we get

$$\sum_{p=1}^{n-3} \frac{\eta_1^{(i-j)p}}{\eta_1^p - 1} = \sum_{p=1}^{n-3} \frac{\eta_1^{-(i-j)p}}{\eta_1^{-p} - 1} = - \sum_{p=1}^{n-3} \frac{\eta_1^{(1-i+j)p}}{\eta_1^p - 1} = j - i + \frac{1}{2} - \frac{n-2}{2}.$$

The formulas for  $\langle \epsilon_i^1, \epsilon_j^1 \rangle$  follow.

Let us fix an integer  $\kappa$  divisible by the order of the semi-simple part of  $\sigma$ . Using Proposition 4 we get that  $\kappa = 2(n-2)$  is such a number. If  $\beta \in H$ , then let us define

$$\beta_0 := \frac{1}{\kappa} (\beta + \sigma(\beta) + \dots + \sigma^{\kappa-1}(\beta)), \quad \beta_{\text{tw}} := \beta - \beta_0.$$

**Lemma 7.** *The following formulas hold*

$$\langle \alpha, \beta_0 \rangle = \text{rk}(\alpha) \deg(\beta) - \text{rk}(\beta) \deg(\alpha) + \text{rk}(\alpha) \text{rk}(\beta)$$

and

$$\langle \beta_0, \alpha \rangle = \text{rk}(\beta) \deg(\alpha) - \text{rk}(\alpha) \deg(\beta) + \text{rk}(\alpha) \text{rk}(\beta) \left( \frac{1}{n-2} - 1 \right).$$

*Proof.* Using that  $\chi_{j,p}(\sigma(E)) = \eta_j^{-p} \chi_{j,p}(E)$ , where  $\eta_j = e^{2\pi i/a_j}$  we get that

$$e^{\pi i \theta} e^{\pi i \rho} \Psi(\beta_0) = \mathbf{i} \text{rk}(\beta) + \left( \text{rk}(\beta) \frac{\pi + \mathbf{i} \gamma}{n-2} + 2\pi \deg(\beta_0) \right) P.$$

Since

$$\Psi(\alpha) = \text{rk}(\alpha) + \left( -\text{rk}(\alpha) \frac{\gamma}{n-2} + 2\pi \mathbf{i} \deg(\alpha) \right) P + \dots,$$

where the dots involve cohomology classes supported on the twisted sectors. Recalling again formula (4) we get

$$\langle \alpha, \beta_0 \rangle = \text{rk}(\alpha) \deg(\beta_0) - \text{rk}(\beta) \deg(\alpha) + \frac{1}{\kappa} \text{rk}(\alpha) \text{rk}(\beta)$$

Since  $\sigma$  is multiplication by the tangent bundle  $T\mathbb{P}_{n-2,2,2}^1$  we get that

$$\deg(\beta_0) = \deg(\beta) + \text{rk}(\beta) \frac{\kappa - 1}{\kappa}.$$

The formula for  $\langle \alpha, \beta_0 \rangle$  follows. The second formula is proved in a similar way, so we omit the argument.  $\square$

**Lemma 8.** *The following formula holds*

$$(\alpha | \sigma^s(\beta_{\text{tw}})) = \sum_{j=1}^3 \sum_{p=1}^{a_j-1} \frac{1}{a_j} \eta_j^{ps} \chi_{j,p}(\alpha) \chi_{j,a_j-p}(\beta).$$

*Proof.* Note that

$$\text{rk}(\beta_{\text{tw}}) = 0, \quad \deg(\beta_{\text{tw}}) = -\text{rk}(\beta)(1 - 1/\kappa),$$

and

$$\Psi(\sigma^s \beta_{\text{tw}}) = -2\pi \mathbf{i} \text{rk}(\beta)(1 - 1/\kappa) P + \sum_{j,p} \Gamma\left(1 - \frac{p}{a_j}\right) \chi_{j,p}(\beta) \eta_j^{-ps} \phi_{j,p}.$$

Recalling again formula (4) we get that the pairing  $(\alpha | \sigma^s(\beta_{\text{tw}}))$  is given by

$$\frac{1}{2\pi} (\Psi(\alpha), (e^{-\pi i \theta} + e^{\pi i \theta}) \Psi(\sigma^s \beta_{\text{tw}})) = \sum_{j,p} (\Psi(\alpha), \phi_{j,p}) \Gamma(p/a_j)^{-1} \chi_{j,p}(\beta) \eta_j^{-ps}.$$

It remains only to use that

$$(\Psi(\alpha), \phi_{j,p}) = \frac{1}{a_j} \Gamma(p/a_j) \chi_{j,a_j-p}(\alpha). \quad \square$$

**4.3. Phase factors.** By definition phase factors are scalar functions that arise when we compose two vertex operators. In our case, the phase factors can be expressed in terms of Givental's symplectic form as follows:

$$\widetilde{\Gamma}^\alpha(\lambda_1)\widetilde{\Gamma}^\beta(\lambda_2) = e^{\Omega(\widetilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \widetilde{\mathbf{f}}_\beta^-(\lambda_2, z))} \cdot \widetilde{\Gamma}^\alpha(\lambda_1)\widetilde{\Gamma}^\beta(\lambda_2) \cdot,$$

and

$$\Gamma^\alpha(t, \lambda_1)\Gamma^\beta(t, \lambda_2) = e^{\Omega(\mathbf{f}_\alpha^+(t, \lambda_1, z), \mathbf{f}_\beta^-(t, \lambda_2, z))} \cdot \Gamma^\alpha(t, \lambda_1)\Gamma^\beta(t, \lambda_2) \cdot,$$

where the normal ordering  $\cdot \cdot$  means that all differentiations should be moved to the right.

**Proposition 9.** *If  $\alpha, \beta \in K^0(\mathbb{P}_{n-2,2,2}^1)$  then the symplectic pairing*

$$\begin{aligned} \Omega(\widetilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \widetilde{\mathbf{f}}_\beta^-(\lambda_2, z)) &= -2\pi i \operatorname{rk}(\alpha) \deg(\beta) + \\ &+ \log \left( (Q\lambda_2^{-1/(n-2)})^{\operatorname{rk}(\alpha)\operatorname{rk}(\beta)} \prod_{s=1}^{\kappa} \left( 1 - \eta^{-s}(\lambda_2/\lambda_1)^{1/\kappa} \right)^{(\alpha|\sigma^s\beta)} \right), \end{aligned}$$

where  $\eta = e^{2\pi i/\kappa}$  and the RHS should be expanded into a Laurent series in  $\lambda_1^{-1}$  in the region  $|\lambda_1| > |\lambda_2|$ .

*Proof.* By definition the symplectic pairing is

$$\sum_{m=0}^{\infty} (-1)^{m+1} (\widetilde{I}_\alpha^{(m)}(\lambda_1), \widetilde{I}_\beta^{(-m-1)}(\lambda_2)).$$

Substituting the explicit formulas for the calibrated periods from Proposition 4, part a) we get

$$\begin{aligned} (14) \quad & -2\pi i \operatorname{rk}(\alpha) \deg(\beta) - \frac{1}{n-2} \operatorname{rk}(\alpha) \operatorname{rk}(\beta) \left( \log \lambda_2 - C_0 + \sum_{\ell=1}^{\infty} \frac{(\lambda_2/\lambda_1)^\ell}{\ell} \right) \\ & - \sum_{j=1}^3 \sum_{p=1}^{a_j-1} \sum_{m=0}^{\infty} \frac{(\lambda_2/\lambda_1)^{m+p/a_j}}{p+ma_j} \chi_{j,p}(\alpha) \chi_{j,a_j-p}(\beta). \end{aligned}$$

Let us compute the sum over  $m$  in (14). We will use the following simple fact: if  $r$  is an arbitrary integer, then

$$\frac{1}{\kappa} \sum_{s=1}^{\kappa} \eta^{(r-l)s} = \begin{cases} 1 & \text{if } l = r + m\kappa \text{ for some integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

Using the above formula we get the following identity:

$$\frac{1}{\kappa} \sum_{l=1}^{\infty} \sum_{s=1}^{\kappa} \eta^{rs} \frac{(\eta^{-s}(\lambda_2/\lambda_1)^{1/\kappa})^l}{l} = \sum_{m=0}^{\infty} \frac{(\lambda_2/\lambda_1)^{m+\frac{r}{\kappa}}}{r+m\kappa}.$$

Let us specialize the above identity to  $r = p\kappa/a_j$ . Since  $\eta^{rs} = e^{2\pi i rs/\kappa} = \eta_j^{ps}$ , we get

$$\frac{1}{a_j} \sum_{l=1}^{\infty} \sum_{s=1}^{\kappa} \eta_j^{ps} \frac{(\eta^{-s}(\lambda_2/\lambda_1)^{1/\kappa})^l}{l} = \sum_{m=0}^{\infty} \frac{(\lambda_2/\lambda_1)^{m+\frac{p}{a_j}}}{p+ma_j}.$$

Using the above formula and Lemma 8, we get that the expression in (14) is equal to

$$-\sum_{\ell=1}^{\infty} \sum_{s=1}^{\kappa} \frac{1}{\ell} (\eta^{-s} (\lambda_2/\lambda_1)^{1/\kappa})^{\ell} (\alpha|\sigma^s \beta_{\text{tw}}) = \log \prod_{s=1}^{\kappa} \left(1 - \eta^{-s} (\lambda_2/\lambda_1)^{1/\kappa}\right)^{(\alpha|\sigma^s \beta_{\text{tw}})}.$$

Recalling Lemma 7, we get

$$(\alpha|\beta_0) = \frac{1}{n-2} \text{rk}(\alpha) \text{rk}(\beta).$$

Note that this formula implies also that  $(\alpha|\sigma^s \beta_0) = (\alpha|\beta_0)$ . Finally, since  $C_0 = (n-2) \log Q$  and

$$-\sum_{\ell=1}^{\infty} \frac{(\lambda_2/\lambda_1)^{\ell}}{\ell} = \log(1 - \lambda_2/\lambda_1) = \log \prod_{s=1}^{\kappa} \left(1 - \eta^{-s} (\lambda_2/\lambda_1)^{1/\kappa}\right)$$

the formula that we want to prove follows. □

**Lemma 10.** *The following formula holds:*

$$\sum_{s=1}^{\kappa} \left(\frac{1}{2} - \frac{s}{\kappa}\right) (\alpha|\sigma^s \beta) = -\langle \alpha, \beta \rangle + \text{rk}(\alpha) \deg(\beta) - \text{rk}(\beta) \deg(\alpha).$$

*Proof.* Since  $\sigma^{-1}$  is multiplication by the dualizing sheaf, we can write Serre's duality as

$$\langle b, a \rangle = -\langle a, \sigma^{-1} b \rangle.$$

Therefore, the intersection pairing

$$(a|b) = \langle a, (1 - \sigma^{-1})b \rangle, \quad a, b \in K^0(\mathbb{P}_{n-2,2,2}^1).$$

Using the above formula let us write the LHS of the identity that we want to prove as

$$(15) \quad \sum_{s=1}^{\kappa} \left(\frac{1}{2} - \frac{s}{\kappa}\right) \langle \alpha, (1 - \sigma^{-1})\sigma^s \beta \rangle = \langle \alpha, -\beta - \frac{1}{2}(\sigma^{\kappa} - 1)\beta + \beta_0 \rangle.$$

Here we wrote  $(1 - \sigma^{-1})\sigma^s = \sigma^s - \sigma^{s-1}$ , split the sum over  $s$  into two sum. After shifting the summation index in the second sum  $s \mapsto s+1$ , most of the terms cancel out and what is left is the RHS of (15).

