

ON MIURA MAPS FOR \mathcal{W} -SUPERALGEBRAS

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ABSTRACT. We prove the injectivity of the Miura maps for \mathcal{W} -superalgebras and the isomorphisms between the Poisson vertex superalgebras obtained as the associated graded of the \mathcal{W} -superalgebras in terms of the Li's filtration and the level 0 Poisson vertex superalgebras associated with the arc spaces of the corresponding Slodowy slices in full generality.

INTRODUCTION

Let \mathfrak{g} be a basic classical (simple) Lie superalgebra over the field of complex numbers \mathbb{C} , $k \in \mathbb{C}$ a complex number, a non-zero even nilpotent element f of \mathfrak{g} , and $\Gamma: \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ a good grading for f . Then a vertex superalgebra $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$, called the (affine) \mathcal{W} -superalgebra, is defined as the quantum Drinfeld-Sokolov reduction of the universal affine vertex superalgebra $V^k(\mathfrak{g})$ of \mathfrak{g} at level k [KRW]. The principal \mathcal{W} -algebras $\mathcal{W}^k(\mathfrak{g})$ associated with simply-laced simple Lie algebras and principal nilpotent elements are known to have the *triality* (T):

- (T1) (Feigin-Frenkel duality) $\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^{L_k(L\mathfrak{g})}$,
- (T2) (GKO construction) $\mathcal{W}^k(\mathfrak{g}) \simeq \text{Com}\left(V^{k'+1}(\mathfrak{g}), V^{k'}(\mathfrak{g}) \otimes L_1(\mathfrak{g})\right)$.

Here $L\mathfrak{g}$ is the Langlands dual Lie algebra of \mathfrak{g} , $L_1(\mathfrak{g})$ is the simple quotient of $V^1(\mathfrak{g})$, $\text{Com}(W, V)$ denotes the coset vertex superalgebra of V by W , and the levels $(k, {}^Lk, k')$ satisfy certain relations, see [FF, ACL] for details.

Recently, Gaiotto-Rapčák [GR] and Procházka-Rapčák [PR] proposed a generalization of the triality for \mathcal{W} -superalgebras and their cosets by affine vertex subalgebras in terms of 4 dimensional $\mathcal{N} = 4$ topologically twisted super Yang-Mills theories. Since the Kazama-Suzuki coset construction of the $\mathcal{N} = 2$ superconformal algebra appears as a very special case, these isomorphisms are expected to be efficient for the study of the representation theory of vertex superalgebras, see [CGN, CL] for a mathematical approach.

In the proofs of the triality (T), a vertex superalgebra homomorphism, called the *Miura map* [A1, FBZ, G1]

$$\mu: \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \rightarrow V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \quad (0.1)$$

plays a fundamental role and its injectivity is very important. (See Section 2 for details.) The situation is expected to be the same for the generalized triality in the sense of [GR, PR]. The injectivity of μ is proved in [F, A2] when \mathfrak{g} is non-super and f is principal, whose proof also applies for an arbitrary f and is proved in [G1] when \mathfrak{g} and f are arbitrary but the level k is generic. Unfortunately, it is not enough for the application to the representation theory of vertex superalgebras based on the triality since the representation theory is rich only for *special levels* out of generic levels. The main theme of the paper is to improve the situation and, especially, to prove the injectivity in full generality.

In Section 1, we extend some results on unipotent algebraic groups to the super settings for the application to the study of \mathcal{W} -superalgebras. We explain the correspondence between formal supergroups and finite dimensional Lie superalgebras

[Ma1, Se] by using the Campbell-Hausdorff formal supergroups. This correspondence gives the one between unipotent algebraic supergroups and finite dimensional nilpotent Lie superalgebras [MO]. This implies that every unipotent algebraic supergroup is an affine superspace (i.e., $\mathbb{C}^{p|q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$) as is well-known in the non-super cases, cf [Mil]. Let G be an unipotent algebraic supergroup with Lie superalgebra \mathfrak{g} . Then we have a left \mathfrak{g} -module structure on the coordinate ring $\mathbb{C}[G]$. We prove that under a certain condition, the Lie superalgebra cohomology $H^n(\mathfrak{g}; \mathbb{C}[G])$ is isomorphic to $\delta_{n,0}\mathbb{C}$. The proof is very standard: we use a filtration on the complex, which gives a spectral sequence $E_1 \simeq H_{\text{dR}}^\bullet(G) \Rightarrow H^\bullet(\mathfrak{g}; \mathbb{C}[G])$ and then use Poincaré lemma $H_{\text{dR}}^n(G) \simeq \delta_{n,0}$ since G is an affine superspace.

In Section 2, after we review the definition of \mathcal{W} -superalgebras and the construction of the Miura maps (0.1), we extend some results of principal \mathcal{W} -algebras to the super cases, following [A1]. We first consider the Li's filtrations [L] on $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ and $V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$, respectively. The associated graded vector superspaces admit a natural structure of Poisson vertex superalgebras (PVAs) and μ induces a PVA homomorphism

$$\bar{\mu}: \text{gr}_F \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \rightarrow \text{gr}_F \left(V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \right).$$

The injectivity of μ reduces to that of $\bar{\mu}$ (Lemma 2.2). Let $\{e, h, f\} \subset \mathfrak{g}$ be an \mathfrak{sl}_2 -triple in the even subsalgebra of \mathfrak{g} containing the nilpotent element f . We write $\mathfrak{g}_{\geq -1/2} = \bigoplus_{j \geq -1/2} \mathfrak{g}_j$, $\mathfrak{g}_+ = \bigoplus_{j > 0} \mathfrak{g}_j$, and denote by G_+ the unipotent algebraic supergroup whose Lie superalgebra is \mathfrak{g}_+ . Then the affine subspace $f + \mathfrak{g}_{\geq -1/2}$ is stable under the Adjoint action of G_+ and, moreover, admits a G_+ -equivalent isomorphism of affine supervarieties

$$\mathcal{S}_f \times G_+ \simeq f + \mathfrak{g}_{\geq -1/2}, \quad (X, g) \mapsto g^{-1}Xg,$$

where $\mathcal{S}_f = f + \mathfrak{g}^e \subset \mathfrak{g}$, called the *Slodowy slice* of f (Proposition 2.5). This is a generalization to the super setting of [A2, eq.(4)] and [GG, Lemma 2.1]. We have a Poisson structure on $\mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]$ (see Section 2.4), which restricts to $\mathbb{C}[\mathcal{S}_f] \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]^{G_+}$. If Γ is a \mathbb{Z} -grading, then the Poisson supervariety \mathcal{S}_f is also obtained as the Hamiltonian reduction of \mathfrak{g} equipped with the Kostant-Kirillov Poisson structure with respect to G_+ and character $\chi = (f| \cdot)$. It turns out that we have an isomorphism $\text{gr}_F \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \simeq \mathbb{C}[J\mathcal{S}_f]$ as PVAs (Proposition 2.8) where $J\mathcal{S}_f$ is the arc space of \mathcal{S}_f (see Section 1.6) and $\mathbb{C}[J\mathcal{S}_f]$ is inherited with the level 0 PVA structure induced by the Poisson structure on $\mathbb{C}[\mathcal{S}_f]$. On the other hand, we identify $\text{gr}_F \left(V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \right) \simeq \mathbb{C}[J(f + (\mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0))]$ and thus

$$\bar{\mu}: \mathbb{C}[J\mathcal{S}_f] \rightarrow \mathbb{C}[J(f + (\mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0))].$$

This map is induced by its finite dimensional analogue

$$\bar{\mu}_{\text{fin}}: \mathbb{C}[\mathcal{S}_f] \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]^{G_+} \rightarrow \mathbb{C}[f + (\mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0)],$$

which is just the restriction map. We prove the injectivity of $\bar{\mu}_{\text{fin}}$ (Proposition 2.9), which immediately implies the injectivity of $\bar{\mu}$ and thus of the original Miura map μ (Theorem 2.1).

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1. UNIPOTENT ALGEBRAIC SUPERGROUP

1.1. Formal supergroups. We review the contravariant equivalence between formal supergroups and Lie superalgebras, following [Se, Ma1]. The results in Section 1.1-1.4 are proved in much more generality in [Ma1]. Given a commutative \mathbb{C} -superalgebra A , we denote by $A = A_{\bar{0}} \oplus A_{\bar{1}}$ the parity decomposition and by \bar{a} the parity of $a \in A$. In particular, given a basis $\{x_\alpha\}_\alpha$ of A , the parity \bar{x}_α is also denoted by $\bar{\alpha}$. The multiplication $m: A \times A \rightarrow A$ is also denoted by $ab = m(a, b)$.

Let $\{x_\alpha\}_{\alpha \in S}$ be a set equipped with parity, i.e., $S = S_{\bar{0}} \sqcup S_{\bar{1}}$ and $\bar{x}_\alpha = \bar{\alpha}$. The ring of (commutative) superpolynomials in the variables $\{x_\alpha\}_{\alpha \in S}$ is the commutative \mathbb{C} -superalgebra

$$\mathbb{C}[x_\alpha \mid \alpha \in S] := \mathbb{C}[x_\alpha \mid \alpha \in S_{\bar{0}}] \otimes \bigwedge_{x_\beta, (\beta \in S_{\bar{1}})},$$

which is the tensor product of the polynomial ring generated by the variables $\{x_\alpha\}_{\alpha \in S_{\bar{0}}}$ and the exterior algebra generated by the variables $\{x_\alpha\}_{\alpha \in S_{\bar{1}}}$.

