Perverse sheaves over real hyperplane arrangements II

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Abstract

Let $\mathcal{H}$ be an arrangement of hyperplanes in $\mathbb{R}^n$ and $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ be the category of perverse sheaves on $\mathbb{C}^n$ smooth with respect to the stratification given by complexified flats of $\mathcal{H}$. We give a description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ in terms of "matrix diagrams", i.e., diagrams formed by vector spaces $E_{A,B}$ labelled by pairs (A, B) of real faces of $\mathcal{H}$ (of all dimensions) or, equivalently, by the cells $iA + B$ of a natural cell decomposition of $\mathbb{C}^n$. A matrix diagram is formally similar to a datum describing a constructible (non-perverse) sheaf but with the direction of one half of the arrows reversed.

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0 Introduction

Let $\mathcal{H}$ be a finite arrangement of linear hyperplanes in $\mathbb{R}^n$. It gives rise to a stratification $\mathcal{S}^{(0)} = \mathcal{S}^{(0)}_{\mathcal{H}}$ of the complex space $\mathbb{C}^n$ into the generic parts of the complexified flats of $\mathcal{H}$, see §1 for a precise definition. Let $\mathbf{k}$ be a field. We denote by $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ the category of perverse sheaves (middle perversity) of $\mathbf{k}$-vector spaces on $\mathbb{C}^n$ which are constructible with respect to $\mathcal{S}^{(0)}$. Such perverse sheaves and their categories are of great importance in several areas, including representation theory of quantum groups [2]. They also provide a large class of nontrivial examples of categories of perverse sheaves.

In [9], we gave a description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ in terms of certain quivers, i.e., diagrams of vector spaces E_A labelled by *faces* A of $\mathcal{H}$. We recall that faces are the locally closed polyhedral cones (of all dimensions) into which $\mathcal{H}$ decomposes $\mathbb{R}^n$. They form a poset which we denote $(\mathcal{C} = \mathcal{C}_{\mathcal{H}}, \leqslant)$.

In this paper we propose an alternative description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ which is extremely simple and appealing. It is given in terms of *matrix diagrams*, see Definition 3.1, i.e., diagrams consisting of :

(0) Vector spaces $E_{A,B}$ labelled by arbitrary pairs of faces $A, B \in \mathcal{C}$.

(1′) A *representation* of $\mathcal{C}$ with respect to the second argument, i.e., a transitive system of linear maps

$$\partial' : E_{A,B_1} \longrightarrow E_{A,B_2}, \;\; B_1 \leqslant B_2.$$

(1″) An *anti-representation* of $\mathcal{C}$ with respect to the first argument, i.e., a transitive system of linear maps

$$\partial'' : E_{A_2,B} \longrightarrow E_{A_1,B}, \;\; A_1 \leqslant A_2.$$

It is required that:

(2) The maps ∂' and ∂'' commute with each other, i.e., unite into a covariant functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Vect}_{\mathbf{k}}$.

(3) If the "product cells" $iA + B_1$ and $iA + B_2$, $B_1 \leqslant B_2$, lie in the same complex stratum, then the corresponding ∂' is an isomorphism. Likewise, if $iA_1 + B$ and $iA_2 + B$, $A_1 \leqslant A_2$, lie in the same complex stratum, then the corresponding ∂'' is an isomorphism.

Using the "Tits product" $A \circ B$ on real cells (see §7B below), one can give (Remark 7.9(a)) a reformulation of the condition (3) in terms involving only real cells.

Our main result, Theorem 3.4, says that $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ is equivalent to the category of data $(E_{A,B}, \partial', \partial'')$ satsfying the above conditions. We also present, in Theorem 7.11, a generalization to arrangements of affine hyperplane with real equations.

The simplest example for Theorem 3.4 is that of $\mathrm{Perv}(\mathbb{C}, 0)$, the category of perverse sheaves on $\mathbb{C}$ with only singularity at 0. Our description identifies it with the category of commutative 3×3-diagrams below with the arrows at the outer rim being isomorphisms:

$$
\begin{tikzcd}
E_{-,+} \arrow[r, "\simeq"] & E_{0,+} & E_{+,+} \arrow[l, "\simeq"'] \\
E_{-,0} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r] & E_{0,0} \arrow[u] \arrow[d] & E_{+,0} \arrow[l] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
E_{-,-} \arrow[r, "\simeq"'] & E_{0,-} & E_{+,-} \arrow[l, "\simeq"]
\end{tikzcd}
\tag{0.1}
$$

Theorem 3.4 is strikingly similar to the much more standard description of *constructible sheaves* on $(\mathbb{C}^n, \mathcal{S}^{(0)})$ in terms of the quasi-regular cell decomposition of $\mathbb{C}^n$ into the product cells $iA + B$. Such a sheaf $\mathcal{G}$ is given by its stalks $G_{A,B}$ at the $iA + B$ and generalization maps γ', γ'' just like in $(1')$ and $(1'')$ but *covariant in both cases.* The condition that $\mathcal{G}$ is indeed $\mathcal{S}^{(0)}$-constructible means, just like in (3), that γ' and γ'' are isomorphisms whenever the source and target correspond to product cells that lie in the same complex stratum.

Dually, an $\mathcal{S}^{(0)}$-*constructible cosheaf* (i.e., from the derived category point of view, a constructible complex Verdier dual to a sheaf) is given by a diagram consisting of the G_{AB} and maps δ', δ'' *contravariant in both cases*, with the same properties.

Our result shows that perverse sheaves, occupying, intuitively, the middle position between sheaves and cosheaves, admit a matching description that is just as simple, by reversing one of the two sets of arrows.

Like a diagram describing a constructible sheaf, a matrix diagram has several "layers" (corresponding to complex strata $L^\circ_{\mathbb{C}}$) with the property that the arrows within each layer are isomorphisms, and therefore give a local system on the corresponding $L^\circ_{\mathbb{C}}$. For a constructiible sheaf $\mathcal{G}$ this local system is just the restriction of $\mathcal{G}$ to $L^\circ_{\mathbb{C}}$. For a perverse sheaf $\mathcal{F}$ this corresponds to the restriction to $L^\circ_{\mathbb{C}}$ of the *hyperbolic restriction* of $\mathcal{F}$ to the closure of the stratum which is the complex flat $L_{\mathbb{C}}$, see [9], §5A. The arrows between different layers describe the way such local systems are glued together.

For example, the outer rim of a diagram in (0.1) represents a local system on $\mathbb{C}\backslash\{0\}$ obtained by restricting the corresponding perverse sheaf from $\mathbb{C}$ to $\mathbb{C}\backslash\{0\}$, while the full diagram can be seen as symbolically representing the complex plane $\mathbb{C}$ itself. Further, the incoming and the outgoing arrows at the middle term resemble the attractive and repulsive trajectories of a hyperbolic vector field on $\mathbb{C} = \mathbb{R}^2$, very much in the spirit of the original philosophy of hyperbolic localization [8, 5].

Our method of proof of Theorem 3.4 is similar to that of [9] but simplified, stripped, so to say, to the bare bones. As in [9], the starting point is the Cousin resolution $\mathcal{E}^\bullet(\mathcal{F})$ of $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ associated to the system of tube cells $\mathbb{R}^n + iA$, $A \in \mathcal{C}$, see §4. The matrix diagram $(E_{A,B})$ corresponding to $\mathcal{F}$ is the linear algebra data describing $\mathcal{E}^\bullet(\mathcal{F})$ as a complex of cellular sheaves on the cell decomposition formed by the cells $iA + B$. That is, the $E_{A,B}$ themselves are (up to sign factors) the stalks of the terms $\mathcal{E}^p = \mathcal{E}^p(\mathcal{F})$. The maps ∂' are the generalization maps describing the sheaf structure on each $\mathcal{E}^p$. The maps ∂'' describe the differentials $d : \mathcal{E}^p \to \mathcal{E}^{p+1}$. The condition $(1')$ express the fact that each $\mathcal{E}^p$ is a sheaf. The condition $(1'')$ expresses the requirement that $d^2 = 0$ in $\mathcal{E}^\bullet(\mathcal{F})$. The remaining conditions in (2) mean that d is a morphism of cellular sheaves. Thus any datum $(E_{A,B}, \partial', \partial'')$ satisfying (1)-(2) always gives a cellular complex $\mathcal{E}^\bullet$. The nontrivial part of Theorem 3.4 is that the condition (3) precisely guarantees that this complex is in fact a perverse sheaf lying in $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$, in particular, that it is constructible with respect to $\mathcal{S}^{(0)}$. A crucial step here is a direct combinatorial identification of the Verdier dual to $\mathcal{E}^\bullet$ (Proposition 4.6 and §5). The proof of Theorem 3.4 is finished in §6.

We would also like to emphasize an important difference between the present description and that of [9]. While the approach of [9] is centered around the "real skeleton" $\mathbb{R}^n \subset \mathbb{C}^n$, the linear algebra data in the present description are directly tied to all the cells of a cell decomposition of the underlying stratified space. Therefore the new approach can be viewed as somewhat bridging the gap between the *geometric definition* of perverse sheaves (via the t-structure on the derived category) and various *combinatorial descriptions* (usually obtained by a judicious choice of extra data). Because of its "local" nature, it may be applicable in a wider range of situations than just hyperplane arrangements. A different "bridging" approach, close to the ideas of MacPherson [11], is developed in [6].