Note that

$$\langle \alpha, (\sigma^{\kappa} - 1)\beta \rangle = \kappa \langle \alpha, (\sigma - 1)\beta_0 \rangle = \kappa(\sigma^{-1} \alpha|\beta_0) = 2 \text{rk}(\alpha) \text{rk}(\beta),$$

where we used Lemma 7 for the last equality. Therefore (15) becomes

$$-\langle \alpha, \beta \rangle - \text{rk}(\alpha) \text{rk}(\beta) + \langle \alpha, \beta_0 \rangle.$$

Recalling Lemma 7 one more time completes the proof. □

Let us discuss the analytic properties of the phase factors. The symplectic pairing  $\Omega(\tilde{\mathbf{f}}_{\alpha}^{+}(\lambda_1, z), \tilde{\mathbf{f}}_{\beta}^{-}(\lambda_2, z))$  is a convergent Laurent series that extends analytically to a multivalued analytic function  $\tilde{\Omega}_{\alpha, \beta}(\lambda_1, \lambda_2)$  in the domain

$$0 < |\lambda_1 - \lambda_2| < \min(|\lambda_1|, |\lambda_2|).$$

Using Proposition 9 and Lemma 10 we get that

$$(16) \quad \widetilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) - \widetilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1) = -2\pi i \left( \langle \alpha, \beta \rangle + \ell(\alpha|\beta) \right)$$

for some integer  $\ell \in \mathbb{Z}$  that depends on how we fix the branches of the functions on the LHS.

The symplectic pairing

$$(17) \quad \Omega(\mathbf{f}_\alpha^+(t, \lambda_1, z), \mathbf{f}_\beta^-(t, \lambda_2, z)) = \sum_{m=0}^{\infty} (-1)^{m+1} (I_\alpha^{(m)}(t, \lambda_1), I_\beta^{(-m-1)}(t, \lambda_2))$$

will be interpreted via its Laurent series expansion in  $\lambda_1^{-1}$ . The radius of convergence will be determined below. Let us express (17) in terms of  $\Omega(\widetilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \widetilde{\mathbf{f}}_\beta^-(\lambda_2, z))$  and the calibration operator  $S(t, Q, z)$ . We have the following conjugation formula (see [6], formula (17)):

$$(18) \quad \widehat{S} e^{\widehat{\mathbf{f}} \widehat{S}^{-1}} = e^{W(\mathbf{f}^+, \mathbf{f}^+)/2} e^{(S\mathbf{f})^\wedge},$$

where  $W(t, \mathbf{q}, \mathbf{q}) = \sum_{k,\ell=0}^{\infty} (W_{k,\ell}(t) q_\ell, q_k)$  is the quadratic form defined by

$$\sum_{k,\ell=0}^{\infty} W_{k,\ell}(t) w^{-k} z^{-\ell} = \frac{S^*(t, Q, w) S(t, Q, z) - 1}{z^{-1} + w^{-1}},$$

where  $*$  denotes transposition with respect to the Poincare pairing, and if  $\mathbf{f} = \sum_{k=0}^{\infty} f_k z^k$ , then  $W(t, \mathbf{f}, \mathbf{f}) := \sum_{k,l=0}^{\infty} (W_{k,l}(t) f_l, f_k)$ . Put

$$(19) \quad W_{\alpha,\beta}(t, \lambda_1, \lambda_2) := W(t, \widetilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \widetilde{\mathbf{f}}_\beta^-(\lambda_2, z)).$$

Since  $\mathbf{f}_\alpha(t, \lambda, z) = S(t, Q, z) \widetilde{\mathbf{f}}_\alpha(\lambda, z)$ , the conjugation formula (18) and the definition of the phase factors, i.e., the fact that the symplectic pairings arise as a composition of vertex operators yield the following relation:

$$(20) \quad \Omega(\mathbf{f}_\alpha^+(t, \lambda_1, z), \mathbf{f}_\beta^-(t, \lambda_2, z)) = \Omega(\widetilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \widetilde{\mathbf{f}}_\beta^-(\lambda_2, z)) + W_{\alpha,\beta}(t, \lambda_1, \lambda_2).$$

On the other hand, since  $S(t, Q, z)$  is a solution to the Dubrovin's connection with respect to  $t$ , we get that the de Rham differential (with respect to  $t \in H$ )

$$dW_{\alpha,\beta}(t, \lambda_1, \lambda_2) = I_\alpha^{(0)}(t, \lambda_1) \bullet I_\beta^{(0)}(t, \lambda_2),$$

where the RHS is identified with an element in  $T_t^* H = H^*$  via the Poincare pairing. Using the divisor equation, we get that  $\lim_{t \rightarrow -\infty} S(t, Q, z) e^{-t_{0,1} P/z} = 1$  where we define  $t \rightarrow -\infty$  to be the limit  $t_{i,p} \rightarrow 0$  ( $(i, p) \neq (0, 1)$ ) and  $\text{Re}(t_{0,1}) \rightarrow -\infty$ . Although the limit of  $W_{\alpha,\beta}(t, \lambda_1, \lambda_2)$  does not exists, it is not hard to analyse the singular term, i.e., if we subtract  $\frac{1}{n-2} t_{0,1} \text{rk}(\alpha) \text{rk}(\beta)$ , then the limit will exists and it is 0. Therefore, we have the following integral formula:

$$(21) \quad W_{\alpha,\beta}(t, \lambda_1, \lambda_2) = \frac{1}{n-2} t_{0,1} \text{rk}(\alpha) \text{rk}(\beta) + \int_{-\infty}^t \left( I_\alpha^{(0)}(t', \lambda_1) \bullet I_\beta^{(0)}(t', \lambda_2) - \frac{1}{n-2} dt'_{0,1} \text{rk}(\alpha) \text{rk}(\beta) \right).$$

Put  $r(t) := \max_i |u_i(t)|$ , where  $\{u_i(t)\}$  is the set of eigenvalues of  $E_{\bullet_t}$ . We claim that the infinite series (17) is convergent for all  $(t, \lambda_1, \lambda_2)$  in the domain

$$D_{\infty}^+ := \{(t, \lambda_1, \lambda_2) \in H \times \mathbb{C}^2 \mid |\lambda_1 - \lambda_2| < |\lambda_1| - r(t), \quad |\lambda_2| < |\lambda_1|\}.$$

Indeed, the period vectors  $I^{(m)}(t, \lambda)$  are solutions to a Fuchsian connection that has poles at  $\lambda = u_i(t)$ . Therefore, their Laurent series expansions at  $\lambda = \infty$  are convergent for  $|\lambda| > r(t)$ . The symplectic pairing  $\Omega(\tilde{f}_{\alpha}(\lambda_1, z), \tilde{f}_{\beta}(\lambda_2, z))$  is convergent in the domain  $|\lambda_1| > |\lambda_2|$  (see Proposition 9). Combining these observations with formulas (20) and (21), we get that the Laurent series expansion at  $\lambda_1 = \infty$  of (17) is convergent for all  $(t, \lambda_1, \lambda_2)$  satisfying  $|\lambda_1| > |\lambda_2|$  and  $|\lambda_1| > r(t)$ . Clearly, the domain  $D_{\infty}^+$  satisfies these inequalities. The reason why we imposed also the inequality  $|\lambda_1 - \lambda_2| < |\lambda_1| - r(t)$  is to guarantee that the straight segment from  $\lambda_1$  to  $\lambda_2$  is outside the disk with center 0 and radius  $r(t)$ , which allows us to make the value of the symplectic pairing (17) depends only on the choice of a reference path from  $(t^{\circ}, \lambda^{\circ})$  to  $(t, \lambda_1)$ . Such a path determines the value of  $I_{\alpha}^{(m)}(t, \lambda_1)$ , while the composition of the reference path and the straight segment from  $(t, \lambda_1)$  to  $(t, \lambda_2)$  provides a reference path that fixes the value of the periods  $I_{\beta}^{(-m-1)}(t, \lambda_2)$ . Let us denote by  $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$  the value of the symplectic pairing (17) at  $(t, \lambda_1, \lambda_2) \in D_{\infty}^+$ . Using formulas (20) and (21) we can extend analytically and define  $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$  for all  $(t, \lambda_1) \in (H \times \mathbb{C})'$  and  $\lambda_2$  sufficiently close to  $\lambda_1$  and  $\lambda_2 \neq \lambda_1$ . Note however that the value of  $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$ , even if we fix a reference path from  $(t^{\circ}, \lambda^{\circ})$  to  $(t, \lambda_1)$ , is determined only up to an integer multiple of  $2\pi i(\alpha|\beta)$  because, according to Proposition 9, the symplectic pairing has a logarithmic singularity at  $\lambda_1 = \lambda_2$  of the form  $(\alpha|\beta)\log(\lambda_1 - \lambda_2)$ .