We denote by $\hat{R}_{p|q}$ a formal power series ring

$$\hat{R}_{p|q} = \mathbb{C}[[x_1, \dots, x_p]] \otimes \bigwedge_{\phi_1, \dots, \phi_q}$$

with p even variables and q odd variables, and by $\hat{\mathfrak{m}} = (x_1, \dots, x_p, \phi_1, \dots, \phi_q) \subset \hat{R}_{p|q}$ its unique maximal ideal. It is complete in the linear topology whose basis of open neighborhoods around 0 is $\{\hat{\mathfrak{m}}^n\}_{n=0}^\infty$. Let $m: \hat{R}_{p|q} \times \hat{R}_{p|q} \rightarrow \hat{R}_{p|q}$ denote the product and $u: \mathbb{C} \rightarrow \hat{R}_{p|q}$ the unit morphism. We denote by $\hat{R}_{p|q} \hat{\otimes} \hat{R}_{p|q}$ the completed tensor product $\hat{R}_{p|q} \hat{\otimes} \hat{R}_{p|q} := \varprojlim_{n,m} \hat{R}_{p|q} / \hat{\mathfrak{m}}^n \otimes \hat{R}_{p|q} / \hat{\mathfrak{m}}^m$. Then we have a natural isomorphism

$$\hat{R}_{p|q} \hat{\otimes} \hat{R}_{p|q} \simeq \mathbb{C} \left[[x_1^{(1)}, \dots, x_p^{(1)}, x_1^{(2)}, \dots, x_p^{(2)}] \right] \otimes \bigwedge_{\phi_1^{(1)}, \dots, \phi_q^{(1)}, \phi_1^{(2)}, \dots, \phi_q^{(2)}}.$$

A *formal supergroup* is a super bialgebra $(\hat{R}_{p|q}, \Delta, \epsilon)$ with continuous homomorphisms

- coproduct $\Delta: \hat{R}_{p|q} \rightarrow \hat{R}_{p|q} \hat{\otimes} \hat{R}_{p|q}$,
- counit $\epsilon: \hat{R}_{p|q} \rightarrow \mathbb{C}$.

It has an unique continuous homomorphism $S: \hat{R}_{p|q} \rightarrow \hat{R}_{p|q}$ such that

$$m \circ (S \otimes 1) \circ \Delta = \epsilon \circ u = m \circ (1 \otimes S) \circ \Delta,$$

called the antipode. Thus formal supergroups are Hopf superalgebras, see [CCF].

Remark 1.1. Let G be an algebraic supergroup with the identity e . Then the germ of the structure sheaf \mathcal{O}_G at e , which we denote by $\mathcal{O}_{G,e}$, is a local ring. Let $\mathfrak{m}_{G,e}$ denote its unique maximal ideal. Then the completion

$$\hat{\mathcal{O}}_{G,e} := \varprojlim \mathcal{O}_{G,e} / \mathfrak{m}_{G,e}^n$$

has a natural structure of formal supergroup.

A morphism of formal supergroups $(\hat{R}_{p|q}, \Delta, \epsilon)$ and $(\hat{R}_{p'|q'}, \Delta', \epsilon')$ is a morphism of Hopf superalgebras

$$F: \hat{R}_{p|q} \rightarrow \hat{R}_{p'|q'}$$

which is continuous in the linear topology. We denote by $\underline{\mathbf{FG}}$ the category of formal supergroups.

1.2. Point distributions of formal supergroups. Let $F = (\hat{R}_{p|q}, \Delta, \epsilon)$ be a formal supergroup and $\hat{\mathfrak{m}} \subset \hat{R}_{p|q}$ its maximal ideal. A *point distribution* is a continuous linear map

$$\xi: \hat{R}_{p|q} \rightarrow \mathbb{C}$$

where \mathbb{C} is inherited with the discrete topology. Let $U_F := \text{Hom}_{\mathbb{C}}^c(\hat{R}_{p|q}, \mathbb{C})$ denote the vector superspace of point distributions. Since any point distribution factors

$$\hat{R}_{p|q} \twoheadrightarrow \hat{R}_{p|q}/\hat{\mathfrak{m}}^\ell \rightarrow \mathbb{C}$$

for some $\ell \in \mathbb{Z}_{\geq 0}$, we have $U_F = \varinjlim_{\ell} \text{Hom}_{\mathbb{C}}(\hat{R}_{p|q}/\hat{\mathfrak{m}}^\ell, \mathbb{C})$. It has an associative algebra structure by the dual maps

$$\Delta^*: U_F \rightarrow U_F \otimes U_F, \quad \epsilon^*: \mathbb{C} \rightarrow U_F$$

of Δ and ϵ , respectively. Note that these maps are well-defined since Δ and ϵ are continuous. Moreover, U_F has a Hopf superalgebra structure $F^\vee = (U_F, m^*, u^*, S^*)$.

Let us describe F^\vee more explicitly. The elements $X^\alpha = x_1^{i_1} \cdots x_p^{i_p} \phi_1^{j_1} \cdots \phi_q^{j_q} \in \hat{R}_{p|q}$, $(\alpha = (i_1, \dots, i_p, j_1, \dots, j_q) \in S = \mathbb{Z}_{\geq 0}^p \times \{0, 1\}^q)$, form a topological basis of $\hat{R}_{p|q}$. We denote by ξ_α , $(\alpha \in S)$, its dual basis of U_F , i.e., ξ_α satisfies $\xi_\alpha(X^\beta) = \delta_{\alpha, \beta}$. Then $\xi_\alpha \otimes \xi_\beta$, $(\alpha, \beta \in S)$, form a basis of $U_F \otimes U_F$. Here $\xi_\alpha \otimes \xi_\beta$ satisfies

$$\xi_\alpha \otimes \xi_\beta(X^{\alpha'} \otimes X^{\beta'}) = (-1)^{\bar{\alpha}'\bar{\beta}} \xi_\alpha(X^{\alpha'}) \xi_\beta(X^{\beta'}).$$

In the following, we also use the symbols $X^\alpha \in \hat{R}_{p|q}$, $(\alpha \in \tilde{S} = \mathbb{Z}_{\geq 0}^p \times \mathbb{Z}_{\geq 0}^q)$, standing for 0 if $\alpha \notin S$ and the notation $\alpha = (\alpha_1, \alpha_2)$, $|\alpha_1| = \sum_{r=1}^p i_r$ where $\alpha_1 = (i_1, \dots, i_p) \in \mathbb{Z}_{\geq 0}^p$ and $|\alpha_2| = \sum_{s=1}^q j_s$ where $\alpha_2 = (j_1, \dots, j_q) \in \mathbb{Z}_{\geq 0}^q$. We define a degree on U_F by $\deg(\xi_\alpha) = |\alpha| := |\alpha_1| + |\alpha_2|$.

Then we have

$$\begin{aligned} \epsilon^*: \mathbb{C} &\rightarrow U_F, & u^*: U_F &\rightarrow \mathbb{C} \\ 1 &\mapsto \xi_0, & \xi_\alpha &\mapsto \delta_{\alpha, 0}. \end{aligned} \tag{1.1}$$

The product $F^*: U_F \times U_F \rightarrow U_F$ satisfies

$$\xi_\alpha \cdot \xi_\beta = (-1)^{\bar{\alpha}\bar{\beta}} \binom{\alpha_1 + \beta_1}{\alpha_1} \xi_{\alpha+\beta} + \text{lower degree terms}, \tag{1.2}$$

where

$$\binom{p}{q} := \prod_{i=1}^{\ell} \binom{p_i}{q_i}, \quad p = (p_1, \dots, p_\ell), \quad q = (q_1, \dots, q_\ell)$$

and that the lower degree terms do not contain constant term $\mathbb{C}\xi_0$. In particular, it follows that the elements ξ_α with $|\alpha| = 1$ generate U_F as a superalgebra. The coproduct m^* is the unique superalgebra homomorphism such that

$$m^*(\xi_\alpha) = \xi_\alpha \otimes 1 + 1 \otimes \xi_\alpha \tag{1.3}$$

for $\alpha \in S$ such that $|\alpha| = 1$.

Lemma 1.2 ([Ma1, Lemma 4.1]). *The bilinear map*

$$[-, -]: U_F \times U_F \rightarrow U_F, \quad (a, b) \mapsto ab - (-1)^{\bar{a}\bar{b}} ba$$

defines a Lie superalgebra structure on the subspace $U_F^1 = \oplus_{|\alpha|=1} \mathbb{C}\xi_\alpha$.

Proof. Since U_F is an associative superalgebra, it suffices to prove that $[-, -]$ preserves the subspace U_F^1 . But it follows from (1.2). \square

Let $\mathcal{U}(U_F^1)$ denote the universal enveloping superalgebra of the Lie superalgebra U_F^1 . Recall that it admits a Hopf superalgebra structure with coproduct

$$\mathcal{U}(U_F^1) \rightarrow \mathcal{U}(U_F^1) \otimes \mathcal{U}(U_F^1), \quad X \mapsto X \otimes 1 + 1 \otimes X, \quad (X \in U_F^1),$$

counit

$$\mathcal{U}(U_F^1) \rightarrow \mathcal{U}(U_F^1)/(U_F^1) \simeq \mathbb{C},$$

and antipode

$$\mathcal{U}(U_F^1) \rightarrow \mathcal{U}(U_F^1), \quad X \mapsto -X, \quad (X \in U_F^1).$$

Here $(U_F^1) \subset \mathcal{U}(U_F^1)$ denotes the left ideal generated by the subspace U_F^1 .

