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A RELATION FOR GROMOV-WITTEN INVARIANTS OF LOCAL CALABI-YAU THREEFOLDS

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Abstract. We compute certain *open* Gromov-Witten invariants for toric Calabi-Yau threefolds. The proof relies on a relation for ordinary Gromov-Witten invariants for threefolds under certain birational transformation, and a recent result of Kwokwai Chan.

1. Introduction

The aim of this paper is to compute genus zero *open* Gromov-Witten invariants for toric Calabi-Yau threefolds, through a relation between ordinary local Gromov-Witten invariants of the canonical line bundle K_S of a projective surface S and the canonical bundle K_{S_n} of a blow-up S_n of S at n points.

The celebrated SYZ mirror symmetry was initiated from the work of Strominger-Yau-Zaslow [19]. It successfully explains mirror symmetry when there is no quantum correction [15, 18]. It also works nicely for toric Fano manifolds [5]. Quantum corrections are involved in this case, which are the open Gromov-Witten invariants counting holomorphic disks bounded by Lagrangian torus fibers. Cho and Oh [7] classified such holomorphic disks and computed the mirror superpotential. However, when the toric manifold is not Fano, the moduli of holomorphic disks may contain bubble configurations, leading to a nontrivial obstruction theory. The only known results are the computations of the mirror superpotentials of Hirzebruch surface $\mathbb{F}_2$ by Fukaya-Oh-Ohta-Ono's [9] using their big machinery, and $\mathbb{F}_2$ and $\mathbb{F}_3$ by Auroux's [1] via wall-crossing technique (see also the excellent paper [2]).

Our first main result Theorem 4.5 identifies genus zero open Gromov-Witten invariants with ordinary Gromov-Witten invariants of another Calabi-Yau threefold. As an illustration, we state its corollary for the case of canonical line bundles of toric surfaces, avoiding technical terms at the moment.

Theorem 1.1 (Corollary of Theorem 4.5). *Let S be a smooth toric projective surface with canonical line bundle K_S, which is itself a toric manifold. Let $L \subset K_S$ be a Lagrangian toric fiber, which is a regular fiber of the moment map on K_S equipped with a toric Kähler form. We denote by $\beta \in \pi_2(K_S, L)$ the class represented by a holomorphic disk whose image lies in a fiber of $K_S \to S$. For any class $\alpha \in H_2(S,\mathbb{Z})$ represented by a curve $C \subset S$, we let $\alpha' \in H_2(\tilde{S},\mathbb{Z})$ be the class represented by the proper transform of C, where $\tilde{S}$ is the blow-up of S at one point.*

Let $n_{\beta+\alpha}$ be the one-point genus zero open Gromov-Witten invariant of (K_S, L) (see Equation (4) for its definition), and $\langle 1\rangle^{K_{\tilde{S}}}_{0,0,\alpha'}$ be genus zero Gromov-Witten invariant of $K_{\tilde{S}}$ (see Equation (2) for its definition). Suppose $\tilde{S}$ is Fano. Then

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$$n_{\beta+\alpha} = \langle 1 \rangle^{K_{\tilde{S}}}_{0,0,\alpha'}.$$

This result is used to derive open Gromov-Witten invariants in a recent paper [4] on the SYZ program for toric Calabi-Yau manifolds. We prove Theorem 4.5 using our second main result stated below and a generalized version of Chan's result [3] relating open and closed Gromov-Witten invariants.

Let S be a smooth projective surface and $X = \mathbf{P}(K_S \oplus \mathcal{O}_S) \to S$ be the fiberwise compactification of the canonical line bundle K_S. Let S_n be the blowup of S at n distinct points, and $W = \mathbf{P}(K_{S_n} \oplus \mathcal{O}_{S_n})$ be the fiberwise compactification of K_{S_n}. We relate certain n-point Gromov-Witten invariants of X to Gromov-Witten invariants of W without point condition.

Theorem 1.2. *Let X and W be defined above. Let $h \in H_2(X,\mathbb{Z})$ be the fiber class of $X \to S$ and $\alpha \in H_2(S,\mathbb{Z})$ viewed as a class in $H_2(X,\mathbb{Z})$ via the zero-section embedding $S = \mathbf{P}(0 \oplus \mathcal{O}_S) \to X$. Then for any $n \geq 0$ we have*

$$\langle [\mathrm{pt}], \cdots, [\mathrm{pt}] \rangle^X_{0,n,\alpha+nh} = \langle 1 \rangle^W_{0,0,\alpha'} \tag{1}$$

where $[\mathrm{pt}] \in H^6(X,\mathbb{Z})$ is the Poincaré dual of the point class, and $\alpha' \in H_2(S_n,\mathbb{Z})$ is the proper transform of α.

Now we outline the proof of Theorem 1.2 in the case $n = 1$. Fix a generic fiber H of X. Let x be the intersection point of H with the divisor at infinity $\mathbf{P}(K_S \oplus 0) \subset X$. We construct a birational map $f : X \xleftarrow{\pi_1} \tilde{X} \overset{\pi_2}{\dashrightarrow} W$ so that π_1 is the blowup at x, and π_2 is a simple flop along $\tilde{H}$, which is the proper image of H under π_1. We compare Gromov-Witten invariants of X and W through the intermediate space $\tilde{X}$. Equality (1) follows from the results of Gromov-Witten invariants under birational transformations listed in Section 2.

We remark that Theorem 1.2 is a corollary of Proposition 3.1 which holds for all genera. They can be generalized to the case when K_S is replaced by other local Calabi-Yau threefolds, as we shall explain in Section 3.

This paper is organized as follows. Section 2 serves as a brief review on definitions and results that we need in Gromov-Witten theory. In Section 3 we prove Theorem 1.2 and its generalization to quasi-projective threefolds. In Section 4 we deal with toric Calabi-Yau threefolds and prove Theorem 4.5. Finally in Section 5 we generalize Theorem 1.2 to $\mathbf{P}^n$-bundles over an arbitrary smooth projective variety.

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2. Gromov-Witten invariants under birational maps

In this section we review Gromov-Witten invariants and their transformation under birational maps.

