# VANISHING THEOREMS FOR REAL ALGEBRAIC CYCLES 

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#### Abstract

We establish the analogue of the Friedlander-Mazur conjecture for Teh's reduced Lawson homology groups of real varieties, which says that the reduced Lawson homology of a real quasi-projective variety $X$ vanishes in homological degrees larger than the dimension of $X$ in all weights. As an application we obtain a vanishing of homotopy groups of the mod-2 topological groups of averaged cycles and a characterization in a range of indices of the motivic cohomology of a real variety as homotopy groups of the complex of averaged equidimensional cycles. We also establish an equivariant Poincare duality between equivariant Friedlander-Walker real morphic cohomology and dos Santos' real Lawson homology. We use this together with an equivariant extension of the mod-2 Beilinson-Lichtenbaum conjecture to compute some real Lawson homology groups in terms of Bredon cohomology.


## Contents

1. Introduction ..... 1
2. Equivariant Homotopy and Cohomology .....
3. Topological Spaces of Cycles ..... 7
4. Poincare Duality ..... 15
5. Compatibility of Cycle Maps ..... 23
6. Vanishing Theorem ..... 31
7. Reduced Topological Cocycles ..... 37
Appendix A. Topological Monoids ..... 44
Appendix B. Tractable Monoid Actions ..... 46
References ..... 49

## 1. Introduction

Let $X$ be a quasi-projective real variety. The Galois group $G=G a l(\mathbb{C} / \mathbb{R})$ acts on $\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)$, the topological group of $q$-cycles on the complexification. Cycles on the real variety $X$ correspond to cycles on $X_{\mathbb{C}}$ which are fixed by conjugation. Inside the topological group of $\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}$ of cycles fixed by conjugation is the topological group $\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}$ of averaged cycles which are the cycles of the form $\alpha+\bar{\alpha}$. The space of reduced cycles on $X$ is the quotient topological group

$$
\mathcal{R}_{q}(X)=\frac{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}}
$$

Homotopy groups of some of the above abelian topological groups are related to classical topological invariants. For example for $X$ a projective real variety

[^0]we obtain the singular homology groups $\pi_{*} \mathcal{R}_{0}(X)=H_{*}(X(\mathbb{R}), \mathbb{Z} / 2)$ Teh05 and $\pi_{*} \mathcal{Z}_{0}\left(X_{\mathbb{C}}\right)^{a v}=H_{*}\left(X_{\mathbb{C}}(\mathbb{C}) / G, \mathbb{Z}\right)$ LLFM03], as well as Bredon homology $\pi_{*} \mathcal{Z}_{0}(X)=$ $H_{*, 0}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z}}\right)$ LF97]. Other homotopy groups are related to classical algebraic geometry invariants. For example $\pi_{0}\left(Z_{r}\left(X_{\mathbb{C}}\right)^{G}\right)$ computes the group of algebraic cycles of dimension r on $X$ modulo real algebraic equivalence FW02a and consequently with $\mathbb{Z} / n$ coefficients equals the Chow group $C H_{r}(X) \otimes \mathbb{Z} / n$ (see Proposition 5.1). However most of these homotopy groups remain a mysterious combination of topological and algebraic information of the real variety $X$.

These homotopy groups are hard to compute and examples are few. Nonetheless an examination of existing computations shows that these homotopy groups are all zero in large degrees. For example, in ( Lam90 ) Lam proves that

$$
\mathcal{R}_{q}\left(\mathbb{P}_{\mathbb{R}}^{n}\right) \simeq \prod_{i=0}^{n-q} K(\mathbb{Z} / 2, i)
$$

In particular $\pi_{k}\left(\mathcal{R}_{q}\left(\mathbb{P}_{\mathbb{R}}^{n}\right)\right)=0$ for $k>n-q$. Similar vanishing results are seen in the computations of LLFM05 for a real variety $X$ with the property that its complexification is the quaternionic projective space (see Example 6.14).

In Teh08 Teh proves a conditional Harnack-Thom type theorem for the homotopy groups of reduced algebraic cycles on $X$ which holds under the assumption that these homotopy groups are all finitely generated and they are zero in high degrees. In the case of divisors he shows that

$$
\pi_{k} \mathcal{R}_{d-1}(X)=0
$$

when $k \geq 3$ for any smooth projective real variety $X$ of dimension $d$.
The main theorem of this paper provides this vanishing in general and should be viewed as a massive generalization of both the classical vanishing of singular homology groups of a manifold in degree larger than the manifold and of the vanishing results discussed above.

Theorem 1.1. Let $X$ be a quasi-projective real variety. Then

$$
\pi_{k} \mathcal{R}_{q}(X)=0
$$

for $k \geq \operatorname{dim} X-q+1$.
In the case of divisors our result improves the previously known vanishing range. The case of real projective space described above shows that the theorem's vanishing range is optimal.

The homotopy groups of reduced algebraic cycles $R_{q}(X)$ define a homology theory for real quasi-projective varieties $X$ introduced in Teh05 which is defined by $R L_{q} H_{n}(X)=\pi_{n-q}\left(\mathcal{R}_{q}(X)\right)$ for $n \geq q$ and called reduced Lawson homology. In this notation our vanishing result reads $R L_{q} H_{n}(X)=0$ for any $n>\operatorname{dim}(X)$. Thus our vanishing result shows that the Friedlander-Mazur conjecture holds for the reduced Lawson homology of real varieties.

The homotopy groups of $\mathcal{R}_{q}(X)$ fit into a long exact sequence

$$
\cdots \rightarrow \pi_{k+1} \mathcal{R}_{q}(X) \rightarrow \pi_{k} \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v} \rightarrow L_{q} H \mathbb{R}_{q-k, q}(X) \rightarrow \pi_{k} \mathcal{R}_{q}(X) \rightarrow \cdots
$$

where $L_{q} H \mathbb{R}_{q-k, q}(X)=\pi_{k} \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}$ is the real Lawson homology introduced by dos Santos in dS03a. As a consequence of Suslin rigidity the homotopy groups of $\mathcal{R}_{q}(X)$ are also related to motivic cohomology of $X$

$$
\cdots \rightarrow \pi_{k} z_{\text {equi }}\left(\mathbb{A}_{\mathbb{C}}^{q}, 0\right)\left(X_{\mathbb{C}} \times \Delta_{\mathbb{C}}^{\bullet}\right)^{a v} \rightarrow H_{\mathcal{M}}^{2 q-k}(X ; \mathbb{Z}(q)) \rightarrow \pi_{k} \mathcal{R}_{q}(X) \rightarrow \cdots
$$

Thus an immediate corollary of the vanishing theorem is an identification of the homotopy groups of the space of averaged cycles and of the complex of averaged equidimensional cycles.

Corollary 1.2. Let $X$ be a smooth quasi-projective real variety. Then for any $k \geq \operatorname{dim} X-q+1$

$$
L_{q} H \mathbb{R}_{q-k, q}(X)=\pi_{k} \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}
$$

and

$$
H_{\mathcal{M}}^{2 q-k}(X ; \mathbb{Z}(q))=\pi_{k} z_{e q u i}\left(\mathbb{A}_{\mathbb{C}}^{q}, 0\right)\left(X_{\mathbb{C}} \times \Delta_{\mathbb{C}}^{\bullet}\right)^{a v}
$$

Theorem 1.1 also implies that the mod-2 homotopy groups of the topological group of average cycles satisfy an optimal vanishing (see also Example 6.19).

Corollary 1.3. Let $X$ be a smooth projective real variety of dimension d. Then

$$
\pi_{n} \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}=0
$$

for $n \geq 2 d-2 p+1$.
An essential ingredient in the proof of our vanishing theorem is the Milnor conjecture proved by Voevodsky in Voe03. The Milnor conjecture relates motivic cohomology and etale cohomology while real morphic cohomology naturally compares with Bredon cohomology. We need to know that these cycle maps are suitably related which is done in Theorem 5.9,

Theorem 1.4. Let $X$ be a smooth quasi-projective real variety. The diagram commutes

where $H^{p-q, q}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2\right)$ denotes Bredon cohomology and $H_{G}^{p}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2\right)$ denotes Borel cohomology.

This suggests that there are possible advantages in replacing the map on Chow groups of real cycles into Borel cohomology with the map into Bredon cohomology since in many respects Bredon cohomology behaves better than Borel cohomology. An application of this idea will be given in a forthcoming paper.

Together with the mod-2 Beilinson-Lichtenbaum conjecture for real and complex varieties (which is a consequence of the Milnor conjecture by [SV00a) we conclude an equivariant Beilinson-Lichtenbaum type theorem for an equivariant extension of Friedlander-Walker's real morphic cohomology groups (see Definition 3.13).

Theorem 1.5. Let $X$ be a smooth quasi-projective real variety and $k>0$. The cycle map

$$
\Phi: L^{q} H \mathbb{R}^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z} / 2^{k}}\right)
$$

is an isomorphism if $r \leq 0$ (and $s \leq q$ ) and an injection if $r=1$ (and $s \leq q$ ).

Using Friedlander-Voevodsky duality for bivariant cycle theory we show in Corollary 4.20 that the equivariant morphic cohomology and real Lawson homology groups are isomorphic through a Poincare duality. As a consequence the equivariant Beilinson-Lichtenbaum says that in a range we may compute the mod- 2 real Lawson homology groups in terms of mod-2 Bredon cohomology. This allows a computation for curves with integral coefficients.

Corollary 1.6. Let $X$ be a smooth real curve. Then

$$
L^{q} H \mathbb{R}^{r, s}(X ; \mathbb{Z}) \rightarrow H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z}}\right)
$$

is an isomorphism for any $q \geq 0, r \leq q$, and $s \leq q$.
The space of reduced cocycles on $X$, related via Poincare duality with the space of reduced cycles, is defined as

$$
\mathcal{R}^{q}(X)=\frac{\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v}}
$$

where $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)$ is the space of algebraic cocycles on $X_{\mathbb{C}}$ and $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}$ agrees with the space of real cocycles introduced by Friedlander-Walker in FW02a (see Proposition 3.7). There is a natural comparison map

$$
\mathcal{R}^{q}(X) \rightarrow \operatorname{Map}\left(X(\mathbb{R}), \mathcal{R}_{0}\left(\mathbb{A}^{q}\right)\right)
$$

and since $\mathcal{R}_{0}\left(\mathbb{A}^{q}\right)=K(\mathbb{Z} / 2, q)$ this provides a natural map

$$
\begin{equation*}
\text { cyc } c_{k}: \pi_{k} \mathcal{R}^{q}(X) \rightarrow H_{\text {sing }}^{q-k}(X(\mathbb{R}) ; \mathbb{Z} / 2) \tag{1.7}
\end{equation*}
$$

which is the cycle map for reduced morphic cohomology groups defined in Teh05. Via Poincare duality the vanishing theorem is equivalent to the statement that $c y c_{k}$ is an isomorphism for $k>q$.

Via the Milnor conjecture over $\mathbb{C}$ and over $\mathbb{R}$ we can deduce an isomorphism $\pi_{k} \mathcal{R}^{q}(X) \rightarrow \pi_{k} \mathcal{R}_{\text {top }}^{q}(X)$ for $k \geq q$. Here $\mathcal{R}_{t o p}^{q}(X)$ is the group of "reduced topological cocycles". For a precise definition see Section 7 but essentially this is a version of the quotient group $\operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G} / \operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{\text {av }}$ which has reasonable homotopical properties (such as fitting into a homotopy fiber sequence involving $\operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}$ and $\left.\operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{a v}\right)$.

The final ingredient for our vanishing theorem is now provided by Corollary 7.14 which shows that for $X$ projective,

$$
\widetilde{\Phi}: \pi_{k} \mathcal{R}^{q}(X) \rightarrow \pi_{k} \mathcal{R}_{t o p}^{q}(X)
$$

agrees with the cycle map $\pi_{k} \mathcal{R}^{q}(X) \rightarrow H_{\text {sing }}^{q-k}(X(\mathbb{R}) ; \mathbb{Z} / 2)$ for $k \geq 2$.
Here is a short outline of the paper. In the second section we review the equivariant homotopy used in the paper. The third section is dedicated to introducing the topological spaces of cycles we study and proving some basic properties that we use and for which we don't find exact references in the literature. In the fourth section we prove a Poincare Duality between equivariant morphic cohomology and real Lawson homology. In the fifth section we discuss the cycle maps from equivariant morphic cohomology and Bredon cohomology and equivariant applications of the Beilinson-Lichtenbaum conjecture. The sixth section is devoted to the proof of our main vanishing Theorem. One of the main technical ingredients of this proof is left for section seven where we reinterpret the cycle map 1.7 from reduced Lawson
homology groups to the singular homology in a manner needed to prove our vanishing theorem. The paper ends with two appendixes where we prove and recollect a few results on topological monoids used in the paper.

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Notation: By a quasi-projective $k$-variety we mean a reduced and separated quasi-projective scheme of finite type over a field $k$. We write $S c h / k$ for the category of quasi-projective $k$-varieties and $S m / k$ for the subcategory of smooth quasiprojective $k$-varieties. Except in section $2, G$ always denotes $G a l(\mathbb{C} / \mathbb{R})$ and $\sigma \in G$ denotes the nontrivial element.

## 2. Equivariant Homotopy and Cohomology

We recall the basic definitions and theorems we need from equivariant homotopy theory. For more details see May96. In this paper we will only work with $G=$ $\mathbb{Z} / 2$, but since no simplification results in the basic definitions, we let $G$ denote an arbitrary finite group. The category $T o p^{G}$ of $G$-spaces consists of compactly generated spaces equipped with a left $G$-action and morphisms are continuous $G$ equivariant maps. If $X$ is a $G$-space and $H \subseteq G$ is a subgroup write $X^{H}$ for the subspace of all points fixed by $H$. The category $T o p_{*}^{G}$ of based $G$-spaces consists of $G$-spaces $X$ together with a $G$-invariant basepoint $x \in X$ and maps are base-point preserving equivariant maps. A space together with a disjoint, invariant base-point will be denoted $X_{+}$.

Equivariant homotopy theory. Let $I$ denote the unit interval with trivial $G$ action. A $G$-homotopy between two equivariant maps $f, g: X \rightarrow Y$ is an equivariant $\operatorname{map} F: X \times I \rightarrow Y$ such that $\left.F\right|_{X \times\{0\}}=f$ and $\left.F\right|_{X \times\{1\}}=g$. An equivariant map $f: X \rightarrow Y$ is an equivariant homotopy equivalence provided there is an equivariant map $g: Y \rightarrow X$ such that both $f \circ g$ and $g \circ f$ are $G$-equivalently homotopic to the identity. An equivariant map $f: X \rightarrow Y$ is a $G$-weak equivalence provided both $f^{H}: X^{H} \rightarrow Y^{H}$ is a non-equivariant weak equivalence for all subgroups $H \subseteq G$. Formally inverting the $G$-weak equivalences gives the homotopy category of $G$ spaces. Similarly inverting the based $G$-weak equivalences between based $G$-spaces we obtain the based $G$-homotopy category. Write $[X, Y]_{G}$ for classes of based maps in the homotopy category of based maps.

A $G$ - $C W$ complex $X$ is a topological union $X=\cup X_{n}$ of $G$-spaces such that $X_{0}$ is a disjoint union of orbits $G / H$ and $X_{n}$ is obtained from $X_{n-1}$ by attaching cells of the form $D^{n} \times G / H$ via attaching maps $\sigma: S^{n-1} \times G / H \rightarrow X_{n-1}$.

The equivariant Whitehead theorem holds for $G-C W$ complexes. That is, if $f: X \rightarrow Y$ is a $G$-equivariant weak equivalence between $G-C W$-complexes then $f$ is a $G$-homotopy equivalence.

A map $A \rightarrow X$ is said to have the homotopy extension property with respect to $Z$ if for any equivariant partial homotopy $H: X \times\{0\} \coprod_{A \times\{0\}} A \times I \rightarrow Z$ there is
an equivariant map $H^{\prime}$ making the diagram below commute


An equivariant cofibration $A \hookrightarrow X$ is an equivariant map which has the homotopy extension property with respect to all $Z$ in $T o p^{G}$. Inclusions of sub-G$C W$ complexes $A \subseteq X$ are equivariant cofibrations.

Let $V$ be a real representation of $G$, write $S^{V}$ for the one-point compactification of $V$. The $V$ th homotopy group of a based $G$-space $X$ is

$$
\pi_{V} X=\left[S^{V}, X\right]_{G}
$$

Note that $S^{V}$ always has at least two fixed points, 0 and $\infty$.
When $G=\mathbb{Z} / 2$ and $V=\mathbb{R}^{p, q}$, where $V=\mathbb{R}^{p+q}$ with $G$ acting trivially on the first $p$-components and on the last $q$-components the $G$ action is multiplication by -1 we use the notation

$$
\pi_{p, q} X=\pi_{\mathbb{R}^{p, q}} X
$$

Borel homology and cohomology. The Borel-equivariant cohomology of $X$ with coefficients in an abelian group $A$ is defined to be the ordinary singular cohomology of the homotopy orbit space of $X$ :

$$
H_{G}^{p}(X ; A)=H^{p}((X \times E G) / G ; A)
$$

Similarly the Borel-equivariant homology is defined to be

$$
H_{p}^{G}(X ; A)=H_{p}((X \times E G) / G ; A)
$$

When $X$ has free $G$-action then $(X \times E G) / G \rightarrow X / G$ is a homotopy equivalence and therefore when $X$ has free $G$-action $H^{p}(X / G ; A) \cong H_{G}^{p}(X ; A)$ and $H_{p}(X / G ; A) \cong H_{p}^{G}(X ; A)$.

Mackey functors. Bredon homology and cohomology take Mackey functors as coefficients. There are several equivalent ways to define a Mackey functor May96. Classically for $G$ a finite group one defines a Mackey functor as follows . Let $\mathcal{F}_{G}$ denote the category of finite $G$-sets as objects and with equivariant set maps as morphisms. A Mackey functor $\underline{M}$ consists of a pair of abelian-group valued functors $\underline{M}=\left(M^{*}, M_{*}\right)$ on $\mathcal{F}_{G}$, with $M^{*}$ contravariant and $M_{*}$ covariant. The functors $M^{*}$ and $M_{*}$ satisfy the following requirements.
(1) $M^{*}, M_{*}$ take the same value on objects and convert disjoint unions of $G$-sets into products of abelian groups.
(2) When

is a pull-back square of finite $G$-sets then

is a commutative square of abelian groups.
Given an abelian group $A$, the constant Mackey functor $\underline{A}$ is the Mackey functor which on objects $\underline{A}(G / K)=A$ and on a map $f: G / H \rightarrow G / K, M^{*}(f)=i d$ and $M_{*}(f)$ is multiplication by the index $[K: H]$.

Bredon homology and cohomology. Bredon cohomology (homology) with coefficients in a Mackey functor $\underline{M}$ is a cohomology (homology) theory $H^{*}(-; \underline{M})$ ) $\left(H_{*}(-; \underline{M})\right.$ graded by $R O(G)$. For $V \in R O(G)$ there is an equivariant EilenbergMaclane space $K(\underline{M}, V)$ which represents the reduced cohomology $\tilde{H}^{V}(X ; \underline{M})$ for a based $G$-space,

$$
\tilde{H}^{V}(X ; \underline{M})=[X, K(\underline{M}, V)]_{G}
$$

When $G=\mathbb{Z} / 2$ then $R O(G)=\mathbb{Z} \oplus \mathbb{Z}$ with generators $\mathbb{R}^{1,0}$ and $\mathbb{R}^{0,1}$. We use the convention that $H^{p, q}(X ; \underline{M})=H^{\mathbb{R}^{p, q}}(X ; \underline{M})$ (and similarly for homology).

If $A$ is an abelian group (with trivial $G$-action) then $H^{p, 0}(X ; \underline{A})=H_{\text {sing }}^{p}(X / G ; A)$. More generally, Borel and Bredon cohomology are related by the natural isomorphism

$$
H^{p, q}(X \times E G ; \underline{A}) \cong H_{G}^{p+q}(X ; A(q))
$$

where $A(q)$ is $A$ with $\sigma$ acting by $(-1)^{q}$ (see [dSLF04, Proposition 1.15]).
Equivariant Dold-Thom theorem. Let $X$ be a compactly generated Hausdorff space. The free abelian group on the points of $X$ is defined to be $\mathcal{Z}_{0}(X)=$ $\left[\coprod_{d} S P^{d}(X)\right]^{+}$, where $S P^{d}(X)$ is the $d$ th symmetric product on $X$ and $(-)^{+}$denotes group completion of the displayed monoid which is topologized via the quotient topology.

The degree homomorphism $\operatorname{deg}: \mathcal{Z}_{0}(X) \rightarrow \mathbb{Z}$ is defined by $\operatorname{deg}\left(\sum n_{i} x_{i}\right)=\sum n_{i}$ is a continuous homomorphism. Write $\mathcal{Z}_{0}(X)_{0}$ for the kernel of this map. Notice that there is an isomorphism of topological groups $\mathcal{Z}_{0}(X) \cong \mathcal{Z}_{0}\left(X_{+}\right)_{0}$.

If $X$ is a $G$-space then the action on $X$ induces a $G$-action on $\mathcal{Z}_{0}(X)$ and $\mathcal{Z}_{0}(X)_{0}$. By [LF97, Corollary 2.9] when $X$ is a $G$ - $C W$ complex so is $\mathcal{Z}_{0}(X)$.

The classical Dold-Thom theorem says that $\pi_{n} \mathcal{Z}_{0}(X)_{0}=\tilde{H}_{n}(X ; \mathbb{Z})$ and the equivariant Dold-Thom theorem proved by Lima-Filho LF97 and dos Santos dS03b says that

$$
\pi_{V} \mathcal{Z}_{0}(X)_{0}=\tilde{H}_{V}(X ; \underline{\mathbb{Z}})
$$

In particular $\mathcal{Z}_{0}\left(S^{V}\right)_{0}$ is an Eilenberg-Maclane space $K(\underline{Z}, V)$.

## 3. Topological Spaces of Cycles

Group completions of monoids. Let $M$ be a compactly generated Hausdorff topological abelian monoid. The naive group completion of $M$ is the quotient of $M \times M$ by the monoid action of $M$ where $M$ acts by $(a, b) \mapsto(m+a, m+b)$. Write $M^{+}$for this abelian group, which is topologized as the quotient of $M \times M$. Recall that $M$ is said to satisfy the cancellation property if $a+m=b+m$ implies that $a=b$ for any $a, b, m \in M$. When $M$ satisfies cancellation then the naive
group completion can be described as $M^{+}=M \times M / \sim$, where $(a, b) \sim(c, d)$ if $a+d=b+c$.

Naive group completion does not generally behave well topologically. For example it may happen that $M^{+}$is not a Hausdorff topological group nor is it clear how homotopy invariants of $M$ and $M^{+}$are related. Friedlander-Gabber [FG93] and Lima-Filho LF93] have studied conditions under which the naive group completion of a topological monoid is a Hausdorff group and $M \rightarrow M^{+}$is a homotopy group completion. All of the topological monoids with which we work are tractable monoids in the sense of Friedlander-Gabber (see Appendix (B) and in particular the naive group completion of these groups are homotopy group completions.

Our main objects of interest are the group completions of submonoids of the Chow monoids of effective algebraic cycles on algebraic varieties. Let $k$ be a field of characteristic 0 and $j: Y \subseteq \mathbb{P}_{k}^{n}$ be a projective $k$-variety. The Chow variety $\mathcal{C}_{q}(Y, j)=\coprod_{d \geq 0} \mathcal{C}_{q, d}(Y, j)$ of effective $q$-dimensional cycles on $Y$ is an (infinite, disjoint) union of projective $k$-varieties. See [Fri91 for details.

Cycle-spaces over $\mathbb{C}$. Let $Z$ be a complex variety. Denote the set of complex points equipped with the analytic topology by $Z(\mathbb{C})^{a n}$. Since there will be little chance for confusion we will often simply write this space as $Z(\mathbb{C})$ with the topology understood. If $j: Y \subseteq \mathbb{P}_{\mathbb{C}}^{n}$ is a projective variety then $\mathcal{C}_{q}(Y, j)(\mathbb{C})$ is a topological monoid and we will generally omit $j$ from the notation since the homeomorphism type of this space is independent of $j$.