Proposition 1.3 ([Ma1, Proposition 4.8]). *The Lie superalgebra homomorphism $U_F^1 \hookrightarrow U_F$ induces a \mathbb{C} -superalgebra homomorphism $\mathcal{U}(U_F^1) \rightarrow U_F$. It is an isomorphism of Hopf superalgebras.*

Proof. It is obvious that we have a \mathbb{C} -superalgebra homomorphism $\mathcal{U}(U_F^1) \rightarrow U_F$. Let ι denote this map. Then it follows from (1.1) and (1.3) that ι is a super bialgebra homomorphism. Since an antipode on a super bialgebra is unique if it exists, ι intertwines the antipodes of $\mathcal{U}(U_F^1)$ and U_F . Thus ι is a homomorphism of Hopf superalgebras. Since the elements ξ_α with $\alpha \in S$ such that $|\alpha| = 1$ generate U_F as a \mathbb{C} -superalgebra, ι is surjective. Now the PBW type theorem for $\mathcal{U}(U_F^1)$ implies that ι is an isomorphism. \square

1.3. Equivalence between Formal supergroups and Lie superalgebras. Let $\underline{\mathbf{LA}}$ denote the category of finite dimensional Lie superalgebras. Taking the universal enveloping superalgebra $\mathfrak{g} \mapsto \mathcal{U}(\mathfrak{g})$ for a Lie superalgebra \mathfrak{g} defines a functor

$$\mathcal{U}: \underline{\mathbf{LA}} \rightarrow \underline{\mathbf{BA}}$$

from $\underline{\mathbf{LA}}$ to the category of super bialgebras $\underline{\mathbf{BA}}$. The functor \mathcal{U} is fully faithful and its quasi-inverse is given by taking the primitive elements:

$$\mathcal{P}: \underline{\mathbf{BA}} \rightarrow \underline{\mathbf{LA}}, \quad (R, \Delta, \epsilon) \mapsto \mathcal{P}(R) := \{a \in R \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}.$$

The Lie superalgebra structure is given by $(a, b) \mapsto ab - (-1)^{\bar{a}\bar{b}}ba$ for $a, b \in \mathcal{P}(R)$. Thus $\underline{\mathbf{LA}}$ is regarded as a full subcategory of $\underline{\mathbf{BA}}$. By Proposition 1.3, the association $F \mapsto F^\vee$ for a formal supergroup F gives a contravariant functor $(-)^\vee: \underline{\mathbf{FG}} \rightarrow \underline{\mathbf{LA}}$.

Theorem 1.4 ([Ma1, Theorem 4.4]). *The functor $(-)^\vee: \underline{\mathbf{FG}} \rightarrow \underline{\mathbf{LA}}$ gives a contravariant equivalence of categories.*

A quasi-inverse is given again by taking the dual vector superspace

$$(-)^\vee: \underline{\mathbf{LA}} \rightarrow \underline{\mathbf{FG}}, \quad \mathcal{U}(\mathfrak{g}) \mapsto \text{Hom}_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), \mathbb{C}), \quad (1.4)$$

where \mathfrak{g} is an arbitrary finite dimensional Lie superalgebra. The super coalgebra (algebra) structure on $\text{Hom}_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), \mathbb{C})$ is induced by the super algebra (coalgebra) structure on $\mathcal{U}(\mathfrak{g})$. In the purely even setting, the equivalence between the categories of formal groups and of finite dimensional Lie algebras is a special case of *Cartier duality* between the category of linearly compact commutative Hopf algebras and the category of cocommutative Hopf algebras, see [D] for details.

The quasi-inverse (1.4) is known naturally isomorphic to the functor of taking the Campbell-Hausdorff formal supergroup. Let \mathfrak{g} denote a finite dimensional Lie superalgebra over \mathbb{C} . By viewing \mathfrak{g} as an affine superscheme, we denote by $\mathbb{C}[\mathfrak{g}]$ its coordinate ring. We identify $\mathbb{C}[\mathfrak{g}]$ with the symmetric superalgebra $S(\mathfrak{g}^*)$ of the dual vector superspace \mathfrak{g}^* of \mathfrak{g} . Then the Lie superbracket $[-, -]$ on \mathfrak{g} induces a superalgebra homomorphism

$$D: S(\mathfrak{g}^*) \rightarrow S(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*), \quad (1.5)$$

called the co(super)bracket of \mathfrak{g} . Consider the completion $\hat{S}(g^*) = \varprojlim_n S(\mathfrak{g}^*)/\mathcal{I}^n$ where \mathcal{I} is the argumentation ideal and the completed tensor product

$$\hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*) = \varprojlim_{n,m} S(\mathfrak{g}^*)/\mathcal{I}^n \otimes S(\mathfrak{g}^*)/\mathcal{I}^m.$$

Then the vector superspace $\text{Hom}_{\mathbb{C}}(\mathfrak{g}^*, \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*))$ admits a Lie superalgebra structure by

$$[f, g] := m \circ (f \otimes g) \circ D, \quad f, g \in \text{Hom}_{\mathbb{C}}(\mathfrak{g}^*, \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*)) \quad (1.6)$$

where m denotes the multiplication of $\hat{S}(\mathfrak{g}^*) \otimes \hat{S}(\mathfrak{g}^*)$. Define a superalgebra homomorphism $\Delta: \hat{S}(\mathfrak{g}^*) \rightarrow \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*)$ by

$$\Delta(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{\substack{(p_i, q_i) > 0 \\ i=1, \dots, n}} \frac{[\iota_1^{p_1}, \iota_2^{q_1}, \dots, \iota_1^{p_n}, \iota_2^{q_n}](X)}{\sum (p_i + q_i) \prod p_i! q_i!}, \quad (X \in \mathfrak{g}^*) \quad (1.7)$$

where the summation (p_i, q_i) is over $\mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$, ι_i denotes the natural inclusion

$$\iota_i: \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathfrak{g}^* \subset \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*),$$

for $i = 1, 2$ and

$$\begin{aligned} & [\iota_1^{p_1}, \iota_2^{q_1}, \dots, \iota_1^{p_n}, \iota_2^{q_n}] \\ &= \begin{cases} \underbrace{[\iota_1, \dots, [\iota_1, [\iota_2, \dots, [\iota_2 \dots [\iota_1, \dots, [\iota_1, [\iota_2, \dots, [\iota_2, \iota_2], \dots, [\iota_2, \iota_2], \dots]]]]]}_{p_1} \underbrace{\dots}_{q_1} \underbrace{\dots}_{p_n} \underbrace{\dots}_{q_n-1}, & (q_n \neq 0), \\ \underbrace{[\iota_1, \dots, [\iota_1, [\iota_2, \dots, [\iota_2 \dots [\iota_1, \dots, [\iota_1, \iota_1], \dots]]]}_{p_1} \underbrace{\dots}_{q_1} \underbrace{\dots}_{p_n-1}, & (q_n = 0). \end{cases} \end{aligned}$$

Then by [Ma1, Proposition 4.11], $(\hat{S}(\mathfrak{g}^*), \Delta, \epsilon)$ defines a formal supergroup where $\epsilon: \hat{S}(\mathfrak{g}^*) \rightarrow \mathbb{C}$ is the obvious counit. Moreover, by the proof of [Ma1, Theorem 4.4], the functor

$$CH: \underline{\mathbf{LA}} \rightarrow \underline{\mathbf{FG}}, \quad \mathfrak{g} \mapsto (\hat{S}(\mathfrak{g}^*), \Delta, \epsilon)$$

is a quasi-inverse of $(-)^{\vee}: \underline{\mathbf{FG}} \rightarrow \underline{\mathbf{LA}}$.

1.4. Unipotent algebraic supergroup and Nilpotent Lie superalgebra. An affine algebraic supergroup G is called *unipotent* if the following equivalent conditions hold:

- The coordinate ring $\mathbb{C}[G]$ is irreducible as a coalgebra, i.e., $\mathbb{C}[G]$ has a unique simple subcoalgebra, which is \mathbb{C} ,
- The isomorphism classes of simple rational G -supermodules are trivial modules $\{\mathbb{C}^{1|0}, \mathbb{C}^{0|1}\}$,

see [Ma2, Definition 2.9]. Let $\underline{\mathbf{Unip-AG}}$ denote the full subcategory of the category of affine algebraic supergroup consisting of unipotent affine algebraic supergroups. Then we have a functor

$$\hat{\mathcal{O}}_{-,e}: \underline{\mathbf{Unip-AG}} \rightarrow \underline{\mathbf{FG}}, \quad G \mapsto \hat{\mathcal{O}}_{G,e},$$

by Remark 1.1 and

$$T_e -: \underline{\mathbf{Unip-AG}} \rightarrow \underline{\mathbf{LA}}, \quad G \mapsto T_e G.$$

The latter one is just taking the Lie superalgebra on the tangent space at the identity e , see [CCF] for details. By construction, we have a natural isomorphism

$$(-)^{\vee} \circ \hat{\mathcal{O}}_{-,e} \simeq T_e -: \underline{\mathbf{Unip-AG}} \rightarrow \underline{\mathbf{LA}}. \quad (1.8)$$

A finite dimensional Lie superalgebra \mathfrak{g} is called *nilpotent* if the descending central series

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}], \quad (n > 0), \quad (1.9)$$

vanishes, i.e., $\mathfrak{g}^n = 0$ for some n . Let $\underline{\text{Nil-LA}} \subset \underline{\text{LA}}$ denote the full subcategory consisting of (finite dimensional) nilpotent Lie superalgebras.