Let X be a smooth projective variety. Let $\overline{M}_{g,n}(X,\beta)$ be the moduli space of stable maps $f:(C;x_1,\cdots x_n)\to X$ with genus $g(C)=g$ and $[f(C)]=\beta\in H_2(X,\mathbb{Z})$. Let $\mathrm{ev}_i:\overline{M}_{g,n}(X,\beta)\to X$ be the evaluation maps at marked points $f\mapsto f(x_i)$. The genus g n-pointed Gromov-Witten invariant for classes $\gamma_i\in H^*(X)$, $i=1,\ldots,n$, is defined as

$$\langle\gamma_1,\cdots,\gamma_n\rangle^X_{g,n,\beta}=\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^n\mathrm{ev}_i^*(\gamma_i).$$

When the expected dimension of $\overline{M}_{g,n}(X,\beta)$ is zero, for instance, when X is a Calabi-Yau threefold and $n=0$, we will be interested primarily in the invariant

$$\langle 1\rangle^X_{g,0,\beta}=\int_{[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}}1 \tag{2}$$

which equals to the degree of the 0-cycle $[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}$ of $\overline{M}_{g,0}(X,\beta)$.

Roughly speaking, the invariant $\langle\gamma_1,\cdots,\gamma_n\rangle^X_{g,n,\beta}$ is a virtual count of genus g curves in the class β which intersect with generic representatives of the Poincaré dual $PD(\gamma_i)$ of γ_i. In particular, if we want to count curves in a homology class β passing through a generic point $x\in X$, we simply take some γ_i to be [pt], the Poincaré dual of a point. In the genus zero case, there is an alternative way to do this counting: let $\pi:\tilde{X}\to X$ be the blow-up of X at one point x; we count curves in the homology class $\pi^!(\beta)-e$, where $\pi^!(\beta)=PD(\pi^*PD(\beta))$ and e is the line class in the exceptional divisor. By the result of Hu [13] (or the result of Gathmann [12] independently), this gives the desired counting:

Theorem 2.1. *([12],[13]) Let $\pi:\tilde{X}\to X$ be the blow-up of X at one point. Let e be the line class in the exceptional divisor. Let $\beta\in H_2(X,\mathbb{Z}),\gamma_1,\cdots,\gamma_n\in H^*(X)$. Then we have*

$$\langle\gamma_1,\cdots,\gamma_n,[\mathrm{pt}]\rangle^X_{0,n+1,\beta}=\langle\pi^*\gamma_1,\cdots,\pi^*\gamma_n\rangle^{\tilde{X}}_{0,n,\pi^!(\beta)-e}$$

where $\pi^!(\beta)=PD(\pi^*PD(\beta))$.

Another result that we need is the transformation of Gromov-Witten invariants under flops.

Let $f:X\dashrightarrow X_f$ be a simple flop between two threefolds along a smooth $(-1,-1)$ rational curve. There is a natural isomorphism

$$\varphi:H_2(X,\mathbb{Z})\longrightarrow H_2(X_f,\mathbb{Z}).$$

Suppose that Γ is an exceptional curve in X and Γ_f is the corresponding exceptional curve in X_f. Then

$$\varphi([\Gamma])=-[\Gamma_f].$$

The following theorem is proved by A.-M. Li and Y.-B. Ruan.

Theorem 2.2. *([17]) Let $f : X \dashrightarrow X_f$ be a simple flop between threefolds and φ be the isomorphism given above. If $\beta \neq m[\Gamma] \in H_2(X, \mathbb{Z})$ for any exceptional curve Γ and $\gamma_i \in H^*(X_f)$, we have*

$$\langle \varphi^*\gamma_1, \cdots, \varphi^*\gamma_n\rangle^X_{g,n,\beta} = \langle \gamma_1, \cdots, \gamma_n\rangle^{X_f}_{g,n,\varphi(\beta)}.$$

3. Gromov-Witten invariants of projectivization of K_S

We are now ready to prove Theorem 1.2 and its generalization to certain quasi-projective threefolds.

Let S be a smooth projective surface. The fiberwise compactification $p : X = \mathbf{P}(K_S \oplus \mathcal{O}_S) \to S$ is a $\mathbf{P}^1$-bundle. We embed S into X as the zero section of the bundle K_S, i.e. $S = \mathbf{P}(0 \oplus \mathcal{O}_S) \subset X$. We denote $S^+ := \mathbf{P}(K_S \oplus 0) \subset X$ be the section at infinity of $p : X \to S$, and let h be the fiber class of p. Then any class $\beta \in H_2(X, \mathbb{Z})$ which is represented by a holomorphic curve can be written as $\alpha + nh$, where n is the intersection number of β with the infinity section S^+, and $p_*(\beta) = \alpha \in H_2(S, \mathbb{Z})$. By Riemann-Roch theorem, the expected dimension of $\overline{M}_{0,n}(X, \beta)$ is $3n$. One has the Gromov-Witten invariant

$$\langle [\mathrm{pt}], \cdots, [\mathrm{pt}]\rangle^X_{0,n,\beta}$$

which counts rational curves in the class β passing through n generic points.

Let $x_1, \cdots, x_n$ be n distinct points in X and $y_i = p(x_i) \in S$. Consider the blow-up $\pi : S_n \to S$ of S along the points $y_1, \cdots, y_n$ with exceptional divisors $e_1, \cdots, e_n$. For $\alpha \in H_2(S, \mathbb{Z})$, we let $\beta' \in H_2(S_n, \mathbb{Z})$ to be the class $\pi^! \alpha - \sum_{i=1}^n e_i$, which is called the strict transform of α. When α is represented by some holomorphic curve C, β' is the class represented by the strict transform of C under the blowup π.

Let $W = \mathbf{P}(K_{S_n} \oplus \mathcal{O}_{S_n})$ be the fiberwise compactification of K_{S_n}. Then β' defined above is a homology class of W since $S_n \subset W$. The moduli space $\overline{M}_{0,0}(W, \beta')$ has expected dimension zero, we get the Gromov-Witten invariant $\langle 1\rangle^W_{0,0,\beta'}$.

Proposition 3.1. *Let S be a smooth projective surface. Denote $p : X = \mathbf{P}(K_S \oplus \mathcal{O}_S) \to S$. Let X_1 be the blowup of X at a point x on the infinity section of $X \to S$. Let $W = \mathbf{P}(K_{S_1} \oplus \mathcal{O}_{S_1})$ where $\pi : S_1 \to S$ is the blowup of S at the point $y = p(x)$. Then W is a simple flop of X_1 along the proper transform $\tilde{H}$ of the fiber H through x.*

Proof. Since $\tilde{H}$ is the proper transform of H under the blowup $\pi_1 : X_1 \to X$ at x, $\tilde{H}$ is isomorphic to $\mathbf{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We have a simple flop $f : X_1 \dashrightarrow X'$ along $\tilde{H}$. Next we show that $X' \cong W$. To this end, we use an alternative way to describe the birational map $f\pi_1^{-1} : X \dashrightarrow X'$.

