THE RIEMANN-ROCH THEOREM WITHOUT DENOMINATORS IN MOTIVIC HOMOTOPY THEORY

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Abstract. The Riemann-Roch theorem without denominators for the Chern class maps on higher algebraic K-groups with values in motivic cohomology groups in the context of motivic homotopy theory is proved.

1. Introduction

In this article, the Riemann-Roch theorem without denominators for the Chern class maps on higher algebraic K-groups with values in motivic cohomology groups in the context of motivic homotopy theory is proved. The Riemann-Roch theorem without denominators for the Chern class maps on the K_0-group of schemes with values in the Chow groups was stated (in full generality) by Grothendieck [SGA6, Exposé XIV, (3.1), p. 670] and was proved in full by Jouanolou [Jo]. Gillet [Gi] extended the theorem (also given the same name) to the Chern class maps for the higher algebraic K-groups with values in cohomology theories satisfying certain axioms. It is not clear at least from the definition that the motivic cohomology theory satisfies these axioms. Levine has proved the theorem for his motivic cohomology groups in [Le, p.174, 3.4.7.Theorem]. We prove the Riemann-Roch theorem without denominators by translating the argument in Gillet [Gi] to the setting of motivic homotopy theory of Morel and Voevodsky [Mo-Vo].

The setting in which we work is provided by the theory of Voevodsky in [Mo-Vo] and in [Vo1]. We denote by $\mathrm{H}(k)$ the $\mathbb{A}^1$-homotopy category over $\operatorname{Spec} k$ in [Vo1, p.585 DEFINITION 3.5]. The homotopy category of pointed spaces ([Vo1, p.586]) is denoted by $\mathrm{H}_\bullet(k)$. Let Sm/k denote the category of schemes which is smooth over $\operatorname{Spec} k$. For a smooth k-scheme X and a locally closed smooth subscheme $Y \subset X$, we let X/Y denote the quotient in the category of Nisnevich sheaves on Sm/k. We often regard it as an object in $\mathrm{H}_\bullet(k)$.

For $n \geq 0$, let $K(\mathbb{Z}(n), 2n)$ denote the Eilenberg-MacLane space as in [Vo1, DEFINITION 6.1, p.597]. We will use the product map $m_{m,n} : K(\mathbb{Z}(m), 2m) \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(m+n), 2(m+n))$ of [Vo1, p.597, bottom] for $m, n \geq 0$. For $n < 0$, we denote by $K(\mathbb{Z}(n), 2n)$ the zero object in the category $\mathrm{H}_\bullet(k)$. For $m, n \in \mathbb{Z}$ with $m < 0$ or $n < 0$, we denote by $m_{m,n}$ the unique map $m_{m,n} : K(\mathbb{Z}(m), 2m) \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(m+n), 2(m+n))$ in $\mathrm{H}_\bullet(k)$.

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From now on we assume that the base field k is perfect. For $i, j \geq 0$ and a pointed space X, we put $H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^i_s \wedge X, K(\mathbb{Z}(j), 2j))$. For $i, j \in \mathbb{Z}$ with $i < 0$ or $j < 0$, we set $H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) = 0$.

Let $\iota : X \hookrightarrow Y$ be a closed immersion of smooth schemes over $\mathrm{Spec}\, k$ of (equi-) codimension d. Let $N = N_{Y/X}$ denote the normal bundle. The main theorem is the following.

Theorem 1.1 (cf. [Gi, THEOREM 3.1, p.235]). *Let $n \geq 0$ and $q \geq 1$. The following diagram is commutative.*

$$\begin{array}{ccc} K_n(X) & \xrightarrow{\iota_!} & K^X_n(Y) \\ {\scriptstyle P^d_q(c(-),c(N))}\big\downarrow & & \big\downarrow{\scriptstyle c_{q+d}} \\ H^{2q-n}_{\mathcal{M}}(X, \mathbb{Z}(q)) & \xrightarrow{\iota_!} & H^{2q-n+2d}_{\mathcal{M}}(Y/(Y \setminus \iota(X)), \mathbb{Z}(q+d)) \end{array}$$

Here $K^X_n(Y)$ is the higher algebraic K-group with supports (defined in Section 2.1), c_q is the Chern class (defined in Section 2.1), $P^d_q(c(-), c(N))$ is the homomorphism which we will define in Section 4.3.2 using universal power series $P^{d,e}_q$ (defined in Section 3.3), and $\iota_!$ are the Gysin maps (defined in Section 2.2.3).

The statement of the theorem is not new. An advantage of giving a formalism of Chern classes and proving the Riemann-Roch theorem in the context of motivic homotopy theory is that we obtain compatibility with localization sequences and universal coefficient theorems easily. Some applications will be given in our other papers.

It is known that the K-groups defined in Section 2.1 is isomorphic to the Thomason-Trobaugh K-groups for smooth k-schemes. There is defined pushforwards (another name for a map related to the Gysin map) for Thomason-Trobaugh K-groups. In Section 6, we show that the Gysin map defined in earlier section using Panin's result and the pushforward map of Thomason and Trobaugh are compatible under the comparison isomorphism. One of the key results is the construction of the isomorphism between K-theory with support which is functorial in some sense.

This article is organized as follows. In Section 2, we first give the definition of Chern classes using Riou's result([Ri2]). The Chern classes in this setting is also defined by Pushin ([Pu]). We then recall the definition of the Gysin maps for motivic cohomology and for K-theory. For this, we follow largely an exposition by Panin ([Pa]). We need the details for our proof of the main result. In Section 3, we collected some definitions of universal power series that often appear in Riemann-Roch type theorems. Section 4 is devoted to the proof. In Section 5.1, we see that, with our definition, it is easy to show that the Riemann-Roch theorem is compatible with the localization sequences. In Section 6, we show that the Gysin map and the pushforward map of Thomason and Trobaugh are compatible.

2. Chern classes and Gysin maps

2.1. **Chern classes.** Let k be a perfect field.

We let Gr denote the infinite Grassmannian as defined in [Ri2, p.3]. For a pointed space X and a non-negative integer n, we let $K_n(X) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^n_s \wedge X, \mathbb{Z} \times \mathrm{Gr})$. If X is a smooth k-scheme, this coincides with the Thomason K-group, that is, the n-th homotopy group of the K-theory spectrum $K(X)$ defined in [TT, 3.1. Definition] ([Vo1, THEOREM 6.5, p.599]). Riou shows ([Ri2, p.4]) that the product

$\mu : (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \to \mathbb{Z} \times \mathrm{Gr}$ defined in [Mo, p.74] induces the usual product of K-theory. We let $\mathrm{ch}_0 : K_n(X) \to H^{-n}_{\mathcal{M}}(X, \mathbb{Z}(0))$ denote the map

$$K_n(X) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^n_s \wedge X, \mathbb{Z} \times \mathrm{Gr}) \to \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^n_s \wedge X, \mathbb{Z})$$

induced by the composite $\mathbb{Z} \times \mathrm{Gr} \to \mathbb{Z} \cong K(\mathbb{Z}(0), 0)$ of the first projection $\mathbb{Z} \times \mathrm{Gr} \to \mathbb{Z}$ and the canonical weak equivalence $K(\mathbb{Z}(0), 0) \cong \mathbb{Z}$. We note that $\mathrm{ch}_0 = 0$ for $n \geq 1$ since $S^n_s \wedge X$ has connected stalks. For a smooth k-scheme Y and a closed subset $X \subset Y$ of the underlying topological space of Y, we let $K^X_n(Y)$ denote the group $K_n(Y/(Y \setminus X))$ where we regard $Y \setminus X$ as an open subscheme of Y. For $n \geq 1$, we define the n-th Chern class $c_n : \mathbb{Z} \times \mathrm{Gr} \to K(\mathbb{Z}(n), 2n)$ to be the map which corresponds, by the isomorphism of Riou [Ri2, Théorème 3.1, p.5], to the natural transform $K_0(-) \to H^{2n}_{\mathcal{M}}(-, \mathbb{Z}(n)) = \mathrm{CH}^n(-)$ of the contravariant functors on Sm/k which is the classical n-th Chern class map $c_n : K_0(X) \to \mathrm{CH}^n(X)$ on each smooth k-scheme X. We also call the induced map $K_m(Z) \to H^{2n-m}_{\mathcal{M}}(Z, \mathbb{Z}(n))$ for a space Z the n-th Chern class map.

2.2. **Gysin maps.** In this section, we recall the definition of the Gysin map for motivic cohomology and for K-theory.

2.2.1. Let X be a smooth k-scheme, and V be a vector bundle over X. We let $\mathrm{Th}(V) = \mathrm{Th}(V/X) = V/(V \setminus \iota(X))$ denote the Thom space of V ([Mo-Vo, p.111, Definition 2.16]) where $\iota : X \to V$ is the zero section of V. The scheme V is canonically embedded in $\mathbb{P}(V \oplus \mathbf{1})$ as an open subscheme. Here $\mathbf{1}$ denotes the trivial bundle of rank one. We identify the boundary $\mathbb{P}(V \oplus \mathbf{1}) \setminus V$ with $\mathbb{P}(V)$. Then the canonical morphism of pointed spaces $\mathbb{P}(V \oplus \mathbf{1})/\mathbb{P}(V) \to \mathrm{Th}(V)$ is an isomorphism in the homotopy category $\mathrm{H}_\bullet(k)$. This is Proposition 2.17.3 of [Mo-Vo, p.112].

2.2.2. Let us recall the construction of the Thom class of a vector bundle. We follow Panin [Pa, THEOREM 3.35, p. 305].

Let Pair/k be the category of pairs (X, U) of a smooth k-scheme X and an open subscheme $U \subset X$. Let A be a ring object in the category $\mathrm{H}_\bullet(k)$. Let $\widetilde{A}$ denote the contravariant functor $Z \mapsto \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^n_s \wedge Z, A)$ from the category $\mathrm{H}_\bullet(k)$ to the category of graded rings. We assume that the contravariant functor $(X, U) \mapsto \widetilde{A}(X/U)$ from the category Pair/k to the category of rings is a ring cohomology theory with a Chern structure c in the sense of Panin [Pa, DEFINITION 3.2, p. 285]. We also assume that, for every smooth k-scheme X and a line bundle $\mathcal{L}$ on X, the element $c(\mathcal{L}) \in \widetilde{A}(X)$ belongs to the subring $A(X) \subset \widetilde{A}(X)$. For a pointed space Z, we denote by $A(Z)$ the ring $\mathrm{Hom}_{\mathrm{H}_\bullet(k)}(Z, A)$. For a morphism $f : Z \to Z'$ in $\mathrm{H}_\bullet(k)$, we denote by f^* the homomorphism $A(Z') \to A(Z)$ supplied by the composition with f.

By [Pa, THEOREM 3.27, p. 302], for a smooth k-scheme X and for a vector bundle V on X of rank n, the Chern classes $c_1(V), \ldots, c_n(V) \in A(X)$ are defined. In this paragraph we recall their characterization. Let $\pi : \mathbb{P}(V) \to X$ be the projective bundle associated to V. We regard the ring $\widetilde{A}(\mathbb{P}(V))$ as a left $\widetilde{A}(X)$-module via the ring homomorphism $\pi^* : \widetilde{A}(X) \to \widetilde{A}(\mathbb{P}(V))$. Let $\xi_V = c(\mathcal{O}_{\mathbb{P}(V)}(-1))$, where $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is the dual to the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on $\mathbb{P}(V)$. (Throughout this article, we will regard line bundles on a scheme X as invertible $\mathcal{O}_X$-modules and vice versa.) By the projective bundle formula [Pa, THEOREM 3.9, p. 290], the ring $\widetilde{A}(\mathbb{P}(V))$ is free of rank n as a left $\widetilde{A}(X)$-module with basis $1, \xi_V, \ldots, \xi_V^{n-1}$.

Hence the subring $A(\mathbb{P}(V)) \subset \widetilde{A}(\mathbb{P}(V))$ is free of rank n as a left $A(X)$-module with basis $1, \xi_V, \ldots, \xi_V^{n-1}$. Then by the construction of the Chern classes, the elements $c_1(V), \ldots, c_n(V) \in A(X)$ are characterized by the formula

$$\xi_V^n + \sum_{i=1}^{n} (-1)^i c_i(V) \xi_V^{n-i} = 0. \tag{2.1}$$

Let V be a vector bundle of rank n over X. Let $W = V \oplus \mathbf{1}$ and let $p : \mathbb{P}(W) \to X$ denote the projection. Since the restriction of the line bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$ to $\mathbb{P}(V)$ is equal to $\mathcal{O}_{\mathbb{P}(V)}(1)$, the pullback homomorphism $A(\mathbb{P}(W)) \to A(\mathbb{P}(V))$ (resp. $\widetilde{A}(\mathbb{P}(W)) \to \widetilde{A}(\mathbb{P}(V))$) is a homomorphism of left $A(X)$-modules (resp. left $\widetilde{A}(X)$-modules) which maps ξ_W to ξ_V. Hence the pullback homomorphism $A(\mathbb{P}(W)) \to A(\mathbb{P}(V))$ (resp. $\widetilde{A}(\mathbb{P}(W)) \to \widetilde{A}(\mathbb{P}(V))$) is surjective and we have a short exact sequence

$$0 \to A(\mathbb{P}(W)/\mathbb{P}(V)) \to A(\mathbb{P}(W)) \to A(\mathbb{P}(V)) \to 0$$

$$(\text{resp. } 0 \to \widetilde{A}(\mathbb{P}(W)/\mathbb{P}(V)) \to \widetilde{A}(\mathbb{P}(W)) \to \widetilde{A}(\mathbb{P}(V)) \to 0). \tag{2.2}$$

Moreover the subgroup $A(\mathbb{P}(W)/\mathbb{P}(V)) \subset A(\mathbb{P}(W))$ (resp. $\widetilde{A}(\mathbb{P}(W)/\mathbb{P}(V)) \subset \widetilde{A}(\mathbb{P}(W))$) is free of rank one as a left $A(X)$-module (resp. a left $\widetilde{A}(X)$-module) with basis $\overline{\mathrm{th}}^{\mathrm{naive}}(V) := \xi_W^n + \sum_{i=1}^n (-1)^i c_i(V) \xi_W^{n-i}$.

