Microlocal properties of sheaves and complex WKB.

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1 Introduction

In this paper we are going to study the following PDE on one unknown function Ψ in two complex variables x, s:

$$-\Psi_{xx} + V(x)\Psi_{ss} = 0, \tag{1}$$

where $V(x)$ is a given entire function which has finitely many zeros.

This equation is related to the Schrödinger equation

$$-h^2\partial_x^2\psi(x,h) + V(x)\psi(x,h) = 0 \tag{2}$$

by means of the Laplace transform $1/h \mapsto \partial_s$. According to resurgent analysis, the analytic behavior of $\Psi(x, s)$ determines quasi-classical asymptotics of solutions of (2).

A multivalued solution Ψ of (1) can be specified by means of prescribing its initial values. Our problem is now as follows. Consider a class of initial value problems for (1) with a fixed type of the analytic behavior of the initial data; we are to find a manifold where solutions of these problems are defined.

1.1 Cauchy problem

We study the Cauchy problem for (1) of the following type. We fix a point $x_0 \in \mathbb{C}$ and prescribe $\Psi(x_0, s) = \psi_0(s)$ and $\frac{\partial \Psi(x,s)}{\partial x}|_{x=x_0} = \psi_1(s)$ as multivalued analytic functions of s. Let us now give a more precise account.

1.1.1 Initial data

Fix an acute angle $\alpha \in (0, \pi/2)$. Let $S_\alpha := (0,\infty)\times(-\alpha, \alpha+2\pi)$ be an open sector of aperture $2\pi+2\alpha$. Let $\pi_{S_\alpha} : S_\alpha \to \mathbb{C}$ be the covering map $\pi_{S_\alpha}(r,\phi) := re^{i\phi}$. The map π_{S_α} induces a complex structure on S_α so that π_{S_α} is a local biholomorphism. The initial conditions are given by two holomorphic functions

$$\psi_0 \text{ and } \psi_1 \text{ on } S_\alpha. \tag{3}$$

1.2 Multi-valued solution to a multi-valued Cauchy problem

We first fix a complex surface $\mathcal{S}$ along with a local biholomorphism $p_{\mathcal{S}} : \mathcal{S} \to \mathbb{C}\times\mathbb{C}$. Let us also fix a map

$$h : S_\alpha \to \mathcal{S} \tag{4}$$

fitting into the following commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "i_{x_0}"] & \mathbb{C}\times\mathbb{C} \\ S_\alpha \arrow[u, "\pi_{S_\alpha}"] \arrow[r, "h"] & \mathcal{S} \arrow[u, "p_{\mathcal{S}}"'] \end{tikzcd}$$

where $i_{x_0} : \mathbb{C} \to \mathbb{C}\times\mathbb{C}$ is given by the formula $i_{x_0}(s) = (x_0, s)$.

The equation (1) gets transferred onto $\mathcal{S}$ by means of a local biholomorphism $p_{\mathcal{S}}$. Call this equation "the transferred equation".

The coordinates (x,s) on $\mathbb{C}\times\mathbb{C}$ give rise to local coordinates on $\mathcal{S}$. Given a function Ψ on $\mathcal{S}$, we then have a well defined derivative $\dfrac{\partial\Psi}{\partial x}$ as a holomorphic function on $\mathcal{S}$.

We say that a solution Ψ of the transferred equation <u>is a solution of the Cauchy problem with initial data (3)</u> on $\mathcal{S}$, if $\Psi\circ h = \psi_0$; $\dfrac{\partial\Psi}{\partial x}\circ h = \psi_1$.

1.3 Formulation of the result

Our main result is a construction of a complex surface $\mathcal{S}$ and a map h as in (4), such that for every choice of the initial data, there exists a unique solution Ψ of the Cauchy problem on $\mathcal{S}$.

We prove (Sec. 3.16) that the surface $\mathcal{S}$ is "extends infinitely in the direction of K", where $K \in \mathbb{C}$ is the following cone:

$$K := \{re^{i\phi}; r \geq 0; -\alpha \leq \phi \leq \alpha\}. \tag{5}$$

Let us give a more precise formulation. Fix a point $x \in \mathbb{C}$ such that $V(x) \neq 0$. Consider a one-dimensional complex manifold $\mathcal{S}^x := p_{\mathcal{S}}^{-1}(x \times \mathbb{C})$, where the projection onto $x \times \mathbb{C}$ gives a local biholomorphism $P^x : \mathcal{S}^x \to \mathbb{C}$. Let $\mathbf{U} \subset \mathbb{C}$ be an open parallelogram whose sides are parallel to vectors $e^{i\alpha}$ and $e^{-i\alpha}$. Let $\sigma : \mathbf{U} \to \mathcal{S}^x$ be a section of P^x. Let also $\mathbf{r}_{-\alpha} \subset K$ be the ray $[0, \infty).e^{-i\alpha}$.