It is well known that a simple flop f is a composite of a blowup and a blowdown. Let $\pi_2 : X_2 \to X_1$ be the blowup of X_1 along $\tilde{H}$ with exceptional divisor $E_2 \cong \tilde{H} \times \mathbf{P}^1$. Because the restriction of normal bundle of E_2 to $\tilde{H}$ is $\mathcal{O}(-1)$, we can blow down X_2 along the $\tilde{H}$ fiber direction of E_2 to get $\pi_3 : X_2 \to X'$. Of course we have $f = \pi_3\pi_2^{-1}$ and $\pi_3\pi_2^{-1}\pi_1^{-1} : X \dashrightarrow X'$.

Notice that the composite $\pi_2^{-1}\pi_1^{-1} : X \dashrightarrow X_2$ can be written in another way. Let $\rho_1 : Z_1 \to X$ be the blowup of X along H with exceptional divisor E'. Let F be the inverse image $\rho^{-1}(x)$. Then $F \cong \mathbf{P}^1$. Next we blow up Z_1 along F to get $\rho_2 : Z_2 \to Z_1$. It is straightforward to verify that $Z_2 = X_2$ and $\rho_1\rho_2 = \pi_1\pi_2$.

Thus we have $\pi_3\pi_2^{-1}\pi_1^{-1} = \pi_3(\rho_1\rho_2)^{-1} : X \dashrightarrow X'$, from which it follows easily that $X' \cong W$. $\square$

Corollary 3.2. *With notations as in the Proposition, and let e_1 be the exceptional curve class of π, we have*

$$\langle 1\rangle^{X_1}_{g,0,\beta} = \langle 1\rangle^{W}_{g,0,\beta'} \tag{3}$$

where $\beta = \alpha + k\tilde{H}$ and $\beta' = \pi^!\alpha - ke_1$ for any nonzero $\alpha \in H_2(S,\mathbb{Z})$.

Proof. From the Proposition, we know there is a flop $f : X_1 \dashrightarrow W$. Applying Theorem 2.2 to the flop f, since $\varphi([\tilde{H}]) = -e_1$, we get

$$\varphi(\beta) = \varphi((\pi_1^!\alpha) + [k\tilde{H}]) = \pi^!\alpha - ke_1 = \beta'.$$

Then (3) follows directly. $\square$

When S_1 is a Fano surface, K_{S_1} is a local Calabi-Yau threefold and curves inside S_1 can not be deformed away from S_1. Indeed any small neighborhood N_{S_1} of S_1 (resp. $N_{S\cup C}$ of $S\cup C$) inside any Calabi-Yau threefold has the same property. Here C is a $(-1,-1)$-curve which intersects S transversely at a single point. Therefore we can define local Gromov-Witten invariants for N_{S_1} and $N_{S\cup C}$. Using a canonical identification,

$$H_2(S_1) \simeq H_2(S) \oplus \mathbb{Z}\langle e_1\rangle \simeq H_2(S\cup C),$$

the above corollary implies that the local Gromov-Witten invariants for local Calabi-Yau threefolds N_{S_1} and $N_{S\cup C}$ are the same. When the homology class in S_1 does not have e_1-component, this becomes simply the local Gromov-Witten invariants for N_S. This last relation for Gromov-Witten invariants of N_{S_1} and N_S was pointed out to us by J.-X. Hu [14] and he proved this result by the degeneration method. This relationship was first observed by Chiang-Klemm-Yau-Zaslow [6] in the case S is $\mathbf{P}^2$ and genus is zero by explicit calculations.

These results can be generalized to the case when K_S is replaced by other local Calabi-Yau threefolds. The illustration of such a generalization is given at the end of this section.

Now we prove Theorem 1.2, that is

$$\langle [\mathrm{pt}],\cdots,[\mathrm{pt}]\rangle^X_{0,n,\beta} = \langle 1\rangle^W_{0,0,\beta'}.$$

Proof of Theorem 1.2. First we assume $n=1$, that is, $\pi : S_1 \to S$ is a blowup of S at one point y with exceptional curve class e_1 and $W = \mathbf{P}(K_{S_1}\oplus\mathcal{O}_{S_1})$. We need to show that

$$\langle [\mathrm{pt}]\rangle^X_{0,1,\beta} = \langle 1\rangle^W_{0,0,\beta'},$$

where $\beta = \alpha + h$ and $\beta' = \pi^!\alpha - e_1$.

Applying Theorem 2.1 to $\pi_1 : X_1 \to X$, and notice that

$$\pi_1^!(\beta) - e = \pi_1^!(\alpha + h) - e = \pi_1^!\alpha + [\tilde{H}],$$

which we denote by β_1, we then have $\langle [\mathrm{pt}]\rangle^X_{0,1,\beta} = \langle 1\rangle^{X_1}_{0,0,\beta_1}$. Next we apply Proposition 3.1 for $k=1$, we get

$$\langle 1\rangle^{X_1}_{0,0,\beta_1} = \langle 1\rangle^W_{0,0,\beta'},$$

which proves the result for $n=1$.

For $n > 1$, we simply apply the above procedure successively. $\square$

In particular, when $S = \mathbf{P}^2$ and $n = 1$, S_1 is the Hirzebruch surface $\mathbb{F}_1$. We use ℓ to denote the line class of $\mathbf{P}^2$. The class of exceptional curve e represents the unique minus one curve in $\mathbb{F}_1$ and $f = \pi^! \ell - e$ is its fiber class. In this case, the corresponding class $\beta' = k\pi^! \ell - e = (k-1)e + kf$. The values of $N_{0,\beta'}$ have been computed in [6]. Starting with $k = 1$, they are $-2, 5, -32, 286, -3038, 35870$. (See Table 1.)

	b	0	1	2	3	4	5	6
a								
0			-2	0	0	0	0	0
1		1	3	5	7	9	11	13
2		0	0	-6	-32	-110	-288	-644
3		0	0	0	27	286	1651	6885
4		0	0	0	0	-192	-3038	-25216
5		0	0	0	0	0	1695	35870
6		0	0	0	0	0	0	-17064

TABLE 1. Invariants of $K_{\mathbb{F}_1}$ for classes $ae + bf$

We remark that Theorem 1.2 can be generalize to quasi-projective threefolds with properties we describe below. Let X be a smooth quasi-projective threefold. Assume there is a distinguished Zariski open subset $U \subset X$, so that U is isomorphic to the canonical line bundle K_S over a smooth projective surface S, and there is a Zariski open subset $S' \subset S$, so that each fiber F of K_S over S' is closed in X. Typical examples of such threefolds include a large class of toric Calabi-Yau threefolds.