The monoid $\mathcal{C}_{q}(Y)(\mathbb{C})^{a n}$ is tractable and therefore the naive group completion is a homotopy group completion. Write

$$
\mathcal{Z}_{q}(Y)=\left(\mathcal{C}_{q}(Y)(\mathbb{C})\right)^{+}
$$

for the naive group group completion of this monoid. Define the filtration $\{0\} \subseteq$ $\cdots \subseteq \mathcal{Z}_{q, \leq d}(Y) \subseteq \mathcal{Z}_{q, \leq d+1}(Y) \subseteq \cdots \subseteq \mathcal{Z}_{q}(Y)$ by

$$
\mathcal{Z}_{q, \leq d}(Y)=\left(\coprod_{d_{1}+d_{2} \leq d} \mathcal{C}_{q, d_{1}}(Y)(\mathbb{C}) \times \mathcal{C}_{q, d_{2}}(Y)(\mathbb{C})\right) / \sim \subseteq \mathcal{Z}_{q}(Y)
$$

By LF93] each $\mathcal{Z}_{q, \leq d}(Y)$ is a closed, compact Hausdorff space and $\mathcal{Z}_{q}(Y)$ has the weak topology with respect to this filtration.

When $U$ is quasi-projective with projectivization $U \subseteq \bar{U}$ then define $\mathcal{Z}_{q}(U)=$ $\mathcal{Z}_{q}(\bar{U}) / \mathcal{Z}_{q}\left(U_{\infty}\right)$ where $U_{\infty}=\bar{U} \backslash U$. The images of $\mathcal{Z}_{q, \leq k}(\bar{U})$ in the quotient $\mathcal{Z}_{q}(U)$ give a filtration by compact subspaces (and $\mathcal{Z}_{q}(U)$ has the weak topology with respect to this filtration). This definition is independent of choice of projectivization LF92, [FG93.

Cycle-spaces over $\mathbb{R}$. Suppose that $Z$ is a real variety. Write the set of $\mathbb{R}$-points equipped with the analytic topology as $Z(\mathbb{R})^{a n}$ or simply $Z(\mathbb{R})$ with the topology understood.

Let $Y$ be a projective real variety. Consider the topological monoid $\mathcal{C}_{q}(Y)(\mathbb{R})$. As explained in the proof of FW02a, Proposition 8.2] (see Proposition B.5), the topological monoid $\mathcal{C}_{q}(Y)(\mathbb{R})$ is tractable and therefore its naive group completion is a homotopy group completion. Write

$$
\mathcal{Z}_{q}(Y)=\left(\mathcal{C}_{q}(Y)(\mathbb{R})\right)^{+}
$$

for the naive group completion.

Suppose that $U$ is a quasi-projective real variety with projectivization $U \subseteq \bar{U}$. Define the topological group of $q$-cycles on the quasi-projective real variety $U$ to be

$$
\mathcal{Z}_{r}(U)=\mathcal{Z}_{r}(\bar{U}) / \mathcal{Z}_{r}(\bar{U} \backslash U)
$$

If $X$ is a real variety and $\pi: X_{\mathbb{C}} \rightarrow X$ is its complexification then $G$ acts on $X_{\mathbb{C}}(\mathbb{C})$ and induces a homeomorphism

$$
X(\mathbb{R}) \stackrel{\cong}{\Longrightarrow} X_{\mathbb{C}}(\mathbb{C})^{G}
$$

In particular if $X$ is a projective real variety then by [Fri91, Proposition 1.1] $\mathcal{C}_{q}\left(X_{\mathbb{C}}\right)=\mathcal{C}_{r}(X)_{\mathbb{C}}$ and so we have the isomorphism of topological monoids

$$
\mathcal{C}_{r}(X)(\mathbb{R}) \stackrel{\cong}{\Longrightarrow} \mathcal{C}_{r}\left(X_{\mathbb{C}}\right)(\mathbb{C})^{G}
$$

Proposition 3.1. Let $U$ be a quasiprojective real variety. Then

$$
\mathcal{Z}_{r}(U) \xrightarrow{\cong} \mathcal{Z}_{r}\left(U_{\mathbb{C}}\right)^{G}
$$

is an isomorphism of topological abelian groups. In particular the group $\mathcal{Z}_{r}(U)$ is independent of projectivization $U \subseteq \bar{U}$.

Proof. For $U$ projective this follows immediately from Proposition A.3. The quasiprojective case now follows by a comparison of short exact sequences of topological abelian groups

where the exactness of the bottom row is a consequence of the Lemma 3.3
Remark 3.2. Let $U$ be a quasi-projective real variety. Since $+: \mathcal{Z}_{k}\left(U_{\mathbb{C}}\right) \times \mathcal{Z}_{k}\left(U_{\mathbb{C}}\right) \rightarrow$ $\mathcal{Z}_{k}\left(U_{\mathbb{C}}\right)$ is closed we see by taking $G$-fixed points that $+: \mathcal{Z}_{k}(U) \times \mathcal{Z}_{k}(U) \rightarrow \mathcal{Z}_{k}(U)$ is a closed map for any real variety $U$.

Lemma 3.3. Let $Y \subseteq X$ be a closed subvariety of a real projective variety. Then

$$
\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)^{G} / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)^{G} \cong\left(\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right) / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)\right)^{G}
$$

and

$$
\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)^{a v} / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)^{a v} \cong\left(\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right) / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)\right)^{a v}
$$

are isomorphisms of topological groups.
Proof. Consider the quotient maps $\pi: \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right) / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)$ and $q: \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)^{G} \rightarrow$ $\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)^{G} / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)^{G}$. Consider the filtration $\left\{\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right\}\right)^{G}$ of $\left(\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right) / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)\right)^{G}$ and the filtration $\left\{q\left(\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G}\right)\right\}$ of $\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)^{G} / \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)^{G}$. These spaces have the weak topology given by these filtrations so it is enough to see that

$$
q\left(\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G}\right) \rightarrow\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right)^{G}
$$

is a homeomorphism for all $d$.
First we show that $\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G} \rightarrow\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right)^{G}$ is surjective. If $[\eta] \in\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right)^{G}$, we can choose a representative $\eta=\sum n_{V} V \in \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}$ such that each $V \nsubseteq Y_{\mathbb{C}}$. Since $\eta-\bar{\eta} \in \mathcal{Z}_{r}\left(Y_{\mathbb{C}}\right)$ (and each $\bar{V} \nsubseteq Y_{\mathbb{C}}$ ) we see that $\eta=\bar{\eta}$ and therefore the map

$$
\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G} \rightarrow\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right)^{G}
$$

is surjective. The map

$$
q \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G} \rightarrow\left(\pi \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}\right)^{G}
$$

is easily seen to be injective and since $\mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)_{\leq d}^{G}$ is compact this map is closed so is a homeomorphism.

The second statement about the topological group of averaged cycles is proved in a similar fashion.

Spaces of algebraic cocycles. In this section we recall the construction of topological monoids of algebraic cocycles [FL92, Fri98]. Let $X, Y$ be quasi-projective real varieties over $k=\mathbb{C}$ or $\mathbb{R}$. Write $\operatorname{Mor}_{k}(X, Y)$ for the set of continuous algebraic maps between $X$ and $Y$. When $X$ is semi-normal then $\operatorname{Mor}_{k}(X, Y)=\operatorname{Hom}(X, Y)$. Friedlander-Walker construct "analytic" topologies on $\operatorname{Mor}_{k}(X, Y)$ in FW01b for $k=\mathbb{C}$ and in FW02a for $k=\mathbb{R}$. The set of continuous algebraic maps with this topology will be written $\operatorname{Mor}_{k}(X, Y)^{a n}$. By [FW02a, Lemma 1.2] $\operatorname{Mor}_{\mathbb{R}}(X, Y)^{a n}=$ $\left(\operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, Y_{\mathbb{C}}\right)^{a n}\right)^{G}$.

When $X, Y$ are projective real varieties and $W, Z$ are projective complex varieties then this topology coincides with the subspace topology induced by the inclusions

$$
\operatorname{Mor}_{\mathbb{C}}(W, Z) \subseteq \operatorname{Map}(W(\mathbb{C}), Z(\mathbb{C}))
$$

and

$$
\operatorname{Mor}_{\mathbb{R}}(X, Y) \subseteq \operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), Y_{\mathbb{C}}(\mathbb{C})\right)^{G}
$$

where $\operatorname{Map}(-,-)$ denotes the space of continuous maps is with compact-open topology. When the domain is only quasi-projective then the analytic topology on the algebraic mapping spaces is no longer the compact-open topology but rather the topology of convergence with bounded degree (see [FL97, Appendix A]).

Let $W$ be a quasi-projective complex variety and $Z$ be a projective complex variety. Write $d=\operatorname{dim} W$. Let $\mathcal{C}_{r}(Z)(W)$ denote the monoid of effective cycles on $W \times Z$ equidimensional or relative dimension $r$ on $W$. This is made into a topological monoid via the subspace topology induced by the inclusion

$$
\mathcal{C}_{r}(Z)(W) \subseteq \mathcal{C}_{d+r}(W \times Z) \stackrel{\text { def }}{=} \frac{\mathcal{C}_{d+r}(\bar{W} \times Z)}{\mathcal{C}_{d+r}\left(W_{\infty} \times Z\right)}
$$

where $W \subseteq \bar{W}$ a projective closure with closed complement $W_{\infty}=\bar{W} \backslash W$. This topology may also be described as follows. Let

$$
\mathcal{E}_{r}(Z)(W) \subseteq \mathcal{C}_{r+d}(\bar{W} \times Z)
$$

denote the constructable submonoid consisting of effective cycles whose restriction to $W \times Z$ is equidimensional of relative dimension $r$ over $W$. By [Fri98, Proposition 1.8] the topology on $\mathcal{C}_{r}(Z)(W)$ (given by the subspace topology above) coincides with the quotient topology given by

$$
\mathcal{C}_{r}(Z)(W)=\frac{\mathcal{E}_{r}(Z)(W)}{\mathcal{C}_{r+d}\left(W_{\infty} \times Z\right)}
$$

Define the topological group of equidimensional cycles of relative dimension r over $W$ as $\mathcal{Z}_{r}(Z)(W)=\left[\mathcal{C}_{r}(Z)(W)\right]^{+}$(where as usual the naive group completion is given the quotient topology). Since $\mathcal{C}_{r}(Z)(W)$ is a tractable monoid the naive group completion is a homotopy group completion.

In Fri91 it is shown that a morphism of varieties $f: W \rightarrow \mathcal{C}_{r}(Z)$ has an associated graph in $\mathcal{Z}_{f} \in \mathcal{C}_{r}(Z)(W)$. By [FL97, Proposition A.1] this defines an
isomorphism of topological monoids $\Gamma: \operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{r}(Z)\right) \rightarrow \mathcal{C}_{r}(Z)(W)$ for any normal, quasi-projective complex variety $W$ by [FL97, Proposition A.1]. Therefore the graph map $\Gamma$ also induces an isomorphism of topological abelian groups

$$
\Gamma: \operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}(Z)\right)^{+} \rightarrow \mathcal{Z}_{0}(W)(Z)
$$

for any normal, quasi-projective variety $W$ and any projective variety $Z$. The composite of $\Gamma$ and the continuous inclusion $\mathcal{Z}_{0}(W)(Z) \subseteq \mathcal{Z}_{\operatorname{dim} W}(W \times Z)$ defines the duality map

$$
\mathcal{D}: \operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}(Z)\right)^{+} \xrightarrow{\Gamma} \mathcal{Z}_{0}(W)(Z) \subseteq \mathcal{Z}_{d}(W \times Z) .
$$

While this is a continuous injective homomorphism it is not a topological embedding (see FL97]).

Lemma 3.4. (c.f. Teh05, Proposition 2.9])
(1) If $W$ is a normal quasi-projective complex variety and $Z_{1} \subset Z_{2}$ is a closed subvariety of a complex projective variety then

$$
\operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}\left(Z_{1}\right)\right)^{+} \subseteq \operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}\left(Z_{2}\right)\right)^{+}
$$

is a closed subspace.
(2) If $U$ is a normal quasi-projective real variety and $Y \subseteq Z$ is a closed subvariety of a projective real variety then

$$
\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}(Y)\right)^{+} \subseteq \operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}(Z)\right)^{+}
$$

is a closed subspace.
Proof. For the first statement it is equivalent to show that $\mathcal{Z}_{r}\left(Z_{1}\right)(W) \subseteq \mathcal{Z}_{r}\left(Z_{2}\right)(W)$ is closed. Using Lemma A. 1 we see that $\mathcal{C}_{r+d}\left(W \times Z_{1}\right) \subset \mathcal{C}_{r+d}\left(W \times Z_{2}\right)$ is closed and since $\mathcal{C}_{r}\left(Z_{1}\right)(W)=\mathcal{C}_{r+d}\left(W \times Z_{1}\right) \cap \mathcal{C}_{r}\left(Z_{2}\right)(W)$ we conclude that $\mathcal{C}_{r}\left(Z_{1}\right)(W) \subset$ $\mathcal{C}_{r}\left(Z_{2}\right)(W)$ is a closed subspace. Write $\pi: \mathcal{C}_{r}\left(Z_{2}\right)(W)^{\times 2} \rightarrow \mathcal{Z}_{r}\left(Z_{2}\right)(W)$ for the quotient. Then $\pi^{-1} \mathcal{Z}_{r}\left(Z_{1}\right)(W)=\mathcal{C}_{r}\left(Z_{1}\right)(W)^{\times 2}+\Delta\left(\mathcal{C}_{r}\left(Z_{2}\right)(W)\right.$ ) (where $\Delta$ denotes the diagonal) is closed by step (1) in Proposition 3.6. Therefore $\mathcal{Z}_{r}\left(Z_{1}\right)(W) \subseteq$ $\mathcal{Z}_{r}\left(Z_{2}\right)(W)$ is closed.

The second statement follows immediately from the first statement together with Proposition A. 3 and FW02a, Lemma 1.2].

Definition 3.5.
(1) Let $W$ be a quasi-projective complex variety. The space of algebraic $q$-cocyles is defined to be

$$
\mathcal{Z}^{q}(W)=\frac{\operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{+}}{\operatorname{Mor}_{\mathbb{C}}\left(W, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\right)^{+}}
$$

(2) Let $U$ be a quasi-projective real variety. The space of real algebraic $q$ cocyles is defined to be

$$
\mathcal{Z}^{q}(U)=\frac{\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q}\right)\right)^{+}}{\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q-1}\right)\right)^{+}}
$$

Proposition 3.6. Let $U$ be a normal quasi-projective real variety then

$$
\frac{\left(\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{a n,+}\right)^{G}}{\left(\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\right)^{a n,+}\right)^{G}} \xlongequal{\cong}\left(\frac{\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{a n,+}}{\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\right)^{a n,+}}\right)^{G}
$$

and

$$
\frac{\left(\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{a n,+}\right)^{a v}}{\left(\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\right)^{a n,+}\right)^{a v}} \xlongequal{\cong}\left(\frac{\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{a n,+}}{\operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\right)^{a n,+}}\right)^{a v}
$$

are isomorphisms of topological groups.
Proof. By Lemma A. 2 and Proposition A. 3 it is enough to show that for $Y^{\prime} \subseteq Y$ a closed subvariety of a projective real variety and $U$ a quasiprojective real variety that

$$
\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)^{G}}{\mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)^{G}} \rightarrow\left(\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)}{\mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)}\right)^{G}
$$

is an isomorphism of topological monoids.
We proceed in several steps.
(1) The map

$$
+: \mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right) \times \mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right) \rightarrow \mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)
$$

is a proper map. Observe that if $\alpha+\beta$ is equidimensional then both $\alpha$ and $\beta$ are equidimensional and therefore

is a pull-back square. Since addition is a proper map on effective cycles we see that it is a proper map for effective cocycles as well.
(2) The map

$$
\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)^{G}}{\mathcal{\mathcal { C }}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)^{G}} \rightarrow\left(\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)}{\mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)}\right)^{G}
$$

is easily seen to be a continuous bijection by an argument similar to the one used in Lemma 3.3
(3) Finally since $\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right) \rightarrow \mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right) / \mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)$ is a closed map by Lemma A. 1 we conclude that the continuous bijection

$$
\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)^{G}}{\mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)^{G}} \rightarrow\left(\frac{\mathcal{C}_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)}{\mathcal{C}_{r}\left(Y_{\mathbb{C}}^{\prime}\right)\left(U_{\mathbb{C}}\right)}\right)^{G}
$$

is a closed map and therefore a topological isomorphism.
The second statement for average cocycles is proved in a similar fashion, using Proposition 3.10 and that $C_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)^{a v} \subseteq C_{r}\left(Y_{\mathbb{C}}\right)\left(U_{\mathbb{C}}\right)$ is closed.

As with the topological group of cycles on a real variety $X$ we may view the topological group of real cocycles as the topological group of cycles on the complexification which are fixed by the Galois action.
Proposition 3.7. Let $X$ be a normal quasi-projective real variety. Then

$$
\mathcal{Z}^{q}(X)=\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}
$$

Proof. This follows from Proposition A. 3 together with the previous proposition since $\operatorname{Mor}_{\mathbb{R}}\left(X, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q}\right)\right)=\operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{G}$.

Remark 3.8. The space $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)$ has the equivariant homotopy type of a $G$ - $C W$ complex (see Corollary B.6).

When $W=X_{\mathbb{C}}$ is the complexification of a quasi-projective real variety and $Z=Y_{\mathbb{C}}$ is the complexification of a projective real variety the graph map is an equivariant morphism. In particular the duality map

$$
\mathcal{D}: \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}_{d}\left(X_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{q}\right)
$$

is an equivariant continuous map.
Definition 3.9. (1) Let $X$ be a projective real variety. Define the topological group of averaged cocycles to be

$$
\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v}=\left\{f+\sigma \cdot f \mid f \in \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)\right\} \subseteq \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)
$$

(2) Let $X$ be a normal projective real variety. Define the topological group of reduced cocycles to be the quotient topological group

$$
\mathcal{R}^{q}(X)=\frac{\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v}}
$$

Lemma 3.10 shows that $\mathcal{R}^{q}(X)$ is a Hausdorff topological group.
In Teh05 Teh defines

$$
\mathcal{R}_{0}(Y)(X)=\frac{\operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(Y_{\mathbb{C}}\right)\right)^{+, G}}{\operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(Y_{\mathbb{C}}\right)\right)^{+, a v}}
$$

and defines the reduced cocycles are defined as

$$
\mathcal{R}^{q}(X)=\frac{\mathcal{R}_{0}\left(\mathbb{P}^{q}\right)(X)}{\mathcal{R}_{0}\left(\mathbb{P}^{q-1}\right)(X)}
$$

for any real normal projective variety X and real projective variety Y .
By Proposition 3.6 this definition and the one above give isomorphic topological groups.

Lemma 3.10. (c.f. Teh05, Proposition 2.4]) Let $X$ be a real projective variety. The subset of averaged cocycles $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v} \subseteq \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)$ is a closed subgroup.
Proof. Write $\bar{f}$ for $\sigma \cdot f$ and $\bar{V}$ for $\sigma \cdot V$.
Suppose that $\left\{\left[f_{n}\right]+\overline{\left[f_{n}\right]}\right\}$ is a sequence in $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v}$ which converges in $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)$. Write $[\gamma]=\lim _{n \rightarrow \infty}\left[f_{n}\right]+\overline{\left[f_{n}\right]}$ for its limit. We need to conclude that $[\gamma]$ is an averaged cocycle.

The set $\left\{\left[f_{n}\right]+\left[\overline{f_{n}}\right]\right\} \cup\{[\gamma]\} \subseteq \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)$ is compact. Applying the duality map to this set yields the compact subset

$$
\left\{\Gamma\left(\left[f_{n}\right]\right)+\Gamma\left(\left[\overline{f_{n}}\right]\right)\right\} \cup\{\Gamma([\gamma])\} \subseteq \mathcal{Z}_{d}\left(X_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{q}\right)
$$

Since this is a compact subset it lies in $\mathcal{Z}_{d, \leq k}\left(X_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{q}\right)$ for some $k$.
The sequence $\left\{\left[g_{n}\right]\right\} \subseteq \mathcal{Z}_{d, \leq k}\left(X_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{q}\right)$ has a convergent subsequence. Write $\left\{\left[g_{n_{i}}\right]\right\}$ for this convergent subsequence and write $\lim _{n_{i} \rightarrow \infty}\left[g_{n_{i}}\right]=[g] \in \mathcal{Z}_{d, \leq k}\left(X_{\mathbb{C}} \times\right.$ $\left.\mathbb{A}_{\mathbb{C}}^{q}\right)$ for its limit. Note that $[g]$ satisfies $[g]+\overline{[g]}=\Gamma(\gamma)$. Since $\Gamma$ is injective and its image consists precisely of equidimensional cycles, we are done if we can find an equidimensional cycle $\left[g^{\prime}\right]$ such that $\left[g^{\prime}\right]+\overline{\left[g^{\prime}\right]}=[g]+\overline{[g]}$.

Choose a representative $\gamma \in \operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{+}$of $[\gamma]$. Choose a representative $g=\sum n_{V} V \in \mathcal{Z}_{d}\left(X_{\mathbb{C}} \times \mathbb{P}_{\mathbb{C}}^{q}\right)$ of $[g]$ such that if $n_{V} \neq 0$ then $V \nsubseteq \mathbb{P}^{q-1} \times X$. Since $[g]+\overline{[g]}=\Gamma([\gamma]) \in \mathcal{Z}_{d}\left(X_{\mathbb{C}} \times \mathbb{A}_{\mathbb{C}}^{q}\right)$ we see that $g+\bar{g}=\sum\left(n_{V}+n_{\bar{V}}\right) V=\Gamma(\gamma)+h$
where $h \in \mathcal{Z}_{d}\left(X_{\mathbb{C}} \times \mathbb{P}_{\mathbb{C}}^{q-1}\right)$. Write $h=\Sigma m_{W} W$. Since $V \nsubseteq \mathbb{P}^{q-1}$ whenever $n_{V} \neq 0$ we see that if $m_{W} \neq 0$ then a term of $-m_{W} W$ must appear in $\Gamma(\gamma)$. In particular $h$ is equidimensional. Consequently $g+\bar{g}$ is equidimensional.

If $n_{V}+n_{\bar{V}} \neq 0$ then $V$ is equidimensional. Define

$$
g^{\prime}=\sum_{n_{V}+n_{\bar{V}} \neq 0} n_{V} V
$$

Since $g^{\prime}$ is an equidimensional cycle there is an $f \in\left(\operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{+}\right)^{G}$ such that $\Gamma(f)=g^{\prime}$. Since $\Gamma([f]+\overline{[f]})=\left[g^{\prime}\right]+\overline{\left[g^{\prime}\right]}=[g]+\overline{[g]}=\Gamma([c])$ and $\Gamma$ is injective, we conclude that $[c]=[f]+\overline{[f]}$.

A continuous algebraic map $f: W \rightarrow V$ between two complex varieties induces a continuous map $f: W(\mathbb{C}) \rightarrow V(\mathbb{C})$. Friedlander-Lawson [FL92, Proposition 4.1] show that this defines a continuous map

$$
\begin{equation*}
\Phi: \mathcal{Z}^{r}(W) \rightarrow \operatorname{Map}\left(W(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{r}\right)\right) \tag{3.11}
\end{equation*}
$$

where the mapping space between two topological spaces is given with the compactopen topology. If $Y$ is a real variety this provides a continuous equivariant comparison map

$$
\Phi: \mathcal{Z}^{r}\left(Y_{\mathbb{C}}\right) \rightarrow \operatorname{Map}\left(Y_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{r}\right)\right)
$$

of topological abelian groups.
Definition 3.12. (Real Morphic Cohomology) Friedlander-Walker FW02a define real morphic cohomology of a quasi-projective real variety by

$$
L^{q} H \mathbb{R}^{n}(X)=\pi_{2 q-n} \mathcal{Z}^{q}(X)
$$

for $2 q-n \geq 0$.
We will be using an equivariant extension of their theory for normal quasiprojective real varieties defined below.
Definition 3.13. (Equivariant Morphic Cohomology) Let $X$ be a normal quasiprojective variety. Then the equivariant morphic cohomology is (in equivariant homotopy indexing notation)

$$
L^{q} H \mathbb{R}^{k, r}(X)=\pi_{q-k, q-r} \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)
$$

for $q-k, q-r \geq 0$.
By Proposition 3.7 we see that

$$
L^{q} H \mathbb{R}^{n}(X)=\pi_{2 q-n} \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}
$$

so Friedlander-Walker's real morphic cohomology groups are a part of the equivariant morphic cohomology, $L^{q} H \mathbb{R}^{q-r, q}(X)=\pi_{r, 0} \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)=L^{q} H \mathbb{R}^{2 q-r}(X)$.