We prove that

Theorem 1.1 *There exists a set* $\Gamma \subset \mathbb{C}$ *satisfying:*

1) for every point $s \in \mathbb{C}$*, the intersection* $(s - K) \cap \Gamma$ *is at most finite,*

2) $\mathbf{U} \subset (\mathbf{U} + K)\backslash(\Gamma + \mathbf{r}_{-\alpha})$*;*

3) σ *extends uniquely onto* $(\mathbf{U} + K)\backslash(\Gamma + \mathbf{r}_{-\alpha})$*.*

This theorem is proved in Sec.3.16: it easily follows from Theorem 3.12, as explained after its formulation.

In [V83] a similar problem was treated non-rigorously.

Our construction of $\mathcal{S}$, as well as the proof of the above Theorem 1.1, are based on sheaf-theoretical methods [KS]. The relation between linear PDEs and sheaves is well known and consitutes the subject of Algebraic Analysis.

In the next subsection, we will briefly describe the idea of our sheaf-theoretic approach.

1.4 Introducing sheaves

We start with introducing a covering space X of $\mathbb{C}$, and defining the so-called action function on X.

1.4.1 A covering space X

Let **TP** be the set of zeros of $V(x)$ – "turning points" of $V(x)$. We assume throughout the paper that **TP** is finite. We also assume $x_0 \notin \mathbf{TP}$. Let X be the universal covering of $\mathbb{C}\backslash TP$. We can choose a determination of $\sqrt{V(x)}$ and its primitive $S(x) = \int^x \sqrt{V(\xi)}d\xi$ on X. It will be more convenient for us to use the notation $z := S(x)$. Since $dS(x)$ is nowhere vanishing on X, we can use z as a local coordinate on X. As above, we denote by s the coordinate on $\mathbb{C}$, so that (z, s) are local coordinates on $X \times \mathbb{C}$.

Equation (1) gets transfered onto $X \times \mathbb{C}$ and in the coordinates (z, s) it looks as follows:

$$-\Psi_{zz} + \Psi_{ss} + \text{l.o.t.} \; = \; 0 \tag{6}$$

where l.o.t. stands for a differential operator of order ≤ 1 applied to Ψ. We now pass to a sheaf-theoretical consideration.

1.4.2 Solution sheaf and its singular support

Let Sol be the solution sheaf of (6). According to [KS, Th.11.3.3], the singular support of Sol is of a very special form which is determined by the highest order term of (6) (see Sec. 3.2 for more details). More specifically, let $(z, s, \zeta dz + \sigma ds)$ be local coordinates on $T^*(X \times \mathbb{C})$. Then

$$S.S.\, Sol \ \subset \ \Omega_X \ := \{(z, s, \zeta dz + \sigma ds) \ : \ \zeta = \sigma \text{ or } \zeta = -\sigma\}. \tag{7}$$

It turns out that this condition contains enough information on Sol in order to deal with solving the Cauchy problem. In fact, at this stage, we abstact from our PDE, and only remember that its solution sheaf has its singular support as specified.

1.4.3 Initial value problem in sheaf-theoretical terms

Choose and fix a preimage $\mathbf{x}_0 \in X$ of x_0. Define a map $g : S_\alpha \to X \times \mathbb{C}$ by setting $g(\tilde{s}) := (\mathbf{x}_0, \pi_{S_\alpha}(\tilde{s}))$. Cauchy-Kowalewski theorem implies that the initial conditions (3) are in 1-to-1 correspondence with elements of $\Gamma(S_\alpha, g^{-1}Sol)$, see Sec. 3.3 for more detail.

As explained in the same Sec., the latter group can be identified with $R^0\operatorname{Hom}_{X\times\mathbb{C}}(Rg_!\mathbb{Z}_{S_\alpha}[-2], Sol)$. Therefore, the initial data (3) can be interpreted as a map

$$m_\psi : Rg_!\mathbb{Z}_{S_\alpha}[-2] \to Sol, \tag{8}$$

see (22).