Theorem 1.2 still holds for such threefolds, provided that the blow-up of the surface S mentioned above at a generic point is Fano. Since we will not use this generalization in the paper, we only sketch the proof.

First we construct a partial compactification $\bar{X}$ of X. Given a generic point $x \in U$, we have a unique fiber through x, say H. Let $\{y\} = H \cap S$. Take a small open neighborhood $y \in V$, we compactify K_V along the fiber by adding a section at infinity as we did before. We call the resulting variety by $\bar{X}$.

The Gromov-Witten invariant $\langle [\mathrm{pt}] \rangle^{\bar{X}}_{0,1,\beta}$ is well defined. Indeed, let $\beta \in H_2(\bar{X}, \mathbb{Z})$; and suppose $\beta = \alpha + [H]$ for some α in $H_2(S, \mathbb{Z})$. The moduli space of genus zero stable maps to $\bar{X}$ representing β and passing through the generic point x is compact since S is Fano. Then the invariants can be defined as before.

To show the equality $\langle [\mathrm{pt}] \rangle^{\bar{X}}_{0,1,\beta} = \langle 1 \rangle^{\tilde{S}}_{0,0,\beta'}$, we construct a birational map $f : \bar{X} \dashrightarrow W$ as in the proof of Theorem 1.2. Let $\tilde{S} \subset W$ be the image of S. Then $\tilde{S}$ is the blowup of S at y. Let $\beta' \in H_2(\tilde{S}, \mathbb{Z})$ be the strict transform of α. Since $\tilde{S}$ is Fano, we can define local Gromov-Witten invariant $\langle 1 \rangle^{\tilde{S}}_{0,0,\beta'}$. The equality follows directly as in the proof of Theorem 1.2.

4. Toric Calabi-Yau threefolds

In this section we study open Gromov-Witten invariants of a toric Calabi-Yau manifold and prove our main Theorem 4.5. As an application, we show that certain open Gromov-Witten invariants for toric Calabi-Yau threefolds can be computed via local mirror symmetry.

First we recall the standard notations. Let N be a lattice of rank 3, M be its dual lattice, and Σ_0 be a strongly convex simplicial fan supported in $N_{\mathbb{R}}$, giving rise to a toric variety $X_0 = X_{\Sigma_0}$. (Σ_0 is 'strongly convex' means that its support $|\Sigma_0|$ is convex and does not contain a whole line through the origin.) Denote by $v_i \in N$ the primitive generators of rays of Σ_0, and denote by D_i the corresponding toric divisors for $i = 0, \ldots, m-1$, where $m \in \mathbb{Z}_{\geq 3}$ is the number of such generators.

Calabi-Yau condition for X_0: There exists $\underline{\nu} \in M$ such that $(\underline{\nu}\,,\, v_i) = 1$ for all $i = 0, \ldots, m-1$.

By fixing a toric Kaehler form ω on X_0, we have a moment map $\mu : X_0 \to P_0$, where $P_0 \subset M_{\mathbb{R}}$ is a polyhedral set defined by a system of inequalities

$$(v_j\,,\,\cdot) \geq c_j$$

for $j = 0, \ldots, m-1$ and suitable constants $c_j \in \mathbb{R}$. (Figure 2 shows two examples of toric Calabi-Yau varieties.)

Let $L \subset X_0$ be a regular fiber of μ, and $\pi_2(X_0, L)$ be the group of disk classes. For $b \in \pi_2(X_0, L)$, the most important classical quantities are the area $\int_b \omega$ and the Maslov index $\mu(b)$. By [7], $\pi_2(X_0, L)$ is generated by basic disk classes β_i for $i = 0, \ldots, m-1$, where each β_i corresponds to the ray generated by v_i.

Other than these two classical quantities, one has the one-pointed genus-zero open Gromov-Witten invariant associated to b defined by Fukaya-Oh-Ohta-Ono [8] as follows. Let $\overline{M}_1(X_0, b)$ be the moduli space of stable maps from bordered Riemann surfaces of genus zero with one boundary marked point to X_0 in the class b, and denote by $[\overline{M}_1(X_0, b)]$ its virtual fundamental class. One has the evaluation map $\mathrm{ev} : \overline{M}_1(X_0, b) \to L$. The one-pointed open Gromov-Witten invariant associated to b is defined as

$$n_b := \int_{[\overline{M}_1(X_0,b)]} \mathrm{ev}^*[\mathrm{pt}] \tag{4}$$

where $[\mathrm{pt}] \in H^n(L)$ is the Poincaré dual of a point in L. Since the expected dimension of $\overline{M}_1(X_0, b)$ is $\mu(b)+n-2$ and $\mathrm{ev}^*[\mathrm{pt}]$ is of degree n, n_b is non-zero only when $\mu(b) = 2$.

To investigate genus zero open Gromov-Witten invariants of a toric Calabi-Yau manifold X_0, we'll need the following simple lemma for rational curves in toric varieties:

Lemma 4.1. *Let Y be a toric variety which admits $\nu \in M$ such that ν defines a holomorphic function on Y whose zeros contain all toric divisors of Y. Then the image of any non-constant holomorphic map $u : \mathbf{P}^1 \to Y$ lies in the toric divisors of Y. In particular this holds for a toric Calabi-Yau variety.*

Proof. Denote the holomorphic function corresponding to $\nu \in M$ by f. Then $f \circ u$ gives a holomorphic function on $\mathbf{P}^1$, which must be a constant by maximal principle. $f \circ u$ cannot be constantly non-zero, or otherwise the image of u lies in $(\mathbb{C}^\times)^n \subset Y$, forcing u to be constant. Thus $f \circ u \equiv 0$, implying the image of u lies in the toric divisors of Y.