We put $\overline{\mathrm{th}}(V) = c_n(\mathcal{O}_{\mathbb{P}(W)}(1) \otimes p^*(V)) \in A(\mathbb{P}(W))$. The restriction of the vector bundle $\mathcal{O}_{\mathbb{P}(W)}(1) \otimes p^*(V)$ to $\mathbb{P}(V)$ is equal to $\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^*(V)$, where $\pi : \mathbb{P}(V) \to X$ is the canonical projection. Since there is a canonical inclusion $\mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^*(V)$ whose quotient is vector bundles on $\mathbb{P}(V)$ whose cokernel is a vector bundle of rank $n-1$, it follows from the Whitney sum formula (which can be proved using the splitting principle) that the pullback of $\overline{\mathrm{th}}(V)$ to $A(\mathbb{P}(V))$ is zero. Hence $\overline{\mathrm{th}}(V)$ belongs to the subgroup $A(\mathbb{P}(W)/\mathbb{P}(V)) = A(\mathbb{P}(W)/(\mathbb{P}(W) \setminus X))$ of the group $A(\mathbb{P}(W))$. We define the class $\mathrm{th}(V) \in A(\mathrm{Th}(V))$ as the image of the element $\overline{\mathrm{th}}(V)$ under the isomorphism of Section 2.2.1 identifying $A(\mathbb{P}(W)/\mathbb{P}(V))$ with $A(\mathrm{Th}(V))$.

2.2.3. The Gysin map for K-theory and for the motivic cohomology are defined as follows. Let $\iota : X \to Y$ be a closed embedding of smooth k-schemes. Then there is a canonical isomorphism in the homotopy category $H_\bullet(k)$ of the form $Y/(Y \setminus \iota(X)) \cong \mathrm{Th}(N)$ where $N = N_{Y/X}$ is the normal bundle. (This is Theorem 2.23 of [Mo-Vo, p.115].) Consider the following diagram

$$A(S_s^n \wedge X) \xrightarrow{p^*} A(S_s^n \wedge \mathbb{P}) \xrightarrow{-\cup \overline{\mathrm{th}}(N)} A(S_s^n \wedge \mathbb{P}) \xleftarrow{f^*} A(S_s^n \wedge \mathrm{Th}(N)).$$

Here the notation is as follows:

- $\mathbb{P} = \mathbb{P}(N \oplus \mathbf{1})$.
- $f : \mathbb{P} \to \mathrm{Th}(N)$ is the composite of the quotient map $\mathbb{P} \to \mathbb{P}/\mathbb{P}(N)$ and the isomorphism $\mathbb{P}/\mathbb{P}(N) \cong \mathrm{Th}(N)$ of Section 2.2.1.
- The map $- \cup \overline{\mathrm{th}}(N)$ is the composite

$$A(S_s^n \wedge \mathbb{P}) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S_s^n \wedge \mathbb{P}, A) \xrightarrow{-\wedge \overline{\mathrm{th}}(N)} \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S_s^n \wedge \mathbb{P} \wedge \mathbb{P}, A \wedge A) \xrightarrow{\alpha} \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S_s^n \wedge \mathbb{P}, A)$$

  where the map $- \wedge \overline{\mathrm{th}}(N)$ is supplied by the smash product with $\overline{\mathrm{th}}(N) \in A(\mathbb{P}) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(\mathbb{P}, A)$ and the map α is supplied by the composition with the diagonal map $\mathbb{P} \to \mathbb{P} \wedge \mathbb{P}$ and the product map $A \wedge A \to A$.

Since $\overline{\mathrm{th}}(N) \in A(\mathbb{P})$ belongs to the subgroup $A(\mathbb{P}/\mathbb{P}(N))$, the image of $\overline{\mathrm{th}}(N)$ under the pullback $A(\mathbb{P}) \to A(\mathbb{P}(N))$ is zero. Hence the composite

$$A(S_s^n \wedge \mathbb{P}) \xrightarrow{-\cup\overline{\mathrm{th}}(N)} A(S_s^n \wedge \mathbb{P}) \to A(S_s^n \wedge \mathbb{P}(N))$$

is zero. Thus the image of the map $-\cup\overline{\mathrm{th}}(N)$ is contained in the image of the homomorphism f^*. It follows from the short exact sequence (2.2) that the homomorphism f^* is injective. We then define the map $\iota_!$ to be the unique homomorphism which makes the diagram

$$\begin{array}{ccc} A(S_s^n \wedge X) & \xrightarrow{p^*} & A(S_s^n \wedge \mathbb{P}) \\ {\scriptstyle \iota_!}\downarrow & & \downarrow{\scriptstyle -\cup\overline{\mathrm{th}}(N)} \\ A(S_s^n \wedge \mathrm{Th}(N)) & \xrightarrow{f^*} & A(S_s^n \wedge \mathbb{P}) \end{array} \tag{2.3}$$

commutative.

In our application, we take $A = \mathbb{Z} \times \mathrm{Gr}$ or we take $A = \prod_{n\geq 0} A_n$ where $A_n = K(\mathbb{Z}(n), 2n)$. In these cases, the Chern structures of the contravariant functor $(X, U) \mapsto \widetilde{A}(X/U) := \bigoplus_n \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S_s^n \wedge (X/U), A)$ from Pair/k are given in [Pa, §3.8]. We recall that, for a line bundle L, the Chern class $c(L)$ in K-theory is given by the class $[\mathcal{O}] - [L]$, and in motivic cohomology by the class of the divisor associated with L. We check the three properties in [Pa, DEFINITION 2.1, p. 269], the three properties in [Pa, DEFINITION 2.13, p. 280], and the three properties in [Pa, DEFINITION 3.2, p. 285]. The property 1 in [Pa, DEFINITION 2.1, p. 269] follows from the surjectivity of $A(X) \to A(U)$ for any object (X, U) in Pair/k. The property 2 in [Pa, DEFINITION 2.1, p. 269] follows from [Mo-Vo, Lemma 2.27]. The property 3 in [Pa, DEFINITION 2.1, p. 269], the three properties in [Pa, DEFINITION 2.13, p. 280], and the properties 1 and 3 in [Pa, DEFINITION 3.2, p. 285] are not difficult to check. The property 2 in [Pa, DEFINITION 3.2, p. 285] in the case where $A = \mathbb{Z} \times \mathrm{Gr}$ follows from the short exact sequence $0 \to K_n(X) \to K_n(X \times \mathbb{P}^1) \to K_n(X \times \mathbb{A}^1) \to 0$ given by Quillen's localization sequence and the Poincaré duality [TT, 3.21. Theorem, p. 328] and the coincidence of the map $K_n(X) \to K_n(X \times \mathbb{P}^1)$ in the short exact sequence above with the map $x \mapsto p^*x \cup \xi$ where $p : X \times \mathbb{P}^1 \to X$ is the first projection and $\xi = c(\mathcal{O}(-1))$. The property 2 in [Pa, DEFINITION 3.2, p. 285] in the case where $A = \prod_{n\geq 0} A_n$ follows from [Vo-Su-Fr, Chapter 5, Proposition 3.5.1]. In the latter case where $A = \prod_{n\geq 0} A_n$, for a vector bundle V on a smooth k-scheme X, it follows from the formula (2.1) in Section 2.2.2 that the Chern class map $c_i(V)$ for each $i \geq 1$ is equal to the image $c_i([V])$ of the class $[V]$ of V in $K_0(X)$ under the map $c_i : K_0(X) \to H^{2i}_{\mathcal{M}}(X, \mathbb{Z}(i))$ introduced in Section 2.1, since we have $c(\mathcal{L}) = c_1([\mathcal{L}])$ for a line bundle $\mathcal{L}$ on X by the definition of the Chern structure. In particular we have $\overline{\mathrm{th}}(N) \in A_d(\mathbb{P})$. Hence the map $\iota_! : A(S_s^n \wedge X) \to A(S_s^n \wedge \mathrm{Th}(N))$ defined above sends $A_n(S_s^n \wedge X)$ into $A_{n+d}(S_s^n \wedge \mathrm{Th}(N))$.

Lemma 2.1. *When $A = \mathbb{Z} \times \mathrm{Gr}$ or $A = \prod_{n\geq 0} K(\mathbb{Z}(n), 2n)$, the Gysin map $\iota_!$ is an isomorphism.*

Proof. When $A = \mathbb{Z} \times \mathrm{Gr}$ (resp. $A = \prod_{n\geq 0} K(\mathbb{Z}(n), 2n)$), a direct computation shows that $\overline{\mathrm{th}}(N) = (-1)^d \overline{\mathrm{th}}^{\mathrm{naive}}(N) \cdot [\mathcal{O}_{\mathbb{P}}(d)]$ (resp. $\overline{\mathrm{th}}(N) = (-1)^d \overline{\mathrm{th}}^{\mathrm{naive}}(N)$). Hence in these two cases, the composite $A(S_s^n \wedge X) \xrightarrow{p^*} A(S_s^n \wedge \mathbb{P}) \xrightarrow{-\cup\overline{\mathrm{th}}(N)} A(S_s^n \wedge \mathbb{P})$

is injective and its image is equal to the kernel of the pullback homomorphism $A(S^n_s \wedge \mathbb{P}) \to A(S^n_s \wedge \mathbb{P}(N))$. This proves the claim. $\square$

3. Universal power series

We define some universal power series in this section. We use the following notation. For $m, n \geq 0$, let $\mathbb{Z}[[x;y]]_{m,n}$ denote the ring $\mathbb{Z}[[x_1, \ldots, x_m; y_1, \ldots, y_n]]$ of formal power series in $m+n$ indeterminates $x_1, \ldots, x_m, y_1, \ldots, y_n$. We define the degree of a monomial in $\mathbb{Z}[[x;y]]_{m,n}$ by putting $\deg(x_i) = \deg(y_j) = 1$. For an integer q, we say that an element f in $\mathbb{Z}[[x;y]]_{m,n}$ is homogeneous of degree q if f is written as a $\mathbb{Z}$-linear combination of monomials of degree q. For $1 \leq i \leq n$ (resp. $1 \leq j \leq n$), let c_i (resp. c'_j) denote the i-th (resp. j-th) elementary symmetric polynomial with variables $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_n$). Then the ring $\mathbb{Z}[[c;c']]_{m,n} = \mathbb{Z}[[c_1, \ldots, c_m; c'_1, \ldots, c'_n]]$ of formal power series in $m+n$ indeterminates $c_1, \ldots, c_m, c'_1, \ldots, c'_n$ is a subring of $\mathbb{Z}[[x;y]]_{m,n}$. For an integer q, we say that an element f in $\mathbb{Z}[[c;c']]_{m,n}$ is homogeneous of degree q if f is homogeneous of degree q as an element in $\mathbb{Z}[[x;y]]_{m,n}$.

For integers m, n, m', n' with $m \geq m' \geq 0$ and $n \geq n' \geq 0$, we regard the ring $\mathbb{Z}[[x;y]]_{m',n'}$ as the quotient of $\mathbb{Z}[[x;y]]_{m,n}$ by the ideal generated by $x_{m'+1}, \ldots, x_m$, $y_{n'+1}, \ldots, y_n$. Under the quotient map $\mathbb{Z}[[x;y]]_{m,n} \twoheadrightarrow \mathbb{Z}[[x;y]]_{m',n'}$, the element c_i (resp. c'_j) maps to c_i (resp. c'_j) if $i \leq m'$ (resp. $j \leq n'$) and maps to 0 otherwise. We denote by $\mathbb{Z}[[x;y]]$ the projective limit $\varprojlim_{m,n\geq 0} \mathbb{Z}[[x;y]]_{m,n}$ where the transition map is the quotient map $\mathbb{Z}[[x;y]]_{m,n} \twoheadrightarrow \mathbb{Z}[[x;y]]_{m',n'}$. We denote by $\mathbb{Z}[[c;c']]$ the subring $\varprojlim_{m,n\geq 0} \mathbb{Z}[[c;c']]_{m,n}$ of $\mathbb{Z}[[x,y]]$. For an integer q, we say that an element $f = (f_{m,n})_{m,n}$ in $\mathbb{Z}[[x;y]]$ (resp. in $\mathbb{Z}[[c;c']]$) is homogeneous of degree q if $f_{m,n}$ is homogeneous of degree q for any $m, n \geq 1$ (resp. if f is homogeneous of degree q as an element in $\mathbb{Z}[[x;y]]$). We note that if $0 \neq f \in \mathbb{Z}[[c;c']]$ is homogeneous of degree q, then $q \geq 0$ and f belongs to the subring $\mathbb{Z}[c_1, \ldots, c_q; c'_1, \ldots, c'_q]$ of $\mathbb{Z}[[c;c']]$.