**Remark 11.** The definition of the domain  $D_{\infty}^+$  in [17], Section 4.4, involves one more inequality  $|\lambda_2| > r(t)$ . This inequality is necessary only if we take also the Laurent series expansion of (17) at  $\lambda_2 = \infty$ .

**4.4. Phase form.** The *phase 1-form* is by definition the 1-form

$$\mathcal{W}_{\alpha, \beta}(t, \xi) = I_{\alpha}^{(0)}(t, 0) \bullet I_{\beta}^{(0)}(t, \xi),$$

where the parameter  $\xi$  is sufficiently small and the RHS is interpreted as an element in  $T_t^*H = H^*$  via the Poincare pairing. Note that we are using the linear structure on  $H$  to trivialize the tangent bundle  $TH = H \times H$ . The form is defined for all  $t \in H'$ , where  $H'$  is defined to be the set of points where  $E_{\bullet_t}$  is invertible. Recall the reference point  $(t^{\circ}, \lambda^{\circ})$  for  $H \times \mathbb{C}$  used in the definition of the periods. We fix  $t^{\circ} - \lambda^{\circ} \mathbf{1} \in H$  to be a reference point. The value of the form depends on the choice of a reference path in  $H'$ .

Suppose that  $C \subset H'$  is a simple loop based at the point  $t - \lambda_1 \mathbf{1} \in H'$  that goes around a generic point on the discriminant. Let  $\varphi \in \mathbb{K}$  be the reflection vector corresponding to the simple loop corresponding to the composition of  $C$  and a reference path from  $t^{\circ} - \lambda^{\circ} \mathbf{1}$  to  $t - \lambda_1 \mathbf{1}$ . Then the following

formula holds:

$$(22) \quad \Omega_{w(\alpha), w(\beta)}(t, \lambda_1, \lambda_2) - \Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2) = 2\pi i(\alpha|\varphi)\langle\varphi, \beta\rangle + \int_{t' \in C} \mathcal{W}_{\alpha, \beta}(t', \lambda_2 - \lambda_1),$$

where  $w$  is the monodromy transformation corresponding to the closed loop  $C$  and  $\lambda_2$  is sufficiently close to  $\lambda_1$ . Here sufficiently close means the following: the line segment  $[(t', \lambda_1 - \lambda_2), (t', 0)]$ , when  $t'$  varies along  $C$ , sweeps out a fat loop in  $H \times \mathbb{C}$ , which does not intersect the discriminant. The proof of (22) is given in [14], Section 4.4 in the settings of singularity theory. It generalizes in our case too using results from [16] and [17]. For the reader's convenience, we give a self-contained proof.

The following result is proved in [16], Lemma 15.

**Proposition 12.** *The phase form is weighted-homogeneous of weight 0, i.e.,*

$$(\xi \partial_\xi + L_E) \mathcal{W}_{\alpha, \beta}(s, \xi) = 0,$$

where  $L_E$  is the Lie derivative with respect to the vector field  $E$ .

*Proof.* Note that

$$\mathcal{W}_{\alpha, \beta}(s, \xi) = (dI_\alpha^{(-1)}(s, 0), I_\beta^{(0)}(s, \xi)).$$

It is easy to check that  $\mathcal{W}_{\alpha, \beta}$  is a closed 1-form, so using the Cartan's magic formula  $L_E = d\iota_E + \iota_E d$ , where  $\iota_E$  is the contraction by the vector field  $E$ , we get

$$L_E \mathcal{W}_{\alpha, \beta} = d((\theta + 1/2)I_\alpha^{(-1)}(s, 0), I_\beta^{(0)}(s, \xi)) = -d(I_\alpha^{(-1)}(s, 0), (\theta - 1/2)I_\beta^{(0)}(s, \xi)).$$

We used that  $\theta$  is skew-symmetric with respect to the residue pairing and that

$$\iota_E dI_\alpha^{(-1)}(s, 0) = E I_\alpha^{(-1)}(s, 0) = (\theta + 1/2)I_\alpha^{(-1)}(s, 0),$$

where the last equality is obtained by substituting  $\lambda = 0$  in the differential equation  $\nabla_{\partial/\partial\lambda}^{(-1)} I_\alpha^{(-1)}(t, \lambda) = 0$ . Furthermore, using the Leibnitz rule we get

$$(23) \quad L_E \mathcal{W}_{\alpha, \beta} = -(dI_\alpha^{(-1)}(s, 0), (\theta - 1/2)I_\beta^{(0)}(s, \xi)) - (I_\alpha^{(-1)}(s, 0), (\theta - 1/2)dI_\beta^{(0)}(s, \xi)).$$

The first pairing on the RHS of (23), using the differential equation  $(\theta - 1/2)I^{(0)}(s, \xi) = (\xi \partial_\xi + E)I^{(0)}(s, \xi)$ , becomes

$$(24) \quad (dI_\alpha^{(-1)}(s, 0), (\xi \partial_\xi + E)I_\beta^{(0)}(s, \xi)) = \xi \partial_\xi \mathcal{W}_{\alpha, \beta}(s, \xi) + (AI_\alpha^{(0)}(s, 0), E \bullet I_\beta^{(1)}(s, \xi)),$$

where  $A = \sum_i (\phi_i \bullet) dt_i$  and we used that the periods are solutions to the second structure connection. Similarly, using the skew-symmetry of  $\theta$  and the differential equation  $dI_\beta^{(0)} = -AI_\beta^{(1)}$ , we transform the 2nd pairing on the RHS of (23) into

$$(25) \quad ((\theta + 1/2)I_\alpha^{(-1)}(s, 0), AI_\beta^{(1)}(s, \xi)) = -(E \bullet I_\beta^{(0)}(s, 0), AI_\alpha^{(1)}(s, \xi)).$$



On the other hand, since the Frobenius multiplication is commutative, the commutator  $[A, E\bullet] = 0$ , so the terms (25) and (24) add up to  $\xi \partial_\xi \mathcal{W}_{\alpha,\beta}(s, \xi)$ .  $\square$

Proposition 12 yields the following identity

$$(26) \quad \partial_{\lambda_2} \mathcal{W}_{\alpha,\beta}(t', \lambda_2 - \lambda_1) = d_{t'} \left( \frac{1}{\lambda_2 - \lambda_1} \left( (\theta + 1/2) I_\alpha^{(-1)}(t', 0), I_\beta^{(0)}(t', \lambda_2 - \lambda_1) \right) \right),$$

where  $d_{t'}$  is the de Rham differential on  $H$ .

**Proposition 13.** *The following formula holds:*

$$\partial_{\lambda_1} \Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = \frac{1}{\lambda_1 - \lambda_2} (I_\alpha^{(0)}(t, \lambda_1), (\lambda_2 - E\bullet) I_\beta^{(0)}(t, \lambda_2)).$$

*Proof.* The de Rham differentials with respect to  $t$  of both sides coincide with the 1-form  $\partial_{\lambda_1} \mathcal{W}_{\alpha,\beta}(t - \lambda_1 \mathbf{1}, \lambda_2 - \lambda_1)$ , where for the LHS we used formulas (20) and (21) and for the RHS we used formula (26), the equation  $(\lambda_2 - E\bullet) I_\beta^{(0)}(t, \lambda_2) = (\theta + 1/2) I_\beta^{(-1)}(t, \lambda_2)$ , and the symmetry  $\mathcal{W}_{\alpha,\beta}(t - \lambda_1 \mathbf{1}, \lambda_2 - \lambda_1) = \mathcal{W}_{\beta,\alpha}(t - \lambda_2 \mathbf{1}, \lambda_1 - \lambda_2)$ . It is enough to prove the formula for  $(t, \lambda_1, \lambda_2) \in D_\infty^+$ , because the difference of the values of both the RHS and the LHS at two different points  $(t^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)})$  and  $(t^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)})$  with  $\lambda_2^{(1)} - \lambda_1^{(1)} = \lambda_2^{(2)} - \lambda_1^{(2)} =: \xi$  is given by the same path integral  $\int_{t^{(1)} - \lambda_1^{(1)} \mathbf{1}}^{t^{(2)} - \lambda_1^{(2)} \mathbf{1}} \partial_\xi \mathcal{W}_{\alpha,\beta}(t', \xi)$ . Using formulas (20) and (21), we get

$$\lim_{t \rightarrow -\infty} \partial_{\lambda_1} \Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = \partial_{\lambda_1} \Omega(\tilde{\mathbf{f}}_\alpha^+(\lambda_1, z), \tilde{\mathbf{f}}_\alpha^-(\lambda_2, z)) = \frac{1}{\kappa} \sum_{s=1}^{\kappa} (\alpha | \sigma^s \beta) \frac{\lambda_2^{1/\kappa} \lambda_1^{-1}}{\eta^s \lambda_1^{1/\kappa} - \lambda_2^{1/\kappa}}.$$

Note that

$$\lim_{t \rightarrow -\infty} \left( \frac{1}{\lambda_1 - \lambda_2} (I_\alpha^{(0)}(t, \lambda_1), (\lambda_2 - E\bullet) I_\beta^{(0)}(t, \lambda_2)) \right) = \frac{1}{\lambda_1 - \lambda_2} \left( \tilde{I}_\alpha^{(0)}(\lambda_1), (\lambda_2 - \rho) \tilde{I}_\beta^{(0)}(\lambda_2) \right).$$

Therefore, we need only to prove that

$$(27) \quad \frac{1}{\lambda_1 - \lambda_2} \left( \tilde{I}_\alpha^{(0)}(\lambda_1), (\lambda_2 - \rho) \tilde{I}_\beta^{(0)}(\lambda_2) \right) = \frac{1}{\kappa} \sum_{s=1}^{\kappa} (\alpha | \sigma^s \beta) \frac{\lambda_2^{1/\kappa} \lambda_1^{-1}}{\eta^s \lambda_1^{1/\kappa} - \lambda_2^{1/\kappa}}.$$

