# LAURENT PHENOMENON FOR LANDAU–GINZBURG POTENTIAL

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Abstract. We prove that the Laurent polynomial $W = x + y + \frac{1}{xy}$ enjoys an excessive Laurent phenomenon: there are infinitely many birational coordinate changes that send $W$ to a Laurent polynomial, and there is a recursive way to produce them as consecutive mutations. Then we show that the Laurent polynomials obtained by our construction (as well as their Newton polytopes) are in one-to-one correspondence with Markov triples i.e. with natural solutions of the equation $a^2 + b^2 + c^2 = 3abc$.

## 1. Introduction

Let us first briefly recall the results of [1]. Let $S \subset \mathbb{Z}^2$ be the set of primitive vectors in $\mathbb{Z}^2$, i.e. vectors with coprime coordinates. For a vector $u \in S$ we define a piecewice linear mutation to be an automorphism of the set $\mathbb{Z}^2$ given by the formula:

$$\mu_u^t : v \mapsto v + \max(\langle u, v\rangle, 0)u,$$

where $\langle u, v\rangle$ is a antisymmetric bilinear form on $\mathbb{Z}^2$, normalized by $\langle(1, 0),(0, 1)\rangle = 1$.

For a vector $u \in S$ we define a mutation in the direction $u$ as a birational automorphism of $\mathbb{P}^2$ given by the formula:

$$\mu_{(m,n)} : x^a y^b \mapsto x^a y^b (1 + x^n y^{-m})^{an-bm}$$

There is a *tropicalisation* map that associates a piecewise-linear automorphism of $f^t \in PL(\mathbb{Z}^2)$ to every birational transformation $f \in \operatorname{Aut}\mathbb{C}(x, y)$ (we aditionally assume that $f$ preserves the volume form $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$). In particular, the piecewise-linear transformations $\mu^t_{(m,n)}$ are the tropicalisations of the birational transformations $\mu_{(m,n)}$.

The geometric meaning of the tropicalization is the following. Suppose we have a toric surface $X$ given by the fan $T$. Then $T' = \mu_v^t(T)$ is another fan, defining toric surface $X'$. Let $D_v$ be the toric divisor on $X$ corresponding to the vector $v$, and $s$ is the point on $D_v$ with coordinate $-1$. Let $D'_{-v}$ be the toric divisor on $X'$ corresponding to the vector $-v$, and $s'$ is the point on $D'_{-v}$ with coordinate $-1$. Then by the results of [1], there is a surface $\widetilde{X}$ and maps

$$\pi : \widetilde{X} \to X,$$
$$\pi' : \widetilde{X} \to X',$$

where $\pi$ is the blow-up of $X$ at $s$, and $\pi'$ is the blow-up of $X'$ at $s'$. This gives a resolution of birational isomorphism

$$\mu_v = \pi' \circ \pi^{-1}.$$

Moreover strict transform of toric divisors from $X$ to $\widetilde{X}$ equals strict transform of toric divisors from $X'$. The correspondence between toric divisors is given by the map $\mu^t$. Namely we have:

$$\pi^*_{st} D_t = \pi'^*_{st} D_{\mu^t_v(t)},$$

where $\pi^*_{st}$ denotes strict transform.

## 2. Mutations

2.1. **Properties of potential.** Consider a toric surface $X$ with rational function $F$, called potential. Let us introduce a curve $C$ defined by the formula:

$$C - \sum_t n_t D_t = (F),$$

where $\sum_t n_t D_t$ is the part of $(F)$ supported on toric divisors. The open toric orbit has specific toric coordinates $x, y$, which we use as rational coordinates on $X$. We denote $D_t$ the divisor corresponding to the ray $t \in \mathbb{Z}^2$, as well

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as all its strict transforms. If $t=(a,b)$, then the function $\frac{x^b}{y^a}$ gives a rational function $D_t \to \mathbb{P}^1$, which we call the canonical coordinate. We consider it up to taking its inverse. Each toric divisor has the point, where canonical coordinate equals $-1$. We denote the set of all such points by $\Omega$.

To such a pair $(X,C)$ we associate a set of vectors $V \subset \mathbb{Z}^2$ with multiplicities, which will encode the way the curve $C$ intersects toric divisors. If the curve $C$ intersects divisor corresponding to a vector $v$ transversally, then vector $v$ enters $V$ the number of times equal to the multiplicity of intersection. If the intersection of $C$ with such divisor is not transversal, then we count the correct multiplicities using blow-ups. Let $s \in D_v \subset X$ be a point where the canonical coordinate equals $-1$, and $C$ intersects $D_v$ in $s$. Then we make a blow-up of $X$ in $s$, and we denote $E_1$ the exceptional curve of the blow-up. Then we blow-up the point of intersection of $E_1$ and the strict transform of $D_v$, and we denote $E_2$ the exceptional curve of the blow-up. We continue by induction, so that $E_k$ is the exceptional curve of the blow-up at intersection of $E_{k-1}$ and the strict transform of $D_v$. We denote $n_k$ the index of intersection of the strict transform of $C$ with the curve $E_k \backslash (E_k \cap E_{k+1})$. In the last formula we just remove one point of intersection of $E_k$ with $E_{k+1}$. Of course, there will be only finite number of $E_k$ which intersect $C$, so we need to consider only finite number of blow-ups. Then vector $kv$ enters set $V$ with multiplicity $n_k$.