In dS03a dos Santos defines real Lawson homology.
Definition 3.14. (Real Lawson Homology) For any quasi-projective real variety $X$, the real Lawson homology is defined by

$$
L_{q} H \mathbb{R}_{n, m}(X)=\pi_{n-q, m-q} \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)
$$

for $n-q, m-q \geq 0$.
Definition 3.15. Let $X$ be a quasi-projective real variety. The following definitions are taken from LLFM03.
(1) Define the space of averaged cycles $\mathcal{Z}_{q}(X)^{a v}$ to be

$$
\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}=\operatorname{Im}(N) \subseteq \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}
$$

so $\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v} \subseteq \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}$ is the subgroup generated by cycles of the form $Z+\bar{Z}$ and given the subspace topology. By Remark 3.2 this is a closed subgroup. Here $N: \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)$ is defined by $N(Z)=Z+\bar{Z}$.
(2) Define the space of reduced cycles $\mathcal{R}_{q}(X)$ to be the quotient group

$$
\mathcal{R}_{q}(X)=\frac{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}}
$$

Remark 3.16. These spaces all have the homotopy type of a $C W$-complex (see Corollary B.6).

Teh Teh05] defines the reduced real Lawson homology of $X$ to be

$$
R L_{q} H_{n}(X)=\pi_{n-q} \mathcal{R}_{q}(X)
$$

for $n \geq q$. According to Lemma 3.3 this definition coincides with the definition given in Teh05] in the case of a quasi-projective variety.

Example 3.17. Let $X$ be a projective real variety.
(1) LLFM03, Lemma 8.4] The space of averaged zero-cycles computes the singular homology of the quotient of analytic space of complex points

$$
\pi_{k} \mathcal{Z}_{0}\left(X_{\mathbb{C}}\right)^{a v}=H_{k}(X(\mathbb{C}) / G ; \mathbb{Z})
$$

(2) By the equivariant Dold-Thom theorem dS03b the space of fixed zerocycles computes (a portion of) Bredon cohomology

$$
\pi_{k} \mathcal{Z}_{0}\left(X_{\mathbb{C}}\right)^{G}=H_{k, 0}(X(\mathbb{C}) ; \underline{\mathbb{Z}})
$$

(3) Teh05, Proposition 2.7] The space of reduced real cycles computes the singular homology with $\mathbb{Z} / 2$ coefficients of the analytic space of real points

$$
\pi_{k} \mathcal{R}_{0}(X)=H_{k}(X(\mathbb{R}) ; \mathbb{Z} / 2)
$$

## 4. Poincare Duality

In this section we use the duality for bivariant cycle homology in FV00 to establish a duality between Lawson homology of a real variety and real morphic cohomology. This together with the duality between Lawson homology and morphic cohomology [FL97] gives an equivariant duality between the algebraic cocycle spaces and algebraic cycle spaces for the complexification of a real variety.

The material and methods used here closely parallel [FW03, Section 3] where Friedlander-Walker reformulate Lawson homology and morphic cohomology for complex varieties.
Recognition Principle. Let $F(-)$ be a presheaf sets (respectively simplicial sets, or abelian groups) on $S c h / \mathbb{R}$. If $T$ is a topological space then define $F(T)$ by the filtered colimit

$$
F(T)=\operatorname{colim}_{T \rightarrow V(\mathbb{R})} F(V)
$$

In particular we obtain a simplicial set (respectively a bisimplicial set, or simplicial abelian group) by

$$
\begin{equation*}
d \mapsto F\left(\Delta_{t o p}^{d}\right) \tag{4.1}
\end{equation*}
$$

We record an analogue of the recognition principle [FW03, Theorem 2.3] which is needed to move the duality for bivariant cycle homology to a duality for real Lawson homology and morphic cohomology. Friedlander-Walker's proof in the complex case uses the uad-topology which is essentially due to Deligne.

Definition 4.2. (1) A continuous map of topological spaces $f: S \rightarrow T$ is said to satisfy cohomological descent if for any sheaf $A$ of abelian groups on $T$ the natural map

$$
H^{*}(T, A) \rightarrow H^{*}\left(N_{T}(S), f^{*} A\right)
$$

is an isomorphism. Here $N_{T}(S) \rightarrow T$ is the Cech nerve of $f$, i.e. $N_{T}(S)$ is the simplicial space which in degree $n$ is the $n+1$-fold fiber product of $S$ over $T$. A map $f: S \rightarrow T$ is said to be of universal cohomological descent provided the pullback $S \times_{T} T^{\prime} \rightarrow T^{\prime}$ along any continuous map $T^{\prime} \rightarrow T$ is again of cohomological descent.
(2) The uad-topology on $S c h_{\mathbb{R}}$ is the Grothendieck topology associated to the pretopology generated by collections $\left\{U_{i} \rightarrow X\right\}$ such that $\coprod U_{i}(\mathbb{R})^{a n} \rightarrow$ $X(\mathbb{R})^{a n}$ is a surjective map of universal cohomological descent.
Example 4.3. (1) A proper and surjective map of real varieties $X \rightarrow Y$ which induces a surjective map of real points is a uad-cover. Indeed, in this case $X(\mathbb{R})^{a n} \rightarrow Y(\mathbb{R})^{a n}$ is a proper surjective map of topological spaces, and therefore is a map of universal cohomological descent (see [Del74, 5.3.5]).
(2) Any Nisnevich cover is a uad-cover. Any $c d h$-cover is a uad-cover. In particular every real variety $X$ is locally smooth in the uad topology because resolution of singularities implies there is a $c d h$-cover $X^{\prime} \rightarrow X$, with $X^{\prime}$ smooth.
(3) Unlike the complex case not every etale-cover is a uad-cover (e.g. Spec $\mathbb{C} \rightarrow$ $\operatorname{Spec} \mathbb{R}$ is an etale cover but not a uad-cover).

Here is the recognition principle.
Theorem 4.4 ([FW03, Theorem 2.2]). Suppose that $F \rightarrow G$ is a natural transformation of presheaves of abelian groups on $S c h_{\mathbb{R}}$. If $F_{u a d} \rightarrow G_{u a d}$ is an isomorphism of uad-sheaves, then

$$
F\left(\Delta_{t o p}^{\bullet}\right) \rightarrow G\left(\Delta_{t o p}^{\bullet}\right)
$$

is a homotopy equivalence of simplicial abelian groups.
Proof. Friedlander-Walker's proof given in [W03] works by changing the space $X(\mathbb{C})^{a n}$ associated with a complex variety with the space $Y(\mathbb{R})^{a n}$ associated to a real variety together with the fact that $Y(\mathbb{R})^{a n}$ may be triangulated.

Corollary 4.5. Suppose that $f: F \rightarrow G$ is a map of presheaves of simplicial abelian groups such that $F(V) \rightarrow G(V)$ is a homotopy equivalence for any smooth $V$. Then the map of simplicial abelian groups $\operatorname{diag} F\left(\Delta_{\text {top }}^{\bullet}\right) \rightarrow \operatorname{diag} G\left(\Delta_{\text {top }}^{\bullet}\right)$ is a homotopy equivalence.

Poincare Duality. Let $X$ be a variety over a field $k$ of characteristic zero. Recall the presheaf $z_{\text {equi }}(X, r)(-)$ of equidimensional $r$-cycles. This is the unique $q f h$-sheaf on $S c h / k$ such that for a normal variety $U$ the group $z_{\text {equi }}(X, r)(U)$ is the free abelian group on closed, irreducible subvarieties $V \subseteq U \times_{k} X$ which are equidimensional of relative dimension $r$ over some irreducible component of $U$.

If $X$ and $Y$ are real varieties then $G=G a l(\mathbb{C} / \mathbb{R})$ acts on the group $z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(U_{\mathbb{C}}\right)$ by $\sigma \cdot\left[V \subseteq U_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}}\right]=\left[\sigma V \subseteq U_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}}\right]$.
Lemma 4.6. Let $X$ and $U$ be real varieties. Then

$$
z_{e q u i}(X, r)(U) \xrightarrow{\pi^{*}}\left(z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(U_{\mathbb{C}}\right)\right)^{G}
$$

is a natural isomorphism where $\pi:\left(U \times_{\mathbb{R}} X\right)_{\mathbb{C}} \rightarrow U \times_{\mathbb{R}} X$.
Proof. It suffices to check this for $U$ normal, since normalization is a $q f h$-cover. By SV00b, Lemma 2.3.2] $\pi^{*}: \operatorname{Cycl}(U \times X) \rightarrow \operatorname{Cycl}\left((U \times X)_{\mathbb{C}}\right)^{G}$ is an isomorphism, where $C y c l(W)$ denotes the group of cycles on $W$. We are done if we see that $f: V \rightarrow U$ is equidimensional if and only if $\tilde{f}: V_{\mathbb{C}} \rightarrow U_{\mathbb{C}}$ is equidimensional. By Gro66, Proposition 13.3.8] if $f$ is equidimensional then so is $\tilde{f}$. Suppose that $\tilde{f}$ : $V_{\mathbb{C}} \rightarrow U_{\mathbb{C}}$ is equidimensional. Since $U_{\mathbb{C}}$ is normal, $\tilde{f}: V_{\mathbb{C}} \rightarrow U_{\mathbb{C}}$ is an open mapping and for all $v^{\prime} \in V$ the local rings $\mathcal{O}_{V_{\mathbb{C}}, v^{\prime}}$ are equidimensional by Gro66, Corollaire 14.4.6]. By Gro65, Corollaire 2.6.4, Proposition 7.1.3] the map $f: V \rightarrow U$ is open and $\mathcal{O}_{V, v}$ is equidimensional for all $v \in V$ since $U_{\mathbb{C}} \rightarrow U$ is faithfully flat and therefore $f$ is equidimensional.

In the proof of [FW02a, Proposition 2.4] it is shown that for any presheaf $F(-)$ of sets on $S c h / \mathbb{C}$ and any topological space $T$ the natural map

$$
\operatorname{colim}_{T \rightarrow V(\mathbb{R})} F\left(V_{\mathbb{C}}\right) \xrightarrow{\cong} \operatorname{colim}_{T \rightarrow U(\mathbb{C})} F(U)
$$

is an isomorphism. In the first indexing set $V$ ranges over real varieties and in the second $U$ ranges over complex varieties.

In particular $z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} T\right)$ may be computed via the filtered colimit

$$
z_{e q u i}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} T\right)=\operatorname{colim}_{T \rightarrow V(\mathbb{R})} z_{e q u i}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} V_{\mathbb{C}}\right)
$$

which equips $z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} T\right)$ with an action of $G$. Filtered colimits commute with fixed points and so
$z_{\text {equi }}(X, r)\left(Y \times_{\mathbb{R}} T\right) \rightarrow\left(z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} T\right)\right)^{G}=\operatorname{colim}_{T \rightarrow V(\mathbb{R})}\left(z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)\left(Y_{\mathbb{C}} \times_{\mathbb{C}} V_{\mathbb{C}}\right)\right)^{G}$
is an isomorphism.
For X projective we have the natural isomorphism of presheaves (in fact of $q f h$ sheaves) of abelian groups on $S c h_{\mathbb{R}}$ (see [SV00b, Lemma 4.4.14]

$$
z_{\text {equi }}(X, r)(-) \cong \operatorname{Mor}_{\mathbb{R}}\left(-, \mathcal{C}_{r}(X)\right)^{+}
$$

The following is the real analogue of [FW03, Proposition 3.1].
Proposition 4.7. Let $T$ be a compactly generated Hausdorff topological space and $X$ a quasi-projective real variety. There is a natural map of abelian groups

$$
z_{e q u i}(X, r)(T) \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}(X)\right)
$$

given by sending $\left(f: T \rightarrow U(\mathbb{R}), \alpha \in z_{\text {equi }}(X, r)(U)\right)$ to the function $\left.t \mapsto \alpha\right|_{f(t)}$.
This map is contravariant for continuous maps of compactly-generated Hausdorff spaces $T^{\prime} \rightarrow T$, covariant for proper maps $X \rightarrow X^{\prime}$ and contravariant for flat maps $X^{\prime} \rightarrow X$ (with a shift in dimension).

When $X$ is a projective real variety the induced map of simplicial abelian groups

$$
z_{e q u i}(X, r)\left(\Delta_{\text {top }}^{\bullet}\right) \rightarrow \operatorname{Sing} . \mathcal{Z}_{r}(X)
$$

is the natural homotopy equivalence

$$
\left[\operatorname{Sing}_{\bullet}\left(\mathcal{C}_{r}(X)^{a n}\right)\right]^{+} \xrightarrow{\simeq} \operatorname{Sing} . \mathcal{Z}_{r}(X)
$$

More generally, for a quasi-projective real variety $X$ with projectivization $X \subset \bar{X}$ this map fits into a comparison of homotopy fiber sequences


Therefore the map

$$
z_{\text {equi }}(X, r)\left(\Delta_{\text {top }}^{\bullet}\right) \rightarrow \operatorname{Sing} . \mathcal{Z}_{r}(X)
$$

is a natural weak equivalence for any quasi-projective real variety $X$.
Proof. The map

$$
z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)(T)=\operatorname{colim}_{T \rightarrow W(\mathbb{C})} z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)(W) \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)\right)
$$

given sending $\left(f: T \rightarrow W(\mathbb{C}), \alpha \in z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)(W)\right)$ to the function $t \mapsto \alpha_{\mid f(t)}$ is shown to be well-defined in [FW03, Proposition 3.1] and to satisfy the stated naturality properties. Observe that if $W=V_{\mathbb{C}}$ is the complexification of a real variety then $\overline{\alpha_{\mid f(t)}}=\bar{\alpha}_{\mid \overline{f(t)}}$. Therefore composing with the natural isomorphism

$$
\operatorname{colim}_{T \rightarrow V(\mathbb{R})} z_{e q u i}\left(X_{\mathbb{C}}, r\right)\left(V_{\mathbb{C}}\right) \xrightarrow{\cong} \operatorname{colim}_{T \rightarrow W(\mathbb{C})} z_{e q u i}\left(X_{\mathbb{C}}, r\right)(W)
$$

gives a well-defined equivariant map

$$
z_{e q u i}\left(X_{\mathbb{C}}, r\right)(T) \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)\right)
$$

which by taking fixed points induces the map
$z_{\text {equi }}(X, r)(T)=z_{\text {equi }}\left(X_{\mathbb{C}}, r\right)(T)^{G} \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}\left(X_{\mathbb{C}}\right)\right)^{G}=\operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}(X)\right)$,
which is the map of the proposition and satisfies the stated naturality properties.
When $X$ is a projective real variety and $T$ is a compact Hausdorff space, the map $z_{\text {equi }}^{\text {eff }}\left(X_{\mathbb{C}}, r\right)(T) \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{C}_{r}\left(X_{\mathbb{C}}\right)\right)$ is an isomorphism by FW02b Corollary 4.3]. Since this is an equivariant map, taking fixed points yields the isomorphism of monoids

$$
z_{e q u i}^{e f f}(X, r)(T) \stackrel{\cong}{\rightrightarrows} \operatorname{Hom}_{c t s}\left(T, \mathcal{C}_{r}(X)\right)
$$

Therefore the map

$$
z_{\text {equi }}(X, r)\left(\Delta_{t o p}^{\bullet}\right) \xrightarrow{\cong}\left[\operatorname{Hom}_{\text {cts }}\left(\Delta_{\text {top }}^{\bullet}, \mathcal{C}_{r}(X)^{a n}\right)\right]^{+} \rightarrow \operatorname{Sing} . \mathcal{Z}_{r}(X)
$$

is a homotopy equivalence by Quillen's theorem [FM94, App Q] on homotopy group completions of simplicial abelian monoids.

Finally the diagram (4.8) commutes by the naturality properties of the map $z_{\text {equi }}(X, r)(T) \rightarrow \operatorname{Hom}_{c t s}\left(T, \mathcal{Z}_{r}(X)\right)$. By FV00, 5.12,8.1], Proposition 4.12 (homotopy invariance), and Theorem4.4 (recognition principle) the upper row of the diagram (4.8) is a homotopy fiber sequence. Comparing the upper and lower homotopy fiber sequence yields the final statement of the proposition.

Proposition 4.9. For a quasi-projective real variety $U$, projective real variety $Y$, and compact Hausdorff space $T$, there is a natural map of abelian groups

$$
z_{\text {equi }}(Y, 0)\left(U \times_{\mathbb{R}} T\right) \rightarrow \operatorname{Hom}_{c t s}\left(T, \operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}(Y)\right)^{+}\right)
$$

given by sending $(f, \alpha)$ to the function $t \mapsto \alpha_{\mid f(t)}$.
Proof. The map

$$
z_{\text {equi }}\left(Y_{\mathbb{C}}, 0\right)\left(U_{\mathbb{C}} \times_{\mathbb{C}} T\right) \rightarrow \operatorname{Hom}_{\text {cts }}\left(T, \operatorname{Mor}_{\mathbb{C}}\left(U_{\mathbb{C}}, \mathcal{C}_{0}\left(Y_{\mathbb{C}}\right)\right)^{+}\right)
$$

from [FW03, Proposition 3.3] is equivariant and therefore taking fixed points induces the natural map of abelian groups

$$
z_{e q u i}(Y, 0)\left(U \times_{\mathbb{R}} T\right) \rightarrow \operatorname{Hom}_{c t s}\left(T, \operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}(Y)\right)^{+}\right)
$$

Let $Y$ be a projective real variety and $U$ a normal quasi-projective real variety of dimension $d$ with projectivization $U \subseteq X$ and closed complement $X_{\infty}=X \backslash U$. Write $\mathcal{E}_{r}(Y)(U) \subseteq \mathcal{C}_{r+d}\left(Y \times_{\mathbb{R}} X\right)$ for the submonoid consisting of those cycles of dimension $r+d$ on $Y \times X$ whose restriction to $U$ is equidimensional of relative dimension $r$ over $U$. This is a constructable embedding. This can be seen by arguing as in Fri98 for the complex case. The subspace topology on this monoid agrees with the quotient topology $\mathcal{C}_{r}(Y)(U)=\mathcal{E}_{r}(Y)(U) / \mathcal{C}_{r+d}\left(Y \times X_{\infty}\right)$ by the same reasoning as in [Fri98, Proposition 1.8]. The topological group of equidimensional cycles is the naive groups completion $\mathcal{Z}_{r}(Y)(U)=\mathcal{C}_{r}(Y)(U)^{+}$. Since these are tractable monoids, they are related to equidimensional cocycles via the homotopy fiber sequence

$$
\mathcal{Z}_{r+d}\left(Y \times X_{\infty}\right) \rightarrow\left(\mathcal{E}_{r}(Y)(U)\right)^{+} \rightarrow \mathcal{Z}_{r}(Y)(U)
$$

Define the presheaf $e(U, Y, r)(-)$ to be the pull-back of presheaves


We have for each quasi-projective real variety $V$ the short exact sequence of abelian groups

$$
0 \rightarrow z_{\text {equi }}\left(Y \times X_{\infty}, r+d\right)(V) \rightarrow(e(U, Y, r)(V))^{+} \rightarrow z_{\text {equi }}(Y, r)(U \times V) \rightarrow 0
$$

Proposition 4.10. Let $Y$ be a projective real variety, $U$ a normal quasi-projective real variety, and $T$ a compact Hausdorff space. Then

$$
e(U, Y, r)(T) \xrightarrow{\cong} \operatorname{Hom}_{c t s}\left(T, \mathcal{E}_{r}(Y)(U)\right)
$$

is an isomorphism.
Proof. Observe that if $V$ is a quasi-projective real variety then the isomorphism $\operatorname{Mor}_{\mathbb{R}}\left(V, \mathcal{C}_{r+d}(Y \times X)\right) \cong z_{\text {equi }}^{\text {eff }}(Y \times X, r+d)(V)$ restricts to give the isomorphism $\operatorname{Mor}_{\mathbb{R}}(V, \mathcal{E}(Y)(U)) \cong e(U, Y, r)(V)$. Here if $E \subseteq W$ is a constructable subset then $\operatorname{Mor}(V, E) \subseteq \operatorname{Mor}(V, W)$ is the subset consisting of those continuous algebraic maps whose image is contained in $E$.

The isomorphism $e(U, Y, r)(T) \xrightarrow{\cong} \operatorname{Hom}_{c t s}\left(T, \mathcal{E}_{r}(Y)(U)\right)$ now follows as in FW01b, Corollary 4.3].

Proposition 4.11. Let $U$ be a normal quasi-projective real variety and $Y$ a projective real variety. The map of simplicial abelian groups from Proposition 4.9

$$
z_{e q u i}(Y, 0)\left(U \times \Delta_{t o p}^{\bullet}\right) \rightarrow \text { Sing. }\left(\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}(Y)\right)^{+}\right)
$$

is a homotopy equivalence.
Proof. By proposition 4.10 we have $e(U, Y, r)\left(\Delta_{\text {top }}^{\bullet}\right) \cong \operatorname{Sing} . \mathcal{E}_{r}(Y)(U)$. Now by taking group completions, tractability of the monoid $\mathcal{E}_{r}(Y)(U)$ and Quillen's theorem [FM94, App Q] we conclude that

$$
e(U, Y, r)\left(\Delta_{t o p}^{\bullet}\right)^{+} \xrightarrow{\simeq} \operatorname{Sing} \bullet\left(\mathcal{E}_{r}(Y)(U)^{+}\right)
$$

is a homotopy equivalence.
We conclude the proposition by comparing homotopy fiber sequences of simplicial abelian groups


The left arrow is a homotopy equivalence by Proposition 4.7, we have just seen that the middle map is a homotopy equivalence, the right horizontal maps induce a surjection on $\pi_{0}$ and so we conclude that $z_{\text {equi }}(Y, r)\left(U \times \Delta_{\text {top }}^{\bullet}\right) \rightarrow \operatorname{Sing} . \mathcal{Z}^{r}(Y)(U)$ is a homotopy equivalence.

Proposition 4.12. The presheaves $z_{\text {equi }}(X, r)\left(\Delta_{\text {top }}^{\bullet} \times-\right)$ are homotopy invariant in the sense that the map of complexes

$$
z_{e q u i}(X, r)\left(\Delta_{t o p}^{\bullet}\right) \rightarrow z_{e q u i}(X, r)\left(\Delta_{t o p}^{\bullet} \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{1}\right)
$$

is a quasi-isomorphism.
Proof. The same argument as in [FW01a, Lemma 1.2].
The duality theorem for bivariant cycle theory [FV00, Theorem 7.4] says that for real varieties $X, U$ with $U$ smooth of dimension $d$, the natural inclusion

$$
\begin{equation*}
\mathcal{D}: z_{\text {equi }}(X, r)\left(U \times_{\mathbb{R}}-\right) \hookrightarrow z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)(-) \tag{4.13}
\end{equation*}
$$

induces a quasi-isomorphism of complexes

$$
\mathcal{D}: z_{\text {equi }}(X, r)\left(U \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}\right) \xrightarrow{\simeq} z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)\left(\Delta_{\mathbb{R}}^{\bullet}\right) .
$$

Proposition 4.14. For a smooth real variety $U$ and a quasi-projective real variety $X$ the map

$$
z_{e q u i}(X, r)\left(U \times_{\mathbb{R}} \Delta_{t o p}^{\bullet}\right) \xrightarrow{\mathcal{D}} z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)\left(\Delta_{t o p}^{\bullet}\right)
$$

is a quasi-isomorphism.
Proof. Consider the commutative diagram

$$
\begin{gathered}
z_{\text {equi }}(X, r)\left(U \times_{\mathbb{R}} \Delta_{\text {top }}^{\bullet}\right) \xrightarrow{\mathcal{D}} \quad z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)\left(\Delta_{t o p}^{\bullet}\right) \\
\pi^{*} \downarrow \\
z_{\text {equi }}(X, r)\left(U \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet} \times_{\mathbb{R}} \Delta_{t o p}^{\bullet}\right) \xrightarrow{\mathcal{D}} z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)\left(\Delta_{\mathbb{R}}^{\bullet} \times_{\mathbb{R}} \Delta_{\text {top }}^{\bullet}\right) .
\end{gathered}
$$

The vertical arrows are quasi-isomorphisms by homotopy invariance. The bottom right arrow is a quasi-isomorphism by Corollary 4.5 since

$$
z_{\text {equi }}(X, r)\left(U \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet} \times W\right) \rightarrow z_{\text {equi }}\left(X \times_{\mathbb{R}} U, r+d\right)\left(\Delta_{\mathbb{R}}^{\bullet} \times_{\mathbb{R}} W\right)
$$

is a quasi-isomorphism for all smooth real varieties $W$ by [FV00, Theorem 7.4] and therefore the top horizontal map is a quasi-isomorphism as well.

