CATEGORICAL ENTROPIES ON SYMPLECTIC MANIFOLDS

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ABSTRACT. In this paper, being motivated by symplectic topology, we study categorical entropy. Specifically, we prove inequalities between categorical entropies of functors on a category and its localisation. We apply the inequalities to symplectic topology to prove equalities between categorical entropies on wrapped, partially wrapped, and compact Fukaya categories if the functors are induced by the same compactly supported symplectic automorphisms. We also provide a practical way to compute the categorical entropy of symplectic automorphisms by using Lagrangian Floer theory if their domains satisfy a type of Floer-theoretical duality. Our main examples of symplectic manifolds satisfying the duality conditions are the plumbings of T^*S^n along a tree. Moreover, for symplectic automorphisms of Penner type, we prove that our computation of categorical entropy becomes a computation by simple linear algebra.

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1. Introduction

Let $\mathscr{C}$ be a triangulated category, and let $\Phi : \mathscr{C} \to \mathscr{C}$ be an exact endofunctor on $\mathscr{C}$. Then, the pair $(\mathscr{C}, \Phi)$ forms a categorical dynamical system. Dimitrov, Haiden, Katzarkov, and Kontsevich [DHKK14] defined an invariant measuring the complexity of the dynamical system $(\mathscr{C}, \Phi)$. The invariant is called the *categorical entropy* of Φ. After [DHKK14] introduced the notion, the categorical entropy has drawn many interests, see [Kik17, Fan18, KT19, Ouc20, KST20, KO20, FFO21, Ike21] for example. Some of the listed references also mentioned, explicitly and implicitly, about the categorical entropy in a symplectic topological setting. In the current paper, we study the notion of categorical entropy, motivated by the viewpoint of symplectic topology.

In order to explain our motivation, let us consider the following situation: Let M be a symplectic manifold, and let $\phi : M \to M$ be a symplectic automorphism. It is well-known that ϕ induces an auto-equivalence Φ on the Fukaya category of M. There are many natural questions arising from the situation. For example, "ϕ induces a topological dynamical system. Then, what is the relation between the topological entropy of ϕ and the categorical entropy of Φ?" Or, "many properties of topological entropy are known in the literature. Does categorical entropy satisfy similar properties?"

In the current paper, we study categorical entropy motivated by the following two questions:

Question 1. If ϕ is an exact symplectic (more precisely, Liouville) automorphism on a Weinstein manifold W, ϕ induces an auto-equivalence on each of the compact, wrapped, and partially wrapped Fukaya categories of W. Then, what relations do the categorical entropies of the induced functors have?

Question 2. The categorical entropy is not easy to compute. In the above situation, can we compute the categorical entropy of Φ *practically* from the symplectic topology?

The first half of the paper, motivated by Question 1, studies the categorical entropies on a category and its localisation. The second half provides a way to compute the categorical entropy of a symplectic automorphism ϕ where ϕ is defined on a Weinstein manifold of a specific type. The rest of this introduction section contains more details on our results.

To answer Question 1, in the first half of the paper, we prove Theorem 1.1, which is stated/proven in purely categorical way.

Theorem 1.1 (= Theorem 3.10)**.** *Let $\mathscr{C}$ be a triangulated (or pretriangulated $\mathscr{A}_\infty$/dg) category, and let $\mathscr{D}$ be a triangulated full subcategory of $\mathscr{C}$. Let us assume that a functor $\Phi_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ preserves $\mathscr{D}$ so that there is an induced functor*

$$\Phi_{\mathscr{C}/\mathscr{D}} : \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}.$$

When $h_t(\Phi)$ means the categorical entropy of a functor Φ, the following inequalities hold:

$$h_t(\Phi_{\mathscr{C}/\mathscr{D}}) \leq h_t(\phi_{\mathscr{C}}) \leq \max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}), h_t(\Phi_{\mathscr{D}})\}.$$

Since the compact Fukaya category is a full subcategory of the wrapped Fukaya category, and since the wrapped Fukaya category is a quotient (i.e., localisation) of any partially wrapped Fukaya category, we can apply Theorem 1.1 in order to answer Question 1. One of its applications is as follows:

Theorem 1.2 (= Theorem 3.20)**.** *If $h_t(\Phi_{\mathscr{W}(W)}), h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}) \geq 0$ (prerequisites can be dropped when $t = 0$), then*

$$h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W,\Lambda)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}), \tag{1.1}$$

where $\Phi_{\mathscr{W}(W)}$ (resp. $\Phi_{\mathscr{W}(W,\Lambda)}$) denotes the functor induced from a compactly supported exact symplectic automorphism ϕ: $W \to W$, on the wrapped Fukaya category $\mathscr{W}(W)$ of W (resp. partially wrapped Fukaya category $\mathscr{W}(W,\Lambda)$ of W with a stop Λ.) The first equality in (1.1) *holds without the condition $h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}) \geq$* 0.

Note that we can use Equation (1.1) to compute the categorical entropy of $\Phi_{\mathscr{W}(W)}$ as follows: One can choose a stop Λ making the corresponding partially wrapped Fukaya category $\mathscr{W}(W,\Lambda)$ smooth and proper. Then by applying [DHKK14, Theorem 2.6], the categorical entropy of the functor $\Phi_{\mathscr{W}(W)}$ can be computed via *Lagrangian Floer theory*, as

$$h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda)}) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^k_{\mathscr{W}(W,\Lambda)}(G,\Phi^n_{\mathscr{W}(W,\Lambda)}(G))e^{-kt}, \tag{1.2}$$

where G is a split-generator of $\mathscr{W}(W,\Lambda)$, which is described in [CRGG17] and [GPS18b].

The first part consists of two sections, Sections 2 and 3. Every subsection in the first part, except Section 3.5, deals with the category theory, and Section 3.5 applies the category-theoretic results to symplectic topology.

In most of the second part, we concentrate on symplectic manifolds of a specific type. The symplectic manifolds we are interested in are plumbings of T^*S^n where $n \geq 3$, along trees.

Before discussing more details, let us explain the main reason why we consider the plumbing spaces. First of all, Abouzaid and Smith [AS12] (resp. Chantraine, Rizell, Ghiggini, and Golovko[CRGG17] and Ganatra, Pardon, and Shende [GPS18b]) gave a split-generator of the compact (resp. wrapped) Fukaya category. Let S (resp. L) denote the split-generator. Then, it is known that S and L satisfy a kind of *duality* (for the formal statement of the duality, see the conditions in Theorem 1.4.)

We note that, as mentioned above, [DHKK14, Theorem 2.6] proved that if a category $\mathscr{C}$ is smooth and proper, then the categorical entropy of a functor $\Phi : \mathscr{C} \to \mathscr{C}$ can be computed relatively easily, from the morphism spaces of $\mathscr{C}$ as in (1.2). Unfortunately, it is not expected that one can apply the result in a symplectic setting unless introducing a stop because, usually, the compact (resp. wrapped) Fukaya category is not smooth (resp. proper.) However, from the duality between S and L, we prove a result similar to [DHKK14, Theorem 2.6]. In other words, we prove the following theorems:

Theorem 1.3 (= Theorem 4.13 and Theorem 4.14)**.** *Let ϕ be a Liouville automorphism on the plumbing space of T^*S^n along a tree for some $n \geq 3$. Let $\Phi_{\mathscr{F}}, \Phi_{\mathscr{W}}$ denote the functors on the compact and wrapped Fukaya category induced from ϕ. Then, the following hold:*

$$h_t(\Phi_{\mathscr{F}}) = \lim_{m\to\infty}\frac{1}{m}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^{n+k}\left(\phi^m(S),L\right)e^{kt}, \tag{1.3}$$

$$h_t(\Phi_{\mathscr{W}}) = \lim_{m\to\infty}\frac{1}{m}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^{n-k}\left(S,\phi^m(L)\right)e^{kt}, \tag{1.4}$$

$$h_t(\Phi_{\mathscr{F}}) = h_{-t}\left(\Phi_{\mathscr{W}}^{-1}\right), h_t(\Phi_{\mathscr{W}}) = h_{-t}\left(\Phi_{\mathscr{F}}^{-1}\right). \tag{1.5}$$

Moreover, if $h_t(\Phi_{\mathscr{F}}), h_t(\Phi_{\mathscr{W}}) \geq 0$ (these are not necessary when $t = 0$), and ϕ is compactly supported, then

$$h_t(\Phi_{\mathscr{F}}) = h_t(\Phi_{\mathscr{W}}). \tag{1.6}$$

One can generalize Theorem 1.3 to Weinstein manifolds W satisfying a given duality below:

Theorem 1.4 (= Theorem 4.15)**.** *Let us assume that a Weinstein manifold W has sets of Lagrangians $\{S_i\}_{1\leq i\leq k}$ and $\{L_i\}_{1\leq i\leq k}$ such that*

- *$\{S_i\}_{1\leq i\leq k}$ (resp. $\{L_i\}_{1\leq i\leq k}$) generates the compact (resp. wrapped) Fukaya category of W,*
- *the morphism space in the wrapped Fukaya category $\mathrm{Hom}^*(S_i,L_j)$ is non-zero only if $i = j$ and is one-dimensional when $i = j$,*
- *$\bigoplus_{i,j}\mathrm{Hom}^*(S_i,S_j)$ is non-negatively graded,*
- *$\bigoplus_{i,j}\mathrm{Hom}^*(L_i,L_j)$ is non-positively graded, and*
- *$\dim\mathrm{Hom}^0(S_i,S_i) = 1 = \dim\mathrm{Hom}^0(L_i,L_i)$ for all $1 \leq i \leq k$.*

Let $S := \bigoplus_i S_i$, $L := \bigoplus_i L_i$. If ϕ is a Liouville automorphism on W, then Equations (1.3)–(1.5) *hold for ϕ. Also, if $h_t(\Phi_{\mathscr{F}}), h_t(\Phi_{\mathscr{W}}) \geq 0$ (these are not necessary when $t = 0$), and ϕ is compactly supported, then Equation* (1.6) *hold.*

In Section 5, the second section of the second half, we give examples of symplectic automorphisms whose categorical entropy are computed by a simple way. More specifically, we focus on symplectic automorphisms of *Penner type* (see Definition 5.2,) which is constructed by a higher dimensional generalization of Penner's construction [Pen88] of pseudo-Anosov surface automorphisms. In his thesis [Lee21], the last-named author proved that symplectic automorphisms of the Penner type satisfy geometric stability. We expected that one could easily compute the categorical entropy of the Penner type because of the geometric stability. The expectation turns out to be true: one can compute the categorical entropy by simple linear algebra. More precisely, we prove the following:

Theorem 1.5 (=Theorem 5.8)**.** *Let ϕ be a symplectic automorphism of Penner type, defined on a plumbing space of T^*S^n along a tree for some $n \geq 3$. Then, the Lagrangian Floer homology between $\phi(S)$ and L defines a matrix M_ϕ such that the logarithm of the spectral radius of M_ϕ is the categorical entropy of ϕ.*

This fact gives us infinitely many examples of functors having positive entropies.

In the last section, we offer an application and an example. More specifically, we give a counterexample of *Gromov–Yomdin type* equality in Section 6.1. Section 6.2 provides an application of Theorem 1.4 on some spaces different from the plumbing spaces.

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2. Twisted complex formulation of the categorical entropy

In Section 2, we recall the notion of categorical entropy by [DHKK14], and redefine it using twisted complexes in the setting of A_∞ and dg categories. While the original approach uses a single split-generator of a category, we show that the definition extends to multiple split-generators. See Proposition 2.13. It has some practical advantages; see Sections 3, 4, or 5, for example. We will also compare the categorical entropies of functors with their opposites/inverses in Proposition 2.14.

First, we set the notation. Let $\mathscr{C}$ be a (strictly unital or cohomologically unital) A_∞ or dg category, which is $\mathbb{Z}$-graded (or $\mathbb{Z}/2$-graded) and k-linear for some coefficient field k. We write $\hom^*_{\mathscr{C}}(A,B)$ for the morphism complex of the objects $A, B \in \mathscr{C}$ with the differential d, and $\mathrm{Hom}^*_{\mathscr{C}}(A,B)$ for the cohomology of $\hom^*_{\mathscr{C}}(A,B)$. Equivalences of A_∞/dg categories are quasi-equivalences, which we denote by $\simeq$. We will also use the symbol $\simeq$ when two objects (resp. morphisms) are homotopy equivalent (resp. homotopic). We write $A[n]$ for the n-shift of an object $A \in \mathscr{C}$, which can be thought as $k[n] \otimes A$. We write $\{G_1, \dots, G_m\}_{\mathscr{C}}$ (or $\{G_1, \dots, G_m\}$ if the category $\mathscr{C}$ is clear from the context) for the full A_∞/dg subcategory of $\mathscr{C}$ consisting $G_1, \dots, G_m \in \mathscr{C}$. The reader can refer to [Sei08b] for more details.

Now, we recall the notion of twisted complex.

Definition 2.1. *Let $\mathscr{C}$ be a dg category.*

(1) *A* twisted complex $\tilde{K} = \left[(K_i[d_i])_{i=1}^n, (f_{ij})_{1\leq j<i\leq n}\right]$ in $\mathscr{C}$ *consists of objects* $K_i \in \mathscr{C}$*, shifts* $d_i \in \mathbb{Z}$*, and degree* 1 *morphisms*

$$f_{ij}\colon K_j[d_j] \to K_i[d_i]$$

for $i > j$*, satisfying*

$$df_{ij} + \sum_{i>l>j} f_{il} \circ f_{lj} = 0. \tag{2.1}$$

If we define

$$K := K_1[d_1] \oplus K_2[d_2] \oplus \ldots \oplus K_n[d_n],$$

and the strictly lower triangular degree 1 matrix

$$f := (f_{ij})\colon K \to K,$$

then Equation 2.1 becomes

$$df + f \circ f = 0.$$

(2) *A* (homogeneous) morphism $\tilde{\alpha}\colon [(K_i[d_i]), (f_{ij})] \to [(L_i[e_i]), (g_{ij})]$ *of twisted complexes is a (homogeneous) morphism* $\alpha\colon K \to L$ *with the grading* $|\tilde{\alpha}| := |\alpha|$ *and the differential*

$$d\tilde{\alpha} := d\alpha + g \circ \alpha - (-1)^{|\alpha|}\alpha \circ f.$$

Hence, the morphism space between the twisted complexes $\tilde{K} := [(K_i[d_i]), (f_{ij})]$ *and* $\tilde{L} := [(L_i[e_i]), (g_{ij})]$ *is given by*

$$\hom^*(\tilde{K}, \tilde{L}) = \bigoplus_{i,j} \hom^*(K_i, L_j)[e_j - d_i] \tag{2.2}$$

as a graded vector space, equipped with the above differential.

(3) *The* d-shift *of* $[(K_i[d_i]), (f_{ij})]$ *is* $[(K_i[d_i + d]), (f_{ij}[d])]$.

(4) *The* cone of a closed degree zero morphism $\tilde{\alpha}$*, denoted by* $\mathrm{Cone}(\tilde{\alpha})$*, is the twisted complex*

$$\left[(K_i[d_i+1], L_j[e_j]), \begin{pmatrix} f[1] & 0 \\ -\alpha & g \end{pmatrix}\right].$$

(5) *A closed degree zero morphism* $\tilde{\alpha}$ *is a* homotopy equivalence *if* $\mathrm{Cone}(\tilde{\alpha}) \simeq 0$*, i.e.,* $1_{\mathrm{Cone}(\tilde{\alpha})} = d\tilde{h}$ *for some* $\tilde{h}$.

(6) *We write* $\mathrm{Tw}(\mathscr{C})$ *for* the dg category of twisted complexes in $\mathscr{C}$.

(7) *For any* $\tilde{L} \in \mathrm{Tw}(\mathscr{C})$*, we call* $\tilde{K} = [(K_i[d_i]), (f_{ij})]$ a twisted complex for $\tilde{L}$ *if* $\tilde{K}$ *and* $\tilde{L}$ *are homotopy equivalent in* $\mathrm{Tw}(\mathscr{C})$.

Remark 2.2. *If* $\mathscr{C}$ *is an* A_∞*-category, we can similarly define twisted complexes in* $\mathscr{C}$ *and* $\mathrm{Tw}(\mathscr{C})$*. However, Equation 2.1 becomes*

$$\sum_{r=1}^{\infty} \mu^r_{\Sigma(\mathscr{C})}(f, \ldots, f) = 0$$

where $\Sigma(\mathscr{C})$ *is the additive enlargement of* $\mathscr{C}$*, which is obtained by enlarging* $\mathscr{C}$ *by finite direct sums and shifts, and* $\mu^r_{\Sigma(\mathscr{C})}$ *are* A_∞*-operations of* $\Sigma(\mathscr{C})$*. See* [Sei08b] *for more details.*

Remark 2.3. *We will mostly use uppercase letters with tilde for twisted complexes, and lowercase letters with tilde for the morphisms between twisted complexes. When we write* $\tilde{K} \in \mathrm{Tw}(\mathscr{C})$*, it will be understood that* $\tilde{K}$ *is an explicit twisted complex given by*

$$\tilde{K} = \left[(K_i[d_i])_{i=1}^n, (f_{ij})_{1\leq j<i\leq n}\right]$$

for some $K_i \in \mathscr{C}$*,* $d_i \in \mathbb{Z}$*, and degree 1 morphisms* $f_{ij}\colon K_j \to K_i$ *for* $i > j$ *satisfying (2.1). Note that* $\tilde{K}$ *contains shifts* d_i*'s as extra information.*

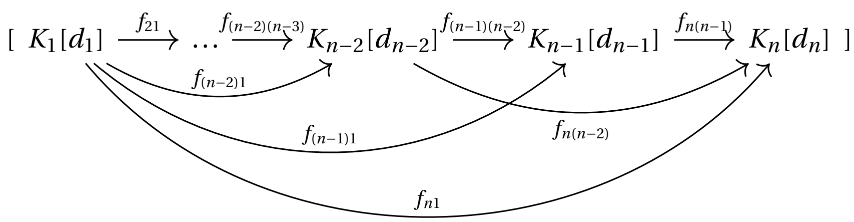
Remark 2.4.

(1) Another notation for the twisted complex

$$\tilde{K} = \left[(K_i[d_i])_{i=1}^n, (f_{ij})_{1 \le j < i \le n} \right]$$

is

(2.3)



(2) A twisted complex $\tilde{K} \in \mathrm{Tw}(\mathscr{C})$ can be seen as an iterated cone in $\mathrm{Tw}(\mathscr{C})$. Order of taking cone can be chosen freely after properly shifting K_i's. One example of iterated cones corresponding to $\tilde{K}$ in (2.3) is

$$\mathrm{Cone}(K_1[d_1 - 1] \to \ldots \to \mathrm{Cone}(K_{n-2}[d_{n-2} - 1] \xrightarrow{(f_{(n-1)(n-2)}\ f_{n(n-2)})} \mathrm{Cone}(K_{n-1}[d_{n-1} - 1] \xrightarrow{f_{n(n-1)}} K_n[d_n]))).$$

(3) A twisted complex $\tilde{K} \in \mathrm{Tw}(\mathscr{C})$ can be expressed as

$$\tilde{K} = \left[(\tilde{K}_i[d_i])_{i=1}^n, (\tilde{f}_{ij})_{1 \le j < i \le n} \right]$$

where $\tilde{K}_i \in \mathrm{Tw}(\mathscr{C})$, and the morphisms $\tilde{f}_{ij} \colon \tilde{K}_j[d_j] \to \tilde{K}_i[d_i]$ for $i > j$ satisfy

$$d\tilde{f}_{ij} + \sum_{i > l > j} \tilde{f}_{il} \circ \tilde{f}_{lj} = 0.$$

In this case, we regard this twisted complex as in (2.3), where each K_i is replaced by the actual twisted complex $\tilde{K}_i = [(L_p^i[e_p^i]), (g_{pq}^i)] \in \mathrm{Tw}(\mathscr{C})$, and each f_{ij} is replaced by the arrows between twisted complexes coming from $\tilde{f}_{ij}$.