3.1. For an integer $q \geq 1$, let Q_q be the universal polynomial in [SGA6, Chap. 1, §3]. We regard it as an element in $\mathbb{Z}[[c;c']]$. The element Q_q is characterized by the following property. For any $m, n \geq 1$, the image of Q_q under the projection $\mathbb{Z}[[c;c']] \twoheadrightarrow \mathbb{Z}[[c;c']]_{m,n}$ is the coefficient of t^q in the formal power series

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 + (x_i + y_j)t}{(1 + x_i t)(1 + y_j t)} \in \mathbb{Z}[[x;y]]_{m,n}[[t]]$$

with coefficients in $\mathbb{Z}[[x;y]]_{m,n}$. The element Q_q is homogeneous of degree q.

3.2. Let m, n be integers. Consider the element

$$(1 + \sum_{i\geq 1} c_i t^i)^n (1 + \sum_{j\geq 1} c'_j t^j)^m (1 + \sum_{q\geq 1} Q_q(c,c') t^q) \in \mathbb{Z}[[c;c']][[t]].$$

For $q \geq 1$, we define $R_{m,n,q} \in \mathbb{Z}[[c;c']]$ to be the coefficient of t^q in the element above. It is homogeneous of degree q.

3.3. Let d be a non-negative integer. For $0 \leq p \leq d$, we let

$$S^{d,(p)}(x_1) = \prod_{\substack{I \subset \{1,\ldots,d\} \\ |I|=p}} (1 + (x_1 - \sum_{j\in I} y_j)t) \in 1 + t\mathbb{Z}[[x;y]]_{1,d}[[t]].$$

Let e, m be integers with $m \geq 0$ and $m \geq e$. We let

$$S^{d,(p),e,m} = \frac{\prod_{i=1}^{m} S^{d,(p)}(x_i)}{(S^{d,(p)}(0))^{m-e}} \in 1 + t\mathbb{Z}[[x;y]]_{m,d}[[t]].$$

Then the element $S^{d,(p),e,m}$ belongs to the subset $1+t\mathbb{Z}[[c;c']]_{m,d}[[t]]$ of $1+t\mathbb{Z}[[x;y]]_{m,d}[[t]]$. The system of elements $(S^{d,(p),e,m})_m$ as m varies gives an element in $1+t\mathbb{Z}[[c;c']]_d[[t]]$ where $\mathbb{Z}[[c;c']]_d$ is a shorthand for $\varprojlim_m \mathbb{Z}[[c;c']]_{m,d}$. We denote the element by $S^{d,(p),e}$. The following lemma follows immediately from the definition of the formal power series $S^{d,(p),e}$ so the proof is omitted.

Lemma 3.1. *Let us write $S^{d,(p)}(0) = 1 + \sum_{q \geq 1} c_q^{(p)} t^q$, where $c_q^{(p)}$ is an element in $\mathbb{Z}[[c;c']]_{0,d} = \mathbb{Z}[[c'_1, \ldots, c'_d]]$ which is homogeneous of degree q. Then $S^{d,(p),e}$ is equal to*

$$1 + \sum_{q \geq 1} R_{e,\binom{d}{p},q}(c; c_1^{(p)}, c_2^{(p)}, \ldots) t^q.$$

For $q \geq 0$, we let $P_q^{d,e} \in \mathbb{Z}[[c;c']]_d$ denote the coefficient of t^q in $((\prod_{p=0}^{d} (S^{d,(p),e})^{(-1)^p}) - 1)/c'_d$. For the divisibility by c'_d, see [Fu-La, p.44–45]. By construction, $P_q^{d,e}$ is homogeneous of degree $q-d$. In particular we have $P_q^{d,e} = 0$ for $q < d$.

4. Proof of Theorem 1.1

4.1. One can use the argument in the proof of [Ri1, Théorème v.13, p. 159] to show that there is a degree-preserving isomorphism

$$\mathbb{Z}[[c;c']] \cong \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(\mathrm{Gr} \times \mathrm{Gr}, \prod_{q \geq 0} K(\mathbb{Z}(q), 2q)). \tag{4.1}$$

Here, the degree q part of the right hand side of (4.1) is $\mathrm{Hom}_{\mathrm{H}_\bullet(k)}(\mathrm{Gr}\times\mathrm{Gr}, K(\mathbb{Z}(q), 2q))$. In Section 3.2, we constructed, for each $m, n, q \in \mathbb{Z}$ with $q \geq 1$, an element $R_{m,n,q} \in \mathbb{Z}[[c;c']]$. This element defines a map $\mathrm{Gr} \times \mathrm{Gr} \to K(\mathbb{Z}(q), 2q)$ via the isomorphism above. We define a map $R : (\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr}) \to \prod_{q \geq 1} K(\mathbb{Z}(q), 2q)$ as the map whose restriction to $(\{m\}\times\mathrm{Gr})\times(\{n\}\times\mathrm{Gr}) = \mathrm{Gr}\times\mathrm{Gr}$ for each $m, n \in \mathbb{Z}$ is equal to the map $(R_{m,n,q})_q : \mathrm{Gr} \times \mathrm{Gr} \to \prod_{q \geq 1} K(\mathbb{Z}(q), 2q)$.

Lemma 4.1. *Let the notations be as above. We have*

$$R = c_\bullet \circ \mu$$

in $\mathrm{Hom}_{\mathrm{H}_\bullet(k)}((\mathbb{Z} \times \mathrm{Gr}) \times (\mathbb{Z} \times \mathrm{Gr}), \prod_{q \geq 1} K(\mathbb{Z}(q), 2q))$. *Here we put* $c_\bullet = (c_q)_{q \geq 1}$.

Proof. One can show using the argument in the proof of [Ri1, Theoreme v.9, p. 157] that the canonical map

$$\mathrm{Hom}_{\mathrm{H}_\bullet(k)}((\mathbb{Z}\times\mathrm{Gr})\times(\mathbb{Z}\times\mathrm{Gr}), K(\mathbb{Z}(q), 2q)) \to \mathrm{Hom}(K_0(-)\times K_0(-), H^{2q}_{\mathcal{M}}(-, \mathbb{Z}(q)))$$

is an isomorphism for each $q \geq 0$. Here the latter is the set of morphisms in the category of presheaves on Sm/k. Hence it suffices to prove that the morphisms of functors $K_0(-)\times K_0(-) \to \prod_{q \geq 1} H^{2q}_{\mathcal{M}}(-, \mathbb{Z}(q))$ induced by R and by $c_\bullet \circ \mu$ coincide. Let us use the same symbol to denote the morphism of functors.

We endow $\prod_{q\geq 1} H^{2q}_{\mathcal{M}}(X,\mathbb{Z}(q))$ for each X in Sm$/k$ with the structure of abelian group by canonically identifying it with the multiplicative group

$$\left\{ 1+\sum_{q\geq 1} a_q t^q \;\middle|\; a_q \in H^{2q}_{\mathcal{M}}(X,\mathbb{Z}(q)) \right\}.$$

Then both R and $c_\bullet \circ \mu$ can be regarded as bilinear morphisms of presheaves of abelian groups.

Thus, to check the equality, we are reduced to the case of a pair of classes of vector bundles, and moreover, by splitting principle, a pair of classes of line bundles. That is, we need only verify that $c_\bullet([L_1 \otimes L_2]) = R(c_1(L_1), c_1(L_2))$ for line bundles L_1 and L_2 over a smooth k-scheme. This follows immediately from the construction of the elements $(R_{m,n,q})_{m,n}$. □

Lemma 4.1 shows that the map R factors through the canonical map $(\mathbb{Z}\times \mathrm{Gr}) \times (\mathbb{Z}\times\mathrm{Gr}) \to (\mathbb{Z}\times\mathrm{Gr})\wedge(\mathbb{Z}\times\mathrm{Gr})$ by [Ri1, Lemme III.33, p. 96]. By abuse of notation, we denote by R the map $(\mathbb{Z}\times\mathrm{Gr})\wedge(\mathbb{Z}\times\mathrm{Gr}) \to \prod_{q\geq 0} K(\mathbb{Z}(q), 2q)$ induced by R.

4.2.

4.2.1. Given $q, d \geq 0$, let

$$P^d_q : \mathbb{Z}\times\mathrm{Gr}\times\mathrm{Gr} \to K(\mathbb{Z}(q-d), 2(q-d))$$

denote the map whose restriction to $\{e\}\times\mathrm{Gr}\times\mathrm{Gr} \cong \mathrm{Gr}\times\mathrm{Gr}$ for each integer e is equal to the composite

$$\mathrm{Gr}\times\mathrm{Gr} \to \prod_{i\geq 0} K(\mathbb{Z}(i), 2i) \to K(\mathbb{Z}(q-d), 2(q-d))$$

where the first map is the map corresponding to the element $P^{d,e}_q \in \mathbb{Z}[[c;c']]_d$ via the isomorphism (4.1) (here we regard $\mathbb{Z}[[c;c']]_d$ as a subring of $\mathbb{Z}[[c;c']]$ via the inclusion $\mathbb{Z}[[c;c']]_d \to \mathbb{Z}[[c;c']]$ which sends c_i to c_i for $i \geq 0$ and c'_j to c'_j for $1 \leq j \leq d$) and the second map is the projection to the $(q-d+1)$-st factor (resp. the unique map to the final object $K(\mathbb{Z}(q-d), 2(q-d))$) if $q \geq d$ (resp. if $q < d$). By Lemma 3.1 the map P^d_q factors though the canonical map $\mathbb{Z}\times\mathrm{Gr}\times\mathrm{Gr} \to (\mathbb{Z}\times\mathrm{Gr})\wedge\mathrm{Gr}$. By abuse of notation, we denote by P^d_q the map $(\mathbb{Z}\times\mathrm{Gr})\wedge\mathrm{Gr} \to K(\mathbb{Z}(q-d), 2(q-d))$ induced by P^d_q.

4.2.2. Let $d \geq 1$. Let $\mathrm{Gr}_d = \varinjlim_r \mathrm{Gr}_{d,r}$ denote the Grassmannian which classifies vector bundles of rank d. Let $U_d \to \mathrm{Gr}_d$ denote the universal bundle. We let $p_d : \mathbb{P}_d = \mathbb{P}(U_d \oplus \mathbf{1}) \to \mathrm{Gr}_d$ denote the projection. Let $Q_d = (p_d^* U_d) \otimes \mathcal{O}_{\mathbb{P}_d}(1)$. We write $[Q_d] : \mathbb{P}_d \to \mathbb{Z}\times\mathrm{Gr}$ for a map in $\mathrm{H}_\bullet(k)$ representing the class of the vector bundle $[Q_d] \in K_0(\mathbb{P}_d)$.

4.2.3. We have three maps:

$$\begin{aligned}
\alpha_1 &= (\mathrm{id}_{\mathbb{Z}\times\mathrm{Gr}}) \circ \mathrm{pr}_{\mathbb{Z}\times\mathrm{Gr}} : (\mathbb{Z}\times\mathrm{Gr})\wedge\mathbb{P}_d \to \mathbb{Z}\times\mathrm{Gr},\\
\alpha_2 &= \mathrm{pr}_{\mathrm{Gr}} \circ [Q_d] \circ \mathrm{pr}_{\mathbb{P}_d} : (\mathbb{Z}\times\mathrm{Gr})\wedge\mathbb{P}_d \to \mathrm{Gr},\\
\alpha_3 &= c_d(Q_d) \circ \mathrm{pr}_{\mathbb{P}_d} : (\mathbb{Z}\times\mathrm{Gr})\wedge\mathbb{P}_d \to K(\mathbb{Z}(d), 2d).
\end{aligned}$$

Here we write pr with a subscript to denote the projection to the factor. We define a map $A_d^q : (\mathbb{Z}\times \mathrm{Gr})\wedge \mathbb{P}_d \to K(\mathbb{Z}(q),2q)$ as the composite

$$\begin{aligned}
&(\mathbb{Z}\times \mathrm{Gr}_d)\wedge \mathbb{P}_d \xrightarrow{(\alpha_1,\alpha_2,\alpha_3)} (\mathbb{Z}\times \mathrm{Gr})\wedge \mathrm{Gr}\wedge K(\mathbb{Z}(d),2d)\\
&\xrightarrow{(P_q^d\circ \mathrm{pr}_{(\mathbb{Z}\times\mathrm{Gr})\wedge\mathrm{Gr}},\mathrm{id}_{K(\mathbb{Z}(d),2d)})} K(\mathbb{Z}(q-d),2(q-d))\wedge K(\mathbb{Z}(d),2d)\\
&\xrightarrow{m_{q-d,d}} K(\mathbb{Z}(q),2q).
\end{aligned}$$

4.2.4.

