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Derived categories for complex-analytic manifolds

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Abstract

We construct a twist-closed enhancement of the derived category of coherent sheaves on a smooth compact complex-analytic manifold by means of DG-category of dbar-superconnections.

1 Introduction

The derived category of coherent sheaves is known to be a meaningful homological invariant of an algebraic variety. The basic motivation for the authors of the present paper was the wish to understand to which extent the derived category is a good invariant for complex analytic manifolds.

Let X be a compact complex analytic manifold and $\mathcal{D}^b_{\mathrm{coh}}(X)$ the derived category of $\mathcal{O}_X$-modules with bounded coherent cohomology. There are some indications that this category is not as good as it is in the algebraic case. First, a result of Verbitsky [V1] implies that $\mathcal{D}^b_{\mathrm{coh}}(X)$ are equivalent for all K3 surfaces X with no curves. Hence, the derived category does not 'feel' the complex geometry of the generic K3 surface. Note that, in the case of projective varieties, a conjecture (cf. [Kaw], [R]) states that there is only a finite number of smooth projective varieties derived equivalent to a given one. It is known that there is at most countably many algebraic varieties in a given class of derived equivalence ([AT]).

Second, a wonderful property of the derived category of coherent sheaves on a smooth proper algebraic variety is that it satisfies a property similar to Brown representability. Namely, the category is *saturated*, i.e all bounded cohomological functors with values in vector spaces are representable (see [BK1], [BVdB]). It was shown in [BVdB] that for the derived category of a smooth compact complex surface with no curves (say, a generic K3) this property does not hold. It was also conjectured that if the derived category is saturated then the variety is (an analytification of) an algebraic space. Recently, the conjecture was proved by B. Toën and M. Vaquié ([TV2]), though they used an a priori stronger version of saturatedness.

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It was probably these facts that lead to a currently rather typical opinion that the derived category was not a meaningful homological invariant in the complex analytic case. Formally speaking this was correct. However, there was some ambiguity in what sort of natural structure is reasonable to fix when considering derived categories. In particular, it was mentioned quite a time ago that it made sense to consider triangulated categories together with *enhancements*, a sort of enrichments of the categories with a DG-structures [BK2]. We will show that some good (twist-closed) enhancements are relevant to complex-analytic geometry.

We construct a particular nice DG-enhancement, $\mathcal{E}_X$, of $\mathcal{D}^b_{\mathrm{coh}}(X)$ in this paper. Its objects are DG-modules over the DG-algebra of Dolbeault forms. Our first principal result is that the homotopy category of this DG-category is equivalent to $\mathcal{D}^b_{\mathrm{coh}}(X)$.

An important property of the category $\mathcal{E}_X$ is that it is *twist-closed*, which means that the functors constructed via twisted complexes in the category are representable. Twisted complexes are solutions of the Maurer-Cartan equation with values in the (degree one) endomorphisms of objects in the DG-category. Note that twist-closed categories are pre-triangulated, i.e so-called one-sided twisted complexes are representable, but it is crucial for the construction of moduli in complex-analytic geometry to have representability of all twisted complexes.

In the sequel to this paper, we will show that solutions of the Maurer-Cartan equation in $\mathcal{C}_X$ are 'well-parameterized' if we consider moduli of appropriate *simple* objects, thus giving a parametrization of the corresponding representation objects. By patching the parameter spaces for all simple objects, we construct a coarse moduli space of simple objects in the category. The moduli space represents the relevant functor of points on the category of Stein varieties.

It is worthwhile to mention here that the first construction of the moduli spaces of simple objects in the derives category of coherent sheaves on a projective variety is due to M. Inaba [Ina]. He has shown that the moduli functor for this case is represented by an algebraic space. M. Liebich constructed a moduli stack of appropriate objects in the derived category of a proper variety [Lieb].

Toën and Vaquié constructed a similar moduli of simple objects (and, more generally, a higher stack of all objects) in a *saturated* DG-category [TV1]. As we have already discussed, the derived category of a complex-analytic manifold is not, in general, saturated. Hence, their result is not applicable in the complex-analytic case.

2 DG-enhancements

In this section we recall some facts on DG-categories and introduce the notion of twist-closed DG-categories.

DG-categories can be considered as a particular class of A_∞-categories. The theory has a direct generalization to the A_∞-case. We avoid this more general context here, because natural enhancements which one meets in complex geometry have a DG-structure.

Let $\mathcal{E}$ be an additive DG-category over a field. This means that we have direct sums of objects, morphisms between any two objects constitute a $\mathbb{Z}$-graded complex of vector spaces over the field, and Leibnitz rule for the composition of morphisms is satisfied. We will denote the space of degree i morphisms in $\mathcal{E}$ by $\mathbb{R}\mathrm{hom}^i$.

Two objects $A, B \in \mathcal{E}$ are said to be DG-isomorphic if there exists an invertible degree zero closed morphism $f \in \mathbb{R}\mathrm{hom}^0(A, B)$. Accordingly, one defines DG-isomorphism of functors.

We assume that the category is equipped with an equivalence $T : \mathcal{E} \to \mathcal{E}$, called translation functor, together with a DG-isomorphism of functors:

$$\mu : \mathrm{id} \to T,$$

where μ has degree -1.

If $\mathcal{E}$ is a DG-category, then its *homotopy category* $\mathcal{H}o\mathcal{E}$ is defined as a category with the same objects as in $\mathcal{E}$, but morphisms are the degree zero homology of complexes of morphisms in $\mathcal{E}$.

We now explain the ideology (based on [BK2]) of twisted complexes and functors which they represent.