Proposition 1.5 ([MO]). *The functor $T_e -: \underline{\text{Unip-AG}} \rightarrow \underline{\text{Nil-LA}}$ gives an equivalence of categories.*

Let \mathfrak{g} be a finite dimensional Lie superalgebra and $D: \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ denote the cobracket. Then \mathfrak{g} is nilpotent if and only if the map D^n defined by

$$D^n = (D \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-1}) \circ D^{n-1}, \quad n \geq 1,$$

vanishes, i.e., $D^n = 0$ for some $n \geq 0$. Thus the coproduct (1.7) of the corresponding formal group $CH(\mathfrak{g})$ stabilize the supersymmetric algebra $S(\mathfrak{g}^*)$:

$$\Delta: S(\mathfrak{g}^*) \rightarrow S(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*). \quad (1.10)$$

Equivalently, $\text{Spec}(S(\mathfrak{g}^*))$ has a structure of affine algebraic supergroup. By Proposition 1.5 and (1.8), $\text{Spec}(S(\mathfrak{g}^*))$ is a unipotent affine algebraic supergroup and, conversely, any unipotent affine algebraic supergroup is obtained in this way. Since $\text{Spec}(S(\mathfrak{g}^*))$ is an affine superspace, we obtain the following assertion.

Corollary 1.6. *Any unipotent affine algebraic supergroup is an affine superspace $\mathbb{C}^{p|q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$.*

1.5. Lie superalgebra cohomology. Let G be an affine algebraic supergroup with Lie superalgebra $\mathfrak{g} = T_e G$. For any commutative superalgebra A and an A -point $g \in G(A)$, we have a right multiplication

$$R_g: G(A) \rightarrow G(A), \quad h \mapsto hg.$$

This induces a left \mathfrak{g} -module structure on the coordinate ring

$$R: \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}[G]). \quad (1.11)$$

We call G positively graded if \mathfrak{g} is $\mathbb{Z}_{>0}$ -graded:

$$\mathfrak{g} = \bigoplus_{n>0} \mathfrak{g}_n, \quad [\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}.$$

In this case, \mathfrak{g}^* is naturally $\mathbb{Z}_{<0}$ -graded wt: $\mathfrak{g}^* = \bigoplus_{n<0} \mathfrak{g}_n^*$ where $\mathfrak{g}_n^* = \text{Hom}_{\mathbb{C}}(\mathfrak{g}_{-n}, \mathbb{C})$ for $n < 0$. We extend the grading on \mathfrak{g}^* to $S(\mathfrak{g}^*)$ multiplicatively: $\text{wt}(ab) = \text{wt}(a) + \text{wt}(b)$. Then the cobracket D in (1.5) preserves the grading and so does the coproduct $\Delta: \mathbb{C}[G] \rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G]$ by (1.7).

Proposition 1.7. *For a positively graded unipotent affine algebraic supergroup G ,*

$$H^n(\mathfrak{g}; \mathbb{C}[G]) \simeq \delta_{n,0} \mathbb{C}.$$

Proof. Although the assertion is well-known, we include a proof for the completeness of the paper. Since \mathfrak{g} is nilpotent, we have

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supset \cdots \supset \mathfrak{g}^m = 0$$

for some $m \in \mathbb{Z}_{\geq 0}$. We write $N = \dim \mathfrak{g}$ and take a basis $\{v_\alpha\}_{\alpha=1}^N$ of \mathfrak{g} such that the first $\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ elements gives a basis of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, and the next $\dim [\mathfrak{g}, \mathfrak{g}]/[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ elements gives a basis of $[\mathfrak{g}, \mathfrak{g}]/[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ etc. Let $c_{\alpha\beta}^\gamma$ denote the structure constants of \mathfrak{g} , i.e., $[v_\alpha, v_\beta] = \sum_\gamma c_{\alpha\beta}^\gamma v_\gamma$. Note that $c_{\alpha\beta}^\gamma$ is non-zero only if $\gamma \geq \alpha, \beta$. The Chevalley-Eilenberg complex of \mathfrak{g} with coefficients in $\mathbb{C}[G]$ is

$$C^\bullet(\mathfrak{g}; \mathbb{C}[G]) = \mathbb{C}[G] \otimes S(\Pi \mathfrak{g}^*),$$

$$d = \sum_{\alpha} (-1)^{\bar{\alpha}} R(v_{\alpha}) \otimes \varphi^{\alpha} - \frac{1}{2} \sum_{\alpha, \beta, \gamma} (-1)^{\bar{\alpha}\bar{\gamma}} c_{\alpha\beta}^{\gamma} \varphi_{\gamma} \varphi^{\alpha} \varphi^{\beta}. \quad (1.12)$$

Here $\Pi \mathfrak{g}^*$ denotes the parity-reversed superspace of \mathfrak{g}^* , spanned by $\{\varphi^{\alpha}\}_{\alpha=1}^N$ with parity $\bar{\varphi}^{\alpha} = \bar{\alpha} + 1 \in \mathbb{Z}_2$. The symbols φ^{α} in the differential d is the multiplication by φ^{α} and φ_{α} is the contracting operator for φ^{α} , i.e., $\varphi_{\alpha}(\varphi^{\beta}) = \delta_{\alpha,\beta}$, which has the same parity as φ^{α} , (see e.g. [DK]). By (1.10), we may take $\mathbb{C}[\mathfrak{g}]$ as the coordinate ring $\mathbb{C}[G]$ of G . Let $\{x_{\alpha}\}_{\alpha=1}^N$ denote the linear coordinates of \mathfrak{g} corresponding to the basis $\{v_{\alpha}\}_{\alpha=1}^N$. Then $\mathbb{C}[\mathfrak{g}] = \mathbb{C}[x_{\alpha} \mid \alpha = 1, \dots, N]$. By (1.7), (1.11) is expressed as

$$R(v_{\alpha}) = \sum_{\beta} \left(\epsilon^{(2)} \frac{\partial}{\partial x_{\alpha}^{(2)}} \Delta(x_{\beta}) \right) \frac{\partial}{\partial x_{\beta}},$$

where

$$\epsilon^{(2)} \frac{\partial}{\partial x_{\alpha}^{(2)}} \Delta(x_{\beta}) := \sum x_{\beta(1)} \epsilon \left(\frac{\partial}{\partial x_{\alpha}} x_{\beta(2)} \right)$$

and $\Delta(x_{\beta}) = \sum x_{\beta(1)} \otimes x_{\beta(2)}$. Thus the map R preserves the grading. Extend the grading on $\mathbb{C}[G]$ to $C^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$ by $\text{wt}(\varphi^{\alpha}) = \text{wt}(x_{\alpha})$. Then we have the grading decomposition

$$C^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) = \bigoplus_{n \leq 0} C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) \quad (1.13)$$

and d preserves the grading. Define a degree for each monomial by

$$\deg(x_{\alpha}) = 1 = \deg(\varphi^{\alpha}), \quad \deg(ab) = \deg(a) + \deg(b)$$

and $F^p C^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$ by the subspace spanned by all the monomials of degree greater than or equal to p . Then (1.12) implies $d: F^p C^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) \rightarrow F^{p+1} C^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$. Hence, there is a spectral sequence $E_r \Rightarrow H^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$ such that

$$E_1 = H(\text{gr}_F C^{\bullet}(\mathfrak{g}; \mathbb{C}[G]); \text{gr } d).$$

The filtration gives a filtration on each $C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$ by

$$F^p C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) = C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) \cap F^p C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G]).$$

It is of finite length and satisfies $C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G]) = \cup_p F^p C_n^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$. Therefore, the spectral sequence E_r converges. Since the differential on $\text{gr}_F C^{\bullet}(\mathfrak{g}; \mathbb{C}[G])$ is

$$\text{gr } d = \sum_{\alpha=1}^N (-1)^{\bar{\alpha}} \frac{\partial}{\partial x_{\alpha}} \otimes \varphi^{\alpha},$$

the complex $(\text{gr}_F C^{\bullet}(\mathfrak{g}; \mathbb{C}[G]), \text{gr } d)$ is the (algebraic) de Rham complex of G . By Corollary 1.6,

$$H^n(\text{gr}_F C^{\bullet}(\mathfrak{g}; \mathbb{C}[G]), \text{gr } d) \simeq H_{\text{dR}}^n(\mathbb{C}^{p|q})$$

for some $p, q \in \mathbb{Z}_{\geq 0}$. Then by Poincaré lemma below, we have $H_{\text{dR}}^n(\mathbb{C}^{p|q}) \simeq \delta_{n,0} \mathbb{C}$. It follows that the spectral sequence collapses at $r = 1$:

$$\text{gr}_F H^n(\mathfrak{g}; \mathbb{C}[G]) \simeq E_1^n \simeq \delta_{n,0} \mathbb{C}.$$

This completes the proof. \square

The following assertion is proved by Kostant in the analytic setting [Ko, Theorem 4.6] and is also well-known in the algebraic setting.

Lemma 1.8 (Poincaré lemma). $H_{\text{dR}}^n(\mathbb{C}^{p|q}) \simeq \delta_{n,0} \mathbb{C}$.