1.4.4 Semi-orhogonal decomposition of $Rg_!\mathbb{Z}_{S_\alpha}[-2]$.

Let $\mathbf{D}(X\times\mathbb{C})$ be the bounded derived category of sheaves of abelian groups on $X\times\mathbb{C}$. Let $\mathcal{C} \subset \mathbf{D}(X\times\mathbb{C})$ be the full triangulated subcategory consisting of all objects whose singular support is contained in Ω_X as in (7). Let ${}^\perp\mathcal{C} \subset \mathbf{D}(X \times \mathbb{C})$ be the so-called left semi-orthogonal complement to $\mathcal{C}$, i.e. a full subcategory consisting of all objects Y such that $R\operatorname{hom}(Y, X) = 0$ for all $X \in \mathcal{C}$. We prove

Theorem 1.2 *1) There exists the following distinguished triangle in* $\mathbf{D}(X \times \mathbb{C})$*:*

$$\to Rg_!\mathbb{Z}_{S_\alpha}[-2] \xrightarrow{i_\Phi} \Phi \to \delta \xrightarrow{+1}$$

where $\Phi \in \mathcal{C}$*,* $\delta \in {}^\perp\mathcal{C}$ *("semi-orthogonal decomposition");*

2) Stalks of Φ *at every point of* $X \times \mathbb{C}$ *have no negative cohomology.*

This theorem coincides (up-to slight reformulations) with Theorem 3.2. The object Φ and the map $i_\Phi : Rg_!\mathbb{Z}_{S_\alpha}[-2] \to \Phi$ are constructed in Sec 3.6-3.13. The bulk of the paper (Sec. 4–Sec. 6) is devoted to showing that the constructed object Φ and a map i_Φ satisfy the above theorem.

It is well known that the distinguished triangle in part 1 of Th.1.2 , if exists, is unique up to a unique isomorphism, meaning that Φ is defined uniquely. It also follows that the precomposition with i_Φ:

$$i_\Phi : \circ - \;:\; R^0\operatorname{Hom}_{X\times\mathbb{C}}(\Phi, Sol) \;\to\; R^0\operatorname{Hom}(Rg_!\mathbb{Z}_{S_\alpha}[-2], Sol)$$

is an isomorphism of groups. This implies that the map m_ψ, cf. (8), uniquely factors as follows:

$$Rg_!\mathbb{Z}_{S_\alpha}[-2] \;\to \Phi \;\stackrel{m_\psi}{\to}\; Sol.$$

Let $\Phi_0 := \tau_{\leq 0}\Phi$. Condition 2) of Theorem 1.2 implies that Φ_0 is a sheaf of abelian groups. We have a composition

$$(m_\psi)_0 \;:\; \Phi_0 \to \Phi \to Sol.$$

1.4.5 Étale space of Φ_0 and solving the initial data problem

Let $\mathcal{S}'$ be the étale space of Φ_0. We have a local homeomorphism $p_{\mathcal{S}'} : \mathcal{S}' \to X \times \mathbb{C}$ so that we have a unique complex structure on $\mathcal{S}'$ making $p_{\mathcal{S}'}$ into a local biholomorphism. It turns out, that the map $(m_\psi)_0$ gives rise to a solution of the transferred equation on $\mathcal{S}'$. Indeed, every such a solution can be equivalently described as an element in $\Psi \in \Gamma(\mathcal{S}'; p_{\mathcal{S}'}^{-1}Sol)$. We also have a canonical section $\rho \in \Gamma(\mathcal{S}'; p_{\mathcal{S}'}^{-1}\Phi_0)$ (by the construction of the étale space); the map $(m_\psi)_0$ induces a map $\nu : p_{\mathcal{S}'}^{-1}\Phi_0 \to p_{\mathcal{S}'}^{-1}Sol$, and we set $\Psi := \nu(\rho)$.

It is now straigtforward (Sec. 3.5.2) to prove that thus constructed solution Ψ is a solution on $\mathcal{S}'$ of the Cauchy problem with the initial data (3).

By choosing an appropriate connected component $\mathcal{S}$ of $\mathcal{S}'$ we finish the construction.

2 Conventions and Notations

Throughout the paper, we fix an acute angle $\alpha \in (0, \pi/2)$.

2.1 Various subsets of $\mathbb{C}$

We introduce the following subsets of $\mathbb{C}$:

— K is the closed cone consisting of all complex numbers whose argument belongs to $[-\alpha, \alpha]$, including 0;

— $\mathbf{r}_\alpha := e^{i\alpha}.[0, \infty)$; $\mathbf{r}_{-\alpha} := e^{-i\alpha}.[0, \infty)$;

2.2 Sector S_α

We set $S_\alpha := (0, \infty) \times (-\alpha; \alpha + 2\pi)$. Let $\pi_{S_\alpha} : S_\alpha \to \mathbb{C}$ be the map given by $\pi_{S_\alpha}(r, \phi) := re^{i\phi}$. It is clear that π_{S_α} is a local homeomorphism, whence a structure of 1-dimensional complex manifold on S_α. Complex analysts call S_α an open sector with aperture $2\pi + 2\alpha$.

2.3 Potential $V(x)$. Stokes lines. Assumptions

Throughout the paper, we fix an entire function $V(x)$ on $\mathbb{C}$. We assume that $V(x)$ has only finitely many zeros which are traditionally called 'turning points'.