2.2. **The case of $\mathbb{P}^2$.** We consider a Laurent polynomial $W = x + y + \frac{1}{xy}$.

The curve defined by the equation $W=0$ is an elliptic curve, intersecting toric divisors at toric points. Let us consider a toric surface $X_0$ given by fan:

$$(2,-1),(1,-1),(0,-1),(-1,-1),(-1,0),(-1,1),(-1,2),(0,1),(1,0).$$

This surface is a blow-up of $\mathbb{P}^2$ at 6 points, and the strict transform of $W=0$ is the smooth elliptic curve $C_0$ that intersects transversally 3 toric divisors $D_{(2,-1)}, D_{(-1,-1)}, D_{(-1,2)}$. In particular, the set $V$ for the pair $(X_0,C_0)$ is $V_0 = \{(2,-1),(-1,-1),(-1,2)\}$.

By analogy with cluster mutations, we define the seed to be a triple $(X,F,(u,v,w))$, where $X$ is a toric surface, $F$ is a rational function on $X$, called potential, and $(u,v,w)$ is a triple of vectors in $\mathbb{Z}^2$. The seed can be mutated in either of three directions $u$,$v$ or $w$. The cluster mutation $\mu_u$ in the direction of $u$ is defined as:

$$u' = \mu_u^{seed}(u) = -u,$$
$$v' = \mu_u^{seed}(v) = \mu_u^t(v),$$
$$w' = \mu_u^{seed}(w) = \mu_u^t(w).$$

$X'$ is the toric surface, whose fan is obtained from the fan of $X$ by applying $\mu_u^t$. The function $F'$ is the pull-back of $F$ under birational isomorphism $\mu_v$. Note, that if compose mutation in direction $u$ with mutation in direction $-u$, then we obtain the seed, which is related to the original seed by the action of a unipotent element of $SL(2,\mathbb{Z})$.

We choose initial seed $(X_0,W,V_0)$, and then we start to apply mutations in different directions. In this way we obtain the set of seeds.

**Theorem 1.** *The function $F$ in all the seeds is a Laurent polynomial.*

*Proof.* Given a seed $(X,F,(u,v,w))$ we can define curve $C$ by the equation

$$C - \Sigma_t n_t D_t = (F),$$

where $\Sigma_v n_v D_v$ is the part corresponding to toric divisors. Recall, that there is surface $\widetilde{X}$ and maps

$$\pi : \widetilde{X} \to X,$$
$$\pi' : \widetilde{X} \to X',$$

where $\pi$ is a blow-up of the point on $D_v \subset X$, and $\pi'$ is a blow-up of the point on $D_{v'} \subset X'$.

For the seed $(X',F',(u',v',w'))$ we have the curve $C'$ given by

$$C' - \Sigma_t n'_t D_t = (F').$$

Now we prove the following

**Lemma 2.**
- *The intersection of $C'$ with toric divisors belongs to the set $\Omega$;*
- *If $t \in \{u',v',w'\}$, then the intersection index $k'_t$ of $C'$ with $D_t$ is such that $k'_t \geqslant n'_t$;*
- *$C'$ is an effective divisor;*
- *$C'$ is the union of an elliptic curve $A'$ and rational curves;*
- *$A'$ can only intersect 3 toric divisors $D_{u'}$,$D_{v'}$ and $D_{w'}$, and the intersection is transversal.*

*Proof.* Note that the statement is true for the initial seed. We have

$$C_0 - \sum_{t\in T} D_t = (W).$$

Now we suppose that the statement of the lemma is true for $(X, F, (u, v, w))$, and we verify it for $(X', F', (u', v', w'))$. The birational transformation $\mu_u : X \to X'$ is decomposed as a blow-up $\pi$ and blow-down $\pi'$. Let $E$ be the exceptional curve of $\pi$, and $E'$ the exceptional curve of $\pi'$. After blowing up $\pi$ at the intersection of $C$ and $D_u$ we have:

$$(\pi^* F) = \pi^*_{st} C + (k_u - n_u)E + \Sigma_v n_v D_v.$$

The divisor $C'' = \pi^* C + (k_u - n_u)E$ is effective, because $k_u \geqslant n_u$. From the other side

$$(\pi^* F') = (\pi^* F) = (\pi')^*_{st} C' + (k'_{-u} - n_{-u})E' + \Sigma_{v'} n_{v'} D_{v'}. \tag{3}$$

In [1] we proved, that canonical coordinates on toric divisors are preserved by $\mu_u$. It implies that the set $\Omega \subset X$ of points with coordinate $-1$ maps by $\mu_u$ to the corresponding set on $X'$, except for points on divisors $D_u, D_{-u}$, where we are making blow-ups. But $C'$ can intersect $D_u, D_{-u}$ only at the set $\Omega$, which proves the first statement of the lemma.