Proof. The algebraic de Rham complex of $\mathbb{C}^{p|q}$ is

$$C_{\text{dR}}^\bullet(\mathbb{C}^{p|q}) = \mathbb{C}[\mathbb{C}^{p|q}] \otimes S(\Pi\mathbb{C}^{p|q}).$$

We write $\mathbb{C}[\mathbb{C}^{p|q}] = \mathbb{C}[x_1, \dots, x_p, \theta_1, \dots, \theta_q]$ where x_i , (resp. θ_j), are even, (resp. odd), variables, and $S(\Pi\mathbb{C}^{p|q}) = \mathbb{C}[dx_1, \dots, dx_p, d\theta_1, \dots, d\theta_q]$ where dx_i , (resp. $d\theta_j$), are odd, (resp. even), variables. Then the differential is

$$d = \sum_{i=1}^p \frac{\partial}{\partial x_i} \otimes dx_i - \sum_{j=1}^q \frac{\partial}{\partial \theta_j} \otimes d\theta_j.$$

Thus by Künneth formula, $H_{\text{dR}}^\bullet(\mathbb{C}^{p|q}) \simeq (H_{\text{dR}}^\bullet(\mathbb{C}^{1|0}))^{\otimes p} \otimes (H_{\text{dR}}^\bullet(\mathbb{C}^{0|1}))^{\otimes q}$. Therefore, it suffices to show the assertion in the cases $\mathbb{C}^{1|0}$ and $\mathbb{C}^{0|1}$ respectively. The first one is the usual Poincaré lemma. For the second one, note that the d -closed forms are linear combinations of $1 \otimes d\theta^n$, ($n \geq 0$), which are exact if and only if for $n > 0$ since

$$d(-\theta \otimes d\theta^{p-1}) = 1 \otimes d\theta^p.$$

This completes the proof. \square

1.6. Superscheme of formal arcs. Let $\underline{\text{SSch}}$ denote the category of superschemes over \mathbb{C} . An object

$$D := \text{Spec}(\mathbb{C}[[t]]),$$

of $\underline{\text{SSch}}$ is called the formal disc.

Proposition 1.9 ([KV, Proposition 4.2.1]). *Let X be a superscheme over \mathbb{C} . The contravariant functor*

$$\underline{\text{SSch}} \rightarrow \underline{\text{Set}}, \quad Y \mapsto \text{Hom}_{\underline{\text{SSch}}}(Y \widehat{\times} D, X)$$

is represented by a superscheme JX , that is,

$$\text{Hom}_{\underline{\text{SSch}}}(Y \widehat{\times} D, X) \simeq \text{Hom}_{\underline{\text{SSch}}}(Y, JX)$$

for any object Y of $\underline{\text{SSch}}$. Here $Y \widehat{\times} D$ is the completion of $Y \times D$ with respect to the subsuperscheme $Y \widehat{\times} \{0\}$.

The superscheme JX in the above proposition is called the *superscheme of formal arcs* in X or the *arc space* of X . The association $X \mapsto JX$ gives a functor:

$$J\text{-} : \underline{\text{SSch}} \rightarrow \underline{\text{SSch}}. \quad (1.14)$$

The arc space JX of X has a canonical projection $\pi_X : JX \rightarrow X$ satisfying the following functoriality: for any morphism $f : X \rightarrow Y$ of superschemes, the morphism $Jf : JX \rightarrow JY$ makes the following diagram commutative:

$$\begin{array}{ccc} JX & \xrightarrow{Jf} & JY \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y. \end{array}$$

If X is an affine superscheme with

$$\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_N] / (f_1, \dots, f_M)$$

where $\mathbb{C}[x_1, \dots, x_N]$ is the ring of superpolynomials in the variables x_1, \dots, x_N and $f_1, \dots, f_M \in \mathbb{C}[x_1, \dots, x_N]$, then JX is again an affine superscheme with

$$\mathbb{C}[X] = \mathbb{C}[x_{1,(n)}, \dots, x_{N,(n)} \mid n < 0] / (f_{1,(n)}, \dots, f_{M,(n)} \mid n < 0).$$

$$\sum_{n < 0} f_{j(n)} z^{-n-1} = f_j(x_1(z), \dots, x_N(z))$$

with $x_i(z) = \sum_{n < 0} x_{i(n)} z^{-n-1}$. In this case, the canonical projection π_X is given by

$$\mathbb{C}[X] \rightarrow \mathbb{C}[JX], \quad x_i \mapsto x_{i(-1)}.$$

Later, we use the following properties of the functor (1.14).

Lemma 1.10 (cf. [A1]). *For superschemes X and Y , the following holds:*

- (1) $X \simeq Y$ implies $JX \simeq JY$.
- (2) $J(X \times Y) \simeq JX \times JY$.

Let G be an algebraic supergroup with Lie superalgebra $\mathfrak{g} = T_e G$. The arc space JG of G has a structure of group superscheme. The Lie superalgebra on the tangent space $T_e JG$ is $J\mathfrak{g} := \mathfrak{g}[[t]]$. The following proposition can be proved in the same way as Proposition 1.7.

Proposition 1.11. *For a positively graded unipotent affine algebraic supergroup G ,*

$$H^n(J\mathfrak{g}; \mathbb{C}[JG]) \simeq \delta_{n,0} \mathbb{C}.$$

2. \mathcal{W} -SUPERALGEBRAS AND MIURA MAPS

2.1. \mathcal{W} -superalgebras. We review here the definition of (affine) \mathcal{W} -superalgebras introduced by Kac, Roan and Wakimoto [KRW]. Given a vertex superalgebra V , (see [Ka2] for details) we denote by $Y(A, z) = A(z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$ the field corresponding to an element $A \in V$ and set $\int A(z) dz = A_{(0)}$. Given $A, B \in V$, we denote by $A(z)B(z)$: the normally ordered product and by $A(z)B(w) \sim \sum_{n \geq 0} \frac{C_n(w)}{(z-w)^{n+1}}$ the operator product expansion (OPE) of $A(z)$ and $B(z)$.

Let \mathfrak{g} be a basic classical simple Lie superalgebra (over \mathbb{C}) and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the parity decomposition. The Lie superalgebra \mathfrak{g} admits a non-degenerate even invariant bilinear form κ . Since such forms are all proportional, we may write $\kappa = k\kappa_0$ for some $k \in \mathbb{C}$ where $\kappa_0 = (\cdot, \cdot)$ is the one satisfying $\kappa_0(\theta, \theta) = 2$ for the highest root θ of \mathfrak{g}_0 , see [Ka1, Mu]. Let $f \in \mathfrak{g}_0$ be a non-zero nilpotent element. A good grading of \mathfrak{g} with respect to f is a $\frac{1}{2}\mathbb{Z}$ -grading on \mathfrak{g}

$$\Gamma: \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$$

such that $f \in \mathfrak{g}_{-1}$ and the adjoint action ad_f of f is injective $\mathfrak{g}_j \hookrightarrow \mathfrak{g}_{j-1}$ for $j \geq 1/2$ and surjective $\mathfrak{g}_j \twoheadrightarrow \mathfrak{g}_{j-1}$ for $j \leq 1/2$, (see [EK, H] for the classification). We fix a basis x_α , ($\alpha \in I = \{1, \dots, \dim \mathfrak{g}\}$) of \mathfrak{g} such that each x_i is of homogeneous parity and Γ -grading. Then we have $I = \sqcup_j I_j$ where $I_j = \{\alpha \mid x_\alpha \in \mathfrak{g}_j\}$. Let $c_{\alpha, \beta}^\gamma$ denote the structure constants, i.e., $[x_\alpha, x_\beta] = \sum_\gamma c_{\alpha, \beta}^\gamma x_\gamma$.

Let $V^k(\mathfrak{g})$ denote the universal affine vertex superalgebra of \mathfrak{g} at level k , which is generated $X(z)$, ($X \in \mathfrak{g}$), satisfying the OPEs

$$X(z)Y(z) \sim \frac{[X, Y](z)}{(z-w)} + \frac{k(X, Y)}{(z-w)^2}, \quad X, Y \in \mathfrak{g}.$$

We define a conformal grading on $V^k(\mathfrak{g})$ by $\Delta(u) = 1 - j$ for $u \in \mathfrak{g}_j$. Let $F_{\text{ch}}(\mathfrak{g}_+)$ be the charged fermion vertex superalgebra associated with $\mathfrak{g}_+ := \bigoplus_{j > 0} \mathfrak{g}_j$. It is generated by fields $\varphi_\alpha(z), \varphi^\alpha(z)$, ($\alpha \in I_+ := \sqcup_{j > 0} I_j$) of parity reversed to x_α , satisfying the OPEs

$$\varphi_\alpha(z)\varphi^\beta(w) \sim \frac{\delta_{\alpha, \beta}}{z-w}, \quad \varphi_\alpha(z)\varphi_\beta(w) \sim 0 \sim \varphi^\alpha(z)\varphi^\beta(w), \quad \alpha, \beta \in I_+.$$

We define a conformal grading on $F_{\text{ch}}(\mathfrak{g}_+)$ by $\Delta(\varphi_\alpha) = 1 - j$, $\Delta(\varphi^\alpha) = j$ for $\alpha \in I_j$, and a degree on $F_{\text{ch}}(\mathfrak{g}_+) = \bigoplus_{n \in \mathbb{Z}} F_{\text{ch}}^n$ by

$$\deg(\varphi^\alpha(z)) = 1 = -\deg(\varphi_\alpha(z)), \quad \alpha \in I_+,$$

$$\deg(\partial A(z)) = \deg(A(z)), \quad \deg(: A(z)B(z) :) = \deg(A(z)) + \deg(B(z)).$$