Lemma 4.15. Let $Y$ be a projective real variety and $U$ a smooth real variety. The following diagram commutes

where the vertical maps are the ones from Proposition 4.9 and Proposition 4.7 .
Proof. By [FW03, Proposition 3.3] the diagram of equivariant maps of simplicial sets

commutes. Taking fixed points yields the result.
Write

$$
z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q / q-1}, 0\right)(U)=\operatorname{coker}\left(z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q-1}, 0\right)(U) \rightarrow z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q}, 0\right)(U)\right)
$$

for the cokernel of the map of presheaves induced by $\mathbb{P}_{\mathbb{R}}^{q-1} \subseteq \mathbb{P}_{\mathbb{R}}^{q}$.
Proposition 4.16. Let $U$ be a smooth real variety of dimension $d$. The sequence of natural maps of complexes below consist of quasi-isomorphisms.

$$
\begin{align*}
z_{\text {equi }}\left(\mathbb{A}_{\mathbb{R}}^{q}, 0\right)\left(U \times_{\mathbb{R}} \Delta_{\text {top }}^{\bullet}\right) & \leftarrow z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q / q-1}, 0\right)\left(U \times_{\mathbb{R}} \Delta_{\text {top }}^{\bullet}\right) \rightarrow  \tag{4.17}\\
& \rightarrow \frac{\operatorname{Sing}_{\bullet}\left(\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q}\right)\right)^{+}\right.}{\operatorname{Sing}_{\bullet}\left(\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q-1}\right)\right)^{+}\right.} \rightarrow \operatorname{Sing}_{\bullet} \mathcal{Z}^{q}(U)
\end{align*}
$$

Proof. That the first map of diagram (4.17) is a quasi-isomorphism follows from consideration of the comparison diagram


The vertical arrows are all quasi-isomorphisms by Proposition 4.14 and by Proposition 4.7 Because $\mathcal{C}_{k}(V)$ is a tractable monoid, the bottom row is homotopy equivalent to a short exact sequence of simplicial abelian groups and therefore the top rows are as well. It follows immediately that the first arrow of diagram4.17is a quasi-isomorphism. The second arrow of diagram (4.17) is a quasi-isomorphism by

Proposition 4.11 and the last arrow of the diagram is a quasi-isomorphism because $\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{n}\right)\right)$ is a tractable monoid.

Definition 4.18. If $k<0$ then define $\mathcal{Z}_{k}(X)$ to be $\mathcal{Z}_{0}\left(X \times \mathbb{A}^{-k}\right)$.
We can now conclude the duality for real morphic cohomology and real Lawson homology.

Corollary 4.19. Let $U$ be a smooth real variety of dimension $d$. Then

$$
\mathcal{Z}^{q}\left(U_{\mathbb{C}}\right)^{G} \xrightarrow{\mathcal{D}} \mathcal{Z}_{d}\left(\mathbb{A}_{\mathbb{C}}^{q} \times_{\mathbb{C}} U_{\mathbb{C}}\right)^{G} \simeq \mathcal{Z}_{d-q}\left(U_{\mathbb{C}}\right)^{G}
$$

is a natural homotopy equivalence.
In particular it induces the natural isomorphism

$$
L^{q} H \mathbb{R}^{n}(U) \stackrel{( }{\cong} L_{d-q} H \mathbb{R}_{d-n, d}(U)
$$

Proof. This follows from Proposition 4.14, Lemma 4.15, Proposition 4.7, Proposition 4.16, and homotopy invariance dS03a, Proposition 4.15]. Indeed these show that the following diagram is commutative and the left hand maps are homotopy equivalences,

$$
\begin{aligned}
& z_{\text {equi }}\left(\mathbb{A}_{\mathbb{R}}^{q}, 0\right)\left(U \times \Delta_{\text {top }}^{\bullet}\right) \leftarrow z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q / q-1}, 0\right)\left(U \times \Delta_{\text {top }}^{\bullet}\right) \longrightarrow \\
& \downarrow_{\mathcal{D}} \quad \downarrow^{\mathcal{D}} \\
& z_{\text {equi }}\left(\mathbb{A}_{\mathbb{R}}^{q} \times U, d\right)\left(\Delta_{t o p}^{\bullet}\right) \longleftarrow z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q / q-1} \times U, d\right)\left(\Delta_{t o p}^{\bullet}\right) \longrightarrow \\
& \begin{aligned}
& \longrightarrow \text { Sing。 }\left(\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q}\right)\right)^{+}\right. \\
& \operatorname{Sing}\left(\operatorname{Mor}_{\mathbb{R}}\left(U, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{R}}^{q-1}\right)\right)^{+}\right.
\end{aligned} \longrightarrow \operatorname{Sing} \bullet \mathcal{Z}^{q}(U)
\end{aligned}
$$

Therefore the right hand map is also a homotopy equivalence
Combining Friedlander-Lawson's duality between Lawson homology and morphic cohomology over $\mathbb{C}$ and the duality over $\mathbb{R}$ immediately gives an equivariant duality theorem.

Corollary 4.20. Let $U$ be a smooth real variety of dimension $d$. The sequence of maps

$$
\mathcal{Z}^{q}\left(U_{\mathbb{C}}\right) \rightarrow \mathcal{Z}_{d}\left(U_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^{q}\right) \leftarrow \mathcal{Z}_{d-q}\left(X_{\mathbb{C}}\right)
$$

consists of $G$-equivariant homotopy equivalences. In particular

$$
L^{q} H \mathbb{R}^{n, m}(U) \stackrel{\cong}{\Longrightarrow} L_{d-q} H \mathbb{R}_{d-n, d-m}(U)
$$

for all smooth quasi-projective real varieties $U$.
Remark 4.21. A smooth $G$-manifold $M$ equipped such that the action of $G$ on its tangent bundle makes it into a real $n$-bundle satisfies an equivariant Poincare duality,

$$
\mathcal{P}: H^{p, q}(M ; \underline{Z}) \stackrel{\cong}{\rightrightarrows} H_{n-p, n-q}(M ; \underline{\mathbb{Z}}) .
$$

In a forthcoming paper we prove that the duality $\mathcal{D}$ is compatible under the cycle maps with the duality $\mathcal{P}$.

## 5. Compatibility of Cycle Maps

Generalized cycle maps. Let $X$ be a smooth real variety. The generalized cycle map relates motivic cohomology and etale cohomology,

$$
c y c: \mathbb{H}_{\mathcal{M}}^{2 q-k, q}(X ; \mathbb{Z} / 2) \rightarrow H_{e t}^{2 q-k}\left(X ; \mu_{2}^{\otimes q}\right)
$$

By [Cox79] the etale cohomology of a real variety is equal to the Borel equivariant cohomology of its space of complex points,

$$
H_{e t}^{2 q-k}\left(X ; \mu_{2}^{\otimes q}\right) \cong H_{\mathbb{Z} / 2}^{2 q-k}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2\right)
$$

On the other hand morphic cohomology and motivic cohomology agree with finite coefficients (see Proposition 5.1).

Combining the generalized cycle map in morphic cohomology and the comparison map between Bredon and Borel equivariant cohomology

$$
L^{q} H \mathbb{R}^{q-k, q}(X ; \mathbb{Z} / 2) \rightarrow H^{q-k, q}\left(X_{\mathbb{R}}(\mathbb{C}) ; \underline{Z} / 2\right) \rightarrow H_{\mathbb{Z} / 2}^{2 q-k}(X(\mathbb{C}) ; \mathbb{Z} / 2)
$$

together with the isomorphism $\mathbb{H}_{\mathcal{M}}^{2 q-k, q}(X ; \mathbb{Z} / 2) \stackrel{\cong}{\cong} L^{q} H \mathbb{R}^{q-k, q}(X ; \mathbb{Z} / 2)$ gives another map

$$
\mathbb{H}_{\mathcal{M}}^{2 q-k, q}(X ; \mathbb{Z} / 2) \cong L^{q} H \mathbb{R}^{q-k, q}(X ; \mathbb{Z} / 2) \rightarrow H_{\mathbb{Z} / 2}^{2 q-k}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2\right) \cong H_{e t}^{2 q-k}\left(X ; \mu_{2}^{\otimes q}\right)
$$

In this section we verify that these two potentially different cycle maps are equal and we explore a few consequences. In particular this allows compatibility of cycle maps allows us to conclude that $L^{q} H \mathbb{R}^{q-p, q}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{q-p, q}\left(X_{\mathbb{R}}(\mathbb{C}) ; \mathbb{Z} / 2^{k}\right)$ is an isomorphism for $p \geq q$ and for any smooth $X$.

Before continuing, we show that motivic cohomology and morphic cohomology for real varieties agree with finite coefficients. This is a well-known to the experts, but because of the lack of a good reference we prove it below.

Proposition 5.1. Let $X$ be a smooth real variety. Then for any $n>0$

$$
\mathbb{H}_{\mathcal{M}}^{2 q-k, q}(X ; \mathbb{Z} / n) \xrightarrow{\cong} L^{q} H \mathbb{R}^{q-k, q}(X ; \mathbb{Z} / n)
$$

Proof. We show that the natural map of simplicial abelian groups

$$
z_{\text {equi }}\left(\mathbb{A}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet}\right) \otimes \mathbb{Z} / n \rightarrow z_{\text {equi }}\left(\mathbb{A}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet} \times \Delta_{\text {top }}^{\bullet}\right) \otimes \mathbb{Z} / n
$$

is a quasi-isomorphism which implies the result by Proposition 4.11 and Proposition 4.12. Write $F(U)$ for the presheaf

$$
U \mapsto \pi_{k}\left(z_{\text {equi }}\left(\mathbb{A}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet} \times U\right) \otimes \mathbb{Z} / n\right)
$$

on $S c h / \mathbb{R}$ and $F_{0}(U)$ for the constant presheaf

$$
U \mapsto \pi_{k}\left(z_{\text {equi }}\left(\mathbb{A}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet}\right) \otimes \mathbb{Z} / n\right)
$$

Restricted to $S m / \mathbb{R}$ these are homotopy invariant presheaves with transfers. Recall FW02a, Lemma 3.8] that if $F(-)$ is a homotopy invariant presheaf with transfers, $Y$ is smooth, and $y \in Y(\mathbb{R})$ then $F\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right) \rightarrow F(\mathbb{R})$ is an isomorphism, where $\mathcal{O}_{Y, y}^{h}$ is the Henselization of the local ring $\mathcal{O}_{Y, y}$.

Let $H(-)$ denote either the kernel or the cokernel of the natural transformation $F_{0}(-) \rightarrow F(-)$. Let $Y$ be a quasi-projective real variety and $\gamma \in H(Y)$. Let $\tilde{Y} \rightarrow Y$ be a $c d h$-cover with $\tilde{Y}$ smooth (in particular it is a uad-cover). Since
$H\left(\operatorname{Spec} \mathcal{O}_{\tilde{Y}, y}^{h}\right)=0$ for any $y \in \tilde{Y}(\mathbb{R})$ there are finitely many etale maps $\tilde{Y}_{k} \rightarrow \tilde{Y}$ such that $\left.\gamma\right|_{\tilde{Y}_{k}}=0$ and $\coprod \tilde{Y}_{k} \rightarrow \tilde{Y}$ is a uad-cover.

Therefore $H_{u a d}=0$ and $\left(F_{0}\right)_{u a d} \rightarrow F_{u a d}$ is an isomorphism. By Theorem4.4 we conclude that

$$
\pi_{k}\left(z_{\text {equi }}\left(\mathbb{A}_{\mathbb{R}}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet}\right) \otimes \mathbb{Z} / n\right) \rightarrow \pi_{k}\left(z_{\text {equi }}\left(\mathbb{A}_{\mathbb{R}}^{q}, 0\right)\left(X \times \Delta_{\mathbb{R}}^{\bullet} \times \Delta_{\text {top }}^{\bullet}\right) \otimes \mathbb{Z} / n\right)
$$

is an isomorphism.
An application of the Bousfield-Friedlander spectral sequence finishes the proof.

Friedlander-Walker introduce in FW02a the equivalence relation of real algebraic equivalence. Briefly two cycles $\alpha, \beta$ on a real variety $X$ are real algebraically equivalent provided there is a smooth real curve $C$, two real points $c_{0}, c_{1}$ in the same analytic connected component of $C(\mathbb{R})$, and a cycle $\gamma$ on $X \times C$ such that $\alpha=\left.\gamma\right|_{c_{0}}$ and $\beta=\left.\gamma\right|_{c_{1}}$. Since $L^{q} H \mathbb{R}^{q, q}(X)$ is the group of codimension $q$ cycles on $X$ modulo real algebraic equivalence we obtain the following corollary.

Corollary 5.2. Let $X$ be a smooth real variety and $0 \leq r \leq \operatorname{dim}(X)$. Rational equivalence and real algebraic equivalence yield the same equivalence relation on the group of $r$-cycles on $X$ with finite coefficients.

$$
\begin{aligned}
& \text { Recall that } z_{\text {equi }}\left(\mathbb{P}^{q / q-1}, 0\right)(U)=z_{\text {equi }}\left(\mathbb{P}^{q}, 0\right)(U) / z_{\text {equi }}\left(\mathbb{P}^{q-1}, 0\right)(U) \text {. Write } \\
& \qquad \begin{aligned}
\mathbb{Z} / 2(q)(X) & =\left(z_{\text {equi }}\left(\mathbb{P}_{\mathbb{R}}^{q / q-1}, 0\right)\left(X \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}\right) \otimes \mathbb{Z} / 2\right)[-2 q] \\
\mathbb{Z} / 2(q)^{s t t}(X) & =\operatorname{Sing}_{\bullet}\left(\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G}\right)[-2 q] \\
\mathbb{Z} / 2(q)^{\text {top }}(X) & =\operatorname{Hom}_{\text {cts }}\left(X_{\mathbb{C}}(\mathbb{C}) \times \Delta_{\text {top }}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}[-2 q] \\
\mathbb{Z} / 2(q)^{\text {Bor }}(X) & =\operatorname{Hom}_{\text {cts }}\left(X_{\mathbb{C}}(\mathbb{C}) \times E G \times \Delta_{\text {top }}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}[-2 q]
\end{aligned}
\end{aligned}
$$

where we identify a simplicial abelian group with its associated bounded above cochain complex. These form presheaves of cochain complexes on $S m / \mathbb{R}$. These chain complexes compute respectively motivic cohomology, real morphic cohomology, Bredon cohomology, and Borel cohomology. Note that $\mathbb{Z} / 2(q), \mathbb{Z} / 2(q)^{t o p}$ and $\mathbb{Z} / 2(q)^{\text {Bor }}$ are in fact complexes of etale sheaves on $(S m / \mathbb{R})$.

There are natural maps between these complexes,

$$
\mathbb{Z} / 2(q)(X) \xrightarrow{\rho} \mathbb{Z} / 2(q)^{\text {sst }}(X) \xrightarrow{\Phi} \mathbb{Z} / 2(q)^{\text {top }}(X) \xrightarrow{\psi} \mathbb{Z} / 2(q)^{\text {Bor }}(X)
$$

obtained as follows. From Proposition 4.9 and the projection $\Delta_{\mathbb{R}}^{\bullet} \rightarrow \operatorname{Spec} \mathbb{R}$ we obtain

$$
z_{\text {equi }}\left(\mathbb{P}^{q / q-1}, 0\right)\left(X \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}\right) \rightarrow \operatorname{Sing} \mathcal{Z}^{q}\left(X \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}\right)=\operatorname{Sing} \mathcal{Z}^{q}\left(X_{\mathbb{C}} \times_{\mathbb{C}} \Delta_{\mathbb{C}}^{\bullet}\right)^{G}
$$

which induces $\mathbb{Z} / 2(q)(X) \rightarrow \mathbb{Z} / 2(q)^{\text {sst }}(X)$. The second map $\Phi$ is the map (3.11) and the third map $\psi$ is induced by the projection $X_{\mathbb{C}}(\mathbb{C}) \times E G \rightarrow X_{\mathbb{C}}(\mathbb{C})$.

Nisnevich hypercohomology and descent. These cohomology theories may be computed as Nisnevich hypercohomology groups. This allows us to view these cycle maps as maps in a derived category where we can use a computation of SV00a.

Say that a cartesian square

is a distinguished Nisnevich square provided the map $Y \xrightarrow{f} X$ is etale, $i: U \subseteq X$ is an open embedding, and $f:(Y \backslash V) \rightarrow(X \backslash U)$ is an isomorphism. The Nisnevich topology is the Grothendieck topology on $S m / k$ generated by covers of the form $U \coprod Y \rightarrow X$ where $U \subseteq X$ and $f: Y \rightarrow X$ form part of a distinguished square as above.

Given a presheaf of chain complexes $F$ and a closed $i: A \subseteq B$ and open complement $j: U \subseteq B$ define

$$
F(B)_{A}=\operatorname{cone}\left(F(B) \xrightarrow{j^{*}} F(U)\right)[-1],
$$

which fits into the exact triangle

$$
F(B)_{A} \rightarrow F(B) \xrightarrow{j^{*}} F(U)
$$

Say that a presheaf $F(-)$ of chain complexes satisfies Nisnevich descent provided that for a distinguished square as in (5.3) the square

is homotopy cartesian. Recall this means that this square induces the MayerVietoris exact triangle (in the derived category of abelian groups):

$$
F(X) \rightarrow F(Y) \oplus F(U) \rightarrow F(V)
$$

Equivalently, it means that $F(Y)_{Z^{\prime}} \rightarrow F(X)_{Z}$ is an isomorphism in the derived category of abelian groups where $Z=X \backslash U$ and $Z^{\prime}=Y \backslash V$.

When a presheaf of chain complexes $F(-)$ (with $F(\emptyset)=0$ ) satisfies Nisnevich descent then the Nisnevich hypercohomology of a smooth $X$ with coefficients in $F$ is computed as

$$
H^{p}(F(X))=H^{p}\left(F_{N i s}(X)\right)=\mathbb{H}_{N i s}^{p}\left(X ; F_{N i s}\right)
$$

(see for example CTHK97, Theorem 7.5.1] for presheaves of chain complexes, Nis89 for descent in the case of presheaves of spectra, BG73 for descent in the Zariski topology).

Note that

$$
\begin{aligned}
\mathbb{H}_{N i s}^{2 q-p}\left(X ;\left(\mathbb{Z} / 2(q)^{s s t}\right)_{N i s}\right) & =L^{q} H^{q-p, q}(X ; \mathbb{Z} / 2) \\
\mathbb{H}_{N i s}^{2 q-p}\left(X ; \mathbb{Z} / 2(q)^{t o p}\right) & =H^{q-p, q}\left(X_{\mathbb{R}}(\mathbb{C}) ; \underline{\mathbb{Z} / 2}\right) \\
\mathbb{H}_{\text {Nis }}^{2 q-p}\left(X ; \mathbb{Z} / 2(q)^{\text {Bor }}\right) & =H_{\mathbb{Z} / 2}^{2 q-p}(X(\mathbb{C}) ; \mathbb{Z} / 2)
\end{aligned}
$$

In the first case this follows because the motivic complex $\mathbb{Z} / 2(q)$ satisfies Nisnevich descent and $\mathbb{Z} / 2(q)(X) \rightarrow \mathbb{Z} / 2(q)^{\text {sst }}(X)$ is a quasi-isomorphism of chain complexes for all smooth $X$.

Given $i: A \subseteq B$ a closed subvariety with open complement $j: U \subseteq B$ and write $C(j)$ for the mapping cone of $j: U(\mathbb{C}) \subseteq B(\mathbb{C})$. Then by a comparison of exact triangles we see that

$$
\mathbb{Z} / 2(q)^{t o p}(B)_{A} \simeq \operatorname{Hom}_{c t s_{*}}\left(C(j) \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}[-2 q]
$$

and

$$
\mathbb{Z} / 2(q)^{B o r}(B)_{A} \simeq \operatorname{Hom}_{c t s_{*}}\left(C(j) \wedge E G_{+} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}[-2 q]
$$

Let $F(-)$ denote either $\mathbb{Z} / 2(q)^{\text {top }}(-)$ or $\mathbb{Z} / 2(q)^{\text {Bor }}(-)$ and let

be a distinguished Nisnevich square in $S m / \mathbb{R}$ then

is an equivariant homotopy pushout diagram of $G$-spaces (see for example [DI04] ). Therefore $C\left(j^{\prime}\right) \xrightarrow{\simeq} C(j)$ is an equivariant homotopy equivalence. Consequently $F(X)_{Z} \rightarrow F(Y)_{Z^{\prime}}$ is an isomorphism in the derived category of abelian groups and therefore

is homotopy cartesian. This means that both $\mathbb{Z} / 2(q)^{\text {top }}(-)$ and $\mathbb{Z} / 2(q)^{\text {Bor }}(-)$ satisfy Nisnevich descent.

Compatibility of cycle maps. We are now ready to show that two cycle maps discussed in the beginning of this section are the same map.

Lemma 5.4. Suppose that $V$ is a quasi-projective complex variety considered as a real variety. Then $\pi_{k} \mathbb{Z} / 2(q)^{B o r}(V)=H_{\text {sing }}^{2 q-k}(V(\mathbb{C}) ; \mathbb{Z} / 2)$.

Proof. If $V$ is a complex variety then $\left(V \times_{\mathbb{R}} \mathbb{C}\right)(\mathbb{C})=V(\mathbb{C}) \amalg V(\mathbb{C})$ and $G$ acts by interchanging the factors. In particular $G$ acts freely on $V_{\mathbb{C}}(\mathbb{C})$ and $\left(V_{\mathbb{C}}(\mathbb{C}) \times\right.$ $E G) / G \rightarrow V_{\mathbb{C}}(\mathbb{C}) / G=V(\mathbb{C})$ is a vector-bundle which immediately implies that $\pi_{k} \mathbb{Z} / 2(q)^{\text {Bor }}(V)=H_{\text {sing }}^{2 q-k}(V(\mathbb{C}) ; \mathbb{Z} / 2)$.

Write $\pi_{0}:(S m / \mathbb{R})_{e t} \rightarrow(S m / \mathbb{R})_{N i s}$ for the canonical map of sites.
Proposition 5.5. The complex of etale sheaves $\pi_{0}^{*} \mathbb{Z} / 2(q)^{\text {Bor }}$ on $(S m / \mathbb{R})_{\text {et }}$ is canonically quasi-isomorphic to $\mu_{2}^{\otimes q}$.

Proof. Write $\mathcal{H}^{i}$ for the etale sheafification of the $i$ th cohomology presheaf of $\mathbb{Z} / 2(q)^{\text {Bor }}$. First we show that $\mathcal{H}^{i}=0$ for $i \neq 0$. It is enough to show that for each real variety $X$ and $\gamma \in H^{i}\left(\left[X_{\mathbb{C}}(\mathbb{C}) \times E G\right] / G ; \mathbb{Z} / 2\right)$ that we can find an etale covering $\left(U_{j} \rightarrow X\right)$ such that $\left.\gamma\right|_{U_{j}}=0$ for each $j$. The map $Y=X_{\mathbb{C}} \rightarrow X$ is an etale cover for any real variety $X$. Write $\gamma^{\prime}=\left.\gamma\right|_{Y}$. By the previous Lemma $\left.H^{i}\left(Y_{\mathbb{C}}(\mathbb{C}) \times E G\right] / G ; \mathbb{Z} / 2\right)=H_{\text {sing }}^{i}(Y(\mathbb{C}) ; \mathbb{Z} / 2)$. Since $Y$ has an etale cover $U_{j} \rightarrow Y$ such that $\left.\gamma^{\prime}\right|_{U_{j}}=0$ for each $j$ (see e.g. [Mil80, Lemma III.3.15] ) we conclude that $\mathcal{H}^{i}=0$ for $i \neq 0$.