A twisted complex in a DG-category $\mathcal{E}$ is a pair $T = (E, \alpha)$, where E is an object in $\mathcal{E}$ and a *twisting cochain* α is in $\mathbb{R}\mathrm{hom}^1(E, E)$ and satisfies Maurer-Cartan equation:

$$\mathrm{d}\alpha + \alpha^2 = 0.$$

Any twisted complex $T = (E, \alpha)$ defines a contravariant functor

$$h^T : \mathcal{E} \to \mathrm{C}^{\cdot}(\mathrm{Vect}),$$

where $\mathrm{C}^{\cdot}(\mathrm{Vect})$ stands for the category of complexes of vector spaces. The functor is defined on $A \in \mathcal{E}$ by:

$$A \mapsto \mathbb{R}\mathrm{hom}^{\cdot}(A, E),$$

but the differential in $\mathbb{R}\mathrm{hom}^{\cdot}(A, E)$ is twisted by α:

$$\mathrm{d}_\alpha = \mathrm{d}_{\mathbb{R}\mathrm{hom}^{\cdot}(A,E)} + \alpha.$$

The Maurer-Cartan equation for α implies that $\mathrm{d}_\alpha^2 = 0$.

We will also consider twisted complexes of a particular form. A twisted complex $T = (E, \alpha)$ is called *one-sided*, if the object E is decomposed into a finite direct sum $E = \oplus E_i$ and α is strictly upper triangular with respect to this decomposition.

The basic example of a one-sided twisted complex is obtained by taking E to be of the form

$$E = E_1 \oplus E_2 \tag{1}$$

onesided

and assuming α to be strictly upper triangular with respect to this decomposition, i.e. to belong to $\mathbb{R}\mathrm{hom}^1(E_1, E_2)$, which is a subspace in $\mathbb{R}\mathrm{hom}^1(E, E)$. Note that α^2 is automatically zero for such α. Therefore, the Maurer-Cartan equation reduces to $\mathrm{d}\alpha = 0$. Thus α can be interpreted as a closed degree zero morphism $E_1 \to E_2[1]$.

In the following definition, DG-categories are assumed to be additive and equipped with translation functors.

Definition 2.1 *(i) A DG-category is called* twist-closed *if h^T is representable for any twisted complex T.*

(ii) A DG-category is called pre-triangulated *if h^T is representable for any one-sided twisted complex T.*

The following theorem was proved in [BK2].

Theorem 2.2 *If $\mathcal{E}$ is pre-triangulated, then $\mathcal{H}o\mathcal{E}$ is naturally triangulated.*

The idea behind this theorem is that the twisting cochain α in a one-sided twisted complex of the form (1) defines a morphism $\tilde{\alpha} : E_1 \to E_2[1]$ in $\mathcal{H}o\mathcal{E}$ and the object that represents the functor h^T gives a cone of $\tilde{\alpha}$ in the homotopy category. Thus the basic hereditary problem of the axiomatics of triangulated categories that the cones of morphisms are not canonical is resolved by lifting morphisms in a triangulated category to closed morphisms in an appropriate DG-category.

Another reason why DG-context looks more suitable is that the pre-triangulatedness is a property of a DG-category, while to make a category triangulated one has to put an extra structure on the category (to fix a class of exact triangles). The price to pay for transferring into the DG-world is that one has to consider DG-categories up to an appropriate equivalence relation, i.e. there is always a variety of equivalent choices for DG-categories "representing" a given triangulated category.

Definition 2.3 *If $\mathcal{D}$ is a triangulated category, then a pre-triangulated category $\mathcal{E}$ together with an equivalence of triangulated categories $\mathcal{H}o\mathcal{E} \rightarrow \mathcal{D}$ is said to be an* enhancement *of $\mathcal{D}$. The category $\mathcal{D}$ is then said to be enhanced. A functor between two DG-categories is said to be an* quasi-equivalence *if it induces an equivalence of the corresponding homotopy categories.*

It is clear from definitions that a twist-closed DG-category is pre-triangulated, hence its homotopy category is naturally triangulated. It will be crucial for our further constructions to have enhancements which are twist-closed.

NB! The twist-closedness is not preserved under quasi-equivalences.

The following example shows that a standard enhancement of the derived category of coherent sheaves on an algebraic variety is not twist-closed.

2.4 Example. Let X be an algebraic variety with the structure sheaf $\mathcal{O}_X$ and $\mathcal{D} = \mathcal{D}^b_{\mathrm{coh}}(X)$ the derived category of complexes $\mathcal{O}_X$-modules with bounded coherent cohomology. Consider the DG-category $\mathcal{E} = \mathrm{I}(X)$ of bounded below complexes of injective $\mathcal{O}_X$-modules with bounded coherent cohomology. By definition, this is a full subcategory in the DG-category $C^{\cdot}(\mathcal{O}_X - \mathrm{mod})$ of complexes of $\mathcal{O}_X$-modules. It is known that $\mathrm{I}(X)$ (not the $C^{\cdot}(\mathcal{O}_X - \mathrm{mod})$) is an enhancement of $\mathcal{D}^b_{\mathrm{coh}}(X)$ [BK2], [BLL].

Let E be a complex in $\mathrm{I}(X)$ with the differential d, such that some term of E is not a coherent $\mathcal{O}_X$-module (this is a typical object). Consider the twisted complex $T = (E, \alpha)$ where $\alpha = -\mathrm{d}$. One can show that the functor h^T is not representable. Indeed, it is represented by the object in $C^{\cdot}(\mathcal{O}_X - \mathrm{mod})$ which is a complex with the same graded components as E and with trivial differential. Its cohomology is not coherent.