Remark 2.5. *Given a twisted complex*

$$\tilde{K} = \left[(K_i[d_i])_{i=1}^n, (f_{ij})_{1 \le j < i \le n} \right] \in \mathrm{Tw}(\mathscr{C}),$$

its (essential) image under an A_∞ (or dg) functor $\Phi \colon \mathrm{Tw}(\mathscr{C}) \to \mathrm{Tw}(\mathscr{D})$ is given by

$$\Phi(\tilde{K}) = \left[(\Phi(K_i)[d_i])_{i=1}^n, (\tilde{g}_{ij})_{1 \le j < i \le n} \right] \in \mathrm{Tw}(\mathscr{D}),$$

for some $\tilde{g}_{ij} \colon \Phi(K_j)[d_j] \to \Phi(K_i)[d_i]$, where $\Phi(\tilde{K})$ is regarded as in Remark 2.4(3). See [Sei08b, Section (3m)] *for more details. This will be useful later when we study twisted complexes under the repetitive application of an A_∞/dg functor.*

Next, we will define the length of a twisted complex, which will be used in the definition of categorical entropy.

Definition 2.6. *Let $\mathscr{C}$ be an A_∞ (or dg) category. Consider a twisted complex*

$$\tilde{K} = \left[(K_i[d_i])_{i=1}^n, (f_{ij})_{1 \le j < i \le n} \right] \in \mathrm{Tw}(\mathscr{C}).$$

(1) Components of $\tilde{K}$ *is the collection*

$$\{K_i[d_i] \mid i = 1, \ldots, n\}.$$

Given a full dg subcategory $\mathscr{D}$ of $\mathscr{C}$, $\mathscr{D}$-components of $\tilde{K}$ *is the collection*

$$\{K_i[d_i] \mid K_i \in \mathscr{D}\}.$$

(2) *For a given $t \in \mathbb{R}$,* the length of $\tilde{K}$ (at t) *is defined as*

$$\mathrm{len}_t \tilde{K} := \sum_{i=1}^{n} e^{d_i t}.$$

If $t = 0$, then the length of $\tilde{K}$ is just n. Given full dg subcategory $\mathscr{D}$ of $\mathscr{C}$, the length of $\tilde{K}$ (at t) with respect to $\mathscr{D}$ *is defined as*

$$\mathrm{len}_{t,\mathscr{D}} \tilde{K} := \sum_{K_i \in \mathscr{D}} e^{d_i t}.$$

When $\mathscr{D}$ consists of a single object D, we will write D-components of $\tilde{K}$ (resp. $\mathrm{len}_{t,D}\tilde{K}$*) for $\mathscr{D}$-components of $\tilde{K}$ (resp.* $\mathrm{len}_{t,\mathscr{D}}\tilde{K}$*) in short. When $t = 0$, we will sometimes just write* $\mathrm{len}\,\tilde{K}$ *(resp.* $\mathrm{len}_{\mathscr{D}}\tilde{K}$*) for* $\mathrm{len}_t\tilde{K}$ *(resp.* $\mathrm{len}_{t,\mathscr{D}}\tilde{K}$*).*

Remark 2.7. *If $\mathscr{C}$ is $\mathbb{Z}/2$-graded, the length of a twisted complex only makes sense if $t = 0$. The same will be true for the complexity (Definition 2.9) and the categorical entropy (Definition 2.11).*

Now, we define the notion of (split-closed) pretriangulated A_∞/dg categories and (split-)generators, in order to define the *categorical entropy* later.

Definition 2.8. *Let $\mathscr{C}$ be an A_∞ (or dg) category.*

(1) *An A_∞ (or dg) category $\mathscr{C}$ is* pretriangulated*, if there is a quasi-equivalence $\mathscr{C} \simeq \mathrm{Tw}(\mathscr{C})$. Hence,* $\mathrm{Tw}(\mathscr{C})$ *is also called* the pretriangulated closure *of $\mathscr{C}$.*
(2) *We write* $\mathrm{Perf}(\mathscr{C})$ *for* the split-closed pretriangulated closure *of $\mathscr{C}$, which is obtained by splitting direct summands in* $\mathrm{Tw}(\mathscr{C})$.
(3) *We say $G_1, \ldots, G_m \in \mathscr{C}$* generate *(resp.* split-generate) $\mathscr{C}$ if

$$\mathscr{C} \subset \mathrm{Tw}(\{G_1, \ldots, G_m\}) \ \left(\text{resp.} \mathscr{C} \subset \mathrm{Perf}(\{G_1, \ldots, G_m\})\right).$$

We call $G_1, \ldots, G_m$ *generators (resp.* split-generators) of $\mathscr{C}$.

Note that the length of a twisted complex is not invariant under homotopy equivalence, but the following definition is an invariant.

Definition 2.9. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $G_1, \ldots, G_m, C \in \mathscr{C}$. Given $t \in \mathbb{R}$, the* complexity of C (at t) with respect to $G_1, \ldots, G_m$ *is defined as*

$$\delta_t(G_1, \ldots, G_m; C) := \inf \{\mathrm{len}_t \tilde{K} \mid \tilde{K} \in \mathrm{Tw}(\{G_1, \ldots, G_m\}) \textit{ and } \tilde{K} \simeq C \oplus C' \textit{ for some } C' \in \mathscr{C}\}.$$

Note that if C is not split-generated by $G_1, \ldots, G_m$, then $\delta_t(G_1, \ldots, G_m; C) = \infty$. Also, when $t = 0$, the complexity is zero if and only if $C \simeq 0$.

Remark 2.10. *The complexity $\delta_t(G; C)$ in* [DHKK14] *is defined on a triangulated category $\mathscr{T}$ using a (right) Postnikov system*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C_{n-1} & \rightarrow \ \ldots \ \rightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C \oplus C' \\
 & {\scriptstyle +1}\nwarrow \ \swarrow & & & & {\scriptstyle +1}\nwarrow \ \swarrow & & {\scriptstyle +1}\nwarrow \ \swarrow & \\
 & G[d_n] & & & & G[d_2] & & G[d_1] &
\end{array}$$

rather than a twisted complex. In the A_∞/dg enhancement $\mathscr{C}$ of $\mathscr{T}$ (i.e., $H^0(\mathscr{C}) \simeq \mathscr{T}$), any such system can be seen as an iterated cone of $G[d_i]$'s, which provides a twisted complex quasi-isomorphic to $C \oplus C'$ as in Remark 2.4(2) *(after setting $K_i[d_i] = G[d_i]$), and vice versa.*

We will now present the definition of categorical entropy for functors on A_∞/dg categories, which is originally given in [DHKK14] on triangulated categories.

Definition 2.11. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category split-generated by a single object G, and $\Phi\colon \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor. Then, for a given $t \in \mathbb{R}$,* the categorical entropy of Φ *is defined as*

$$h_t(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G; \Phi^n(G)) \in \{-\infty\} \cup \mathbb{R}.$$

The categorical entropy of Φ is well-defined and independent of the choice of the split-generator G of $\mathscr{C}$, see [DHKK14]. We also note that $h_t(\Phi) = -\infty$ if $\Phi^n(G) \simeq 0$ for some n.

Remark 2.12.

(1) *We can also define the categorical entropy for an A_∞/dg functor $\Phi\colon \mathscr{C} \to \mathscr{C}$ on any A_∞/dg category $\mathscr{C}$ by considering its pretriangulated closure $\mathrm{Tw}(\mathscr{C})$ and the induced functor $\Phi\colon \mathrm{Tw}(\mathscr{C}) \to \mathrm{Tw}(\mathscr{C})$.*

(2) *Since the notion of categorical entropy, Definition 2.11, is defined for A_∞/dg functors on A_∞/dg categories with a split-generator, we assume that every A_∞/dg category admits a split-generator throughout this paper in the context of categorical entropy.*

(3) *As we remarked earlier, if $\mathscr{C}$ is $\mathbb{Z}/2$-graded, the categorical entropy only makes sense when $t = 0$. However, we can consider $\mathscr{C}$ as a 2-periodic $\mathbb{Z}$-graded category to be able to talk about the categorical entropy when $t \neq 0$, in which case, $h_{t\neq 0}(\Phi) = -\infty$ for any Φ.*

Definition 2.11 defines the categorical entropy using the length with respect to a split-generator. The following proposition shows that one can also compute the categorical entropy from the length with respect to a finite collection of split-generators.

Proposition 2.13. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category split-generated by $G_1, \dots, G_m$, and define $G := G_1 \oplus \dots \oplus G_m$. Let $\Phi\colon \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor. Then, for a given $t \in \mathbb{R}$, the categorical entropy of Φ can be given by*

$$h_t(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G_1, \dots, G_m; \Phi^n(G)).$$

Proof. First, note that G is a split-generator for $\mathscr{C}$. Let $\tilde{K} \in \mathrm{Tw}(\{G_1, \dots, G_m\})$ be an arbitrary twisted complex for $\Phi^n(G) \oplus A$ for some $A \in \mathscr{C}$. Then one can replace each G_i in $\tilde{K}$ with G to get a twisted complex $\tilde{K}' \in \mathrm{Tw}(\{G\})$ for $\Phi^n(G) \oplus A \oplus B$ for some $B \in \mathscr{C}$. Note that

$$\mathrm{len}_t \tilde{K}' = \mathrm{len}_t \tilde{K}.$$

This shows

$$\delta_t(G; \Phi^n(G)) \le \delta_t(G_1, \dots, G_m; \Phi^n(G)).$$

On the other hand, let $\tilde{L} \in \mathrm{Tw}(\{G\})$ be an arbitrary twisted complex for $\Phi^n(G) \oplus C$ for some $C \in \mathscr{C}$. Then if we replace each G in $\tilde{L}$ with the equivalent twisted complex $[G_1 \xrightarrow{0} \dots \xrightarrow{0} G_m]$, we get a twisted complex $\tilde{L}' \in \mathrm{Tw}(\{G_1, \dots, G_m\})$ for $\Phi^n(G) \oplus C$. Note that

$$\mathrm{len}_t \tilde{L}' = m \times \mathrm{len}_t \tilde{L}.$$

This shows

$$\delta_t(G_1, \dots, G_m; \Phi^n(G)) \le m \delta_t(G; \Phi^n(G)).$$

By applying $\lim_{n\to\infty} \frac{1}{n} \log$ to the inequalities

$$\delta_t(G; \Phi^n(G)) \le \delta_t(G_1, \dots, G_m; \Phi^n(G)) \le m \delta_t(G; \Phi^n(G)),$$

we get

$$h_t(\Phi) \le \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G_1, \dots, G_m; \Phi^n(G)) \le h_t(\Phi).$$

□

We finish this section by the following proposition regarding an opposite and an inverse functor.

Proposition 2.14. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\Phi\colon \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor. For any $t \in \mathbb{R}$, the following hold;*

(1) $h_t(\Phi) = h_{-t}(\Phi^{\mathrm{op}})$, where Φ^{op} is an induced functor on $\mathscr{C}^{\mathrm{op}}$.
(2) If Φ is an auto-equivalence, and $\mathscr{C}$ is saturated (i.e., is smooth, proper, and admits a split-generator), then $h_t(\Phi^{-1}) = h_{-t}(\Phi)$.

Proof. Let G be a split-generator of $\mathscr{C}$. If

$$\tilde{K} = \left[(G[d_1], \ldots, G[d_n]), (f_{ij})\right] \in \mathrm{Tw}(\{G\}_{\mathscr{C}})$$

is a twisted complex for $\Phi^n(G) \oplus C$ for some $C \in \mathscr{C}$, then

$$\tilde{K}' = \left[(G^{\mathrm{op}}[-d_n], \ldots, G^{\mathrm{op}}[-d_1]), ((f_{n+1-j,n+1-i})^{\mathrm{op}})\right] \in \mathrm{Tw}(\{G^{\mathrm{op}}\}_{\mathscr{C}^{\mathrm{op}}})$$

is a twisted complex for $(\Phi^n(G) \oplus C)^{\mathrm{op}} = (\Phi^{\mathrm{op}})^n(G^{\mathrm{op}}) \oplus C^{\mathrm{op}}$, where $(G[d_i])^{\mathrm{op}}$ is viewed as the $(-d_i)$-shift of G^{op} in $\mathscr{C}^{\mathrm{op}}$. Since G^{op} split-generates $\mathscr{C}^{\mathrm{op}}$, this implies the first statement.

Next, assume $\mathscr{C}$ is saturated. Then it is known by [DHKK14] that

$$h_t(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_k \left(\dim(\mathrm{Hom}^k_{\mathscr{C}}(G, \Phi^n(G)))\right) e^{-kt}.$$

Since $\mathscr{C}^{\mathrm{op}}$ is also saturated, we have

$$\begin{aligned} h_{-t}(\Phi) = h_t(\Phi^{\mathrm{op}}) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_k \left(\dim(\mathrm{Hom}^k_{\mathscr{C}^{\mathrm{op}}}(G^{\mathrm{op}}, (\Phi^{\mathrm{op}})^n(G^{\mathrm{op}})))\right) e^{-kt} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_k \left(\dim(\mathrm{Hom}^k_{\mathscr{C}}(\Phi^n(G), G))\right) e^{-kt} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_k \left(\dim(\mathrm{Hom}^k_{\mathscr{C}}(G, (\Phi^{-1})^n(G)))\right) e^{-kt} = h_t(\Phi^{-1}). \end{aligned}$$

□

3. Categorical entropy of functors descended to quotients

The main theorem of Section 3, Theorem 3.10, is a comparison of categorical entropies of A_∞/dg functors on an A_∞/dg category, its full A_∞/dg subcategory, and their quotient. The motivation of Theorem 3.10 comes from the fact that the wrapped Fukaya category is a quotient of any partially wrapped Fukaya category by linking disks. One can find another motivation from some properties of topological entropy, which we explain in Section 3.4.

In Section 3.1, we prepare for stating and proving Theorem 3.10. In Section 3.2, we prove Theorem 3.10. In Section 3.3, we investigate the case of categories with admissible subcategories and get a comparison result Corollary 3.15, which follows from Theorem 3.10. In Section 3.5, we apply Theorem 3.10 to symplectic topology and compare the categorical entropies of functors on wrapped and partially wrapped Fukaya categories in Theorem 3.20.

Remark 3.1. *We will be mostly working with dg categories and dg functors, but the results and proofs will apply to A_∞-categories and A_∞-functors because of the following argument: Any A_∞-category is quasi-equivalent to a dg category by Yoneda embedding. For any dg categories $\mathscr{C}$ and $\mathscr{D}$, we also have (see* [Fao17] *and* [COS19]*)*

$$\mathrm{Hom}_{A_\infty}(\mathscr{C}, \mathscr{D}) \simeq \mathrm{RHom}_{\mathrm{dg}}(\mathscr{C}, \mathscr{D}) \simeq \mathrm{Hom}_{\mathrm{dg}}(\mathscr{C}', \mathscr{D})$$

where $\mathrm{Hom}_{A_\infty}(\mathscr{C}, \mathscr{D})$ (resp. $\mathrm{Hom}_{\mathrm{dg}}(\mathscr{C}, \mathscr{D})$) is the space of A_∞ (resp. dg) functors $\mathscr{C} \to \mathscr{D}$, $\mathrm{RHom}_{\mathrm{dg}}(\mathscr{C}, \mathscr{D})$ is the derived hom-space, and $\mathscr{C}'$ is a cofibrant dg category quasi-equivalent to $\mathscr{C}$. This means that any A_∞-functor between A_∞-categories can be considered as a dg functor between dg categories.

3.1. **Quotient of an A_∞/dg category.** First, we define the notion of the dg quotient of a dg category.

Definition 3.2 ([Dri04])**.** *Let $\mathscr{C}$ be a dg category.*

(1) If $\mathscr{D}$ is a full dg subcategory of $\mathscr{C}$, then the dg quotient *$\mathscr{C}/\mathscr{D}$ is obtained from $\mathscr{C}$ by adding a degree -1 morphism $\varepsilon_D\colon D \to D$ for each $D \in \mathscr{D}$ freely (in algebra level), such that*

$$d\varepsilon_D = 1_D.$$

(2) The dg localisation functor

$$l\colon \mathscr{C} \to \mathscr{C}/\mathscr{D}$$

is the dg functor sending everything to itself.

It is easy to check that $D \simeq 0$ in $\mathscr{C}/\mathscr{D}$ for all $D \in \mathscr{D} \subset \mathscr{C}$.

Remark 3.3. *We can define the quotient of an A_∞-category and the A_∞-localisation functor using the dg quotient and dg localisation functor via Remark 3.1. For a more direct approach, see* [LO06].

Next proposition shows that the quotient of a pretriangulated A_∞/dg category is again pretriangulated.

Proposition 3.4. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\mathscr{D}$ be its full A_∞ (or dg) subcategory. Then the quotient $\mathscr{C}/\mathscr{D}$ is also pretriangulated, and quasi-equivalent to $\mathscr{C}/\mathrm{Tw}(\mathscr{D})$.*

Proof. Assume $\mathscr{C}$ and $\mathscr{D}$ are dg categories for simplicity as in Remark 3.1. Observe that $\mathscr{C}/\mathrm{Tw}(\mathscr{D})$ is the dg quotient of $\mathscr{C}/\mathscr{D}$ by $\mathrm{Tw}(\mathscr{D})$. In $\mathscr{C}/\mathscr{D}$, we have $D \simeq 0$ for all $D \in \mathscr{D} \subset \mathscr{C}$. Therefore, $E \simeq 0$ for all $E \in \mathrm{Tw}(\mathscr{D}) \subset \mathscr{C}$. Hence, taking quotient by $\mathrm{Tw}(\mathscr{D})$ does not have any effect, and we get

$$\mathscr{C}/\mathscr{D} \simeq \mathscr{C}/\mathrm{Tw}(\mathscr{D}) \simeq \mathrm{Tw}(\mathscr{C}/\mathscr{D}),$$

where the second equivalence is by [Dri04]. This in particular shows that $\mathscr{C}/\mathscr{D}$ is pretriangulated. □

In Section 3, we compare the categorical entropies of functors $\Phi_{\mathscr{C}}, \Phi_{\mathscr{D}}, \Phi_{\mathscr{C}/\mathscr{D}}$ on $\mathscr{C}, \mathscr{D}$ and $\mathscr{C}/\mathscr{D}$, respectively, described in the following definition.

Definition 3.5. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Let $\Phi_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor satisfying $\Phi_{\mathscr{C}}(\mathscr{D}) \subset \mathscr{D}$. Then we define two functors:*

(1) $\Phi_{\mathscr{D}}\colon \mathscr{D} \to \mathscr{D}$ is the restriction of $\Phi_{\mathscr{C}}$ on $\mathscr{D}$, i.e.,

$$\Phi_{\mathscr{D}} := \Phi_{\mathscr{C}}|_{\mathscr{D}}.$$

(2) $\Phi_{\mathscr{C}/\mathscr{D}}\colon \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}$ is the unique (up to isomorphism) A_∞-functor satisfying

$$\Phi_{\mathscr{C}/\mathscr{D}} \circ l \simeq l \circ \Phi_{\mathscr{C}} \tag{3.1}$$

where $l\colon \mathscr{C} \to \mathscr{C}/\mathscr{D}$ is the localisation functor.