It is enough to check the above formula when  $(\alpha, \beta)$  is a pair of basis vectors, that is,  $\alpha = \phi_{i,p}$  and  $\beta = \phi_{j,q}$ . To begin with, we need to find explicit formulas for the intersection pairing and  $\sigma$  in the basis  $\{\phi_{i,p}\}$ . The intersection pairing can be computed easily using the formulas in Lemma 2. Namely, we have

$$(\phi_{0,0} | \phi_{0,0}) = \frac{1}{n-2}, \quad (\phi_{i,p} | \phi_{i,a_i-p}) = \frac{\sin(\pi p/a_i)}{\pi a_i} \quad (1 \leq i \leq 3, 1 \leq p \leq a_i - 1)$$

and all other pairings between the bases vectors are 0. In particular, the kernel of the intersection pairing is spanned by  $\phi_{0,1}$ . Since the classical monodromy operator  $\sigma$  is defined by analytic continuation of the periods in counterclockwise direction, using formula (13), we get

$$\sigma(\phi_{0,0}) = \phi_{0,0} + 2\pi i \frac{1}{n-2} \phi_{0,1},$$

$$\sigma(\phi_{0,1}) = \phi_{0,1},$$

$$\sigma(\phi_{i,p}) = \eta_i^{-p} \phi_{i,p} \quad (1 \leq i \leq 3, 1 \leq p \leq a_i - 1).$$

The proof of (27) is a straightforward computation. Let us consider only the case when  $\alpha = \phi_{i,p}$  and  $\beta = \phi_{i,a_i-p}$  leaving the rest of the cases as an exercise. The LHS of (27) is

$$\frac{1}{\lambda_1 - \lambda_2} \frac{\lambda_1^{-p/a_i}}{\Gamma(1 - p/a_i)} \frac{\lambda_2^{p/a_i}}{\Gamma(p/a_i)} = \frac{\sin(\pi p/a_i)}{\pi a_i} \frac{(\lambda_2/\lambda_1)^{p/a_i}}{\lambda_1 - \lambda_2}.$$

The RHS of (27), since  $(\alpha|\sigma^s\beta) = \eta_i^{ps}(\phi_{i,p}|\phi_{i,a_i-p})$ , is

$$\frac{\sin(\pi p/a_i)}{\pi a_i} \frac{1}{\kappa} \sum_{s=1}^{\kappa} \eta_i^{ps} \frac{\lambda_2^{1/\kappa} \lambda_1^{-1}}{\eta^s \lambda_1^{1/\kappa} - \lambda_2^{1/\kappa}} = \frac{\sin(\pi p/a_i)}{\pi a_i} \lambda_1^{-1} \frac{1}{\kappa} \sum_{s=1}^{\kappa} \sum_{m=1}^{\infty} \eta^{(-m+p\kappa/a_i)s} (\lambda_2/\lambda_1)^{m/\kappa},$$

where we took the Laurent series expansion at  $\lambda_1 = \infty$  and used that  $\eta_i^{ps} = \eta^{ps\kappa/a_i}$ . Let us exchange the order of the sums over  $s$  and  $m$ . The sum over  $s$  is not zero if and only if  $m/\kappa = p/a_i + \ell$  for some integer  $\ell \geq 0$ . Therefore, the above sum, i.e., the RHS of (27) is

$$\frac{\sin(\pi p/a_i)}{\pi a_i} \lambda_1^{-1} \sum_{\ell=0}^{\infty} (\lambda_2/\lambda_1)^{\ell+p/a_i} = \frac{\sin(\pi p/a_i)}{\pi a_i} \frac{(\lambda_2/\lambda_1)^{p/a_i}}{\lambda_1 - \lambda_2},$$

which is precisely the same as the LHS of (27).  $\square$

Under the same notation as in formula (22), let us introduce the following bilinear form:

$$I_C(\alpha, \beta) := \int_{t' \in C} \mathcal{W}_{\alpha, \beta}(t', \lambda_2 - \lambda_1) - \Omega_{w(\alpha), w(\beta)}(t, \lambda_1, \lambda_2) + \Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2).$$

We have to prove that  $I_C(\alpha, \beta) = -2\pi i(\alpha|\varphi)\langle\varphi, \beta\rangle$ . Using the differential equations of the second structure connection, it is easy to check that  $I_C(\alpha, \beta)$  is independent of  $t$ . Proposition 13 and formula (26) imply that  $I_C(\alpha, \beta)$  is also independent of  $\lambda_1$  and  $\lambda_2$ . Therefore, we may assume that the loop  $C$  is in a neighborhood of  $t - u_i(t)\mathbf{1}$ , for some  $1 \leq i \leq n+1$ , where  $u_i(t)$  ( $1 \leq i \leq n+1$ ) is the set of canonical coordinates, i.e., the eigenvalues of the operator  $E_{\bullet_t}$ . For homotopy reasons, we can also assume that the path  $C$  has a parametrization  $t' = t + (x - \lambda_1)\mathbf{1}$ , where the parameter  $x$  varies counterclockwise along a small loop  $\gamma$  that encloses the points  $\lambda_1 - u_i$  and  $\lambda_2 - u_i$ .

Suppose that formula (22) is proved in the special case when  $\beta = \varphi$  and  $(\alpha|\varphi) = 0$ , that is,

$$(28) \quad \int_{t' \in C} \mathcal{W}_{\alpha, \varphi}(t', \lambda_2 - \lambda_1) = -2\Omega_{\alpha, \varphi}(t, \lambda_1, \lambda_2).$$

The general case can be deduced in the following way: Let us decompose

$$\alpha =: \alpha' + (\alpha|\varphi)\varphi/2, \quad \beta =: \beta' + (\beta|\varphi)\varphi/2,$$

where  $(\alpha'|\varphi) = (\beta'|\varphi) = 0$ . Note that

$$w(\alpha) = \alpha' - (\alpha|\varphi)\varphi/2, \quad w(\beta) =: \beta' - (\beta|\varphi)\varphi/2.$$

The quantity  $I_C(\alpha, \beta)$  is bilinear in  $\alpha$  and  $\beta$ , so substituting the above formulas, after a short computation, we get

$$(29) \quad I_C(\alpha, \beta) = \int_C \mathcal{W}_{\alpha', \beta'}(t', \lambda_2 - \lambda_1) + \frac{1}{4}(\alpha|\varphi)(\beta|\varphi) \int_C \mathcal{W}_{\varphi, \varphi}(t', \lambda_2 - \lambda_1) +$$

$$(30) \quad \frac{1}{2}(\alpha|\varphi) \int_C \mathcal{W}_{\varphi, \beta'}(t', \lambda_2 - \lambda_1) + \frac{1}{2}(\beta|\varphi) \int_C \mathcal{W}_{\alpha', \varphi}(t', \lambda_2 - \lambda_1) +$$

$$(31) \quad (\alpha|\varphi)\Omega_{\varphi, \beta'}(t, \lambda_1, \lambda_2) + (\beta|\varphi)\Omega_{\alpha', \varphi}(t, \lambda_1, \lambda_2).$$

The first integral in (29) is 0 because the cycles  $\alpha'$  and  $\beta'$  are invariant with respect to the local monodromy, which is equivalent to the fact that the periods  $I_{\alpha'}^{(0)}(t', 0)$  and  $I_{\beta'}^{(0)}(t', \lambda_2 - \lambda_1)$  are analytic in  $t'$  in a neighborhood of the point  $t - u_i(t)\mathbf{1}$ . The second integral in (29), since it coincides with  $I_C(\varphi, \varphi)$ , does not depend on  $(t, \lambda_1, \lambda_2)$ , so we may set  $\lambda_2 = \lambda_1$ . By definition,

$$\int_C \mathcal{W}_{\varphi, \varphi}(t', 0) = \int_{\gamma} (I_{\varphi}^{(0)}(t, \lambda_1 - x), I_{\varphi}^{(0)}(t, \lambda_1 - x)),$$

where  $\gamma$  is the small loop in the  $x$ -plane mentioned above based at  $x = 0$  and enclosing  $\lambda_1 - u_i$  and  $\lambda_2 - u_i$ . On the other hand, the period has the following expansion

$$I_{\varphi}^{(0)}(t', \xi) = \frac{\sqrt{2}}{\sqrt{\xi - u_i(t')}} \left( e_i + O(\xi - u_i) \right),$$

where  $e_i = \frac{du_i}{\sqrt{\Delta_i}}$  are the normalized idempotent, that is,  $(e_i, e_j) = \delta_{i,j}$ . Clearly, only the leading order term contributes, so

$$(32) \quad \int_C \mathcal{W}_{\varphi, \varphi}(t', 0) = \int_{\gamma} \frac{2dx}{\lambda_1 - u_i - x} = -4\pi i.$$

Both integrals in (30) can be computed with formula (28), that is, the first integral is  $-2\Omega_{\beta', \varphi}(t, \lambda_2, \lambda_1)$  and the second one is  $-2\Omega_{\alpha', \varphi}(t, \lambda_1, \lambda_2)$ . We get the following formula:

$$I_C(\alpha, \beta) = -\pi i(\alpha|\varphi)(\beta|\varphi) + (\alpha|\varphi) \left( \Omega_{\varphi, \beta'}(t, \lambda_1, \lambda_2) - \Omega_{\beta', \varphi}(t, \lambda_2, \lambda_1) \right).$$

Recalling formulas (20) and (16), we finally get

$$I_C(\alpha, \beta) = -\pi i(\alpha|\varphi)(\beta|\varphi) - 2\pi i(\alpha|\varphi)\langle \varphi, \beta' \rangle.$$

Substituting  $\beta' = \beta - (\beta|\varphi)\varphi/2$  and using  $\langle \varphi, \varphi \rangle = 1$ , we get  $I_C(\alpha, \beta) = -2\pi i(\alpha|\varphi)\langle \varphi, \beta \rangle$ , which is what we had to prove.