When $X_{\mathbb{C}}$ is connected then $H^{0}\left(\left[X_{\mathbb{C}}(\mathbb{C}) \times E G\right] / G ; \mathbb{Z} / 2\right)=\mathbb{Z} / 2$. More generally if $X=\amalg X_{i}$ is the disjoint union of $c$ connected real varieties then $H^{0}\left(\left[X_{\mathbb{C}}(\mathbb{C}) \times\right.\right.$ $E G] / G ; \mathbb{Z} / 2)=\mathbb{Z} / 2^{\times c}$. This shows $\mathcal{H}^{0}=\mathbb{Z} / 2=\mu_{2}^{q}$.

Finally since $\mathcal{H}^{i}=0$ for $i \neq 0$ we have canonical isomorphisms

$$
\mathbb{Z} / 2=\operatorname{Hom}_{e t}\left(\mathcal{H}^{0}, \mu_{2}^{\otimes q}\right)=\operatorname{Hom}_{D^{-}\left((S m / \mathbb{R})_{e t}\right)}\left(\pi^{*} \mathbb{Z} / 2(q)^{B o r}, \mu_{2}^{\otimes q}\right)
$$

Recall [SV00a, Section 6] that there is an injective etale resolution $0 \rightarrow \mu_{2}^{\otimes q} \rightarrow J^{\bullet}$ such that $\pi_{0 *} J^{\bullet}$ is a complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves. Proposition 5.5 gives a canonical map $\pi_{0}^{*} \mathbb{Z} / 2(q)^{\text {Bor }} \rightarrow$ $J^{\bullet}$ and by adjointness we obtain a map

$$
\mathbb{Z} / 2(q)^{\text {Bor }} \rightarrow \mathbb{R}\left(\pi_{0}\right)_{*} \mu_{2}^{\otimes q}=\left(\pi_{0}\right)_{*} J^{\bullet}
$$

Consider the following sequence of maps of complexes of Nisnevich sheaves

$$
\begin{equation*}
\mathbb{Z} / 2(q) \rightarrow\left(\mathbb{Z} / 2(q)^{s s t}\right)_{N i s} \xrightarrow{\Phi} \mathbb{Z} / 2(q)^{\text {top }} \rightarrow \mathbb{Z} / 2(q)^{\text {Bor }} \rightarrow \mathbb{R}\left(\pi_{0}\right)_{*} \mu_{2}^{\otimes q} \tag{5.6}
\end{equation*}
$$

The complex of Nisnevich sheaves with transfers $B_{2}(q)$ is defined in SV00a, Section 6] to be the truncation

$$
B_{2}(q)=\tau_{\leq q}\left(\pi_{0}\right)_{*} J^{\bullet}=\tau_{\leq q}\left(\mathbb{R} \pi_{0 *} \mu_{2}^{\otimes q}\right)
$$

in particular $\mathbb{H}_{N i s}^{p}\left(X ; B_{2}(q)\right)=H_{e t}^{p}\left(X ; \mu_{2}^{\otimes q}\right)$ for $p \leq q$ and all smooth $X$. Since the cohomology sheaves of $\mathbb{Z} / 2(q)$ (and therefore of $\mathbb{Z} / 2(q)^{s s t}$ as well) vanish in degrees $i>q$ and so the sequence of maps (5.6) factors through the truncations,

$$
\begin{equation*}
\mathbb{Z} / 2(q) \rightarrow\left(\mathbb{Z} / 2(q)^{s s t}\right)_{N i s} \rightarrow \tau_{\leq q} \mathbb{Z} / 2(q)^{t o p} \rightarrow \tau_{\leq q} \mathbb{Z} / 2(q)^{\text {Bor }} \rightarrow B_{2}(q) \tag{5.7}
\end{equation*}
$$

Remark 5.8. It is important to note that the composites 5.6 and 5.7 are non-trivial. This can be seen, for example, by evaluating on Spec $\mathbb{C}$. The map $\mathbb{Z} / 2(q)(X) \rightarrow$ $\left(\mathbb{Z} / 2(q)^{s s t}\right)_{N i s}(X)$ is a quasi-isomorphism for any smooth real variety $X$ by Proposition 5.1. The comparison map $\left(\mathbb{Z} / 2(q)^{s t t}\right)_{N i s}(\mathbb{C}) \rightarrow \mathbb{Z} / 2(q)^{t o p}(\mathbb{C})$ is an equality. By Proposition 5.17 below, for any $X$, the map $\mathbb{Z} / 2(q)^{\text {top }}(X) \rightarrow \mathbb{Z} / 2(q)^{B o r}(X)$ induces an isomorphism on cohomology in degrees $p \leq q$. Finally since $\mathbb{Z} / 2(q)^{B o r} \rightarrow$ $\mathbb{R} \pi_{0 *} \mu_{2}^{\otimes q}$ is obtained as the adjoint of a quasi-isomorphism and Spec $\mathbb{C}$ is an etale point (of $(S m / \mathbb{R})_{e t}$ ) the $\operatorname{map} \mathbb{Z} / 2(q)^{B o r}(\mathbb{C}) \rightarrow \mathbb{R} \pi_{0 *} \mu_{2}^{\otimes q}(\mathbb{C})$ cannot be zero.

Write $D^{-}(N i s)$ (respectively $\left.D^{-}(N S w T / \mathbb{R})\right)$ for the derived category of bound above complexes of Nisnevich sheaves (respectively Nisnevich sheaves with transfers). Write $D M^{-}(\mathbb{R}) \subseteq D^{-}(N S w T / \mathbb{R})$ for the full subcategory consisting of complexes with homotopy invariant Nisnevich cohomology sheaves.

Theorem 5.9. Let $X$ be a smooth real variety. The diagram commutes


Proof. By SV00a, Corollary 6.11.1] and the vanishing of the cohomology sheaves of $\mathbb{Z} / 2(q)$ above degree $q$,

$$
\mathbb{Z} / 2=\operatorname{Hom}_{D M^{-}(\mathbb{R})}\left(\mathbb{Z} / 2(q), B_{2}(q)\right)=\operatorname{Hom}_{D M^{-}(\mathbb{R})}\left(\mathbb{Z} / 2(q), \pi_{0 *} J^{\bullet}\right)
$$

Also by [SV00a, Lemma 6.5] the inclusion of bi-complexes

$$
\operatorname{Hom}_{N i s}\left(\mathbb{Z} / 2(q), \pi_{0 *} J^{\bullet}\right) \subseteq \operatorname{Hom}_{N S W T}\left(\mathbb{Z} / 2(q), \pi_{0 *} J^{\bullet}\right)
$$

is an equality. Therefore

$$
\mathbb{Z} / 2=\operatorname{Hom}_{D M^{-}(\mathbb{R})}\left(\mathbb{Z} / 2(q), \pi_{0 *} J^{\bullet}\right)=\operatorname{Hom}_{D^{-}(N i s)}\left(\mathbb{Z} / 2(q), \pi_{0 *} J^{\bullet}\right)
$$

Finally, the map $\mathbb{Z} / 2(q) \rightarrow \pi_{0 *} J^{\bullet}$ obtained from (5.6) is not trivial by Remark 5.8 and so we conclude that it must be the cycle map.

Applications and computations. As a result of the compatibility of cycle maps we can conclude some Beilinson-Lichtenbaum type theorems for morphic cohomology which we need to prove the vanishing theorem. We also use these to make a few computations of equivariant morphic cohomology.
Corollary 5.11. Let $X$ be a smooth real variety. The map

$$
\Phi: L^{q} H \mathbb{R}^{q-k, q}\left(X ; \mathbb{Z} / 2^{n}\right) \rightarrow H^{q-k, q}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z} / 2^{n}}\right)
$$

is an isomorphism for $q \leq k$ and a monomorphism for $q=k+1$.
Proof. Consider the commutative diagram (5.10). By SV00a the Milnor conjecture, proved by Voevodsky Voe03, implies that cyc is an isomorphism for $k \geq q$ and an injection for $k=q-1$. This immediately implies the statement for injectivity. If $k \geq q$ then since $c y c$ is an isomorphism we conclude that $H_{G}^{2 q-k}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2\right) \rightarrow$ $H_{e t}^{2 q-k}\left(X ; \mu_{2}^{\otimes q}\right)$ is a surjective map between finitely dimensional $\mathbb{Z} / 2$-vector spaces. By Cox79] these are isomorphic $\mathbb{Z} / 2$-vector spaces and therefore the map is an isomorphism. Since $H^{q-k, q}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z}}\right) \rightarrow H_{G}^{2 q-k}\left(X_{\mathbb{C}}(X) ; \mathbb{Z} / 2\right)$ is also an isomorphism for $k \geq q$ by Proposition 5.17 we conclude that $\Phi$ is also an isomorphism for $k \geq q$. This yields the result for $\mathbb{Z} / 2$-coefficients.

We have the following diagram of distinguished triangles in $D^{-}(N i s)$ :


To see that the bottom row is a triangle in $D^{-}(N i s)$ it is enough to check that the map on cohomology sheaves $\mathcal{H}^{q}\left(\tau_{\leq q} \mathbb{Z} / 4(q)^{t o p}\right) \rightarrow \mathcal{H}^{q}\left(\tau_{\leq q} \mathbb{Z} / 2(q)^{\text {top }}\right)$ is a surjection. This follows from the surjectivity of the composition

$$
\mathcal{H}^{q}(\mathbb{Z}(q)) \rightarrow \mathcal{H}^{q}\left(\tau_{\leq q} \mathbb{Z} / 4(q)^{t o p}\right) \rightarrow \mathcal{H}^{q}\left(\tau_{\leq q} \mathbb{Z} / 2(q)^{t o p}\right)
$$

which is a consequence of the local vanishing of $\mathbb{Z}(q)$ and the quasi-isomorphisms $\mathbb{Z} / 2(q) \rightarrow B_{2}(q)$ and $\tau_{\leq q} \mathbb{Z} / 2(q)^{t o p} \rightarrow B_{2}(q)$. Now the conclusion follows from the long exact sequence in hypercohomology associated to the diagram. Using induction on $n$ we conclude that $\Phi:\left(\mathbb{Z} / 2^{n}(q)^{s s t}\right)_{N i s} \rightarrow \tau_{\leq q} \mathbb{Z} / 2^{n}(q)^{t o p}$ is a quasiisomorphism.

Corollary 5.12. Let $X$ be a smooth complex variety. For any $n>0$ the map

$$
\Phi: L^{q} H^{2 q-k}\left(X ; \mathbb{Z} / 2^{n}\right) \rightarrow H_{\text {sing }}^{2 q-k}\left(X(\mathbb{C}) ; \mathbb{Z} / 2^{n}\right)
$$

is an isomorphism for any $q \leq k$ and a monomorphism for $q=k+1$.
Proof. Follows immediately from Corollary 5.11 by viewing $X$ as real variety.
Corollary 5.13. Let $X$ be a smooth real variety and $k>0$. The cycle map

$$
\Phi: L^{q} H \mathbb{R}^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2^{k}\right)
$$

is an isomorphism if $r \leq 0$ (and $s \leq q$ ) and an injection if $r=1$ (and $s \leq q$ ).
Proof. Write $F_{q}=\operatorname{hofib}\left(\mathcal{Z}^{q} / 2^{k}\left(X_{\mathbb{C}}\right) \rightarrow \operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z} / 2_{0}^{k}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)\right.$ ) for the homotopy fiber of the cycle map. The homotopy fiber construction is equivariant and yields an equivariant homotopy fiber sequence

$$
F_{q} \rightarrow \mathcal{Z}^{q} / 2^{k}\left(X_{\mathbb{C}}\right) \rightarrow \operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z} / 2_{0}^{k}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)
$$

By Corollary 5.12 and Corollary5.11both $\pi_{k}\left(F_{q}\right)=0$ and $\pi_{k}\left(F_{q}^{G}\right)=0$ for $k \geq q-1$. Therefore $\Omega^{q-1} F_{q}$ is equivariantly weakly contractible for $q \geq 1$ and if $q=0$ then $F_{0}$ is equivariantly contractible. The result follows now from the long exact sequence of homotopy groups applied to the equivariant homotopy fiber sequence

$$
\Omega^{q-1} F_{q} \rightarrow \Omega^{q-1} \mathcal{Z}^{q} / 2^{k}\left(X_{\mathbb{C}}\right) \rightarrow \Omega^{q-1} \operatorname{Map}\left(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z} / 2_{0}^{k}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)
$$

Corollary 5.14. Let $X$ be a smooth real curve. Then

$$
L^{q} H \mathbb{R}^{r, s}(X ; \mathbb{Z}) \rightarrow H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z}}\right)
$$

is an isomorphism for any $q \geq 0, r \leq q$, and $s \leq q$.
Proof. By Poincare duality for real Lawson homology and equivariant morphic cohomology and Remark 4.21, $L^{q} H^{r, s}(X ; \mathbb{Z}) \xrightarrow{\cong} H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z}}\right)$ for $q \geq 1$. By Corollary 5.13. $L^{0} H \mathbb{R}^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right)$ is an isomorphism for $r, s \leq 0$. When $A$ is an abelian group and 2 is invertible in $A$ then a transfer argument shows that

$$
L^{0} H \mathbb{R}^{r, s}(X ; A) \stackrel{\cong}{\rightrightarrows} H^{r, s}(X ; \underline{A})
$$

This isomorphism and the one with mod- $2^{k}$ coefficients give the result of the corollary.
Corollary 5.15. Let $X$ be a smooth real surface. Then for any $k>0$

$$
L^{q} H \mathbb{R}^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{r, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \underline{\mathbb{Z} / 2^{k}}\right)
$$

is an isomorphism for $q=0$ and $r, s \leq 0$ and it is an injection for $r=1$ and $s \leq 1$. Moreover $L^{1} H \mathbb{R}^{1, s}\left(X ; \mathbb{Z} / 2^{k}\right)=0$ for $s \leq-2$.

Recall that $\pi_{0} \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G}$ is the group of codimension $q$ cycles on $X$ modulo real algebraic equivalence.

Corollary 5.16. Let $X$ be a smooth real variety of dimension $d$. Then for any $k>0$

$$
L^{1} H \mathbb{R}^{r, s}\left(X ; \mathbb{Z} / 2^{k}\right) \rightarrow H^{r, s}\left(X ; \underline{\mathbb{Z} / 2^{k}}\right)
$$

is an isomorphism for any $r \leq 0$ and $s \leq 1$ and it is an injection for $r=1$ and $s \leq 1$. Moreover

$$
\begin{aligned}
& L^{1} H \mathbb{R}^{1,1}\left(X ; \mathbb{Z} / 2^{k}\right)=C H^{1}(X) \otimes \mathbb{Z} / 2^{k} \subseteq H^{1,1}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2^{k}\right) \\
& L^{1} H \mathbb{R}^{1, s}\left(X ; \mathbb{Z} / 2^{k}\right)=0 \quad \text { for } s \leq-2
\end{aligned}
$$

Proof. All statements follow immediately from Corollary 5.13 except the last one. The first part of the last statement follows from Proposition 5.1. For rest of the last statement, by Corollary 5.13 together with Proposition 5.17 we have

$$
L^{1} H \mathbb{R}^{1, s}\left(X ; \mathbb{Z} / 2^{k}\right) \hookrightarrow H^{1, s}\left(X_{\mathbb{C}}(\mathbb{C}) ; \mathbb{Z} / 2^{k}\right) \hookrightarrow H_{G}^{1+s}\left(X_{\mathbb{C}}(\mathbb{C}) ; A\right)=0
$$

for $1+s<0$.
We finish this section with the computation used in Corollary 5.11 that Bredon and Borel cohomology agree in the range relevant to the Beilinson-Lichtenbaum conjecture.

Proposition 5.17. Let $W$ be a G-CW complex, $M$ a $G$-module, and $\underline{M}$ the associated constant Mackey functor. The map

$$
H^{m, q}(W ; \underline{M}) \rightarrow H^{m, q}(W \times E G ; \underline{M})
$$

is an isomorphism for $m \leq 0$ and it is an injection for $m=1$.
In particular for $q \leq p$ the map

$$
H^{q-p, q}(W ; \underline{\mathbb{Z} / 2}) \rightarrow H_{\mathbb{Z} / 2}^{2 q-p}(W, \mathbb{Z} / 2)
$$

is an isomorphism and an injection for $p=q-1$.
Proof. Define $\tilde{E} G=\operatorname{colim}_{n} S^{0, n}$. This space fits into a homotopy cofiber sequence

$$
E G_{+} \rightarrow S^{0} \rightarrow \tilde{E} G
$$

and $\tilde{E} G / G \simeq S^{1,0} \wedge B G$.
First we consider the case that $G$ acts trivially on $W$. From the previous homotopy cofiber sequence we obtain the homotopy cofiber sequence

$$
\begin{equation*}
W_{+} \wedge E G_{+} \rightarrow W_{+} \rightarrow W_{+} \wedge \tilde{E} G \tag{5.18}
\end{equation*}
$$

Since $G$ acts trivially on $W$ we have that $\left(W_{+} \wedge \tilde{E} G\right) / G=W_{+} \wedge \tilde{E} G / G \simeq W_{+} \wedge S^{1,0} \wedge$ $B G$ and therefore $\tilde{H}^{k, 0}\left(W_{+} \wedge \tilde{E} G ; \underline{M}\right)=\tilde{H}_{\text {sing }}^{k-1}\left(W_{+} \wedge B G ; M\right)=0$ if $k \leq 1$. Notice that $S^{0,1} \wedge \tilde{E} G=S^{0,1} \wedge \operatorname{colim}_{n} S^{0, n}=\operatorname{colim}_{n} S^{0, n+1} \cong \tilde{E} G$. Since $\tilde{E} G \cong S^{0,1} \wedge \tilde{E} G$ is an equivariant equivalence this induces an isomorphism

$$
\tilde{H}^{k, s}\left(W_{+} \wedge \tilde{E} G ; \underline{M}\right) \cong \tilde{H}^{k, 0}\left(W_{+} \wedge \tilde{E} G ; \underline{M}\right)
$$

for all $s$ and therefore $\tilde{H}^{k, s}\left(W_{+} \wedge \tilde{E} G, \underline{M}\right)=0$ for $k \leq 1$ for all $s$. Now from the long exact sequence associated to the cofiber sequence (5.18) it follows that for all $q$ the $\operatorname{map} H^{m, q}(W ; \underline{M}) \rightarrow H^{m, q}(W \times E G ; \underline{M})$ is an isomorphism for $m \leq 0$ and an injection for $m=1$.

Consider now a general $G-C W$ complex $W$ and consider the quotient $W / W^{G}$. Since $G$ acts freely on the based space $W / W^{G}$ we have the isomorphism

$$
\tilde{H}^{s, t}\left(X / X^{G} ; \underline{M}\right) \stackrel{\cong}{\rightrightarrows} \tilde{H}^{s, t}\left(X / X^{G} \wedge E G_{+} ; M\right)
$$

Applying the five lemma to the comparison of long exact sequences obtained from the cofiber sequences $W_{+}^{G} \rightarrow W_{+} \rightarrow W / W^{G}$ and $\left(W^{G} \times E G\right)_{+} \rightarrow(W \times E G)_{+} \rightarrow$ $W / W^{G} \wedge E G_{+}$yields the result.

## 6. Vanishing Theorem

Let $X$ be a real variety. Continue to write $G=\mathbb{Z} / 2$ and $\sigma \in G$ for the nontrivial element. Recall that the reduced real cycle group is defined to be the quotient topological group

$$
\mathcal{R}_{q}(X)=\frac{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}_{q}\left(X_{\mathbb{C}}\right)^{a v}}
$$

This is the free $\mathbb{Z} / 2$-module generated by closed subvarieties $Z \subseteq X$ such that both $Z$ and $Z_{\mathbb{C}}$ are irreducible. In particular, if $X$ is a complex variety viewed as a variety over $\mathbb{R}$ then $\mathcal{R}_{q}(X)=0$.

Reduced real Lawson homology of $X$ is defined by the homotopy groups of this space,

$$
R L_{q} H_{n}(X)=\pi_{n-q} \mathcal{R}_{q}(X)
$$

In this section we prove our main theorem which we state now.
(Theorem 6.10). Let $X$ be a quasi-projective real variety. Then

$$
\pi_{k} \mathcal{R}_{n}(X)=R L_{n} H_{k+n}(X)=0
$$

for $k \geq \operatorname{dim} X-n+1$.
To avoid difficulties with point-set topology below we work simplicially. Note that if $X$ is a $G$-space then Sing。 $X$ is a $G$-simplicial set and $\operatorname{Sing} .\left(X^{G}\right)=(\operatorname{Sing} \bullet X)^{G}$. If $A_{\bullet}$ is a $G$-simplicial set then $\left|A_{\bullet}^{G}\right|=\left|A_{\bullet}\right|^{G}$ (see for example Dug05, Lemma A.5]).

Definition 6.1. (1) Let $W$ be a $G$-space. Write

$$
\widetilde{\mathcal{Z}}_{t o p}^{q}(W)=\operatorname{Hom}_{c t s}\left(W \times \Delta_{t o p}^{\bullet}, \mathcal{Z}_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)
$$

and

$$
\widetilde{\mathcal{Z}}^{q} / 2_{t o p}(W)=\operatorname{Hom}_{c t s}\left(W \times \Delta_{t o p}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)
$$

These are simplicial abelian groups and $G$ acts on them by the standard formula $(\sigma f)(x)=\sigma f(\sigma x)$.
(2) Let $X$ be a normal quasi-projective real variety. Write

$$
\widetilde{\mathcal{Z}}^{q}\left(X_{\mathbb{C}}\right)=\operatorname{Sing} \bullet \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)
$$

and

$$
\widetilde{\mathcal{Z}}^{q} / 2\left(X_{\mathbb{C}}\right)=\operatorname{sing} . \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)
$$

We have

$$
\pi_{k} \widetilde{\mathcal{Z}}_{\text {top }}^{q}\left(X_{\mathbb{C}}(\mathbb{C})\right)^{G} \cong H^{q-k, q}\left(X_{\mathbb{C}}(\mathbb{C}), \underline{\mathbb{Z}}\right)
$$

and

$$
\pi_{k} \widetilde{\mathcal{Z}}^{q}\left(X_{\mathbb{C}}\right)^{G} \cong L^{q} H \mathbb{R}^{2 q-k}(X ; \mathbb{Z})
$$

and similarly for the mod-2 groups. In particular if $X$ is a complex variety viewed as a real variety then $X_{\mathbb{C}}(\mathbb{C})=X(\mathbb{C}) \amalg X(\mathbb{C})$ (with $G$-action switching the factors) and so

$$
\pi_{k} \widetilde{\mathcal{Z}}_{\text {top }}^{q}\left(X_{\mathbb{C}}(\mathbb{C})\right)^{G} \cong H_{\text {sing }}^{2 q-k}(X(\mathbb{C}), \underline{\mathbb{Z}}) \quad \text { and } \quad \pi_{k} \widetilde{\mathcal{Z}}^{q}\left(X_{\mathbb{C}}\right)^{G} \cong L^{q} H^{2 q-k}(X ; \mathbb{Z})
$$

and similarly for the mod- 2 groups.
The comparison maps (3.11) of simplicial abelian groups

$$
\Phi: \widetilde{\mathcal{Z}}^{q}\left(X_{\mathbb{C}}\right) \rightarrow \widetilde{\mathcal{Z}}_{\text {top }}^{q}\left(X_{\mathbb{C}}(\mathbb{C})\right) \quad \text { and } \quad \Phi: \widetilde{\mathcal{Z}}^{q} / 2\left(X_{\mathbb{C}}\right) \rightarrow \widetilde{\mathcal{Z}}^{q} / 2_{\text {top }}\left(X_{\mathbb{C}}(\mathbb{C})\right)
$$

are $G$-equivariant for any quasi-projective real variety $X$.
If $M_{\bullet}$ is a simplicial $G$-module write $N=1+\sigma: M_{\bullet} \rightarrow M_{\bullet}$ and define $M_{\bullet}^{a v}$ to be

$$
M_{\bullet}^{a v}=\operatorname{Im}(N)=\operatorname{Im}\left(1+\sigma: M_{\bullet} \rightarrow M_{\bullet}\right)
$$

Definition 6.2. Let $W$ be a $G$-space. Define the group of reduced topological cocycles (of codimension $q$ ) to be the quotient simplicial abelian group

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W)=\frac{\operatorname{Hom}_{c t s}\left(W \times \Delta_{t o p}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{G}}{\operatorname{Hom}_{c t s}\left(W \times \Delta_{t o p}^{\bullet}, \mathcal{Z} / 2_{0}\left(\mathbb{A}_{\mathbb{C}}^{q}\right)\right)^{a v}}=\frac{\widetilde{\mathcal{Z}}^{q} / 2_{t o p}(W)^{G}}{\widetilde{\mathcal{Z}}^{q} / 2_{t o p}(W)^{a v}}
$$

To relate the space of reduced algebraic cocycles with the reduced topological cocycles we introduce the following auxiliary simplicial set for $X$ a quasi-projective normal real variety:

$$
\widetilde{\mathcal{R}}^{q}(X)=\frac{\widetilde{\mathcal{Z}^{q}} / 2\left(X_{\mathbb{C}}\right)^{G}}{\widetilde{\mathcal{Z}}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}}
$$

Proposition 6.3. Let $X$ be a normal quasi-projective real variety. The following diagrams commute and the horizontal rows are short exact sequences of simplicial abelian groups (and therefore in particular the horizontal rows are homotopy fiber sequences of simplicial sets)

and


Proof. These diagrams commute because $\Phi$ is a $G$-homomorphism.
Whenever $M$ is a $G$-module whose underlying abelian group is 2-torsion then the sequence of abelian groups $0 \rightarrow M^{G} \rightarrow M \xrightarrow{N} M^{a v} \rightarrow 0$ is exact.