Remark 3.6. *The existence and uniqueness of $\Phi_{\mathscr{C}/\mathscr{D}}$ follow from the universal property of A_∞/dg localisation (see e.g.* [Toe07]*): Consider the A_∞/dg functor $l \circ \Phi_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}/\mathscr{D}$. Since $(l \circ \Phi_{\mathscr{C}})(D) \simeq 0$ for all $D \in \mathscr{D}$, by the universal property, there exists a unique A_∞-functor $\Phi_{\mathscr{C}/\mathscr{D}}$ (not necessarily a dg functor) up to isomorphism satisfying (3.1).*

The following proposition about the split-generation of quotients will be useful when comparing categorical entropies in the next subsection.

Proposition 3.7. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Assume that $G \oplus D$ split generates $\mathscr{C}$, and D split generates $\mathscr{D}$ for some $G \in \mathscr{C}$ and $D \in \mathscr{D}$. Then G split-generates $\mathscr{C}/\mathscr{D}$.*

Proof. Let $C \in \mathscr{C}/\mathscr{D}$. Then we have $C \in \mathscr{C}$, as $\mathscr{C}$ and $\mathscr{C}/\mathscr{D}$ have the same objects. Hence, there exists $C' \in \mathscr{C}$ such that $C \oplus C' \in \mathrm{Tw}(\{G \oplus D\}_{\mathscr{C}})$. Then, by applying the localisation functor l to $C \oplus C' \in \mathscr{C}$, we get

$$l(C \oplus C') = C \oplus C' \in \mathrm{Tw}(\{l(G \oplus D)\}_{\mathscr{C}/\mathscr{D}}) = \mathrm{Tw}(\{G \oplus D\}_{\mathscr{C}/\mathscr{D}}) \simeq \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$$

since $D \simeq 0$ in $\mathscr{C}/\mathscr{D}$, where the second C and C' are in $\mathscr{C}/\mathscr{D}$. This proves that G split-generates $\mathscr{C}/\mathscr{D}$. □

The following lemma will be the main ingredient of Theorem 3.10.

Lemma 3.8. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Assume that $G \oplus D$ split-generates $\mathscr{C}$, and D split-generates $\mathscr{D}$ for some $G \in \mathscr{C}$ and $D \in \mathscr{D}$. Let $X \in \mathscr{C}$, and $\tilde{L} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ be a twisted complex for $l(X) \oplus B$ for some $B \in \mathscr{C}/\mathscr{D}$, i.e.,*

$$\tilde{L} \simeq l(X) \oplus B$$

where $l\colon \mathscr{C} \to \mathscr{C}/\mathscr{D}$ is the localisation functor. Then there exist a twisted complex $\tilde{K} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ for $X \oplus A$ for some $A \in \mathscr{C}$, i.e.,

$$\tilde{K} \simeq X \oplus A$$

such that the G-components of $\tilde{K}$ are the same as the G-components of $\tilde{L}$. In particular,

$$\mathrm{len}_{t,G}\tilde{K} = \mathrm{len}_t \tilde{L}.$$

Proof. Assume that $\mathscr{C}$ and $\mathscr{D}$ are dg categories for simplicity following Remark 3.1. Before we start the proof, note that for any $X, Y \in \mathscr{C}$, we have the isomorphism by [Ver77, Ver96] (which is restated in [Dri04])

$$l\colon \varinjlim_{(Y \to Z) \in Q_Y} \mathrm{Hom}^*_{\mathscr{C}}(X, Z) \xrightarrow{\sim} \mathrm{Hom}^*_{\mathscr{C}/\mathscr{D}}(l(X), l(Y)), \tag{3.2}$$

where Q_Y is the (filtered) category of morphisms $f\colon Y \to Z$ in $\mathscr{C}$ such that $\mathrm{Cone}(f)$ is homotopy equivalent to an object in $\mathscr{D}$. This shows, for any $\beta \in \mathrm{Hom}^*_{\mathscr{C}/\mathscr{D}}(l(X), l(Y))$, there exists $Z \in \mathscr{C}$ such that

(i) $Z \simeq \mathrm{Cone}(E \to Y)$ for some $E \in \mathscr{D}$, and
(ii) there is a morphism $\alpha \in \mathrm{Hom}^*_{\mathscr{C}}(X, Z)$ so that $l(\alpha) \simeq \beta$.

With the above arguments, we can prove the lemma by induction on the number of components of $\tilde{L} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$.

For the base step, let us assume that $\tilde{L}$ has no components, i.e., $\tilde{L} = 0$. Note that $\tilde{L} \simeq l(X) \oplus B \simeq l(X \oplus A_0)$ for some $A_0 \in \mathscr{C}$ since l is essentially surjective. Then, $X \oplus A_0 \in \mathscr{C}$ is in the kernel of l. It means that $X \oplus A_0 \oplus A_1$ is homotopy equivalent to an object in $\mathscr{D}$ for some $A_1 \in \mathscr{C}$ (see e.g. [Kra10]). Since D split-generates $\mathscr{D}$, there exists $A_2 \in \mathscr{D} \subset \mathscr{C}$ such that $X \oplus A_0 \oplus A_1 \oplus A_2$ is homotopy equivalent to an object $\tilde{K} \in \mathrm{Tw}(\{D\}_{\mathscr{C}})$, which has no G-components.

For the inductive step, let us assume the induction hypothesis, i.e., we assume that the lemma holds for any $X \in \mathscr{C}$ and $\tilde{L} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ satisfying $\tilde{L} \simeq l(X) \oplus B$ for some $B \in \mathscr{C}/\mathscr{D}$, whenever $\tilde{L}$ has n many G-components.

Now we consider $X \in \mathscr{C}$ and $\tilde{L} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ such that $\tilde{L} \simeq l(X) \oplus B$ for some $B \in \mathscr{C}/\mathscr{D}$, and $\tilde{L}$ has $(n+1)$ many G-components. We note that, as mentioned in Remark 2.4(2), the twisted complex $\tilde{L}$ can be written as an iterated cone. Thus, there is a twisted complex $\tilde{T} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ and $d \in \mathbb{Z}$ such that

$$\tilde{L} = \mathrm{Cone}\left(\tilde{T} \xrightarrow{\beta} G[d]\right), \tag{3.3}$$

where $\beta \in \mathrm{Hom}^0_{\mathscr{C}/\mathscr{D}}\left(\tilde{T}, G[d]\right)$, and $\tilde{T}$ has n many G-components. Since the localisation functor $l\colon \mathscr{C} \to \mathscr{C}/\mathscr{D}$ is essentially surjective, there exists $W \in \mathscr{C}$ such that $l(W) \simeq \tilde{T}$. Moreover, by the induction hypothesis, there exist $\tilde{S} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ and $A' \in \mathscr{C}$ such that $\tilde{S} \simeq W \oplus A'$, and $\tilde{S}$ and $\tilde{T}$ have the same G-components.

From the isomorphism (3.2), one can show that there is an object $Z_1 \in \mathscr{C}$ such that

(i)' $Z_1 \simeq \mathrm{Cone}(E_1 \to G[d])$ for some $E_1 \in \mathscr{D}$, and
(ii)' there is a morphism $\alpha \in \mathrm{Hom}^0_{\mathscr{C}}(W, Z_1)$ so that $l(\alpha) \simeq \beta$.

Now, we consider the cone of $\alpha : W \to Z_1$. Let

$$Y := \mathrm{Cone}\left(W \xrightarrow{\alpha} Z_1\right) \in \mathscr{C}.$$

Then, one can easily observe that Y is a lift of $\tilde{L}$ with respect to the localisation functor l, i.e., $l(Y) \simeq \tilde{L}$. Hence,

$$l(Y) \simeq l(X) \oplus B \simeq l(X \oplus A_0)$$

for some $A_0 \in \mathscr{C}$, since l is essentially surjective. This shows that there is a homotopy equivalence $\delta : l(X \oplus A_0) \to l(Y)$. Consequently, from (3.2), one obtains $Z_2 \in \mathscr{C}$ such that

(i)" $Z_2 \simeq \mathrm{Cone}(E_2 \to Y)$ for some $E_2 \in \mathscr{D}$, and
(ii)" there is a morphism $\gamma \in \mathrm{Hom}^0_{\mathscr{C}}(X \oplus A_0, Z_2)$ so that $l(\gamma) \simeq \delta$.

Let

$$E_3 := \mathrm{Cone}\left(X \oplus A_0 \xrightarrow{\gamma} Z_2\right) \in \mathscr{C}.$$

Then, since $l(\gamma) \simeq \delta$ is a homotopy equivalence,

$$l(E_3) = l\left(\mathrm{Cone}(\gamma)\right) \simeq \mathrm{Cone}(\delta) \simeq 0.$$

In other words, E_3 is a direct summand of an object in $\mathscr{D}$ (see e.g. [Kra10]), i.e., $E_3 \oplus A'' \in \mathscr{D}$ for some $A'' \in \mathscr{C}$.

Collecting all the above results, we get

$$\begin{aligned} X \oplus A_0 &\simeq \mathrm{Cone}\left(Z_2[-1] \to E_3[-1]\right) \\ &\simeq \mathrm{Cone}\left(\mathrm{Cone}(E_2 \to Y)[-1] \to E_3[-1]\right) \\ &\simeq \mathrm{Cone}\left(\mathrm{Cone}\left(E_2 \to \mathrm{Cone}(W \xrightarrow{\alpha} Z_1)\right)[-1] \to E_3[-1]\right) \\ &\simeq \mathrm{Cone}\left(\mathrm{Cone}\left(E_2 \to \mathrm{Cone}\left(W \xrightarrow{\alpha} \mathrm{Cone}(E_1 \to G[d])\right)\right)[-1] \to E_3[-1]\right). \end{aligned} \tag{3.4}$$

As mentioned in Remark 2.4(2), one can convert the iterated cone (3.4) to a twisted complex for $X \oplus A_0$

$$X \oplus A_0 \simeq [E_2[1] \to W[1] \to E_1[1] \to G[d] \to E_3[-1]].$$

Here, the notation for the twisted complex is as in Remark 2.4(1). Note that $E_1, E_2, E_3 \oplus A'' \in \mathscr{D}$, and since D split-generates $\mathscr{D}$, there exist $A_1, A_2, A_3 \in \mathscr{D} \subset \mathscr{C}$ such that

$$E_1 \oplus A_1, \quad E_2 \oplus A_2, \quad E_3 \oplus A'' \oplus A_3 \quad \in \mathrm{Tw}(\{D\}_{\mathscr{C}}).$$

Also, recall that $\tilde{S} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ satisfies $\tilde{S} \simeq W \oplus A'$ for some $A' \in \mathscr{C}$, and $\tilde{S}$ has the same G-components as $\tilde{T}$. Set $A := A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A' \oplus A'' \in \mathscr{C}$. Then, define $\tilde{K} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ by

$$\begin{aligned} X \oplus A &\simeq \left[(E_2 \oplus A_2)[1] \to (W \oplus A')[1] \to (E_1 \oplus A_1)[1] \to G[d] \to (E_3 \oplus A'' \oplus A_3)[-1]\right] \\ &\simeq \left[(E_2 \oplus A_2)[1] \to \tilde{S}[1] \to (E_1 \oplus A_1)[1] \to G[d] \to (E_3 \oplus A'' \oplus A_3)[-1]\right] := \tilde{K}. \end{aligned}$$

Hence, the G-components of $\tilde{K}$ consist of the G-components of $\tilde{S}[1]$ and $G[d]$. We also know that the G-components of $\tilde{L}$ consist of the G-components of $\tilde{T}[1]$ and $G[d]$ by (3.3). Since G-components of $\tilde{S}$ and $\tilde{T}$ are the same, this proves the induction step. □

3.2. **Comparison of categorical entropies of $\Phi_{\mathscr{C}}, \Phi_{\mathscr{D}}$, and $\Phi_{\mathscr{C}/\mathscr{D}}$.** Now, we are ready to compare the categorical entropies of an A_∞/dg functor $\Phi_{\mathscr{C}}$ on a pretriangulated A_∞/dg category $\mathscr{C}$, its restriction $\Phi_{\mathscr{D}}$ to a full pretriangulated A_∞/dg subcategory $\mathscr{D}$ of $\mathscr{C}$, and its descent $\Phi_{\mathscr{C}/\mathscr{D}}$ to the quotient $\mathscr{C}/\mathscr{D}$.

It is easy to prove the next proposition, which is the first result comparing the categorical entropies of $\Phi_{\mathscr{C}}$ and $\Phi_{\mathscr{C}/\mathscr{D}}$.

Proposition 3.9. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Let $\Phi_{\mathscr{C}}: \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor satisfying $\Phi_{\mathscr{C}}(\mathscr{D}) \subset \mathscr{D}$, and $\Phi_{\mathscr{C}/\mathscr{D}}: \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}$ be the induced functor as in Definition 3.5. Then, for any $t \in \mathbb{R}$, we have*

$$h_t(\Phi_{\mathscr{C}/\mathscr{D}}) \leq h_t(\Phi_{\mathscr{C}}).$$

Proof. Assume that $\overline{G} := G \oplus D$ split-generates $\mathscr{C}$, and D split-generates $\mathscr{D}$ for some $G \in \mathscr{C}$ and $D \in \mathscr{D}$. Then by Proposition 3.7, G split-generates $\mathscr{C}/\mathscr{D}$. For a fixed $n \in \mathbb{N}$, let $\tilde{K} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ be a twisted complex for $\Phi^n_{\mathscr{C}}(\overline{G}) \oplus C$ for some $C \in \mathscr{C}$. Applying the localisation functor l to $\tilde{K}$, and replacing D with 0, we get a twisted complex $\tilde{L} \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ for

$$l(\Phi^n_{\mathscr{C}}(\overline{G}) \oplus C) \simeq \Phi^n_{\mathscr{C}/\mathscr{D}}(l(\overline{G})) \oplus l(C) \simeq \Phi^n_{\mathscr{C}/\mathscr{D}}(G) \oplus l(C).$$

Moreover, the G-components of $\tilde{K}$ are preserved in the new twisted complex $\tilde{L}$. Thus,

$$\mathrm{len}_t \tilde{L} = \mathrm{len}_{t,G} \tilde{K} \leq \mathrm{len}_t \tilde{K}.$$

This shows that

$$\delta_t(G; \Phi^n_{\mathscr{C}/\mathscr{D}}(G)) \leq \delta_t(G, D; \Phi^n_{\mathscr{C}}(\overline{G})),$$

which implies that

$$h_t(\Phi_{\mathscr{C}/\mathscr{D}}) \leq h_t(\Phi_{\mathscr{C}})$$

by Proposition 2.13. □

Proposition 3.9 gives a lower bound for the entropy of $\Phi_{\mathscr{C}}$. The following, which is one of our main theorems, gives an upper bound.

Theorem 3.10. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Let $\Phi_{\mathscr{C}}: \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor satisfying $\Phi_{\mathscr{C}}(\mathscr{D}) \subset \mathscr{D}$, and let $\Phi_{\mathscr{D}}: \mathscr{D} \to \mathscr{D}$ and $\Phi_{\mathscr{C}/\mathscr{D}}: \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}$ be the induced functors as in Definition 3.5. Then, for any $t \in \mathbb{R}$, we have*

$$h_t(\Phi_{\mathscr{C}/\mathscr{D}}) \leq h_t(\Phi_{\mathscr{C}}) \leq \max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}), h_t(\Phi_{\mathscr{D}})\}.$$

The idea of the proof is roughly as follows: Given the setting of Proposition 3.7, where $\mathscr{C}/\mathscr{D}$ is split-generated by $G \in \mathscr{C}$, $\mathscr{D}$ is split-generated by $D \in \mathscr{D}$, and $\mathscr{C}$ is split-generated by $\overline{G} := G \oplus D$, the initial goal is to achieve the inequality

$$\delta_t(G, D; \Phi^n_{\mathscr{C}}(\overline{G})) \leq \delta_t(G; \Phi^n_{\mathscr{C}/\mathscr{D}}(G)) + \delta_t(D; \Phi^n_{\mathscr{D}}(D)) \tag{3.5}$$

for sufficiently large n, which would imply the inequality $h_t(\Phi_{\mathscr{C}}) \leq \max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}), h_t(\Phi_{\mathscr{D}})\}$.

An attempt would be to use Lemma 3.8: For any twised complex $\tilde{L}_n \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ for $\Phi^n_{\mathscr{C}/\mathscr{D}}(G) \oplus B_n$ for some $B_n \in \mathscr{C}/\mathscr{D}$, there is a twisted complex $\tilde{K}_n \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ for $\Phi^n_{\mathscr{C}}(\overline{G}) \oplus A_n$ for some $A_n \in \mathscr{C}$ such that $\mathrm{len}_{t,G} \tilde{K}_n = \mathrm{len}_t \tilde{L}_n$. Then we have

$$\delta_t(G, D; \Phi^n_{\mathscr{C}}(\overline{G})) \leq \mathrm{len}_t \tilde{K}_n = \mathrm{len}_{t,G} \tilde{K}_n + \mathrm{len}_{t,D} \tilde{K}_n = \mathrm{len}_t \tilde{L}_n + \mathrm{len}_{t,D} \tilde{K}_n.$$

We can choose $\tilde{L}_n$ so that $\mathrm{len}_t \tilde{L}_n$ is close to $\delta_t(G; \Phi^n_{\mathscr{C}/\mathscr{D}}(G))$. Assuming that $\mathrm{len}_{t,D} \tilde{K}_n$ approaches $R_n \in [0, \infty]$ as $\mathrm{len}_t \tilde{L}_n$ approaches $\delta_t(G; \Phi^n_{\mathscr{C}/\mathscr{D}}(G))$, we get

$$\delta_t(G, D; \Phi^n_{\mathscr{C}}(\overline{G})) \leq \delta_t(G; \Phi^n_{\mathscr{C}/\mathscr{D}}(G)) + R_n.$$

However, we have no control over R_n via Lemma 3.8. In particular, we cannot relate R_n to $\delta_t(D;\Phi^n_{\mathscr{D}}(D))$ using the lemma to achieve (3.5).

The solution of this problem is the observation that the following also gives the entropy:

$$\frac{1}{n}\log\left(\frac{\delta_t(G,D;\Phi^{2n}_{\mathscr{C}}(\overline{G}))}{\delta_t(G,D;\Phi^{n}_{\mathscr{C}}(\overline{G}))}\right)\xrightarrow{n\to\infty} h_t(\Phi_{\mathscr{C}}). \tag{3.6}$$

For that reason, we want to create a sequence of twisted complexes $\tilde{K}_i\in \mathrm{Tw}(\{G,D\}_{\mathscr{C}})$ for $\Phi^{2^iN}_{\mathscr{C}}(\overline{G})\oplus A_i$ for some $A_i\in\mathscr{C}$ such that approximately

$$\frac{\delta_t(G,D;\Phi^{2^{i+1}N}_{\mathscr{C}}(\overline{G}))}{\delta_t(G,D;\Phi^{2^{i}N}_{\mathscr{C}}(\overline{G}))}\le\frac{\mathrm{len}_t\tilde{K}_{i+1}}{\mathrm{len}_t\tilde{K}_{i}}\le \delta_t(G;\Phi^{2^iN}_{\mathscr{C}/\mathscr{D}}(G))+\delta_t(D;\Phi^{2^iN}_{\mathscr{D}}(D)) \tag{3.7}$$

holds for a large N. Similar to (3.5), this would also imply Theorem 3.10 by (3.6). The first inequality is approximately true for infinitely many i's by a basic fact about sequences (given in the following Lemma 3.11), since $\delta_t(G,D;\Phi^{2^iN}_{\mathscr{C}}(\overline{G}))\le \mathrm{len}_t\tilde{K}_i$.