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1 Generalities on arrangements

We keep the notations and conventions of [9] which we recall for the reader's convenience. Thus:

$\mathcal{H}$ is a finite arrangement of linear hyperplanes in $\mathbb{R}^n$, assumed *central*, i.e., $\bigcap_{H\in\mathcal{H}} H = \{0\}$. We choose, once and for all, a linear equation $f_H : \mathbb{R}^n \to \mathbb{R}$ for any $H \in \mathcal{H}$. We denote

$$\mathrm{sgn} : \mathbb{R} \longrightarrow \{+, -, 0\}$$

the standard sign function.

$(\mathcal{C} = \mathcal{C}_{\mathcal{H}}, \leqslant)$ is the poset of *faces* of $\mathcal{H}$, ordered by inclusion of the closures. By definition, $x, y \in \mathbb{R}^n$ lie in the same face iff $\mathrm{sgn}\, f_H(x) = \mathrm{sgn}\, f_H(y)$ for each $H \in \mathcal{H}$. For a face C and $H \in \mathcal{H}$ we denote

$$s_H(C) = \mathrm{sgn}\big(f_H|_C\big).$$

The faces are locally closed polyhedral subsets of $\mathbb{R}^n$ forming a disjoint union (stratification) of $\mathbb{R}^n$. Each face is a topological cell, i.e., is homeomorphic to $\mathbb{R}^p$ for some p.

If $A, B \in \mathcal{C}$, and $p > 0$, the notation $A <_p B$ means that $A \leqslant B$ and $\dim(B) = \dim(A) + p$.

A *flat* of $\mathcal{H}$ is any intersection of hyperplanes from $\mathcal{H}$. This is understood to include $\mathbb{R}^n$ itself (the intersection of the empty set of hyperplanes).

For any subset $A \subset \mathbb{R}^n$ we denote by $\mathbb{L}(A)$ the $\mathbb{R}$-linear span of A and denote

$$\pi_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{L}(A)$$

the projection.

For any subspace $L \subset \mathbb{R}^n$ we denote $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{C}^n$ its complexification. In particular we have the complexified arrangement $\mathcal{H}_{\mathbb{C}} = \{H_{\mathbb{C}}, H \in \mathcal{H}\}$ in $\mathbb{C}^n$.

The space $\mathbb{C}^n$ is equipped with the *complex stratification* ($\mathbb{C}$-*stratification* for short) $\mathcal{S}^{(0)}$ whose strata are the generic parts of complexified flats, i.e., the locally closed subvarieties ($\mathbb{C}$-*strata*)

$$L^{\circ}_{\mathbb{C}} \;=\; L_{\mathbb{C}} \setminus \bigcup_{H \not\supset L} H_{\mathbb{C}},$$

where L is a flat of $\mathcal{H}$. It also has the stratification (decomposition) $\mathcal{S}^{(2)}$ into the *product cells* $C + iD$, $C, D \in \mathcal{C}$. This decomposition refines $\mathcal{S}^{(0)}$.

We will also use the *intermediate* (or *Björner-Ziegler*) stratification $\mathcal{S}^{(1)}$ of $\mathbb{C}^n$ into strata $[C, D]$ parametrized by intervals $C \leqslant D$ in $\mathcal{C}$. By definition (see [4] and [9] §2D), $x + iy \in \mathbb{C}^n$ lies in $[C, D]$, if:

$$y \in C, \quad x \in \pi_C^{-1}(\pi_C(D)). \tag{1.1}$$

We consider each $\mathcal{S}^{(i)}$, $i = 0, 1, 2$, as a poset of strata ordered by inclusion of closures.

Ir is known (see [9] Prop. 2.10) that, denoting by $\prec$ the relation of refinement of stratifications, we have

$$\mathcal{S}^{(2)} \prec \mathcal{S}^{(1)} \prec \mathcal{S}^{(0)}.$$

In particular, we have the equivalence relations $\equiv_{\mathcal{S}^{(0)}}$ and $\equiv_{\mathcal{S}^{(1)}}$ on $\mathcal{S}^{(2)} = \mathcal{C} \times \mathcal{C}$ describing when the two product cells lie in one $\mathcal{S}^{(0}$-stratum (i.e., $\mathbb{C}$-stratum) or in one $\mathcal{S}^{(1)}$-stratum. For simplicity we write $\equiv$ for $\equiv_{\mathcal{S}^{(0)}}$. We now describe these equivalence relations more explicitly, starting with $\equiv$.

For a face $C \in \mathcal{C}$ we denote

$$\mathcal{H}^C \;=\; \big\{H \in \mathcal{H} \,|\, H \supset C\big\} \tag{1.2}$$

the set of hyperplanes from $\mathcal{H}$ containing H.

Proposition 1.3. *We have $A + iB \equiv C + iD$, if and only if*

$$\mathcal{H}^A \cap \mathcal{H}^B = \mathcal{H}^C \cap \mathcal{H}^D.$$

Proof: Let

$$\mathcal{H}_{\mathbb{C}}^{A+iB} = \{H_{\mathbb{C}} \in \mathcal{H}_{\mathbb{C}} | \, H_{\mathbb{C}} \supset A + iB\}.$$

Then $A + iB \equiv C + iD$ if and only if $\mathcal{H}_{\mathbb{C}}^{A+iB} = \mathcal{H}_{\mathbb{C}}^{C+iD}$. But for $H \in \mathcal{H}$, we have $H_{\mathbb{C}} \in \mathcal{H}_{\mathbb{C}}^{A+iB}$ if and only if $H \in \mathcal{H}^A \cap \mathcal{H}^B$. Indeed, let $f = f_H : \mathbb{R}^n \to \mathbb{R}$ be the linear equaltion of H. Then

$$f^{\mathbb{C}} : \mathbb{C}^n \longrightarrow \mathbb{C}, \quad f^{\mathbb{C}}(x + iy) = f(x) + if(y), \;\; x, y \in \mathbb{R}^n,$$

is a $\mathbb{C}$-linear equaltion of $H_{\mathbb{C}}$. So if $x, y \in \mathbb{R}^n$ and $f^{\mathbb{C}}(x + iy) = 0$, the $f(x) = f(y) = 0$. □

We now describe more explicitly the relation $\equiv_{\mathcal{S}^{(1)}}$, i.e., the way how an $\mathcal{S}^{(1)}$-stratum is decomposed into product cells.

Proposition 1.4. *Let $C \in \mathcal{C}$. Introduce an equivalence relation $\sim_C$ as the equivalence closure of the following relation $\approx_C$:*

$$B_1 \approx_C B_2 \quad \textit{iff} \quad B_1 \leqslant B_2 \textit{ and } B_1 + iC \equiv B_2 + iC.$$

Then, equivalence classes $\mathcal{B}$ of $\sim_C$ are on bijection with $\mathcal{S}^{(1)}$-strata $[C, D]$, $D \geqslant C$. More precisely, each such stratum consists of product cells $B + iC$, $B \in \mathcal{B}$ for some $\sim_C$-class $\mathcal{B}$.

Proof: By definition (the first condition in (1.1)), each $[C, D]$ is the union of the $B + iC$ where B runs over some subset $\mathcal{B} \subset \mathcal{C}$. We prove that $\mathcal{B}$ is in fact an equivalence class of $\sim_C$.

We first prove that each such $\mathcal{B}$ is a union of equivalence classes of $\sim_C$. For this it suffices to show that

$$B_1 \approx_C B_2 \quad \Rightarrow \quad B_1 + iC \equiv_{\mathcal{S}^{(1)}} B_2 + iC.$$

Indeed, suppose $B_1 \leqslant B_2$ and $B_1 + iC \equiv B_2 + iC$. Then, in the notation of (1.2),

$$\mathcal{H}^{B_1} \supset \mathcal{H}^{B_2} \quad \text{and} \quad \mathcal{H}^{B_1} \cap \mathcal{H}^C = \mathcal{H}^{B_2} \cap \mathcal{H}^C. \tag{1.5}$$

The condition $B_1 + iC \equiv_{\mathcal{S}^{(1)}} B_2 + iC$ that we need to prove, means that B_1 and B_2 lie in the same region of the form $\pi_C^{-1}(\pi_C(D))$, i.e., that

$$s_H(B_1) = s_H(B_2) \quad \text{for each } H \in \mathcal{H}^C. \tag{1.6}$$

By (1.5), for $H \in \mathcal{H}^C$ we have $s_H(B_1) = 0$ iff $s_H(B_2) = 0$. On the other hand, since $B_1 \leqslant B_2$, the difference between $s_H(B_1)$ and $s_H(B_2)$ can only be that $s_H(B_2) \in \{\pm\}$ while $s_H(B_1) = 0$. But this is impossible by (1.5). So $B_1 + iC \equiv_{\mathcal{S}^{(1)}} B_2 + iC$ as claimed.

We now prove that $\mathcal{B}$ is a single equivalence class of $\sim_C$. For this we note that the equivalence relation on $\mathcal{B}$ generated by $\leqslant$ (inclusion of closure of faces) has a single equivalence class. Indeed, $[C, D]$ is known to be a cell (in particular, connected) decomposed into product cells $B + iC$, $B \in \mathcal{B}$. So the relation of inclusion of closures on these cells generates but a single equvalence class.