For the proof of (28), the most delicate step is to reduce the proof to the case when the loop  $C$  belongs to a neighborhood of a point  $t - u_i(t)\mathbf{1}$ , such that  $|u_i(t)| > |u_j(t)|$  for all  $i \neq j$ . In order to do this, we need to use the *homotopy invariance property* of  $I_C(\alpha, \beta)$  and the so-called *Painlevé property* of a semi-simple Frobenius manifold. Let us state the homotopy invariance property. Suppose that  $h : [0, 1] \times [0, 1] \rightarrow H'$  is a continuous map, such that  $\gamma_s = h(s, \cdot) : [0, 1] \rightarrow H'$  is a simple loop around

the discriminant for all  $s \in [0, 1]$ . Let  $C_0 = \gamma_0$  and  $C_1 = B^{-1} \circ \gamma_1 \circ B$ , where  $B$  is the path  $B(s) = h(0, s)$  for  $s \in [0, 1]$ . Then  $I_{C_0}(\alpha, \beta) = I_{C_1}(\alpha, \beta)$ . For the proof, we need only to notice that the difference

$$I_{C_0}(\alpha, \beta) - I_{C_1}(\alpha, \beta) = \int_{[0,1] \times [0,1]} h^*(d\mathcal{W}_{\alpha, \beta}).$$

It is easy to check that the phase form  $\mathcal{W}_{\alpha, \beta}$  is closed, so the above integral must be 0. Let us state the Painleve property. Let us assume that  $t$  is a semi-simple point, such that the canonical coordinates give an embedding of a neighborhood  $U$  of  $t$  in the configuration space

$$\mathfrak{M}_{n+1} := \{u \in \mathbb{C}^{n+1} \mid u_i \neq u_j \text{ for } i \neq j\}.$$

The Painleve property says that the Frobenius structure on  $U \subset \mathfrak{M}_{n+1}$  can be extended along any path in  $\mathfrak{M}_{n+1}$  except for a set of paths that correspond to the points of an analytic hypersurface in the universal cover of  $\mathfrak{M}_{n+1}$ . The Painleve property is very difficult to prove. It is a corollary of a result of Malgrange about the analytic continuation of the solutions of the Schlesinger equations (see [12]). We refer to [15] for a self-contained proof and for further references. Therefore, for every given  $i$  ( $1 \leq i \leq n+1$ ), we can choose a generic point  $u' \in \mathfrak{M}_{n+1}$ , such that the  $i$ -th coordinate  $u'_i$  of  $u'$  has the largest absolute value and there exists a path  $A : [0, 1] \rightarrow \mathfrak{M}_{n+1}$  connecting  $u(t)$  and  $u'$ , such that the Frobenius structure extends along it. Clearly, if we have a simple loop  $\gamma \subset \mathbb{C}$ , based at  $\lambda_1$ , around  $u_i(t) = A_i(0)$ , we can extend it to a continuous family  $\gamma_s$  ( $0 \leq s \leq 1$ ), such that  $\gamma_s \subset \mathbb{C}$  is a simple loop around  $A_i(s)$ . Therefore,  $h(s_1, s_2) := A(s_1) - \gamma_{s_1}(s_2)\mathbf{1}$  is a homotopy which allows us to replace the simple loop  $C(s) := t - \gamma(s)\mathbf{1}$  around  $t - u_i(t)\mathbf{1}$  with a simple loop around the point  $t' - u_i(t')\mathbf{1}$ , where  $t' = A(1)$ . By construction  $u_i(t') = A_i(1) = u'_i$  has the largest absolute value among all coordinates of  $u'$ .

Let us prove (28) assuming the following conditions: the path  $C$  is a sufficiently small loop in a neighborhood of a generic point on the discriminant with parametrization  $t' = t + (x - \lambda_1)\mathbf{1}$ , where  $x$  varies along a small loop  $\gamma$  based at 0 enclosing  $\lambda_1 - u_i$  and  $\lambda_2 - u_i$ , and  $|u_i(t)| > |u_j(t)|$  for all  $i \neq j$ . We still have the freedom to choose  $\lambda_1$  and  $\lambda_2$  as we wish provided that they are sufficiently close to  $u_i(t)$ . First, we choose a small disk  $\Delta$  with center 0, such that,

$$|u_i(t)| > |u_j(t) + x|, \quad \forall x \in \Delta, \quad \forall j \neq i.$$

Let us choose  $\lambda_1$  and  $\lambda_2$  to be on the line passing through 0 and  $u_i(t)$ , such that,  $|\lambda_1| > |\lambda_2| > |u_i(t)|$  and  $\lambda_1 - u_i(t), \lambda_2 - u_i(t) \in \Delta$ . Note that  $(t, \lambda_1, \lambda_2) \in D_\infty^+$  and that for homotopy reasons we may assume that the loop  $\gamma \subset \Delta$ . Using integration by parts and that  $I_\varphi^{(-n-1)}(t, \lambda_2 - x)$  vanishes at  $x = \lambda_2 - u_i$ , we get for all  $x \in \Delta$

$$(33) \quad \int_{\lambda_2 - u_i}^x (I_\alpha^{(0)}(t, \lambda_1 - y), I_\varphi^{(0)}(t, \lambda_2 - y)) dy = \sum_{n=0}^{\infty} (-1)^{n+1} (I_\alpha^{(n)}(t, \lambda_1 - x), I_\varphi^{(-n-1)}(t, \lambda_2 - x)),$$

where the RHS is interpreted via its Laurent series expansion at  $\lambda_1 = \infty$  and the integration is along the straight segment  $[\lambda_2 - u_i, x]$ . The period  $I_\alpha^{(0)}(t, \lambda_1 - y) = I_\alpha^{(0)}(t + y\mathbf{1}, \lambda_1)$  is analytic at  $\lambda_1 = u_i(t) + y$ , because the condition  $(\alpha|\varphi) = 0$  implies that it is invariant with respect to the local monodromy. Since all other singularities are at the points  $\lambda_1 = u_j(t) + y$  for  $j \neq i$  and  $|u_j(t) + y| < |u_i(t)| < |\lambda_1|$ , we get that the Laurent series expansion at  $\lambda_1 = \infty$  of  $I_\alpha^{(0)}(t, \lambda_1 - y)$  is convergent for all  $|\lambda_1| > |u_i(t)|$  and it depends analytically on  $y \in \Delta$ . Since  $I_\varphi^{(0)}(t, \lambda_2 - y)$  has at most a pole of order  $\frac{1}{2}$  at  $y = \lambda_2 - u_i$ , we get that the LHS of (33) has the form  $(\lambda_2 - u_i - x)^{1/2} f(x, \lambda_1)$  for some function  $f$  holomorphic for all  $x \in \Delta$  and  $|\lambda_1| > |u_i|$ . Therefore, since the loop  $\gamma \subset \Delta$ , we get

$$-2(\lambda_2 - u_i)^{1/2} f(0) = \int_\gamma (I_\alpha^{(0)}(t, \lambda_1 - x), I_\varphi^{(0)}(t, \lambda_2 - x)) dx = \int_\gamma \mathcal{W}_{\alpha, \varphi}(t', \lambda_2 - \lambda_1).$$

It remains only to note that the RHS of (33) at  $x = 0$ , since the point  $(t, \lambda_1, \lambda_2) \in D_\infty^+$ , is precisely  $\Omega_{\alpha, \varphi}(t, \lambda_1, \lambda_2)$ .

## 5. THE SYMPLECTIC SPACE FORMALISM

We would like to recall Givental's symplectic space formalism, which we will need in the proof of Theorem 5.

**5.1. Quantizing symplectic transformation.** Suppose that  $A : \mathcal{H} \rightarrow \mathcal{H}$  is an infinitesimal symplectic transformation

$$\Omega(Af, g) + \Omega(f, Ag) = 0, \quad \forall f, g \in \mathcal{H}.$$

If  $\mathbf{f} \in \mathcal{H}$ , then let us decompose it as

$$\mathbf{f} = \sum_{k=0}^{\infty} \sum_{i,a} (p_{i,a,k} \phi^{i,a}(-z)^{-k-1} + q_{i,a,k} \phi_{i,a} z^k),$$

where  $\{\phi_{i,a}\}$  is the basis of  $H$  that we fixed already and  $\{\phi^{i,a}\}$  is its dual basis with respect to the Poincare pairing. The numbers  $p_{i,a,k}$  and  $q_{i,a,k}$  can be viewed as coordinates of  $\mathbf{f}$ . Under the natural trivialization of the tangent bundle  $T\mathcal{H} = \mathcal{H} \times \mathcal{H}$  the symplectic pairing  $\Omega$  defines a symplectic form on  $\mathcal{H}$  which coincides with  $\sum_{i,a,k} dp_{i,a,k} \wedge dq_{i,a,k}$ . In other words,  $\mathcal{H}$  is a symplectic manifold and  $p_{i,a,k}$  and  $q_{i,a,k}$  form a Darboux coordinate system. The linear map  $\mathbf{f} \mapsto A\mathbf{f}$  defines a linear Hamiltonian vector field with Hamiltonian

$$h_A(\mathbf{f}) := \frac{1}{2} \Omega(A\mathbf{f}, \mathbf{f}).$$

We define  $\widehat{A} := \widehat{h}_A$  where quadratic functions are quantized according to the following rules:

$$(q_{i,a,k} q_{j,b,\ell})^\sim := \hbar^{-1} q_{i,a,k} q_{j,b,\ell}, \quad (p_{i,a,k} p_{j,b,\ell})^\sim = \hbar \frac{\partial^2}{\partial q_{i,a,k} \partial q_{j,b,\ell}}.$$

and

$$(q_{i,a,k} p_{j,b,\ell})^\sim = (p_{j,b,\ell} q_{i,a,k})^\sim = q_{i,a,k} \frac{\partial}{\partial q_{j,b,\ell}},$$

Finally, if  $M = e^A$  is a symplectic transformation then we define  $\widehat{M} := e^{\widehat{A}}$ .

**5.2. Givental's higher genus reconstruction.** Suppose that  $t \in H$  is a semi-simple point. The eigenvalues  $\{u_i(t)\}_{1 \leq i \leq N}$  of  $E_{\bullet t}$  form a local coordinate system in which the Frobenius pairing and multiplication take the form

$$(\partial/\partial u_i, \partial/\partial u_j) = \delta_{i,j}/\Delta_j(t), \quad \frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{i,j} \frac{\partial}{\partial u_i}$$

where  $\Delta_j$  ( $1 \leq j \leq N$ ) are functions such that  $\Delta_j(t) \neq 0$ . Let  $e_i(t) := \sqrt{\Delta_i} \partial/\partial u_i \in T_t H = H$  be the normalized idempotents. There exists a unique formal power series

$$R(t, z) = 1 + \sum_{k=1}^{\infty} R_k(t) z^k, \quad R_k(t) \in \text{End}(H)$$

such that the functions  $R(t, z) e_i(t) e^{u_i(t)/z}$  are solutions to the Dubrovin's connection (see [4]).