Here is an example of a twist-closed enhancement.

2.5 Example. Let $\mathcal{D} = \mathcal{D}(\text{mod} - A)$ be the derived category of (right) modules over a finite dimensional associative algebra A of finite global dimension. Consider the DG-category $\mathcal{E} = \mathcal{P}(A)$ consisting of perfect complexes, i.e. finite complexes of finitely generated projective A-modules. This is a full subcategory in the DG-category $C^{\cdot}(\text{mod} - A)$ of all complexes of A-modules. Again $\mathcal{P}(A)$ (and not $C^{\cdot}(\text{mod} - A)$) is an enhancement of $\mathcal{D}(\text{mod} - A)$. Every twisted complex $T = (E, \alpha)$ over this category produces a functor h^T which is representable by the complex E with the differential

$$d_T = d_E + \alpha.$$

This is a perfect complex. Hence the enhancement is twist-closed.

3 The enhancement by super-connections

Let X be a smooth complex-analytic space and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Denote by $\mathcal{D}^b_{\text{coh}}(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with bounded coherent cohomology.

We will construct a twist-closed enhancement of $\mathcal{D}^b_{\text{coh}}(X)$. Our further construction of moduli of objects in DG-categories will be rather sensitive with respect to the property of twist-closedness.

We want to construct a twist-closed DG-category $\mathcal{C} = \mathcal{C}_X$ whose homotopy category is equivalent to $\mathcal{D}^b_{\text{coh}}(X)$. The idea of the construction can be explained via Koszul duality applied to the algebra of differential operators (cf. [Kap]). This goes along the following lines.

3.1 The view-point via Koszul duality

Denote by $\mathcal{A}^{i,j} = \mathcal{A}^{i,j}_X$ the sheaf of smooth complex-valued (i, j)-forms on X. For the sake of simplicity, we will also use the notation $\mathcal{A}^i = \mathcal{A}^i_X = \mathcal{A}^{0,i}$. In particular, $\mathcal{A}^0_X$ is the sheaf of smooth functions on X.

A locally free coherent sheaf on X is given by a smooth vector bundle $\mathcal{E}$ on X with a flat $\bar{\partial}$ connection ∇. One can interpret such a connection as a module over the sheaf of algebras $\mathcal{D}^{\bar{\partial}}_X$ of $\bar{\partial}$-differential operators on X. $\mathcal{D}^{\bar{\partial}}_X$ is a nonhomogeneous quadratic algebra over $\mathcal{A}^0_X$. The quadratic dual algebra to $\mathcal{D}^{\bar{\partial}}_X$ is the sheaf of DG-algebras on X

$$\mathcal{A}_X = \oplus \mathcal{A}^i_X$$

of smooth $(0, i)$-forms on X equipped with Dolbaut differential $\bar{\partial}$. We use notation $\mathcal{A}^+ = \mathcal{A}^+_X = \oplus_{i>0} \mathcal{A}^i_X$ for the positive part of Dolbaut complex. Denote by $\mathcal{A}^\natural_X$ the same graded algebra with no differential. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then $\mathcal{A}^{i,j}(\mathcal{E}) = \mathcal{A}^{i,j}_X(\mathcal{E}) = \mathcal{A}^{i,j}_X \otimes_{\mathcal{O}_X} \mathcal{E}$ denotes the sheaf of smooth (i, j)-forms on X with values in $\mathcal{E}$. Similarly, we put $\mathcal{A}(\mathcal{E}) = \mathcal{A}_X(\mathcal{E}) = \mathcal{A}_X \otimes \mathcal{E}$.

Koszul duality typically states that appropriate derived categories of modules over Koszul dual algebras are equivalent. This is a reason to search for an enhancement of $\mathcal{D}^b_{\mathrm{coh}}(X)$ among DG-categories of appropriate $\mathcal{A}_X$-DG-modules.

For the case of a locally free $\mathcal{O}_X$-module $\mathcal{E}$, the $\bar\partial$-connection $\nabla : \mathcal{E} \to \mathcal{A}^{0,1}(\mathcal{E})$ can be extended to a differential in $\mathcal{A}(\mathcal{E})$. As a result, $\mathcal{A}(\mathcal{E})$ acquires a natural structure of a DG-module over $\mathcal{A}$. This is the module which corresponds to $\mathcal{E}$.

We want to extend this correspondence to arbitrary objects in $\mathcal{D}^b_{\mathrm{coh}}(X)$. It is natural to call $\mathcal{A}$-DG-modules of appropriate form by $\bar\partial$-super-connections (see below). If every coherent sheaf on X has a resolution by locally free $\mathcal{O}_X$-modules, which is true if X is the complexification of an algebraic variety or if X is a surface [S], then it is easy to construct a super-connection corresponding to an arbitrary object in $\mathcal{D}^b_{\mathrm{coh}}(X)$.

In general, locally free resolutions don't exist. A counterexample with a complex torus is due to C. Voisin [Vo]. This makes the issue more subtle. Technically, it will be easier for us to construct a functor in the inverse direction and prove that it is an equivalence.