But if $B_1 \leqslant B_2$ and $B_1 + iC \equiv_{\mathcal{S}^{(1)}} B_2 + iC$, then we have (1.6) and so (1.5) so $B_1 \approx_C B_2$. $\square$

2 Generalities on cellular sheaves and perverse sheaves

We fix a base field $\mathbf{k}$ and denote $\mathrm{Vect}_{\mathbf{k}}$ the category of $\mathbf{k}$-vector spaces. By a *sheaf* we always mean a sheaf of $\mathbf{k}$-vector spaces. For a topological space X we denote by Sh_X the category of sheaves on X and by $D^b\,\mathrm{Sh}_X$ the bounded derived category of Sh_X. For $V \in \mathrm{Vect}_{\mathbf{k}}$ we denote by $\underline{V}_X$ the constant sheaf on X with stalk V.

If $(X, \mathcal{S})$ is a stratified space, then we denote $\mathrm{Sh}_{X,\mathcal{S}}$ the category of $\mathcal{S}$-*constructible* sheaves on X, i.e., sheaves $\mathcal{F}$ which are locally constant, with finite-dimensional stalks, on each stratum of $\mathcal{S}$. We denote by $D^b_{\mathcal{S}}\,\mathrm{Sh}_X \subset D^b\,\mathrm{Sh}_X$ the full subcategory of $\mathcal{S}$-*constructible complexes*, i.e., of complexes $\mathcal{F}$ such that each cohomology sheaf $\underline{H}^i(\mathcal{F})$ lies in $\mathrm{Sh}_{X,\mathcal{S}}$. The triangulated category $D^b_{\mathcal{S}}\,\mathrm{Sh}_X$ has a perfect dualty, the *Verdier duality*, denoted $\mathcal{F} \mapsto \mathbb{D}(\mathcal{F})$.

By a *cell* we mean a topological space σ homeomorphic to $\mathbb{R}^d$ for some d. A *cellular space* is a stratified space $(X.\mathcal{S})$ such that each stratum is a cell. We consider $\mathcal{S}$ as the poset of cells with the order $\leqslant$ given bu inclusion of the closures. A *cellular sheaf*, resp. *cellular complex* on a cellular space $(X, \mathcal{S})$ is an $\mathcal{S}$-constructible sheaf, resp. complex on X. For such a sheaf, resp. complex $\mathcal{F}$ and a cell $j_\sigma : \sigma \hookrightarrow X$ we denote by

$$\mathcal{F}\|_\sigma \;=\; R\Gamma(\sigma, j_\sigma^* \mathcal{F})$$

the stalk of $\mathcal{F}$ at σ. We recall from [9] §1D the concept of a *quasi-regular cellular space* as well as the following fact.

Proposition 2.1. *Let $(X, \mathcal{S})$ be a quasi-regular cellular space. Then $\mathrm{Sh}(X, \mathcal{S})$ is equivalent to the category of representations of the poset $(\mathcal{S}, \leqslant)$, i.e., of data formed by:*

(0) Vector spaces $G_\sigma, \sigma \in \mathcal{S}$.

(1) Linear maps $\gamma_{\sigma,\sigma'} : G_\sigma \to G_{\sigma'}$ given for each $\sigma \leqslant \sigma'$ and satisfying the transitivity relations:

(2) $\gamma_{\sigma,\sigma''} = \gamma_{\sigma',\sigma''} \circ \gamma_{\sigma,\sigma'}$ for any $\sigma \leqslant \sigma' \leqslant \sigma''$.

Explicitly, to a sheaf $\mathcal{G} \in \mathrm{Sh}(X, \mathcal{S})$ there corresponds the datum formed by the stalks $G_\sigma = \mathcal{G}\|_\sigma$ and the generalization maps $\gamma_{\sigma,\sigma'} : \mathcal{G}\|_\sigma \to \mathcal{G}\|_{\sigma'}$. □

We specialise to the situation of §1 and take $X = \mathbb{C}^n$. The stratifications $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are quasi-regular cell decompositions of $\mathbb{C}^n$, while $\mathcal{S}^{(0)}$ is not.

We will use the involution

$$\tau : \mathbb{C}^n \to \mathbb{C}^n, \quad (x + iy) \;\mapsto\; (y + ix). \tag{2.2}$$

This involution is not $\mathbb{C}$-linear but it preserves $\mathcal{H}$ and the stratifications $\mathcal{S}^{(0)}$ (stratum by stratum, i.e., τ preserves each stratum) and $\mathcal{S}^{(2)}$ (as a whole, i.e., τ takes each stratum to another stratum). But it does not preserve $\mathcal{S}^{(1)}$. We denote by $\tau\mathcal{S}^{(1)}$ the new stratification of $\mathbb{C}^n$ whose strata are obtained by applying τ to the strata of $\mathcal{S}^{(1)}$.

Proposition 2.3. *Let $\mathcal{F} \in \mathrm{Sh}(\mathbb{C}^n, \mathcal{S}^{(2)})$. Suppose that $\mathcal{F}$ is both $\mathcal{S}^{(1)}$-constructible and $\tau\mathcal{S}^{(1)}$-constructible. Then $\mathcal{F}$ is $\mathcal{S}^{(0)}$-constructible.*

Proof: By definition, being $\mathcal{S}^{(0)}$-construcible means that for inclusion $B_1 + iC_1 \leqslant B_2 + iC_2$ of (closures of) product cells such that $B_1 + iC_1 \equiv B_2 + iC_2$, the corresponding generalization map

$$\gamma_{B_1+iC_1, B_2+iC_2} : \mathcal{F}\|_{B_1+iC_1} \longrightarrow \mathcal{F}\|_{B_2+iC_2}$$

is an isomorphism. Now, $\mathcal{S}^{(2)} = \mathcal{C} \times \mathcal{C}$ is the product stratification, and, moreover, the stratification of each complexified flat of $\mathcal{H}$ into $\mathcal{S}^{(2)}$-strata is also a product stratification. So it suffices to prove the isomorphicity of $\gamma_{B_1+iC_1,B_2+iC_2}$ in two separate cases, horizontal and vertical:

(1) $B_1 \leqslant B_2$, $C_1 = C_2 = C$, and $B_1 + iC \equiv B_2 + iC$.

(2) $B_1 = B_2 = B$, $C_1 \leqslant C_2$ and $B + iC_1 \equiv B + iC_2$.

Now, inclusions of type (1) generate, by Proposition 1.4, inclusions of $\mathcal{S}^{(2)}$-cells within the same $\mathcal{S}^{(1)}$-cell. Similarly, inclusions of type (2) generate inclusions of $\mathcal{S}^{(2)}$-cells within the same $\tau\mathcal{S}^{(1)}$-cell. So if $\mathcal{F}$ is both $\mathcal{S}^{(1)}$-constructible and $\tau\mathcal{S}^{(1)}$-constructible, then the generalization maps corresponding to (1) and (2) are all isomorphisms hence $\mathcal{F}$ is $\mathcal{S}^{(0)}$-constructible. $\square$

We denote by $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \subset D^b_{\mathcal{S}^{(0)}} \mathrm{Sh}_{\mathbb{C}^n}$ the category of perverse sheaves (with respect to the middle perversity), which are are $\mathcal{S}^{(0)}$-constructible. Explicitly, we normalize the perversity conditions by saying that $\mathcal{F}$ is perverse, if:

(P^-) The sheaf $\underline{H}^p(\mathcal{F})$ is supported on a subvariety of complex codimension $\geqslant p$.

(P^+) If S is a stratum of $\mathcal{S}^{(0)}$ of complex codimension p, then the sheaves $\underline{\mathbb{H}}^i_S(\mathcal{F})$ of hypercohomology with support in S, are zero for $i < p$.

In this normalization, a perverse sheaf reduces, on the open stratum, to a local system in degree 0. As well known , the condition (P^+) for $\mathcal{F}$ is equivalent to (P^-) for the shifted Verdier dual

$$\mathcal{F}^* = \mathbb{D}(\mathcal{F})[-2n].$$

The functor $\mathcal{F} \mapsto \mathcal{F}^*$ is thus a perfect duality on $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$. Since the involution τ preserves $\mathcal{S}^{(0)}$, the pullback functor τ^{-1} preserves the category $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ and so this category has another perfect duality

$$\mathcal{F} \mapsto \mathcal{F}^\tau := \tau^{-1}\mathcal{F}^*. \tag{2.4}$$

3 Matrix diagrams and the main result

Definition 3.1. A *matrix diagram* of type $\mathcal{H}$ is a collection E of the following data:

(M0) Finite-dimensional $\mathbf{k}$-vector spaces $E_{A,B}$ given for any two faces $A, B \in \mathcal{C}$.

(M1) Linear maps

$$\partial' = \partial'_{(A|B_1,B_2)} : E_{A,B_1} \longrightarrow E_{A,B_2}, \text{ given for any faces } A \text{ and } B_1 \leqslant B_2,$$
$$\partial'' = \partial''_{(A_2,A_1|B)} : E_{A_2,B} \longrightarrow E_{A_1,B}, \text{ given for any faces } A_1 \leqslant A_2 \text{ and } B,$$

satisfying the following conditions:

(M2) The maps ∂', ∂'' define a representation of the poset $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ in $\mathrm{Vect}_{\mathbf{k}}$. That is, we have

$$\begin{aligned}
\partial'_{(A|B_2,B_3)} \circ \partial'_{(A|B_1,B_2)} &= \partial'_{(A|B_1,B_3)}, \quad \forall\, A \text{ and } B_1 \leqslant B_2 \leqslant B_3;\\
\partial''_{(A_2,A_1|B)} \circ \partial''_{(A_3,A_2|B)} &= \partial''_{(A_3,A_1|B)}, \quad \forall\, A_1 \leqslant A_2 \leqslant A_3 \text{ and } B;\\
\partial'_{(A_2,A_1|B_2)} \circ \partial''_{(A_2|B_1,B_2)} &= \partial''_{(A_1|B_1,B_2)} \circ \partial'_{(A_2,A_1|B)}, \quad \forall\, A_1 \leqslant A_2 \text{ and } B_1 \leqslant B_2.
\end{aligned}$$

(M3′) If A and $B_1 \leqslant B_2$ are such $B_1 + iA \equiv B_2 + iA$ (that is, these product cells lie in the same complex stratum), then $\partial'_{(A|B_1,B_2)}$ is an isomorphism.