To achieve the second inequality, we create the sequence $\tilde{K}_i\in\mathrm{Tw}(\{G,D\}_{\mathscr{C}})$ in a particular way: Define $\tilde{K}_0$ from $\tilde{L}_N\in\mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ by Lemma 3.8 as we mentioned above, so that $\mathrm{len}_{t,G}\tilde{K}_0=\mathrm{len}_t\tilde{L}_N$. Define $\tilde{K}_{i+1}$ from $\tilde{K}_i$ by applying $\Phi^{2^iN}_{\mathscr{C}}$ to $\tilde{K}_i$, and then by replacing each $\Phi^{2^iN}_{\mathscr{C}}(G)$ in $\tilde{K}_{i+1}$ by $\tilde{K}_i$ and each $\Phi^{2^iN}_{\mathscr{C}}(D)=\Phi^{2^iN}_{\mathscr{D}}(D)$ in $\tilde{K}_{i+1}$ by a twisted complex in $\{D\}_{\mathscr{D}}$ whose length is close to $\delta_t(D;\Phi^{2^iN}_{\mathscr{D}}(D))$. Note that $\mathrm{len}_{t,G}\tilde{K}_i=(\mathrm{len}_t\tilde{L}_N)^{2^i}$ holds. Hence, we approximately have

$$\frac{\mathrm{len}_t\tilde{K}_{i+1}}{\mathrm{len}_t\tilde{K}_i}=\frac{\mathrm{len}_{t,G}\tilde{K}_i\cdot\mathrm{len}_t\tilde{K}_i+\mathrm{len}_{t,D}\tilde{K}_i\cdot\delta_t(D;\Phi^{2^iN}_{\mathscr{D}}(D))}{\mathrm{len}_t\tilde{K}_i}\le(\mathrm{len}_t\tilde{L}_N)^{2^i}+\delta_t(D;\Phi^{2^iN}_{\mathscr{D}}(D)).$$

Since $(\mathrm{len}_t\tilde{L}_N)^{2^i}$ is approximately $\delta_t(G;\Phi^{2^iN}_{\mathscr{C}/\mathscr{D}}(G))$, this proves the second inequality in (3.7), and hence, Theorem 3.10. The rigorous proof of Theorem 3.10 will appear after Lemma 3.11.

Lemma 3.11. *Let $(a_i)_{i=0}^\infty$ and $(a'_i)_{i=0}^\infty$ be sequences of positive real numbers satisfying $a_i\le a'_i$ for all i. Then we have*

$$\liminf_{i\to\infty}\frac{a_{i+1}/a_i}{a'_{i+1}/a'_i}\le 1.$$

Proof. Let us assume the contrary. Then, there exist a real number $M>1$ and a natural number N such that

$$\frac{a_{i+1}/a_i}{a'_{i+1}/a'_i}\ge M$$

for all $i\ge N$. However, for any $i>N$, we have

$$\frac{a_N}{a'_N}\left(\prod_{j=N}^{i-1}\left(\frac{a_{j+1}/a_j}{a'_{j+1}/a'_j}\right)\right)=\frac{a_i}{a'_i}\le 1$$

which gives a contradiction since

$$\lim_{i\to\infty}\left(\prod_{j=N}^{i-1}\frac{a_{j+1}/a_j}{a'_{j+1}/a'_j}\right)=\infty.$$

□

Proof of Theorem 3.10. Assume that $\overline{G}:=G\oplus D$ split-generates $\mathscr{C}$, and D split-generates $\mathscr{D}$ for some $G\in\mathscr{C}$ and $D\in\mathscr{D}$. Then by Proposition 3.7, G split-generates $\mathscr{C}/\mathscr{D}$. The first inequality is Proposition 3.9, thus it is enough to prove the second inequality

$$h_t(\Phi_{\mathscr{C}})\le\max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}),h_t(\Phi_{\mathscr{D}})\}.$$

If $h_t(\Phi_{\mathscr{C}}) = -\infty$, then the above holds. Thus, let us assume that $h_t(\Phi_{\mathscr{C}}) > -\infty$.

We note that, in the rest of the proof, we use the following notation for convenience:

$$\alpha := h_t(\Phi_{\mathscr{C}}), \quad \beta := h_t(\Phi_{\mathscr{C}/\mathscr{D}}), \quad \gamma := h_t(\Phi_{\mathscr{D}}).$$

Moreover, we remark that we will be working with the twisted complexes $\tilde{K}_n \in \mathrm{Tw}(\{G,D\}_{\mathscr{C}})$, $\tilde{L}_n \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$, and $\tilde{M}_n \in \mathrm{Tw}(\{D\}_{\mathscr{D}})$ in the rest of the proof.

Step 1: We will relate the complexity $\delta_t(G,D;\Phi^n_{\mathscr{C}}(\overline{G}))$ to α.

Let ϵ be a fixed positive real number. Then, by Proposition 2.13, there is a natural number N_1 such that for all $n \geq N_1$, the following holds:

$$\left|\frac{1}{n}\log\delta_t(G,D;\Phi^n_{\mathscr{C}}(\overline{G})) - \alpha\right| < \epsilon.$$

Thus, for any $n \geq N_1$, the following holds:

$$e^{n(\alpha-\epsilon)} < \delta_t(G,D;\Phi^n_{\mathscr{C}}(\overline{G})) < e^{n(\alpha+\epsilon)}. \tag{3.8}$$

The second step is to choose twisted complexes $\tilde{L}_n \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ and $\tilde{M}_n \in \mathrm{Tw}(\{D\}_{\mathscr{D}})$, whose lengths are related to β and γ respectively. For the choices, we consider three different cases, the first case is $\beta > -\infty, \gamma > -\infty$, the second case is $\beta = -\infty, \gamma = -\infty$, and the third case is that one of $\{\beta,\gamma\}$ is $-\infty$ and the other is not. We will consider the first case in steps 2 – 4, the second case in step 5, and the third case in step 6.

Step 2: As mentioned above, we assume that $\beta > -\infty, \gamma > -\infty$. Then, for the fixed $\epsilon > 0$ in step 1, there is a natural number N_2 (resp. N_3) such that

$$\left|\frac{1}{n}\log\delta_t(G;\Phi^n_{\mathscr{C}/\mathscr{D}}(G)) - \beta\right| < \epsilon, \text{ for all } n \geq N_2,$$
$$\left|\frac{1}{n}\log\delta_t(D;\Phi^n_{\mathscr{D}}(D)) - \gamma\right| < \epsilon, \text{ for all } n \geq N_3.$$

Moreover, for any $n \geq N_2$ (resp. $n \geq N_3$), there is a twisted complex $\tilde{L}_n \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ (resp. $\tilde{M}_n \in \mathrm{Tw}(\{D\}_{\mathscr{D}})$) such that

- $\tilde{L}_n$ is a twisted complex for $\Phi^n_{\mathscr{C}/\mathscr{D}}(G) \oplus B_n$ for some $B_n \in \mathscr{C}/\mathscr{D}$ (resp. $\tilde{M}_n$ is a twisted complex for $\Phi^n_{\mathscr{D}}(D) \oplus C_n$ for some $C_n \in \mathscr{D}$),
- the following hold:

$$\mathrm{len}_t\tilde{L}_n < e^{n(\beta+\epsilon)}, \tag{3.9}$$
$$\mathrm{len}_t\tilde{M}_n < e^{n(\gamma+\epsilon)}. \tag{3.10}$$

Step 3: We fix a sufficiently large integer N such that

$$N \geq \max\{N_1, N_2, N_3\}.$$

Then, we will construct a sequence of twisted complexes $\tilde{K}_i \in \mathrm{Tw}(\{G,D\}_{\mathscr{C}})$ (which will be compared to $\delta_t(G,D;\Phi^{2^iN}_{\mathscr{C}}(\overline{G}))$ as described right above of Lemma 3.11) so that

(i) $\tilde{K}_i$ is a twisted complex for $\Phi^{2^iN}_{\mathscr{C}}(\overline{G}) \oplus A_i$ for some $A_i \in \mathscr{C}$,

(ii) the following equalities hold:

$$\mathrm{len}_{t,G}\tilde{K}_i = (\mathrm{len}_t \tilde{L}_N)^{2^i}, \tag{3.11}$$

$$\mathrm{len}_t \tilde{K}_{i+1} = \mathrm{len}_{t,G}\tilde{K}_i \cdot \mathrm{len}_t \tilde{K}_i + \mathrm{len}_{t,D}\tilde{K}_i \cdot \mathrm{len}_t \tilde{M}_{2^i N}. \tag{3.12}$$

We will choose $\tilde{K}_i$ inductively.

For the base case $i = 0$, observe that

$$\tilde{L}_N \simeq \Phi^N_{\mathscr{C}/\mathscr{D}}(G) \oplus B_N \simeq \Phi^N_{\mathscr{C}/\mathscr{D}}(l(\overline{G})) \oplus B_N \simeq l(\Phi^N_{\mathscr{C}}(\overline{G})) \oplus B_N.$$

Then, by Lemma 3.8, there exists a twisted complex $\tilde{K}_0 \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ such that

- $\tilde{K}_0$ is a twisted complex for $\Phi^N_{\mathscr{C}}(\overline{G}) \oplus A_0$ for some $A_0 \in \mathscr{C}$,
- $\mathrm{len}_{t,G}\tilde{K}_0 = \mathrm{len}_t \tilde{L}_N$.

We note that the last item is (3.11) for $i = 0$.

In order to choose $\tilde{K}_i$ for all $i \in \mathbb{N}$ inductively, let us assume that there is $\tilde{K}_i \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ satisfying (i)–(ii) above. One can apply $\Phi^{2^i N}_{\mathscr{C}}$ to the twisted complex $\tilde{K}_i$ to obtain a twisted complex

$$\tilde{K}'_{i+1} \in \mathrm{Tw}(\{\Phi^{2^i N}_{\mathscr{C}}(G), \Phi^{2^i N}_{\mathscr{C}}(D)\}_{\mathscr{C}})$$

such that

- $\tilde{K}'_{i+1}$ is a twisted complex for $\Phi^{2^i N}_{\mathscr{C}}(\Phi^{2^i N}_{\mathscr{C}}(\overline{G}) \oplus A_i) = \Phi^{2^{i+1} N}_{\mathscr{C}}(\overline{G}) \oplus \Phi^{2^i N}_{\mathscr{C}}(A_i)$,
- $\mathrm{len}_{t,\Phi^{2^i N}_{\mathscr{C}}(G)}\tilde{K}'_{i+1}$ (resp. $\mathrm{len}_{t,\Phi^{2^i N}_{\mathscr{C}}(D)}\tilde{K}'_{i+1}$) is equal to $\mathrm{len}_{t,G}\tilde{K}_i$ (resp. $\mathrm{len}_{t,D}\tilde{K}_i$).

We would like to construct a twisted complex $\tilde{K}_{i+1} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ by modifying $\tilde{K}'_{i+1}$ as follows: The components of $\tilde{K}'_{i+1}$ are shifts of $\Phi^{2^i N}_{\mathscr{C}}(G)$ and $\Phi^{2^i N}_{\mathscr{C}}(D)$. We replace each $\Phi^{2^i N}_{\mathscr{C}}(D)$ in the components of $\tilde{K}'_{i+1}$ with

$$\Phi^{2^i N}_{\mathscr{C}}(D) \oplus C_{2^i N} = \Phi^{2^i N}_{\mathscr{D}}(D) \oplus C_{2^i N}$$

then replace it with the equivalent twisted complex $\tilde{M}_{2^i N} \in \mathrm{Tw}(\{D\}_{\mathscr{D}}) \subset \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$. Similarly, we replace each $\Phi^{2^i N}_{\mathscr{C}}(G)$ in the components of $\tilde{K}'_{i+1}$ with

$$\Phi^{2^i N}_{\mathscr{C}}(G) \oplus \Phi^{2^i N}_{\mathscr{C}}(D) \oplus A_i = \Phi^{2^i N}_{\mathscr{C}}(\overline{G}) \oplus A_i$$

then replace it with the equivalent twisted complex $\tilde{K}_i \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$.

Since the replacements can be understood as taking direct sums with $C_{2^i N}$'s and $\Phi^{2^i N}_{\mathscr{C}}(D) \oplus A_i$'s, we get a twisted complex $\tilde{K}_{i+1} \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$ for $\Phi^{2^{i+1} N}_{\mathscr{C}}(\overline{G}) \oplus A_{i+1}$ for some $A_{i+1} \in \mathscr{C}$ after the replacements. Also, the following hold:

$$\mathrm{len}_{t,G}\tilde{K}_{i+1} = \mathrm{len}_{t,\Phi^{2^i N}_{\mathscr{C}}(G)}\tilde{K}'_{i+1} \cdot \mathrm{len}_{t,G}\tilde{K}_i = \mathrm{len}_{t,G}\tilde{K}_i \cdot \mathrm{len}_{t,G}\tilde{K}_i = (\mathrm{len}_t \tilde{L}_N)^{2^i} \cdot (\mathrm{len}_t \tilde{L}_N)^{2^i} = (\mathrm{len}_t \tilde{L}_N)^{2^{i+1}},$$

and

$$\mathrm{len}_t \tilde{K}_{i+1} = \mathrm{len}_{t,\Phi^{2^i N}_{\mathscr{C}}(G)}\tilde{K}'_{i+1} \cdot \mathrm{len}_t \tilde{K}_i + \mathrm{len}_{t,\Phi^{2^i N}_{\mathscr{C}}(D)}\tilde{K}'_{i+1} \cdot \mathrm{len}_t \tilde{M}_{2^i N} = \mathrm{len}_{t,G}\tilde{K}_i \cdot \mathrm{len}_t \tilde{K}_i + \mathrm{len}_{t,D}\tilde{K}_i \cdot \mathrm{len}_t \tilde{M}_{2^i N}.$$

Hence, (i)–(ii) holds for $\tilde{K}_{i+1}$. This concludes the construction of the sequence of twisted complexes $\tilde{K}_i \in \mathrm{Tw}(\{G, D\}_{\mathscr{C}})$.

Step 4: We will prove Theorem 3.10 for the first case, i.e., the case of $\beta > -\infty, \gamma > -\infty$. If we assume that $\beta \geq \gamma$, then we would like to prove that

$$h_t(\Phi_{\mathscr{C}}) = \alpha \leq \beta = \max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}) = \beta, h_t(\Phi_{\mathscr{D}}) = \gamma\}.$$

Based on this, we will find a contradiction under the assumptions that $\beta \geq \gamma$ and $\alpha > \beta$ using Lemma 3.11.

Consider the sequences

$$a_i := \delta_t(G, D; \Phi_{\mathscr{C}}^{2^i N}(\overline{G})), \quad a'_i := \mathrm{len}_t \tilde{K}_i.$$

Obviously, we have $a_i \le a'_i$ for all i by the definition of the complexity. In order to study the sequence

$$\frac{a_{i+1}/a_i}{a'_{i+1}/a'_i},$$

we note that

$$e^{2^i N(\alpha-\epsilon)} < a_i < e^{2^i N(\alpha+\epsilon)}, \qquad e^{2^{i+1} N(\alpha-\epsilon)} < a_{i+1} < e^{2^{i+1} N(\alpha+\epsilon)},$$

from (3.8). This induces that

$$e^{2^i N(\alpha-3\epsilon)} < \frac{a_{i+1}}{a_i} < e^{2^i N(\alpha+3\epsilon)}. \tag{3.13}$$

We also note that for any i,

$$\begin{aligned}\frac{a'_{i+1}}{a'_i} &= \mathrm{len}_{t,G}\tilde{K}_i + \frac{\mathrm{len}_{t,D}\tilde{K}_i \cdot \mathrm{len}_t \tilde{M}_{2^i N}}{\mathrm{len}_t \tilde{K}_i} \\ &\le \mathrm{len}_t(\tilde{L}_N)^{2^i} + \mathrm{len}_t \tilde{M}_{2^i N} \\ &< e^{2^i N(\beta+\epsilon)} + e^{2^i N(\gamma+\epsilon)} \\ &= (1 + e^{2^i N(\gamma-\beta)}) e^{2^i N(\beta+\epsilon)} \\ &\le 2 e^{2^i N(\beta+\epsilon)},\end{aligned} \tag{3.14}$$

from (3.9)–(3.12) and the assumption that $\beta \ge \gamma$.

Since we assume that $\alpha > \beta$, by choosing a sufficiently small $\epsilon > 0$ and a sufficiently large N, one can have

$$N(\alpha - \beta - 4\epsilon) > 1.$$

Then, (3.13) and (3.14) show that

$$\frac{a_{i+1}/a_i}{a'_{i+1}/a'_i} > \frac{1}{2} e^{2^i N(\alpha-3\epsilon) - 2^i N(\beta+\epsilon)} = \frac{1}{2} e^{2^i N(\alpha-\beta-4\epsilon)}$$

for all i. This implies

$$\lim_{i\to\infty} \frac{a_{i+1}/a_i}{a'_{i+1}/a'_i} = \infty$$

which contradicts with Lemma 3.11. Thus, $\alpha \le \beta$ if $\beta \ge \gamma$.

If we assume that $\gamma > \beta$ instead of $\beta \ge \gamma$, then we should show that $\alpha \le \gamma$. Under the assumption that $\alpha > \gamma$, the same arguments give a contradiction again.

To sum up, steps 2–4 prove Theorem 3.10 for the first case, i.e., $\beta > -\infty$ and $\gamma > -\infty$.

Step 5: Here, we will consider the second case where

$$\beta = \gamma = -\infty.$$

We will show that $\alpha = -\infty$ by contradiction. Since $\beta = h_t(\Phi_{\mathscr{C}/\mathscr{D}}) = -\infty$, for any $R \in \mathbb{R}$, there is a $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$,

$$\frac{1}{n} \log \delta_t(G; \Phi_{\mathscr{C}/\mathscr{D}}^n(G)) < R.$$

Furthermore, there is a twisted complex $\tilde{L}_n \in \mathrm{Tw}(\{G\}_{\mathscr{C}/\mathscr{D}})$ such that for any $\epsilon > 0$,

$$\mathrm{len}_t \tilde{L}_n < e^{n(R+\epsilon)}. \tag{3.9$'$}$$

Similarly, there is $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$, there exists a twisted complex $\tilde{M}_n \in \mathrm{Tw}(\{D\}_{\mathscr{D}})$ so that for any $\epsilon > 0$,

$$\mathrm{len}_t \tilde{M}_n < e^{n(R+\epsilon)}. \tag{3.10$'$}$$

We can choose $R < \alpha$. Then, by repeating the arguments in steps 2–4 with slight modifications, one can prove Theorem 3.10 for the second case. The modifications are using (3.9′) and (3.10′), instead of (3.9) and (3.10), and are replacing both of β and γ with R.

Step 6: Here, we will consider the third case, i.e., only one of $\{\beta, \gamma\}$ is $-\infty$. For convenience, let us assume that $\beta = -\infty < \gamma$. Then, the arguments in steps 2–4 will work after slight modification, as we did in step 5. The slight modifications are using (3.9′) instead of (3.9), and replacing β with a sufficiently small R.

