Abstract. In section 1 we show that differential operators $L_{n_{D3}}$ of type $D3$ annihilate periods for two pencils of hypersurfaces $1 - tf_n = 0$ in 3-torus given by levels of Laurent polynomials $f_n$, in particular $L_n$ are of Picard–Fuchs type. However these operators are not associated to any Fano threefolds of first series. In section 2 we prove that they annihilate regularized $G$-series (generating functions for 1-point Gromov–Witten invariants) of two Fano threefolds of degrees 28 and 30 with second Betti numbers 2 and 3, so they indeed come from symplectic geometry of Fano threefolds. In section 3 we demonstrate the mirror symmetry between two deformation classes of smooth Fano threefolds $Y_{28}$, $Y_{30}$ from section 2 and two Laurent polynomials $f_{14}$, $f_{15}$ from section 1.

Let $t$ be a coordinate on 1-dimensional torus $T = \text{Spec } \mathbb{C}[t, t^{-1}]$ and $D = \frac{d}{dt}$.

Definition 0.1 ([13]). Operators of type $nD3$ is a family of linear differential operators

$$L(b_1, b_2, b_3, b_4, b_5) = D^3 - t \cdot b_1 D(D + 1)(2D + 1) - t^2 \cdot (D + 1)(b_2 D(D + 2) + 4b_3) -$$

$$- t^3 \cdot b_4(D + 1)(D + 2)(2D + 3) - t^4 \cdot b_5(D + 1)(D + 2)(D + 3)$$

with parameters $b_i$.

Operator $L_b$ of type $nD3$ has a unique power series solution $F_b$ with initial condition $F_b(0) = 1$:

$$F_b = 1 + \frac{t^2}{2} + (5b_1 b_3 + 2b_4) \frac{t^3}{9} + \ldots$$

We consider two particular operators $L_n$ of type $nD3$ (from [13, 11]):

$$L_{14} = L_{(1,59,16,68,80)},$$

$$L_{15} = L_{(1,43,12,78,216)}.$$  

with solutions

$$F_{14}(t) = 1 + 8t^2 + 24t^3 + 240t^4 + 1440t^5 + 11960t^6 + 89040t^7 + \ldots$$

$$F_{15}(t) = 1 + 6t^2 + 24t^3 + 162t^4 + 1080t^5 + 7620t^6 + 55440t^7 + \ldots$$

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Original definition has the other basis $a_{01}, a_{02}, a_{03}, a_{11}, a_{12}$ for parameter space $\mathbb{A}^5$. Bases $a$ and $b$ are equivalent over $\mathbb{Z}$: $b_1 = a_{11}, b_2 = a_{12} + 2a_{01} - a_{11}^2, b_3 = a_{01}, b_4 = a_{02} - a_{01}a_{11}, b_5 = a_{03} - a_{01}^2; a_{01} = b_3, a_{02} = b_4 + b_1b_3, a_{03} = b_5 + b_3^2, a_{11} = b_1, a_{12} = b_2 - 2b_3 + b_1^2$.
1. Laurent polynomials

Consider the Laurent polynomials $f_n$:

\begin{align}
(1.1) \quad f_{14} &= x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + \frac{1}{xy} + xz + \frac{1}{yz} \\
(1.2) \quad f_{15} &= x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + \frac{1}{y} + yz
\end{align}

Proposition 1.3 (Dutch trick cor Utrecht trick, see e.g. [15]). Let $\omega = \prod_{k=1}^d \frac{dx_k}{2\pi i x_k}$ be a logarithmic volume form on the torus $X = (\mathbb{C}^*)^d = \text{Spec} \mathbb{C}[x_1, x_1^{-1}, \cdots, x_d, x_d^{-1}]$. For Laurent polynomial $w(x) \in \mathbb{C}[X]$ define regular constant term series as $G_w^n(t) = \int_{|x| = 1} \omega_x \frac{w(x)}{1-tw(x)} = \sum_{k \geq 0} < w^k > t^k$, where $< w >$ is a constant term of Laurent polynomial $w$. Let $\omega_t = \text{Res}_{t-w=0} \omega_x \frac{w(x)}{1-tw(x)}$ be a fiberwise residue of $\frac{w(x)}{1-tw(x)}$. Then $G_w^n(t)$ is a solution to Picard–Fuchs equation for the pencil of hypersurfaces $1-tw(x) = 0$ equipped with volume form form $\omega_t$.