Let $\Phi(\mathfrak{g}_{1/2})$ be the neutral Fermion vertex superalgebra associated with $\mathfrak{g}_{1/2}$, which is generated by fields $\Phi_\alpha(z)$, ($\alpha \in I_{1/2}$) satisfying the OPEs

$$\Phi_\alpha(z)\Phi_\beta(w) \sim \frac{\chi([x_\alpha, x_\beta])}{z-w}, \quad \alpha, \beta \in I_{\frac{1}{2}},$$

where $\chi(x) = (f, x)$ for $x \in \mathfrak{g}$. We define a conformal grading on $\Phi(\mathfrak{g}_{1/2})$ by $\Delta(\Phi_\alpha) = 1/2$ for all $\alpha \in I_{1/2}$. Define a \mathbb{Z} -graded vertex superalgebra by

$$C_k^\bullet(\mathfrak{g}, f; \Gamma) = V^k(\mathfrak{g}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \otimes F_{\text{ch}}^\bullet(\mathfrak{g}_+),$$

with cohomological grading given by the degree on $F_{\text{ch}}(\mathfrak{g}_+)$ and a differential $d_{(0)}$ by

$$\begin{aligned} d(z) = \sum_{\alpha \in I_+} : ((-1)^{\bar{\alpha}} x_\alpha(z) + \Phi_\alpha(z) + \chi(x_\alpha)) \varphi^\alpha(z) : \\ - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in I_+} (-1)^{\bar{\alpha}\bar{\gamma}} c_{\alpha, \beta}^\gamma : \varphi_\gamma(z) \varphi^\alpha(z) \varphi^\beta(z) : \end{aligned}$$

where $\Phi_\alpha = 0$ for $\alpha \notin I_{1/2}$. Then $(C_k^\bullet(\mathfrak{g}, f; \Gamma), d_{(0)})$ forms a cochain complex, called the *BRST complex*. The vertex superalgebra obtained as the cohomology $H(C_k^\bullet(\mathfrak{g}, f; \Gamma), d_{(0)})$ is called the (affine) \mathcal{W} -superalgebra associated with $(\mathfrak{g}, f, k, \Gamma)$ and denoted by $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$. By [KW1, Theorem 4.1], we have

$$\mathcal{W}^k(\mathfrak{g}, f; \Gamma) = H^0(C_k^\bullet(\mathfrak{g}, f; \Gamma), d_{(0)}).$$

Since $d_{(0)}$ preserves the conformal grading on $C_k^\bullet(\mathfrak{g}, f; \Gamma)$ induced by those on each component $V^k(\mathfrak{g})$, $\Phi(\mathfrak{g}_{1/2})$ and $F_{\text{ch}}^\bullet(\mathfrak{g}_+)$, $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ has an induced conformal grading, which is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -grading.

2.2. Miura map. Define a field

$$J^u(z) = u(z) + \sum_{\alpha, \beta \in I_+} (-1)^{\bar{\alpha}} c_{u, \beta}^\alpha : \varphi_\alpha(z) \varphi^\beta(z), \quad u \in \mathfrak{g}.$$

Let $C_+ \subset C_k^\bullet(\mathfrak{g}, f; \Gamma)$ denote the vertex subalgebra generated by $J^u(z)$, ($u \in \mathfrak{g}_+$), and $\varphi_\alpha(z)$, ($\alpha \in I_+$), and $C_- \subset (C_k^\bullet(\mathfrak{g}, f; \Gamma))$ denote the one generated by $J^u(z)$, ($u \in \mathfrak{g}_{\leq 0} := \bigoplus_{j \leq 0} \mathfrak{g}_j$), $\Phi_\alpha(z)$, ($\alpha \in I_{1/2}$), and $\varphi^\alpha(z)$, ($\alpha \in I_+$). By [KW1, KW2], $C_\pm \subset C_k^\bullet(\mathfrak{g}, f; \Gamma)$ are subcomplexes and give a decomposition

$$C_k^\bullet(\mathfrak{g}, f; \Gamma) \simeq C_+^\bullet \otimes C_-^\bullet.$$

Moreover, we have $H^n(C_+^\bullet, d_{(0)}) \simeq \delta_{n,0}\mathbb{C}$. Thus we have an isomorphism

$$\mathcal{W}^k(\mathfrak{g}, f; \Gamma) \simeq H^0(C_-^\bullet, d_{(0)}).$$

Since C_- is $\mathbb{Z}_{\geq 0}$ -graded as a complex, it follows that $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ is a vertex subalgebra of C_-^0 . By [KRW, Theorem 2.4, (c)], we have

$$J^u(z)J^v(w) \simeq \frac{J^{[u,v]}(w)}{(z-w)} + \frac{\tau_k(u, v)}{(z-w)^2}, \quad u, v \in \mathfrak{g}_{\leq 0},$$

where

$$\tau_k(u, v) = k(u, v) + \frac{1}{2}(\kappa_{\mathfrak{g}}(u, v) - \kappa_{\mathfrak{g}_0}(u, v))$$

and $\kappa_{\mathfrak{g}}$, (resp. $\kappa_{\mathfrak{g}_0}$) denotes the Killing form of \mathfrak{g} , (resp. \mathfrak{g}_0). It follows that

$$C_-^0 \simeq V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}).$$

Since $V^{\tau_k}(\mathfrak{g}_{<0}) \otimes \Phi(\mathfrak{g}_{1/2}) \subset V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ is a (vertex superalgebra) ideal, we have a natural surjection

$$V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \rightarrow \left(V^{\tau_k}(\mathfrak{g}_{\leq 0}) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}) \right) / \left(V^{\tau_k}(\mathfrak{g}_{<0}) \otimes \Phi(\mathfrak{g}_{1/2}) \right) \simeq V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}).$$

The restriction to the subalgebra $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ is called the *Miura map* [FBZ, A1, G1]

$$\mu: \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \rightarrow V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}}). \quad (2.1)$$

Note that $V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{\frac{1}{2}})$ has a conformal grading defined by

$$\Delta(J^u(z)) = 1, \quad (u \in \mathfrak{g}_0), \quad \Delta(\Phi_\alpha(z)) = \frac{1}{2}, \quad (\alpha \in I_{\frac{1}{2}}),$$

and that μ preserves the conformal grading Δ .

Theorem 2.1. *The Miura map μ is injective.*

The above theorem is proved in [F, A1] when \mathfrak{g} is purely even and f is principal. But their proofs apply for an arbitrary nilpotent element. It is also proved in [G1] when the level k is generic. Our proof is a slight generalization of the one in [A1] together with necessary supergeometry.

2.3. Li's filtration. By [L], given a vertex superalgebra V , the subspaces $F^p V$, ($p \in \mathbb{Z}$), spanned by

$$a_{(-n_1-1)}^1 a_{(-n_2-1)}^2 \cdots a_{(-n_r-1)}^r |0\rangle$$

with $a^1, a^2, \dots, a^r \in V$, $n_i \geq 0$, $n_1 + n_2 + \cdots + n_r \geq p$, form a descending filtration $F^\bullet V$ of V satisfying the following properties:

- $F^p V_{(n)} F^q V \subset F^{p+q-n-1} V$, ($p, q \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$),
- $F^p V_{(n)} F^q V \subset F^{p+q-n} V$, ($p, q \in \mathbb{Z}_{\geq 0}$, $n \geq 0$).

Let

$$\mathrm{gr}_F V := \bigoplus_{p \in \mathbb{Z}} \mathrm{gr}_F^n V, \quad \mathrm{gr}_F^n V = F^p V / F^{p+1} V$$

denote the associated graded vector superspace and $\sigma_p: F^p V \rightarrow \mathrm{gr}_F^n V$ the canonical projection. By [L], $\mathrm{gr}_F V$ has a Poisson vertex superalgebra structure by

$$\begin{aligned} \partial \sigma_p(a) &= \sigma_{p+1}(\partial a), \quad \sigma_p(a) \sigma_q(b) = \sigma_{p+q}(a_{(-1)} b), \\ \{\sigma_p(a)_\lambda \sigma_q(b)\} &= \sum_{n \geq 0} \frac{1}{n!} \sigma_{p+q-n}(a_{(n)} b) \lambda^n. \end{aligned}$$

(see e.g. [FBZ, L] for the definition of Poisson vertex superalgebras and e.g. [BDK, Su] in terms of λ -brackets). It is obvious that any homomorphism $\eta: V \rightarrow W$ of vertex superalgebras induces a homomorphism of Poisson vertex superalgebras

$$\bar{\eta}: \mathrm{gr}_F V \rightarrow \mathrm{gr}_F W, \quad \sigma_p(a) \mapsto \sigma_p(\eta(a)).$$

Lemma 2.2. *Let V and W be vertex superalgebras equipped with conformal $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -gradings and $\eta: V \rightarrow W$ be a homomorphism of vertex superalgebras preserving the conformal gradings. Then η is injective if $\bar{\eta}$ is injective.*

Proof. Let $V = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_\Delta$, (resp. $W = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} W_\Delta$) denote the conformal grading on V , (resp. W). It is immediate from the definition that the Li's filtration $F^\bullet V$ defines a filtration on each V_Δ by setting

$$F^p V_\Delta = V_\Delta \cap F^p V.$$

Moreover, it is of finite length, i.e., $V_\Delta \cap F^p V = 0$ for $p \gg 0$. Since η preserves the conformal gradings, $\mathrm{gr}_F \eta$ is restricted to

$$\mathrm{gr}_F \eta: \mathrm{gr}_F V_\Delta \rightarrow \mathrm{gr}_F W_\Delta, \quad \Delta \geq 0.$$

It is straightforward to show the injectivity of F on V_Δ inductively from that of $\text{gr}_F \eta$ on $\text{gr}_F V_\Delta$ starting with the subspace $\text{gr}_F^N V_\Delta$ such that $\text{gr}_F^n V_\Delta = 0$ for $n > N$. \square

By [A1, Section 3], the Poisson vertex superalgebra $\text{gr}_F V^{\tau_k}(\mathfrak{g}_0) \otimes \Phi(\mathfrak{g}_{1/2})$ is isomorphic to the algebra of differential superpolynomials

$$S^\partial(\mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}}) = \mathbb{C}[\partial^n x_\alpha \mid \alpha \in I_0 \sqcup I_{\frac{1}{2}}, n \geq 0]$$

with λ -brackets

$$\begin{aligned} \{x_{\alpha\lambda}x_\beta\} &= [x_\alpha, x_\beta], \quad \alpha, \beta \in I_0, \\ \{x_{\alpha\lambda}x_\beta\} &= \chi([x_\alpha, x_\beta]), \quad \alpha, \beta \in I_{\frac{1}{2}}, \\ \{x_{\alpha\lambda}x_\beta\} &= 0, \quad \alpha \in I_0, \beta \in I_{\frac{1}{2}}. \end{aligned}$$

Thus the Miura map μ in (2.1) induces a homomorphism of Poisson vertex superalgebras

$$\bar{\mu}: \text{gr}_F \mathcal{W}^k(\mathfrak{g}, f, \Gamma) \rightarrow S^\partial(\mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}}). \quad (2.2)$$

Lemma 2.2 implies the following.