When $\beta > -\infty = \gamma$, the same logic works. □

We end this subsection by pointing out that under some assumptions, the categorical entropies of $\Phi_{\mathscr{C}}$ and $\Phi_{\mathscr{C}/\mathscr{D}}$ agree with each other:

Corollary 3.12. *Let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. Let $\Phi_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor satisfying $\Phi_{\mathscr{C}}(\mathscr{D}) \subset \mathscr{D}$, and let $\Phi_{\mathscr{D}}\colon \mathscr{D} \to \mathscr{D}$ and $\Phi_{\mathscr{C}/\mathscr{D}}\colon \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}$ be the induced functors as in Definition 3.5. If $\Phi_{\mathscr{C}/\mathscr{D}}$ is a quasi-equivalence and $h_0(\Phi_{\mathscr{D}}) = 0$, then*

$$h_0(\Phi_{\mathscr{C}/\mathscr{D}}) = h_0(\Phi_{\mathscr{C}}).$$

Remark 3.13. *Not every triangulated category has an A_∞/dg enhancement. However, all the results from Section 2 (except Proposition 2.14*(2)*) until here hold for triangulated categories and exact functors between them. When $\mathscr{C}$ and $\mathscr{D}$ are triangulated categories, $\mathscr{C}/\mathscr{D}$ denotes the Verdier quotient. The proofs can be modified by replacing twisted complexes by iterated cones as in Remark 2.4*(2).

3.3. **Categorical entropy via admissible subcategories.** In this subsection, let $\mathscr{C}$ be a pretriangulated A_∞ (or dg) category, and $\mathscr{D}$ be its full pretriangulated A_∞ (or dg) subcategory. In order to make an inequality in Theorem 3.10 an equality, we will assume that $\mathscr{D}$ is *admissible*. See Definition 3.14 for the definition of admissible subcategory. Our motivation of the assumption will be given in the next subsection. Note that the results in this subsection also hold for triangulated categories and exact functors between them, as pointed out in Remark 3.13.

Definition 3.14.

(1) *The* right (resp. left) orthogonal complement $\mathscr{D}^\perp$ *(resp. ${}^\perp\mathscr{D}$) of $\mathscr{D}$ is the full pretriangulated A_∞ (or dg) subcategory of $\mathscr{C}$, consisting of objects $K \in \mathscr{C}$ such that*

$$\mathrm{Hom}^*(L, K) = 0, \forall L \in \mathscr{D} \ \text{ (resp. } \mathrm{Hom}^*(K, L) = 0, \forall L \in \mathscr{D}).$$

(2) *A full pretriangulated A_∞ (or dg) subcategory $\mathscr{D}$ of a pretriangulated A_∞ (or dg) category $\mathscr{C}$ is said to be* right-admissible (resp. left-admissible) *if for any $L \in \mathscr{C}$, there is a distinguished triangle*

$$L' \to L \to L'' \to L'[1]$$

for some $L' \in \mathscr{D}$ and $L'' \in \mathscr{D}^\perp$ (resp. for some $L' \in {}^\perp\mathscr{D}$ and $L'' \in \mathscr{D}$) and such a triangle is unique up to unique isomorphism.

It is an easy consequence of Definition 3.14 that, for any right-admissible (resp. left-admissible) subcategory $\mathscr{D}$ of $\mathscr{C}$, its right (resp. left) orthogonal complement $\mathscr{D}^\perp$ (resp. ${}^\perp\mathscr{D}$) is left-admissible (resp. right-admissible) and the left (resp. right) orthogonal complement ${}^\perp(\mathscr{D}^\perp)$ (resp. $({}^\perp\mathscr{D})^\perp$) of $\mathscr{D}^\perp$ (resp. ${}^\perp\mathscr{D}$) is $\mathscr{D}$.

Let us consider the composition of the inclusion $\mathscr{D}^{\perp} \hookrightarrow \mathscr{C}$ (resp. ${}^{\perp}\mathscr{D} \hookrightarrow \mathscr{C}$) and the localization functor $l : \mathscr{C} \to \mathscr{C}/\mathscr{D}$, which we will denote by $l_{\mathscr{D}^{\perp}}$ (resp. $l_{{}^{\perp}\mathscr{D}}$). It was remarked in [Dri04] that a subcategory $\mathscr{D}$ of $\mathscr{C}$ is right-admissible (resp. left-admissible) if and only if $\mathscr{D}$ is split-closed and the functor $l_{\mathscr{D}^{\perp}} : \mathscr{D}^{\perp} \to \mathscr{C}/\mathscr{D}$ (resp. $l_{{}^{\perp}\mathscr{D}} : {}^{\perp}\mathscr{D} \to \mathscr{C}/\mathscr{D}$) is an equivalence.

Now assume that an A_∞/dg subcategory $\mathscr{D}$ of an A_∞/dg category $\mathscr{C}$ is right-admissible and hence that $\mathscr{D}^{\perp}$ is a left-admissible subcategory of $\mathscr{C}$. Then the above observations imply that the functor

$$l_{\mathscr{D}} : \mathscr{D} = {}^{\perp}(\mathscr{D}^{\perp}) \to \mathscr{C}/(\mathscr{D}^{\perp})$$

is an equivalence. Let us denote by

$$p : \mathscr{C} \to \mathscr{D} \ \ (\text{resp. } q : \mathscr{C} \to \mathscr{D}^{\perp})$$

the right (resp. left) adjoint of the inclusion

$$i : \mathscr{D} \to \mathscr{C} \ \ (\text{resp. } j : \mathscr{D}^{\perp} \to \mathscr{C})$$

whose existence can be proven using the definition of admissibility. We note that

$$p(L) \simeq L' \ \ (\text{resp. } q(L) \simeq L''[-1]),$$

where L, L', L'' are the same as in Definition 3.14.

Furthermore assume that $\Phi_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an A_∞/dg functor. Then we define its induced functor $\Phi_{\mathscr{D}^{\perp}} : \mathscr{D}^{\perp} \to \mathscr{D}^{\perp}$ by

$$\Phi_{\mathscr{D}^{\perp}} = q \circ \Phi_{\mathscr{C}} \circ j.$$

Analogously, for a left-admissible A_∞/dg subcategory $\mathscr{D}$ of $\mathscr{C}$, one can define the induced functor $\Phi_{{}^{\perp}\mathscr{D}} : {}^{\perp}\mathscr{D} \to {}^{\perp}\mathscr{D}$.

Then we are ready to prove the following corollary of Theorem 3.10.

Corollary 3.15. *Let $\mathscr{D}$ be a right-admissible (resp. left-admissible) A_∞ (or dg) subcategory of $\mathscr{C}$. Let $\Phi_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ be an A_∞ (or dg) functor satisfying $\Phi_{\mathscr{C}}(\mathscr{D}) \subset \mathscr{D}$. Then, for any $t \in \mathbb{R}$, we have*

$$h_t(\Phi_{\mathscr{C}}) = \max\{h_t(\Phi_{\mathscr{D}}), h_t(\Phi_{\mathscr{D}^{\perp}})\} \ \ (\text{resp. } h_t(\Phi_{\mathscr{C}}) = \max\{h_t(\Phi_{\mathscr{D}}), h_t(\Phi_{{}^{\perp}\mathscr{D}})\}). \tag{3.15}$$

Proof. Let us consider the case that $\mathscr{D}$ is right-admissible only since the proof for the other case is similar.

First we show that, for any $t \in \mathbb{R}$,

$$h_t(\Phi_{\mathscr{C}}) \geq h_t(\Phi_{\mathscr{D}}). \tag{3.16}$$

For that purpose, take a split-generator D for $\mathscr{D}$ and a split-generator E for $\mathscr{D}^{\perp}$. Then $C := D \oplus E$ is a split-generator for $\mathscr{C}$.

It is easy to see that for any positive integer m, we have

$$\delta_t\left(C; \Phi_{\mathscr{C}}^m(C)\right) \geq \delta_t\left(p(C); (p \circ \Phi_{\mathscr{C}}^m)(C)\right). \tag{3.17}$$

But, for the right hand side of (3.17), we have

$$p(C) \simeq D,$$

and

$$\begin{aligned}(p \circ \Phi_{\mathscr{C}}^m)(C) &= p\left((\Phi_{\mathscr{D}}^m)(D) \oplus (\Phi_{\mathscr{C}}^m(E)\right) \\ &= \Phi_{\mathscr{D}}^m(D) \oplus (p \circ \Phi_{\mathscr{C}}^m)(E).\end{aligned}$$

Hence, we have, for any $t \in \mathbb{R}$,

$$\begin{aligned}\delta_t\left(p(C); (p \circ \Phi_{\mathscr{C}}^m)(C)\right) &= \delta_t\left(D; \Phi_{\mathscr{D}}^m(D) \oplus (p \circ \Phi_{\mathscr{C}}^m)(E)\right) \\ &\geq \delta_t\left(D; \Phi_{\mathscr{D}}^m(D)\right).\end{aligned} \tag{3.18}$$

The inequalities (3.17) and (3.18) prove the assertion (3.16).

On the other hand, we have

$$l(L) \simeq l\big(\mathrm{Cone}(L''[-1] \to L')\big) \simeq l(L''[-1]),$$
$$(l \circ j \circ q)(L) \simeq l(L''[-1]),$$

where L, L' and L'' are the same as in Definition 3.14. Thus, $(l \circ j \circ q)(L) \simeq l(L)$.

Now we show that

$$h_t(\Phi_{\mathscr{D}^\perp}) = h_t(\Phi_{\mathscr{C}/\mathscr{D}}), \forall t \in \mathbb{R}. \tag{3.19}$$

For a proof, observe that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{D}^\perp & \xrightarrow{l_{\mathscr{D}^\perp}} & \mathscr{C}/\mathscr{D} \\ \downarrow \Phi_{\mathscr{D}^\perp} & & \downarrow \Phi_{\mathscr{C}/\mathscr{D}} \\ \mathscr{D}^\perp & \xrightarrow[l_{\mathscr{D}^\perp}]{} & \mathscr{C}/\mathscr{D}. \end{array}$$

Indeed, we have, for any $K \in \mathscr{D}^\perp$,

$$\begin{aligned} (\Phi_{\mathscr{C}/\mathscr{D}} \circ l_{\mathscr{D}^\perp})(K) &= (\Phi_{\mathscr{C}/\mathscr{D}} \circ l \circ j)(K) \\ &\simeq (l \circ \Phi_{\mathscr{C}} \circ j)(K) \\ &\simeq (l \circ j \circ q \circ \Phi_{\mathscr{C}} \circ j)(K) \\ &= (l \circ j \circ \Phi_{\mathscr{D}^\perp})(K) \\ &= (l_{\mathscr{D}^\perp} \circ \Phi_{\mathscr{D}^\perp})(K). \end{aligned}$$

Here we used (3.3) for the third equality.

Hence, two functors $\Phi_{\mathscr{D}^\perp} : \mathscr{D}^\perp \to \mathscr{D}^\perp$ and $\Phi_{\mathscr{C}/\mathscr{D}} : \mathscr{C}/\mathscr{D} \to \mathscr{C}/\mathscr{D}$ are identified on object level via the equivalence

$$l_{\mathscr{D}^\perp} : \mathscr{D}^\perp \to \mathscr{C}/\mathscr{D}.$$

Since the categorical entropy depends on how a functor acts on a generator iteratively, this proves the assertion.

Now, Theorem 3.10 and (3.19) say that, for any $t \in \mathbb{R}$, we have

$$h_t(\Phi_{\mathscr{D}^\perp}) \le h_t(\Phi_{\mathscr{C}}) \le \max\{h_t(\Phi_{\mathscr{D}}), h_t(\Phi_{\mathscr{D}^\perp})\}. \tag{3.20}$$

The desired equality (3.15) follows from (3.16) and (3.20). □

Remark 3.16. *Corollary 3.15 is also proved in a recent work* [Kim22, Proposition 2.8]. *We note that Theorem 3.10 can be applied to any full subcategory, but Corollary 3.15 cannot.*

3.4. **Motivation: Comparison with properties of the topological entropy.** Before moving on to the symplectic topology side of this paper, we compare the properties of categorical entropy, described in Section 3, to those of the topological entropy. In particular, we are interested in the the following properties For more details, we refer the reader to [Gro87, Section 1.6].

(i) Take two continuous self mappings of compact space, say $f_i : X_i \to X_i$ for $i = 1, 2$, and let $F : X_1 \to X_2$ be a continuous mapping such that $F \circ f_1 = f_2 \circ F$. If F is onto, then

$$\text{the topological entropy of } f_1 \ge \text{ the topological entropy of } f_2.$$

(ii) Let $f : X \to X$ be a continuous self mapping, and let us assume that there are two subsets X_1 and X_2 of X such that
$$X_1 \cup X_2 = X, f(X_i) \subset X_i.$$
It is known that
$$\text{the topological entropy of } f = \max_{i=1,2}\{\text{the topological entropy of } f|_{X_i}\}.$$

(iii) Let $f : X \to X$ be a continuous self mapping, and let us assume that there is a subset $Y \subset X$ such that the restriction of f onto Y is a self mapping on Y. Then,
$$\text{the topological entropy of } f \geq \text{the topological entropy of } f|_Y.$$

One can easily observe that Proposition 3.9, or the first inequality of Theorem 3.10, is the counterpart of (i) for the categorical entropy. The localisation functor plays the role of F in (i).

Similarly, the second inequality of Theorem 3.10 is a counterpart of (ii) for the categorical entropy, but that is not a *full* counterpart of (ii). More precisely, we have an inequality in Theorem 3.10, but there is an equality in (ii). Moreover, there is no counterpart of (iii) in Section 3.2.

We roughly explain the reason why we could not have counterparts of (ii) and (iii) in Section 3.2. First, we note that, for measuring the topological entropy of f in (iii), one needs to fix a metric g on X. One also use the metric $g|_Y$ that is the restriction of g onto Y, in order to measure the topological entropy of $f|_Y$. Since one uses the same metric, (iii) holds obviously. See [Gro87, Section 1.6] for more details.

Remark 3.17. *To be more precise, we note that the notion of entropy measured by using a metric is the* geometric *entropy, not topological entropy. However, it is well-known that the geometric and topological entropy coincide if the domain space is compact.*

However, for measuring the categorical entropy, one should choose a split-generator $\overline{G}$ for an A_∞/dg category $\mathscr{C}$, and a split-generator D for a full A_∞/dg subcategory $\mathscr{D}$ of $\mathscr{C}$. Since the relation between D and $\overline{G}$ is arbitrary, it seems that proving the counterpart of (iii) is not an easy task.

Similarly, in order to turn Theorem 3.10 to
$$h_t(\Phi_{\mathscr{C}}) = \max\{h_t(\Phi_{\mathscr{C}/\mathscr{D}}), h_t(\Phi_{\mathscr{D}})\}, \tag{3.21}$$
i.e., the counterpart of (ii), we need to prove
$$h_t(\Phi_{\mathscr{D}}) \leq h_t(\Phi_{\mathscr{C}}).$$
This is the counterpart of (iii). Thus, we could not have the counterpart of (ii) by the same reason.

From the above argument, we can expect that if there is a nice relation between D and $\overline{G}$, it maybe possible to achieve Equation (3.21). Based on it, we assumed that $\mathscr{D}$ is admissible in Corollary 3.15, and we proved the equality.

Remark 3.18. *The property (iii) of topological entropy can be induced from (ii) by setting $X_1 = Y$ and $X_2 = X$. We write (ii) and (iii) separately in order to emphasize the difference between topological and categorical entropies.*

3.5. **Application: Categorical entropy on wrapped and partially wrapped Fukaya categories.** We discuss one direct application of the previous results on Fukaya categories.

Let W be a Weinstein manifold of finite type and Λ be a Legendrian stop in its ideal boundary $\partial_\infty W$. Write $\mathscr{W}(W)$ for the pretriangulated closure of its wrapped Fukaya category of W, which is an A_∞-category generated by the Lagrangian cocores $G_1, \ldots, G_m$. Also, we denote $\mathscr{W}(W, \Lambda)$ as the pretriangulated closure of the partially wrapped Fukaya category of (W, Λ). We assume $2c_1(W) = 0$ so that relevant categories are

$\mathbb{Z}$-graded with respect to the choice of a complex volume form. If $2c_1(W) \neq 0$, the categories will be $\mathbb{Z}/2$-graded, in which case, we can only talk about the categorical entropy when $t = 0$. See Section 4.1.1 for a brief introduction on wrapped Fukaya categories of Weinstein manifolds. Define

$$G := G_1 \oplus \ldots \oplus G_m.$$

Let $D_1, \ldots, D_r$ be linking disks corresponding to Λ, and let $\mathscr{D}$ be the full pretriangulated A_∞-subcategory of $\mathscr{W}(W, \Lambda)$ generated by $D_1, \ldots, D_r$. Define

$$D := D_1 \oplus \ldots \oplus D_r \quad \text{and} \quad \overline{G} := G \oplus D.$$

By [GPS20], we can write

$$\mathscr{W}(W) \simeq \mathscr{W}(W, \Lambda)/\mathscr{D}.$$

Then there is the localisation functor

$$l \colon \mathscr{W}(W, \Lambda) \to \mathscr{W}(W).$$

Note that $\overline{G}$ split-generates $\mathscr{W}(W, \Lambda)$, G split-generates $\mathscr{W}(W)$, and D split-generates $\mathscr{D}$.

Let $\phi \colon W \to W$ be a compactly supported exact symplectic automorphism, suitably graded so that it acts as an identity on D_i's. See Subsection 4.1.3 for a detailed discussion. Then there are induced functors

$$\Phi_{\mathscr{W}(W,\Lambda)} \colon \mathscr{W}(W, \Lambda) \to \mathscr{W}(W, \Lambda), \quad \text{and} \quad \Phi_{\mathscr{W}(W)} \colon \mathscr{W}(W) \to \mathscr{W}(W),$$

satisfying

$$\Phi_{\mathscr{W}(W)} \circ l \simeq l \circ \Phi_{\mathscr{W}(W,\Lambda)}.$$

Proposition 3.19. *Let $\phi \colon W \to W$ be a compactly supported exact symplectic automorphism. Then, the induced functor $\Phi_{\mathscr{D}} \colon \mathscr{D} \to \mathscr{D}$ satisfies*

$$h_t(\Phi_{\mathscr{D}}) \leq 0.$$

Proof. Since $\phi(D_i) = D_i$ for any i, $\Phi_{\mathscr{D}}$ is the identity functor. The result follows. □

Theorem 3.20. *Let $\phi \colon W \to W$ be a compactly supported exact symplectic automorphism, and $\Lambda \subset \partial_\infty W$ is a stop.*

(1) If we have $h_t(\Phi_{\mathscr{W}(W)}) \geq 0$ for a given $t \in \mathbb{R}$, then

$$h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda)}).$$

(2) If, moreover, we have $h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}) \geq 0$, then

$$h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W,\Lambda)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}).$$

(3) In particular, the above assumptions hold for $t = 0$ and we have

$$h_0(\Phi_{\mathscr{W}(W)}) = h_0(\Phi_{\mathscr{W}(W,\Lambda)}) = h_0(\Phi^{-1}_{\mathscr{W}(W,\Lambda)}) = h_0(\Phi^{-1}_{\mathscr{W}(W)}).$$

Proof. The first item (1) is a direct corollary of Theorem 3.10. To be more precise, let $\mathscr{C} := \mathscr{W}(W, \Lambda)$, and let $\mathscr{D}$ be the category defined as above. Then, Proposition 3.19 and Theorem 3.10 complete the proof of (1).