Theorem 1.4. For $n = 14, 15$ we have $G_{f_n} = F_n$. In particular, differential operators $L_n$ are Picard–Fuchs differential operators for level families $1-tf_n = 0$ of Laurent polynomials $f_n$.

The proof is straightforward once one knows the formulation: functions $G_{f_n}$ satisfy some differential equation, and each coefficient of $G_f$ gives a linear relation for coefficients of the differential equation.

2. Fano varieties

2.1. Introduction. An important symplectic invariant of Fano variety $Y$ is its regularized $G$-series $RG_Y \in \mathbb{Q}[[t]]$ (see [23]). If $Y$ is a Fano threefold of first series then $RG_Y$ is the unique analytic solution of the unique linear differential operator $L_Y$ from 5-parametric family $nD3$ with the initial condition $RG_Y(0) = 1$. In [11] Golyshev specified 17 points in the parameter space of $D3$'s that correspond to Fano threefolds of first series. He pointed out to me two other interesting differential operators [13] and [14] in family $nD3$. These equations are also listed in [11]. In [13] Lian and Yau realized $F_{14}$ and $F_{15}$ as periods for two families of K3 surfaces. In the first chapter we find Laurent polynomials $f_{14}(x_1, x_2, x_3)$ and $f_{15}(x_1, x_2, x_3)$ such that $F_n = \frac{1}{(2\pi i)^3} \int_{|x_i| = 1} \frac{dx_1 dx_2 dx_3}{1-tf_n x_1 x_2 x_3}$ i.e. we realized $F_n$ as periods of hypersurfaces $1-tf_n = 0$ in 3-dimensional torus $\text{Spec} \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]$. Laurent polynomials $f_n$ are associated with the singular toric Fano 3-folds $X_{2n}$ that admit smoothing $Y_{2n}$ in the unique deformation class. In this article we prove that $F_n$ coincides with $RG_{Y_{2n}}$:

Theorem 2.1. Series $RG_{Y_{2n}}$ is annihilated by differential operator $L_n$: $RG_{Y_{2n}} = F_n$.

We note that Fano threefolds $Y_{28}$ and $Y_{30}$ are not of first series ($b_2(Y_{28}) = 2$ and $b_2(Y_{30}) = 3$), but nevertheless they turn out to be quantum minimal: the minimal degree in $D$ of differential operators annihilating their $RG$-series is 3, and not naively expected 4 and 5.

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\textsuperscript{2}Fano variety is a smooth variety $Y$ with an ample anticanonical line bundle $\det T_Y$. Fano threefold $Y$ is said to be of first series if its second Betti number is equal to one: $b_2(Y) = \dim H^2(Y, \mathbb{Q}) = 1$. Iskovskikh showed that there are exactly 17 deformation classes of such varieties, and his classification was redone with different methods by Mukai and Ciliberto-Miranda-Lopez. We refer the reader to the textbook [12] for the details on algebraic geometry of Fano threefolds.
2.2. **G-series.** Take $D = t \frac{d}{dt}$ where $t$ is a coordinate on one-dimensional torus $T = \text{Spec} \ C[t, t^{-1}]$.

Let $\ast$ be quantum multiplication on cohomologies of Fano variety $Y$.

Define *quantum differential equation* as a trivial vector bundle over $T$ with fibre $H^r(Y)$ and connection

\[(2.2)\quad D \Phi = c_1(Y) \ast \Phi\]

where $\Phi(t) \in H^r(Y)$.

Let $G_Y(t) = [pt] + \sum_{n \geqslant 1} G^{(n)} t^n$ be the unique power series solution of (2.2) with initial condition $G_Y(0) = [pt]$, where $[pt] \in H^{2 \dim Y}(Y, \mathbb{Z})$ is the cohomology class Poincaré-dual to the class of the point. Define series

\[(2.3)\quad G_Y(t) = \int_Y G_Y(t) = 1 + \sum_{n \geqslant 1} g_n(Y) t^n\]

and

\[(2.4)\quad R G_Y(t) = 1 + \sum_{n \geqslant 1} g_n(Y) n! t^n\]

**Remark 2.5.** String equation implies $g_1(Y) = 0$.

**Proposition 2.6 (\cite{Mir}).** If $Y$ is Fano threefold of first series then $R G_Y$ is annihilated by unique normalized differential operator of type $D3$.