For a toric Calabi-Yau variety X_0, $(\underline{\nu}\,,\, v_i) = 1 > 0$ for all $i = 0, \ldots, m-1$ implies that the meromorphic function corresponding to $\underline{\nu}$ indeed has no poles. □

As a consequence to the above lemma, we have the following

Proposition 4.2. *Assume the notations introduced above. For a disk class $b \in \pi_2(X_0, L)$ which has Maslov index two, $\overline{M}_1(X_0, b)$ is empty unless*

(1) *$b = \beta_i$ for some i; or*
(2) *$b = \beta_i + \alpha$, where the corresponding toric divisor D_i is compact and $\alpha \in H_2(X_0, \mathbb{Z})$ is represented by a rational curve.*

Proof. By Theorem 11.1 of [8], $\overline{M}_1(X_0, b)$ is empty unless $b = \sum_i k_i\beta_i + \sum_j \alpha_j$ where $k_i \in \mathbb{Z}_{\geq 0}$ and each $\alpha_j \in H_2(X_0, \mathbb{Z})$ is realized by a holomorphic sphere. Since X_0 is Calabi-Yau, every α_j has Chern number zero. Thus

$$2 = \mu(b) = \sum_i k_i \mu(\beta_i) = \sum_i 2k_i$$

where $\mu(b)$ denotes the Maslov index of b. Thus $b = \beta_i + \alpha$ for some $i = 0, \ldots, m-1$ and $\alpha \in H_2(X_0, \mathbb{Z})$ is realized by some chains Q of non-constant holomorphic spheres in X_0.

Now suppose that $\alpha \neq 0$, and so Q is not a constant point. By Lemma 4.1, Q must lie inside $\bigcup_{i=0}^{m-1} D_i$. Q must have non-empty intersection with the holomorphic disk representing $\beta_i \in \pi_2(X_0, L)$ for generic L, implying some components of Q lie inside D_i and have non-empty intersection with the torus orbit $(\mathbb{C}^\times)^2 \subset D_i$. But if D_i is non-compact, then the fan of D_i (as a toric manifold) is simplicial convex incomplete, and so D_i is a toric manifold satisfying the condition of Lemma 4.1. Then Q has empty intersection with the open orbit $(\mathbb{C}^\times)^2 \subset D_i$, which is a contradiction. □

It was shown in [7, 8] that $n_b = 1$ for basic disc classes $b = \beta_i$. The remaining task is to compute n_b for $b = \beta_i + \alpha$ with nonzero $\alpha \in H_2(X_0)$. In this section we prove Theorem 4.5, which relates n_b to certain closed Gromov-Witten invariants, which can then be computed by usual localization techniques.

Suppose we would like to compute n_b for $b = \beta_i + \alpha$, and without loss of generality let's take $i = 0$ and assume that D_0 is a compact toric divisor. We construct a toric compactification X of X_0 as follows. Let v_0 be the primitive generator corresponding to D_0, and we take Σ to be the refinement of Σ_0 by adding the ray generated by $v_\infty := -v_0$ (and then completing it into a convex fan). We denote by $X = X_\Sigma$ the corresponding toric variety, which is a compactification of X_0. We denote by $h \in H_2(X, \mathbb{Z})$ the fiber class of X, which has the property that $h \cdot D_0 = h \cdot D_\infty = 1$ and $h \cdot D = 0$ for all other irreducible toric divisors D. Then for $\alpha \in H_2(X_0, \mathbb{Z})$, we have the ordinary Gromov-Witten invariant $\langle [\mathrm{pt}] \rangle^X_{0,1,h+\alpha}$.

When $X_0 = K_S$ for a toric Fano surface S and D_0 is the zero section of $K_S \to S$, by comparing the Kuranishi structures on moduli spaces, it was shown by K.-W. Chan [3] that the open Gromov-Witten invariant n_b indeed agrees with the closed Gromov-Witten invariant $\langle [\mathrm{pt}] \rangle^X_{0,1,h+\alpha}$:

Proposition 4.3 ([3])**.** *Let $X_0 = K_S$ for a toric Fano surface S and X be the fiberwise compactification of X_0. Let $b = \beta_i + \alpha$ with $\beta_i \cdot S = 1$ and $\alpha \in H_2(S, \mathbb{Z})$. Then*

$$n_b = \langle [\mathrm{pt}] \rangle^X_{0,1,h+\alpha}.$$

Indeed his proof extends to our setup without much modification, and for the sake of completeness we show how it works:

Proposition 4.4 (slightly modified from [3])**.** *Let X_0 be a toric Calabi-Yau manifold and X be its compactification constructed above. Let $b = \beta_i + \alpha$ with $\beta_i \cdot S = 1$ and $\alpha \in H_2(S, \mathbb{Z})$, and we assume that all rational curves in X representing α are contained in X_0. Then*

$$n_b = \langle [\mathrm{pt}] \rangle^X_{0,1,h+\alpha}.$$

Proof. For notation simplicity let $M_{\mathrm{op}} := \overline{M}_1(X_0, b)$ be the open moduli and $M_{\mathrm{cl}} := \overline{M}_1(X, h+\alpha)$ be the corresponding closed moduli. By evaluation at the marked point we have a $\mathbf{T}$-equivariant fibration

$$\mathrm{ev} : M_{\mathrm{op}} \to \mathbf{T}$$

whose fiber at $p \in \mathbf{T} \subset X_0$ is denoted as $M^{\mathrm{ev}=p}_{\mathrm{op}}$. Similarly we have a $\mathbf{T}_{\mathbb{C}}$-equivariant fibration

$$\mathrm{ev} : M_{\mathrm{cl}} \to \bar{X}$$

whose fiber is $M^{\mathrm{ev}=p}_{\mathrm{cl}}$. By the assumption that all rational curves in X representing α is contained in X_0, one has