(M3″) Similarly, if $A_1 \leqslant A_2$ and B are such that $B + iA_1 \equiv B + iA_2$, then $\partial''_{(A_2,A_1|B)}$ is an isomorphism.

We denote by $\mathcal{M}_{\mathcal{H}}$ the category of matrix diagrams of type $\mathcal{H}$. This category is abelian and has a perfect duality

$$E \mapsto E^*, \quad (E^*)_{A,B} = (E_{B,A})^* \tag{3.2}$$

("hermitian conjugation") with the maps ∂' for E^* being the dual of the ∂'' for E and the ∂'' for E^* being dual to the ∂' for E.

Remark 3.3. We note that any matrix diagram is "symmetric" in the following weak sense. Since $B + iA$ and $A + iB$ lie in the same complex stratum by Proposition 1.3, we have an isomorphism (non-canonical) $E_{A,B} \simeq E_{B,A}$. It is given by the monodromy of the "layer" (system of isomorphic maps ∂', ∂'') of the matrix diagram corresponding to this complex stratum.

Our main result is as follows.

Theorem 3.4. *We have mutually quasi-inverse equivalences of categories*

$$\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \underset{\mathbb{G}}{\overset{\mathbb{E}}{\rightleftarrows}} \mathcal{M}_{\mathcal{H}}$$

taking the twisted Verdier duality $\mathcal{F} \mapsto \mathcal{F}^\tau$ to the duality (3.2).

4 From a perverse sheaf to a matrix diagram: the Cousin complex

Here we construct a functor $\mathbb{E} : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \to \mathcal{M}_{\mathcal{H}}$. We use the general analysis of perverse sheaves on $(\mathbb{C}^n, \mathcal{H})$ in terms of their Cousin complexes [9].

For a face $A \in \mathcal{C}$ we denote $\lambda_A : \mathbb{R}^n + iA \hookrightarrow \mathbb{C}^n$ the embedding of the corresponding "tube cell".

Proposition 4.1. *Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$. Then:*

(a) The complex $\lambda_A^! \mathcal{F}$ reduces to a single sheaf $\widetilde{\mathcal{E}}_A = \widetilde{\mathcal{E}}_A(\mathcal{F})$ in degree $\mathrm{codim}(A)$.

(b) The complex $\lambda_{A*}\widetilde{\mathcal{E}}_A(\mathcal{F})$ also reduces to a single sheaf $\mathcal{E}_A = \mathcal{E}_A(\mathcal{F})$ supported on $\mathbb{R}^n + i\overline{A}$.

(c) $\mathcal{E}_A(\mathcal{F})$, considered as a sheaf on $\mathbb{R}^n + i\overline{A}$, is the pullback, with respect to the projection to $\mathbb{R}^n$, of a sheaf on $\mathbb{R}^n$, constructible with respect to the stratification $\mathcal{C}$.

Proof: For $A = \{0\}$, this is Prop. 4.9(a) of [9]. For an arbitrary A this follows by further applying Prop. 3.10 and Cor. 3.22 from [9]. □

Further, the standard coboundary maps on the sheaves of cohomology with support give the *Cousin complex*

$$\mathcal{E}^\bullet(\mathcal{F}) = \left\{ \bigoplus_{\mathrm{codim}(A)=0} \mathcal{E}_A(\mathcal{F}) \xrightarrow{\delta} \bigoplus_{\mathrm{codim}(A)=1} \mathcal{E}_A(\mathcal{F}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0(\mathcal{F}) \right\} \tag{4.2}$$

which is a complex of sheaves on $\mathbb{C}^n$ canonically isomorphic to $\mathcal{F}$ in $D^b\,\mathrm{Sh}_{\mathbb{C}^n}$, see [9], Cor. 4.11. The grading in $\mathcal{E}^\bullet(\mathcal{F})$ is by $\mathrm{codim}(A)$.

Further, the matrix elements of δ which are morphisms of sheaves

$$\delta_{A_2,A_1} : \mathcal{E}_{A_2}(\mathcal{F}) \longrightarrow \mathcal{E}_{A_1}(\mathcal{F}), \quad \mathrm{codim}(A_2) = \mathrm{codim}(A_1) - 1,$$

are nonzero only if $A_1 <_1 A_2$. The condition $\delta^2 = 0$ means that the δ_{A_2,A_1} anticommute with each other. That is, for any faces $A_1, A_2 \neq A_2', A_3$ such that $A_1 \leqslant_1 A_2, A_2' \leqslant_1 A_3$ (a commutative square in $\mathcal{C}$ as a category), we have

$$\delta_{A_2,A_1} \circ \delta_{A_3,A_2} = -\delta_{A_2',A_1} \circ \delta_{A_3,A_2'} : \mathcal{E}_{A_3}(\mathcal{F}) \longrightarrow \mathcal{E}_{A_1}(\mathcal{F}).$$

This anticommutativity can be converted to commutativity in a standard way by "introducing signs". More precisely, for any cell σ let

$$\mathrm{or}(\sigma) \;=\; H_c^{\dim(\sigma)}(\sigma, \mathbf{k})$$

be the 1-dimensional orientation vector space of σ. Note that $\mathrm{or}(\sigma)^{\otimes 2}$ is canonically identified with $\mathbf{k}$. In particular, every face A being a cell, we have the space $\mathrm{or}(A)$. For any $A_1 <_1 A_2$ we have a canonical isomorphism

$$\psi_{A_1,A_2} : \mathrm{or}(A_1) \longrightarrow \mathrm{or}(A_2)$$

which is the matrix element of the differential in the cellular cochain complex of $\overline{A}_2$ with coefficients in $\mathbf{k}$. For $A_1 \leqslant_1 A_2, A_2' \leqslant_1 A_3$ as above, the isomorphisms ψ anticommute. So we get the following:

Proposition 4.3. *(a) The morphisms*

$$\partial_{A_2,A_1} = \delta_{A_2,A_1} \otimes \psi^{-1}_{A_1,A_2} : \mathcal{E}_{A_2}(\mathcal{F}) \otimes_{\mathbf{k}} \mathrm{or}(A_2) \longrightarrow \mathcal{E}_{A_1}(\mathcal{F}) \otimes_{\mathbf{k}} \mathrm{or}(A_1)$$

satisfy the commutativity constraints. That is, for any $A_1 \leqslant_1 A_2, A_2' \leqslant_1 A_3$ as above,

$$\partial_{A_2,A_1} \circ \partial_{A_3,A_2} \;=\; \partial_{A_2',A_1} \circ \partial_{A_3,A_2'}.$$

(b) The maps ∂_{A_2,A_1}, $A_1 <_1 A_2$ extend to a representation of the poset $\mathcal{C}^{\mathrm{op}}$ in $\mathrm{Sh}_{\mathbb{C}^n}$ which takes A to $\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A)$. In other words, for any $A_1 <_p A_2$, $p \geqslant 1$, we have a morphism of sheaves

$$\partial_{A_2,A_1} : \mathcal{E}_{A_2}(\mathcal{F}) \otimes \mathrm{or}(A_2) \longrightarrow \mathcal{E}_{A_1}(\mathcal{F}) \otimes \mathrm{or}(A_1)$$

defined as

$$\partial_{A_2,A_1} \;=\; \partial_{A_1',A_1} \circ \partial_{A_2',A_1'} \circ \cdots \circ \partial_{A_2,A_{p-1}'},$$

for any chain $A_1 <_1 A_1' <_1 \cdots <_1 A_{p-1}' <_1 A_2$, the result being independent on the choice of such chain. These morphisms satisfy the transitivity condition for any $A_1 <_p A_2 <_q A_3$. $\square$

Let now $A, B \in \mathcal{C}$ be two faces. We associate to $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ the vector space

$$E_{A,B} = E_{A,B}(\mathcal{F}) \;:=\; (\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A))_{B+i0},$$

the stalk of $\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A)$ at the cell $B+i0 \subset \mathbb{R}^n + iA$. Because of Proposition 4.1(c), we have a canonical identification

$$E_{A,B}(\mathcal{F}) \;\simeq\; (\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A))_{B+iA'}, \quad A' \leqslant A \tag{4.4}$$

with the stalk at $B + iA'$ for any $A' \leqslant A$.

If we have faces A and $B_1 \leqslant B_2$, then we define

$$\partial'_{(A|B_1,B_2)} : E_{A,B_1}(\mathcal{F}) \longrightarrow E_{A,B_2}(\mathcal{F})$$

to be the generalization map of the cellular sheaf $\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A)$ from $B_1 + i0$ to $B_2 + i0$.

If we have faces $A_1 \leqslant A_2$ and B, then we define

$$\partial''_{(A_1,A_2|B)} : E_{A_2,B}(\mathcal{F}) \longrightarrow E_{A_1,B}(\mathcal{F})$$

to be the map of stalks at $B + i0$ induced by the morphism of sheaves ∂_{A_2,A_1}.