Let $M = \overline{M}_{0, 1}(Y, \beta)$ be the moduli space of stable maps $\phi : (C, p) \to Y$ from rational curves $C$ with marked point $p$ to $Y$ of degree $\phi_* [C] = \beta \in H_2(Y, \mathbb{Z})$, $\psi_1$ be the first Chern class of the tautological line bundle on $M$, and $ev_1 : M \to Y$ be the evaluation map $ev_1(\phi, C, p) = \phi(p) \in Y$.

Consider Givental’s $J$-function

\[(2.7)\quad J_Y(t_i; z) = \prod_i t_i^{a_i D_i} \sum_{\beta \in H_2(Y)} \prod_i \frac{t_i^{\beta \cdot D_i}}{z^{c_1(Y) \cdot \beta}} \cdot (ev_1)_* \frac{z}{z - \psi_1},\]

where $D_i$ is a base of $H^2(Y, \mathbb{Z})$ and $c_1(Y) = \sum a_i D_i$.

Next proposition is a corollary of Givental’s theorem that relates solutions of quantum differential equation and $J$-function.

**Proposition 2.8.** $G$-series equals to the fundamental term of Givental’s $J$-series:

\[G_Y = \int_{[Y]} J_Y(t_i = t^{a_i}, z = 1) \cup [pt].\]

2.3. **The threefolds.** Define degree of Fano variety $Y$ to be its anticanonical degree

\[\text{deg}(Y) = (-K_Y)^{\dim Y} = \int_{[Y]} c_1(Y)^{\dim Y},\]

Euler number as topological Euler characteristic $\chi(Y) = \int_{[Y]} c_{\dim Y}$ and Picard number $\rho(Y)$ as $\text{rk} \ H^2(Y, \mathbb{Z})$. 

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2.3.1. **Degree 30.**

**Definition 2.9.** Fano threefolds $Y_{30}$ are blowups of a curve of bidegree $(2, 2)$ on $W$, where $W$ is a hyperplane section of bidegree $(1, 1)$ of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. This deformation class of varieties has number 13 in table 3 of [13], it has degree 30, $\chi = 8$ and $\rho = 3$.

**Proposition 2.10.** $Y_{30}$ is a complete intersection of three numerically effective divisors of tridegrees $(1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Description [24],[10] shows that $Y_{30}$ is a complete intersection of sections of numerically effective line bundles in a smooth toric Fano variety. This allows us to compute $G$-series $G_{Y_{30}}$ using Givental’s theorem [10].

So $G_{Y_{30}}$ is given by the pullback of hypergeometric series from three-dimensional torus

$$G_{Y_{30}} = e^{-3t} \sum_{a,b,c \geq 0} \frac{(a+b)!(a+c)!(b+c)!}{a!b!c!3^3} t^{a+b+c} = 1 + 3t^2 + 4t^3 + \frac{27}{4}t^4 + 9t^5 + \ldots$$

2.3.2. **Degree 28.**

**Definition 2.12.** Fano threefolds $Y_{28}$ are blowups of a twisted quartic on a 3-dimensional quadric $Q$. This deformation class of varieties has number 21 in table 2 of [13], it has degree 28, $\chi = 6$ and $\rho = 2$.

Realize $Q$ as a linear section of 4-dimensional quadric $Gr(2, 4) \subset \mathbb{P}^5$ parametrizing lines in $\mathbb{P}^3$. Let $U$ be the universal vector bundle of rank 2 on $Gr(2, 4)$, and $\mathcal{O}_{Gr}(1) = \det U^*$ be the ample generator of Picard group on this Grassmannian.

**Proposition 2.13.** Consider a generic section of rank 5 vector bundle $(U^* \otimes \mathcal{O}_{\mathbb{P}^4}(1))^\otimes 2 \oplus \mathcal{O}_{Gr}(1)$ on the 8-fold $Gr(2, 4) \otimes \mathbb{P}^4$. Then it is a smooth Fano threefold from deformation class of $Y_{28}$.