The conditions in Sec 2.3.2 below will be also assumed throughout the paper.

2.3.1 Stokes curves and further assumptions

Let $w \in \mathbb{C}$, $V(w) = 0$ be a k-fold zero of $V(x)$. We define an α-Stokes curve $z(t)$, $0 \le t < C$, emanating from w as follows:

—$z(t)$ is a smooth curve with $z(0) = w$ and $-V(z)(dz/dt)^2 \in e^{2i\alpha}\mathbb{R}_{>0}$.

The following facts are well known.

1) There are exactly $k + 2$ α-Stokes curves emanating from w.

2) One can choose C (to be a positive real number or $+\infty$) in such a way that either $z(C) := \lim_{t \to C}$ coincides with another turning point of $V(x)$, or $z(C) = \infty$. In the latter case we say that the Stokes curve terminates at infinity.

2.3.2 Further assumptions

We will assume the following properties of $V(z)$.

a) All α- and $(-\alpha)$-Stokes curves terminate at infinity.

b) Every α-Stokes curve intersects only finitely many $-\alpha$-Stokes curves, and every $(-\alpha)$-Stokes curve intersects only finitely many α-Stokes curves.

It is well known in the complex WKB theory that for every entire $V(x)$ with finitely many zeros one can find an α satisfying these assumptions.

2.4 Universal cover X

Let $\mathcal{U}$ be the complement in $\mathbb{C}$ to the (finite) set of turning points of the potential $V(x)$. α-Stokes curves split $\mathcal{U}$ into regions called $\underline{\alpha\text{-Stokes regions}}$; similarly, one can define $-\alpha$-regions. Throughout the paper, we denote by X the universal cover of $\mathcal{U}$, and by $p_X : X \to \mathcal{U} \to \mathbb{C}$ the covering map.

2.5 Initial point x_0

We fix a point $x_0 \in X$. We assume that $\underline{p_X(x_0) \text{ does not belong to any of } \alpha\text{- or } -\alpha\text{-Stokes lines.}}$

2.6 Action function on X

Fix a choice of $\sqrt{V(x)}$ on $\mathcal{U}$ and a function

$$z : X \to \mathbb{C} \quad : \quad dz(x) = \sqrt{V(x)}dx. \tag{9}$$

It follows that dz is nowhere vanishing, i.e. z is a local coordinate near every point of X. The function z has the meaning of the action function. We use the notation z because z will play the role of a local coordinate on X. The function z should not be confused with the map map $p_X : X \to \mathbb{C}$.

2.7 Subdivision of X into α-strips

Let $\mathbf{P} \subset \mathcal{U}$ be a closed α-Stokes region on $\mathcal{U}$, that is, $\mathbf{P}$ is one of the regions into which the complex plane $\mathbb{C}$ is subdivided by α-Stokes curves.

Let us now switch to the universal cover $p : X \to \mathcal{U}$. It follows that $p^{-1}\mathbf{P}$ splits into a disjoint union of its connected components $\mathbf{P} = \coprod_{\gamma \in \Gamma_{\mathbf{P}}} P_\gamma$, where $p : P_\gamma \xrightarrow{\sim} \mathbf{P}$. Call each such P_γ (for every α-Stokes region $\mathbf{P}$) $\underline{\text{an } \alpha\text{-strip}}$. It follows that the function z maps each α-strip homeomorphically into $\underline{\text{a generalized strip on } \mathbb{C}}$, i.e. a subset of $\mathbb{C}$ of one of the following types, fig. 1. Here the removed points ζ_t, ζ_b correspond to the turning points of $V(x)$.

Throughout the paper α-strips will be denoted by means of the letter P with different subscripts. We will often identify α strips with their images in $\mathbb{C}$ under z.