Proposition 4.5. *The system $E(\mathcal{F}) = (E_{A,B}(\mathcal{F}), \partial', \partial'')$ is a matrix diagram of type $\mathcal{H}$. We have therefore a functor*

$$\mathbb{E} : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \longrightarrow \mathcal{M}_{\mathcal{H}}, \quad \mathcal{F} \mapsto E(\mathcal{F}).$$

Proof: We first prove the conditions $(M2)$ of Definition 3.1 of a matrix diagram. The first condition in $(M2)$ follows from the fact that $\mathcal{E}_A(\mathcal{F}) \otimes \mathrm{or}(A)$ is a cellular sheaf amd so its generalization maps are transitive. The second condition in $(M2)$ follows from Proposition 4.3(b). Finally, the third condition in $(M2)$ follows from the fact that ∂_{A_2,A_1} is a morphism of cellular sheaves and so the maps it induces on the stalks, commute with the generalization maps.

Let us prove the condition $(M3')$ of Definition 3.1. By construction, $\mathcal{E}_A = \mathcal{E}_A(\mathcal{F})$ is locally constant on the intersection of each stratum of $\mathbb{R}^n + i\overline{A}$ (i.e., of each $\mathbb{R}^n + iA'$, $A' \leqslant A$ with each $\mathbb{C}$-stratum. So if $B_1 + iA \equiv B_2 + iA$ and $B_1 \leqslant B_2$, then the generalization map on the stalks

$$\gamma_{B_1+iA,B_2+iA} : (\mathcal{E}_A)_{B_1+iA} \longrightarrow (\mathcal{E}_A)_{B_2+iA}$$

is an isomorphism. But in virtue of (4.4), this map is identified with

$$\partial'_{(A|B_1,B_2)} : E_{A,B_1} \longrightarrow E_{A,B_2}$$

and so the latter map is an isomorphism, proving $(3')$.

The property $(M3'')$ for $E(\mathcal{F})$ will follow from $(M3')$ for $E(\mathcal{F}^\tau)$ if we prove the following fact.

Proposition 4.6. *The system $E(\mathcal{F}^\tau)$ is identified with the dual system to $E(\mathcal{F})$, that is, $E_{A,B}(\mathcal{F}^\tau)$ is identified with $(E_{B,A}(\mathcal{F}))^*$ so that the maps ∂' (resp. ∂'') for $E(\mathcal{F}^\tau)$ are identified with the duals to the ∂'' (resp. ∂') for $E(\mathcal{F})$.*

This will be done in the next section.

5 Verdier duality and the Cousin complex

In this section we rewrite the Cousin complex $\mathcal{E}^\bullet(\mathcal{F})$ in a way manifestly compatible with Verdier duality. We start by one more general remark on cellular sheaves. .

Let $(X, \mathcal{S})$ be a quasi-regular cellular space with cell embeddings denoted $j_\sigma : \sigma \hookrightarrow X$. Let $\mathcal{G}$ be a cellular sheaf on X given by the linear algebra data $(G_\sigma, \gamma_{\sigma,\sigma'})$ of Proposition 2.1. Then, these data give a complex in the derived category $D^b \operatorname{Sh}_X$:

$$\bigoplus_{\dim(\sigma)=0} j_{\sigma!}\underline{G_\sigma}_\sigma \longrightarrow \bigoplus_{\dim(\sigma)=1} j_{\sigma!}\underline{G_\sigma}_\sigma[1] \longrightarrow \cdots \tag{5.1}$$

whose total object is $\mathcal{G}$, see [9] (1.12). We say that (5.1) is a resolution of $\mathcal{G}$. Note that given just vector spaces G_σ, the datum of such a complex is equivalent to the datum of transitive $\gamma_{\sigma,\sigma'}$, i.e., of a cellular sheaf with these stalks.

We apply this to the sheaf $\mathcal{E}_A(\mathcal{F})$ on the cellular space formed by $\mathbb{C}^n$ with the stratification $\mathcal{S}^{(2)}$ into product cells $B + iA$. Given any such cell, we have a commutative diagram of embeddings

$$\begin{tikzcd} B + iA \arrow[r, "k^{BA}"] \arrow[d, "l^{BA}"'] \arrow[dr, "\varepsilon^{BA}"] & \mathbb{R}^n + iA \arrow[d, "\lambda_A"] \\ B + i\mathbb{R}^n \arrow[r, "\kappa_B"'] & \mathbb{C}^n. \end{tikzcd}$$

Proposition 5.2. *let V be a $\mathbf{k}$-vector space. Then we have a canonical identification*

$$\lambda_{A*}\, k_!^{BA}\, \underline{V}_{B+iA} \simeq \kappa_{B!}\, l_*^{BA}\, \underline{V}_{B+iA}.$$

Proof: The stalk of either of these sheaves at $x + iy \in \mathbb{C}^n$ is

$$\begin{cases} V, & \text{if } x \in B \text{ and } y \in \overline{A}, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

We will denote the sheaf in the proposition by $\varepsilon_{!*}^{BA}\, \underline{V}_{B+iA}$ and refer to it as a *1-cell sheaf of* $(!*)$*-type*. We similarly denote 1-cell sheaves of $(*!)$-type as

$$\varepsilon_{*!}^{BA}\, \underline{V}_{B+iA} \;=\; \lambda_{A!}\, k_*^{BA}\, \underline{V}_{B+iA} \;\simeq\; \kappa_{B*}\, l_!^{BA}\, \underline{V}_{B+iA}.$$

Proposition 5.3. *Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ and $A \in \mathcal{C}$. Then the sheaf $\mathcal{E}_A(\mathcal{F})$ has a resolution (in $D^b\,\mathrm{Sh}_{\mathbb{C}^n}$) of the form*

$$\bigoplus_{\dim(B)=0} \varepsilon_{!*}^{BA}\, \underline{E_{AB}}_{B+iA} \longrightarrow \bigoplus_{\dim(B)=1} \varepsilon_{!*}^{BA}\, \underline{E_{AB}}_{B+iA}[1] \longrightarrow \cdots$$

with the differentials given by the maps ∂'.

Proof: This is an instance of (5.1). It simply reflects the fact that $\mathcal{E}_A$ is the sheaf on $\mathbb{C}^n$ coming from the sheaf on $\mathbb{R}^n + i\overline{A}$ which is pulled back from the $\mathcal{C}$-constrructible (cellular) sheaf on $\mathbb{R}^n$ with stalks E_{AB} and generalization maps ∂'. $\square$

Corollary 5.4. *Any $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ has a canonical resolution (in $D^b\,\mathrm{Sh}_{\mathbb{C}^n}$) in the form of the double complex*

$$\begin{array}{ccc}
\displaystyle\bigoplus_{\substack{\mathrm{codim}(A)=0\\ \dim(B)=0}} \varepsilon_{!*}^{BA}\, \underline{E_{AB} \otimes \mathrm{or}(A)}_{B+iA} & \longrightarrow & \displaystyle\bigoplus_{\substack{\mathrm{codim}(A)=0\\ \dim(B)=1}} \varepsilon_{!*}^{BA}\, \underline{E_{AB} \otimes \mathrm{or}(A)}_{B+iA}[1] \longrightarrow \cdots \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{\substack{\mathrm{codim}(A)=1\\ \dim(B)=0}} \varepsilon_{!*}^{BA}\, \underline{E_{AB} \otimes \mathrm{or}(A)}_{B+iA} & \longrightarrow & \displaystyle\bigoplus_{\substack{\mathrm{codim}(A)=1\\ \dim(B)=1}} \varepsilon_{!*}^{BA}\, \underline{E_{AB} \otimes \mathrm{or}(A)}_{B+iA}[1] \longrightarrow \cdots \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

with the horizontal differentials given by the maps ∂' and the vertical differentials given by the ∂''.

Proof: This is just the Cousin complex written in terms of 1-cell sheaves (of (!∗)-type). □

We now prove Proposition 4.6. For this, we apply the shifted Verdier duality to the double complex in Corollary 5.4 and note the three standard facts:

- $\mathbb{D}$ interchanges $f_!$ and f_*.
- For a cell σ of real dimension d and a finite-dimensional $\mathbf{k}$-vector space V, we have $\mathbb{D}(\underline{V}_\sigma) = \underline{V^* \otimes \mathrm{or}(\sigma)}_\sigma[d]$.
- $\mathrm{or}(\sigma)^{\otimes 2} \simeq \mathbf{k}$ canonically.

We conclude that $\mathcal{F}^*$ has a resolution in $D^b\,\mathrm{Sh}_{\mathbb{C}^n}$ in the form of the double complex

(5.5)
$$\begin{array}{ccccc}
\bigoplus\limits_{\substack{\dim(A)=0\\ \mathrm{codim}(B)=0}} \varepsilon^{BA}_{*!}\, \underline{E^*_{AB}\otimes \mathrm{or}(B)}_{B+iA} & \longrightarrow & \bigoplus\limits_{\substack{\dim(A)=0\\ \mathrm{codim}(B)=1}} \varepsilon^{BA}_{*!}\, \underline{E^*_{AB}\otimes \mathrm{or}(B)}_{B+iA} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\bigoplus\limits_{\substack{\dim(A)=1\\ \mathrm{codim}(B)=0}} \varepsilon^{BA}_{*!}\, \underline{E^*_{AB}\otimes \mathrm{or}(B)}_{B+iA}[1] & \longrightarrow & \bigoplus\limits_{\substack{\mathrm{codim}(A)=1\\ \dim(B)=1}} \varepsilon^{BA}_{!*}\, \underline{E_{AB}\otimes \mathrm{or}(A)}_{B+iA}[1] & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}$$

with the horizontal differentials given by the duals to the ∂'' for $E(\mathcal{F})$ and the vertical differentials given by the duals of the ∂' for $E(\mathcal{F})$. It corresponds, therefore, to the dual system $E(\mathcal{F})^*$.