If $h_t(\Phi_{\mathscr{W}(W)}) \geq 0$ and if $h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}) \geq 0$, we have $h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda)})$ and $h_{-t}(\Phi^{-1}_{\mathscr{W}(W,\Lambda)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W)})$ because of (1). Thus, in order to prove (2), it is enough to prove $h_t(\Phi_{\mathscr{W}(W)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W)})$.

For that, we fix a Lefschetz fibration $\pi : W \to \mathbb{C}$. We note that the existence of π is proven in [GP17]. And, we set $\Lambda_0 := \pi^{-1}(-\infty)$. Then, as mentioned in [GPS18b], $\mathscr{W}(W, \Lambda_0)$ is a smooth and proper category. Thus, we have

$$h_t(\Phi_{\mathscr{W}(W)}) = h_t(\Phi_{\mathscr{W}(W,\Lambda_0)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W,\Lambda_0)}) = h_{-t}(\Phi^{-1}_{\mathscr{W}(W)}),$$

by Proposition 2.14.

Assumptions of (1) and (2) holds for $t = 0$ because $\Phi_{\mathscr{W}(W)}$ is a quasi-equivalence. Hence, (3) holds. □

4. ENTROPIES OF EXACT SYMPLECTIC AUTOMORPHISMS

The main goal of Section 4 is to prove Theorem 1.3, or equivalently, Theorem 4.13 and Theorem 4.14. Section 4.1 briefly reviews preliminaries. Also, we fix our Weinstein manifolds in Section 4.1, which are plumbing spaces of T^*S^n. In Section 4.2, we prove Theorem 4.13, i.e., the categorical entropy of any compactly supported symplectic automorphism can be computed in terms of Lagrangian Floer (co)homology. And this leads us to the proof of Theorem 4.14, studying relationships between categorical entropies on the compact and wrapped Fukaya categories, in Section 4.3.

4.1. **Setup.**

4.1.1. *Preliminaries on compact and wrapped Fukaya categories.* We briefly introduce the notion of compact Fukaya category and wrapped Fukaya category of a Weinstein manifold. See [Sei08b, GPS20] for more details. Let $(W, d\lambda)$ be a Weinstein manifold of finite type such that $2c_1(W) = 0$. Let us fix a quadratic complex volume form η^2 on W. Then the Lagrangian Grassmannian bundle $\mathrm{Gr}(TW) \to W$, whose fiber at $p \in W$ is the Lagrangian Grassmannian $\mathrm{Gr}(T_pW)$, admits a squared phase map $\alpha : \mathrm{Gr}(TW) \to S^1$.

Every Lagrangian submanifold L of W has a natural section $\beta_L : L \to \mathrm{Gr}(TW)$ mapping $p \in L$ to $T_pL \in \mathrm{Gr}(TW)$. We say that a pair $(L, \widetilde{\gamma}_L)$(or L itself) is a graded Lagrangian if L is a Lagrangian and $\widetilde{\gamma}_L : L \to \mathbb{R}$ is a lifting of $\gamma_L := \alpha \circ \beta_L : L \to S^1 = \mathbb{R}/\mathbb{Z}$.

To deal with pin structures, we fix $b \in H^2(W, \mathbb{Z}/2)$ called a *background class*. We say that L is relatively pin with respect to b if the second Stiefel-Whitney class $w_2(L)$ coincides with $b|_L$. We will specify a particular choice of b whenever it is neccessary.

Definition 4.1.

(1) *Let $\mathscr{W}(W)$ be the wrapped Fukaya category of W whose objects are exact Lagrangian submanifolds L of W with a grading and a relatively pin structure such that L is tangent to the Liouville vector field outside a compact subset of W, Let us call such a Lagrangian submanifold L* admissible.
(2) *Let $\mathscr{F}(W)$ be the full subcategory of $\mathscr{W}(W)$ consisting of closed Lagrangians.*

By abuse of notation, let us denote $\mathrm{Tw}\mathscr{W}(W)$ *and* $\mathrm{Tw}\mathscr{F}(W)$ *by* $\mathscr{W}(W)$ *and* $\mathscr{F}(W)$, *respectively.*

4.1.2. *Plumbing spaces $P_n(T)$.* Now, we will fix Weinstein manifolds that we will work with in the later parts of the paper.

Let T be a tree and let $V(T)$ denote the set of vertices of T. First, we introduce the following:

Definition 4.2. *For a tree T, let $P_n(T)$ denote the Weinstein manifold obtained by plumbing multiple copies of T^*S^n along a tree T. See, e.g.,* [EL17b] *for a more detailed definition.*

One can set the following notation:

Definition 4.3.

(1) *By definition, for any vertex $v \in V(T)$, there is a Lagrangian sphere corresponding to v. Let S_v denote the Lagrangian sphere corresponding to v.*
(2) *Let L_v denote a Lagrangian cocore of S_v. In particular, L_v is a Lagrangian disk such that*

$$|L_v \cap S_w| = \begin{cases} 1, & \text{if } v = w, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.4.

(i) *To be more precise, the choice of "*Lagrangian cocore*" in Definition 4.3 is not unique. In fact, when W is a Weinstein manifold, for any smooth point p of a Lagrangian skeleton of W, there is a Lagrangian disk which transversally intersects the Lagrangian skeleton at p. This is because, after a proper modification of the Liouville structure, the new Liouville structure near the smooth point p is exactly the same as the standard Liouville structure of a cotangent bundle. We note that the zero section of the cotangent bundle is a small neighborhood of p in the Lagrangian skeleton. For more details, we refer the reader to* [CRGG17, Section 9.1]. *Also, see* [GPS18a, Sections 1.1 and 7.1].

(ii) *Also, for a tree T, $P_n(T)$ admits a Weinstein structure whose Lagrangian skeleton is the union of the zero sections, i.e.,*

$$\cup_{v\in V(T)} S_v.$$

Thus, based on (i), for any point $p \in S_v$ except the plumbing points, there is a Lagrangian cocore L_p. Moreover, if $p_1, p_2 \in S_v$, then it is easy to check that L_{p_1} and L_{p_2} are Hamiltonian isotopic. Thus, the Lagrangian cocore L_v in Definition 4.3 is well defined up to Hamiltonian isotopy.

For a given tree T, since $c_1(P_n(T)) = 0$ (see [EL17b] for $n = 2$), $P_n(T)$ has quadratic complex volume forms. Furthermore, if $n \geq 2$, then since $H^1(P_n(T), \mathbb{Z}) = 0$, any two quadratic complex volume forms define equivalent gradings on Fukaya categories. If $n \geq 3$, then $H^2(P_n(T), \mathbb{Z}/2) = 0$ and the background class b is zero. From now on, we will work with $n \geq 3$. Then, for every $v \in V(T)$, both S_v and L_v have a unique pin structure and they are gradable. Their gradings will be explained in Lemma 4.5.

As mentioned before, it was shown in [EL17a] that $\mathscr{W}(P_n(T))$ and $\mathscr{F}(P_n(T))$ are Koszul dual to each other for $n \geq 3$. Moreover, [AS12, Section 4.2] shows the following lemma.

Lemma 4.5. *If two vertices $v, w \in V(T)$ are connected by an edge in T, we write $v \sim w$. Then, there are gradings for S_v and L_v for all $v \in V(T)$ and Floer data for $\mathscr{W}(P_n(T))$ so that the following hold:*

(1) *For each $v \in V(T)$,*

$$\hom^*(L_v, S_v) = \begin{cases} k\langle \overline{p}_v\rangle \simeq k & \text{if } * = 0, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently,

$$\hom^*(S_v, L_v) = \begin{cases} k\langle p_v\rangle \simeq k & \text{if } * = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{p_v\} = S_v \cap L_v$ and $\overline{p}_v$ denotes the same intersection point regarded as a morphism from L_v to S_v.

(2) *For any $v, w \in V(T)$ such that $v \neq w$,*

$$\hom^*(S_v, L_w) = \hom^*(L_v, S_w) = 0.$$

(3) *For any $v \in V(T)$,*

$$\hom^*(S_v, S_v) = \begin{cases} k & \text{if } * = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

(4) *For any $v, w \in V(T)$ such that $v \sim w$, there exists an integer $s_{vw} \in \{1, \dots, n-1\}$ such that*

$$\hom^*(S_v, S_w) = \begin{cases} k & * = s_{vw}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s_{vw} + s_{wv} = n.$$

(5) *For any $v, w \in V(T)$ such that $v \neq w, v \nsim w$,*

$$\hom^*(S_v, S_w) = 0.$$

(6) *For any $v, w \in V(T)$, $\mathrm{Hom}^*(S_v, S_w)$ is non-negatively graded and $\dim \mathrm{Hom}^0(S_v, S_v) = 1$.*

(7) For any $v, w \in V(T)$, $\mathrm{Hom}^*(L_v, L_w)$ is non-positively graded and $\dim \mathrm{Hom}^0(L_v, L_v) = 1$.

In the rest of our paper, our morphism spaces of the wrapped Fukaya categories of plumbing spaces will be as in Lemma 4.5.

Remark 4.6. *We recall the reason why we care those specific spaces given in Section 4.1. If W is a plumbing space of T^*S^n with $n \geq 3$ along a tree, then there are many known facts about relations between the wrapped and compact Fukaya category of W, for example, the "*Koszul duality*" mentioned in* [EL17a]. *Thus, the plumbing spaces can be good starting points to study the connection between entropies on different Fukaya categories of the same Weinstein manifold.*

4.1.3. *Symplectic automorphism.* In this subsection, we discuss auto-equivalences on the compact and wrapped Fukaya categories induced by a symplectic automorphism.

Let $(W, d\lambda)$ be a Weinstein manifold of finite type such that $2c_1(W) = 0$. Let a quadratic complex volume form η^2 on W, and let $b \in H^2(W, \mathbb{Z}/2)$ be given so that the wrapped Fukaya category $\mathscr{W}(W)$ and the compact Fukaya category $\mathscr{F}(W)$ make sense.

For an exact symplectic automorphism ϕ on W to induce autoequivalences on $\mathscr{W}(W)$ and $\mathscr{F}(W)$, ϕ is required to satisfy the following three conditions:

(1) $\phi^*\lambda - \lambda = dh$ for some compactly supported function $h : W \to \mathbb{R}$.
(2) $\phi^*\eta^2 = f\eta^2$ for some non-vanishing complex-valued function $f : W \to \mathbb{C}^*$ which admits a lifting $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) : W \to \mathbb{R} \times \mathbb{R}$ where $\mathbb{R} \times \mathbb{R} \to \mathbb{C}^*, (s,t) \mapsto e^{2\pi(s+it)}$ is regarded as a universal covering of $\mathbb{C}^*$.
(3) $\phi^* b = b$.

In particular, we use the following terminology used in [Sei08a] for exact symplectic automoprhisms satisfying the condition (1) above.

Definition 4.7. *A Liouville automorphism of $(W, d\lambda)$ is an exact symplectic automorphism ϕ on $(W, d\lambda)$ satisfying the first condition (1) above.*

Note that any compactly supported exact symplectic automorphism of W is a Liouville automorphism of W.

Now let L be a Lagrangian submanifold of W tangent to the Liouville vector field outside a compact subset of W. This is equivalent to saying that $\lambda|_L$ vanishes outside the compact subset. Therefore, the first condition (1) means that $\phi(L)$ is tangent to the Liouville vector field outside a compact subset of W as well.

The second condition (2) implies that ϕ defines a graded automorphism $(\phi, \gamma^\#)$ in the sense of [Sei08b, Chapter (12i)]. Indeed, $\gamma^\# : \mathrm{Gr}(TW) \to \mathbb{R}$ can be given by $\widetilde{f}_2 \circ \mathscr{P}$ where $\mathscr{P} : \mathrm{Gr}(TW) \to W$ is the projection. This allows us to send a graded Lagrangian $(L, \widetilde{\gamma}_L)$ to another graded Lagrangian $(\phi(L), \widetilde{\gamma}_L \circ \phi^{-1} + \gamma^\# \circ \beta_L \circ \phi^{-1})$.

Furthermore, the third condition (3) ensures that for a Lagrangian L relatively pin with respect to b, $\phi(L)$ is again relatively pin with respect to b since $b|_{\phi(L)} = (\phi^{-1})^* b|_{\phi(L)} = (\phi^{-1})^*(b|_L) = (\phi^{-1})^* w_2(L) = w_2(\phi(L))$. Furthermore, the pull-back $(\phi^{-1})^*$ sends a relative pin structure on L to one on $\phi(L)$.

In summary, if the conditions (1), (2) and (3) are satisfied, then ϕ induces well-defined auto-equivalences on both $\mathscr{W}(W)$ and $\mathscr{F}(W)$ by sending an admissible Lagrangian L to another admissible Lagrangian $\phi(L)$ with a grading and a relative pin structure described above. Passing these to the pretriangulated closure, we define the following notations.

Definition 4.8. *Let $\Phi_{\mathscr{F}(W)}$ and $\Phi_{\mathscr{W}(W)}$ be the auto-equivalence induced by ϕ on $\mathscr{F}(W)$ and $\mathscr{W}(W)$, respectively.*

Throughout the paper, for any Weinstein domain having exactly one boundary component(, which holds when its dimension is greater than or equal to 4), and any compactly supported Liouville automorphism ϕ on its completion W, we will choose a graded automorphism $(\phi, \gamma^\#)$ in a specific way once the second condition (2) is satisfied. Indeed, we consider the graded automorphism $(\phi, \gamma^\#)$ such that $\gamma^\# = 0$ on $\mathscr{P}^{-1}(W \setminus \mathrm{Supp}(\phi)) \subset \mathrm{Gr}(TW)$, which makes sense since $\mathrm{Supp}(\phi)$ is assumed to be compact and ϕ is the identity on the complement $W \setminus \mathrm{Supp}(\phi)$.

Note that the second condition (2) holds for any ϕ if $H^1(W, \mathbb{Z}) = 0$ and the third condition (3) holds for any ϕ if b is chosen to be zero. In particular, in the case of plumbings, observe that $P_n(T)$ satisfies $H^1(P_n(T), \mathbb{Z}) = 0$ for any $n \geq 2$ and any tree T. Furthermore, since $H^2(P_n(T), \mathbb{Z}/2) = 0$ for any $n \geq 3$ and any tree T as mentioned above, the background class b is necessarily 0 for the case $n \geq 3$. Therefore, any Liouviille automorphism of $P_n(T)$ defines equivalences on Fukaya categories of $P_n(T)$.

4.2. **Entropies of Liouville automorphisms.** Let $n \geq 3$. We note that the generation results for both $\mathscr{F}(P_n(T))$ and $\mathscr{W}(P_n(T))$ are well-known. For $\mathscr{F}(P_n(T))$, see [AS12, Theorem 1.1]. More precisely, it was shown that $\mathscr{F}(P_n(T))$ is generated by the spheres $\{S_v | v \in V(T)\}$. Hence we can take $S = \bigoplus_{v \in V(T)} S_v$ as a split-generator of $\mathscr{F}(P_n(T))$. On the other hand, $\mathscr{W}(P_n(T))$ is generated by the cocores $\{L_v | v \in V(T)\}$, and so we can take $L = \bigoplus_{v \in V(T)} L_v$ as a split-generator of $\mathscr{W}(P_n(T))$.

As seen above, the endomorphism algebra $\hom^*(S,S) = \bigoplus_{v,w} \hom^*(S_v, S_w)$ is non-negatively graded and furthermore $\hom^0(S_v, S_v) = k\langle e_{S_v} \rangle$ for a cohomological unit e_{S_v}, which we will call the identity morphism of S_v. On the other hand, the endomorphism algebra $\mathrm{Hom}^*(L,L) = \bigoplus_{v,w} \mathrm{Hom}^*(L_v, L_w)$ is known to be non-positively graded for certain gradings of L_v, and $\mathrm{Hom}^0(L_v, L_v)$ is generated by the strict unit. By using a minimal model for $\hom^*(L,L)$ (See [Sei08b, Section (1i)]), we may assume that $\hom^*(L_v, L_v)$ is also non-positively graded. This means that every morphism of $\hom^0(L_v, L_v)$ is a cocycle and is a multiple of a cohomological unit $e_{L_v} \in \hom^0(L_v, L_v)$, which we will call the identity morphism of L_v. Note that $\{L_v | v \in V(T)\}$ still generates $\mathscr{W}(P_n(T))$ even after $\bigoplus_{v,w} \hom^*(L_i, L_j)$ is perturbed into a minimal model.

Before stating Theorem 4.10, observe that Definition 2.6, Equation (2.2), and Lemma 4.5 imply the following:

Lemma 4.9.

1. *Let* $\mathscr{S} \in \mathrm{Tw}\{S_v | v \in V(T)\}$. *For any* $v \in V(T)$ *and* $t \in \mathbb{R}$, *the following equality holds:*
$$\mathrm{len}_{t,S_v} \mathscr{S} = \sum_{k \in \mathbb{Z}} \dim \hom^{n+k}(\mathscr{S}, L_v)\, e^{kt}.$$
2. *Let* $\mathscr{L} \in \mathrm{Tw}\{L_v | v \in V(T)\}$. *For any* $v \in V(T)$ *and* $t \in \mathbb{R}$, *the following equality holds:*
$$\mathrm{len}_{t,L_v} \mathscr{L} = \sum_{k \in \mathbb{Z}} \dim \hom^{n-k}(S_v, \mathscr{L})\, e^{kt}.$$

Theorem 4.10.

1. ([AS12, Lemma 2.9]) *For any admissible closed Lagrangian* E *of* $P_n(T)$, *there is a twisted complex* $\mathscr{E} \in \mathrm{Tw}\{S_v | v \in V(T)\}$ *for* E, *in which none of the arrows are multiples of the identity morphisms.*
2. *For any admissible Lagrangian* E *of* $P_n(T)$, *there is a twisted complex* $\mathscr{E} \in \mathrm{Tw}\{L_v | v \in V(T)\}$ *for* E, *in which none of the arrows are multiples of the identity morphisms.*

Proof. Refer to [AS12, Lemma 2.9] for a proof of the first statement (1).

Let us now prove the second statement (2). As mentioned above, any Lagrangian E is quasi-isomorphic to a twisted complex $\mathscr{F} \in \mathrm{Tw}\{L_v | v \in V(T)\}$. It suffices to show that any twisted complex $\mathscr{F} \in \mathrm{Tw}\{L_v | v \in V(T)\}$ is quasi-isomorphic to a twisted complex $\mathscr{E} \in \mathrm{Tw}\{L_v | v \in V(T)\}$, in which none of the arrows are nonzero multiples the identity morphisms.

Let $\mathscr{F} = \left[(L_{v_i}[d_i])_{i=1}^k, (f_{ij})\right]$ be given for some $k \in \mathbb{Z}_{\geq 1}$, $v_i \in V(T)$, $d_i \in \mathbb{Z}$ and $f_{ij} \in \hom^1(L_{v_j}[d_j], L_{v_i}[d_i])$ for $i > j$. Since $\hom^*(L_{v_j}, L_{v_i})$ is non-positively graded, if f_{ij} is nonzero then, we have

$$d_i \leq d_j - 1. \tag{4.1}$$

Hence we may assume that the components $L_{v_1}[d_1], \dots, L_{v_k}[d_k]$ of $\mathscr{F}$ are ordered in such a way that

$$d_i \leq d_j, \forall 1 \leq j < i \leq k. \tag{4.2}$$

We will show the assertion by an induction on the length k of $\mathscr{F}$.

Consider the subset J of $\{1, \dots, k\}$ defined by

$$J = \{1 \leq j \leq k | f_{ij} \text{ is a nonzero multiple of the identity morphism for some } i > j\}.$$

If J is empty, then $\mathscr{F}$ is the desired twisted complex.