Proposition 4.2. *Let $q\geq 0$. The diagram*

$$\begin{array}{ccc}
(\mathbb{Z}\times \mathrm{Gr})\wedge \mathbb{P}_d & \xrightarrow{B} & (\mathbb{Z}\times \mathrm{Gr})\wedge(\mathbb{Z}\times \mathrm{Gr})\\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle R_q=(R_{m,n,q})_{m,n}}\\
(\mathbb{Z}\times \mathrm{Gr})\wedge \mathbb{P}_d & \xrightarrow{A_d^q} & K(\mathbb{Z}(q),2q)
\end{array}$$

is commutative. Here the map B is

$$(\mathbb{Z}\times \mathrm{Gr})\wedge \mathbb{P}_d \xrightarrow{(\mathrm{id}_{\mathbb{Z}\times\mathrm{Gr}}\circ \mathrm{pr}_{\mathbb{Z}\times\mathrm{Gr}},\, c_d^K(Q_d)\circ \mathrm{pr}_{\mathbb{P}_d})} (\mathbb{Z}\times \mathrm{Gr})\wedge(\mathbb{Z}\times \mathrm{Gr}).$$

Proof. Let $Q_d^\vee$ denote the dual of Q_d. We write $\wedge^\bullet Q_d^\vee$ for the Koszul complex associated to the dual $s^\vee : Q_d^\vee \to \mathcal{O}_{\mathbb{P}_d}$ of the section $s:\mathcal{O}_{\mathbb{P}_d}\to Q_d$ which is the composite of the tautological section $\mathcal{O}_{\mathbb{P}_d}\to p^*(U_d\oplus \mathbf{1})\otimes \mathcal{O}_{\mathbb{P}_d}(1)$ and the projection $p^*(U_d\oplus\mathbf{1})\otimes\mathcal{O}_{\mathbb{P}_d}(1)\to U_d\otimes \mathcal{O}_{\mathbb{P}_d}(1)$. By the formal computation using the splitting principle, we have

$$c_d^K(Q_d)=\sum_{p=0}^{d}(-1)^p[\wedge^p Q_d^\vee]\in K_0(\mathbb{P}_d).$$

Let $c_q^{(p)}\in \mathbb{Z}[[c'_1,\dots,c'_d]]$ be the polynomial introduced in Section 3.3. Let $x_1,\dots,x_d$ be the Chern roots of Q_d. Then the set of Chern roots of $\wedge^p Q_d^\vee$ is

$$\{-x_{i_1}-x_{i_2}\cdots -x_{i_p} | 1\leq i_1<\cdots<i_p\leq d\}$$

for $p=0,\dots,d$. By the formal computation using the splitting principle, we have

$$c_q(\wedge^p Q_d^\vee)=c_q^{(p)}(c_1(Q_d),\dots,c_d(Q_d)).$$

Therefore it follows from Lemma 3.1 and the construction of $P_q^{d,e}$ that

$$R_q\circ B = c_d(Q_d)\cdot[P_q^d\circ(\alpha_1,\alpha_2)].$$

The right hand side equals the map A_d^q by definition. This proves the proposition. $\square$

4.3.

4.3.1. Let X be a smooth k-scheme and let V be a vector bundle of rank d on X. Then the class $[V]\in K_0(X)$ of V gives a map $[V]:X\to \mathbb{Z}\times\mathrm{Gr}$ in $\mathrm{H}_\bullet(k)$. Let Z be a pointed space. (Later, we put $Z=S_s^n\wedge X$ for some n.) For a map $z:Z\to\mathbb{Z}\times\mathrm{Gr}$ in $\mathrm{H}_\bullet(k)$, we write $P_q^d(c(z),c(V))$ for the composite

$$Z\times X\xrightarrow{(z,[V])}\mathbb{Z}\times\mathrm{Gr}\times\mathbb{Z}\times\mathrm{Gr}\to\mathbb{Z}\times\mathrm{Gr}\times\mathrm{Gr}\xrightarrow{P_q^d} K(\mathbb{Z}(q-d),2(q-d)) \qquad (4.2)$$

where the second map is the canonical projection to the first, the second, and the fourth factor. By abuse of notation, we let $P^d_q(c(z), c(V))$ denote the map $Z \wedge X \to K(\mathbb{Z}(q-d), 2(q-d))$ induced by $P^d_q(c(z), c(V)) : Z \times X \to K(\mathbb{Z}(q-d), 2(q-d))$.

4.3.2. Let $Q_V = p^*V \otimes \mathcal{O}_{\mathbb{P}}(1)$. The following is a corollary to Proposition 4.2.

Corollary 4.3. *Let the notations be as above. We put $\mathbb{P} = \mathbb{P}(V \oplus \mathbf{1})$. Let $q \geq 0$. The diagram*

$$\begin{tikzcd}
Z \wedge \mathbb{P} \arrow[r, "{(z, c^K_d(Q_V))}"] \arrow[d, "="'] & (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \arrow[d, "{R_q = (R_{m,n,q})_{m,n}}"] \\
Z \wedge \mathbb{P} \arrow[r, "{c_d(Q_V) \cdot P^d_q(c(z), c(Q_V))}"] & K(\mathbb{Z}(q), 2q)
\end{tikzcd}$$

is commutative. Here the horizontal map at the bottom is the composite

$$\begin{aligned}
&Z \wedge \mathbb{P} \xrightarrow{\mathrm{id} \wedge (\mathrm{id}, c_d(Q))} Z \wedge \mathbb{P} \wedge K(\mathbb{Z}(d), 2d) \\
&\xrightarrow{P^d_q(c(z), c(p^*V)) \wedge \mathrm{id}} K(\mathbb{Z}(q-d), 2(q-d)) \wedge K(\mathbb{Z}(d), 2d) \\
&\xrightarrow{m_{q-d,d}} K(\mathbb{Z}(q), 2q).
\end{aligned}$$

Proof. The vector bundle V is the pullback of U_d by a morphism $X \to \mathrm{Gr}_d$. As Chern classes are compatible with pullbacks, the claim follows from Proposition 4.2. $\square$

4.4. Let us suppose that we are given a vector bundle V of rank d on X. Let $p : \mathbb{P} = \mathbb{P}(V \oplus \mathbf{1}) \to X$ denote the projection. Later, we will take V to be the normal bundle of some closed immersion $\iota : X \hookrightarrow Y$.

4.4.1. Consider the following diagram

$$\begin{tikzcd}
K_n(X) \arrow[r, "{P^d_\bullet(c(-), c(V))}"] \arrow[d, "p^*"'] & \prod_{q \geq 1} H^{2(q-d)-n}_{\mathcal{M}}(X, \mathbb{Z}(q-d)) \arrow[d, "p^*"'] \\
K_n(\mathbb{P}) \arrow[r, "{P^d_\bullet(c(-), c(p^*V))}"] & \prod_{q \geq 1} H^{2(q-d)-n}_{\mathcal{M}}(\mathbb{P}, \mathbb{Z}(q-d)).
\end{tikzcd} \tag{4.3}$$

Here for $q \geq 0$ the q-th component of the map $P^d_\bullet(c(-), c(V))$ (resp. $P^d_\bullet(c(-), c(p^*V))$) is the map which sends $z \in K_n(X) = \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(S^n_s \wedge X, \mathbb{Z} \times \mathrm{Gr})$ to the composite

$$S^n_s \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} S^n_s \wedge X \wedge X \xrightarrow{P^d_q(c(z), c(V))} K(\mathbb{Z}(q), 2q)$$

$$(resp.\ S^n_s \wedge \mathbb{P} \xrightarrow{\mathrm{id} \wedge \Delta} S^n_s \wedge \mathbb{P} \wedge \mathbb{P} \xrightarrow{P^d_q(c(z), c(p^*V))} K(\mathbb{Z}(q), 2q))$$

considered as an element in $H^{2q-n}_{\mathcal{M}}(X, \mathbb{Z}(q))$ (resp. in $H^{2q-n}_{\mathcal{M}}(\mathbb{P}, \mathbb{Z}(q))$), where $\Delta : X \to X \wedge X$ is the diagonal map and the map $P^d_q(c(z), c(V))$ (resp. $P^d_q(c(z), c(p^*V))$) is the map (4.2) for $Z = S^n_s \wedge X$ (resp. for $Z = S^n_s \wedge \mathbb{P}$). The commutativity of the diagram follows immediately from the definitions.

4.4.2. Let $Q_V = p^*V \otimes \mathcal{O}_{\mathbb{P}}(1)$. Consider the following diagram:

$$\begin{array}{ccc} K_n(\mathbb{P}) & \xrightarrow{P^d_\bullet(c(-),c(Q_V))} & \prod_{q\geq 1} H^{2(q-d)-n}_{\mathcal{M}}(\mathbb{P},\mathbb{Z}(q-d)) \\ {\scriptstyle -\cup c^K_d(Q_V)}\downarrow & & \downarrow{\scriptstyle -\cup c_d(Q_V)} \\ K_n(\mathbb{P}) & \xrightarrow{c_\bullet} & \prod_{q\geq 1} H^{2q-n}_{\mathcal{M}}(\mathbb{P},\mathbb{Z}(q)) \end{array} \tag{4.4}$$

Here the map $P^d_\bullet(c(-),c(Q_V))$ is defined in a manner similar to that of $P^d_\bullet(c(-),c(p^*V))$. The square is commutative by Corollary 4.3.

Lemma 4.4. *Let the notations be as above. We have*

$$p^*(P^d_\bullet(c(-),c(V)))\cup c_d(Q_V) = (P^d_\bullet(c(p^*(-),c(Q_V)))\cup c_d(Q_V).$$

Proof. The argument below uses Chern polynomials. See for example [Fu, Section 3.2, p.50].

Using the splitting principle, we may and will assume that $V = L_1\oplus\cdots\oplus L_d$ is the sum of line bundles. The Chern polynomial of Q_V is $c_t(Q_V) = \prod_{i=1}^d(1 + c_1(p^*L_i\otimes\mathcal{O}(1))t) = \prod_{i=1}^d((1+c_1(p^*L_i)t)+c_1(\mathcal{O}(1))t)$. The Chern polynomial of p^*V is $c_t(p^*V) = \prod_{i=1}^d(1+c_1(p^*L_i))$. For the claim, it suffices to show that $c_t(p^*V)c_d(Q_V) = c_t(Q_V)c_d(Q_V)$.

Let $\xi = \xi_{V\oplus\mathbf{1}} = c(\mathcal{O}_{\mathbb{P}(V\oplus\mathbf{1})}(-1))$ (see Section 2.2.2). Then $c_t(Q_V)-c_t(p^*V)$ is a polynomial in ξ with no constant term. Therefore it suffices to show that $\xi c_d(Q_V) = 0$. By the definition of c_d, we have $\xi c_d(Q_V) = \xi^{d+1}+(-1)^d\xi^d(\sum c_1(p^*L_i))+\cdots+\xi c_1(p^*L_1)\cdots c_1(p^*L_d)$. On the other hand, by equation (2.1), we have $\xi^{d+1}+\sum_{i=1}^{d+1}(-1)^ic_i(p^*(N\oplus 1))\xi^{d+1-i} = 0$. Now use the Whitney sum formula to express the Chern classes of $p^*(V\oplus 1)$ in terms of the Chern classes of p^*V and of p^*1. Since we know that the Chern classes of p^*1 are zero, the claim follows. □

We will use the following corollary.

Corollary 4.5. *We have*

$$p^*(P^d_\bullet(c(-),c(V)))\cup c_d(Q_V) = c_\bullet(p^*(-)\cup c^K_d(Q_V)). \tag{4.5}$$

Proof. This follows from the lemma using the commutativity of the diagram (4.3). □

4.5. Proof of Theorem 1.1.

4.5.1. Let $f_1 : \mathbb{P}(V\oplus\mathbf{1})\to\mathbb{P}(V\oplus\mathbf{1})/\mathbb{P}(V)$ be the quotient map. Consider the commutative diagram

$$\begin{array}{ccc} K_n(\mathbb{P}) & \xrightarrow{c_\bullet} & \prod_{q\geq 1} H^{2q-n}_{\mathcal{M}}(\mathbb{P},\mathbb{Z}(q)) \\ {\scriptstyle f_1^*}\uparrow & & \uparrow{\scriptstyle f_1^*} \\ K_n(\mathbb{P}/\mathbb{P}(V)) & \xrightarrow{c_\bullet} & \prod_{q\geq 1} H^{2q-n}_{\mathcal{M}}(\mathbb{P}/\mathbb{P}(V),\mathbb{Z}(q)). \end{array}$$

Recall that the two f_1^*'s are injective.