In particular the underlying sequences of simplicial abelian $G$-modules in the first diagram form short exact sequences of simplicial abelian groups.

In the second diagram the horizontal rows form short exact sequences by definition of $\widetilde{R}^{q}(-)$ and $\widetilde{R}_{\text {top }}^{q}(-)$.

By definition we have

$$
\left(\text { Sing. } \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)\right)^{a v}=\operatorname{Im}\left(\text { Sing. } \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) \xrightarrow{N} \operatorname{Sing} \bullet \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)\right)
$$

There is a natural map $i:\left(\operatorname{Sing} \bullet \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)\right)^{a v} \rightarrow \operatorname{Sing} \bullet\left(\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}\right)$ which is simply

$$
i(f+\bar{f})=f+\bar{f}
$$

for a continuous map $f: \Delta_{\text {top }}^{d} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)$. The map $i:\left(\operatorname{Sing}, \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)\right)^{a v} \rightarrow$ Sing. $\left(\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}\right)$ induces a map

$$
\begin{equation*}
\bar{i}: \widetilde{\mathcal{R}}^{q}(X) \rightarrow \text { Sing. } \mathcal{R}^{q}(X) \tag{6.6}
\end{equation*}
$$

Lemma 6.7. Let $X$ be a normal real projective variety. The map (6.6) of simplicial abelian groups

$$
\widetilde{\mathcal{R}}^{q}(X) \rightarrow \operatorname{Sing} \bullet \mathcal{R}^{q}(X)
$$

is a homotopy equivalence.
Proof. By Proposition A.5 the maps $\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) / \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}$ and $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v} / 2 \mathcal{Z}^{q}(X)^{G} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}$ are isomorphisms of topological groups. Therefore both

$$
0 \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v} \rightarrow 0
$$

and

$$
0 \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G} \rightarrow \mathcal{R}^{q}(X) \rightarrow 0
$$

are short exact sequences of topological abelian groups. These groups all have the homotopy type of a $C W$-complex and therefore these sequences are homotopy fiber sequences [Teh05]. Applying Sing. to these homotopy fiber sequence and comparing with the homotopy fiber sequences of the top rows of 6.4 and 6.5 gives commutative diagrams of homotopy fiber sequences of simplicial sets:

and


From the first diagram we see that $i: \widetilde{\mathcal{Z}}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v} \xrightarrow{\simeq} \operatorname{Sing} .\left(\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}\right)$ is a weak equivalence of simplicial sets and consequently from the second diagram we conclude that

$$
\widetilde{\mathcal{R}}^{q}(X) \xrightarrow{\simeq} \operatorname{Sing} \bullet \mathcal{R}^{q}(X)
$$

is a weak equivalences of simplicial abelian groups and therefore is a homotopy equivalence of simplicial sets.

We now prove our main theorem.
Theorem 6.10. Let $X$ be a quasi-projective real variety of dimension $d$. Then

$$
R L_{n} H_{n+k}(X)=\pi_{k} \mathcal{R}_{n}(X)=0
$$

for $k \geq d-n+1$.

Proof. We first consider the case when $X$ is a smooth projective real variety.
In case $n=d=\operatorname{dim}(X)$ we have that

$$
\mathcal{R}_{d}(X)=\frac{\mathcal{Z}_{d}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}_{d}\left(X_{\mathbb{C}}\right)^{a v}}=\mathbb{Z} / 2^{\times c}
$$

where $c$ denotes the number of irreducible components of $X$ which are not defined over $\mathbb{C}$.

Therefore $\pi_{0}\left(\mathcal{R}_{d}(X)\right)=\mathbb{Z} / 2^{\times c}$ and $\pi_{i}\left(\mathcal{R}_{d}(X)\right)=0$ for $i>0$.
Consider the comparison of homotopy fiber sequences (6.4) for $q>0$.


By the Milnor conjecture (see Corollary 5.12) the comparison map $\Phi$ induces an isomorphism on $\pi_{k}$ for $k \geq q$ and induces an injection for $k=q-1$. By Corollary 5.11 the map $\Phi^{G}$ induces an isomorphism on $\pi_{k}$ for $k \geq q$ and induces an injection for $k=q-1$. We now conclude by the 5 -lemma that $\Phi^{a v}$ induces an isomorphism on $\pi_{k}$ for $k \geq q+1$. When $k=q$ we have the comparison diagram:

and so $\Phi^{a v}$ induces an injection for $k=q$.
Considering now the comparison diagram (6.5) and using the five-lemma we have that $\bar{\Phi}$ induces an isomorphism on $\pi_{k}$ for $k \geq q+2$ and an injection for $k=q+1$.

By Corollary 7.14 $\pi_{k} \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(X_{\mathbb{C}}(\mathbb{C})\right)=H^{q-k}(X(\mathbb{R}), \mathbb{Z} / 2)$ for $k \geq 2$. In particular $\pi_{k} \widetilde{\mathcal{R}}^{q}(X)=0$ for $k \geq q+1$, when $q \geq 1$. By the homotopy equivalences $\widetilde{\mathcal{R}}^{q}(X) \simeq$ Sing. $\mathcal{R}^{q}(X)$ (see Lemma 6.7) and the duality Teh05, Theorem 5.14] between reduced cycle and reduced cocycle spaces $\mathcal{R}^{q}(X) \xrightarrow{\simeq} \mathcal{R}_{d-q}(X)$ the vanishing $\pi_{k} \widetilde{\mathcal{R}}^{q}(X)=0$ for $k \geq q+1$ is equivalent to the vanishing $\pi_{k} \mathcal{R}_{n}(X)=0$ for $k \geq \operatorname{dim} X-n+1$.

Now let $X$ be a smooth quasi-projective variety and let $X \subseteq \bar{X}$ be a projective closure with closed complement $Z=\bar{X} \backslash X$. The result follows from the projective case and the long exact sequence in homotopy groups induced by the homotopy fiber sequence

$$
\mathcal{R}_{n}(Z) \rightarrow \mathcal{R}_{n}(\bar{X}) \rightarrow \mathcal{R}_{n}(X)
$$

Finally let $X$ be an arbitrary quasi-projective variety. There is an increasing filtration of closed subvarieties

$$
\varnothing=X_{-1} \subseteq X_{0} \subseteq X_{1} \subseteq \cdots \subseteq X_{d}=X
$$

such that $X_{i+1} \backslash X_{i}$ is smooth and $\operatorname{dim} X_{i}=i$. We proceed by induction, the case $i=0$ is done. Consider the long exact sequence which arises from the homotopy fiber sequence

$$
\mathcal{R}_{n}\left(X_{i}\right) \rightarrow \mathcal{R}_{n}\left(X_{i+1}\right) \rightarrow \mathcal{R}_{n}\left(X_{i+1} \backslash X_{i}\right)
$$

Since the result holds for $X_{i}$ by induction and for $X_{i+1} \backslash X_{i}$ because it is smooth we obtain the result for $X_{i+1}$.

Remark 6.13. If $X$ is a projective smooth real variety of dimension $d$ it is proved in Teh05, Theorem 6.7] that $\pi_{k}\left(\mathcal{R}_{d-1}(X)\right)=0$ for any $k \geq 3$. Theorem 6.10 in case $n=d-1$ improves this vanishing bound.

Example 6.14. If $n=0$ and if $X$ has no real points then $\mathcal{R}_{0}(X)=0$ and so

$$
R L_{0} H_{*}(X)=\pi_{*} \mathcal{R}_{0}(X)=H_{*}^{\text {sing }}(X(\mathbb{R}), \mathbb{Z} / 2)=0
$$

Let $\mathbb{P}(\mathbb{H})$ denote the space of complex lines in the quaternions $\mathbb{H}=\mathbb{C} \oplus j \mathbb{C}$ where $j^{2}=-1$. Multiplication by $j$ defines an involution on $\mathbb{P}(\mathbb{H})$ and write $Q$ for the corresponding 1-dimensional real curve. We know that $Q$ is the smooth real curve $X^{2}+Y^{2}+Z^{2}=0$ in $\mathbb{P}_{\mathbb{R}}^{2}$. This is the Severi-Brauer variety corresponding to the non-trivial element of $\operatorname{Br}(\mathbb{R})=\mathbb{Z} / 2$ and has no real points. This means $\mathcal{R}_{0}(Q)=0$ and $\mathcal{R}_{1}(Q)=\mathbb{Z} / 2$. Thus in this case,

$$
0=R L_{0} H_{0}(Q)=R L_{0} H_{1}(Q)=H_{0}(Q(\mathbb{R}), \mathbb{Z} / 2)
$$

and

$$
\mathbb{Z} / 2=R L_{1} H_{1}(Q)
$$

Let $X=S P^{2 d+1}(Q)$ be the smooth projective real variety given by an odd symmetric power of $Q$. Because $X_{\mathbb{C}}=\mathbb{P}_{\mathbb{C}}\left(\mathbb{H}^{d+1}\right)$, we have $\mathcal{R}^{2 q}(X)=\mathbb{Z} / 2$ and $\mathcal{R}^{2 q+1}(X)=0$ for any $2 q \leq 2 d+1$ (see [LLFM05, Theorem 2.3]). This implies that the only nonzero reduced Lawson homology groups of $X$ are $R L_{2 r+1} H_{2 r+1}(X)=$ $\mathbb{Z} / 2$ for any $r \leq d$. Notice that in this case $\operatorname{dim}(X)=2 d+1$.

These computations show that the vanishing in the above theorem is best possible, even in the case of a real variety with no real points.

Example 6.15. According to Lam90, $R L_{r} H_{n}\left(\mathbb{P}_{\mathbb{R}}^{d}\right)=\mathbb{Z} / 2$ for any $0 \leq r \leq n \leq d$ and $R L_{r} H_{n}\left(\mathbb{P}_{\mathbb{R}}^{d}\right)=0$ for any $n>d$.

We also obtain the following vanishing result.
Corollary 6.16. Let $X$ be a smooth projective real variety of dimension $d$. Then

$$
\pi_{n} \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}=0
$$

for $n \geq 2 d-2 p+1$.
Proof. By the Corollaries 5.12, 5.11, A.5 and 4.20 we conclude that

$$
\pi_{n} \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}}=0
$$

for $n \geq 2 d-2 p+1$ from the long exact sequence in homotopy groups induced by the short exact sequence Teh08, Proposition 4.3]

$$
\begin{equation*}
0 \rightarrow \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}} \rightarrow \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)} \rightarrow \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}} \rightarrow 0 \tag{6.17}
\end{equation*}
$$

Consider the short exact sequences of topological abelian groups

$$
0 \rightarrow \frac{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}} \rightarrow \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}} \rightarrow \frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}} \rightarrow 0
$$

Multiplication by 2 induces a homeomorphism

$$
\mathcal{R}_{p}(X)=\frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}}{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}} \cong \frac{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}
$$

and plugging the vanishing for homotopy groups of $\mathcal{R}_{p}(X)$ into the above exact sequence yields the result.

Remark 6.18. Using the same arguments as in Theorem 6.10 shows that the vanishing in Corollary 6.16 holds for any quasi-projective real variety.

Example 6.19. Let $X=\mathbb{P}_{\mathbb{R}}^{d}$. Then

$$
\pi_{n}\left(\frac{\mathcal{Z}_{p}\left(\mathbb{P}_{\mathbb{C}}^{d}\right)^{a v}}{2 \mathcal{Z}_{p}\left(\mathbb{P}_{\mathbb{C}}^{d}\right)^{a v}}\right)=0
$$

for any $n \geq 2 d-2 p+1$. If $p=d$, then

$$
\pi_{2 d-2 p}\left(\frac{\mathcal{Z}_{d}\left(\mathbb{P}_{\mathbb{C}}^{d}\right)^{a v}}{2 \mathcal{Z}_{d}\left(\mathbb{P}_{\mathbb{C}}^{d}\right)^{a v}}\right)=\mathbb{Z} / 2
$$

If $p=0$ then, for any real projective variety $X$,

$$
\pi_{*}\left(\frac{\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}}\right)=H_{*}(X(\mathbb{C}) / G, \mathbb{Z} / 2)
$$

These computations show that the vanishing bound of Corollary 6.16 is the best possible. For these computations see [LLFM05].

The following corollary shows that in a range the morphic cohomology of a real variety $X$ can be computed by the homotopy groups of average cycles on $X$.

Corollary 6.20. Let $X$ be a real quasi-projective variety. Then

$$
\pi_{q}\left(\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G}\right) \simeq \pi_{q}\left(\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}\right)
$$

for any $q \geq \operatorname{dim}(X)-p+1$.
Proof. This follows from Theorem 6.10 together with the long exact sequence of homotopy groups associated to the homotopy fiber sequence

$$
0 \rightarrow \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v} \rightarrow \mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{G} \rightarrow \mathcal{R}_{p}(X) \rightarrow 0
$$

Example 6.21. (1) In case of divisors $p=\operatorname{dim}(X)-1$, Corollary 6.20 and Teh08, Proposition 6.2] show that

$$
\pi_{q}\left(\mathcal{Z}_{p}\left(X_{\mathbb{C}}\right)^{a v}\right)=0
$$

for any $q \geq 2$.
(2) In the case of zero-cycles $p=0$, we get

$$
H_{k, 0}(X(\mathbb{C}), \underline{\mathbb{Z}}) \simeq H_{k}(X(\mathbb{C}) / G, \mathbb{Z})
$$

for any $k \geq \operatorname{dim}(X)+1$.
We conclude this section by observing that the vanishing theorem also shows that motivic cohomology of a real variety can be computed in a range via the complex of averaged equidimensional cycles on the complexification.

Let $X$ and $Y$ be a quasi-projective real varieties. The group of reduced equidimensional cycles is defined to be the quotient group

$$
r_{e q u i}(Y, r)(X)=\frac{z_{e q u i}\left(Y_{\mathbb{C}}, r\right)\left(X_{\mathbb{C}}\right)^{G}}{z_{e q u i}\left(Y_{\mathbb{C}}, r\right)\left(X_{\mathbb{C}}\right)^{a v}}
$$

It is essentially a consequence of Suslin rigidity that the complex of reduced equidimensional cycles computes the reduced Lawson homology.

Proposition 6.22. Let $X$ be a quasi-projective real variety.
(1) The diagram

$$
r_{\text {equi }}\left(\mathbb{A}^{q}, q\right)\left(X \times \Delta_{\text {top }}^{\bullet}\right) \stackrel{\simeq}{\leftrightarrows} r_{\text {equi }}\left(\mathbb{P}^{q / q-1}, q\right)\left(X \times \Delta_{\text {top }}^{\bullet}\right) \stackrel{\simeq}{\leftrightarrows} \operatorname{Sing} . \mathcal{R}^{q}(X)
$$

consists of homotopy equivalences of simplicial sets.
(2) The map

$$
r_{e q u i}\left(\mathbb{P}^{q / q-1}, 0\right)\left(X \times \Delta^{\bullet}\right) \xrightarrow{\simeq} r_{e q u i}\left(\mathbb{P}^{q / q-1}, 0\right)\left(X \times \Delta^{\bullet} \times \Delta_{\text {top }}^{\bullet}\right)
$$

is a homotopy equivalence of simplicial sets.
Proof. The proof is similar to other proofs in this paper so we only provide a sketch. First observe that the simplicial abelian group of reduced equidimensional cycles may be computed as

$$
r_{e q u i}(Y, r)(X)=\frac{\left(z_{\text {equi }}\left(Y_{\mathbb{C}}, r\right)\left(X_{\mathbb{C}}\right) \otimes \mathbb{Z} / 2\right)^{G}}{\left(z_{e q u i}\left(Y_{\mathbb{C}}, r\right)\left(X_{\mathbb{C}}\right) \otimes \mathbb{Z} / 2\right)^{a v}}
$$

Using Proposition 4.11 and the appropriate analogues of the homotopy fiber sequences (6.4) and (6.5) we see that

$$
r_{e q u i}\left(\mathbb{P}^{q}, 0\right)\left(X \times \Delta_{\text {top }}^{\bullet}\right) \rightarrow \frac{\left(\operatorname{Sing} . \operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{+} / 2\right)^{G}}{\left(\operatorname{Sing} \bullet \operatorname{Mor}_{\mathbb{C}}\left(X_{\mathbb{C}}, \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\right)^{+} / 2\right)^{a v}}
$$

is a homotopy equivalence. The first part follows now in a similar fashion as Proposition4.16. The second part follows from the fact that both over $\mathbb{C}$ and over $\mathbb{R}$ with finite coefficients motivic cohomology agrees with morphic cohomology.

Corollary 6.23. Let $X$ be a quasi-projective real variety. Then

$$
H_{\mathcal{M}}^{p}(X ; \mathbb{Z}(q))=\pi_{2 q-p} z_{e q u i}\left(\mathbb{A}_{\mathbb{C}}^{q}, 0\right)\left(X_{\mathbb{C}} \times_{\mathbb{C}} \Delta_{\mathbb{C}}^{\bullet}\right)^{a v}
$$

for $q-1 \geq p$

## 7. Reduced Topological Cocycles

For typographical simplicity throughout this section we write $\mathcal{Z}=\mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}$. This section is devoted to the computation that $\pi_{k} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)=H_{\text {sing }}^{q-k}\left(W^{G} ; \mathbb{Z} / 2\right)$, for $k \geq 2$, where $\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)$ is the quotient simplicial abelian group

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W)=\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)^{G}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)^{a v}}
$$

and $W$ is a based finite $G$ - $C W$ complex.
The idea is to reduce to the case of trivial action. Before doing this we sketch what happens when $G$ acts trivially on $W$. By [LLFM03, Proposition 8.3] the short exact sequence

$$
0 \rightarrow \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{a v} \rightarrow \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{G} \rightarrow \mathcal{R}_{0}\left(S^{q, q}\right)_{0} \rightarrow 0
$$

is a fibration sequence (in fact principle fibration sequence) of topological spaces. Applying $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet},-\right)$ to this sequence yields a homotopy fiber sequence of simplicial sets. Now we compare the homotopy fiber sequences of simplicial abelian groups
where $\mathcal{H}^{q}(W)=\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top,+, }}^{\bullet}, \mathcal{R}_{0}\left(S^{q, q}\right)_{0}\right)$. We will see that when $W$ has trivial $G$-action then the left vertical arrow induces an isomorphism on $\pi_{k}$ for $k \geq 1$ (see Corollary 7.9). Therefore $\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)=\pi_{i} \mathcal{H}^{q}(W)=H_{\text {sing }}^{q-i}(W ; \mathbb{Z} / 2)$ for $i \geq 2$ when $W$ has trivial $G$-action.

For a based $G$-CW complex $W$ and a topological $G$-module $Z$, write

$$
\operatorname{Hom}_{c t s_{*}}(W, Z)_{0}^{G}
$$

for the set of based equivariant maps which are equivariantly homotopic to the 0-map (via a based homotopy).
Lemma 7.1. Let $W$ be a based $G$ - $C W$ complex and let $Z$ be a topological $G$-module. The simplicial set

$$
d \mapsto \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{d}, Z\right)_{0}^{G}
$$

is the path-connected component of the vertex $0 \in \operatorname{Hom}_{\text {cts }_{*}}\left(W \wedge \Delta_{\text {top,+}}^{\bullet}, Z\right)^{G}$.
Proof. A vertex $g \in \operatorname{Hom}_{\text {cts }_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, Z\right)^{G}$ is in the same path component as the 0-map if and only if there is a 1-simplex $F \in \operatorname{Hom}_{\text {cts* }}\left(W \wedge \Delta_{t o p,+}^{1}, Z\right)^{G}$ such that $F(0)=0$ and $F(1)=g$. This happens exactly when $g \in \operatorname{Hom}_{\text {cts }}(W, Z)_{0}^{G}$.

A $d$-simplex, $f: W \wedge \Delta_{t o p,+}^{d} \rightarrow Z$ is in the path-component of 0 if and only if its restriction to a vertex is in the path component of 0 . Since $\Delta_{\text {top, }+}^{d}$ is equivariantly contractible and the restriction $\left.f\right|_{W \wedge\{v\}_{+}}$to a vertex is equivariantly homotopic to the constant map 0 we conclude that $f$ itself is equivariantly homotopic to 0 .

Definition 7.2. Let $W$ be a based $G-C W$ complex.
(1) Define $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)_{0}^{a v}$ to be the path-connected component of the vertex 0 in the simplicial set $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{\bullet}, \mathcal{Z}\right)^{a v}$.
(2) Define

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0}=\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)_{0}^{G}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)_{0}^{a v}}
$$

here the quotient is in the category of simplicial abelian groups.
Restricting to $W^{G}$ gives rise to the comparison map

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} \rightarrow \widetilde{\mathcal{R}}_{t o p}^{q}(W) \rightarrow \operatorname{Hom}_{c t s_{*}}\left(W^{G} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q}\right)\right)
$$

Note that $\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \rightarrow \pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)$ is an isomorphism for $i \geq 2$ and an injection for $i=0,1$. To compute $\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)$ we will show that $\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \rightarrow \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W^{G}\right)_{0}$ is an isomorphism. The surjectivity is easy but the injectivity takes some work.

Proposition 7.3. Let $i: A \hookrightarrow W$ be an equivariant cofibration between based $G$-CW-complexes and let $Z$ be a topological $G$-module. Then

$$
i^{*}: \operatorname{Hom}_{c t s_{*}}(W, Z)_{0}^{G} \rightarrow \operatorname{Hom}_{c t s_{*}}(A, Z)_{0}^{G}
$$

is surjective.
Proof. Suppose that $f: A \rightarrow Z$ is a based equivariant map which is based equivariantly homotopic to the 0-map. Let $H: A \wedge I_{+} \rightarrow Z$ be an equivariant homotopy such that $H(-, 0)=0$ and $H(-, 1)=f$.

By the homotopy extension property of cofibrations, an equivariant map $H^{\prime}$ making the diagram below commute exists


The restriction of $f^{\prime}=H^{\prime}(-, 1)$ to $A$ is equal to $f$ and $H^{\prime}$ is an equivariant homotopy between $f^{\prime}$ and the 0-map.

Corollary 7.4. Let $i: A \hookrightarrow W$ be an equivariant cofibration between based $G-C W$ complexes. The induced map

$$
i^{*}: \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \rightarrow \widetilde{\mathcal{R}}_{\text {top }}^{q}(A)_{0}
$$

is a surjection.
Proof. Consider the square


By the previous proposition, the top horizontal arrow is surjective. The vertical arrows are surjective by definition and therefore the bottom horizontal arrow is also surjective.

For a based $C W$-complex $W$ and a topological abelian group $Z$ we will write $\operatorname{Hom}_{c t s_{*}}(W, Z)_{0}$ for the set of based continuous maps which are based homotopic to the 0-map. Note that the simplicial set

$$
d \mapsto \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{d}, Z\right)_{0}
$$

is the path-connected component of the vertex $0 \in \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)$ (for example consider $W$ and $Z$ with trivial $G$-action and apply Lemma 7.11). If $Z$ and $W$ have a $G$-action write $\left(\operatorname{Hom}_{c t s_{*}}(W, Z)_{0}\right)^{a v}$ for the image of $N=1+\sigma$. This set consists of maps $h: W \rightarrow Z$ which can be written as $h=f+\bar{f}$ where $f$ is a continuous map which is nonequivariantly homotopic to 0 .