3.2 The construction of the enhancement

Let M be a bounded (left) DG-module over $\mathcal{A}$. If d is a differential in M, m an element in M and ω an element in $\mathcal{A}$, then the Leibnitz rule is satisfied:

$$\mathrm{d}(\omega \cdot m) = \bar\partial\omega \cdot m + (-1)^{\deg \omega}\omega \cdot \mathrm{d}m \qquad (2)$$

By forgetting the differential in M, we obtain a module, $M^\natural$, over $\mathcal{A}^\natural$. If M is a left $\mathcal{A}^\natural$-module, then it is naturally endowed with structure of *right* $\mathcal{A}^\natural$-module by the standard formula:

$$m \cdot \omega := (-1)^{\deg m \cdot \deg \omega}\omega \cdot m.$$

Define the objects of the category $\mathcal{C} = \mathcal{C}_X$ to be DG-modules M over $\mathcal{A}$ for which $M^\natural$ are locally free graded modules of finite rank over $\mathcal{A}^\natural$. If M is in $\mathcal{C}$, then it is, in particular, a bounded graded complex of locally free $\mathcal{A}^0$-modules. We say that objects of $\mathcal{C}$ are $\bar\partial$-*super-connections* or simply by *super-connections* in compliance with Quillen's terminology in [Q].

Let M and N are in $\mathcal{C}$. Consider them as right $\mathcal{A}^\natural$-modules. Then the complex $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)$ of local right module homomorphisms $\mathcal{H}om_{\mathcal{A}^\natural}(M^\natural, N^\natural)$ is endowed with a left $\mathcal{A}^\natural$-module structure which comes from the left module structure on N and with the standard differential (coming from the differentials in M and N). Hence, $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)$ is an object in $\mathcal{C}$ too. Note that elements $\phi \in \mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)$, $\omega \in \mathcal{A}$ and $m \in M$ satisfy the sign rule:

$$\phi(\omega \cdot m) = (-1)^{\deg \phi \cdot \deg \omega}\omega \cdot \phi(m). \qquad (3)$$

We define homomorphisms in $\mathcal{C}$ as the graded complex of global sections of $\mathcal{H}om^{\cdot}(M, N)$:

$$\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(M, N) := \Gamma(X, \mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)).$$

Proposition 3.3 *$\mathcal{C}$ is a twist-closed DG-category.*

Proof. The translation functor is defined by the shift of grading of DG-$\mathcal{A}$-modules. Hence, we need to prove that every twisted complex in $\mathcal{C}$ is representable.

Let M be in $\mathcal{C}$, d the differential in M and $\alpha \in \mathbb{R}\mathrm{Hom}^1(M,M)$ a solution of the Maurer-Cartan equation. Since α satisfies the sign rule gradhom 3, then $\mathrm{d}+\alpha$ satisfies the Leibnitz rule (leibnitz 2). Moreover, $(\mathrm{d}+\alpha)^2 = 0$. Hence M_α, the $\mathcal{A}^\natural$-module $M^\natural$ with new differential $\mathrm{d}+\alpha$, is a DG-$\mathcal{A}$-module. Clearly, it lies in $\mathcal{C}$ and is the representation object for the twisted complex defined by the pair (M,α).

Corollary 3.4 *Category $\mathcal{C}$ is pre-triangulated. The homotopy category $\mathcal{H}o(\mathcal{C})$ is triangulated.*

For a twisted complex (M,α), we will denote by M_α the representation object constructed in the proof of the proposition.

Any object in $\mathcal{C}_X$ is clearly a complex of $\mathcal{O}_X$-modules. If M and N are in $\mathcal{C}$, then a closed morphism in $\mathbb{R}\mathrm{Hom}^0(M,N)$ obviously defines a morphism of complexes of the corresponding $\mathcal{O}_X$-modules. Homotopy-equivalent closed morphisms define trivial morphisms in $\mathcal{D}^b_{\mathrm{coh}}(X)$, because the derived category factors through the homotopy category of complexes of $\mathcal{O}_X$-modules. Hence we obtain a functor:

$$\Phi : \mathcal{H}o(\mathcal{C}_X) \to D^b(\mathcal{O}_X - \mathrm{mod}). \qquad (4)$$ functor

The rest of this section is devoted to proving that $\mathcal{C}_X$ defines an enhancement of $\mathcal{D}^b_{\mathrm{coh}}(X)$.

equiv **Theorem 3.5** *Let X be a smooth compact complex-analytic space. Then Φ is a triangulated equivalence between the homotopy category $\mathcal{H}o(\mathcal{C}_X)$ and $\mathcal{D}^b_{\mathrm{coh}}(X)$.*

The strategy of the proof is as follows. First, we will show that functor Φ is fully faithful, i.e retains homomorphisms between any two objects in $\mathcal{H}o(\mathcal{C}_X)$. Second, we will show that any super-connection is a complex of $\mathcal{O}_X$-modules with bounded coherent cohomology. Third, we will prove that any object in $\mathcal{D}^b_{\mathrm{coh}}(X)$ is quasi-isomorphic to a super-connection. The proof is relied on the description of the local structure of super-connections.

3.6 Local structure of super-connections

Every finite complex of locally free $\mathcal{O}_X$-modules $E^\cdot$ yields a super-connection by taking tensor product of complexes

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} E^\cdot.$$

Here we will prove the main technical statement which claims that any super-connection is locally gauge-equivalent to a super-connection of this kind. We believe that this fact is of independent interest.

Let M be a super-connection. Since $M^\natural$ is locally free over $\mathcal{A}^\natural$, it can be non-canonically presented in the form:

$$M^\natural = \mathcal{A}^\natural \otimes \mathcal{E}^\cdot \qquad (5)$$ decompmodule

where $\mathcal{E}^{\cdot} = M/\mathcal{A}^{+}M$ is a finitely graded locally free $\mathcal{A}^0_X$-module. To have this presentation, choose a splitting for the projection $M \to \mathcal{E}^{\cdot}$ and use the $\mathcal{A}^{\natural}$-module structure of $M^{\natural}$. Every $\mathcal{E}^i$ can be understood as a smooth complex vector bundle on X. Consider the (non-canonical) bigrading of $M^{\natural}$ where $\mathcal{A}^i \otimes \mathcal{E}^j$ has bidegree (i, j). The standard grading of $\mathcal{A}^i \otimes \mathcal{E}^{\cdot}$ in M is the total degree $(i + j)$.