$$M^{\mathrm{ev}=p}_{\mathrm{op}} = M^{\mathrm{ev}=p}_{\mathrm{cl}}.$$

There is a Kuranishi structure on $M^{\mathrm{ev}=p}_{\mathrm{cl}}$ which is induced from that on M_{cl} (please refer to [11] and [10] for the definitions of Kuranishi structures). Transversal multisections of the Kuranishi structures give the virtual fundamental cycles $[M_{\mathrm{op}}] \in H_n(X_0, \mathbb{Q})$ and $[M^{\mathrm{ev}=p}_{\mathrm{op}}] \in H_0(\{p\}, \mathbb{Q})$. In the same way we obtain the virtual fundamental cycles $[M_{\mathrm{cl}}] \in H_{2n}(X, \mathbb{Q})$ and $[M^{\mathrm{ev}=p}_{\mathrm{cl}}] \in H_0(\{p\}, \mathbb{Q})$. By taking the multisections to be $\mathbf{T}_{\mathbb{C}}$- ($\mathbf{T}$-) equivariant so that their zero sets are $\mathbf{T}_{\mathbb{C}}$- ($\mathbf{T}$-) invariant,

$$\deg[\overline{M}^{\mathrm{ev}=p}_{\mathrm{cl/op}}] = \deg[\overline{M}_{\mathrm{cl/op}}]$$

and thus it remains to prove that the Kuranishi structures on $M^{\mathrm{ev}=p}_{\mathrm{cl}}$ and $M^{\mathrm{ev}=p}_{\mathrm{op}}$ are the same.

Let $[u_{\mathrm{cl}}] \in M^{\mathrm{ev}=p}_{\mathrm{cl}}$, which corresponds to an element $[u_{\mathrm{op}}] \in M^{\mathrm{ev}=p}_{\mathrm{op}}$. $u_{\mathrm{cl}} : (\Sigma, q) \to X$ is a stable holomorphic map with $u_{\mathrm{cl}}(q) = p$. Σ can be decomposed as $\Sigma_0 \cup \Sigma_1$, where $\Sigma_0 \cong \mathbf{P}^1$ such that $u_*[\Sigma_0]$ represents h, and $u_*[\Sigma_1]$ represents α. Similarly the domain of u_{op} can be docomposed as $\Delta \cup \Sigma_1$, where $\Delta \subset \mathbb{C}$ is the closed unit disk.

We have the Kuranishi chart $(V_{\mathrm{cl}}, E_{\mathrm{cl}}, \Gamma_{\mathrm{cl}}, \psi_{\mathrm{cl}}, s_{\mathrm{cl}})$ around $u_{\mathrm{cl}} \in M^{\mathrm{ev}=p}_{\mathrm{cl}}$, where we recall that $E_{\mathrm{cl}} \oplus \mathrm{Im}(D_{u_{\mathrm{cl}}}\bar{\partial}) = \Omega^{(0,1)}(\Sigma, u^*_{\mathrm{cl}}TX)$ and $V_{\mathrm{cl}} = \{\bar{\partial} f \in E; f(q) = p\}$. On the other hand let $(V_{\mathrm{op}}, E_{\mathrm{op}}, \Gamma_{\mathrm{op}}, \psi_{\mathrm{op}}, s_{\mathrm{op}})$ be the Kuranishi chart around $u_{\mathrm{op}} \in M^{\mathrm{ev}=p}_{\mathrm{op}}$.

Now comes the key: since the obstruction space for the deformation of $u_{\mathrm{cl}}|_{\Sigma_0}$ is 0, E_{cl} is of the form $0 \oplus E' \subset \Omega^{(0,1)}(\Sigma_0, u_{\mathrm{cl}}|^*_{\Sigma_0}TX) \times \Omega^{(0,1)}(\Sigma_1, u_{\mathrm{cl}}|^*_{\Sigma_1}TX)$. Similarly E_{op} is of the form $0 \oplus E'' \subset \Omega^{(0,1)}(\Delta, u_{\mathrm{op}}|^*_{\Delta}TX) \times \Omega^{(0,1)}(\Sigma_1, u_{\mathrm{op}}|^*_{\Sigma_1}TX)$. But since $D_{u_{\mathrm{cl}}|_{\Sigma_1}}\bar{\partial} = D_{u_{\mathrm{op}}|_{\Sigma_1}}\bar{\partial}$, E' and E'' can be taken as the same subspace! Once we do this, it is then routine to see that $(V_{\mathrm{cl}}, E_{\mathrm{cl}}, \Gamma_{\mathrm{cl}}, \psi_{\mathrm{cl}}, s_{\mathrm{cl}}) = (V_{\mathrm{op}}, E_{\mathrm{op}}, \Gamma_{\mathrm{op}}, \psi_{\mathrm{op}}, s_{\mathrm{op}})$. □

Theorem 4.5. *Let X_0 be a toric Calabi-Yau threefold and denote by S the union of its compact toric divisors. Let L be a Lagrangian torus fiber and $b = \beta + \alpha \in \pi_2(X_0, L)$,* *where $\alpha \in H_2(S)$ is represented by a rational curve and $\beta \in \pi_2(X_0, L)$ is one of the basic disk classes.*

Given this set of data, there exists a toric Calabi-Yau threefold W_0 with the following properties:

(1) *W_0 is birational to X_0.*
(2) *Let $S_1 \subset W_0$ be the union of compact divisors of W_0. Then S_1 is the blowup of S at one point.*
(3) *Denote by $\alpha' \in H_2(S_1)$ the class of strict transform of the rational curve representing $\alpha \in H_2(S)$. Assume that every rational curve representative of α' in W_0 lies in S_1. Then the open Gromov-Witten invariant n_b of (X_0, L) is equal to the ordinary Gromov-Witten invariant $\langle 1 \rangle_{0,0,\alpha'}^{W_0}$ of W_0, that is,*

$$n_b = \langle 1 \rangle_{0,0,\alpha'}^{W_0}$$

In particular for $X_0 = K_S$, W_0 is $K_{\tilde{S}}$ by this construction, and so we obtain Theorem 1.1 as its corollary.

Proof. We first construct the toric variety W_0. To begin with, let D_∞ be the toric divisor corresponding to v_∞. Let $x \in X$ be one of the torus-fixed points contained in D_∞. First we blow up x to get X_1, whose fan Σ_1 is obtained by adding the ray generated by $w = v_\infty + u_1 + u_2$ to Σ, where v_∞, u_1 and u_2 are the normal vectors to the three facets adjacent to x. There exists a unique primitive vector $u_0 \neq w$ such that $\{u_0, u_1, u_2\}$ generates a simplicial cone in Σ_1 and u_0 corresponds to a compact toric divisor of X_1: If $\{v_0, u_1, u_2\}$ spans a cone of Σ_1, then take $u_0 = v_0$; otherwise since Σ_1 is simplicial, there exists a primitive vector $u_0 \subset \mathbb{R}\langle v_0, u_1, u_2\rangle$ with the required property. Now $\langle u_1, u_2, w\rangle_{\mathbb{R}}$ and $\langle u_1, u_2, u_0\rangle_{\mathbb{R}}$ form two adjacent simplicial cones in Σ_1, and we may employ a flop to obtain a new toric variety W, whose fan Σ_W contains the adjacent cones $\langle w, u_0, u_1\rangle_{\mathbb{R}}$ and $\langle w, u_0, u_2\rangle_{\mathbb{R}}$. (See Figure 1).