On the other hand, we can form the Cousin resolution of $\mathcal{F}^*$ but using the real, not imaginary tube cells $\kappa_B : B + i\mathbb{R}^n \hookrightarrow \mathbb{C}^n$. This gives the sheaves

$$\check{\mathcal{E}}_B(\mathcal{F}^*) \;=\; \kappa_{B*}\,\kappa_B^!\,\mathcal{F}^*[\mathrm{codim}(B)]$$

and the resolution

$$\check{\mathcal{E}}^\bullet(\mathcal{F}^*) \;=\; \left\{ \bigoplus_{\mathrm{codim}(B)=0} \check{\mathcal{E}}_B(\mathcal{F}^*) \longrightarrow \bigoplus_{\mathrm{codim}(B)=1} \check{\mathcal{E}}_B(\mathcal{F}^*) \longrightarrow \cdots \right\}$$

of $\mathcal{F}^*$. Writing out each $\mathcal{E}_B(\mathcal{F}^*)$ in terms of 1-cell sheaves of type $(*!)$, we get a double complex of the form (5.5) which is a resolution of $\mathcal{F}^*$. But the Cousin resolution of $\mathcal{F}^*$ with respect to the cells $B + i\mathbb{R}^n$ is the same as the Cousin resolution of $\tau^{-1}\mathcal{F}^*$ with respect to the the cells $\mathbb{R}^n + iA$. We conclude that the complex (5.5), associated to $E(\mathcal{F})^*$ must reduce, after applying τ, to the complex of Corollary 5.4 describing $E(\mathcal{F}^*)$. This means that the linear algebra data underlying the two complexes must be identified, i.e., $E(\mathcal{F})^* \simeq E(\mathcal{F}^\tau)$.

This finishes the proof of Propositions 4.6 and 4.5.

6 From a matrix diagram to a perverse sheaf

We now construct a functor

$$\mathbb{G} : \mathcal{M}_{\mathcal{H}} \longrightarrow \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$$

by reversing the procedure used to extract the matrix diagram $E(\mathcal{F})$ from the Cousin complex $\mathcal{E}^\bullet(\mathcal{F})$.

Let $E = (E_{A,B}, \partial', \partial'') \in \mathcal{M}_{\mathcal{H}}$ be given. For each face $A \in \mathcal{C}$ we form the cellular shef $\mathcal{E}_A = \mathcal{E}_A(E)$ on $(\mathbb{C}^n, \mathcal{S}^{(2)})$ which is supported on $\mathbb{R}^n + i\overline{A}$ and pulled there from the cellular sheaf on $(\mathbb{R}^n, \mathcal{C})$ with stalks $E_{A,B} \otimes \mathrm{or}(A)$ and generalization maps $\partial' \otimes \mathrm{Id}$. In other words, $\mathcal{E}_A$ is constant on each $B + i\overline{A}$ and

$$\mathcal{E}_A|_{B+i\overline{A}} = \underline{E_{A,B} \otimes \mathrm{or}(A)}_{B+i\overline{A}} \tag{6.1}$$

Further, the commuting maps ∂'' in E give, after tensoring with the $\mathrm{or}(A)$, anticommuting morphisms of sheaves

$$\delta_{A_2,A_1} : \mathcal{E}_{A_2}(E) \longrightarrow \mathcal{E}_{A_1}(E), \quad A_1 <_1 A_2,$$

and so we can form the complex of sheaves

$$\mathcal{E}^\bullet(E) = \left\{ \bigoplus_{\mathrm{codim}(A)=0} \mathcal{E}_A(E) \xrightarrow{\delta} \bigoplus_{\mathrm{codim}(A)=1} \mathcal{E}_A(E) \xrightarrow{\delta} \cdots \right\}.$$

Proposition 6.2. *The complex $\mathcal{E}^\bullet(E)$ is $\mathcal{S}^{(0)}$-constructible.*

Proof: By definition, $\mathcal{E}^\bullet = \mathcal{E}^\bullet(E)$ is $\mathcal{S}^{(2)}$-constructible. By Proposition 2.3, it suffices to prove that it is both $\mathcal{S}^{1)}$-constructible and $\tau\mathcal{S}^{(1)}$-constructible.

Let us first prove that $\mathcal{E}^\bullet$ is $\mathcal{S}^{1)}$-constructible. By Proposition 1.4 this is equivalent to the following condition:

(Q) If $C_1 \leqslant C_2$ and D are such that $C_1 + iD \equiv C_2 + iD$, then the generalization map
$$\gamma_{C_1+iD, C_2+iD} : \mathcal{E}^\bullet\|_{C_1+iD} \longrightarrow \mathcal{E}^\bullet\|_{C_2+iD}$$
is a quasi-isomorphism of complexes.

We claim that γ_{C_1+iD,C_2+iD} is in fact an isomorphism, not just a quasi-isomorphism of complexes. More precisely, we claim that for any summand $\mathcal{E}_A$ of $\mathcal{E}^\bullet$, the corresponding generalization map

$$\gamma^{\mathcal{E}_A}_{C_1+iD,C_2+iD} : \mathcal{E}_A\|_{C_1+iD} \longrightarrow \mathcal{E}_A\|_{C_2+iD}$$

is an isomorphism of vector spaces. To see this, note that by construction, see (6.1), we have for any C, D:

$$\mathcal{E}_A\|_{C+iD} = \begin{cases} E_{A,C} \otimes \mathrm{or}(A), & \text{if } D \leqslant A, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

So

$$\gamma^{\mathcal{E}_A}_{C_1+iD,C_2+iD} = \begin{cases} \partial' \otimes \mathrm{Id} : E_{A,C_1} \otimes \mathrm{or}(A) \to E_{A,C_2} \otimes \mathrm{or}(A), & \text{if } D \leqslant D, \\ 0 : 0 \longrightarrow 0, & \text{otherwise.} \end{cases}$$

So the fact that it is an isomorphism, follows from condition (3') of Definition 3.1 of matrix diagram and the next lemma.

Lemma 6.4. *Let C_1, C_2 and D be such that $C_1 + iD \equiv C_2 + iD$ lie in the same complex stratum. Let $A \geqslant D$. Then $C_1 + iA \equiv C_2 + iA$.*

Proof: By Proposition 1.3 we have

$$\mathcal{H}^{C_1} \cap \mathcal{H}^D = \mathcal{H}^{C_2} \cap \mathcal{H}^D. \tag{6.5}$$

If $A \geqslant D$, then $\mathcal{H}^A \subset \mathcal{H}^D$, so intersecting (6.5) with $\mathcal{H}^A$, we get $\mathcal{H}^{C_1} \cap \mathcal{H}^A = \mathcal{H}^{C_2} \cap \mathcal{H}^A$, i.e., that $C_1 + iA \equiv C_2 + iA$. This proves Lemma 6.4 and the condition (Q).

We now prove that $\mathcal{E}^\bullet(E)$ is $\tau\mathcal{S}^{(1)}$-constructible. For this it suffices to prove that the shifted Verdier dual $\mathcal{E}^\bullet(E)^*$ is $\tau\mathcal{S}^{(1)}$-constructible. But writing $\mathcal{E}^\bullet(E)$ as the total object of the double complex as in Corollary 5.4, and applying the duality term by term, we find that $\mathcal{E}^\bullet(E)^*$ is the total object of a double complex as in (5.5) which is the same as the complex of sheaves $\check{\mathcal{E}}^\bullet(E^*)$ corresponding to the dual matrix diagram E^* and the system of tube cells $B + i\mathbb{R}^n$ obtained from the system of the $\mathbb{R}^n + iA$ by applying τ. So

$$\mathcal{E}^\bullet(E)^* \simeq \check{\mathcal{E}}^\bullet(E^*) = \tau^{-1}\mathcal{E}^\bullet(E^*) \tag{6.6}$$

is $\tau\mathcal{S}^{(1)}$-constructible because $\mathcal{E}^\bullet(E^*)$ is $\mathcal{S}^{(1)}$-constructible by the above. This finishes the proof of Proposition 6.2.

Proposition 6.7. *The complex $\mathcal{E}^\bullet(E)$ is perverse. We have therefore a functor*

$$\mathbb{G} : \mathcal{M}_{\mathcal{H}} \longrightarrow \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}), \quad E \mapsto \mathcal{E}^\bullet(E).$$

Proof: We first prove the condition (P^-) of perversity: that $\underline{H}^p(\mathcal{E}^\bullet(E))$ is supported on a complex submanifold of complex codimension $\geqslant p$. By construction, the pth term $\mathcal{E}^p(E) = \bigoplus_{\mathrm{codim}(A)=p} \mathcal{E}_A(E)$ is supported on the union of the $\mathbb{R}^n + \overline{A}$ where A runs over faces of $\mathcal{H}$ of real codimension p. So $\mathcal{E}^p(E)$ and therefore $\underline{H}^p(\mathcal{E}^\bullet(E))$ is supported on the union of $\mathbb{R}^n + iL$ where L runs over flats of $\mathcal{H}$ of real codimension p. But since, by Proposition 6.2, $\mathcal{E}^\bullet(E)$ is $\mathcal{S}^{(0)}$-constructible, Supp $\underline{H}^p(\mathcal{E}^\bullet(A))$ is a complex manifold, in fact, a finite-union of $\mathbb{C}$-linear subspaces $M \subset \mathbb{C}^n$. But if such an M lies in $\mathbb{R}^n + iL$, it must lie in $L + iL = L_{\mathbb{C}}$ which has complex codimension p. This proves (P^-) for $\mathcal{E}^\bullet(E)$.