The differential $D = \mathrm{d}_M$ for this module has a decomposition with respect to this bigrading:

$$D = \gamma + \nabla + \sum_{i \geq 2} \beta_i. \tag{6}$$

expansion

Here γ is the component of bidegree $(0, 1)$, ∇ the component of bidegree $(1, 0)$, and β_i is the component of degree $(i, 1 - i)$. Leibnitz rule implies that γ is an endomorphism of M of total degree 1, hence it satisfies the sign rule (3). Therefore, γ is fully defined by $\mathcal{A}^0_X$-module homomorphisms $\gamma_j : \mathcal{E}^j \to \mathcal{E}^{j+1}$. ∇ can be understood as a set of $\bar{\partial}$-connections ∇_j on $\mathcal{E}^j$. An important issue is that these connections are not necessarily flat, which prevents us at this stage from considering $\mathcal{E}^j$ as holomorphic vector bundles. By Leibnitz rule again, components β_i are completely defined by maps $\mathcal{E}^j \to \mathcal{A}^i \otimes \mathcal{E}^{j-i+1}$. In this notation, the bigraded components of the condition

$$D^2 = 0$$

reads as a sequence of equations:

$$\gamma^2 = 0,$$

$$[\gamma, \nabla] = 0,$$

$$\nabla^2 + [\gamma, \beta_2] = 0,$$

$$[\nabla, \beta_2] + [\gamma, \beta_3] = 0,$$

$$[\nabla, \beta_3] + \beta_2^2 + [\gamma, \beta_4] = 0,$$

and so on.

Note that if all β_i's are zero, then these equations are equivalent to requiring ∇ to give holomorphic structure on all bundles $\mathcal{E}^j$'s and γ to be a differential in a complex of holomorphic vector bundles.

Now, we choose a point x_0 in X and an open neighborhood of x_0 in analytic topology on X. We consider local *gauge transformations* of the form:

$$D' = e^{-\phi} D e^{\phi} \tag{7}$$

gauge

with ϕ an $\mathcal{A}^{\natural}$-module endomorphism of $M^{\natural}$, which has degree 0 with respect to the canonical grading. Clearly, ϕ is defined by its values on $\mathcal{E}^{\cdot}$. Thus, we interpret ϕ as an element of $\mathrm{Hom}^0_{\mathcal{A}^0}(\mathcal{E}^{\cdot}, \mathcal{A}^{\natural} \otimes \mathcal{E}^{\cdot})$. The *gauge parameter* ϕ is said to be *strict* if it has a decomposition

$$\phi = \phi_1 + \phi_2 + \ldots \tag{8}$$

phidecompos

where ϕ_i's are homomorphisms in $\mathrm{Hom}^0_{\mathcal{A}^0}(\mathcal{E}^{\cdot}, \mathcal{A}^i \otimes \mathcal{E}^{\cdot})$ over the neighborhood of x_0. The corresponding gauge transformation is also said to be *strict*.

Transformation (7) for components of D' reads:

$$\gamma' = \gamma,$$

i.e. γ does not change,

$$\nabla' = \nabla + [\gamma, \phi_1],$$

$$\beta_2' = \beta_2 + [\gamma, \phi_2] + \frac{1}{2}\gamma\phi_1^2 + \frac{1}{2}\phi_1^2\gamma - \phi_1\gamma\phi_1 + [\nabla, \phi_1],$$

etc.

The following theorem confirms that every superconnection is locally isomorphic to a complex of holomorphic vector bundles.

local **Theorem 3.7** *Any super-connection M over an open neighborhood of a point x_0 in X can be transformed, at the price of shrinking the neighborhood of x_0, by a strict gauge transformation of the form (7) to the form (6) with all β_i's being zero. Moreover, if D holomorphically depend on some parameters w_i's, then one can choose ϕ_i's to be holomorphic in w_i's too.*

Proof. We proceed by induction on dimension of X. If dimension is 0 or 1, then all β_i's are automatically zero. So we have the base of the induction. Assuming the fact is true in dimension n, consider $n+1$ dimensional neighborhood $\tilde{U}$ of the point in X which is of the form:

$$\tilde{U} = U \times U_1,$$

where U is an n-dimensional polydisc and U_1 a 1-dimensional complex disc. Let $z_1, \ldots, z_n$ be holomorphic coordinates on U and z a coordinate on U_1. We can decompose any form in $\mathcal{A}^{0,i}_{\tilde{U}}$ into the sum

$$\omega = \omega_{i,0} + \omega_{i-1,1}$$

with

$$\omega_{i,0} = \sum_{|I|=i} f_I \mathrm{d}\bar{z}_I$$

and

$$\omega_{i-1,1} = \sum_{|I|=i-1} f_I \mathrm{d}\bar{z}_I \mathrm{d}\bar{z}$$

where f_I are smooth functions on $\tilde{U}$ and

$$\mathrm{d}\bar{z}_I = \prod_{j \in I} \mathrm{d}\bar{z}_j$$

is an (ordered) product where j runs over an ordered subset $I \subset \{1, \ldots, n\}$.