4.5.2. Let us recall the setup. Let $\iota : X \hookrightarrow Y$ be a closed immersion of smooth k-schemes of codimension d. Let $N = N_{Y/X}$ be the normal bundle and let $\mathbb{P} = \mathbb{P}(N \oplus \mathbf{1})$ where $\mathbf{1}$ denotes the trivial bundle of rank one. We let $p : \mathbb{P} \to X$ denote the canonical projection.

Let $Q = p^*N \otimes \mathcal{O}_{\mathbb{P}}(1)$ and $\overline{\mathrm{th}}_K = \overline{\mathrm{th}}(N) \in A(\mathbb{P})$ where we take $A = \mathbb{Z} \times \mathrm{Gr}$. By definition we have $\overline{\mathrm{th}}_K = c_d^K(Q) \in \mathrm{Hom}_{\mathrm{H}_\bullet(k)}(\mathbb{P}, \mathbb{Z} \times \mathrm{Gr})$ where c_d^K is the Chern class for the cohomology theory $A = \mathbb{Z} \times \mathrm{Gr}$ with the Chern structure given in [Pa, §3.8]. We let $\overline{\mathrm{th}}_{\mathcal{M}} = c_d(Q) \in H^{2d}_{\mathcal{M}}(\mathbb{P}, \mathbb{Z}(d))$ where c_d is the Chern class (Section 2.1) with values in motivic cohomology.

Let $f_2 : \mathbb{P}(N \oplus 1)/\mathbb{P}(N) \cong \mathrm{Th}(N) \cong Y/(Y \setminus X)$ denote the Thom isomorphism. Consider the commutative diagram

$$\begin{array}{ccc}
K_n(\mathbb{P}/\mathbb{P}(N)) & \xrightarrow{c_\bullet} & \prod_{q \geq 1} H^{2q-n}_{\mathcal{M}}(\mathbb{P}/\mathbb{P}(N), \mathbb{Z}(q)) \\
\cong\uparrow & & \cong\uparrow \\
K_n(Y/(Y \setminus X)) & \xrightarrow{c_\bullet} & \prod_{q \geq 1} H^{2q-n}_{\mathcal{M}}(Y/(Y \setminus X), \mathbb{Z}(q))
\end{array}$$

where the vertical arrows are the isomorphisms induced by f_2.

By the definition of the Gysin map $\iota_!$ given in Section 2.2.3, the composite maps $p^*(-) \cup \overline{\mathrm{th}}_K$ and $p^*(-) \cup \overline{\mathrm{th}}_{\mathcal{M}}$ are equal to $f_1^* \circ f_2 \circ \iota_!$ for K-theory and for motivic cohomology theory respectively. Since $f_1^* \circ f_2$ is injective, the theorem follows from Corollary 4.5 applied with $V = N$.

This finishes the proof of Theorem 1.1. $\square$

5. Miscellaneous results

We collect in this section some facts which may be useful in application. They will be referred to in our other papers.

5.1. Localization sequences. Let Y be a smooth k-scheme. Let $U \subset Y$ be an open subscheme, and let $X = Y \setminus U$ be the closed subscheme of Y with the reduced structure. Consider the diagram

$$\begin{array}{ccc}
K_n(Y) & \xrightarrow{c_\bullet} & \bigoplus_{q \geq 1} H^{2q-n}_{\mathcal{M}}(Y, \mathbb{Z}(q)) \\
{\scriptstyle (1)}\downarrow & & \downarrow{\scriptstyle (4)} \\
K_n(U) & \xrightarrow{c_\bullet} & \bigoplus_{q \geq 1} H^{2q-n}_{\mathcal{M}}(U, \mathbb{Z}(q)) \\
{\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle (5)} \\
K^X_{n-1}(Y) & \xrightarrow{c_\bullet} & \bigoplus_{q \geq 1} H^{2q-(n-1)}_{\mathcal{M}}(Y, U; \mathbb{Z}(q)) \\
{\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (6)} \\
K_{n-1}(Y) & \xrightarrow{c_\bullet} & \bigoplus_{q \geq 1} H^{2q-(n-1)}_{\mathcal{M}}(Y, \mathbb{Z}(q))
\end{array}$$

where the two columns are the localization sequences. The diagram is obviously commutative.

Corollary 5.1. *Suppose that X is smooth over* $\operatorname{Spec} k$ *and $X \subset Y$ is of equi-codimension d. Then the following diagram is commutative:*

$$\begin{array}{ccc}
K_n(Y) & \xrightarrow{c_\bullet} & \bigoplus_{q\geq 1} H^{2q-n}_{\mathcal{M}}(Y,\mathbb{Z}(q)) \\
{\scriptstyle (1)}\big\downarrow & & \big\downarrow{\scriptstyle (4)} \\
K_n(U) & \xrightarrow{c_\bullet} & \bigoplus_{q\geq 1} H^{2q-n}_{\mathcal{M}}(U,\mathbb{Z}(q)) \\
{\scriptstyle (2)'}\big\downarrow & & \big\downarrow{\scriptstyle (5)'} \\
K_{n-1}(X) & \xrightarrow{P^d_\bullet(c(-),c(N))} & \bigoplus_{q\geq 1} H^{2(q-d)-(n-1)}_{\mathcal{M}}(X,\mathbb{Z}(q-d)) \\
{\scriptstyle (3)'}\big\downarrow & & \big\downarrow{\scriptstyle (6)'} \\
K_{n-1}(Y) & \xrightarrow{c_\bullet} & \bigoplus_{q\geq 1} H^{2q-(n-1)}_{\mathcal{M}}(Y,\mathbb{Z}(q)).
\end{array}$$

Here the map (2)' (resp. (3)', (5)', (6)') is the composite $\iota_!\circ(2)$ (resp. $(3)\circ(\iota_!)^{-1}$, $\iota_!\circ(5)$, $(6)\circ(\iota_!)^{-1}$). (Note that the two Gysin maps $\iota_!$ are isomorphisms by Lemma 2.1).

Proof. This follows immediately from Theorem 1.1. □

5.2.

Lemma 5.2. *Let k be a perfect field and let X be a scheme which is smooth over* $\operatorname{Spec} k$*. Then the composite of the canonical homomorphism $H^0(X,\mathcal{O}_X^\times)\to K_1(X)$ and the Chern class map $c_1 : K_1(X)\to H^1_{\mathcal{M}}(X,\mathbb{Z}(1))$ is bijective.*

Proof. Since the composite $H^0(X,\mathcal{O}_X^\times)\to K_1(X)\xrightarrow{c_1} H^1_{\mathcal{M}}(X,\mathbb{Z}(1))\cong H^0(X,\mathcal{O}_X^\times)$ is functorial in X, it comes, by Yoneda's lemma, from an endomorphism $\alpha:\mathbb{G}_{m,k}\to\mathbb{G}_{m,k}$ of the multiplicative group scheme $\mathbb{G}_{m,k}$ over $\operatorname{Spec} k$. Note that α is equal to the n-th-power map for some integer n. Let us apply Corollary 5.1 for $X=\mathbb{A}^1_k$ and $U=\mathbb{G}_{m,k}$; we have the commutative diagram

$$\begin{array}{ccc}
K_1(\mathbb{G}_{m,k}) & \xrightarrow{c_1} & H^1_{\mathcal{M}}(\mathbb{G}_{m,k},\mathbb{Z}(1)) \\
{\scriptstyle \partial}\big\downarrow & & {\scriptstyle \partial}\big\downarrow \\
K_0(k) & \xrightarrow{\mathrm{ch}_0} & H^0_{\mathcal{M}}(k,\mathbb{Z}(0))=\mathbb{Z}
\end{array}$$

where ch_0 is the map defined in Section 2.1. Since the pullback map $K_0(\mathbb{A}^1_k)\to K_0(\mathbb{G}_{m,k})$ is an isomorphism and the group $H^2_{\mathcal{M}}(\mathbb{A}^1_k,\mathbb{Z}(1))$ is zero, the two vertical maps ∂ in the diagram above are surjective. Since the map ch_0 is an isomorphism, the composite $K_1(\mathbb{G}_{m,k})\xrightarrow{\partial}K_0(k)\xrightarrow{\mathrm{ch}_0}H^0_{\mathcal{M}}(k,\mathbb{Z}(0))=\mathbb{Z}$ is surjective. Since the canonical map $H^0(\mathbb{G}_{m,k},\mathcal{O}^\times_{\mathbb{G}_{m,k}})\to K_1(\mathbb{G}_{m,k})$ is an isomorphism, the image of the composite $K_1(\mathbb{G}_{m,k})\xrightarrow{c_1}H^1_{\mathcal{M}}(\mathbb{G}_{m,k},\mathbb{Z}(1))\xrightarrow{\partial}H^0_{\mathcal{M}}(k,\mathbb{Z}(0))$ is equal to $n\mathbb{Z}$. So we have $\mathbb{Z}=n\mathbb{Z}$, hence $n\in\{\pm1\}$. This proves the claim. □

5.3.

Lemma 5.3. *(cf.p.229 LEMMA 2.25* [Gi]*) Let X be a smooth k-scheme Let $i,j\geq 1$ be positive integers. Let $n\geq 1$. For an element $\gamma\in K_n(X)$, we have $c_i(\gamma)c_j(\gamma)=0$ in $H^{2(i+j)-n}_{\mathcal{M}}(X,\mathbb{Z}(i+j))$.*

Proof. An element in $K_n(X)$ is represented by a morphism $\gamma : S^n \wedge X \to \mathbb{Z} \times \mathrm{Gr}$ in $H_\bullet(k)$. The class $c_i(\gamma)c_j(\gamma)$ is represented by the composition $m(c_i \wedge c_j)\Delta\gamma$ in the following diagram:

$$\begin{array}{ccccc} S^n \wedge X & \xrightarrow{\Delta_{S^n\wedge X}} & (S^n \wedge X) \wedge (S^n \wedge X) & & \\ \gamma\big\downarrow & & \gamma\wedge\gamma\big\downarrow & & \\ \mathbb{Z} \times \mathrm{Gr} & \xrightarrow{\Delta} & (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) & \xrightarrow{m(c_i\wedge c_j)} & K(\mathbb{Z}(i+j), 2(i+j)). \end{array}$$

Here m is the product map (see Section 1), and Δ and $\Delta_{S^n\wedge X}$ are the diagonal maps. The diagram is obviously commutative. The claim that the map $\Delta_{S^n\wedge X}$ is nullhomotopic is reduced to the case X is a point, i.e., to the claim that the diagonal map $S^n \to S^n \wedge S^n$ is nullhomotopic if $n \geq 1$. This can be checked easily. Hence the composition $m(c_i \wedge c_j)\Delta\gamma$ is nullhomotopic. □

5.3.1. Using this lemma, we see that many terms of $P^d_q(c(-), c(N))$ in Theorem 1.1 are zero when $n \geq 1$. Let us give an explicit description. We refer to Section 3, especially Section 3.3, for some notation and abbreviation used here. Recall that the map $P^d_q(c(-), c(N))$ is defined using the power series $P^{d,e}_q$. From Lemma 5.3, we know that the terms involving c_ic_j are zero. So we only need to determine the monomials with one c_l for some $1 \leq l \leq d$ and the monomials with no c. We put

$$T^d_n(c') = T^d_n(c'_1, \ldots, c'_d) = {c'_d}^{-1} \sum_{I\subset\{1,\ldots,d\}} (-1)^{|I|}(\sum_{j\in I} y_j)^n.$$

Then we have

$$\begin{aligned} &P^{d,e}_q(c, c') \\ &= P^{d,e}_q(0, c') + \sum_{l=1}^{q} c_l \left\{ c'_d \sum_{r=1}^{q-l} P^{d,e}_r(0, c') \binom{q-r-1}{l-1} T^d_{q-r-l}(c') + \binom{q-1}{l-1} T^d_{q-l}(c') \right\} \end{aligned}$$

for $q \geq 1$ and $e \geq 0$. The computation is long and is omitted.

6. Compatibility with the pushforward of Thomason and Trobaugh

A comparison isomorphism between the K-groups of Morel-Voevodsky and of Thomason-Trobaugh is given in [Mo-Vo, p.140, Theorem 3.13]. We will show that the Gysin map defined in earlier sections is compatible with the pushforward map of Thomason-Trobaugh under this comparison isomorphism. The main result of this section is Proposition 6.9.

While the comparison isomorphism is known for K-groups of smooth k-schemes, it is not stated for K-theory with support. Since we need the isomorphisms for K-theory with support with some functoriality, we will give the construction of the comparison isomorphism for K-theory with support in detail.

6.1. Simplicial presheaves. We collect some known results from homotopy theory of simplicial presheaves in this section. These will be used in the construction of the comparison isomorphism. We did not find a reference for Lemma 6.2 and the proof is given.

6.1.1. Let $\mathcal{C}$ be a category with a final object. We let $\mathrm{sPre}_*\,\mathcal{C}$ denote the category of pointed simplicial presheaves on $\mathcal{C}$.