Let  $\mathcal{D}_{\text{pt}}(\hbar, \mathbf{t})$  be the total descendant potential of a point (before the dilaton shift). Let us denote by  $\mathcal{D}_{\text{pt}}^{(i)}(\hbar, \mathbf{q})$  the formal series obtained from  $\mathcal{D}_{\text{pt}}(\hbar, \mathbf{t})$  via the substitutions

$$\hbar \mapsto \hbar \Delta_i, \quad t_k := q_k(u_i) + \delta_{k,1}$$

where  $q_k = \sum_{i,a} q_{i,a,k} \partial_{i,a}$  is identified with a flat vector field acting on the function  $u_i$  by derivation. According to Givental's higher genus reconstruction [4], proved by Teleman [19], the total descendant potential of the orbifold  $\mathbb{P}_{n-2,2,2}^1$  is given by the following formula

$$\mathcal{D}(\hbar, \mathbf{q}) = e^{F^{(1)}(t)} (S(t, Q, z)^{-1})^\sim (R(t, z))^\sim \prod_{i=1}^N \mathcal{D}_{\text{pt}}^{(i)}(\hbar, \mathbf{q}),$$

where  $F^{(1)}(t)$  is the genus-1 potential without descendants. It is known that both operator series  $S(t, Q, z)$  and  $R(t, z)$  are symplectic, so their quantization makes sense.

**5.3. Conjugation by  $S$ .** We already discussed the conjugation by  $S$  of the vertex operators (see formula (18)). We get

$$(\widetilde{\Gamma}^\alpha(\lambda) \otimes \widetilde{\Gamma}^{-\alpha}(\lambda)) (\widehat{S}^{-1} \otimes \widehat{S}^{-1}) = (\widehat{S}^{-1} \otimes \widehat{S}^{-1}) e^{W_{\alpha,\alpha}(t,\lambda,\lambda)} (\Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda)).$$

Using formula (20), we get

$$(34) \quad W_{\alpha,\beta}(t, \lambda, \lambda) = \lim_{\mu \rightarrow \lambda} \left( \Omega_{\alpha,\beta}(t, \lambda, \mu) - \widetilde{\Omega}_{\alpha,\beta}(\lambda, \mu) \right), \quad (t, \lambda, \lambda) \in D_\infty.$$

Using the definition of  $W_{k,\ell}$  we get

$$\partial_\lambda W_{\alpha,\alpha}(t, \lambda, \lambda) = -(I_\alpha^{(0)}(t, \lambda), I_\alpha^{(0)}(t, \lambda)) + (\widetilde{I}_\alpha^{(0)}(\lambda), \widetilde{I}_\alpha^{(0)}(\lambda)).$$

Recalling the explicit formulas for the calibrated periods, Lemma 7, and Lemma 8 we get

$$\partial_\lambda W_{\alpha,\alpha}(t, \lambda, \lambda) = -(I_\alpha^{(0)}(t, \lambda), I_\alpha^{(0)}(t, \lambda)) + \left( (\alpha|\alpha) + \frac{1}{n-2} \text{rk}(\alpha) \text{rk}(\alpha) \right) \lambda^{-1}.$$

**5.4. Conjugation by  $R$ .** Let  $(t, \lambda)$  be a point in a neighborhood of a generic point on the discriminant. In other words  $\lambda$  is sufficiently close to  $u := u_i(t)$ . Let  $\beta = \epsilon' - \epsilon''$  ( $\epsilon', \epsilon'' \in \mathcal{E}$ ) be the corresponding vanishing cycle. Put  $\alpha := (\epsilon' + \epsilon'')/2$ . Then we have the following factorizations (see [6], Proposition 4)

$$\Gamma^{\epsilon'}(t, \lambda) \otimes \Gamma^{-\epsilon'}(t, \lambda) = e^{K_{\alpha,\beta}(t,\lambda)} (\Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda)) (\Gamma^{\beta/2}(t, \lambda) \otimes \Gamma^{-\beta/2}(t, \lambda)),$$

and

$$\Gamma^{\epsilon''}(t, \lambda) \otimes \Gamma^{-\epsilon''}(t, \lambda) = e^{-K_{\alpha,\beta}(t,\lambda)} (\Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda)) (\Gamma^{-\beta/2}(t, \lambda) \otimes \Gamma^{\beta/2}(t, \lambda)),$$

where

$$K_{\alpha,\beta}(t, \lambda) = - \int_{t-u\mathbf{1}}^{t-\lambda\mathbf{1}} \mathcal{W}_{\alpha,\beta},$$

where the integration path is along the straight segment between the points  $t - \lambda\mathbf{1}$  and  $t - u\mathbf{1}$ .

Let us define the vertex operators

$$\Gamma_i^\pm := \exp \left( \sum_{n \in \mathbb{Z}} (-z \partial_\lambda)^n \frac{e_i}{\sqrt{2(\lambda - u_i)}} \right),$$

where  $e_i = \sqrt{\Delta_i} \partial / \partial u_i$  are the normalized idempotents. Then the following formula holds (see [6], page 490):

$$(\Gamma^{\pm\beta/2} \otimes \Gamma^{\mp\beta/2})(\widehat{R} \otimes \widehat{R}) = e^{V_{\beta,\beta}} (\widehat{R} \otimes \widehat{R}) (\Gamma_i^\pm \otimes \Gamma_i^\mp),$$

where

$$V_{\beta,\beta} = - \lim_{\epsilon \rightarrow 0} \int_{t-(u_i+\epsilon)\mathbf{1}}^{t-\lambda\mathbf{1}} \left( \mathcal{W}_{\beta/2,\beta/2} + \frac{du_i}{2u_i} \right),$$

where the limit is taken along a straight segment connecting  $t - \lambda\mathbf{1}$  and  $t - u_i\mathbf{1}$ .

Using the factorization and the conjugation by  $R$  formulas we get the following important identity

$$(\Gamma^\varphi \otimes \Gamma^{-\varphi})(\widehat{R} \otimes \widehat{R}) = \frac{c_a^i(t, \lambda)}{c_\varphi^i(t, \lambda)} (\Gamma^{\text{sign}(\alpha|\varphi)\alpha} \otimes \Gamma^{-\text{sign}(\alpha|\varphi)\alpha})(\widehat{R} \otimes \widehat{R}) (\Gamma_i^{\text{sign}(\beta|\varphi)} \otimes \Gamma_i^{-\text{sign}(\beta|\varphi)}),$$

where  $\text{sign}(x)$  is the sign of the number  $x$ ,  $\varphi \in \{\pm\epsilon', \pm\epsilon''\}$ , and the coefficients  $c_a^i(t, \lambda)$  for  $a \in K^0(\mathbb{P}_{n-2,2,2}^1)$  are defined by

$$c_a^i(t, \lambda) := \lim_{\epsilon \rightarrow 0} \exp \left( \int_{t-(u_i+\epsilon)\mathbf{1}}^{t-\lambda\mathbf{1}} \left( \mathcal{W}_{a,a} + (a|\beta)^2 \frac{du}{2u} \right) \right).$$

**5.5. Proof of Theorem 5.** The proof is similar to the proof of Theorem 1.5 in [18]. Let us write the total descendant potential as  $\mathcal{D}(\hbar, \mathbf{q}) = e^{F^{(1)}(t)} S(\widehat{t, Q, z})^{-1} \mathcal{A}_t(\hbar; \mathbf{q})$ . The formal series  $\mathcal{A}_t$  coincides with the so called *total ancestor potential* of  $\mathbb{P}_{n-2,2,2}^1$  (see [4, 5]). Substituting this formula in the Hirota quadratic equations of the extended D-Toda hierarchy and conjugating with  $\widehat{S}^{-1}$  we reduce the proof to proving that the total ancestor potential satisfies the following Hirota quadratic equations: for every  $m \in \mathbb{Z}$  the 1-form

$$(35) \quad \frac{d\lambda}{\lambda} \left( \Gamma^\#(t, \lambda) \otimes \Gamma(t, \lambda) \right) \left( \sum_{\epsilon \in \mathcal{E}} b_\epsilon(t, \lambda) \Gamma^\epsilon(t, \lambda) \otimes \Gamma^{-\epsilon}(t, \lambda) \right) (\tau \otimes \tau)$$

computed at

$$(36) \quad \Omega(w_t, \mathbf{q}' - \mathbf{q}'') = m\hbar^{1/2}$$

is regular in  $\lambda$ . Here the notation is as follows

$$w_t := S(t, Q, z) \phi^{0,0}(-z)^{-1},$$

the vertex operator (with coefficients in the algebra of differential operators)

$$\Gamma(t, \lambda) := \exp \left( (\mathbf{f}_\varphi(t, \lambda) - w_t) \hbar^{1/2} \partial_x \right) \exp \left( \hbar^{-1/2} x v_t \right),$$

where  $v_t := S(t, Q, z) \phi_{0,0}$  and  $\varphi := 1 - L$ , and the coefficients

$$b_\epsilon(t, \lambda) = \widetilde{b}_\epsilon(\lambda) e^{W_{\epsilon, \epsilon}(t, \lambda, \lambda)},$$

where the notation  $W_{\epsilon, \epsilon}$  is the same as in (19). We claim that the coefficients  $b_\epsilon(t, \lambda)$  are compatible with the monodromy representation, i.e., the analytic continuation of  $b_\epsilon(t, \lambda)$  along a closed loop  $C$  around the discriminant coincides with  $b_{w(\epsilon)}(t, \lambda)$ , where  $w$  is the monodromy transformation corresponding to the loop  $C$ . Let us first prove the following simple lemma.