We will apply similar decompositions to vector bundle valued forms, to D, ∇, etc. In particular, we have:

$$\nabla = \nabla_{1,0} + \nabla_{0,1},$$

where

$$\nabla_{1,0} = \bar{\partial}_U + A_{1,0},$$

where $A_{1,0} = \sum f_i \mathrm{d}z_i$ and $f_i \in \mathrm{Hom}^0(\mathcal{E}^\cdot, \mathcal{E}^\cdot)$, and

$$\nabla_{0,1} = \bar{\partial}_z + f\mathrm{d}z$$

where again $f \in \mathrm{Hom}^0(\mathcal{E}^\cdot, \mathcal{E}^\cdot)$.

We can find a form g such that

$$g^{-1}\nabla_{0,1} g = \bar{\partial}_z$$

Thus, without loss of generality, we can assume that $\nabla_{0,1} = \bar{\partial}_z$. Now, let us prove by induction on k that there exists a gauge transformation $\phi(w_i)$, such that $\phi_k = \phi_{k,0}$ for all $k \geq 1$ and for the transformed connection $D' = e^{-\phi} D e^{\phi}$ the forms β'_k have $(\beta'_k)_{k,1} = 0$ for all $k \geq 1$. From the transformation rule for D, we get the equation required to annihilate $(\beta'_1)_{1,1}$:

$$(\beta_1)'_{1,1} = (\beta_1)_{1,1} + [\nabla_{0,1}, (\phi_1)_{1,0}] = 0$$

This equation has a solution for $(\phi_1)_{1,0}$. Moreover, Cauchy formula for the solution shows that $(\phi_1)_{1,0}$ holomorphically depends on parameters w_i's. We can perform gauge transformation with $\phi = (\phi_1)_{1,0}$ and kill $(\beta_1)_{1,1}$. Now, if $(\beta_s)_{s,1} = 0$ for $s < k$, then

$$(\beta'_k)_{k,1} = (\beta_k)_{k,1} + [\nabla_{0,1}, (\phi_k)_{k,0}]$$

This equation also has a solution for $(\phi_k)_{k,0}$ holomorphic in w_i's. Again we can perform a gauge transformation with $\phi = (\phi_k)_{k,0}$ and kill $(\beta_k)_{k,1}$.

Once sufficiently many steps of this procedure performed, we transform our supper-connection to the form:

$$D = \gamma + \nabla_{1,0} + \nabla_{0,1} + \sum (\beta_i)_{i+1,0}$$

where

$$\nabla_{0,1} = \bar{\partial}_z$$

Now the equation

$$(D^2)_{*,1} = 0$$

implies

$$\bar{\partial}_z \gamma = 0$$

$$\bar{\partial}_z(\nabla_{1,0}) = 0$$

$$\bar{\partial}_z \beta_k = 0$$

This means that $D = D_{1,0} + \bar{\partial}_z$, where $D_{1,0}$ holomorphically depends on z. We can use inductive hypothesis and find gauge transformation which kills β_k.

Corollary 3.8 *Every superconnection has coherent cohomology.*

3.9 Fully faithfulness

Let M and N be super-connections. We regard them as complexes of $\mathcal{O}_X$-modules and consider the complex of derived local homomorphisms $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M,N)$, which is the object in $D(\mathcal{O}_X - \mathrm{mod})$ too.

Fix a complex $I(N)$ of injective $\mathcal{O}_X$-modules together with a quasi-isomorphism $N \to I(N)$. It induces a morphism of complexes

$$\mu : \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(M,N) \to \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(M,I(N)).$$

Consider the composite ϕ of the natural map $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M,N) \to \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(M,N)$ with μ. Since injective sheaves are acyclic with respect to the functor $\mathcal{H}om(\mathcal{U},-)$, for every $\mathcal{O}_X$-module $\mathcal{U}$ (cf. [KS]), we have a functorial isomorphism in $D(\mathcal{O}_X - \mathrm{mod})$:

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M,N) \simeq \mathcal{H}om^{\cdot}(M,I(N)). \qquad (9)$$ rhom

Hence we have a commutative diagram:

$$\begin{array}{ccc} & \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(M,N) & \\ \nearrow & & \searrow \mu \\ \mathcal{H}om^{\cdot}_{\mathcal{A}}(M,N) & \xrightarrow{\ \phi\ } & \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M,N) \end{array}$$

rhomiso **Lemma 3.10** *ϕ induces a quasi-isomorphism $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M,N) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M,N)$.*

Proof. We need to show that ϕ induces an isomorphism of cohomology sheaves. This is a local statement. Hence, we can use theorem (3.7) and replace M and N by Dolbeaux bicomplexes of finite complexes $\mathcal{E}^{\cdot}_1$ and $\mathcal{E}^{\cdot}_2$ of locally free $\mathcal{O}_X$-modules:

$$M = \mathcal{A} \otimes \mathcal{E}^{\cdot}_1, \quad N = \mathcal{A} \otimes \mathcal{E}^{\cdot}_2.$$

Then the complex $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M,N)$ is isomorphic to the Dolbeaux complex of the complex of sheaves of local homomorphisms $\mathcal{A} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{\cdot}_1, \mathcal{E}^{\cdot}_2)$.

If $\mathcal{E}^{\cdot}_1$ and $\mathcal{E}^{\cdot}_2$ both consist of single locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$, then this complex is obviously quasi-isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2)$. On the other hand, M and N are quasi-isomorphic to, respectively, $\mathcal{E}_1$ and $\mathcal{E}_2$, hence $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M,N) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}_2)$, i.e. the statement of the lemma is clear for this case.

We shall use now the induction on the length the complexes $\mathcal{E}^{\cdot}_1$ and $\mathcal{E}^{\cdot}_2$. If one of them, say $\mathcal{E}^{\cdot}_1$, has length greater than one, then we decompose it into exact triangle

$$\mathcal{E}'^{\cdot}_1 \to \mathcal{E}^{\cdot}_1 \to \mathcal{E}''^{\cdot}_1$$

where $\mathcal{E}'^{\cdot}_1$ and $\mathcal{E}''^{\cdot}_1$ are locally free complexes of shorter length. Let

$$M' \to M \to M'' \qquad (10)$$ MMM

be the corresponding decomposition of the object M.