Corollary 2.3. *The injectivity of $\bar{\mu}$ implies that of μ .*

2.4. Arc space of Slodowy slice. Consider the Li's filtration $F^\bullet C_-^\bullet$ on the complex C_-^\bullet . The associated graded vector superspace $\bar{C}_-^\bullet = \text{gr}_F C_-^\bullet$ is again a complex with differential $\bar{d}_{(0)}$. Note that $\text{gr}_F C_-^\bullet$ is algebra of differential superpolynomials

$$\text{gr}_F C_-^\bullet \simeq S^\partial(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{\frac{1}{2}}) \otimes S^\partial(\Pi \mathfrak{g}_+^*),$$

equipped with the λ -brackets

$$\begin{aligned} \{u_\lambda v\} &= [u, v], \quad (u, v \in \mathfrak{g}_{\leq 0}), \\ \{u_\lambda v\} &= \chi([u, v]), \quad (u, v \in \mathfrak{g}_{\frac{1}{2}}), \\ \{\varphi_\lambda^\alpha u\} &= \sum_{\beta \in I_+} c_{u,\beta}^\alpha \varphi^\beta, \quad (\alpha \in I_+, u \in \mathfrak{g}_{\leq 0}), \\ \{u_\lambda v\} &= 0 = \{\varphi_\lambda^\alpha \varphi^\beta\}, \quad (u \in \mathfrak{g}_{\leq 0}, v \in \mathfrak{g}_{\frac{1}{2}}, \alpha, \beta \in I_+) \end{aligned}$$

where $\{\varphi^\alpha\}_{\alpha \in I_+} \subset \mathfrak{g}_+^*$ is the dual basis of $\{x_\alpha\}_{\alpha \in I_+}$ of \mathfrak{g}_+ with reversed parity. The differential $Q = \bar{d}_{(0)}$ is given by

$$Q(u) = \begin{cases} \sum_{\alpha \in I_+} \varphi^\alpha ((-1)^{\bar{\alpha}} [x_\beta, u]_{\leq 0} - (-1)^{\bar{u}} ([x_\alpha, u]_{1/2} + \chi([x_\alpha, u]_{\geq 1}))), & (u \in \mathfrak{g}_{\leq 0}), \\ \sum_{\alpha \in I_+} \chi([x_\alpha, u]) \varphi^\alpha, & (u \in \mathfrak{g}_{1/2}), \end{cases} \quad (2.3)$$

$$Q(\varphi^\alpha) = -\frac{1}{2} \sum_{\beta, \gamma \in I_+} (-1)^{\bar{\alpha}\bar{\gamma}} c_{\alpha,\beta}^\gamma \varphi^\beta \varphi^\gamma, \quad \alpha \in I_+, \quad (2.4)$$

$$Q(\partial a) = \partial Q(a), \quad Q(ab) = Q(a)b + (-1)^{\bar{a}} aQ(b). \quad (2.5)$$

Here we have used the projections

$$\mathfrak{g} \rightarrow \mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_{\geq 1}, \quad u \mapsto (u_{\leq 0}, u_{1/2}, u_{\geq 1}).$$

We will interpret the complex $(\text{gr}_F C_-^\bullet, Q)$ geometrically. To this end, let us first consider the finite analogue: the the quotient graded superspace

$$\text{gr}_F^{\text{fin}} C_-^\bullet := \text{gr}_F C_-^\bullet / (a\partial(b) \mid a, b \in \text{gr}_F C_-^\bullet).$$

It is easy to check that for a vector superspace A ,

$$S^\partial(A)/(a\partial(b) \mid a, b \in A) \simeq S(A),$$

where $S(A)$ is the symmetric superalgebra of A . Thus we have

$$\mathrm{gr}_F^{\mathrm{fin}} C_- \simeq S(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2}) \otimes S(\Pi \mathfrak{g}_+^*).$$

Let $\mathrm{gr}_F C_- \rightarrow \mathrm{gr}_F^{\mathrm{fin}} C_-$, $(a \mapsto [a])$, denote the canonical projection. It is naturally a Poisson superalgebra by

$$[a][b] = [ab], \quad \{[a], [b]\} = [\{a_\lambda b\}_{\lambda=0}].$$

The Poisson superalgebra $\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet$ is called the *Zhu's C_2 -algebra* of $\mathrm{gr}_F C_-^\bullet$. By (2.5), the differential Q induces a differential Q^{fin} on $\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet$, which is determined by the formulas (2.3)-(2.5) with u replaced by $[u]$, e.g.,

$$Q^{\mathrm{fin}}([\varphi^\alpha]) = -\frac{1}{2} \sum_{\beta, \gamma \in I_+} (-1)^{\bar{\alpha}\bar{\gamma}} c_{\alpha, \beta}^\gamma [\varphi^\beta][\varphi^\gamma], \quad \alpha \in I_+.$$

Then the complex $(\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet, Q^{\mathrm{fin}})$ is identified with the Chevalley-Eilenberg complex of \mathfrak{g}_+ with coefficients in $S(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2})$ since differential Q^{fin} defines a left \mathfrak{g}_+ -module structure on $S(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2})$ by

$$x_\alpha \cdot u = \begin{cases} (-1)^{\bar{\alpha}} [x_\beta, u]_{\leq 0} - (-1)^{\bar{u}} ([x_\alpha, u]_{1/2} + \chi([x_\alpha, u]_{\geq 1})), & (u \in \mathfrak{g}_{\leq 0}), \\ \chi([x_\alpha, u]), & (u \in \mathfrak{g}_{1/2}), \end{cases}$$

for $\alpha \in I_+$. Let G_+ be the unipotent algebraic supergroup whose Lie superalgebra is \mathfrak{g}_+ . The right G_+ -action

$$(f + \mathfrak{g}_{\geq -1/2}) \times G_+ \rightarrow (f + \mathfrak{g}_{\geq -1/2}), \quad (f + X, g) \mapsto g^{-1}(f + X)g,$$

induces a left \mathfrak{g}_+ -module structure on the coordinate ring $\mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]$. Then the isomorphism of \mathbb{C} -superalgebras

$$S(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2}) \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}],$$

$$u \mapsto \begin{cases} -(-1)^{\bar{u}}(u|?), & u \in \mathfrak{g}_{\leq 0}, \\ (u|?), & u \in \mathfrak{g}_{1/2}, \end{cases}$$

is a \mathfrak{g}_+ -homomorphism. Thus we have proved the following.

Lemma 2.4. *The complex $(\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet, Q^{\mathrm{fin}})$ is quasi-isomorphic to the Chevalley-Eilenberg complex of \mathfrak{g}_+ with coefficients in $\mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]$.*

By Jacson-Morozov theorem, there exists an \mathfrak{sl}_2 -triple $\{e, h, f\} \subset \mathfrak{g}_0$ containing f . Set $\mathfrak{g}^e = \{X \in \mathfrak{g} \mid [e, X] = 0\}$. The subvariety $\mathcal{S}_f = f + \mathfrak{g}^e \subset \mathfrak{g}$ is called the *Slodowy slice* of \mathfrak{g} associated with f .

Proposition 2.5. *We have an isomorphism*

$$\xi: \mathcal{S}_f \times G_+ \simeq f + \mathfrak{g}_{\geq -\frac{1}{2}}, \quad (X, g) \mapsto g^{-1}Xg \quad (2.6)$$

of affine supervarieties.

Proof. It suffices to show that ξ gives an isomorphism of all the A -valued points for an arbitrary commutative \mathbb{C} -superalgebra A . Let $f + X \in f + \mathfrak{g}^e(A)$ and $g \in G_+(A)$. Let $Y \in \mathfrak{g}_+(A)$ denote the element corresponding to g by the isomorphism $\mathfrak{g}_+ \simeq G_+$. It satisfies

$$Z := g^{-1}(f + X)g = \sum_{n \geq 0} \frac{1}{n!} [\cdots [f + X, \underbrace{Y, \dots, Y}_n] \cdots].$$

For $u \in \mathfrak{g}$, decompose it as $u = \sum_p u_p$ by the grading Γ . Then we have

$$Z_p = \begin{cases} 0, & (p < -1), \\ f, & (p = -1), \\ X_p + [f, Y_{p+1}] + R_p, & (p \geq -\frac{1}{2}), \end{cases}$$

where R_p is the term determined by X_i , ($i < p$), and Y_j , ($j < p + 1$). Now, the assertion follows from the decomposition

$$\mathfrak{g}_p = \mathfrak{g}_p^e \oplus [f, \mathfrak{g}_{p+1}], \quad p \geq -1/2,$$

which is an immediate consequence of the definition of the good grading Γ . \square

Note that the Poisson superalgebra structure on $S(\mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{1/2}) \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]$ restricts to $\mathbb{C}[\mathcal{S}_f] \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]^{G_+}$. Thus \mathcal{S}_f is a Poisson supervariety. We remark that if Γ is a \mathbb{Z} -grading, then \mathcal{S}_f is the Poisson supervariety obtained as the Hamiltonian reduction of \mathfrak{g} with respect to the Adjoint G_+ -action and an infinitesimal character $\chi = (f| -) : \mathfrak{g}_+ \rightarrow \mathbb{C}$.