**Lemma 14.** *The coefficient  $\widetilde{b}_\epsilon(\lambda)$  can be computed by the following formula:*

$$\frac{\lambda}{\widetilde{b}_\epsilon(\lambda)} = \lim_{\mu \rightarrow \lambda} \left( (\mu - \lambda) e^{\widetilde{\Omega}_{\epsilon, -\epsilon}(\lambda, \mu)} \right) e^{2\pi i \langle \epsilon, \epsilon_1^3 \rangle}.$$

*Proof.* Let us first point out that, thanks to the explicit formulas in Section 4.2, the factor

$$e^{2\pi i \langle \epsilon, \epsilon_1^3 \rangle} = \begin{cases} -1 & \text{if } \epsilon = \pm \epsilon_1^3, \\ 1 & \text{otherwise.} \end{cases}$$

The rest of the proof is an explicit computation using Proposition 9. Let us consider first the case  $\epsilon = \epsilon_1^3$ . We have  $\text{rk}(\epsilon) = \deg(\epsilon) = 0$  and  $\sigma(\epsilon) = -\epsilon$ . Therefore, according to Proposition 9, we have

$$e^{\widetilde{\Omega}_{\epsilon, -\epsilon}(\lambda, \mu)} = \prod_{s=1}^{\kappa} (1 - \eta^{-s} (\mu/\lambda)^{1/\kappa})^{(-1)^{s+1}} = \frac{1 - \frac{\mu}{\lambda}}{\left(1 - \frac{\sqrt{\mu}}{\sqrt{\lambda}}\right)^2} = \frac{\lambda - \mu}{(\sqrt{\lambda} - \sqrt{\mu})^2},$$



where we used that  $\kappa = 2(n-2)$ . Using the above formula we get that the limit in the identity that we want to prove is  $4\lambda = \lambda/\widetilde{b}_{\epsilon_1^3}(\lambda)$ . The computation in the case  $\epsilon = \epsilon_1^2$  is identical. Finally, suppose that  $\epsilon = \epsilon_i^1$ . By definition  $\text{rk}(\epsilon) = 1$  and  $\deg(\epsilon) = \frac{1}{2} + \frac{i}{n-2}$ . Recalling Proposition 4 we get that the pairing  $(\epsilon_i^1 | \sigma^s(\epsilon_i^1)) = 1$  only if  $s$  is an integer multiple of  $n-2$ , and it is otherwise equal to 0. Therefore, according to Proposition 9, we have

$$e^{\widetilde{\Omega}_{\epsilon,-\epsilon}(\lambda,\mu)} = -\eta_1^i \frac{\mu^{1/(n-2)}}{Q} \left(1 - (\mu/\lambda)^{1/(n-2)}\right)^{-1} = -\frac{\eta_1^i}{Q(\mu^{-1/(n-2)} - \lambda^{-1/(n-2)})}.$$

Using the above formula we compute

$$\lim_{\mu \rightarrow \lambda} \left( (\mu - \lambda) e^{\widetilde{\Omega}_{\epsilon,-\epsilon}(\lambda,\mu)} \right) = \frac{(n-2)\eta_1^i}{Q} \lambda^{1+1/(n-2)} = \frac{\lambda}{\widetilde{b}_{\epsilon_i^1}(\lambda)}. \quad \square$$

Recalling (20) we get the following formula:

$$(37) \quad \frac{\lambda}{b_\epsilon(t, \lambda)} = \lim_{\mu \rightarrow \lambda} \left( (\mu - \lambda) e^{\Omega_{\epsilon,-\epsilon}(t, \lambda, \mu)} \right) e^{2\pi i \langle \epsilon, \epsilon_1^3 \rangle}.$$

Now we can prove the claim about the monodromy invariance. Suppose that  $C$  is a simple loop around the discriminant and that  $\alpha$  is the corresponding reflection vector. The analytic continuation of  $\lambda/b_\epsilon(t, \lambda)$  along the path  $C$  is given by

$$\lim_{\mu \rightarrow \lambda} (\mu - \lambda) \exp \left( \Omega_{\epsilon,-\epsilon}(t, \lambda, \mu) + \int_C \mathcal{W}_{\epsilon,-\epsilon}(t', \mu - \lambda) \right) e^{2\pi i \langle \epsilon, \epsilon_1^3 \rangle}.$$

We have to prove that the above limit coincides with

$$\lim_{\mu \rightarrow \lambda} (\mu - \lambda) \exp \left( \Omega_{w(\epsilon), -w(\epsilon)}(t, \lambda, \mu) \right) e^{2\pi i \langle w(\epsilon), \epsilon_1^3 \rangle},$$

where  $w(x) = x - (\alpha|x)\alpha$  is the reflection representing the monodromy transformation along the loop  $C$ . In other words we have to prove that

$$(38) \quad \exp \left( I_C(\epsilon, -\epsilon) + 2\pi i \langle \epsilon, \epsilon_1^3 \rangle - 2\pi i \langle w(\epsilon), \epsilon_1^3 \rangle \right) = 1.$$

According to formula (22),  $I_C(\epsilon - \epsilon) = 2\pi i (\alpha|\epsilon) \langle \alpha, \epsilon \rangle$ , so

$$I_C(\epsilon, -\epsilon) + 2\pi i \langle \epsilon, \epsilon_1^3 \rangle - 2\pi i \langle w(\epsilon), \epsilon_1^3 \rangle = 2\pi i (\alpha|\epsilon) \langle \alpha, \epsilon_1^3 + \epsilon \rangle.$$

Both  $(\alpha|\epsilon)$  and  $\langle \alpha, \epsilon_1^3 + \epsilon \rangle$  are integers as one can see directly from our explicit formulas for the Euler pairing in Section 4.2, so the identity in (38) is true and the claim about the monodromy invariance of the coefficients  $b_\epsilon(t, \lambda)$  is complete.

Note that the 1-form (35) is invariant under the entire monodromy group. Indeed the monodromy transformations up to translations by terms of the type  $r(1-L)$  ( $r \in \mathbb{Z}$ ) act on the set  $\mathcal{E}$  via permutations. Since

$$(\Gamma(t, \lambda)^\# \otimes \Gamma(t, \lambda))(\Gamma^{r\varphi} \otimes \Gamma^{-r\varphi}) = e^{2\pi i \Omega(w_t, \mathbf{q}' - \mathbf{q}'') \hbar^{-1/2}} \Gamma(t, \lambda)^\# \otimes \Gamma(t, \lambda)$$

the substitution (36) eliminates the contributions of the translation terms. The monodromy invariance implies that the 1-form (35) is a formal power series in  $\mathbf{q}' + \mathbf{1}$  and  $\mathbf{q}'' + \mathbf{1}$  whose coefficients are formal Laurent series in  $\hbar^{1/2}$ , whose coefficients are formal Laurent series in  $\lambda^{-1}$ . By definition the regularity condition means that the Laurent series must have only positive powers of  $\lambda$ , i.e., it is polynomial. On the other hand the total ancestor potential is a *tame* asymptotical function (see [6], Proposition 5). This implies that the coefficient in front of each monomial in  $\hbar^{1/2}$ ,  $\mathbf{q}' + \mathbf{1}$ , and  $\mathbf{q}'' + \mathbf{1}$  must be a polynomial expression in the period vectors  $I_\epsilon^{(m)}(t, \lambda)$  ( $m \in \mathbb{Z}$ ,  $\epsilon \in \mathcal{E}$ ). Therefore each coefficient is a meromorphic function in  $\lambda \in \mathbb{C}$  with possible poles only at the canonical coordinates  $u_i(t)$ . In order to prove the polynomiality we need to check that the 1-form (35) does not have a pole at  $\lambda = u_i$ .

Let us pick  $\lambda$  sufficiently close to  $u := u_i(t)$  and fix a reference path that determines the values of all period vectors. Suppose that  $\beta := \epsilon' - \epsilon''$  is a reflection vector corresponding to the simple loop around  $\lambda = u_i$ . The only terms that could have a pole at  $\lambda = u_i$  are

$$(39) \quad \frac{d\lambda}{\lambda} \left( b_{\epsilon'}(t, \lambda) \Gamma^{\epsilon'}(t, \lambda) \otimes \Gamma^{-\epsilon'}(t, \lambda) + b_{\epsilon''}(t, \lambda) \Gamma^{\epsilon''}(t, \lambda) \otimes \Gamma^{-\epsilon''}(t, \lambda) \right) \mathcal{A}_t \otimes \mathcal{A}_t$$

and

$$\frac{d\lambda}{\lambda} \left( b_{\epsilon'}(t, \lambda) \Gamma^{-\epsilon'}(t, \lambda) \otimes \Gamma^{\epsilon'}(t, \lambda) + b_{\epsilon''}(t, \lambda) \Gamma^{-\epsilon''}(t, \lambda) \otimes \Gamma^{\epsilon''}(t, \lambda) \right) \mathcal{A}_t \otimes \mathcal{A}_t.$$

We claim that both are regular at  $\lambda = u_i$ . Let us prove this statement for the first term. The argument for the second term is similar. Let us compose the vertex operator expression in (39) with  $\widehat{R} \otimes \widehat{R}$ . Recalling the formulas from Section 5.4 we get the composition of

$$(40) \quad c_\alpha^i(t, \lambda) \Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda) (\widehat{R} \otimes \widehat{R})$$

and

$$(41) \quad \frac{d\lambda}{\lambda} \left( \frac{b_{\epsilon'}(t, \lambda)}{c_{\epsilon'}^i(t, \lambda)} \Gamma_i^+ \otimes \Gamma_i^- + \frac{b_{\epsilon''}(t, \lambda)}{c_{\epsilon''}^i(t, \lambda)} \Gamma_i^+ \otimes \Gamma_i^- \right)$$

Note that the term (40) is regular at  $\lambda = u_i$ , because  $\alpha$  is invariant with respect to the local monodromy around  $\lambda = u_i$ , so all periods  $I_\alpha^{(m)}(t, \lambda)$  are analytic at  $\lambda = u_i$ . In order to finish the proof we just need to check that up to a constant the operator (41) coincides with

$$\frac{d\lambda}{\sqrt{\lambda - u_i}} \left( \Gamma_i^+ \otimes \Gamma_i^- - \Gamma_i^- \otimes \Gamma_i^+ \right).$$

If we prove this fact then according to Givental [6] the above operator is the Hirota bilinear operator that defines the Hirota bilinear equations for the KdV hierarchy, i.e., if we apply it to  $\tau \otimes \tau$  where  $\tau$  is a tau-function of the KdV then we get an expression regular in  $\lambda$ . The regularity that we would like to prove follows because  $\mathcal{A}_t = \widehat{R} \prod_i \mathcal{D}_{\text{pt}}^{(i)}$  and  $\mathcal{D}_{\text{pt}}^{(i)}$  is a tau-function of KdV as it was conjectured by Witten [20] and proved by Kontsevich [10].