We let $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$ denote the category of pointed simplicial presheaves on $\mathcal{C}$ equipped with the injective model category structure. Recall that the weak equivalence of simplicial presheaves are defined as pointwise (sometimes called sectionwise) weak equivalence, the cofibrations as injections, and the fibrations are defined by the right lifting property. (We note that this is usually called "global injective" model structure on $\mathrm{sPre}_*\,\mathcal{C}$. We do not use the term "global" as this model structure does not resort to topology.)

The model category $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$ is a simplicial model category. For $F, G \in \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$, the simplicial set $\mathrm{Map}_*(F,G)$ is defined by setting $\mathrm{Map}_*(F,G)_n = \mathrm{Hom}_{\mathrm{sPre}_*\,\mathcal{C}}(F \wedge \Delta^n, G)$. The internal hom object will be denoted $\mathrm{MAP}_*(F,G) \in \mathrm{sPre}_*\,\mathcal{C}$. Explicitly, for an object U of $\mathcal{C}$, we have $\mathrm{MAP}_*(F,G)(U) = \mathrm{Map}_{*\mathcal{C}/U}(F|_U, G|_U)$. Here the right hand side uses the simplicial structure of $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}/U$ where $\mathcal{C}/U$ is the category over U, and $F|_U$ and $G|_U$ denote the restriction to $\mathcal{C}/U$.

6.1.2. We use the notation

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

to indicate an adjoint pair. Here F is always the left adjoint and G is the right adjoint. When the categories $\mathcal{C}$ and $\mathcal{D}$ are model categories and the above is a Quillen adjunction, we write

$$LF : \mathrm{Ho}\,\mathcal{C} \leftrightarrows \mathrm{Ho}\,\mathcal{D} : RG$$

for the induced adjunction between the homotopy categories.

6.1.3. We have a Quillen adjunction

$$\mathrm{cst}(-) : \mathrm{sSet}_* \rightleftarrows \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C} : \Gamma_*. \tag{6.1}$$

where cst is the constant presheaf functor and Γ_* is the global section functor (that is, the evaluation at the final object).

Let $F \in \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$. We have a Quillen adjunction

$$F \wedge - : \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C} \rightleftarrows \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C} : \mathrm{MAP}_*(F,-). \tag{6.2}$$

Using these adjunctions, we see that there are isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{sPre}_*\,\mathcal{C}}(G, \mathrm{MAP}_*(\mathrm{cst}(K), F)) \\ &\overset{(1)}{\cong} \mathrm{Hom}_{\mathrm{sPre}_*\,\mathcal{C}}(G \wedge \mathrm{cst}(K), F) \\ &\overset{(2)}{\cong} \mathrm{Hom}_{\mathrm{sSet}_*}(K, \mathrm{Map}_*(G,F)) \end{aligned} \tag{6.3}$$

for $K \in \mathrm{sSet}_*$ and $G, F \in \mathrm{sPre}_*\,\mathcal{C}$, which are functorial in K, G, F. We thus have an adjunction (not necessarily Quillen adjunction):

$$\mathrm{MAP}_*(\mathrm{cst}(-), F) : \mathrm{sSet}_* \leftrightarrows (\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C})^{\mathrm{opp}} : \mathrm{Map}_*(-,F). \tag{6.4}$$

where opp means the opposite category.

Lemma 6.1. *Let $F \in \mathrm{sPre}_*\,\mathcal{C}$ and $U \in \mathcal{C}$. Then there is an isomorphism in* sSet_*

$$F(U) \cong \mathrm{Map}_*(h_{U,+}, F)$$

which is functorial in F and in U. Here $h_{U,+}(-) = h_U(-) \coprod *$ is the pointed presheaf associated to the presheaf h_U represented by U.

Proof. Omitted. $\square$

Lemma 6.2. *Let F be fibrant. Then the functor* $\mathrm{MAP}_*(-, F)$ *sends cofibration to fibration.*

Proof. Let $f : G \to H$ be a cofibration in $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$. We show that the induced map $f^* : \mathrm{MAP}_*(H, F) \to \mathrm{MAP}_*(G, F)$ is a fibration in $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$. Suppose given a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "g"'] & \mathrm{MAP}_*(H,F) \arrow[d, "f^*"] \\ L \arrow[r, "\beta"] & \mathrm{MAP}_*(G,F) \end{tikzcd}$$

in $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$ with g a trivial cofibration. Let $\alpha' : H \wedge K \to F$ and $\beta' : G \wedge L \to F$ denote the maps which correspond to α and β via the adjunction (6.2)(2). We have a commutative diagram:

$$\begin{tikzcd} G \wedge K \arrow[r, "f \wedge \mathrm{id}_K"] \arrow[d, "\mathrm{id}_G \wedge g"'] & H \wedge K \arrow[d, "\alpha'"] \\ G \wedge L \arrow[r, "\beta'"] & F \end{tikzcd}$$

Let J denote the pushout of the diagram

$$H \wedge K \xleftarrow{f \wedge \mathrm{id}_K} G \wedge K \xrightarrow{\mathrm{id}_G \wedge g} G \wedge L.$$

Let $\gamma_1 : H \wedge K \to J$ and $\gamma_2 : G \wedge L \to J$ denote the maps given by the definition of the pushout. The universality of the pushout gives maps $\delta : J \to H \wedge L$ and $\delta' : J \to F$ such that $\delta \circ \gamma_1 = \mathrm{id}_H \wedge g$, $\delta \circ \gamma_2 = f \wedge \mathrm{id}_L$, $\delta \circ \gamma_2 = f \wedge \mathrm{id}_L$, and $\delta' \circ \gamma_2 = \beta'$.

We claim that the map $\delta : J \to H \wedge L$ is a trivial cofibration. We can check this as follows. It is easy to see that the map δ is a monomorphism, hence δ is a cofibration. Let us consider the sequence

$$H \wedge K \xrightarrow{\gamma_1} J \xrightarrow{\delta} H \wedge L.$$

Since the functor

$$- \wedge - : \mathrm{sPre}_*\,\mathcal{C} \times \mathrm{sPre}_*\,\mathcal{C} \to \mathrm{sPre}_*\,\mathcal{C}$$

preserves trivial cofibrations of each variable, the map $\delta \circ \gamma_1 = \mathrm{id}_H \wedge g$ is a trivial cofibration. By considering the sequence $G \wedge L \xrightarrow{\gamma_2} J \xrightarrow{\delta} H \wedge L$, we see in a similar manner that the map $\delta \circ \gamma_2 = f \wedge \mathrm{id}_L$ is also a trivial cofibration.

Since $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$ is left proper, γ_1 and γ_2 are weak equivalences, each being a pushout of a weak equivalence by a cofibration. By the 2-out-of-3 axiom, it follows that δ is a weak equivalence, hence it is a trivial cofibration.

Since we assumed that F is fibrant, from the commutative diagram

$$\begin{tikzcd} J \arrow[r, "\delta'"] \arrow[d, "\delta"'] & F \arrow[d] \\ H \wedge L \arrow[r] & * \end{tikzcd}$$

we obtain a map $\eta : H \wedge L \to F$ satisfying $\eta \circ \delta = \delta'$. Let $\eta' : L \to \mathrm{MAP}_*(H, F)$ denote the map corresponding to η via the adjunction (6.2). It is easy to check that $\eta' \circ g = \alpha$ and $f^* \circ \eta' = \beta$. Hence the map $f^* : \mathrm{MAP}_*(H, F) \to \mathrm{MAP}_*(G, F)$ has the right lifting property with respect to the trivial cofibration $g : K \to L$. Thus the map f^* is a fibration. This finishes the proof. □

Corollary 6.3. *Let F be a fibrant object in* $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$*. Then the functor* $\mathrm{Map}_*(-, F)$ *sends cofibrations to fibrations.*

Proof. This follows from Lemma 6.2 and the adjunction (6.1) by applying the global section functor. □

6.1.4. Let $\mathcal{C}$ be a site with enough points. We also write $\mathcal{C}$ for the underlying category. Let $\mathrm{sPre}_*^{\mathrm{inj,loc}}\,\mathcal{C}$ denote the category $\mathrm{sPre}_*\,\mathcal{C}$ equipped with the local injective model category structure. Recall that the weak equivalences are stalkwise weak equivalences, the cofibrations are injections, and the fibrations are defined by the right lifting property. There is a Quillen adjunction:

$$\mathrm{id} : \mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C} \rightleftarrows \mathrm{sPre}_*^{\mathrm{inj,loc}}\,\mathcal{C} : \mathrm{id}. \tag{6.5}$$

It is also known that $\mathrm{sPre}_*^{\mathrm{inj,loc}}\,\mathcal{C}$ is a left Bousfield localization of $\mathrm{sPre}_*^{\mathrm{inj}}\,\mathcal{C}$ (see [De-Ho-Is, p.24, THEOREM 6.2]).

6.1.5. Let k be a perfect field. Let $\mathcal{C} = (\mathrm{Sm}/k)_{\mathrm{Nis}}$ be the Nisnevich site of smooth k-schemes. Let $\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\,\mathcal{C}$ denote the motivic model category structure on simplicial presheaves. (For the definition we refer to [Ja2]. We use the variant for pointed presheaves.) There is a Quillen adjunction:

$$\mathrm{id} : \mathrm{sPre}_*^{\mathrm{inj,loc}}\,\mathcal{C} \rightleftarrows \mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\,\mathcal{C} : \mathrm{id}. \tag{6.6}$$

6.2. Thomason-Trobaugh K-theory. We collect some facts concerning the Thomason-Trobaugh K-theory. We followed the exposition of Jardine [Ja] whenever possible.

6.2.1. Let X be a scheme. We let $K(X)$ denote the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of $\mathcal{O}_X$-modules. We define the Thomason-Trobaugh K-groups of X to be $K_n^{\mathrm{TT}}(X) = \pi_n(K(X))$ for $n \geq 0$. (Note that we do not consider $K^B(X)$ of [TT, p.360, 6.4 Definition].)

6.2.2. Let $K_{\flat,\mathrm{big}}(X)$ denote the K-theory spectrum of the complicial biWaldhausen category of perfect bounded above complexes of flat $\mathcal{O}_X$-modules on the big Zariski site of X. (We impose a cardinality bound on the objects to avoid set theoretic problems.) They form a presheaf of spectra on the category $\mathcal{C} = (\mathrm{Sm}/k)$ of smooth k-schemes. It is known that $K_{\flat,\mathrm{big}}(X)$ is weakly equivalent to $K(X)$.

For a (Bousfield-Friedlander) spectrum F and $n \geq 0$, we let QF denote the standard stably fibrant model for F in spectra, with level spaces defined by $QF^n = \varinjlim_k \Omega^k F_f^{n+k}$, where F_f is a strictly fibrant model of F in spectra (see [Ja, p.157, Theorem 5]). We may take F_f functorial in F so that QF is also functorial in F.

We obtain a presheaf of spaces, denoted $Q(K)^0(-)$ whose section at X is $Q(K_{\flat,\mathrm{big}}(X))^0$.

We let $Q(K)^0 \to GQ(K)^0$ denote the functorial fibrant replacement in $\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\,\mathcal{C}$. (We sheafify $Q(K)^0$, apply the explicit fibrant replacement functor of [Mo-Vo, p.69, Theorem 1.66], then regard it as a presheaf.) As K-theory satisfies Nisnevich descent ([TT, 10.8 Theorem, p.139]), it follows from a theorem of [Mo-Vo] (we refer to [Ja2, p.457, COROLLARY 1.4] for the details) that $Q(K)^0$ and $GQ(K)^0$ are pointwise weakly equivalent

6.2.3. Let $U \subset Y$ be an open immersion of schemes. We let $X = Y \setminus U$ denote the closed complement (as a topological space). Write $K(Y \,\mathrm{on}\, X) = K_{\flat,\mathrm{big}}(Y \,\mathrm{on}\, X)$ for the K-theory spectrum of the complicial biWaldhausen category of perfect bounded above complexes of flat $\mathcal{O}_Y$-modules that are acyclic when restricted to U.

Let $K(U)^{\mathrm{ac}} = K_{\flat,\mathrm{big}}(U)^{\mathrm{ac}}$ denote the K-theory spectrum of the complicial biWaldhausen category of perfect bounded above complexes of flat $\mathcal{O}_U$-modules that are acyclic.

There is a commutative diagram of spectra

$$\begin{tikzcd} K(Y \,\mathrm{on}\, Z) \arrow[r, "(2)"] \arrow[d, "(1)"'] & K(U)^{\mathrm{ac}} \arrow[d, "(4)"] \\ K(Y) \arrow[r, "(3)"] & K(U) \end{tikzcd}$$

where the map (1) is the forgetful map, the maps (2) (3) are the restriction maps, and (4) the forgetful map. We then obtain by taking $Q(-)^0$ a commutative diagram

$$\begin{tikzcd} Q(K(Y \,\mathrm{on}\, Z))^0 \arrow[r] \arrow[d] & Q(K(U)^{\mathrm{ac}})^0 \arrow[d] \\ Q(K(Y))^0 \arrow[r] & Q(K(U))^0 \end{tikzcd}$$

of spaces.

6.3. In this section and the next, we construct the map α with some subscript. This is roughly one half of the comparison isomorphism.