FIGURE 1. A flop.

W is the compactification of another toric Calabi-Yau W_0 whose fan is constructed as follows: First we add the ray generated by w to Σ_0, and then we flop the adjacent cones $\langle w, u_1, u_2\rangle$ and $\langle u_0, u_1, u_2\rangle$. W_0 is Calabi-Yau because

$$(\underline{\nu}\,,\, w) = 1$$

and a flop preserves this Calabi-Yau condition. Σ_W is recovered by adding the ray generated by v_∞ to the fan Σ_{W_0}.

Now we analyze the transform of classes under the above construction. The class $h \in H_2(X, \mathbb{Z})$ can be written as $h' + \delta$, where $h' \in H_2(X, \mathbb{Z})$ is the class corresponding to the cone $\langle u_1, u_2 \rangle_{\mathbb{R}}$ of Σ and $\delta \in H_2(X_0, \mathbb{Z})$. Let $h'' \in H_2(X_1, \mathbb{Z})$ be the class corresponding to $\{u_1, u_2\} \subset \Sigma_1$, which is flopped to $e \in H_2(W, \mathbb{Z})$ corresponding to the cone $\langle w, u_0 \rangle_{\mathbb{R}}$ of Σ_W. Finally let $\tilde{\delta}, \tilde{\alpha} \in H_2(W, \mathbb{Z})$ be classes corresponding to $\delta, \alpha \in H_2(X_1, \mathbb{Z})$ respectively under the flop. Then $\alpha' = \tilde{\delta} + \tilde{\alpha} - e$ is actually the strict transform of α.

Applying Proposition 4.4 and Theorem 1.2, we obtain the equality

$$n_b = \langle 1 \rangle^{W_0}_{0,0,\alpha'}.$$

□

Finally we give an example to illustrate the open Gromov-Witten invariants.

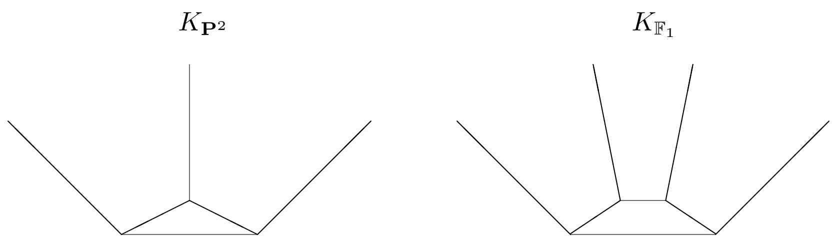

FIGURE 2. Polytope picture for $K_{\mathbf{P}^2}$ and $K_{\mathbb{F}_1}$.

Example 4.6. *Let $X_0 = K_{\mathbf{P}^2}$. There is exactly one compact toric divisor D_0 which is the zero section of $X_0 \to \mathbf{P}^2$. The above construction gives $W_0 = K_{\mathbb{F}_1}$. (Figure 2). Let $\alpha = kl \in H_2(X_0, \mathbb{Z})$, where l is the line class of $\mathbf{P}^2 \subset K_{\mathbf{P}^2}$ and $k > 0$. By Theorem 4.5,*

$$n_{\beta_0 + kl} = \langle 1 \rangle^{W_0}_{0,0,kl-e} = \langle 1 \rangle^{W_0}_{0,0,kf+(k-1)e}$$

where e is the exceptional class of $\mathbb{F}_1 \subset K_{\mathbb{F}_1}$ and f is the fiber class of $\mathbb{F}_1 \to \mathbf{P}^1$. The first few values of these local invariants for $K_{\mathbb{F}_1}$ are listed in Table 1.

5. A generalization to $\mathbf{P}^n$-bundles

In this section we generalize Theorem 1.2 to higher dimensions, that is, to $\mathbf{P}^n$-bundles over an arbitrary smooth projective variety.

Let X be an n-dimensional smooth projective variety. Let F be a rank r vector bundle over X with $1 \leq r < n$. Let $p : W = \mathbf{P}(F \oplus \mathcal{O}_X) \to X$ be a $\mathbf{P}^r$-bundle over X. There are two canonical subvarieties of W, say $W_0 = \mathbf{P}(0 \oplus \mathcal{O}_X)$ and $W_\infty = \mathbf{P}(F \oplus 0)$. We have $W_0 \cong X$.

Let $S \subset X$ be a smooth closed subvariety of codimension $r+1$ with normal bundle N. Let $\pi : \tilde{X} \to X$ be the blowup of X along S with exceptional divisor $E = \mathbf{P}(N)$. Then $F' = \pi^* F \otimes \mathcal{O}_{\tilde{X}}(E)$ is a vector bundle of rank r over $\tilde{X}$. Similar to $p : W \to X$, we let $p' : W' = \mathbf{P}(F' \oplus \mathcal{O}_{\tilde{X}}) \to \tilde{X}$.

It is easy to see that W and W' are birational. We shall construct an explicit birational map $g : W \dashrightarrow W'$. It induces a homomorphism between groups

$$g' : H_2(W, \mathbb{Z}) \to H_2(W', \mathbb{Z}).$$

Let $\beta = h + \alpha \in H_2(W, \mathbb{Z})$ with h the fiber class of W and $\alpha \in H_2(X, \mathbb{Z})$. Then we establish a relation between certain Gromov-Witten invariants of W and W'.

Proposition 5.1. *Let $Y = \mathbf{P}(F_S \oplus 0) \subset W$. For $g : W \dashrightarrow W'$, we have*

$$\langle \gamma_1, \gamma_2, \cdots, \gamma_{m-1}, PD([Y]) \rangle^W_{0,m,\beta} = \langle \gamma'_1, \cdots, \gamma'_{m-1} \rangle^{W'}_{0,m-1,\beta'}.$$

Here γ'_i is the image of γ_i under $H^*(W) \to H^*(W')$ and $\beta' = g'(\beta)$.