Let us first prove that

$$\frac{b_{\epsilon'}(t, \lambda)}{c_{\epsilon'}^i(t, \lambda)} = A' \frac{\lambda}{\sqrt{\lambda - u_i}}$$

and

$$\frac{b_{\epsilon''}(t, \lambda)}{c_{\epsilon''}^i(t, \lambda)} = A'' \frac{\lambda}{\sqrt{\lambda - u_i}}$$

for some constants  $A'$  and  $A''$ . Note that

$$\partial_\lambda W_{\epsilon, \epsilon}(t, \lambda, \lambda) = -(I_\epsilon^{(0)}(t, \lambda), I_\epsilon^{(0)}(t, \lambda)) + \left(1 + \frac{1}{n-2} \text{rk}(\epsilon)^2\right) \lambda^{-1}$$

and

$$\partial_\lambda \log c_\epsilon^i = -(I_\epsilon^{(0)}(t, \lambda), I_\epsilon^{(0)}(t, \lambda)) + (\epsilon|\beta)^2 \frac{1}{2(\lambda - u_i)}.$$

Therefore the above identities are equivalent to

$$\partial_\lambda \log \widetilde{b}_\epsilon(\lambda) = -\frac{1}{n-2} \text{rk}(\epsilon)^2 \lambda^{-1}.$$

It remain only to recall the definition of  $\widetilde{b}_\epsilon(\lambda)$ .

Finally, let us prove that  $A'/A'' = -1$ . Let us compute

$$(42) \quad \log(c_{\epsilon''}^i(t, \lambda)/c_{\epsilon'}^i(t, \lambda)) = \lim_{x \rightarrow 0} \left( \int_{t-\lambda \mathbf{1}}^{t-(u+x)\mathbf{1}} \mathcal{W}_{\epsilon', \epsilon'} + \int_{t-(u+x)\mathbf{1}}^{t-\lambda \mathbf{1}} \mathcal{W}_{\epsilon'', \epsilon''} \right).$$

Let  $C_x$  be a small loop based at  $t - (u+x)\mathbf{1}$  that goes once in a counterclockwise direction around  $t - u\mathbf{1}$ . Note that if we add to the RHS of (42) the integral  $\lim_{x \rightarrow 0} \int_{C_x} \mathcal{W}_{\epsilon', \epsilon'}$ , then the resulting sum of 3 integrals coincides with  $\oint_C \mathcal{W}_{\epsilon', \epsilon'}$ , where  $C$  is a simple loop based at  $t - \lambda \mathbf{1}$  that goes around  $t - u\mathbf{1}$ .

We get

$$\log(c_{\epsilon''}^i(t, \lambda)/c_{\epsilon'}^i(t, \lambda)) = \oint_C \mathcal{W}_{\epsilon', \epsilon'} - \lim_{x \rightarrow 0} \int_{C_x} \mathcal{W}_{\epsilon', \epsilon'},$$

Recalling formula (22) we get that

$$\oint_C \mathcal{W}_{\epsilon', \epsilon'} = \lim_{\mu \rightarrow \lambda} \left( \Omega_{\epsilon'', \epsilon''}(t, \lambda, \mu) - \Omega_{\epsilon', \epsilon'}(t, \lambda, \mu) \right) - 2\pi i (\epsilon'|\beta) \langle \beta, \epsilon' \rangle.$$

Note that (the computation is the same as in the proof of (32))

$$\lim_{x \rightarrow 0} \int_{C_x} \mathcal{W}_{\epsilon', \epsilon'} = -(\epsilon'|\beta)^2 \pi i = -\pi i.$$

Recalling also formula (37) we get

$$\frac{b_{\epsilon'}(t, \lambda)}{b_{\epsilon''}(t, \lambda)} = \lim_{\mu \rightarrow \lambda} \exp \left( \Omega_{\epsilon', \epsilon'}(t, \lambda, \mu) - \Omega_{\epsilon'', \epsilon''}(t, \lambda, \mu) + 2\pi i \langle \epsilon'' - \epsilon', \epsilon_1^3 \rangle \right).$$

Combining the above formulas, we get  $A'/A'' = -e^{2\pi i (\langle \epsilon'' - \epsilon', \epsilon_1^3 \rangle - (\epsilon'|\beta) \langle \beta, \epsilon' \rangle)}$ . We need only to check that the number

$$(43) \quad \langle \epsilon'' - \epsilon', \epsilon_1^3 \rangle - (\epsilon'|\beta) \langle \beta, \epsilon' \rangle$$

is an integer. Since  $\langle \epsilon' | \beta \rangle = 1$  and  $\langle \epsilon', \beta \rangle = \frac{1}{2} - \langle \epsilon', \epsilon'' \rangle$ , we get that (43) coincides with

$$(44) \quad \langle \epsilon'' - \epsilon', \epsilon_1^3 \rangle - \langle \epsilon', \epsilon'' \rangle - \frac{1}{2}.$$

If  $\epsilon'' = \pm \epsilon_1^3$ , then  $\epsilon' \neq \pm \epsilon_1^3$  and using the explicit formulas for the Euler pairing (see Section 4.2), we get  $\langle \epsilon', \epsilon_1^3 \rangle = \langle \epsilon', \epsilon'' \rangle = 0$  and  $\langle \epsilon'', \epsilon_1^3 \rangle = \pm \frac{1}{2}$ . Clearly, (44) is an integer in this case. If  $\epsilon' = \pm \epsilon_1^3$  the argument is identical. Let us assume that both  $\epsilon'$  and  $\epsilon''$  are not proportional to  $\epsilon_1^3$ . Recalling again the explicit formulas for the Euler pairing, we get  $\langle \epsilon', \epsilon_1^3 \rangle = \langle \epsilon'', \epsilon_1^3 \rangle = 0$  and  $\langle \epsilon', \epsilon'' \rangle = \pm \frac{1}{2}$ . Clearly, (44) is an integer in this case too.  $\square$

## REFERENCES

- [1] B. Bakalov, T. Milanov. *W-constraints for the total descendant potential of a simple singularity*. Compositio Math. 149 (2013), no. 5, 840–888.
- [2] B. Dubrovin. *Geometry of 2D topological field theories*. In: “Integrable systems and quantum groups” (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.
- [3] B. Dubrovin. *Painlevé transcendents in two dimensional topological field theory*. arXiv: math/9803107.
- [4] A. Givental. *Semisimple Frobenius structures at higher genus*. Internat. Math. Res. Notices, vol. 23(2001): 1265–1286.
- [5] A. Givental. *Gromov–Witten invariants and quantization of quadratic Hamiltonians*. Mosc. Math. J., vol. 1(2001), 551–568.
- [6] A. Givental.  *$A_{n-1}$ -singularities and  $n$ -KdV hierarchies*. Mosc. Math. J., vol. 3, No. 2(2003): 475–505.
- [7] A. Givental and T. Milanov. *Simple singularities and integrable hierarchies*. The breadth of symplectic and Poisson geometry, 173–201, Progr. Math., 232, Birkhuser Boston, Boston, MA, 2005.
- [8] H. Iritani. *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*. Adv. Math., vol. 222, No. 3(2009): 1016–1079.
- [9] V.G. Kac, *Infinite-dimensional Lie algebras*. 3rd ed., Cambridge Univ. Press, Cambridge, 1990.
- [10] M. Kontsevich. *Intersection theory on the moduli space of curves and the matrix Airy function*. Commun. Math. Phys., vol. 147(1992): 1–23.
- [11] F. ten Kroode and J. van de Leur. *Bosonic and fermionic realization of the affine algebra  $\widehat{\mathfrak{so}}_{2n}$* . Comm. in Alg., vol. 20, no. 11(1992): 3119–3162.
- [12] B. Malgrange. *Sur les deformations isomonodromiques I: Singularites regulieres*. (1982)
- [13] T. Milanov. *The Period map for quantum cohomology of  $\mathbb{P}^2$* . Adv. in Math., vol. 351(2019): 804–869.
- [14] T. Milanov. *The phase factors in singularity theory*. Preprint arXiv:1502.07444.
- [15] T. Milanov. *Lectures on Painleve property for semi-simple Frobenius manifolds*. Preprint arXiv:1702.04323
- [16] T. Milanov. *The total ancestor potential in singularity theory*. In: “B-model Gromov–Witten theory”, 539–571, Trends in Mathematics, E. Clauser and Y. Ruan (eds.), Springer Nature Switzerland AG 2018.
- [17] T. Milanov, Y. Shen, and H.-H. Tseng. *Gromov–Witten theory of Fano orbifold curves, Gamma integral structures and ADE-Toda hierarchies*. Geometry & Topology, vol. 20(2016): 2135–2218.
- [18] T. Milanov and H.-H. Tseng. *The spaces of Laurent polynomials, Gromov–Witten theory of  $\mathbb{P}^1$ -orbifolds, and integrable hierarchies*. J. Reine Angew. Math., vol. 622(2008): 189–235.
- [19] C. Teleman. *The structure of 2D semi-simple field theories*. Invent. Math., vol. 188, no. 3(2012): 525–588.

- [20] E. Witten. *Two-dimensional gravity and intersection theory on moduli space*. Surv. Diff. Geom., vol. 1(1991): 243–310.

SCHOOL OF MATHEMATICS, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, XUZHOU, JIANGSU 221116, P.R. CHINA

*E-mail address:* chengjp@cumt.edu.cn

KAVLI IPMU (WPI), UTIAS, THE UNIVERSITY OF TOKYO, KASHIWA, CHIBA 277-8583, JAPAN

*E-mail address:* todir.milanov@ipmu.jp