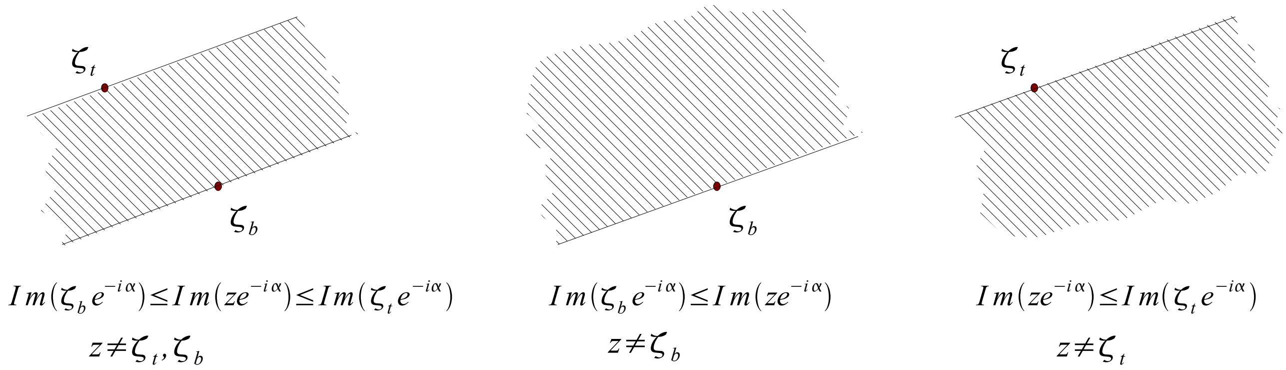

Figure 1**:** Three types of α-strips

2.7.1 Boundary rays

Let P_1, P_2 be α-strips and $P_1 \cap P_2 \neq \emptyset$. Then $\ell = P_1 \cap P_2$ is a ray on X which is identified by means of z with either $\hat{c}(\ell) + e^{i\alpha}.(0, \infty) \subset \mathbb{C}$ or $\hat{c}(\ell) - e^{i\alpha}.(0, \infty) \subset \mathbb{C}$, where $\hat{c}(\ell)$ is a complex number. We denote by $\mathcal{L}^\alpha$ the set of all such rays, to be called <u>boundary α-rays</u>. Every boundary α- ray belongs to the boundaries of exactly two α-strips; the boundary of every α-strip is a disjoint union of boundary α-rays. Boundary α-rays will be often denoted by the letter ℓ with different subscripts.

We say that a boundary α-ray ℓ <u>goes to the left</u> if its image under z is $\hat{c}(\ell) - e^{i\alpha}.(0, \infty)$. Otherwise we say that a boundary α-ray <u>ℓ goes to the right</u>. Accordingly, we get a splitting $\mathcal{L}^\alpha = \mathcal{L}^\alpha_{\text{left}} \sqcup \mathcal{L}^\alpha_{\text{right}}$.

2.7.2 Strips form a tree

Consider a graph whose vertices are α-strips and we join two distinct vertices with an edge if the corresponding strips intersect (along some boundary α-ray). Since X is simply connected, it follows that this graph is a tree.

2.8 $(-\alpha)$-Strips

One has a similar decomposition of X into $(-\alpha)$-strips which are defined based on $-\alpha$-Stokes regions of X. Throughout the paper, $-\alpha$-strips will be denoted by means of the letter Π with different subscripts. Similar to above, every $-\alpha$-strip is homeomorphically mapped under z into a generalized strip whose each boundary ray is parallel to the line $e^{-i\alpha}.\mathbb{R}$. We define boundary $-\alpha$ rays in a similar way (as intersection rays of two $-\alpha$-strips). The function z identifies each boundary ray ℓ with either

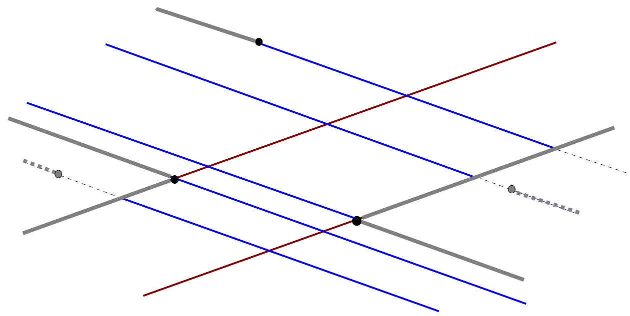

Figure 2: Intersection of an α-strip with several $(-\alpha)$-strips. Thick gray lines indicate branch cuts arising from the many sheets of the projection $X \to \mathbb{C}_x$.

$\hat{c}(\ell) + e^{-i\alpha}.(0,\infty)$ (we then say ℓ goes to the right), or $\hat{c}(\ell) - e^{-i\alpha}.(0,\infty)$ (ℓ goes to the left). We denote the set of all boundary $-\alpha$-rays by $\mathcal{L}^{-\alpha}$. We have a splitting $\mathcal{L}^{-\alpha} = \mathcal{L}^{-\alpha}_{\text{left}} \sqcup \mathcal{L}^{-\alpha}_{\text{right}}$. Bounday $-\alpha$-rays will be denoted by the letter ℓ with various subscripts.

2.9 Interaction of α and $-\alpha$-strips

Choose a (red) α-strip and look at all $(-\alpha)$-strips (blue) that intersect it. These $(-\alpha)$-strips cut the α-strips into parallelograms and two semi-infinite parallelograms, e.g., fig. 2.

2.10 Categories

For a topological space M, we denote by $\mathbf{D}(M)$ the bounded derived category of sheaves of abelian groups on M.