We now justify the use of similar notation for two potentially different simplicial sets. Previously we wrote $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)_{0}^{a v}$ for the path-component
of the vertex $0 \in \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)^{a v}$. We now verify that this pathcomponent can be described explicitly as the image under $N=1+\sigma$ of the pathcomponent of $0 \in \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, Z\right)$. In otherwords $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)_{0}^{a v}=$ $\left(\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top,+}}^{\bullet}, \mathcal{Z}\right)_{0}\right)^{a v}$. This explicit description will be fundamental to our proof of Proposition 7.6 below.

Proposition 7.5. Let $W$ be a based G-CW-complex. The simplicial set

$$
d \mapsto\left(\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{d}, \mathcal{Z}\right)_{0}\right)^{a v}
$$

is the path-connected component of $0 \in \operatorname{Hom}_{\text {cts* }}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)^{a v}$
Proof. First we identify $\left(\operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})_{0}\right)^{a v}$ as the set of vertices of the pathconnected component of $0 \in \operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})^{a v}$. Any $f+\bar{f} \in\left(\operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})_{0}\right)^{a v}$ is in the path component of 0 . Suppose the vertex $h+\bar{h} \in \operatorname{Hom}_{\text {cts* }}(W, \mathcal{Z})^{a v}$ is in the same path component as 0 . This means there is a map of simplicial sets

$$
F: I \rightarrow \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z}\right)^{a v}
$$

such that $F(0)=0$ and $F(1)=h+\bar{h}$. Consider the diagram of simplicial sets,


A surjection between simplicial abelian groups is a fibration and therefore an $F^{\prime}$ exists to make the above square commute.

The map $F^{\prime}(1): W \rightarrow \mathcal{Z}$ is in $\operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})_{0}$ and satisfies

$$
F^{\prime}(1)+\overline{F^{\prime}(1)}=N\left(F^{\prime}(1)\right)=F(1)=h+\bar{h} .
$$

We conclude that $\left(\operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})_{0}\right)^{a v}$ is the set of vertices of the path-connected component of the $0 \in \operatorname{Hom}_{c t s_{*}}(W, \mathcal{Z})^{a v}$.

Now to conclude that the simplicial set $d \mapsto\left(\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top,+}}^{d}, \mathcal{Z}\right)_{0}\right)^{a v}$ is the path-connected component of 0 we need to see that if the restriction $\left.g\right|_{W \wedge\{v\}_{+}}$ of $g \in \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{n}, Z\right)^{a v}$ to a vertex $v \in \Delta_{\text {top }}^{n}$ lies in $\left(\operatorname{Hom}_{c t s_{*}}(W, Z)_{0}\right)^{a v}$ then $g \in\left(\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{n}, Z\right)_{0}\right)^{a v}$. That is, if $g \in \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{n}, Z\right)^{a v}$ and that there is a map $f: W \rightarrow \mathcal{Z}$ which is homotopic to 0 such that the restriction of $g$ to some vertex $v \in \Delta_{\text {top }}^{n}$ satisfies $\left.g\right|_{W \wedge\{v\}_{+}}=f+\bar{f}$ then we need to see that $g$ can be written $g=f^{\prime}+\overline{f^{\prime}}$ for some $f^{\prime}: W \wedge \Delta_{\text {top },+}^{n} \rightarrow \mathcal{Z}$ which is homotopic to 0 . For this we consider the lift $f^{\prime}$ of $g$,


The map $f^{\prime}: W \wedge \Delta_{\text {top },+}^{n} \rightarrow \mathcal{Z}$ satisfies $f^{\prime}+\overline{f^{\prime}}=N\left(f^{\prime}\right)=g$, the restriction of $f^{\prime}$ to $v \in \Delta_{\text {top }}^{n}$ is homotopic to the 0-map and, since $\Delta_{\text {top }}^{n}$ is contractible, $f^{\prime}$ is homotopic to the 0 -map as well.

Therefore we conclude that

$$
d \mapsto\left(\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{d}, \mathcal{Z}\right)_{0}\right)^{a v}
$$

is the path-component of 0 in $\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{d}, \mathcal{Z}\right)^{a v}$.
Proposition 7.6. Let $i: A \hookrightarrow W$ be an equivariant cofibration. Then

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W / A)_{0} \rightarrow \operatorname{ker}\left[\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} \xrightarrow{i^{*}} \widetilde{\mathcal{R}}_{t o p}^{q}(A)_{0}\right]
$$

Proof. Suppose that $[f] \in \operatorname{ker}\left(\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} \xrightarrow{i^{*}} \widetilde{\mathcal{R}}_{\text {top }}^{q}(A)_{0}\right)$ is a $d$-simplex. Then $[f]$ is represented by an equivariant map $f: W \wedge \Delta_{\text {top },+}^{d} \rightarrow \mathcal{Z}$ which is equivariantly homotopic to 0 . Since $i^{*}[f]=0$ this means that $i^{*} f \in \operatorname{Hom}_{c t s_{*}}\left(A \wedge \Delta_{\text {top },+}^{d}, \mathcal{Z}\right)_{0}^{a v}$. Thus there is a continuous map $h: A \wedge \Delta_{\text {top, }+}^{d} \rightarrow \mathcal{Z}$, (nonequivariantly homotopic to 0 ), such that $i^{*} f=\left.f\right|_{A \wedge \Delta_{\text {top },+}^{d}}=h+\bar{h}$. Since $h$ is (nonequivariantly) homotopic to the 0-map, $h: A \wedge \Delta_{\text {top },+}^{d} \rightarrow \mathcal{Z}$ extends to a continuous map $h^{\prime}: W \wedge \Delta_{\text {top },+}^{d} \rightarrow \mathcal{Z}$ which is (nonequivariantly) homotopic to the 0-map.

Explicitely, let $H: A \wedge \Delta_{\text {top },+}^{d} \wedge I_{+} \rightarrow \mathcal{Z}$ be a homotopy such that $H(-, 0)=0$ and $H(-, 1)=h$. By the homotopy extension property for cofibrations, the dotted arrow exists in the diagram


Now $H^{\prime}(-, 1)=h^{\prime}$ is the desired extension of $h, H^{\prime}$ is a homotopy between $h^{\prime}$ and the 0-map and $F \stackrel{\text { def }}{=} f-\left(h^{\prime}+\overline{h^{\prime}}\right)$ represents the same class as $[f]$. Since $\left.F\right|_{A \wedge \Delta_{\text {top },+}^{d}}=0$ the map $F$ defines the map $F^{\prime}: W / A \wedge \Delta_{\text {top },+}^{d} \rightarrow \mathcal{Z}$ such that

$$
F=p^{*} F^{\prime}: W \xrightarrow{p} W / A \wedge \Delta_{t o p,+}^{d} \rightarrow \mathcal{Z}
$$

Therefore

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W / A)_{0} \rightarrow \operatorname{ker}\left(i^{*}: \widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} \rightarrow \widetilde{\mathcal{R}}_{t o p}^{q}(A)_{0}\right)
$$

because $p^{*}\left[F^{\prime}\right]=\left[p^{*} F^{\prime}\right]=[F]=[f]$.
Lemma 7.7. (c.f. LLFM03, Lemma 8.8]) Suppose that $A \bullet$ is a simplicial $G$ module. Then

$$
\left|A_{\bullet}^{a v}\right|=\left|A_{\bullet}\right|^{a v}
$$

Proof. Let $f_{\bullet}: B_{\bullet} \rightarrow C \bullet$ be a map between simplicial sets, then $\left|\operatorname{Im} f_{\bullet}\right|=\operatorname{Im}\left|f_{\bullet}\right|$. The lemma follows since $(-)^{a v}$ is defined to be the image of the map $N=1+\sigma$.

Proposition 7.8. Suppose that $Y=\left|Y_{\bullet}\right|$ is the realization of a based $G$-simplicial set. Then $\mathcal{Z}_{0}(Y)_{0} \rightarrow \mathcal{Z}_{0}(Y)_{0}^{a v}$ and $\mathcal{Z} / 2_{0}(Y)_{0} \rightarrow \mathcal{Z} / 2_{0}(Y)_{0}^{a v}$ are Serre fibrations.
Proof. If $Y$ is a based set and $A$ is an abelian group then define $A \otimes Y=\oplus_{y \in Y \backslash\{*\}} A$. If $Y_{\bullet}$ is a based $G$-simplicial set then $A \otimes Y_{\bullet}$ is a $G$-simplicial set. In case $A=$ $\mathbb{Z}$ or $A=\mathbb{Z} / 2$ we have $\mathcal{Z}\left(\left|Y_{\bullet}\right|\right)_{0}=\left|\mathbb{Z} \otimes Y_{\bullet}\right|$ and $\mathcal{Z} / 2_{0}\left(\left|Y_{\bullet}\right|\right)_{0}=\left|\mathbb{Z} / 2 \otimes Y_{\bullet}\right|$ (see dS03b, McC69] $)$. The map $\mathbb{Z} \otimes Y_{\bullet} \rightarrow\left(\mathbb{Z} \otimes Y_{\bullet}\right)^{a v}$ is a surjection between simplicial abelian groups and so is a fibration of simplicial sets and similarly for $\mathbb{Z} / 2 \otimes Y_{\bullet} \rightarrow$ $\left(\mathbb{Z} / 2 \otimes Y_{\bullet}\right)^{a v}$.

The realization of a Kan fibration is a Serre fibration and therefore both $\mathcal{Z}_{0}(Y)_{0}=$ $\left|\mathbb{Z} \otimes Y_{\bullet}\right| \xrightarrow{N}\left|\left(\mathbb{Z} \otimes Y_{\bullet}\right)^{a v}\right|=\mathcal{Z}_{0}(Y)_{0}^{a v}$ and $\mathcal{Z} / 2_{0}(Y)_{0}=\left|\mathbb{Z} / 2 \otimes Y_{\bullet}\right| \xrightarrow{N}\left|\left(\mathbb{Z} / 2 \otimes Y_{\bullet}\right)^{a v}\right|=$ $\mathcal{Z} / 2_{0}(Y)_{0}^{a v}$ are Serre fibrations of topological spaces.

Below we apply this proposition to the cases $Y=S^{q, q}$ and $Y=S^{q, q} \wedge \mathbb{Z} / 2_{+}$ so we make explicit that these are realizations of $G$-simplicial sets. We consider $\mathbb{Z} / 2$ as a simplicial set in the usual way. The simplicial set $S^{1,0}$ is the ordinary $S^{1}$ with trivial action. The simplicial set $S^{0,1}$ is the simplicial whose nondegenerate simplices are two vertices $\{0\}$ and $\{\infty\}$ and two 1 -simplices. The $G$-action fixes the vertices and switches the 1 -simplices. The realization of this simplicial set is the usual $S^{0,1}$. Now $S^{p, q}$ is the $G$-simplicial set $S^{p, q}=\left(S^{1,0}\right)^{\wedge p} \wedge\left(S^{0,1}\right)^{\wedge q}$ and its realization is the usual $S^{p, q}$.

Lemma 7.9. Let $W$ be a based $G$-space with trivial action. Suppose that $Z$ is a topological $G$-module such that $Z \xrightarrow{N} Z^{a v}$ is a Serre fibration. Then

$$
\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, Z\right)_{0}^{a v}=\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, Z^{a v}\right)_{0}
$$

Proof. Since $W$ and $\Delta_{t o p}^{d}$ have trivial actions,

$$
\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}(Y)_{0}\right)_{0}^{a v} \subseteq \operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}(Y)_{0}^{a v}\right)_{0}
$$

and we wish to see that it is onto.
Suppose that $f: W \wedge \Delta_{t o p,+}^{d} \rightarrow Z^{a v}$ is a map which is homotopic to 0 . Let $H$ be a homotopy between 0 and $f$ and let $H^{\prime}$ be a lift of $H$,

which exists because the right-hand vertical map is a fibration. Finally the map $f^{\prime}(-)=H^{\prime}(-, 1)$ satisfies $f^{\prime}+\bar{f}^{\prime}=f$ and $H^{\prime}$ is a homotopy between 0 and $f^{\prime}$.

Proposition 7.10. Suppose that $W$ has trivial $G$-action. Then for all $n \geq 0$ and all $q \geq 0$,

$$
\widetilde{\mathcal{R}}_{t o p}^{q}\left(W \wedge \mathbb{Z} / 2_{+}\right)_{0}=\{0\}
$$

Proof. First recall dS03b, Lemma 2.4] that given a finite $G$-set $Z$ then there is a $G$-homeomorphism

$$
\operatorname{Hom}_{*}\left(Z_{+}, \mathcal{Z}_{0}(Y)_{0}\right) \stackrel{\left(\mathcal{Z}_{0}\right.}{ }\left(Y \wedge Z_{+}\right)_{0}
$$

defined by $f \mapsto \sum_{z \in Z} f(z) \wedge z$. This yields

$$
\begin{aligned}
\widetilde{\mathcal{R}}_{t o p}^{q}\left(W \wedge \mathbb{Z} / 2_{+}\right)_{0} & =\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \mathbb{Z} / 2_{+} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}\right)_{0}^{G}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \mathbb{Z} / 2_{+} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}\right)_{0}^{a v}}= \\
& =\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top,+}}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}\right)_{0}^{G}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}\right)_{0}^{a v}}= \\
& =\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{G}\right)_{0}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top },+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{a v}\right)_{0}}
\end{aligned}
$$

where the last equality follows from Lemma 7.9 and Proposition 7.8 because $W$ has trivial $G$-action. But since the action of $G$ on $S^{q, q} \wedge \mathbb{Z} / 2_{+}$is free we have the isomorphism

$$
\mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{a v} \cong\left(\mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{G}\right.
$$

and therefore

$$
\widetilde{\mathcal{R}}_{t o p}^{q}\left(W \wedge \mathbb{Z} / 2_{+}\right)_{0}=\frac{\operatorname{Hom}_{c t s_{*}}\left(S^{n} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{G}\right)_{0}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{\text {top,+}}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q} \wedge \mathbb{Z} / 2_{+}\right)_{0}^{a v}\right)_{0}}=\{0\}
$$

Recall that the action of $G$ on a based set $(Y, *)$ is said to be free if $Y^{G}=*$.
Corollary 7.11. Suppose that $W$ is a based finite $G-C W$ complex with free $G$ action. Then

$$
\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0}=\{0\} .
$$

Proof. First we observe that $\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1}\right)_{0} \rightarrow \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n}\right)_{0}$ is an isomorphism for any $n$. Indeed since $W$ is a free $G-C W$ complex $W_{n} / W_{n-1}$ is a wedge of spheres of the form $S^{n} \wedge \mathbb{Z} / 2_{+}$. By the previous proposition $\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1} / W_{n}\right)_{0}=\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(\vee S^{n+1} \wedge\right.$ $\left.\mathbb{Z} / 2_{+}\right)_{0}=\{0\}$.

By Proposition [7.6, $\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1} / W_{n}\right)_{0} \rightarrow \operatorname{ker}\left(\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1}\right)_{0} \rightarrow \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n}\right)_{0}\right)$ is surjective and therefore $\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1}\right)_{0} \subseteq \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n}\right)_{0}$. By Corollary 7.4 this map is onto as well and therefore $\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n+1}\right)_{0}=\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{n}\right)_{0}$. Since $W=W_{N}$ for large $N$ we conclude that $\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0}=\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W_{0}\right)_{0}=\{0\}$.

Corollary 7.12. Suppose that $W$ is a finite $G$ - $C W$-complex. Then

$$
\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \xlongequal{\cong} \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W^{G}\right)_{0}
$$

is an isomorphism of simplicial abelian groups.
Proof. Consider the cofibration sequence $W^{G} \hookrightarrow W \rightarrow W / W^{G}$. The space $W / W^{G}$ has a free $G$-action and so Proposition 7.6 and Corollary 7.11 imply that

$$
\{0\}=\widetilde{\mathcal{R}}_{t o p}^{q}\left(W / W^{G}\right)_{0} \rightarrow \operatorname{ker}\left[\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} \rightarrow \widetilde{\mathcal{R}}_{t o p}^{q}\left(W^{G}\right)_{0}\right]
$$

is surjective. Therefore $\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \subseteq \widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W^{G}\right)_{0}$. Since it is also a surjection by Corollary 7.4 it is an isomorphism.

For a based $G-C W$ complex $W$ define

$$
\mathcal{H}^{q}(W)=\operatorname{Hom}_{c t s_{*}}\left(W^{G} \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{R}_{0}\left(S^{q, q}\right)_{0}\right)
$$

The homotopy groups of $\mathcal{H}^{q}(W)$ compute singular cohomology of the fixed point space, $\pi_{k} \mathcal{H}^{q}(W)=H_{\text {sing }}^{q-k}\left(W^{G}, \mathbb{Z} / 2\right)$.

Theorem 7.13. Let $W$ be a finite $G$ - $C W$-complex. Then

$$
\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \rightarrow \pi_{i} \mathcal{H}^{q}(W)=H_{\text {sing }}^{q-i}\left(W^{G} ; \mathbb{Z} / 2\right)
$$

is an isomorphism for $i \geq 2$ and an injection for $i=0,1$.

Proof. Since $\mathcal{H}^{q}(W)=\mathcal{H}^{q}\left(W^{G}\right)$ and $\widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0}=\widetilde{\mathcal{R}}_{\text {top }}^{q}\left(W^{G}\right)_{0}$ by Corollary 7.12 we immediately reduce to the case that $W=W^{G}$. Since $G$ acts trivially on $W$, Lemma 7.9 and Proposition 7.8 give

$$
\begin{aligned}
\widetilde{\mathcal{R}}_{t o p}^{q}(W)_{0} & =\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}\right)_{0}^{G}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}\right)_{0}^{a v}}= \\
& =\frac{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{G}\right)_{0}}{\operatorname{Hom}_{c t s_{*}}\left(W \wedge \Delta_{t o p,+}^{\bullet}, \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{a v}\right)_{0}}
\end{aligned}
$$

By [LLFM03, Proposition 8.3] the short exact sequence

$$
0 \rightarrow \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{a v} \rightarrow \mathcal{Z} / 2_{0}\left(S^{q, q}\right)_{0}^{G} \rightarrow \mathcal{R}_{0}\left(S^{q, q}\right)_{0} \rightarrow 0
$$

is a principle fibration sequence.
Finally comparing homotopy fiber sequences of simplicial abelian groups

yields the result.

Corollary 7.14. Let $W$ be a finite $G$ - $C W$-complex. Then

$$
\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W) \rightarrow \pi_{i} \mathcal{H}^{q}(W)=H_{\text {sing }}^{q-i}\left(W^{G} ; \mathbb{Z} / 2\right)
$$

is an isomorphism for $i \geq 2$.
Proof. The map $\pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)_{0} \rightarrow \pi_{i} \widetilde{\mathcal{R}}_{\text {top }}^{q}(W)$ is an isomorphism for $i \geq 2$ and an injection for $i=0,1$

## Appendix A. Topological Monoids

In this appendix we collect a few simple results on topological monoids. By topological monoid we will mean a compactly generated Hausdorff topological abelian monoid (and similarly for the phrase topological group). An abelian monoid $M$ is said to have the cancellation property if for any $n, m, p \in M n+p=m+p$ implies $m=n$.

Lemma A.1. Suppose that $M$ is a topological monoid with the cancellation property. If $+: M \times M \rightarrow M$ is closed and $N \subseteq M$ is a closed submonoid then the quotient map $\pi: M \rightarrow M / N$ is closed.

Proof. Suppose that $V \subseteq M$ is closed. Then since $\pi: M \rightarrow M / N$ is a quotient map to see that $\pi V$ is closed it is enough to see that $\pi^{-1} \pi V \subseteq M$ is closed. But $\pi^{-1} \pi V=(V+N) \cap(M+N)$ which is closed.
Lemma A.2. Let $M$ be a topological monoid with the cancellation property and let $N \subseteq M$ be a submonoid. Suppose that $M / N$ is a topological monoid, $M^{+}$is a topological group and $N^{+}$is closed. Then the isomorphism of groups

$$
\left(\frac{M}{N}\right)^{+} \rightarrow \frac{M^{+}}{N^{+}}
$$

is an isomorphism of topological groups.

Proof. The map $M \rightarrow M^{+} \rightarrow M^{+} / N^{+}$sends $N$ to 0 and so we obtain the monoid homomorphism $M / N \rightarrow M^{+} / N^{+}$which is continuous. This induces the continuous group homomorphism $\phi:(M / N)^{+} \rightarrow M^{+} / N^{+}$.

On the other hand the topological monoid quotient map $M \rightarrow M / N$ induces the continuous group homomorphism $M^{+} \rightarrow(M / N)^{+}$. Since $N^{+}$is mapped to 0 it induces the continuous group homomorphism $\psi: M^{+} / N^{+} \rightarrow(M / N)^{+}$.

The continuous maps $\psi$ and $\phi$ are easily seen to be inverse to each other.
Recall that if $A$ is a topological monoid with $G$-action we write $A^{a v} \subseteq A$ for the image of $N=1+\sigma$, so $A^{a v} \subseteq A$ is the topological submonoid consisting of elements of the form $a+\sigma a$.

Proposition A.3. Suppose that $M$ is a Hausdorff topological abelian monoid with the cancellation property and that $M^{+}$is a Hausdorff group. Suppose that G acts on $M$. Then the isomorphism of groups

$$
\left(M^{G}\right)^{+} \xrightarrow{\cong}\left(M^{+}\right)^{G}
$$

is an isomorphism of topological groups. If $\left(M^{+}\right)^{a v} \subseteq M^{+}$is closed then

$$
\left(M^{a v}\right)^{+} \xrightarrow{\cong}\left(M^{+}\right)^{a v}
$$

is an isomorphism of topological groups.
Proof. We just have to show that the "identity" map

$$
\left(M^{+}\right)^{G} \rightarrow\left(M^{G}\right)^{+}
$$

is continuous.
The group completion $M^{+}$is topologized as the quotient

$$
M \times M \xrightarrow{q} M^{+} .
$$

where $q(a, b)=a-b$. The map $q: q^{-1}\left(M^{+}\right)^{G} \rightarrow\left(M^{+}\right)^{G}$ is again a quotient map since $\left(M^{+}\right)^{G}$ is closed.

Consider the map $M \times M \xrightarrow{i d \times \sigma \pi_{2}} M \times M \times M \xrightarrow{\Delta \times i d} M^{\times 4} \xrightarrow{+} M \times M$ which sends $(a, b) \mapsto(a, b, \sigma b, \sigma b) \mapsto(a+\sigma b, b+\sigma b)$. This is a continuous map. Its restriction to $q^{-1}\left(M^{+}\right)^{G}$ is a continuous map $q^{-1}\left(M^{+}\right)^{G} \rightarrow M^{G} \times M^{G}$ which induces the identity map on quotients

$$
\left(M^{+}\right)^{G} \rightarrow\left(M^{G}\right)^{+}
$$

and therefore this is a continuous map.
The second statement for averaged cycles is proved in a similar fashion.
Lemma A.4. Let $M$ be a topological monoid with the cancellation property. Suppose that $M^{+}$is a topological group and $2 M^{+}$is closed in $M^{+}$. Then

$$
\frac{M}{2 M} \rightarrow \frac{M^{+}}{2 M^{+}}
$$

is an isomorphism of topological abelian groups.
Proof. For $m \in M$ write $[m$ ] for its image in $M / 2 M$. This quotient monoid is a group since $[m]+[m]=0$.

For $(m, n) \in M^{+}$write $[m, n]$ for its image in $M^{+} / 2 M^{+}$. The map $M / 2 M \rightarrow$ $M^{+} / 2 M^{+}$which sends $[m]$ to $[m, 0]$ is continuous because $M \rightarrow M / 2 M$ is a quotient. It is an injection because if $[m, 0]=[0,0]$ then there is $\left(2 n, 2 n^{\prime}\right)$ such that
$m+2 n^{\prime}=2 n$ which says that $[m]=0$. It is surjective since $[m, n]=[m+n, n+n]=$ $[m+n, 0]$. The inverse $M^{+} / 2 M^{+} \rightarrow M / 2 M$ is continuous since it is the map $[m, n] \mapsto[m+n, 0]=[m+n, n+n]$.