Now assume that J is non-empty. We will find another twisted complex for $\mathscr{F}$ whose length is strictly less than that of $\mathscr{F}$. For that purpose, let $j_0 = \max J$ and consider the twisted complex $\mathscr{F}_0$ given by

$$\mathscr{F}_0 = \left[(L_{v_i}[d_i])_{i=j_0}^k, (f_{ij})\right]$$

and the twisted complex $\mathscr{F}_1$ given by

$$\mathscr{F}_1 = \left[(L_{v_i}[d_i])_{i=1}^{j_0-1}, (f_{ij})\right]$$

so that we can express $\mathscr{F}$, using the notation in Remark 2.4(1), as the twisted complex

$$\mathscr{F} = [\mathscr{F}_1 \to \mathscr{F}_0].$$

Hence, it suffices to find a twisted complex $\mathscr{E}_0$ for $\mathscr{F}_0$ whose length is strictly less than that of $\mathscr{F}_0$ and replace $\mathscr{F}_0$ by $\mathscr{E}_0$ in the above twisted complex.

For that purpose, let $i_0 > j_0$ be an integer such that f_{i_0,j_0} is a nonzero multiple of the identity morphism, which means that $d_{i_0} = d_{j_0} - 1$. But, due to (4.1), this means that there is no nonzero chain of arrows connecting $L_{v_{j_0}}[d_{j_0}]$ to $L_{v_{i_0}}[d_{i_0}]$ other than the above arrow f_{i_0,j_0}. Furthermore, the maximality of j_0 in J and (4.2) imply that there is no nonzero arrow from $L_{v_j}[d_j]$ to both $L_{v_{j_0}}[d_{j_0}]$ and $L_{v_{i_0}}[d_{i_0}]$ for any $j > j_0$. Therefore, $\mathscr{F}_0$ can be written as

$$\mathscr{F}_0 = [\mathscr{E}_1 \to \mathscr{E}_0]$$

where

$$\mathscr{E}_0 = \left[(L_{v_i}[d_i])_{j_0 < i \leq k, i_0 \neq i \neq j_0}, (f_{ij})\right],$$

and

$$\mathscr{E}_1 = \left[L_{v_{j_0}}[d_{j_0}] \xrightarrow{f_{i_0,j_0}} L_{v_{i_0}}[d_{i_0}]\right].$$

But the twisted complex $\mathscr{E}_1$ is quasi-isomorphic to zero as f_{i_0,j_0} is a nonzero multiple of the identity morphism. Hence $\mathscr{F}_0$ is quasi-isomorphic to $\mathscr{E}_0$, whose length is strictly less than that of $\mathscr{F}_0$ as desired.

This induction process eventually stops since the number of components of a twisted complex is finite. □

Remark 4.11. *The second statement (2) of Theorem 4.10 can be generalized to any cohomological unital pretriangulated A_∞ (or dg) category $\mathscr{C}$ generated by $G_1, \dots, G_m$ satisfying*

- $\mathrm{Hom}^*_{\mathscr{C}}(G_i, G_j)$ *is non-positively graded and*
- $\dim \mathrm{Hom}^0_{\mathscr{C}}(G_i, G_i) = 1$ *for all* $i = 1, \dots, m$

in the sense that, any object of $\mathscr{C}$ is quasi-isomorphic to a twisted complex built from $G_1, \dots, G_m$, in which none of arrows are multiples of the identity morphisms. Here, by an identity morphism, we mean a strict unit for G_i for some $i = 1, \dots, k$ after replacing $\bigoplus_{i,j=1}^k \hom_{\mathscr{C}}(G_i, G_j)$ with a minimal model.

We remark that the twisted complex mentioned in Theorem 4.10, (1) is not only an object of $\mathscr{F}(P_n(T))$, but also an object of $\mathscr{W}(P_n(T))$. In the statement/proof of Theorem 4.10, we consider the twisted complexes as an object in $\mathscr{W}(P_n(T))$ and the hom-spaces below are morphism spaces of $\mathscr{W}(P_n(T))$.

For each $m \in \mathbb{N}$, let $\mathscr{S}_m \in \mathrm{Tw}\{S_v | v \in V(T)\}$ be a twisted-complex for $\phi^m(S)$ satisfying the property in Theorem 4.10 (1), and let $\mathscr{L}_m \in \mathrm{Tw}\{L_v | v \in V(T)\}$ be a twisted-complex for $\phi^m(L)$ satisfying the property in Theorem 4.10 (2). Then we have the following:

Lemma 4.12. *Let ϕ be a Liouville automorphism of $P_n(T)$. For any $m \in \mathbb{N}$ and $v \in V(T)$, the following hold:*

(1) The cochain complex $\hom^*(\mathscr{S}_m, L_v)$ *has the zero differential map.*
(2) The cochain complex $\hom^*(S_v, \mathscr{L}_m)$ *has the zero differential map.*

Proof. Since the proof for the second statement is similar to that of the first statement, we will prove the first statement only.

The twisted complex $\mathscr{S}_m$ may be written as

$$\mathscr{S}_m = \left[(S_{v_i}[d_i])_{i=1}^a, (f_{ij}) \right]$$

where ($f_{ij} \in \hom^1(S_{v_j}[d_j], S_{v_i}[d_i])$, $i > j$) are the arrows in the twisted complex $\mathscr{S}_m$.

Since the components of $\mathscr{S}_m$ that contribute to $\hom^*(\mathscr{S}_m, L_v)$ are of the form $S_v[d]$, we collect all the indices $1 \le i_1 < \cdots < i_{\mathrm{len}_{S_v}\mathscr{S}_m} \le a$ such that $v_{i_k} = v$. Then the morphism space $\hom^*(\mathscr{S}_m, L_v)$ is given by

$$\hom^*(\mathscr{S}_m, L_v) = \bigoplus_{1 \le k \le \mathrm{len}_{S_v}\mathscr{S}_m} \hom^*(S_v[d_{i_k}], L_v) \simeq \bigoplus_{1 \le k \le \mathrm{len}_{S_v}\mathscr{S}_m} k[-d_{i_k} - n]$$

as a graded vector space.

The differential from a summand $\hom^*(S_v[d_{i_k}], L_v) \simeq k[-d_{i_k} - n]$ to itself vanishes since it is one-dimensional. Now suppose that the differential from a summand $\hom^*(S_v[d_{i_l}], L_v) \simeq k[-d_{i_l} - n]$ to $\hom^*(S_v[d_{i_k}], L_v) \simeq k[-d_{i_k} - n]$ does not vanish for some $1 \le k < l \le \mathrm{len}_{S_v}\mathscr{S}_m$. First of all, since the differential increases degree by 1, it means that

$$d_{i_k} = d_{i_l} + 1. \tag{4.3}$$

On the other hand, it also implies that there is a sequence $i_k = j_0 < j_1 < \cdots < j_p = i_l$ for some $p \in \mathbb{Z}_{\ge 1}$ such that the arrow $f_{j_{q+1}, j_q} \in \hom^1(S_{v_{j_q}}[d_{j_q}], S_{v_{j_{q+1}}}[d_{j_{q+1}}])$ is nonzero for all $0 \le q < p$. But this means that

$$d_{j_{q+1}} = d_{j_q} + \deg f_{j_{q+1}, j_q} - 1 \ge d_{j_q}.$$

for all $0 \le q < p$. Indeed, $\deg f_{j_{q+1}, j_q} \ge 1$ since f_{j_{q+1}, j_q} is not a multiple of the identity morphism by our assumption and any homogeneous element of $\hom^*(S_v, S_w)$ for $v, w \in V(T)$ has a positive degree unless it is a multiple of the identity morphism. As a result, we have $d_{i_l} \ge d_{i_k}$, which contradicts to (4.3). ☐

Now we are ready to prove the following:

Theorem 4.13. *Let ϕ be a Liouville automorphism of $P_n(T)$. For any $t \in \mathbb{R}$, the following hold:*

$$h_t\left(\Phi_{\mathscr{F}(P_n(T))}\right) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{k \in \mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\phi^m(S), L\right) e^{kt},$$

$$h_t\left(\Phi_{\mathscr{W}(P_n(T))}\right) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{k \in \mathbb{Z}} \dim \mathrm{Hom}^{n-k}\left(S, \phi^m(L)\right) e^{kt}.$$

Proof. Since the proof for the second equality is similar to that of the first equality, we will prove the first one only.

Recall that the categorical entropy $h_t(\Phi_{\mathscr{F}(P_n(T))})$ measures the exponential growth rate of

$$\inf\left\{\sum_{v\in V(T)} \mathrm{len}_{t,S_v}\widetilde{\mathscr{S}_m} \,\middle|\, \widetilde{\mathscr{S}_m} \simeq \phi^m(S)\oplus C \text{ for some } \widetilde{\mathscr{S}_m}\in \mathrm{Tw}\{S_v|v\in V(T)\}, C\in\mathscr{F}(P_n(T))\right\}. \tag{4.4}$$

Let $\mathscr{S}_m$ be a twisted complex for $\phi^m(S)$ given right above of Lemma 4.12. Then, for any twisted complex $\widetilde{\mathscr{S}_m}$ quasi-isomorphic to $\phi^m(S)\oplus C$ for some $C\in\mathscr{F}(P_n(T))$, we have

$$\begin{aligned}
\mathrm{len}_{t,S_v}\widetilde{\mathscr{S}_m} &= \sum_{k\in\mathbb{Z}} \dim \hom^{n+k}\left(\widetilde{\mathscr{S}_m}, L_v\right) e^{kt} \quad (\because \text{Lemma 4.9})\\
&\geq \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\phi^m(S)\oplus C, L_v\right) e^{kt}\\
&\geq \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\phi^m(S), L_v\right) e^{kt}\\
&= \sum_{k\in\mathbb{Z}} \dim \hom^{n+k}\left(\mathscr{S}_m, L_v\right) e^{kt} \quad (\because \text{Lemma 4.12})\\
&= \mathrm{len}_{t,S_v}\mathscr{S}_m \quad (\because \text{Lemma 4.9}).
\end{aligned}$$

This means that $\mathscr{S}_m$ is a twisted complex quasi-isomorphic to $\phi^m(S)$ giving the infimum in (4.4).

Therefore, the categorical entropy $h_t(\Phi_{\mathscr{F}(P_n(T))})$ is computed by the exponential growth rate of

$$\sum_v \mathrm{len}_{t,S_v}\mathscr{S}_m = \sum_v\sum_{k\in\mathbb{Z}} \dim\hom^{n+k}(\mathscr{S}_m, L_v)e^{kt} = \sum_v\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\phi^m(S), L_v\right)e^{kt} = \sum_{k\in\mathbb{Z}} \mathrm{Hom}^{n+k}\left(\phi^m(S), L\right)e^{kt}$$

as asserted. □

4.3. **Equality between categorical entropies.** Let $n\geq 3$. We prove that, for any compactly supported exact symplectic automorphism ϕ on $P_n(T)$, their induced functors $\Phi_{\mathscr{W}(P_n(T))}$ and $\Phi_{\mathscr{F}(P_n(T))}$ have the same categorical entropy under certain conditions.

Theorem 4.14. *Let ϕ be a Liouville automorphism of $P_n(T)$. Then we have*

(1) For any $t\in\mathbb{R}$, the following equality hold:

$$h_t\left(\Phi_{\mathscr{F}(P_n(T))}\right) = h_{-t}\left(\Phi^{-1}_{\mathscr{W}(P_n(T))}\right), h_t\left(\Phi_{\mathscr{W}(P_n(T))}\right) = h_{-t}\left(\Phi^{-1}_{\mathscr{F}(P_n(T))}\right)$$

(2) If ϕ is further compactly supported, then, for any $t\in\mathbb{R}$ such that $h_t\left(\Phi_{\mathscr{F}(P_n(T))}\right)\geq 0$ and $h_t\left(\Phi_{\mathscr{W}(P_n(T))}\right)\geq 0$, the following equality holds:

$$h_t\left(\Phi_{\mathscr{F}(P_n(T))}\right) = h_t\left(\Phi_{\mathscr{W}(P_n(T))}\right).$$

In particular, since both $h_0\left(\Phi_{\mathscr{F}(P_n(T))}\right)\geq 0$ and $h_0\left(\Phi_{\mathscr{W}(P_n(T))}\right)\geq 0$ always hold, the following equality holds:

$$h_0\left(\Phi_{\mathscr{F}(P_n(T))}\right) = h_0\left(\Phi_{\mathscr{W}(P_n(T))}\right).$$

Proof. Theorem 4.13 implies that, for any $t\in\mathbb{R}$,

$$\begin{aligned}
h_t\left(\Phi_{\mathscr{F}(P_n(T)}\right) &= \lim_{m\to\infty}\frac{1}{m}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^{n+k}\left(\phi^m(S), L\right)e^{kt}\\
&= \lim_{m\to\infty}\frac{1}{m}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^{n+k}\left(S, \phi^{-m}(L)\right)e^{kt}\\
&= \lim_{m\to\infty}\frac{1}{m}\log\sum_{k\in\mathbb{Z}}\dim\mathrm{Hom}^{n-k}\left(S, \phi^{-m}(L)\right)e^{k\cdot(-t)}\\
&= h_{-t}\left(\Phi^{-1}_{\mathscr{W}(P_n(T))}\right).
\end{aligned}$$

The other equality can be proved similarly. This completes the proof of the first statement.

Then the second statement follows from Theorem 3.20 and the first statement. □

More generally, Theorems 4.13 and 4.14 hold for any Weinstein manifold W and any Liouville automorphism ϕ of W if W satisfies the conditions in the following theorem.

Theorem 4.15. *Assume that a Weinstein manifold W has sets of Lagrangians $\{S_i\}_{1\le i\le k}$ and $\{L_i\}_{1\le i\le k}$ such that*

- *$\{S_i\}_{1\le i\le k}$ (resp. $\{L_i\}_{1\le i\le k}$) generates the compact (resp. wrapped) Fukaya category of W,*
- *the morphism space in the wrapped Fukaya category $\hom(S_i, L_j)$ is non-zero only if $i = j$ and is one-dimensional when $i = j$,*
- *$\bigoplus_{i,j} \hom^*(S_i, S_j)$ is non-negatively graded,*
- *$\bigoplus_{i,j} \hom^*(L_i, L_j)$ is non-positively graded, and*
- *$\dim \mathrm{Hom}^0(S_i, S_i) = 1 = \dim \mathrm{Hom}^0(L_i, L_i)$ for all $1 \le i \le k$.*

Let $S := \bigoplus_{i=1}^k S_i, L := \bigoplus_{i=1}^k L_i$. If ϕ is a Liouville automorphism of W, then the following holds:

(1) For any $t \in \mathbb{R}$,

$$h_t\left(\Phi_{\mathscr{F}(W)}\right) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\phi^m(S), L\right) e^{kt},$$
$$h_t\left(\Phi_{\mathscr{W}(W)}\right) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n-k}\left(S, \phi^m(L)\right) e^{kt}.$$

(2) For any $t \in \mathbb{R}$,

$$h_t\left(\Phi_{\mathscr{F}(W)}\right) = h_{-t}\left(\Phi^{-1}_{\mathscr{W}(W)}\right), h_t\left(\Phi_{\mathscr{W}(W)}\right) = h_{-t}\left(\Phi^{-1}_{\mathscr{F}(W)}\right)$$

(3) If ϕ is further compactly supported, then, for any $t \in \mathbb{R}$ such that $h_t\left(\Phi_{\mathscr{F}(W)}\right) \ge 0$ and $h_t\left(\Phi_{\mathscr{W}(W)}\right) \ge 0$,

$$h_t\left(\Phi_{\mathscr{F}(W)}\right) = h_t\left(\Phi_{\mathscr{W}(W)}\right).$$

In particular, if $t = 0$, the condition always holds. Thus,

$$h_0\left(\Phi_{\mathscr{F}(W)}\right) = h_0\left(\Phi_{\mathscr{W}(W)}\right).$$

Proof. In the proofs of Theorems 4.13 and 4.14, we only used the properties of $\mathscr{F}(P_n(T))$ and $\mathscr{W}(P_n(T))$ which are assumed in Theorem 4.15. Thus, the proofs also work for a Weinstein manifold W satisfying the assumptions. □

Remark 4.16. *The first statement (1) of Theorem 4.15 holds not just for autoequivalences induced by a Liouville automorphism of W, but also for any endo-functor $\Phi_{\mathscr{F}}$ on $\mathscr{F}(W)$ in the sense that, for any $t \in \mathbb{R}$,*

$$h_t(\Phi_{\mathscr{F}}) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\Phi^m_{\mathscr{F}}(S), L\right) e^{kt},$$

Similarly, for any endo-functor $\Phi_{\mathscr{W}}$ on $\mathscr{W}(W)$ and for any $t \in \mathbb{R}$, we also have

$$h_t(\Phi_{\mathscr{W}}) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n-k}\left(S, \Phi^m_{\mathscr{W}}(L)\right) e^{kt}.$$

5. ENTROPIES OF SYMPLECTIC AUTOMORPHISMS OF PENNER TYPE

In this section, we focus on the functor of specific type and the computation of its categorical entropy. In fact, we only consider the categorical entropies at $t = 0$ since the categorical entropy of specific functor in this case is related with a spectral radius of some matrix. Then, it can be easily computed by simple linear algebra.

Let ϕ be a compactly supported exact symplectic automorphism on $P_n(T)$, $n \geq 3$. Then, since the induced functors on the compact and wrapped Fukaya categories have the same categorical entropies for $t = 0$ by Theorem 4.14, we simply say that ϕ, not the induced functors, has a categorical entropy. More precisely,

Definition 5.1. *If ϕ is any compactly supported exact symplectic automorphism on $P_n(T)$, then the* categorical entropy of ϕ *is*

$$h_{cat}(\phi) := h_0\left(\Phi_{\mathscr{F}(P_n(T))}\right) = h_0\left(\Phi_{\mathscr{W}(P_n(T))}\right).$$

The result of Section 4 says that

$$h_{cat}(\phi) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^k\left(\phi^m(S), L\right).$$

Even if we have the above result, $h_{cat}(\phi)$ is not easy to compute because measuring the exponential growth rate of $\dim \mathrm{Hom}^*\left(\phi^m(S), L\right)$ is not an easy task in general. In this section, we prove that, if ϕ is of a symplectic automorphism of specific type on $P_n(T)$, its categorical entropy is related to the spectral radius of the matrix M_ϕ whose entries are given by the dimensions of Hom spaces. The specific type will be defined in Section 5.1. These results will more precisely stated in Theorem 5.8.

5.1. **Symplectic automorphisms of Penner type.** Section 4 cares any compactly supported exact symplectic automorphism on $P_n(T)$, but as mentioned in the beginning of Section 5, here we care symplectic automorphisms of a specific type, called *Penner type*.

In order to define the notion of Penner type, we consider the following: Let T be a tree, and let $V(T)$ be the set of vertices. Then, by choosing a vertex $v_0 \in V(T)$, one can define the following:

$$\begin{aligned} V_+(T) &:= \{v \in V(T) \mid v \text{ is connected to } v_0 \text{ by even number of edges.}\} \\ V_-(T) &:= \{v \in V(T) \mid v \text{ is connected to } v_0 \text{ by odd number of edges.}\} \end{aligned} \tag{5.1}$$

The above $V_+(T)$ and $V_-(T)$ are disjoint decomposition of $V(T)$.

Definition 5.2. *Let τ_v denote the Dehn twist along S_v.* A symplectic automorphism $\phi : P_n(T) \to P_n(T)$ is of Penner type *if either ϕ or ϕ^{-1} is a product of*

- *positive powers of τ_v if $v \in V_+(T)$, and*
- *negative powers of τ_w if $w \in V_-(T)$.*

Remark 5.3.