2.10.1 Sub-categories $\mathcal{C}^Y$; ${}^\perp\mathcal{C}^Y$

Let Y be a one dimensional complex manifold equipped with a local biholomorphism $z : Y \to \mathbb{C}$. For example, $Y = X$.

We then refer to points of $T^*(Y \times \mathbb{C})$ as follows $(y, s, \zeta dz, \sigma ds)$, where $y \in Y$, $s \in \mathbb{C}$ and $\zeta, \sigma \in \mathbb{C}$, so that $(y, s) \in Y \times \mathbb{C}$ and (ζ, σ) define the following real 1-form on $Y \times \mathbb{C}$:

$$(\zeta dz + \overline{\zeta} d\overline{z} + \sigma ds + \overline{\sigma} d\overline{s})/2.$$

Let us fix a closed subset $\Omega_Y \subset T^*(Y \times \mathbb{C})$ to consist of all points (y, s, ζ, σ), where $\zeta = \pm\sigma$.

We denote by $\mathcal{C}^Y \subset \mathbf{D}(Y \times \mathbb{C})$ the full triangulated subcategory consisting of all objects F with $SS(F) \subset \mathcal{C}^Y$. We denote by ${}^\perp\mathcal{C}^Y \subset \mathbf{D}(Y \times \mathbb{C})$ the full subcategory consisting of all objects G such that $R\hom(G, F) = 0$ for all $F \in \mathcal{C}^Y$.

2.11 Sheaves

Let Y be a topological space endowed with a continuous map $z : Y \to \mathbb{C}$. If $Y \subset X$, then we always assume that $z : Y \to \mathbb{C}$ is the restriction of the action function $z : X \to \mathbb{C}$. We define the following sheaves on $Y \times \mathbb{C}$:

$$\Lambda_Y^{K+} := \mathbb{Z}_{\{(y,s)|s+z(y)\in K};\quad \Lambda_Y^{K-} := \mathbb{Z}_{\{(y,s)|s-z(y)\in K\}}.$$

3 Statement of the problem and Main resuts

We start this section with giving a precise formulation for the problem of analytic continuation of solutions to (1). It turns out to be more convenient to transfer this PDE to $X \times \mathbb{C}$ by means of the covering map $p_X : X \to \mathbb{C}$.

Next, we give a sheaf-theoretical reformulation of the probem, and explain how the solution (i.e. a complex surface $\mathcal{S}$ along with a local biholomorphism $p_{\mathcal{S}} : \mathcal{S} \to X \times \mathbb{C}$) can be deduced from of a certain semi-orthogonal decomposition Theorem 3.2. The rest of this section is devoted to proving basic properties of $\mathcal{S}$ modulo Theorem 3.2, namely Hausdorffness and infinite continuabilty in the direction of K, which are the main results of this paper. To this end we need an explicit construction of the distinguished triangle of the semi-orthogonal decomposition in Theorem 3.2. This triangle is obtained via combining four other distinguished triangles.

It now remains to prove Theorem 3.2, which is now reduced to showing that each of the above mentioned four triangles (and hence the combined triangle) gives a semi-orthogonal decomposition. This is done in the rest of the paper.

3.1 Transfer of the equation $-\Psi_{xx} + V(x)\Psi_{ss} = 0$ to $X \times \mathbb{C}$

Our main equation (1) can be transferred to $X \times \mathbb{C}$ via the covering map $p \times \mathrm{Id}_{\mathbb{C}} : X \times \mathbb{C} \to \mathcal{U} \times \mathbb{C}$. We will use the action function z on X as in (9). Recall that z is a local coordinate near every point of X. Our notation is summarized in fig.3.

It is easy to see that the transferred equation has the following form

$$-\Psi_{zz} + \Psi_{ss} + l.o.t = 0, \tag{10}$$

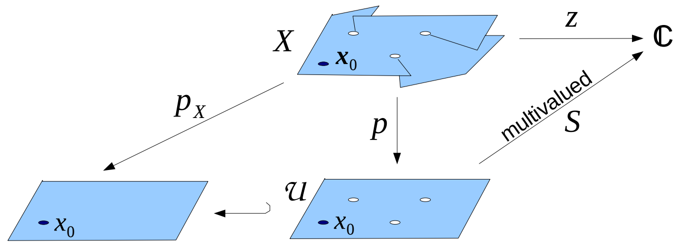

Figure 3

where l.o.t stands for the differential operator of order ≤ 1 applied to Ψ.

Let Sol be the sheaf of solutions of our transferred equation: Sol is a sheaf of abelian groups on $X \times \mathbb{C}$.