Proposition A.5. Let $X$ be a normal quasi-projective real variety.
(1) The continuous homomorphism $N: \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{\text {av }}$ induces an isomorphism of topological groups

$$
\frac{\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)}{\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G}} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}
$$

(2) The continuous homomorphism $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}$ induces an isomorphism of topological groups

$$
\frac{\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v}}{2 \mathcal{Z}^{q}(X)^{G}} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}
$$

Proof. Addition is a closed map for the monoid $\mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(X_{\mathbb{C}}\right)$ (see the proof of Proposition (3.6) and therefore we conclude by Lemma A. 1 that addition is also closed for both the effective cocycles $\mathcal{C}^{q}\left(X_{\mathbb{C}}\right)=\mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(X_{\mathbb{C}}\right) / \mathcal{C}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(X_{\mathbb{C}}\right)$ and closed for $\mathcal{C}^{q} / 2\left(X_{\mathbb{C}}\right)$.

By Lemma A.4 the $\operatorname{map} \mathcal{C}^{q} / 2\left(X_{\mathbb{C}}\right) \xrightarrow{\cong} \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)$ is an isomorphism of topological groups and therefore addition is closed for $\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)$. In particular $N=1+$ $\sigma: \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)$ is closed. Since $\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)$ is 2-torsion $\operatorname{ker}(N)=$ $\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G}$. It now follows that

$$
\frac{\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)}{\mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{G}} \xrightarrow{N} \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}
$$

is an isomorphism of topological groups. For the second statement we need to conclude that the continuous bijection $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{a v} / 2 \mathcal{Z}^{q}\left(X_{\mathbb{C}}\right)^{G} \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}$ has a continuous inverse. Write $g$ for the inverse. Then

is commutative and each map except possibly $g$ is continuous. By the first part of the proposition the composition $\mathcal{Z}^{q}\left(X_{\mathbb{C}}\right) \rightarrow \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right) \xrightarrow{N} \mathcal{Z}^{q} / 2\left(X_{\mathbb{C}}\right)^{a v}$ is a quotient map and therefore $g$ is continuous. We conclude that the map is an isomorphism of topological groups.

## Appendix B. Tractable Monoid Actions

We recall Friedlander-Gabber's notion of tractability for a topological monoid.
Definition B.1. FG93
(1) If $M$ is a Hausdorff topological monoid which acts on a topological space $T$, the action is said to be tractable if $T$ is the topological union of inclusions

$$
\varnothing=T_{-1} \subseteq T_{0} \subseteq T_{1} \subseteq \cdots
$$

such that for each $n \geq 0$ the inclusion $T_{n-1} \subseteq T_{n}$ fits into a push-out of $M$-equivariant maps (with $R_{0}$ empty)

where the upper horizontal map is induced by a cofibration $R_{n} \hookrightarrow S_{n}$ of Hausdorff spaces.

The monoid $M$ is said to be tractable if the diagonal action of $M$ on $M \times M$ is tractable.
(2) If in addition $M, T$ are $G$-spaces say that the action of $M$ on $T$ is equivariantly tractable if the action map is $G$-equivariant, the $R_{n} \hookrightarrow S_{n}$ are equivariant cofibrations between $G$-spaces, and the pushout squares (B.2) are $G$-equivariant.

Fixed points of an equivariant cofibration is a cofibration and fixed points preserve pushouts along a closed inclusion. Therefore if $T$ is an equivariantly tractable $M$-space then it is in particular a tractable $M$-space and $T^{G}$ is a tractable $M^{G}$ space.

The most important feature of tractability is that the naive group completion $M \rightarrow M^{+}$of a tractable monoid is a homotopy group completion [FG93.

It is useful to know that all of our topological groups have reasonable equivariant homotopy types. Below we observe that this is the case by using essentially the same reasoning as in [FW01b, Proposition 2.5]. The essential topological property used here is that Hironaka's triangulation theorem implies that the complexification of a real variety may be equivariantly triangulated (see for example KW03, Theorem 1.3]).

Proposition B.3. Suppose that $T$ is a tractable $M$-space. If $R_{n}, S_{n}$ have the homotopy type of a $C W$-complex then so does $T / M$ Suppose that $T$ is an equivariantly tractable $M$-space. If $R_{n}$, and $S_{n}$ have the equivariant homotopy type of a $G$-CW complex then $T / M$ has the equivariant homotopy type of a $G-C W$ complex.

Proof. We prove the second statement, the first follows in the same manner by discarding equivariant considerations. Modding out by the $M$-action in ( $\overline{\mathrm{B} .2}$ ) we obtain equivariant pushout-squares


By induction and homotopy invariance of pushouts along $G$-cofibrations we see that $T_{n} / M$ has the homotopy type of a $G-C W$ complex and that $T_{n-1} / M \rightarrow T_{n} / M$ is a $G$-cofibration. By Wan80, Theorem 4.9] we conclude that $\operatorname{colim}_{n} T_{n} / M$ has the homotopy type of a $G$ - $C W$ complex. We are done since $T / M=\operatorname{colim}_{n} T_{n} / M$ by the proof of [Fri98, Lemma 1.3].

Proposition B.4. Let $\mathcal{E} \subseteq X$ be a constructable subset of a real projective variety.
(1) The space $\mathcal{E}_{\mathbb{C}}$ has the homotopy type of a G-CW complex.
(2) Suppose that $\mathcal{F} \hookrightarrow \mathcal{E}$ is a closed constructable embedding. Then $\mathcal{F}_{\mathbb{C}} \hookrightarrow \mathcal{E}_{\mathbb{C}}$ is an equivariant cofibration.
Proof. Let $\overline{\mathcal{E}_{\mathbb{C}}}, \overline{\mathcal{F}_{\mathbb{C}}}$ be closures (in $X_{\mathbb{C}}$ ) of $\mathcal{E}_{\mathbb{C}}$ and $\mathcal{F}_{\mathbb{C}}$. There is an equivariant triangulation of $\bar{E}_{\mathbb{C}}$ so that $\overline{\mathcal{F}}_{\mathbb{C}}$ and $\overline{\mathcal{E}}_{\mathbb{C}} \backslash \mathcal{E}_{\mathbb{C}}$ are subcomplexes KW03, Theorem 1.3]. Then $\mathcal{E}_{\mathbb{C}}$ and $\mathcal{F}_{\mathbb{C}}$ are unions of open simplices. The deformation retract of $\mathcal{E}_{\mathbb{C}}$ onto a subsimplicial complex given in the proof of FW01b, Proposition 2.5] is an equivariant retract onto a $G$-simplicial complex (and similarly for $\mathcal{F}_{\mathbb{C}}$ ) which gives the first statement. The construction of a deformation retract onto $\mathcal{F}_{\mathbb{C}}$ of an open neighborhood $U$ of $\mathcal{F}_{\mathbb{C}}$ in $\mathcal{E}_{\mathbb{C}}$ given in [FL97, Lemma C1] works equivariantly which shows that $\mathcal{F}_{\mathbb{C}} \hookrightarrow \mathcal{E}_{\mathbb{C}}$ is a cofibration.

As shown in [FG93, Proposition 1.3] the Chow monoids associated to complex varieties are tractable and in [FL97, Proposition C.3] these results are extended to certain constructable submonoids of Chow monoids. Their proofs work equivariantly to give the equivariant analogue of their result. A submonoid $N \subseteq M$ is said to be full if whenever $m+m^{\prime} \in N$ then both $m, m^{\prime} \in N$. The condition below on $\mathcal{E} \subseteq \mathcal{C}_{k}(X)$ in the proposition is satisfied if $\mathcal{E}$ is Zariski closed or if it is a full submonoid.

Proposition B.5. Let $X$ be a projective real variety and $\mathcal{E} \subseteq \mathcal{C}_{k}(X)$ by a constructable submonoid such that $+: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a Zariski closed mapping. Then
(1) $\mathcal{E}_{\mathbb{C}}$ is an equivariantly tractable monoid.
(2) $\mathcal{E}_{\mathbb{C}}^{+}$has the homotopy type of a $G-C W$ complex.
(3) If $\mathcal{F} \subseteq \mathcal{E}$ is a closed constructable embedding then $\mathcal{E}_{\mathbb{C}}$ is tractable as an $\mathcal{F}_{\mathbb{C}}$-space and $\mathcal{E}_{\mathbb{C}} / \mathcal{F}_{\mathbb{C}}$ is an equivariantly tractable monoid.
(4) Suppose that $\mathcal{F} \hookrightarrow \mathcal{E}$ is a closed constructable embedding. Then $\mathcal{F}^{+} \subseteq \mathcal{E}^{+}$ is closed and the sequence

$$
0 \rightarrow \mathcal{F}_{\mathbb{C}}^{+} \rightarrow \mathcal{E}_{\mathbb{C}}^{+} \rightarrow\left(\mathcal{E}_{\mathbb{C}} / \mathcal{F}_{\mathbb{C}}\right)^{+} \rightarrow 0
$$

is an equivariant short exact sequence of groups of spaces of the homotopy type of a $G-C W$ complex.

Proof. In [FL97, Proposition C.3] the monoid $\mathcal{E}_{\mathbb{C}}$ is shown to be tractable as follows. Write $\mathcal{E}(d)=\mathcal{E}_{\mathbb{C}} \cap \mathcal{C}_{k, d}\left(X_{\mathbb{C}}\right)$ and let $\nu: \mathbb{N}^{2} \rightarrow \mathbb{N}$ be a bijection such that if $a \leq c$, $b \leq d$ then $\nu(a, b) \leq \nu(c, d)$. Define

$$
\begin{gathered}
S_{n}=\mathcal{E}\left(a_{n}\right) \times \mathcal{E}\left(b_{n}\right), \text { where } \nu\left(a_{n}, b_{n}\right)=n \\
R_{n}=\operatorname{Im}\left(\bigcup_{c \geq 0} \mathcal{E}\left(a_{n}-c\right) \times \mathcal{E}\left(b_{n}-c\right) \times \mathcal{E}(c) \rightarrow \mathcal{E}\left(a_{n}\right) \times \mathcal{E}\left(b_{n}\right)\right) \subseteq S_{n}
\end{gathered}
$$

and

$$
T_{n}=\operatorname{Im}\left(\left(\bigcup_{\nu(a, b) \leq n} \mathcal{E}(a) \times \mathcal{E}(b)\right) \times \mathcal{E}_{\mathbb{C}} \rightarrow \mathcal{E}_{\mathbb{C}} \times \mathcal{E}_{\mathbb{C}}\right)
$$

The spaces $R_{n}, S_{n}$, and $T_{n}$ are $G$-spaces, fit into the appropriate pushout squares, and $R_{n} \hookrightarrow S_{n}$ is a closed constructable embedding since addition is closed on $\mathcal{E}_{\mathbb{C}}$ therefore by Proposition B. 4 the $R_{n} \hookrightarrow S_{n}$ are equivariant cofibrations. This shows that $\mathcal{E}$ is equivariantly tractable. The third item follows from similar consideration of [FL97, Proposition C.3]. The second item follows by applying Proposition B.3.

For the last part write $R_{n}^{\prime}, S_{n}^{\prime}$, and $T_{n}^{\prime}$ for the spaces above giving the tractability of $\mathcal{F}_{\mathbb{C}}$. Then $R_{n}^{\prime} \subseteq R_{n}$ and $S_{n}^{\prime} \subseteq S_{n}$ are cofibrations. Considering the comparison of pushouts

$$
T_{n}^{\prime} / \mathcal{F}_{\mathbb{C}}=T_{n-1}^{\prime} / \mathcal{F}_{\mathbb{C}} \bigcup_{R_{n}^{\prime}} S_{n}^{\prime} \rightarrow T_{n} / \mathcal{E}_{\mathbb{C}}=T_{n-1} / \mathcal{E}_{\mathbb{C}} \bigcup_{R_{n}} S_{n}
$$

we see by induction that $T_{n}^{\prime} / \mathcal{F}_{\mathbb{C}} \hookrightarrow T_{n} / \mathcal{E}_{\mathbb{C}}$ is a cofibration and in particular is closed. Therefore $\mathcal{F}_{\mathbb{C}}^{+}=\operatorname{colim}_{n} T_{n}^{\prime} / \mathcal{F}_{\mathbb{C}} \subseteq \operatorname{colim}_{n} T_{n} / \mathcal{E}_{\mathbb{C}}=\mathcal{E}_{\mathbb{C}}^{+}$is a closed subspace FP90, Proposition A.5.5]. Finally $\left.\mathcal{E}_{\mathbb{C}} / \mathcal{F}_{\mathbb{C}}\right)^{+}=\mathcal{E}_{\mathbb{C}}^{+} / \mathcal{F}_{\mathbb{C}}^{+}$by Lemma A.2 which gives the displayed exact sequence.

Spaces of algebraic cycles and algebraic cocycles on complex varieties are shown to have $C W$-structures or homotopy type of $C W$-spaces in LF92, FW01b and in Teh05 for real varieties.

Corollary B.6. Let $U$ be a quasi-projective real variety. Then the spaces $\mathcal{Z}_{k}\left(U_{\mathbb{C}}\right)$, $\mathcal{Z} / \ell_{k}\left(U_{\mathbb{C}}\right), \mathcal{Z}^{q}\left(U_{\mathbb{C}}\right), \mathcal{Z}^{q} / \ell\left(U_{\mathbb{C}}\right)$ all have the homotopy type of a $G-C W$ complex. The spaces $\mathcal{Z}_{k}\left(U_{\mathbb{C}}\right)^{a v}, \mathcal{Z} / \ell_{k}\left(U_{\mathbb{C}}\right)^{a v}, \mathcal{R}_{k}(U), \mathcal{Z}^{q}\left(U_{\mathbb{C}}\right)^{a v}, \mathcal{Z}^{q} / \ell\left(U_{\mathbb{C}}\right)^{a v}$, and $\mathcal{R}^{q}\left(U_{\mathbb{C}}\right)$ all have the homotopy type of a $C W$-complex.

Proof. That $\mathcal{Z}_{k}\left(U_{\mathbb{C}}\right)$ has the homotopy type of a $G$ - $C W$ complex follows immediately from the previous proposition. Let $U \subset \bar{U}$ be a projectivization. Write $\mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)=\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)+\mathcal{C}_{d}\left(\mathbb{P}_{\mathbb{C}}^{q} \times \bar{U}_{\mathbb{C}}\right)$. This is a closed constructable submonoid $\mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right) \subseteq \mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)$ and $\left(\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right) / \mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)\right)^{+} \cong \mathcal{Z}^{q}\left(U_{\mathbb{C}}\right)$ has the equivariant homotopy type of a $G$ - $C W$ complex. Since $\left(\ell \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)\right)^{+} \subseteq$ $\mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{+}$is closed we easily see that $\left(\ell \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)\right)^{+}=\ell\left(\mathcal{C}_{k}\left(U_{\mathbb{C}}\right)\right)^{+} \subseteq \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{+}$. Therefore $\mathcal{Z} / \ell_{k}\left(U_{\mathbb{C}}\right) \cong\left(\mathcal{C}_{k}\left(U_{\mathbb{C}}\right) / \ell \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)\right)^{+}$has the equivariant homotopy type of a $G$ $C W$ complex. Similarly one sees that $\mathcal{Z}^{q} / \ell\left(U_{\mathbb{C}}\right) \cong\left(\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right) / \mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)+\right.$ $\left.\ell \mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)\right)^{+}$has the equivariant homotopy type of a $G$ - $C W$ complex.

The monoid inclusions $\ell \mathcal{C}_{k}\left(U_{\mathbb{C}}\right) \cap \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{a v} \subseteq \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{a v} \subseteq \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{G} \subseteq \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)$ are all closed and so $\mathcal{Z}_{k}\left(U_{\mathbb{C}}\right)^{a v}, \mathcal{R}_{k}(U) \cong\left(\mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{G} / \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{a v}\right)^{+}$, and $\mathcal{Z} / \ell_{k}\left(U_{\mathbb{C}}\right)^{a v} \cong$ $\left(\mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{a v} / \ell \mathcal{C}_{k}\left(U_{\mathbb{C}}\right) \cap \mathcal{C}_{k}\left(U_{\mathbb{C}}\right)^{a v}\right)^{+}$all have the homotopy type of a $C W$-complex.

Similarly $\mathcal{Z}^{q}\left(U_{\mathbb{C}}\right)^{a v} \cong\left(\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)^{a v} / \mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right) \cap \mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)\right)^{+}, \mathcal{Z}^{q} / \ell\left(U_{\mathbb{C}}\right)^{a v} \cong$ $\left(\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)^{a v} /\left(\mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)+\ell \mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)\right) \cap \mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)^{a v}\right)^{+}$, and $\mathcal{R}^{q}\left(U_{\mathbb{C}}\right) \cong$ $\left(\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)^{G} / \mathcal{F}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q-1}\right)\left(U_{\mathbb{C}}\right)^{G}+\mathcal{E}_{0}\left(\mathbb{P}_{\mathbb{C}}^{q}\right)\left(U_{\mathbb{C}}\right)^{a v}\right)^{+}$all have the homotopy type of a $C W$ complex.

## References

[BG73] Kenneth S. Brown and Stephen M. Gersten, Algebraic K-theory as generalized sheaf cohomology, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 266-292. Lecture Notes in Math., Vol. 341. MR MR0347943 (50 \#442)
[Cox79] David A. Cox, The étale homotopy type of varieties over R, Proc. Amer. Math. Soc. 76 (1979), no. 1, 17-22. MR MR534381 (80f:14009)
[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn, The Bloch-Ogus-Gabber theorem, Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 31-94. MR MR1466971 (98j:14021)
[Del74] Pierre Deligne, Théorie de Hodge. III, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5-77. MR MR0498552 (58 \#16653b)
[DI04] Daniel Dugger and Daniel C. Isaksen, Topological hypercovers and $\mathbb{A}^{1}$-realizations, Math. Z. 246 (2004), no. 4, 667-689. MR MR2045835 (2005d:55026)
[dS03a] Pedro F. dos Santos, Algebraic cycles on real varieties and $\mathbb{Z} / 2$-equivariant homotopy theory, Proc. London Math. Soc. (3) 86 (2003), no. 2, 513-544. MR MR1971161 (2004c:55026)
[dS03b] , A note on the equivariant Dold-Thom theorem, J. Pure Appl. Algebra 183 (2003), no. 1-3, 299-312. MR MR1992051 (2004b:55021)
[dSLF04] Pedro F. dos Santos and Paulo Lima-Filho, Quaternionic algebraic cycles and reality, Trans. Amer. Math. Soc. 356 (2004), no. 12, 4701-4736 (electronic). MR MR2084395 (2005m:55018)
[Dug05] Daniel Dugger, An Atiyah-Hirzebruch spectral sequence for $K R$-theory, $K$-Theory 35 (2005), no. 3-4, 213-256 (2006). MR MR2240234 (2007g:19004)
[FG93] Eric M. Friedlander and Ofer Gabber, Cycle spaces and intersection theory, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 325-370. MR MR1215970 (94j:14010)
[FL92] Eric M. Friedlander and H. Blaine Lawson, Jr., A theory of algebraic cocycles, Ann. of Math. (2) 136 (1992), no. 2, 361-428. MR MR1185123 (93g:14013)
[FL97] Eric M. Friedlander and H. Blaine Lawson, Duality relating spaces of algebraic cocycles and cycles, Topology 36 (1997), no. 2, 533-565. MR MR1415605 (97k:14007)
[FM94] Eric M. Friedlander and Barry Mazur, Filtrations on the homology of algebraic varieties, Mem. Amer. Math. Soc. 110 (1994), no. 529, x+110, With an appendix by Daniel Quillen. MR MR1211371 (95a:14023)
[FP90] Rudolf Fritsch and Renzo A. Piccinini, Cellular structures in topology, Cambridge Studies in Advanced Mathematics, vol. 19, Cambridge University Press, Cambridge, 1990. MR MR1074175 (92d:55001)
[Fri91] Eric M. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology, Compositio Math. 77 (1991), no. 1, 55-93. MR MR1091892 (92a:14005)
[Fri98] , Algebraic cocycles on normal, quasi-projective varieties, Compositio Math. 110 (1998), no. 2, 127-162. MR MR1602068 (2000a:14024)
[FV00] Eric M. Friedlander and Vladimir Voevodsky, Bivariant cycle cohomology, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 138-187. MR MR1764201
[FW01a] Eric M. Friedlander and Mark E. Walker, Comparing K-theories for complex varieties, Amer. J. Math. 123 (2001), no. 5, 779-810. MR MR1854111 (2002i:19004)
[FW01b] , Function spaces and continuous algebraic pairings for varieties, Compositio Math. 125 (2001), no. 1, 69-110. MR MR1818058 (2001m:14033)
[FW02a] _ Semi-topological K-theory of real varieties, Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), Tata Inst. Fund. Res. Stud. Math., vol. 16, Tata Inst. Fund. Res., Bombay, 2002, pp. 219-326. MR MR1940670 (2003h:19005)
[FW02b] , Semi-topological K-theory using function complexes, Topology 41 (2002), no. 3, 591-644. MR MR1910042 (2003g:19005)
[FW03] , Rational isomorphisms between $K$-theories and cohomology theories, Invent. Math. 154 (2003), no. 1, 1-61. MR MR2004456 (2004j:19002)
[Gro65] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231. MR MR0199181 (33 \#7330)
[Gro66] , Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255. MR MR0217086 (36 \#178)
[KW03] Max Karoubi and Charles Weibel, Algebraic and Real K-theory of real varieties, Topology 42 (2003), no. 4, 715-742. MR MR1958527 (2004c:19004)
[Lam90] T.-K. Lam, Spaces of real algebraic cycles and homotopy theory, Ph.D. thesis, SUNY Stony Brook, 1990.
[LF92] Paulo Lima-Filho, Lawson homology for quasiprojective varieties, Compositio Math. 84 (1992), no. 1, 1-23. MR MR1183559 (93j:14007)
[LF93] , Completions and fibrations for topological monoids, Trans. Amer. Math. Soc. 340 (1993), no. 1, 127-147. MR MR1134758 (94a:55009)
[LF97] P. Lima-Filho, On the equivariant homotopy of free abelian groups on $G$-spaces and $G$-spectra, Math. Z. 224 (1997), no. 4, 567-601. MR MR1452050 (98i:55014)
[LLFM03] H. Blaine Lawson, Paulo Lima-Filho, and Marie-Louise Michelsohn, Algebraic cycles and the classical groups. I. Real cycles, Topology 42 (2003), no. 2, 467-506. MR MR1941445 (2003m:14013)
[LLFM05] H. Blaine Lawson, Jr., Paulo Lima-Filho, and Marie-Louise Michelsohn, Algebraic cycles and the classical groups. II. Quaternionic cycles, Geom. Topol. 9 (2005), 11871220 (electronic). MR MR2174264 (2007a:14013)
[May96] J. P. May, Equivariant homotopy and cohomology theory, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR MR1413302 (97k:55016)
[McC69] M. C. McCord, Classifying spaces and infinite symmetric products, Trans. Amer. Math. Soc. 146 (1969), 273-298. MR MR0251719 (40 \#4946)
[Mil80] James S. Milne, Étale cohomology, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR MR559531 (81j:14002)
[Nis89] Ye. A. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic $K$-theory, Algebraic $K$-theory: connections with geometry and topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 279, Kluwer Acad. Publ., Dordrecht, 1989, pp. 241-342. MR MR1045853 (91c:19004)
[SV00a] Andrei Suslin and Vladimir Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117-189. MR MR1744945 (2001g:14031)
[SV00b] , Relative cycles and Chow sheaves, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 10-86. MR MR1764199
[Teh05] Jyh-Haur Teh, A homology and cohomology theory for real projective varieties, 2005.
[Teh08] , Harnack-Thom theorem for higher cycle groups and Picard varieties, Trans. Amer. Math. Soc. 360 (2008), no. 6, 3263-3285. MR MR2379796
[Voe03] Vladimir Voevodsky, Motivic cohomology with Z/2-coefficients, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59-104. MR MR2031199 (2005b:14038b)
[Wan80] Stefan Waner, Equivariant homotopy theory and Milnor's theorem, Trans. Amer. Math. Soc. 258 (1980), no. 2, 351-368. MR MR558178 (82m:55016a)
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