1. *This type of symplectic automorphisms is studied by the last named author in* [Lee21]. *In* [Lee21], *it is proved that symplectic automorphisms of Penner type satisfy a geometric stability. Because of the geometric stability, we will show that one can easily compute the categorical entropy of Penner type by a simple linear algebra.*
2. *We would like to point out that two sets $V_+(T)$ and $V_-(T)$ defined in Equation* (5.1) *are dependent on the choice of v_0, but Definition 5.2 is independent from the choice. In the rest of this section, we assume there is an arbitrary chosen v_0.*
3. *We note that the original Penner construction* [Pen88] *in surface theory has one more condition than Definition 5.2. The extra condition is the following Condition (P).*

(P) every $v \in V(T)$ appears in the product.
We omit the condition (P) since the results in the rest of the paper hold without the condition (P). But, if ϕ does not satisfy (P), we expect that ϕ does not induce a pseudo-Anosov auto-equivalence in the sense of [FFH+21]. *For more details on surface theory, we refer the eager reader to* [FM12].

For the future use, we set the following notations. First, let us denote an ordered sequence of vertices $v_i \in V(T)$ by $(v_s, v_{s-1}, \dots, v_1)$.

Definition 5.4.

(1) *For each $v \in V(T)$, let σ_v be defined as follows:*

$$\sigma_v = \begin{cases} 1 & \text{if } v \in V_+(T), \\ -1 & \text{if } v \in V_-(T). \end{cases}$$

(2) *Let $J = (v_s, v_{s-1}, \dots, v_1)$ be an ordered sequence of vertices $v_i \in V(T)$. Then, ϕ_J (resp. ϕ_{-J}) is the symplectic automorphism of Penner type defined as follows:*

$$\phi_J = \tau_{v_s}^{\sigma_{v_s}} \circ \tau_{v_{s-1}}^{\sigma_{v_{s-1}}} \circ \cdots \circ \tau_{v_1}^{\sigma_{v_1}}, \quad \phi_{-J} = \tau_{v_s}^{-\sigma_{v_s}} \circ \tau_{v_{s-1}}^{-\sigma_{v_{s-1}}} \circ \cdots \circ \tau_{v_1}^{-\sigma_{v_1}}.$$

We note that by Definition 5.2, for any ϕ of Penner type, there is an ordered sequence J such that $\phi = \phi_J$ or $\phi = \phi_{-J}$. However, by changing a choice of v_0, one can always assume that $\phi = \phi_J$; see Remark 5.3, (2). In the rest of the paper, we assume that $\phi = \phi_J$ for some J.

5.2. **Construction of a twisted complex.** In Section 4, we constructed a twisted complex in Theorem 4.10 whose nonzero arrows are not a multiple of the identity morphism. This property of the twisted complex helped to prove Theorem 4.14. Now, we construct a twisted complex with the same property, by using Seidel's long exact sequence [Sei03] in the wrapped Fukaya category, which will help us to prove the main theorem of this section.

Lemma 5.5.

(1) *In the wrapped Fukaya category $\mathcal{W}(P_n(T))$, $\tau_v(S_w)$ is isomorphic to the following twisted complex:*

$$\tau_v(S_w) \simeq \begin{cases} S_w[1-n] & v = w, \\ \left[(S_v[1-s_{vw}], S_w), f\right] & v \sim w, \\ S_w & \text{otherwise.} \end{cases} \tag{5.2}$$

(2) *In the wrapped Fukaya category $\mathcal{W}(P_n(T))$, $\tau_v^{-1}(S_w)$ is isomorphic to the following twisted complex:*

$$\tau_v^{-1}(S_w) \simeq \begin{cases} S_w[n-1] & v = w, \\ \left[(S_w, S_v[n-s_{vw}-1]), g\right] & v \sim w, \\ S_w & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. All the cases are from the Seidel's long exact sequence and the grading convention in Lemma 4.5. □

Note that f and g in (5.2) and (5.3) can be described explicitly, but we do not since they will not be used. For the future convenience, let

$$\mathcal{S}_{m,v} = \left[(S_w[d_i])_{\substack{w \in V(T) \\ 1 \le i \le \mathrm{len}_{S_w}(\mathcal{S}_{m,v})}}, (f_{ij}) \right] \tag{5.4}$$

denote the twisted complex for $\phi^m(S_v)$.

If we grade S_v's so that $s_{uw} = 1$, or equivalently, $s_{wu} = n-1$ for any $u \in V_+(T)$, $w \in V_-(T)$, then we get the following lemma.

Lemma 5.6. *With the notation given above, the shift d_i of $S_w[d_i]$ in Equation* (5.4) *satisfies that*

$$d_i = l(1-n), \text{ for some } l \in \mathbb{Z}.$$

5.3. **Entropies of symplectic automorphisms of Penner type.** In this subsection, we prove the main theorem of Section 5 (Theorem 5.8). It says that if a symplectic automorphism ϕ is of Penner type, then the categorical entropy $h_{cat}(\phi)$ can be computed by a simple linear algebra.

In order to state Theorem 5.8, we define the notion of spectral radius, first.

Definition 5.7. *Let $F : V \to V$ be a linear operator on a finite dimensional $\mathbb{R}$-vector space V. The* spectral radius $\mathrm{Rad}(F)$ *of F is defined by the maximum of absolute values of the complex eigenvalues of F.*

We state and prove the main theorem of Section 5.

Theorem 5.8. *Let ϕ be a symplectic automorphism of Penner type on $P_n(T)$ with $n \geq 3$ and $M_\phi : \mathbb{R}^{|V(T)|} \to \mathbb{R}^{|V(T)|}$ be a matrix whose uw-entry is given by $\sum_k \dim \mathrm{Hom}^k\left(\phi(S_w), L_u\right)$. Then,*

$$h_{cat}(\phi) = \log Rad(M_\phi).$$

Proof. We note that for convenience, we use the set $V(T)$ of vertices as an index set throughout the proof.

For any $v \in V(T)$, $\tau_v^{\sigma_v}$ is of Penner type. Thus, $M_{\tau_v^{\sigma_v}}$ is defined and it satisfies that

$$\left(M_{\tau_v^{\sigma_v}}\right)_{uw} = \begin{cases} 1 & u = w, \\ 1 & u = v, w \sim v, \\ 0 & \text{otherwise,} \end{cases}$$

because of Lemma 5.5.

We note that for any twisted complex $\mathscr{S} \in \mathrm{Tw}\{S_v | v \in V(T)\}$, Lemma 5.5 gives another twisted complex $\mathscr{S}' \simeq \tau_v^{\sigma_v}(\mathscr{S}) \in \mathrm{Tw}\{S_v | v \in V(T)\}$. Moreover, the following equation holds:

$$M_{\tau_v^{\sigma_v}} \cdot \left(\mathrm{len}_{S_w} \mathscr{S}\right)_{w \in V(T)} = \left(\mathrm{len}_{s_w} \mathscr{S}'\right)_{w \in V(T)}, \text{ where } \left(\mathrm{len}_{S_w} \mathscr{S}\right)_{w \in V(T)}, \left(\mathrm{len}_{s_w} \mathscr{S}'\right)_{w \in V(T)} \in \mathbb{R}^{|V(T)|}. \tag{5.5}$$

Now, we focus on the twisted complex $\mathscr{S}_{m,v}$ defined in Equation (5.4). Lemma 5.6 implies that every arrow of $\mathscr{S}_{m,v}$ is not a scalar multiple of the identity morphism for any $m \in \mathbb{N}, v \in V(T)$. Thus, $\hom^*\left(\mathscr{S}_{m,v}, L_w\right)$ has the zero differential, and

$$\mathrm{len}_{S_w} \mathscr{S}_{m,v} = \dim \hom^*\left(\mathscr{S}_{m,v}, L_w\right) = \dim \mathrm{Hom}^*\left(\phi^m(S_v), L_w\right) \text{ for all } m \in \mathbb{N}, v, w \in V(T). \tag{5.6}$$

Thus, the (w, v)-entry of M_{ϕ^m} is given as

$$\left(M_{\phi^m}\right)_{w,v} = \sum_{k \in \mathbb{Z}} \dim \hom^k\left(\mathscr{S}_{m,v}, L_w\right) = \mathrm{len}_{S_w} \mathscr{S}_{m,v}. \tag{5.7}$$

We note that since ϕ^m is also of Penner type, M_{ϕ^m} is defined.

Let $\phi = \phi_J$ with some $J = (v_s, \ldots, v_1)$, i.e.,

$$\phi_J = \tau_{v_s}^{\sigma_{v_s}} \circ \tau_{v_{s-1}}^{\sigma_{v_{s-1}}} \circ \cdots \circ \tau_{v_1}^{\sigma_{v_1}}.$$

If M_i denotes $M_{\tau_{v_i}^{\sigma_{v_i}}}$, Equations (5.5) and (5.7) conclude that

$$M_\phi = M_s \cdots M_2 \cdot M_1, \quad M_{\phi^m} = M_\phi^m. \tag{5.8}$$

For any $v \in V(T)$, let $e_v \in \mathbb{R}^{|V(T)|}$ be the vector whose v^{th}-entry is 1 and the other entries are zero. And let $\|\cdot\|_1$ mean the L^1-norm on $\mathbb{R}^{|V(T)|}$. Theorem 4.13, Equations (5.6), (5.8), and (5.7) conclude that

$$\begin{aligned} h_{cat}(\phi) &= \lim_{m\to\infty}\frac{1}{m}\log\sum_{v,w\in V(T)}\operatorname{len}_{S_w}\mathscr{S}_{m,v} \\ &= \lim_{m\to\infty}\frac{1}{m}\log\max\left\{\| M_\phi^m\cdot e_v\|_1 \,\middle|\, v\in V(T)\right\} \end{aligned} \tag{5.9}$$

because every entry of M_ϕ is non-negative and $\{e_v | v \in V(T)\}$ is a basis of $\mathbb{R}^{|V(T)|}$. Thus, by Gelfand's formula, the last term in Equation (5.9) is the same as $\log \operatorname{Rad}(M_\phi)$. □

Corollary 5.9. *Let ϕ be a symplectic automorphism of Penner type, and let $n \geq 3$ be an odd integer. Then,*

$$h_{cat}(\phi) \leq h_{top}(\phi),$$

where $h_{top}(\phi)$ means the topological entropy of ϕ.

Proof. When n is odd, it is easy to see that the matrix representation of $\phi_* : H_n(P_n(T)) \to H_n(P_n(T))$ with respect to the basis $\{[S_v]\}_{v\in V(T)}$ and M_ϕ coincide. Moreover, it is well-known that

$$\log \operatorname{Rad}\left(\phi_* : H_n(P_n(T)) \to H_n(P_n(T))\right) \leq h_{top}(\phi).$$

See [Yom87, Gro03, Gro87] for details. Then, Theorem 5.8 completes the proof. □

6. Examples and applications

6.1. **A counterexample to Gromov–Yomdin type equality.** In this subsection, we give a counter-example of the Gromov–Yomdin type equality. The equality claims that the categorical entropy of a functor Φ is the same as the logarithm of the spectral radius of the linear map which Φ induces on the Grothendieck group. For more details on Gromov–Yomdin type equality, we refer the reader to [KO20, Section 1.2]

Let T be the Dynkin diagram of A_3 type. We label the vertices of $T = A_3$ as $V(A_3) = \{1,2,3\}$ so that the vertex 1 (resp. 3) is connected to the vertex 2. For a fixed $n \in \mathbb{N}_{\geq 3}$, let ϕ be the symplectic automorphism given as

$$\phi = \tau_1 \circ \tau_2^{-1} \circ \tau_3 : P_n(A_3) \xrightarrow{\simeq} P_n(A_3).$$

Since S_1, S_2 and S_3 generate the compact Fukaya category of $P_n(A_3)$, $\{[S_1],[S_2],[S_3]\}$ generates the Grothendieck group of $\mathscr{F}(P_n(A_3))$. By using Seidel's long exact sequences for Dehn twist τ_i, one can have a linear map on the Grothendieck group, which is induced from τ_i. The matrix representations for $\tau_1, \tau_2^{-1}, \tau_3$ with respect to the basis $\{[S_1],[S_2],[S_3]\}$ are

$$B_1^+ := \begin{pmatrix} (-1)^{1-n} & 0 & 0 \\ (-1)^{2-n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_2^- := \begin{pmatrix} 1 & (-1)^{n-2} & 0 \\ 0 & (-1)^{n-1} & 0 \\ 0 & 1 & 1 \end{pmatrix}, B_3^+ := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & (-1)^{1-n} \end{pmatrix}.$$

We note that the matrix representations can vary by shifting S_1, S_2, S_3. To be more clear, we should have specified the grading information, but we omit that for convenience.

From the above computations, one can obtain a matrix representation of the linear map induced from ϕ on the Grothendieck group. The resulting matrix is

$$B_\phi = B_1^+ \circ B_2^- \circ B_3^+ = \begin{pmatrix} (-1)^{1-n} & -1 & -1 \\ (-1)^{2-n} & 1+(-1)^{n-1} & 1+(-1)^{n-1} \\ 0 & 1 & 1+(-1)^{1-n} \end{pmatrix}.$$

If n is even, then we have

$$B_\phi = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, by simple computation, the spectral radius of B_ϕ is 1. If Gromov–Yomdin type equality holds, then $h_{cat}(\phi) = 0$.

However, by Theorem 5.8, it is easy to compute $h_{cat}(\phi)$. The result is that

$$h_{cat}(\phi) = 2 + \sqrt{3}.$$

Thus, the above example disproves Gromov–Yomdin type equality.

Remark 6.1.

(1) *We would like to note that $H_*(P_n(A)_3))$ and the Grothendieck group of the compact Fukaya category of $P_n(A_3)$ are equivalent as vector spaces. More specifically, both vector spaces are generated by the same basis $\{[S_1],[S_2],[S_3]\}$ and this fact induces the equivalence between two vector spaces. Moreover, the symplectic automorphism ϕ induces the same linear maps on both vector spaces. Now let us assume that n is even. The spectral radius of the above B_ϕ is 1. However, it is easy to observe that the* topological *entropy of a specific representative of ϕ should be larger than or equal to $h_{cat}(\phi) > 0$. To sum up, one can observe that the topological entropy of (the specific representative of) ϕ is bigger than the logarithm of the spectral radius of the linear map on $H_*(P_n(T))$ induced by ϕ. It means that ϕ should "not" satisfy the condition of Gromov-Yomdin theorem* [Yom87]. *Thus, there is no complex structure on $P_n(A_3)$ making ϕ holomorphic.*

(2) *We also note that* [FFH+21, Section 3.1] *explains a way of computing the categorical entropy of auto-equivalences for a specific case. The specific case is the following: The auto-equivalence is induced from a symplectic automorphism of Penner type $\phi : P_n(A_2) \to P_n(A_2)$, where A_2 is the Dynkin diagram of A_2 type, and where $n \geq 3$ is an odd integer. Also, the triangulated category which the induced functors are defined on is the compact Fukaya category of $P_n(A_2)$.* [FFH+21] *computed the categorical entropy by using the spectral radius as we did in Theorem 5.8.*

6.2. **Cotangent bundles of simply-connected manifolds.** In this subsection, we give an example of Theorem 4.15, different from the plumbing spaces $P_n(T)$. The example Weinstein manifold is T^*Q where Q is a simply-connected closed smooth manifold of dimension n. Moreover, we also show that for any Liouville automorphism ϕ of T^*Q, $h_t(\Phi)$ is a linear map for $t \in \mathbb{R}$ under some topological assumptions on ϕ.

Choose a background class as $b = \pi^* w_2(Q)$, the pullback of the second Stiefel-Whitney class of Q via the projection $\pi : T^*Q \to Q$. Let S be denote the zero section of T^*Q, and let L denote a cotangent fiber $T^*_q Q$ of T^*Q for some $q \in Q$. We may grade S and L in such a way that

$$\hom^*(S,L) = k[-n].$$

Then one can observe that S and L satisfy the conditions in Theorem 4.15 because

- S generates the compact Fukaya category $\mathscr{F}(T^*Q)$ of T^*Q,
- L generates the wrapped Fukaya category $\mathscr{W}(T^*Q)$ of T^*Q,
- $\dim \hom^*(S,L) = 1$,
- $\hom^*(S,S) \simeq C^*(Q)$ is non-negatively graded,
- $\hom^*(L,L) \simeq C_{-*}(\Omega_q Q)$ is non-positively graded and
- $\dim \mathrm{Hom}^0(S,S) = \dim H^0(Q) = 1 = \dim H_0(\Omega_q Q) = \dim \mathrm{Hom}^0(L,L)$.

Let ϕ be any Liouville automorphism of T^*Q such that $\phi^* : H^2(T^*Q;\mathbb{Z}/2) \to H^2(T^*Q;\mathbb{Z}/2)$ satisfies

$$\phi^* b = b. \tag{6.1}$$

Theorem 4.15 implies that the categorical entropies of the auto-equivalences $\Phi_{\mathscr{F}}$ on $\mathscr{F}(T^*Q)$ and $\Phi_{\mathscr{W}}$ on $\mathscr{W}(T^*Q)$ are explicitly given by

$$h_t(\Phi_{\mathscr{F}}) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\Phi_{\mathscr{F}}^m(S), L\right) e^{kt}, \text{ and}$$
$$h_t(\Phi_{\mathscr{W}}) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n-k}\left(S, \Phi_{\mathscr{W}}^m(L)\right) e^{kt}$$

respectively.

Now, we recall

Theorem 6.2 ([FSS08, Nad09, Abo12])**.** *If Q is simply-connected, then every exact closed graded relatively spin Lagrangian P of T^*Q is quasi-isomorphic to a shift of the zero-section in $\mathscr{F}(T^*Q)$.*

This result implies that, $\Phi_{\mathscr{F}}(S)$ is quasi-isomorphic to a shift $S[d_\phi]$ of S for some $d_\phi \in \mathbb{Z}$. Consequently, for any $m \in \mathbb{Z}$, $\Phi_{\mathscr{F}}^m(S)$ is quasi-isomorphic to $S[md_\phi]$. Therefore, by Theorem 1.4, we have

$$\begin{aligned} h_t(\Phi_{\mathscr{F}}) &= \lim_{m\to\infty} \frac{1}{m} \log \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^{n+k}\left(\Phi_{\mathscr{F}}^m(S), L\right) e^{kt} \\ &= \lim_{m\to\infty} \frac{1}{m} \log \dim \mathrm{Hom}^{n+md_\phi}\left(S[md_\phi], L\right) e^{md_\phi t} \\ &= \lim_{m\to\infty} \frac{1}{m} \log e^{md_\phi t} = d_\phi t. \end{aligned}$$

Moreover, observing that $d_{\phi^{-1}} = -d_\phi$, the first statement of Theorem 4.15 implies that we have

$$h_t(\Phi_{\mathscr{W}}) = h_{-t}\left(\Phi_{\mathscr{F}}^{-1}\right) = d_{\phi^{-1}} \cdot (-t) = d_\phi t.$$

Theorem 6.3. *Let ϕ be a Liouville automorphism of T^*Q satisfying* (6.1)*. Then there exists $d_\phi \in \mathbb{Z}$ such that*

$$h_t(\Phi_{\mathscr{F}}) = d_\phi t = h_t(\Phi_{\mathscr{W}}).$$

In particular, Equation (6.1) *is automatically satisfied if the second Stiefel-Whitney class of Q vanishes.*

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