By induction, we know that ϕ induces quasi-isomorphisms:

$$\mathcal{H}om^{\cdot}_{\mathcal{A}}(M', N) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M', N),$$

$$\mathcal{H}om^{\cdot}_{\mathcal{A}}(M'', N) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M'', N)$$

Cohomology sheaves of $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)$ and $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M, N)$ fit into two long exact sequences, obtained by applying the functors $\mathcal{H}om^{\cdot}_{\mathcal{A}}(-, N)$ and $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(-, N)$ to the triangle (10), and ϕ gives a morphism of these long sequences. The quasi-isomorphism for M follows from the lemma on 5 homomorphisms applied to this diagram. This proves the lemma.

By the standard property of local $\mathbb{R}\mathcal{H}om$, one can recover the global homomorphisms by the formula:

$$\mathrm{Hom}_{D(\mathcal{O}_X-\mathrm{mod})}(M, N) = \mathbb{H}^0(X, \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(M, N)),$$

where $\mathbb{H}^i$ stands for the hyper-cohomology of a complex of $\mathcal{O}_X$-modules. In view of the lemma, we can replace the last argument in this equality:

$$\mathrm{Hom}_{D(\mathcal{O}_X-\mathrm{mod})}(M, N) = \mathbb{H}^0(X, \mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)).$$

There is a standard spectral sequence converging to the hypercohomology:

$$\mathrm{H}^i(X, \mathcal{H}om^j_{\mathcal{A}}(M, N)) \to \mathbb{H}^{i+j}(\mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N))$$

Since all the sheaves $\mathcal{H}om^j_{\mathcal{A}}(M, N)$ are fine, cohomology $\mathrm{H}^i(X, \mathcal{H}om^j_{\mathcal{A}}(M, N))$ are trivial for $i \geq 0$. The spectral sequence degenerates, and yields the desired equality that the Hom-space in the derived category is equivalent to the homology of the complex of global sections of $\mathcal{H}om^{\cdot}_{\mathcal{A}}(M, N)$.

3.11 Coherence of cohomology of super-connections

This follows from the theorem (3.7) on the local structure of the super-connections.

3.12 Constructing a super-connection for a coherent sheaf

Let us construct a super-connection that is quasi-isomorphic to a given coherent sheaf.

Let F be a coherent sheaf of $\mathcal{O}_X$-modules. Consider the sheaves $\mathcal{F} = F \otimes_{\mathcal{O}_X} \mathcal{A}^0$ and $\mathcal{F}_\omega = F \otimes_{\mathcal{O}_X} \mathcal{A}_\omega$, where $\mathcal{A}_\omega$ stands for the sheaf of complex-valued real-analytic functions on X regarded as a real manifold. The sheaf $\mathcal{F}_\omega$ is coherent over $\mathcal{A}_\omega$.

Note that the sheaf $\mathcal{A}_\omega$ is identified with the restriction to the diagonal $X \subset X \times \bar{X}$ of the sheaf of holomorphic functions on the product of X with the complex conjugate manifold $\bar{X}$. This implies, in particular, that $\mathcal{A}_\omega$ is coherent.

As it is explained by Atiyah and Hirzebruch in [AH], the famous Grauert's result [G] on existence of a small Stein neighborhood of any real-analytic manifold in its complexification, together with theorems A and B of Cartan, easily implies existence of the finite locally free resolution of an arbitrary coherent $\mathcal{A}_\omega$-module. Fix such a resolution for $\mathcal{F}_\omega$:

$$0 \to \mathcal{E}_\omega^{-n} \to \dots \mathcal{E}_\omega^0 \to \mathcal{F}_\omega \to 0 \tag{11}$$ resolution

By a result of Malgrange [M] (Malgrange) $\mathcal{A}^0$ is a flat sheaf over $\mathcal{A}_\omega$. Hence we can safely tensor the resolution by $\mathcal{A}^0$ over $\mathcal{A}_\omega$ and obtain a locally free resolution $\mathcal{E}^\cdot$ for $\mathcal{F}$ over $\mathcal{A}^0$:

$$0 \to \mathcal{E}^{-n} \to \dots \mathcal{E}^0 \to \mathcal{F} \to 0 \tag{12}$$ resol1

For reader's convenience, we will provide with another, more explicit, construction for the resolution of the form 12 (resol1). The sheaf $\mathcal{F}$ has, locally in a vicinity of every point on X, a finite resolution by free finite $\mathcal{A}^0$-modules. Indeed, the sheaf F has, locally, a resolution by free $\mathcal{O}_X$-modules, which may be taken to be finite as the manifold is smooth. The sheaf $\mathcal{A}_\omega$ is flat over $\mathcal{O}_X$ and $\mathcal{A}^0$ is flat over $\mathcal{A}_\omega$ by Malgrange theorem, hence $\mathcal{A}^0$ is flat over $\mathcal{O}_X$. Thus, we can tensor the resolution by $\mathcal{A}^0$ and get a required local resolution. Every $\mathcal{A}_0$-module which is generated, locally over a compact manifold, by a finite number of sections, is globally generated by a finite number of sections, because $\mathcal{A}_0$ is a fine sheaf. This is clearly applicable to sheaves which locally have free resolutions. Now if we have an epimorphism of two sheaves which have local resolutions, then the kernel is a sheaf which also has such resolutions. Apply this to the epimorphism $(\mathcal{A}^0)^s \to \mathcal{F}$, which exists because $\mathcal{F}$ is generated by a finite number of global sections. We get that the kernel has also local free resolutions and is generated over $\mathcal{A}_0$ by finite number of sections. Hence we can iterate the process and construct the resolution until the kernel becomes locally free. This must happen after a finite number of iterations, because the manifold, being compact, is covered by a finite number of open sets on which a finite free resolution exists. Just take the resolution of the length equal to the maximum of lengths of these free resolutions on this finite number of open sets, then the last kernel will be locally free.