6.3.1. Consider the following setup in sSet_*. Let $X_i, Y_i \,(1 \le i \le 4)$ be pointed spaces with commutative diagrams

$$\begin{tikzcd} X_1 \arrow[r, "x_{12}"] \arrow[d, "x_{13}"'] & X_2 \arrow[d, "x_{24}"] \\ X_3 \arrow[r, "x_{34}"] & X_4, \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[r, "y_{12}"] \arrow[d, "y_{13}"'] & Y_2 \arrow[d, "y_{24}"] \\ Y_3 \arrow[r, "y_{34}"] & Y_4. \end{tikzcd}$$

Suppose given the following commutative diagram in sSet_*:

$$\begin{tikzcd} Y_3 \arrow[r, "y_{34}"] \arrow[d, "="'] & Y_4 \arrow[d, "="'] & Y_2 \arrow[l, "y_{24}"'] \arrow[d, "\gamma"] \\ Y_3 \arrow[r, "y_{34}"] & Y_4 & X_2 \arrow[l, "\beta\circ x_{24}"'] \\ X_3 \arrow[r, "x_{34}"] \arrow[u, "\alpha"] & X_4 \arrow[u, "\beta"] & X_2. \arrow[l, "x_{24}"'] \arrow[u, "="'] \end{tikzcd}$$

We obtain a zigzag of maps in sSet_* from X_1 to Y_1 as follows:

$$X_1 \to X_2 \times_{X_4} X_3 \xrightarrow{\theta} X_2 \times_{Y_4} Y_3 \xleftarrow{\eta} Y_2 \times_{Y_4} Y_3 \leftarrow Y_1$$

Here the first and the last maps are those obtained from the first two diagrams, and the second and the third maps are obtained from the large diagram.

Lemma 6.4. *Suppose*

- *$Y_2 = *$,*
- *γ is a weak equivalence,*
- *X_2, Y_4, and Y_3 are fibrant.*

Then η is a weak equivalence.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} * \times_{Y_4} Y_3 & \xrightarrow{\mathrm{Ho}(\eta)} & X_2 \times_{Y_4} Y_3 \\ \big\downarrow & & \big\downarrow \\ * \times^h_{Y_4} Y_3 & \xrightarrow{\delta} & X_2 \times^h_{Y_4} Y_3 \end{array}$$

in $\mathrm{Ho}(\mathrm{sSet}_*)$. Here the vertical arrows are the canonical morphisms in $\mathrm{Ho}(\mathrm{sSet}_*)$ from the ordinary fiber product to homotopy fiber product. The lower horizontal arrow δ is the map induced by γ. It follows from [Go-Ja, p.313, Lemma VI.1.10] that the vertical arrows are isomorphisms in $\mathrm{Ho}(\mathrm{sSet}_*)$. Since γ is a weak equivalence, the lower horizontal arrow is an isomorphism in $\mathrm{Ho}(\mathrm{sSet}_*)$. This implies that $\mathrm{Ho}(\eta)$ is an isomorphism in $\mathrm{Ho}(\mathrm{sSet}_*)$. □

If we further assume that the diagram of the Y_i's is homotopy cartesian and Y_1 is fibrant, then the zigzag above gives a map $X_1 \to Y_1$ in $\mathrm{Ho}(\mathrm{sSet}_*)$ by replacing each arrow in the opposite direction by its inverse in $\mathrm{Ho}(\mathrm{sSet}_*)$.

6.4. Let $\iota : X \hookrightarrow Y$ be a closed immersion of smooth k-schemes. Let $U = Y \setminus X$ be the open complement.

6.4.1. We let $X_1 = Q(K(Y \operatorname{on} X))^0$, $X_2 = Q(K(U)^{\mathrm{ac}})^0$, $X_3 = Q(K(Y))^0$, and $X_4 = Q(K(U))^0$.

Lemma 6.5. *The canonical map $* \to X_2$ is a weak equivalence.*

Proof. Omitted. See the remark in [TT, p.328, 3.22]. □

6.4.2. We let $Y_1 = \mathrm{Map}_*(Y/U, GQ(K)^0)$, $Y_2 = *$, $Y_3 = \mathrm{Map}_*(h_{Y,+}, GQ(K)^0)$, and $Y_4 = \mathrm{Map}_*(h_{U,+}, GQ(K)^0)$. We obtain a commutative diagram as in Section 6.3 where the maps y_{12} and y_{23} are the canonical maps, y_{13} is the map induced by $Y \to Y/U$, and y_{34} is the map induced by $U \to Y$. From Lemma 6.3, it follows that Y_1, Y_3, Y_4 are fibrant. From the adjunction (6.4), we see that the functor $\mathrm{Map}_*(-, GQ(K)^0)$ is a right adjoint and hence sends cocartesian diagrams in $\mathrm{sPre}_* \mathcal{C}$ to cartesian diagrams in sSet_*. Note that the diagram

$$\begin{array}{ccc} h_{U,+} & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ h_{Y,+} & \longrightarrow & Y/U \end{array}$$

is cocartesian in $\mathrm{sPre}_* \mathcal{C}$. Applying $\mathrm{Map}_*(-, GQ(K)^0)$ gives the diagram in the Y_i's, which is hence a cartesian diagram in sSet_*. This is also homotopy cartesian, since by [Go-Ja, p.313, Lemma VI.1.10], Y_1 is isomorphic to the homotopy fiber product $Y_3 \times^h_{Y_4} *$.

6.4.3. We define the map $\alpha : X_3 \to Y_3$ as the composite

$$Q(K(Y))^0 = Q(K)^0(Y) \xrightarrow{(*)} GQ(K)^0(Y) \cong \mathrm{Map}_*(h_{Y,+}, GQ(K)^0)$$

where the map $(*)$ is that induced by the map $Q(K)^0 \to GQ(K)^0$ of presheaves, and the isomorphism is that of Lemma 6.1. The map $\beta : X_4 \to Y_4$ is defined in a similar manner. We write

$$\alpha_\iota : Q(K(Y \,\mathrm{on}\, X))^0 \to \mathrm{Map}_*(Y/U, GQ(K)^0)$$

for the map in $\mathrm{Ho(sSet_*)}$ obtained from the zigzag of maps. We also write

$$\alpha_\iota : K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) \to \mathrm{Hom}_{\mathrm{Ho(sSet_*)}}(S^n, \mathrm{Map}_*(Y/U, GQ(K)^0))$$

for the homomorphisms induced from the α_ι above for $n \geq 0$.

Lemma 6.6. *Let $f : Y' \to Y$ be a morphism in $\mathcal{C}$ and let $X' = Y' \times_Y X$. Then the diagram*

$$\begin{tikzcd} Q(K(Y \,\mathrm{on}\, X))^0 \arrow[r, "\alpha_\iota"] \arrow[d] & \mathrm{Map}_*(Y/(Y \setminus X), GQ(K)^0) \arrow[d] \\ Q(K(Y' \,\mathrm{on}\, X'))^0 \arrow[r, "\alpha_{\iota'}"] & \mathrm{Map}_*(Y'/(Y' \setminus X'), GQ(K)^0) \end{tikzcd}$$

where $\iota' : X' \to Y'$ is the base change of ι and the vertical maps are the pullback maps, is commutative.

Proof. Recall that the map α_ι was defined using X_i, Y_i for $1 \leq i \leq 4$, θ, and η. Let us write $X_i', Y_i', \theta', \eta'$ for the corresponding objects for the closed immersion ι'. To prove the claim, it suffices to show that the following diagram in sSet_* is commutative:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \times_{X_4} X_3 \arrow[r, "\theta"] \arrow[d] & X_2 \times_{Y_4} Y_3 \arrow[d] & Y_2 \times_{Y_4} Y_3 \arrow[l, "\eta"'] \arrow[d] & Y_1 \arrow[l] \arrow[d] \\ X_1' \arrow[r] & X_2' \times_{X_4'} X_3' \arrow[r, "\theta'"] & X_2' \times_{Y_4'} Y_3' & Y_2' \times_{Y_4'} Y_3' \arrow[l, "\eta'"'] & Y_1'. \arrow[l] \end{tikzcd}$$

Here the vertical maps are the maps induced by the pullback maps. This is straightforward. ☐

Corollary 6.7. *The following diagram is commutative:*

$$\begin{tikzcd} K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) \arrow[r, "\alpha_\iota"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathrm{Ho(sSet_*)}}(S^n, \mathrm{Map}_*(Y/(Y \setminus X), GQ(K)^0)) \arrow[d] \\ K_n^{\mathrm{TT}}(Y' \,\mathrm{on}\, X') \arrow[r, "\alpha_{\iota'}"] & \mathrm{Hom}_{\mathrm{Ho(sSet_*)}}(S^n, \mathrm{Map}_*(Y'/(Y' \setminus X'), GQ(K)^0)) \end{tikzcd}$$

Proof. This follows immediately from the previous lemma. ☐

6.5. In this section, we construct a map γ with some subscript. This is roughly the other one half of the comparison isomorphism.

6.5.1. Let $Z \in \mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}$ be a simplicial sheaf. For an integer $n \geq 0$, we put $K_n^{\mathrm{MV}}(Z) = \mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(S_s^n \wedge Z, \mathbb{Z} \times \mathrm{Gr})$, where S_s^1 is the sheafification of the constant presheaf $\mathrm{cst}(S^1)$. For a smooth k-scheme X, we put $K_n^{\mathrm{MV}}(X) = \mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(S_s^n \wedge h_{X,+}, \mathbb{Z} \times \mathrm{Gr})$.

We define an isomorphism

$$\gamma_Z : K_n^{\mathrm{MV}}(Z) \to \mathrm{Hom}_{\mathrm{sSet}_*}(S^n, \mathrm{Map}_*(Z, GQ(K)^0))$$

as the composite:

$$\begin{aligned}
&K_n^{\mathrm{MV}}(Z)\\
&= \mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(S_s^n \wedge Z, \mathbb{Z} \times \mathrm{Gr}) \overset{(1)}{\cong} \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, \mathbb{Z} \times \mathrm{Gr})\\
&\overset{(2)}{\cong} \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, GQ(K)^0) \overset{(3)}{\cong} \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj}}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, GQ(K)^0)\\
&\overset{(4)}{\cong} \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj}}\mathcal{C}}(\mathrm{cst}(S^n), \mathrm{MAP}_*(Z, GQ(K)^0)) \overset{(5)}{\cong} \mathrm{Hom}_{\mathrm{sSet}_*}(S^n, \mathrm{Map}_*(Z, GQ(K)^0)).
\end{aligned}$$

Let us give the description of the maps (1)-(5) in the following subsections. It is easy to see from the construction that γ_Z is functorial in Z.

6.5.2. The Quillen adjunction ([Ja2, p.452, THEOREM 1.2])

$$(-)^a : \mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1} \leftrightarrows \mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1} : \mathrm{for},$$

where $(-)^a$ is the associated sheaf functor and for is the forgetful functor regarding sheaf as a presheaf, gives an isomorphism

$$\mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}}(L(\mathrm{cst}(S^n) \wedge Z)^a, \mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}}(\mathrm{cst}(S^n) \wedge Z, R(\mathrm{for})(\mathbb{Z} \times \mathrm{Gr})).$$

The object $L(\mathrm{cst}(S^n) \wedge Z)^a$ is canonically weakly equivalent to the sheaf associated to the cofibrant replacement of $\mathrm{cst}(S^n) \wedge Z$. As $\mathrm{cst}(S^n) \wedge Z$ is cofibrant, we only need to check that $(\mathrm{cst}(S^n) \wedge Z)^a = S_s^n \wedge Z$, which follows from the definition of the wedge product. The object $R(\mathrm{for})(\mathbb{Z} \times \mathrm{Gr})$ is weakly equivalent to the fibrant replacement of $\mathbb{Z} \times \mathrm{Gr}$ regarded as a presheaf. Since $R(\mathrm{for})(\mathbb{Z} \times \mathrm{Gr})$ is weakly equivalent to $\mathbb{Z} \times \mathrm{Gr}$ as sheaves, it is so as presheaves also. The isomorphism (1) is defined to be the composite

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(S_s^n \wedge Z, \mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{Hom}_{\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(L(\mathrm{cst}(S^n) \wedge Z)^a, \mathbb{Z} \times \mathrm{Gr})\\
&\cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, R(\mathrm{for})(\mathbb{Z} \times \mathrm{Gr})) \cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, \mathbb{Z} \times \mathrm{Gr}).
\end{aligned}$$

6.5.3. In $\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}$, we have a weak equivalence

$$\mathbb{Z} \times \mathrm{Gr} \simeq Q(K)^0$$

See [Ja, p.176, (17)].

By definition, the canonical map $Q(K)^0 \to GQ(K)^0$ is a weak equivalence. The composition gives the isomorphism (2).