Now, (P^+) is equivalent to (P^-) for $\mathcal{E}^\bullet(E)^*$. By (6.6), we have $\mathcal{E}^\bullet(E)^* = \tau^{-1}\mathcal{E}(E^*)$, and (P^-) for $\mathcal{E}(E^*)$ has just been proved. □

Proposition 6.8. *The functors $\mathbb{E}$ and $\mathbb{G}$ are quasi-inverse to each other and so are equvalences of categories.*

Proof: That $\mathbb{G} \circ \mathbb{E} \simeq \mathrm{Id}$ is clear: the Cousin complex of $\mathcal{F}$ is a representatve of $\mathcal{F}$. Conversely, suppose we start from a matrix diagram $E = (E_{A,B}, \partial', \partial'')$ and form the complex $\mathcal{E}^\bullet = \mathcal{E}^\bullet(E)$ whose data, as a complex of cellular sheaves, is completely equivalent to the bicomplex as in Corollary 5.4, i.e., yo E. We need to prove that the "intrinsic Cousin complex" associated to $\mathcal{E}^\bullet$, is $\mathcal{E}^\bullet$ itself. This argument is elementary and similar to [9], §6.

More precisely, for a face D let $\lambda_D : \mathbb{R}^n + iD \hookrightarrow \mathbb{C}^n$ be, as before, the embedding. It is enough to prove that for any k-vector space V (we will need $V = E_{AB}$)

$$\lambda_{D*}\,\lambda_D^!\big(\varepsilon_{!*}^{BA}\,\underline{V}_{B+iA}\big) \;=\; \begin{cases} 0, & \text{if } D \neq A, \\ \varepsilon_{!*}^{BD}\,\underline{V}_{B+iD}, & \text{if } D = A. \end{cases}$$

This reduces to the case $V = \mathbf{k}$ which is a Cartesian product situation. So we reduce to a statement about the second factor only, that is, denoting by $j_C : C \to \mathbb{R}^n$ the embedding of a face $C \in \mathcal{C}$, that

$$j_D^!\, j_{A*}\,\underline{\mathbf{k}}_A \;=\; 0, \quad \text{if } D \neq A.$$

(of course, the LHS is equal to $\underline{\mathbf{k}}_D$, of $D = A$). But this claim is clear: by Verdier duality, it is equivalent to

$$j_D^*\, j_{A!}\,\underline{\mathbf{k}}_A = 0, \quad \text{if } D \neq A,$$

which is completely obvious, as $j_{A!}\,\underline{\mathbf{k}}_A$ is just the extension of the constant sheaf by 0 from A to $\overline{A}$ and then to $\mathbb{R}^n$. $\square$

This finishes the proof of Theorem 3.4.

7 Examples and complements

A. The 1-dimensional case. Let $n = 1$ and let $\mathcal{H}$ consist of the "hyperplane" $0 \in \mathbb{R}$. The corresponding category $\mathrm{Perv}(\mathbb{C}, 0)$ consists of perverse sheaves on $\mathbb{C}$ with the possible singularity at 0.

The poset $\mathcal{C}$ of faces has 3 elements: $\mathbb{R}_-$, $\{0\}$ and $\mathbb{R}_+$, so a matrix diagram has the form (0.1) or, with the notations for the arrows spelled out,

(7.1)
$$\begin{tikzcd}
E_{-,+} \arrow[r, "\simeq", "\partial''"'] & E_{0,+} & E_{+,+} \arrow[l, "\simeq"', "\partial''"] \\
E_{-,0} \arrow[u, "\partial'"', "\simeq"] \arrow[r, "\partial''"] \arrow[d, "\partial'", "\simeq"'] & E_{0,0} \arrow[u, "\partial'"] \arrow[d, "\partial'"'] & E_{+,0} \arrow[l, "\partial''"'] \arrow[u, "\partial'", "\simeq"'] \arrow[d, "\partial'"', "\simeq"] \\
E_{-,-} \arrow[r, "\partial''", "\simeq"'] & E_{0,-} & E_{+,-} \arrow[l, "\partial''"', "\simeq"]
\end{tikzcd}$$

Theorem 3.4 says therefore that $\mathrm{Perv}(\mathbb{C}, 0)$ is equivalent to the category of diagrams (7.1). Let us compare this with other known descriptioms. The most classical description [1, 7] is in terms of diagrams of vector spaces

$$\Phi \underset{b}{\overset{a}{\rightleftarrows}} \Psi\ , \qquad \mathrm{Id}_\Psi - ab \text{ is invertible.} \tag{7.2}$$

The approach of [9] gives rise to another description, in terms of diagrams of vector spaces

$$\begin{gathered} E_- \underset{\delta_-}{\overset{\gamma_-}{\leftrightarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \\ \gamma_-\delta_- = \mathrm{Id}_{E_-},\ \ \gamma_+\delta_+ = \mathrm{Id}_{E_+}, \\ \gamma_-\delta_+ : E_+ \to E_-,\ \ \gamma_+\delta_- : E_- \to E_+ \quad \text{are invertible.} \end{gathered} \tag{7.3}$$

See [9] §9A for a direct constriction of an equivalence between the categories of diagrams (7.2) and (7.3). Let us explain an equivalence between the categories of diagrams (7.1) and (7.3). Given a diagram as in (7.1), we consider first its middle horizontal part (the 0th row) which gives the straight arrows δ_+, δ_- in

$$\begin{tikzcd} E_{-,0} \arrow[r, "\delta_-=\partial''"'] & E_{0,0} \arrow[l, bend right=60, "\gamma_-"'] \arrow[r, bend right=60, "\gamma_+"'] & E_{+,0} \arrow[l, "\delta_+=\partial''"'] \end{tikzcd} \tag{7.4}$$

The curved arrows $\gamma_\pm$ are defined as the compositions along the corresponding squares in (7.1). That is, γ_- is the composition

$$E_{0,0} \xrightarrow{\partial'} E_{0,+} \xrightarrow{(\partial'')^{-1}} E_{-,+} \xrightarrow{(\partial')^{-1}} E_{-,0},$$

while γ_+ is the composition

$$E_{0,0} \xrightarrow{\partial'} E_{0,-} \xrightarrow{(\partial'')^{-1}} E_{+,-} \xrightarrow{(\partial')^{-1}} E_{+,0}.$$

The commutativity of (7.1) and invertibility of the arrows at its outer rim implies easily that the diagram (7.4) is of the type (7.3). Further, this procedure gives an equvialence between the categories of diagrams (7.1) and (7.3). We leave the verifications to the reader.

B. Comparison with [9]. For a general arrangement $\mathcal{H}$ of linear hyperlplanes in $\mathbb{R}^n$ we gave, in [9], a description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ in terms of "single-indexed" diagrams

$$Q = \big((E_A)_{A\in\mathcal{C}}, (\gamma_{A_1,A_2}, \delta_{A_2,A_1})_{A_1\leqslant A_2}\big),$$

where:

- E_A are finite-dimensional $\mathbf{k}$-vector spaces given for any $A \in \mathcal{C}$.
- $\gamma_{A_1,A_2} : E_{A_1} \to E_{A_2}$, resp. $\delta_{A_2,A_1} : E_{A_2} \to E_{A_1}$ are linear maps forming a representation, resp. an anti-representation of $\mathcal{C}$ on (E_A) and satisfying the axioms of monotonicity ($\gamma_{A_1,A_2}\delta_{A_2,A_1} = \mathrm{Id}$), transitivity and invertibility, see [9].

Let us compare this with the "matrix" description given by Theorem 3.4. By the construction of [9], the partial data (E_A, γ_{A_1,A_2}) are just the linear algebra data describing the cellular sheaf $i_{\mathbb{R}}^! \mathcal{F}[n]$ on $\mathbb{R}^n$, where $i_{\mathbb{R}} : \mathbb{R}^n \hookrightarrow \mathbb{C}^n$ is the embedding. Therefore

$$E_A = E_{0,A}, \;\; \gamma_{A_1,A_2} = \partial''$$

is just one column of the matrix diagram $(E_{A,B}, \partial', \partial'')$ corresponding to $\mathcal{F}$.

Further, any $E_{A,B}$ is, by our construction, the stalk at $B + iA$ of the sheaf $j_{A*}j_A^! \otimes \mathrm{or}(A)[\mathrm{codim}(A)]$, where $j_A : \mathbb{R}^n + iA \hookrightarrow \mathbb{C}^n$ is the embedding. There stalks were described in [9], Cor. 3.22, and we get

$$E_{A,B} = E_{B\circ A}, \tag{7.5}$$

where $B \circ A$ is "the first cell in the direction A visible from B. More precisely, (see [9] Prop. 2.3) $B \circ A$ is the (uniquely defined) cell containing the points

$$(1-\varepsilon)b + \varepsilon a, \quad b \in B, \; a \in A, \; 0 < \varepsilon \ll 1. \tag{7.6}$$

In particular, $E_{A,0}$ is also identified with E_A, and the maps ∂' connecting different $E_{A,0}$, are precisely the δ_{A_2,A_1}, as both appear from the differential in the Cousin complex. So the partial data (E_A, δ_{A_2,A_1}) is just the 0th row of $(E_{A,B})$.