From 3 we deduce that

$$C'' = (\pi')^*_{st} C' + (k'_{-u} - n_{-u})E'.$$

As divisor $C''$ is effective, we have that $k'_{-u} \geqslant n_{-u}$. The intersection index of $C'$ with $D_t$ for $t \notin \{u, -u\}$ is the same as the corresponding intersection of $C$. This implies the second statement of the lemma. Moreover as strict transform of $C'$ is effective, then $C'$ is effective as well.

We also have:

$$C' = \pi'_* \circ (\pi^*_{st} C + (k_v - n_v)E),$$

which implies that $C'$ contains elliptic curve $A' = \pi'_* \circ \pi^*_{st}(A)$, and possibly additional rational curve $\pi'(E)$, which proves the third statement. $C'$ intersects toric divisors

The strict transform $\pi^*_{st}(A)$ only intersects divisors $D_v, D_w$. So divisor $A' = \pi'_* \circ \pi^*_{st}(A)$ can only intersect $D_{u'}, D_{v'}, D_{w'}$. □

This lemma implies that the divisor $F$ defines effective curve on the open toric orbit, in other words it has poles only on the locus of toric divisors. Therefore, $F$ is a Laurent polynomial.

□

**Lemma 4.** *Suppose that $(u, v, w)$ are vectors from the seed in the counter-clock-wise order. Consider the triple of positive integers*

$$(a, b, c) = (\langle u, v\rangle, \langle v, w\rangle, \langle w, u\rangle).$$

*We claim that*

- *$(a, b, c)$ are positive numbers for all the seeds.*
- *$(a, b, c)$ satisfy Markov's equation $a^2 + b^2 + c^2 = abc$*
- *for each positive solution of Markov's equation $a^2 + b^2 + c^2 = abc$ there is a seed with the respective pairings*

*Proof.* For the starting seed $(X_0, W, V_0)$ we have $(a, b, c) = (3, 3, 3)$. Mutation $\mu_u$ sends $(u, v, w)$ to $(v, -u, w + \langle u, w\rangle u)$. The triple $(a, b, c)$ goes to

$$(a, c, ac - b).$$

Note, that transformation $(a, b, c) \mapsto (a, c, ac - b)$ is the same, as the law for producing Markov numbers. This triple verify the formula:

$$a^2 + b^2 + c^2 = abc.$$

For fixed $a, c$ it is a quadratic equation on $b$. So we can find another root by formula: $b' = ac - b$ or $b' = \frac{a^2+c^2}{b}$. The second formula implies that this numbers are always positive. □

This lemma implies, that vectors $(u, v, w)$ from the seed are not colinear. From the other side Lemma 2 implies that elliptic curve $A$ intersects toric divisors only at $D_u, D_v, D_w$. Let $e_u, e_v, e_w$ be the corresponding indexes of intersection. Then the intersection theory on toric surfaces implies, that

$$e_u u + e_v v + e_w w = 0.$$

As we know that $(u, v, w)$ are not colinear, we deduce that $e_u, e_v, e_w$ are non-zero, thus $A$ has non-zero intersection with $D_u, D_v, D_w$. In particular, vectors $(u, v, w)$ can be reconstructed from $(X, F)$.

The potential $W = x + y + \frac{1}{xy}$ can be interpreted as a mirror image of a complex projective plane $\mathbb{CP}^2$. By results of Cho–Oh the Laurent polynomial $W$ equals to the disc-counting function $m_0(L_{Cl})$ for the Clifford torus $L_{Cl}$, a monotone special Lagrangian torus on $\mathbb{P}^2$ given as a central fiber of the moment map $\mathbb{P}^2 \to \Delta$. By theorem 1 and lemma 4 we proved that there are infinitely many birational transformations $f : (x, y) \to (x', y')$ such that a priori rational function $W' = f^*W$ is a Laurent polynomial. Each $W'$ of this kind can also be considered as a non-standard mirror image of $\mathbb{P}^2$.We conjecture that for each $W'$ there exists a monotone special Lagrangian torus $L'$ on $\mathbb{CP}^2$ such that $W' = m_0(L')$ i.e. $W'$ is Fukaya–Oh–Ohta–Ono's generating function for Maslov index 2 holomorphic discs on $\mathbb{P}^2$ with boundary on $L'$.

## References


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