3.2 Singular support of the solution sheaf Sol

It is well known that to every linear PDE on a manifold M one can put into correspondence a $\mathcal{D}_M$-module, where $\mathcal{D}_M$ is the sheaf of differential operators on M; the solution sheaf of the PDE will then match with the solution sheaf of the $\mathcal{D}_M$ module.

In our situation, let us rewrite the equation (10) in the form $L\Psi = 0$ for an appropriate linear differential operator L on $X \times \mathbb{C}$. Define a $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}$ as follows

$$\mathcal{M} = \mathcal{D}_{X\times\mathbb{C}}/\mathcal{D}_{X\times\mathbb{C}}L.$$

We then have an obvious isomorphism

$$Sol \to \mathcal{H}om_{\mathcal{D}_{X\times\mathbb{C}}}(\mathcal{M}; \mathcal{O}_{X\times\mathbb{C}}). \tag{11}$$

Indeed, every solution Ψ of (10) on an open subset $U \subset X \times \mathbb{C}$ gives rise to a $\mathcal{D}_{X\times\mathbb{C}}$-module map

$$l_\Psi : \mathcal{D}_{X\times\mathbb{C}}|_U \to \mathcal{O}_{X\times\mathbb{C}}|_U$$

where $l_\Psi(T) := T\Psi$. Then, for any $T' \in \mathcal{D}_{X\times\mathbb{C}}(U)$, $l_\Psi(T'L) = T'L\Psi = 0$. Hence, l_Ψ descends to a map

$$l_\Psi : \mathcal{M}|_U \to \mathcal{O}_{X\times\mathbb{C}}|_U,$$

which determines the map (11). It is straightforward to see that thus constructed map (11) is in fact an isomorphism of sheaves.

The usefulness of this fact comes from a Kashiwara-Schapira's theorem on singular support of the object

$$R\mathcal{H}om_{\mathcal{D}_{X\times\mathbb{C}}}(\mathcal{M}; \mathcal{O}_{X\times\mathbb{C}}) \in \mathbf{D}(X \times \mathbb{C}) \tag{12}$$

(derived solution sheaf of $\mathcal{M}$). Let us now prove that this object is quasi-isomorphic to Sol.

The object (12) can be conveniently computed by means of the following free resolution $\mathcal{R}$ of $\mathcal{M}$:

$$(\mathcal{R}) \quad : \quad 0 \to \mathcal{D}_{X\times\mathbb{C}} \xrightarrow{\lambda} \mathcal{D}_{X\times\mathbb{C}} \to 0,$$

where the map λ is as follows: $\lambda(T) = TL$. We obtain that the object $\mathcal{H}om_{\mathcal{D}_{X\times\mathbb{C}}}(\mathcal{M}; \mathcal{O}_{X\times\mathbb{C}})$ is represented in $\mathbf{D}^b(X\times\mathbb{C})$ by the two term complex

$$\mathcal{H}om_{\mathcal{D}_{X\times\mathbb{C}}}(\mathcal{R}; \mathcal{O}_{X\times\mathbb{C}})$$

which is the same as

$$0 \to \mathcal{O}_{X\times\mathbb{C}} \xrightarrow{L} \mathcal{O}_{X\times\mathbb{C}} \to 0. \tag{13}$$

It is classically known, e.g. [Sch, Th.3.1.1], that the action of the operator L is locally surjective, meaning that we have a short exact sequence of sheaves

$$0 \to Sol \to \mathcal{O}_{X\times\mathbb{C}} \xrightarrow{L} \mathcal{O}_{X\times\mathbb{C}} \to 0.$$

This means that the complex of sheaves (13) is quasi-isomorphic to Sol so that finally

$$Sol \cong R\,\mathrm{hom}_{\mathcal{D}_{X\times\mathbb{C}}}(\mathcal{M}; \mathcal{O}_{X\times\mathbb{C}}).$$

Kashiwara-Schapira's theorem [KS, Th.11.3.3] says that the singular support of the object (12) equals the characteristic variety of the $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}$. In our situation, this characteristic variety is well-known to be equal to the zero set of the principal symbol of the operator L. This set is

$$\{(z, s, \zeta dz + \sigma ds) \ : \ \zeta = \pm\sigma\} \ \subset \ T^*(X\times\mathbb{C}), \tag{14}$$

which is the same as Ω_X from Sec. 2.10.1. Thus, by Kashiwara-Schapira's theorem, [KS, Th 11.3.3], we conclude that

$$S.S.Sol \ = \ \Omega_X, \quad Sol \in \mathcal{C}^X,$$

where $\mathcal{C}^X$ is defined in Sec. 2.10.1.

3.3 Initial conditions

Let $x_0 \in X$ be an initial point satifying the assumptions from Sec 2.5. Let us pose a Cauchy problem for the equation (10) similar to Sec. 1.2.