The operation $\circ$ was introduced by Tits [12] in 1974 in the context of buildings (which includes arrangements of root hyperplanes) and later, independently, by Björner, Las Vergnas, Sturmfels, White and Ziegler [3] for oriented matroids (which includes all hyperplane arrangements). For simplicity, we will refer to $\circ$ as the *Tits product*. Let us list some of its properties.

Proposition 7.7. *(a) The Tits product $\circ$ is associative (but not commutative).*

(b) Further, $\circ$ is monotone in the second argument (but not in the first one). That is, if $A_1 \leqslant A_2$, then $B \circ A_1 \leqslant B \circ A_2$ for any B.

(c) In the notation of (1.2) *we have*

$$\mathcal{H}^{B \circ A} = \mathcal{H}^B \cap \mathcal{H}^A.$$

In particular (by Proposition 1.3), $B \circ A$ and $A \circ B$ lie in the same complex stratum.

(d) Let $B, A_1, A_2 \in \mathcal{C}$ be such that $A_1 \leqslant A_2$. Then the following are equivalent:

(di) $B + iA_1$ and $B + iA_2$ lie in the same complex stratum.

(dii) $B \circ A_1 = B \circ A_2$.

Proof: Parts (a) and (b) are well known, see, e.g., [9] Props. 2.3(a) and 2.7(a). Part (c) follows at once from (7.6). Let us prove (d). Suppose (di) holds. Then, by Proposition 1.3,

$$\mathcal{H}^B \cap \mathcal{H}^{A_1} = \mathcal{H}^B \cap \mathcal{H}^{A_2}. \tag{7.8}$$

Since $A_1 \leqslant A_2$, we have $B \circ A_1 \leqslant B \circ A_2$ by (b). But (7.8) means, in virtue of (c), that $\mathcal{H}^{B \circ A_1} = \mathcal{H}^{B \circ A_2}$, and this implies that $\dim(B \circ A_1) = \dim(B \circ A_2)$. Therefore we must have $B \circ A_1 = B \circ A_2$, that is, (dii) holds.

Conversely, suppose (dii) holds. Then, by (c), we have (7.8) so $B + iA_1$ and $B + iA_2$ lie in the same complex stratum by Proposition 1.3, that is, (di) holds. □

Remarks 7.9. (a) Part (c) of Proposition 7.7 can be compared, via (7.5), with Remark 3.3. That is, even though $B \circ A \neq A \circ B$ in general, $E_{B \circ A}$ is isomorphic to $E_{A \circ B}$.

(b) Part (d) of Proposition 7.7 means that the conditions in the axioms $(M3')$ and $(M3'')$ of a matrix diagram can be formulated in terms of the Tits product. More precisely, the conditions that $B + iA_1 \equiv B + iA_2$ (i.e., $B + iA_1$ and $B + iA_2$ lie in the same complex stratum) in in $(M3'')$ directly means that $B \circ A_1 = B \circ A_2$. The condition that $B_1 + iA \equiv B_2 + iA$ in $(M3')$ is equivalent to $A + iB_1 \equiv A + iB_2$, i.e., to $A \circ B_1 = A \circ B_2$.

To summarize, the matrix diagram $(E_{A,B})$ contains the same vector spaces as the single-indexed one (E_A) but with repetitions, being a kind of "Hankel matrix" with respect to the Tits product $\circ$. It is these repetitions that allow us to write the relations among the arrows ∂', ∂'' of a matrix diagram in such a simple, local form: as commutativity of elementary squares.

C. Affine arrangements. Let now $\mathcal{H}$ be a, possibly infinite, arrangement of *affine hyperplanes* in $\mathbb{R}^n$. For any affine hyperplane $H \in \mathcal{H}$ with real affine equation $f_H(x) = a$, where $f_H : \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{R}$-linear nd $a \in \mathbb{R}$, let $\overline{H} \subset \mathbb{R}^n$ be the linear hyperplane with the equation $f_H(x) = 0$. We denote $\overline{\mathcal{H}}$ the linear arrangement of the hyperplanes $\overline{H}$, $H \in \mathcal{H}$ (ignoring possible repetitions) and assume that:

- $\mathcal{H}$ is closed (as a subset in $\mathbb{R}^n$) and locally finite, i.e., any $x \in \mathbb{R}^n$ has a neighborhood meeting only finitely many affine hyperplanes from $\mathcal{H}$.
- $\overline{\mathcal{H}}$ is finite.

The concepts of flats of $\mathcal{H}$, their complexifications and the stratification $\mathcal{S}^{(0)}$ of $\mathbb{C}^n$ into generic parts of complexified flats are defined analogously to the case of linear arrangements. We then have the category $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ of perverse sheaves on $\mathbb{C}^n$ smooth with respect to $\mathcal{S}^{(0)}$. Let us give a modification of Theorem 3.4 to the case of affine arrangements as above.

We denote $\mathcal{S}^{(2)}$ the quasi-regular cell decompostion of $\mathbb{C}^n$ into product cells of the form $iA + B$ with $A \in \overline{\mathcal{C}}$ and $B \in \mathcal{C}$.

Proposition 7.10. *The decomposition $\mathcal{S}^{(2)}$ refines $\mathcal{S}^{(0)}$.*

Proof: This is a consequence of the following obvious remark. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-linear function and $a \in \mathbb{R}$. Denote by $f^{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}$ the complexification of f. Then, for $x, y \in \mathbb{R}^n$ the condition $f^{\mathbb{C}}(x + iy) = a$ is equivalent to $f(x) = a$ amd $f(y) = 0$. $\square$

Theorem 7.11. *The category* $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ *is equivalent to the category of diagrams of finite-dimensional* $\mathbf{k}$*-vector spaces of the form*

$$\begin{gathered} E_{A,B} \in \mathrm{Vect}_{\mathbf{k}},\ A \in \overline{\mathcal{C}},\ B \in \mathcal{C}, \\ \partial' :_{(A|B_1,B_2)}: E_{A,B_1} \longrightarrow E_{A,B_2},\ B_1 \leqslant B_2, \\ \partial''_{(A_1,A_2|B)} : E_{A_1,B} \longrightarrow E_{A_2,B},\ A_1 \leqslant A_2, \end{gathered}$$

such that:

(A2) The maps ∂', ∂'' define a representation of $\overline{\mathcal{C}}^{\mathrm{op}} \times \overline{\mathcal{C}}$ in $\mathrm{Vect}_{\mathbf{k}}$.

(A3) If $iA + B_1$ and $iA + B_2$ lie in the same stratum of $\mathcal{S}^{(0)}$, then $\partial'_{(A|B_1,B_2)}$ is an isomorphism. If $iA_1 + B$ and $iA_2 + B$ lie in the same stratum of $\mathcal{S}^{(0)}$, then $\partial''_{(A_1,A_2|B)}$ is an isomorphism.

Proof: It can be obtained, as in [9] §9B, by an amalgamation argument from the linear case, using the fact that perverse sheaves form a stack of categories. Alternatively, one can perform a direct analysis of the Cousin complex associated to $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ and formed by the sheaves

$$j_{A*} j_A^! \mathcal{F}[\mathrm{codim}(A)], \ A \in \overline{\mathcal{C}}, \quad j_A : \mathbb{R}^n + iA \hookrightarrow \mathbb{C}^n.$$

□

Example 7.12. Consider the arrangement of two points $0, 1$ in $\mathbb{R}$. Then

$$\mathcal{C} \ = \ \{\mathbb{R}_{<0}, 0, (0,1), 1, \mathbb{R}_{>1}\}, \quad \overline{\mathcal{C}} \ = \ \{\mathbb{R}_-, 0, \mathbb{R}_+\},$$

The decomposition $\mathcal{S}^{(2)}$ of $\mathbb{C}$ into the product cells is depicted in Fig. 1.

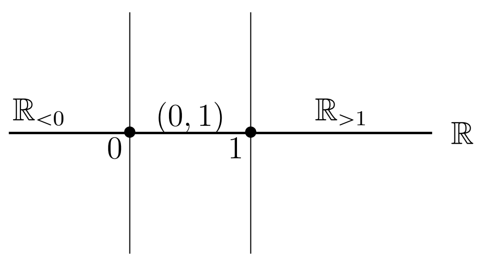

Figure 1: Product cells for the arrangement $\mathcal{H} = \{0, 1\} \subset \mathbb{R}$.

Theorem 7.11 identifies $\mathrm{Perv}(\mathbb{C}, \mathcal{H})$ with the category of commutative diagrams of the form

$$\begin{tikzcd}
E_{<0,+} \arrow[r, "\simeq"] & E_{0,+} & E_{(0,1),+} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & E_{1,+} & E_{>1,+} \arrow[l, "\simeq"'] \\
E_{<0,0} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r] & E_{0,0} \arrow[u] \arrow[d] & E_{(0,1),0} \arrow[l] \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & E_{1,0} \arrow[u] \arrow[d] & E_{>1,+} \arrow[l] \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] \\
E_{<0,-} \arrow[r, "\simeq"'] & E_{0,-} & E_{(0,1),-} \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & E_{1,-} & E_{>1,-} \arrow[l, "\simeq"]
\end{tikzcd}$$

Such a diagram can be seen as an amalgamation of two diagrams of the form (7.1).

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