Consider abelianization for $Y_{28}$: a complete intersection $A_{28}$ in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^4$ of line bundles $\mathcal{O}(1, 0, 1)$ twice, $\mathcal{O}(0, 1, 1)$ twice and $\mathcal{O}(1, 1, 0)$.

$J$-series of $Y_{28}$ (and hence $G$-series $G_{Y_{28}}$) can be computed using abelian/non-abelian correspondence [3,13] of Bertrand, Ciocan-Fontanine, Kim and Sabbah.

**Proposition 2.14.** $G_{Y_{28}}$ equals $e^{-t}$ times coefficient at $H_2$ of

$$\sum_{a_1,a_2,b \geq 0} t^{a_1+a_2+b} \frac{g(a_1+b,H_1)^2g(a_2+b,H_2)^2g(a_1+a_2,H_1+H_2)g(a_2-a_1)}{g(a_1,H_1)^4g(a_2,H_2)^4b!^5}$$

where $g(d,D) = d! \cdot (1 + h_dD)$ is approximation of $\Gamma$-function, $h_d = \sum_{i=1}^d \frac{1}{i}$ is the harmonic number, and $H_1, H_2$ are positive generators of $H^2(\mathbb{P}^3, \mathbb{Z})$ pulled back to $H^2(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{Z})$ by two projections.

So $G_{Y_{28}} = 1 + 4t^2 + 4t^3 + 10t^4 + 12t^5 + \frac{299}{18}t^6 + \ldots$

We’ll present a detailed proof of these computations in [6].

Denote the $G$-series of $Y_{28}$ as $G_n = G_{Y_{2n}}$.

3. Toric Degenerations and Mirror Symmetry

**Definition 3.1 ([13]).** Let $G_Y(t) = 1 + g_2(Y)t^2 + \ldots$ be the $G$-series of $Y$. Laurent polynomial $w$ is said to be mirror for $Y$ if general element of pencil $1 - tw(x) = 0$ is birational to Calabi–Yau variety and $G_Y(t) = G_w(t)$.

**Remark 3.2.** In particular constant term of $w$ is zero.
Definition 3.3. For toric Fano variety \(X\) let \(\Delta(X)\) be its fan polytope. For lattice polytope \(\Delta\) let \(\mathbb{P}_\Delta\) be the toric Fano variety with fan polytope \(\Delta\). Define vertex Laurent polynomial \(w_\Delta\) as sum of all monomials corresponding to the vertices of \(\Delta\):
\[
(3.4) \quad w_\Delta = \sum_{v \in \text{Vertices}(\Delta)} x^v
\]
For Laurent polynomial \(w\) let \(\Delta(w)\) be its Newton polytope and define \(\mathbb{P}(w) = \mathbb{P}_{\Delta(w)}\)

Theorem 3.5 (Mirror construction for smooth toric Fano varieties \([10]\)). If variety \(X\) is a smooth toric Fano variety then \(w_{\Delta(X)}\) is a mirror for \(X\).

Definition 3.6 (Small toric degeneration \([2]\)). Assume \(X\) is a toric Fano variety that admits only terminal Gorenstein singularities. We say that \(X\) is a small toric degeneration of smooth Fano variety \(Y\) if there exists a flat projective morphism to irreducible curve \(\pi: \mathcal{X} \to C\), such that \(X\) and \(Y\) are isomorphic to some fibers of \(\pi\), and the restriction map \(\text{Pic}(\mathcal{X}) \to \text{Pic}(\mathcal{X}_t)\) is an isomorphism for all \(t \in C\).

Conjecture 3.7 (Small toric degeneration hypothesis \([2]\)). If \(X\) is a small toric degeneration of smooth Fano variety \(Y\), then \(w_{\Delta(X)}\) is a mirror for \(Y\).

Theorem 3.8 (\([8]\)). For \(n = 14, 15\) toric Fano variety \(\mathbb{P}(f_n)\) have terminal Gorenstein singularities and is small toric degeneration of smooth Fano threefold \(Y_{2n}\) in the uniquely determined deformation class (number 2.21 and 3.13 in \([14]\)).

Combination of theorems 1.4 and 2.1 implies

Theorem 3.9. For \(n = 14, 15\) Laurent polynomial \(f_n\) is a mirror for family of smooth Fano threefolds \(Y_{2n}\).

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References