Denote by ∇_F the differential $\bar\partial$ acting on $\mathcal{A} \otimes F$ along the first tensor argument. It makes $\mathcal{A} \otimes F$ into a DG-module over $\mathcal{A}$. It is not a super-connection, but we will construct a super-connection M which is quasi-isomorphic to $\mathcal{A} \otimes F$. Put $M^\natural$ to be $\mathcal{A} \otimes \mathcal{E}^\cdot$ as a graded module over $\mathcal{A}$ with the grading obtained by summation of gradings on $(\mathcal{E}^\cdot)^\natural$ and $\mathcal{A}$. We need to construct a suitable differential on $M^\natural$.

Let γ be the differential in the resolution $\mathcal{E}^\cdot$. We denote also by γ its $\mathcal{A}$-linear extension to $M^\natural$. We denote by one letter γ_0 the map $\mathcal{E}^0 \to F$, its $\mathcal{A}$-linear extension to the map $(\mathcal{A} \otimes \mathcal{E}_0)^\natural \to (\mathcal{A} \otimes F)^\natural$ and the extension to the map $M^\natural \to (\mathcal{A} \otimes F)^\natural$ which is zero on other components $(\mathcal{A} \otimes \mathcal{E}_i)^\natural \to (\mathcal{A} \otimes F)^\natural$, for $i \neq 0$. First, we claim that there exists a system of (non-flat, in general) $\bar\partial$-connections on $\mathcal{E}^i$'s which commute with γ and such that

$$\gamma_0 \nabla = \nabla_F \gamma_0$$

First we construct a connection on $\mathcal{E}_0$ compatible with γ_0. If $\mathcal{E}_0$ is a free $\mathcal{A}^0$-module, i.e. a trivial smooth vector bundle, then take a basis $\{s_i\}$ of its sections and define $\nabla(s_i)$ to be any element in $\mathcal{A}^1 \otimes \mathcal{E}_0$ such that $\gamma_0 \nabla(s_i) = \nabla_F \gamma_0(s_i)$. If $\mathcal{E}_0$ is not a trivial bundle, then consider the sum $S_0 = \mathcal{E}_0 \oplus G_0$ which is a trivial vector bundle. Take, as above, the connection $S_0 \to \mathcal{A}^1 \otimes S_0$ on S_0 compatible with the composite map $S_0 \to \mathcal{E}_0 \xrightarrow{\gamma_0} \mathcal{F}$. The $\mathcal{E}_0$-component of this connection will define a connection on $\mathcal{E}_0$ compatible with γ_0.

The connections on other $\mathcal{E}_i$'s can be constructed consecutively starting from $\mathcal{E}_0$ by the same argument with replacing γ_0 by γ.

Using Leibnitz rule, we extend ∇ to a differential operator (also denoted by ∇) on M^{sharp} which preserves grading by $\mathcal{E}^{\cdot}$, is of degree 1 in grading by $\mathcal{A}$ and commutes with γ.

We want to find the differential in $M^{\natural}$ of the form

$$D = \gamma + \nabla + \sum_{i\geq 2} \beta_i$$

where β_i's are $\mathcal{A}^0$-module endomorphisms of M of degree i in the grading of $\mathcal{A}$ and degree $-i+1$ in the grading of $\mathcal{E}^{\cdot}$.

The equation $D^2 = 0$ implies a series of equations on components of D which can be expressed in terms of the super-commutators:

$$[\gamma, \nabla] = 0$$

$$[\gamma, \beta_2] + [\nabla, \nabla] = 0,$$

$$[\gamma, \beta_3] + [\nabla, \beta_2] = 0,$$

and so on.

Suppose we have already found β_i's for $i = 2, \ldots, k-1$, which satisfy the first $k-1$ equations. Then the k-th equation looks like:

$$[\gamma, \beta_k] + u_k = 0, \tag{13}$$

eqk

where u_k has an expression in terms of γ, ∇ and $\beta_2, \ldots, \beta_{k-1}$. Moreover, one can check that $[\gamma, u_k] = 0$. In order to find β_k satisfying (13) we need to show that the operator of commutation with γ has no cohomology on the global sections of endomorphisms of M.

Let us check that the super-connection constructed in this way is quasi-isomorphic to F. We have the chain maps $\gamma_0 : M \to \mathcal{A} \otimes F$. The horizontal filtration of M gives a spectral sequence which implies that γ_0 is a quasi-isomorphism. The fact that the cohomology of $\mathcal{A} \otimes F$ is F can be checked locally. Take locally a resolution $E^{\cdot}$ of F by free $\mathcal{O}_X$-modules. Consider the bicomplex $\mathcal{A} \otimes E^{\cdot}$. The comparison of the two spectral sequences of the bicomplex gives the result.

Since the functor Φ is fully faithful it follows that its image is a full triangulated subcategory in $\mathcal{D}^b_{\text{coh}}(X)$ that contains all coherent sheaves. Therefore, the essential image coincides with $\mathcal{D}^b_{\text{coh}}(X)$. This concludes the proof of theorem (3.5).

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