Corollary 2.6.

- (1) $H^n(\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet, Q^{\mathrm{fin}}) = 0$ for $n \neq 0$.
- (2) $H^0(\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet, Q^{\mathrm{fin}}) \simeq \mathbb{C}[\mathcal{S}_f]$ as Poisson superalgebras.

Proof. By Lemma 2.4 and Proposition 2.5, we have

$$H^n(\mathrm{gr}_F^{\mathrm{fin}} C_-^\bullet, Q^{\mathrm{fin}}) \simeq H^n(\mathfrak{g}_+; \mathbb{C}[f + \mathfrak{g}_{\geq -1/2}]) \simeq H^n(\mathfrak{g}_+; \mathbb{C}[G_+]) \otimes \mathbb{C}[\mathcal{S}_f].$$

Since G_+ is a positively graded unipotent algebraic supergroup, the assertion follows from Proposition 1.7. \square

Next, we consider the complex $(\mathrm{gr}_F C_-^\bullet, Q)$. Recall that the arc space JG_+ of G_+ is an group superscheme whose Lie superalgebra is $J\mathfrak{g}_+ = \mathfrak{g}_+[[t]]$. It follows from Lemma 2.4 that $(\mathrm{gr}_F C_-^\bullet, Q)$ is quasi-isomorphic to the Chevalley-Eilenberg complex of $J\mathfrak{g}_+$ with coefficients in $\mathbb{C}[f + J\mathfrak{g}_{\geq -1/2}]$ with left $J\mathfrak{g}_+$ -module structure induced by the right JG_+ -action

$$J(f + \mathfrak{g}_{\geq -\frac{1}{2}}) \times JG_+ \rightarrow J(f + \mathfrak{g}_{\geq -\frac{1}{2}}), \quad (f + X, g) \mapsto g^{-1}(f + X)g. \quad (2.7)$$

The following is a generalization of [A1, Theorem 5.7].

Lemma 2.7.

- (1) $H^n(\mathrm{gr}_F C_-^\bullet, Q) = 0$ for $n \neq 0$.
- (2) $H^0(\mathrm{gr}_F C_-^\bullet, Q) \simeq \mathbb{C}[J\mathcal{S}_f]$ as Poisson vertex superalgebras.

Proof. By Lemma 1.10 and Proposition 2.5, we have an isomorphism

$$J\mathcal{S}_f \times JG_+ \simeq J(f + \mathfrak{g}_{\geq -1/2}).$$

Therefore, by using Proposition 1.11, we have

$$\begin{aligned} H^n(\mathrm{gr}_F C_-^\bullet, A) &\simeq H^n(J\mathfrak{g}_+; \mathbb{C}[f + J\mathfrak{g}_{\geq -1/2}]) \\ &\simeq H^n(J\mathfrak{g}_+; \mathbb{C}[JG_+]) \otimes \mathbb{C}[J\mathcal{S}_f] \\ &\simeq \delta_{n,0} \mathbb{C}[J\mathcal{S}_f]. \end{aligned}$$

\square

The following is a generalization of [A1, Theorem 5.8].

Proposition 2.8.

$$\mathrm{gr}_F \mathcal{W}^k(\mathfrak{g}, f, \Gamma) \simeq \mathbb{C}[J\mathcal{S}_f]$$

as Poisson vertex superalgebras.

Proof. The filtration $F^\bullet C_-^\bullet$ induces a filtration on $H(C_-^\bullet, d)$ by

$$\mathcal{F}^n H(C_-^\bullet, d) = \text{Im}(\text{Ker}(d) \cap F^n C_-^\bullet \rightarrow H(C_-^\bullet, d))$$

and thus a spectral sequence $E_1 = H(\text{gr}_F C_-^\bullet, Q) \Rightarrow H(C_-^\bullet, Q)$. Recall that C_-^\bullet is $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded $C_-^\bullet = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geq 0}} C_{-, \Delta}^\bullet$ and that $F^\bullet C_-^\bullet$ induces a filtration on each $C_{-, \Delta}^\bullet$ by

$$F^p C_{-, \Delta}^\bullet = F^p C_-^\bullet \cap C_{-, \Delta}^\bullet.$$

Since it is of finite length, the spectral sequence E_r converges. By Lemma 2.7, E_r collapses at $r = 1$:

$$\text{gr}_F H^\bullet(C_-^\bullet, d) \simeq E_1^\bullet = H^\bullet(\text{gr}_F C_-^\bullet, Q) \simeq \delta_{n,0} \mathbb{C}[\mathcal{S}_f].$$

Therefore, it suffices to show $\text{gr}_F \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \simeq \text{gr}_F H^0(C_-^\bullet, d)$. Since we have a map $F^n \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \rightarrow \mathcal{F}^n H^0(C_-^\bullet, d)$ by construction, we obtain a homomorphism of Poisson vertex superalgebras

$$\text{gr}_F \mathcal{W}^k(\mathfrak{g}, f; \Gamma) \rightarrow \text{gr}_F H^0(C_-^\bullet, d) \simeq \mathbb{C}[\mathcal{JS}_f].$$

By [KW1, Theorem 4.1], $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ is strongly generated by a basis of $\mathfrak{g}^e \simeq f + \mathfrak{g}^e = \mathcal{S}_f$, we have a surjection of Poisson vertex superalgebras

$$\mathbb{C}[\mathcal{JS}_f] \twoheadrightarrow \text{gr}_F \mathcal{W}^k(\mathfrak{g}, f; \Gamma),$$

by [A1, Theorem 10]. Since the composition of the above two homomorphisms is the identity of $\mathbb{C}[\mathcal{JS}_f]$, we obtain the assertion. \square

2.5. Injectivity of Miura map. It follows from Proposition 2.8 and the proof of Corollary 2.6 that the map $\bar{\mu}$ in (2.2) is identified with the composition

$$\mathbb{C}[\mathcal{JS}_f] \hookrightarrow \mathbb{C}[\mathcal{JS}_f] \otimes \mathbb{C}[JG_+] \simeq \mathbb{C}[J(f + \mathfrak{g}_{\geq -1/2})] \twoheadrightarrow \mathbb{C}[J(f + \mathfrak{g}_{\text{ini}})],$$

where $\mathfrak{g}_{\text{ini}} = \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0$. Note that it is the image of the functor J - in (1.14) of its finite analogue

$$\bar{\mu}_{\text{fin}}: \mathbb{C}[\mathcal{S}_f] \hookrightarrow \mathbb{C}[\mathcal{S}_f] \otimes \mathbb{C}[G_+] \simeq \mathbb{C}[f + \mathfrak{g}_{\geq -\frac{1}{2}}] \twoheadrightarrow \mathbb{C}[f + \mathfrak{g}_{\text{ini}}]. \quad (2.8)$$

Since the injectivity of $\bar{\mu}_{\text{fin}}$ implies that of $\bar{\mu}$, the proof of Theorem 2.1 is reduced to the following by Corollary 2.3.

Proposition 2.9. *The map $\bar{\mu}_{\text{fin}}$ is injective.*

Proof. Recall we have a functor from the category of affine supervarieties to the category of affine varieties of taking the reduced variety:

$$X = \text{Spec}(\mathbb{C}[X]) \mapsto X_{\text{red}} := \text{Spec}(\mathbb{C}[X]/(\mathbb{C}[X]_{\bar{1}}))$$

where $(\mathbb{C}[X]_{\bar{1}}) \subset \mathbb{C}[X]$ denotes the ideal generated by the odd subspace of $\mathbb{C}[X]$, see [CCF]. Applying this functor to $\bar{\mu}_{\text{fin}}$, we obtain the Miura map for the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ with the same nilpotent element f . By [G2, Lemma 5.12], it is injective. Note that the map $\bar{\mu}_{\text{fin}}$ preserves the parity. Then it suffices to show that the images of the linear coordinates of the odd part $\mathcal{S}_{f, \bar{1}}$ are linearly independent. Therefore, it suffices to show that the image of the map

$$G_{+, \bar{0}} \times (f + \mathfrak{g}_{\text{ini}, \bar{1}}) \rightarrow f + \mathfrak{g}_{\geq -1/2} \twoheadrightarrow \mathcal{S}_{f, \bar{1}} \quad (2.9)$$

is dense in the Zariski topology. Here the last projection is the composition

$$\begin{aligned} \mathfrak{g}_{\geq -1/2} &= \mathfrak{g}_{\geq -1/2, \bar{0}} \oplus \mathfrak{g}_{\geq -1/2, \bar{1}} \\ &\twoheadrightarrow \mathfrak{g}_{\geq -1/2, \bar{1}} = \mathfrak{g}_{\geq -1/2, \bar{1}}^e \oplus (\mathfrak{g}_{\geq -1/2, \bar{1}} \cap \text{Im}(\text{ad}_f)) \\ &\twoheadrightarrow \mathfrak{g}_{\geq -1/2, \bar{1}}^e. \end{aligned}$$

Since the restriction of (2.9) to

$$\{\exp(e)\} \times (f + \mathfrak{g}_{\text{ini}, \bar{1}}) \rightarrow \mathcal{S}_{f, \bar{1}}$$

is an isomorphism as affine varieties, we obtain the assertion. \square

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