The birational map $g : W \dashrightarrow W'$ we shall construct below can be factored as

$$W \overset{\pi_1^{-1}}{\dashrightarrow} \tilde{W} \overset{f}{\dashrightarrow} W'$$

Here $\pi_1 : \tilde{W} \to W$ is a blowup along a subvariety Y. We make the following assumption:

(A) Let $\beta = h + \alpha \in H_2(W, \mathbb{Z})$ with h the fiber class of W and $\alpha \in H_2(X, \mathbb{Z})$. Every curve C in class β can be decomposed uniquely as $C = H \cup C'$ with H a fiber and C' a curve in X.

It follows that the intersection of C and Y is at most one point. Under this assumption we generalize Theorem 2.1 in a straightforward manner as follows.

Proposition 5.2. *Let the notation be as above. Let E' be the exceptional divisor of π_1. Let e be the line class in the fiber of $E' \to Y$. Suppose the assumption (A) holds, we have*

$$\langle \gamma_1, \gamma_2, \cdots, \gamma_{m-1}, PD([Y]) \rangle^W_{0,m,\beta} = \langle \tilde{\gamma}_1, \cdots, \tilde{\gamma}_{m-1} \rangle^{\tilde{W}}_{0,m-1,\beta_1},$$

where $\tilde{\gamma}_i = \pi_1^* \gamma_i$ and $\beta_1 = \pi^!(\beta) - e$.

The proof of Proposition 5.1 is similar to that of Theorem 1.2.

Proof of Proposition 5.1. Since $g = f\pi_1^{-1}$, applying Proposition 5.2, it suffices to show

$$\langle \tilde{\gamma}_1, \cdots, \tilde{\gamma}_{m-1} \rangle^{\tilde{W}}_{0,m-1,\beta_1} = \langle \gamma'_1, \cdots, \gamma'_{m-1} \rangle^{W'}_{0,m-1,\beta'}$$

for the ordinary flop $f : \tilde{W} \dashrightarrow W'$.

Recall that Y-P. Lee, H-W. Lin and C-L. Wang [16] proved that for an ordinary flop $f : M \dashrightarrow M_f$ of splitting type, the big quantum cohomology rings of M and M_f are isomorphic. In particular, their Gromov-Witten invariants for the corresponding classes are the same. Therefore, the above identity follows. □

In the rest of the section we construct the birational map $g : W \dashrightarrow W'$ in two equivalent ways.

Recall that $S \subset X$ is a subvariety. Let $p_S : Z = W \times_X S \to S$ be the restriction of $p : W \to X$ to S. Then $Z = \mathbf{P}(F_S \oplus \mathcal{O}_S)$ with F_S the restriction of F to S. We denote $Y = Z \cap W_\infty = \mathbf{P}(F_S \oplus 0)$, and $q : Y \to S$ the restriction of p_S to Y. Since Y is a projective bundle over S, we let $\mathcal{O}_{Y/S}(-1)$ be the tautological line bundle over Y. The normal bundle of Y in Z is $N_{Y/Z} = \mathcal{O}_{Y/S}(1)$.

We start with the first construction of g. Let $\pi_1 : \tilde{W} \to W$ be the blowup of W along Y. Since the normal bundle $N_{Y/W}$ is equal to $N_{Y/Z} \oplus N_{Y/W_\infty} = \mathcal{O}_{Y/S}(1) \oplus q^*N$, the exceptional divisor of π_1 is

$$E' = \mathbf{P}(\mathcal{O}_{Y/S}(1) \oplus q^*N).$$

Let $\tilde{Z}$ be the proper transform of Z and $\tilde{Y} = \tilde{Z} \cap E'$. The normal bundle of $\tilde{Z}$ in $\tilde{W}$ is $\tilde{N} = p_S^*N \otimes \mathcal{O}_{\tilde{Z}}(-\tilde{Y})$.

Because $Z' \cong Z$ is a $\mathbf{P}^r$-bundle over S, and the restriction of $\tilde{N}$ to each $\mathbf{P}^r$-fiber of $\tilde{Z}$ is isomorphic to $\mathcal{O}(-1)^{\oplus r+1}$, we have an ordinary $\mathbf{P}^r$-flop $f : \tilde{W} \dashrightarrow \tilde{W}_f$ along $\tilde{Z}$. It can be verified that $\tilde{W}_f = W'$ after decomposing f as a blowup and a blowdown. Finally we simply define g as the composite $f\pi_1^{-1} : W \dashrightarrow W'$.

We describe the second construction of g, from which it is easy to see the relation $\tilde{W}_f = W'$.

We let $\rho_1 : W_1 \to W$ be the blowup of W along Z whose exceptional divisor is denoted by E_1. Because the normal bundle of Z in W is q^*N for $q : Z \to S$, we know

$$E_1 = \mathbf{P}(q^*N) \cong Z \times_S \mathbf{P}(N) = Z \times_S E.$$

Indeed, W_1 is isomorphic to the $\mathbf{P}^r$-bundle $\mathbf{P}(F_1 \oplus \mathcal{O}_{\tilde{X}})$ over $\tilde{X}$ with $F_1 = \pi^*F$. Let Y_1 be the inverse image of Y. Now we let $\rho_2 : W_2 \to W_1$ be the blowup of W_1 along Y_1 with exceptional divisor E_2. Let E_1' be the proper transform of E_1 and $Y_2 = E_1' \cap E_2$. Notice that $E_1' \cong E_1$, and the normal bundle of E_1 is $N_1 = q^*N \boxtimes \mathcal{O}_{E/S}(-1)$, we know the normal bundle of E_1' is $N_1' = N_1 \otimes \mathcal{O}_{E_1'}(-Y_2)$.

Since $E_1' \cong Z \times_S E$ is a $\mathbf{P}^r \times \mathbf{P}^r$-bundle over S, composed with the projection $Z \times_S E \to E$, we see that $E_1' \to E$ is a $\mathbf{P}^r$-bundle. Because the restriction of N_1' to the $\mathbf{P}^r$-fiber of $E_1' \to E$ is isomorphic to $\mathcal{O}(-1)^{\oplus r+1}$, we can blowdown W_2 along these fibers of E_1' to get $\pi_3 : W_2 \to W_3 = \tilde{W}_f$. From this description it is easy to see that $W_3 = W'$.

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