6.5.4. From the Quillen adjunction (6.6), we obtain an isomorphism

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc}}\mathcal{C}}(L(\mathrm{id})(\mathrm{cst}(S^n) \wedge Z), GQ(K)^0)\\
&\cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, R(\mathrm{id})GQ(K)^0).
\end{aligned}$$

Since $GQ(K)^0$ is fibrant in $\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}$, it is also fibrant in $\mathrm{sPre}_*^{\mathrm{inj,loc}}$, hence $R(\mathrm{id})GQ(K)^0$ is canonically weakly equivalent to $GQ(K)^0$. We have $L(\mathrm{id})(\mathrm{cst}(S^n) \wedge$

$Z) \simeq \mathrm{cst}(S^n) \wedge Z$ since $\mathrm{cst}(S^n) \wedge Z$ is cofibrant. We obtain an isomorphism $\mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, GQ(K)^0) \cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc}}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, GQ(K)^0)$.

From the Quillen adjunction (6.5), we obtain an isomorphism $\mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj,loc}}\mathcal{C}}(\mathrm{cst}(S^n)\wedge Z, GQ(K)^0) \cong \mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj}}\mathcal{C}}(\mathrm{cst}(S^n) \wedge Z, GQ(K)^0)$ by an argument similar to that in the previous paragraph. The isomorphism (3) is obtained as the composite of these two isomorphisms.

6.5.5. As seen above, $GQ(K)^0$ is fibrant in $\mathrm{sPre}_*^{\mathrm{inj}}\mathcal{C}$. Using the Quillen adjunction (6.2), we obtain the isomorphism (4).

6.5.6. By Lemma 6.2, we see that $\mathrm{MAP}_*(Z, GQ(K)^0)$ is fibrant. Then from the Quillen adjunction (6.1) we obtain the isomorphism

$$\mathrm{Hom}_{\mathrm{sPre}_*^{\mathrm{inj}}\mathcal{C}}(\mathrm{cst}(S^n), \mathrm{MAP}_*(Z, GQ(K)^0)) \cong \mathrm{Hom}_{\mathrm{sSet}_*}(S^n, \Gamma_*\mathrm{MAP}_*(Z, GQ(K)^0)).$$

By definition, $\Gamma_*\mathrm{MAP}_* = \mathrm{Map}_*$. Thus we obtain (5).

6.6. Deformation to normal cone. The definition of the Gysin map in Section 2.2.3 uses the isomorphism $Y/(Y \setminus \iota(X)) \cong \mathrm{Th}(N)$. Let us recall the setup for deformation to normal cone that leads to this isomorphism. What follows is taken from [Mo-Vo, p.115].

Let $q : B \to X \times \mathbb{A}^1$ denote the blow-up of $\iota(X) \times \{0\}$ in $Y \times \mathbb{A}^1$. We have a canonical closed immersion $f : X \times \mathbb{A}^1 \to B$ which splits q over $\iota(X) \times \mathbb{A}^1$, and a canonical closed immersion $g : X \to B$ which splits q over $X \times \{1\}$. There is a canonical isomorphism $q^{-1}(\iota(X) \times \{0\}) \cong \mathbb{P}(N \oplus \mathbf{1})$ which induces an isomorphism $q^{-1}(\iota(X) \times \{0\}) \setminus f(X \times \{0\}) \cong \mathbb{P}(N \oplus \mathbf{1}) \setminus \mathbb{P}(\mathbf{1})$. Hence we obtain a diagram

$$\begin{array}{ccccc} X & \xrightarrow{1} & X \times \mathbb{A}^1 & \xleftarrow{0} & X \\ \iota\downarrow & & f\downarrow & & \downarrow \\ Y & \xrightarrow{1} & B & \xleftarrow{0} & \mathbb{P} \end{array}$$

where the vertical arrows are closed immersions, the squares are cartesian, and $\mathbb{P} = \mathbb{P}(N \oplus \mathbf{1})$.

6.6.1. In $\mathrm{sShv}_*^{\mathrm{inj,loc},\mathbb{A}^1}\mathcal{C}$, we have $\mathrm{Th}(N) \cong q^{-1}(\iota(X) \times \{0\})/q^{-1}(\iota(X) \times \{0\}) \setminus f(X \times \{0\}))$. Hence we obtain two morphisms

$$\begin{array}{l} 1 : Y/(Y \setminus X) \to B/(B \setminus f(X \times \mathbb{A}^1)) \\ 0 : \mathbb{P}(N \oplus 1)/\mathbb{P}(N) \to B/(B \setminus f(X \times \mathbb{A}^1)) \end{array}$$

induced from the morphisms 0 and 1 in the previous section.

By definition of $\iota_!$, we have the following commutative diagram:

$$\begin{array}{ccccc} K_n^{\mathrm{MV}}(X) & \xleftarrow{1^*} & K_n^{\mathrm{MV}}(X \times \mathbb{A}^1) & \xrightarrow{0^*} & K_n^{\mathrm{MV}}(X) \\ \downarrow\iota_! & & & & \downarrow\iota_! \\ K_n^{\mathrm{MV}}(Y/U) & \xleftarrow{1^*} & K_n^{\mathrm{MV}}(B/(B \setminus (X \times \mathbb{A}^1))) & \xrightarrow{0^*} & K_n^{\mathrm{MV}}(\mathbb{P}/(\mathbb{P} \setminus X)) \end{array} \tag{6.7}$$

where the right vertical $\iota_!$ is the one in the diagram (2.3).

6.6.2. We consider the following diagram:

$$
\begin{tikzcd}
K_n^{\mathrm{TT}}(X) \arrow[d, "\iota_*"'] & K_n^{\mathrm{TT}}(X \times \mathbb{A}^1) \arrow[l, "1^*"'] \arrow[r, "0^*"] \arrow[d, "f_*"] & K_n^{\mathrm{TT}}(X) \arrow[d, "s_*"] \\
K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) & K_n^{\mathrm{TT}}(B \,\mathrm{on}\, X \times \mathbb{A}^1) \arrow[l, "1^*"'] \arrow[r, "0^*"] & K_n^{\mathrm{TT}}(\mathbb{P} \,\mathrm{on}\, X).
\end{tikzcd}
\tag{6.8}
$$

The upper horizontal arrows are isomorphisms since the K-theory of a regular scheme is $\mathbb{A}^1$-invariant ([TT, p.362, 6.8 Proposition]). It follows from the base change theorem ([TT, p.321, 3.18 Proposition], see the remark at the end of proof) that the two squares are commutative.

The lower horizontal arrows are isomorphisms. For 1^*, this can be seen using the following commutative diagram:

$$
\begin{tikzcd}
K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) \arrow[d] & K_n^{\mathrm{TT}}(B \,\mathrm{on}\, X \times \mathbb{A}^1) \arrow[l, "1^*"'] \arrow[d] \\
K_n^{\mathrm{TT}}(X) & K_n^{\mathrm{TT}}(X \times \mathbb{A}^1) \arrow[l, "1^*"']
\end{tikzcd}
$$

where the vertical maps are pullback maps. The vertical maps are isomorphisms by excision ([TT, p.322, 3.19 Proposition]) and the lower horizontal map is an isomorphism by the $\mathbb{A}^1$-invariance. One can show that 0^* is an isomorphism in a similar manner. Hence the claim follows.

6.6.3.

Lemma 6.8. *Let $\mathcal{I}_X$ be the ideal sheaf defining $X \hookrightarrow \mathbb{P}$. Let $E_\bullet$ be a strict perfect complex on $\mathbb{P}$ which is quasi-isomorphic to $\mathcal{O}_{\mathbb{P}}/\mathcal{I}_X$. Then we have the following commutative diagram:*

$$
\begin{tikzcd}
K_n^{\mathrm{TT}}(X) \arrow[r, "p^*"] \arrow[d, "s_*"'] & K_n^{\mathrm{TT}}(\mathbb{P}) \arrow[d, "- \cup E_\bullet"] \\
K_n^{\mathrm{TT}}(\mathbb{P} \,\mathrm{on}\, X) \arrow[r] & K_n^{\mathrm{TT}}(\mathbb{P})
\end{tikzcd}
\tag{6.9}
$$

where the lower horizontal arrow is the forget support map.

Proof. For a bounded above complex of flat $\mathcal{O}_X$-module F, we have a canonical quasi-isomorphism

$$s_* F = p^* F \otimes \mathcal{O}_{\mathbb{P}}/\mathcal{I}_X \simeq \mathrm{Tot}(p^* F \otimes E_\bullet).$$

The claim follows from this. □

6.7. We prove the main comparison result (Proposition 6.9) in this section.

6.7.1. Let $X \in \mathcal{C}$ be a smooth k-scheme. We write $\gamma_X = \gamma_{h_{X,+}}$. We put $\beta_X = \gamma_X^{-1} \circ \alpha_X : K^{\mathrm{TT}}(X) \to K^{\mathrm{MV}}(X)$.

Let $\iota : X \hookrightarrow Y$ be a closed immersion of smooth k-schemes. Let $U = Y \setminus X$ be the open complement. We put $\beta_\iota = \gamma_{Y/U}^{-1} \circ \alpha_\iota : K^{\mathrm{TT}}(Y \,\mathrm{on}\, X) \to K^{\mathrm{MV}}(Y/U)$.

Proposition 6.9. *Let $\iota : X \hookrightarrow Y$ be a closed immersion of smooth k-schemes. For $n \geq 0$, the following diagram is commutative:*

$$\begin{array}{ccc} K_n^{\mathrm{TT}}(X) & \xrightarrow{\beta_X} & K_n^{\mathrm{MV}}(X) \\ {\scriptstyle \iota_*}\big\downarrow & & \big\downarrow{\scriptstyle \iota_!} \\ K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) & \xrightarrow{\beta_\iota} & K_n^{\mathrm{MV}}(Y/(Y \setminus X)). \end{array}$$

Here ι_* is the pushforward for Thomason-Trobaugh K-theory and $\iota_!$ is the Gysin map of Section 2.2.3.

Proof. In view of the commutative diagrams (2.3), (6.7), (6.8), (6.9), and recalling the definition of the Gysin map $\iota_!$, it suffices to prove that, for a strict perfect complex $E_\bullet$ which is quasi-isomorphic to $\mathcal{O}_{\mathbb{P}}/\mathcal{I}_X$ where $\mathcal{I}_X$ is the ideal sheaf defining $X \hookrightarrow \mathbb{P}$, the following diagram is commutative:

$$\begin{array}{ccc} K_n^{\mathrm{TT}}(X) & \xrightarrow{\beta_X} & K_n^{\mathrm{MV}}(X) \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle p^*} \\ K_n^{\mathrm{TT}}(\mathbb{P}) & \xrightarrow{\beta_{\mathbb{P}}} & K_n^{\mathrm{MV}}(\mathbb{P}) \\ {\scriptstyle -\cup E_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle -\cup \mathrm{th}^{\mathrm{MV}}} \\ K_n^{\mathrm{TT}}(\mathbb{P}) & \xrightarrow{\beta_{\mathbb{P}}} & K_n^{\mathrm{MV}}(\mathbb{P}) \\ \big\uparrow & & \big\uparrow \\ K_n^{\mathrm{TT}}(\mathbb{P} \,\mathrm{on}\, X) & \xrightarrow{\beta_s} & K_n^{\mathrm{MV}}(\mathbb{P}/(\mathbb{P} \setminus X)) \\ {\scriptstyle 0^*}\big\uparrow & & \big\uparrow{\scriptstyle 0^*} \\ K_n^{\mathrm{TT}}(B \,\mathrm{on}\, X \times \mathbb{A}^1) & \xrightarrow{\beta_f} & K_n^{\mathrm{MV}}(B/B \setminus (X \times \mathbb{A}^1)) \\ {\scriptstyle 1^*}\big\downarrow & & \big\downarrow{\scriptstyle 1^*} \\ K_n^{\mathrm{TT}}(Y \,\mathrm{on}\, X) & \xrightarrow{\beta_\iota} & K_n^{\mathrm{MV}}(Y/(Y \setminus X)). \end{array}$$

It is easy to see that the first square is commutative. The commutativity of the third, fourth and fifth squares follows from Corollary 6.7 and that $\gamma_{(-)}$ is functorial with respect to pullbacks. The claim for the second square is the following lemma.

Lemma 6.10. *The second square is commutative.*

Proof. The second square with $-\cup \mathrm{th}^{\mathrm{MV}}$ replaced with $-\cup \beta_{\mathbb{P}}(E_\bullet)$ is commutative since by [Ri3, p.240, PROPOSITION 3.2.1] the product structures of K^{TT} and K^{MV} are compatible. Now by the construction of the Chern structure of K^{MV},

$$\beta_{\mathbb{P}}([\mathcal{O}_{\mathbb{P}}/\mathcal{I}_X]) = c_d(\mathcal{O}_{\mathbb{P}}(1) \otimes p^*(N)) = \mathrm{th}^{\mathrm{MV}}$$

where c_d is the Chern class for K^{MV}. The first equality follows by using $[\mathcal{O}_{\mathbb{P}}/\mathcal{I}_X] = \sum_{i=0}^{d}(-1)^i[\wedge^i p^*N \otimes \mathcal{O}_{\mathbb{P}}(-i)]$ in $K_0^{\mathrm{MV}}(\mathbb{P})$. This proves the claim. □

This proves the proposition. □

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