Let $S_\alpha := (0,\infty)\times(-\alpha, \alpha+2\pi)$ and $\pi_{S_\alpha} : S_\alpha \to \mathbb{C}$ be the same as in Sec 2.2. Set $q := \mathrm{Id}_X \times \pi_{S_\alpha} : X\times S_\alpha \to X\times\mathbb{C}$. The equation (10) gets transfered to $X\times S_\alpha$ by means of the map q. The transfered equation is of the form

$$L'\Psi = 0, \tag{15}$$

where Ψ is an unknown function on $X \times S_\alpha$ and L' is a linear differential operator. The solution sheaf of this equation is canonically isomorphic to $q^{-1}Sol$.

Let us fix two holomorphic functions ψ_0, ψ_1 on S_α and pose the initial conditions by requiring

$$\Psi(\mathbf{x}_0, s) = \psi^0(s) \text{ and } \partial_z \Psi(\mathbf{x}_0, s) = \psi^1(s), \quad s \in S_\alpha.$$

Cauchy-Kowalewski theorem implies that there exists a neighborhood

$$U \subset X \times S_\alpha \tag{16}$$

on which there exists a unique solution $\Psi \in \Gamma(U, q^{-1}Sol)$ of our Cauchy problem. We have a natural map

$$\Gamma(U, q^{-1}Sol) \to \Gamma(\mathbf{x}_0 \times S_\alpha, q^{-1}Sol|_{\mathbf{x}_0 \times S_\alpha}) = \Gamma(S_\alpha; g^{-1}Sol),$$

where

$$g : S_\alpha \to X \times \mathbb{C} \quad : \quad g(s) = (\mathbf{x}_0, \pi_{S_\alpha}(s)). \tag{17}$$

Thus, our initial data give rise to an element

$$\psi \in \Gamma(S_\alpha; g^{-1}Sol). \tag{18}$$

3.3.1 Definition of a solution

Let us formulate the definition of a multivalued solution of the initial value problem in the sheaf-theoretical language.

Suppose we are given a complex surface Σ endowed with a local biholomorphism $p_\Sigma : \Sigma \to X \times \mathbb{C}$. We can now transfer our differential equation from $X \times \mathbb{C}$ to Σ. The solution sheaf of the transferred equation is then $Sol_\Sigma := p_\Sigma^{-1}Sol$.

In order to transfer the initial condition (18), let us fix a factorization h of the map g:

$$S_\alpha \xrightarrow{h} \Sigma \xrightarrow{p_\Sigma} X \times \mathbb{C}, \tag{19}$$

where h is a complex-analytic map. We then have

$$\Gamma(S_\alpha; g^{-1}Sol) = \Gamma(S_\alpha; h^{-1}p_\Sigma^{-1}Sol) = \Gamma(S_\alpha; h^{-1}Sol_\Sigma).$$

The initial condition ψ now gives rise to an element $\psi_\Sigma \in \Gamma(S_\alpha; h^{-1}Sol_\Sigma)$.

Let us now formulate the notion of a solution to this problem.

We have a restriction map $\mathbf{res}: \Gamma(\Sigma; Sol_\Sigma) \to \Gamma(S_\alpha; h^{-1}Sol_\Sigma)$, which is defined as follows:

$$\mathbf{res}: \Gamma(\Sigma; Sol_\Sigma) = \hom(\mathbb{Z}_\Sigma; Sol_\Sigma) \to \hom(h^{-1}\mathbb{Z}_\Sigma; h^{-1}Sol_\Sigma) = \hom(\mathbb{Z}_{S_\alpha}; h^{-1}Sol_\Sigma) = \Gamma(S_\alpha; h^{-1}Sol_\Sigma).$$

We call an element $\Psi \in \Gamma(\Sigma; Sol_\Sigma)$ <u>a solution of the initial value problem with the initial data ψ</u>, if $\mathbf{res}(\Psi) = \psi_\Sigma$. Since Sol_Σ is a sub-sheaf of $\mathcal{O}_\Sigma$ (the sheaf of analytic functions), the unicity of analytic continuation implies:

Claim 3.1 *Suppose Σ is connected. For every initial condition ψ, the initial value problem has at most a unique solution.*

3.3.2 Equivalent formulation

One can define a notion of a solution to the initial value problem directly in terms of the initial data ψ^0, ψ^1: we can require that a solution Ψ should satisfy: $\Psi \circ h = \psi^0$; $\frac{\partial \Psi}{\partial z} \circ h = \psi^1$. It is clear that this new notion of a solution coincides with the one from the previous subsection. Indeed, the restriction of Ψ onto the neighborhood U as in (16) must coincide with the solution provided by the Cauchy-Kowalewski theorem.

The notion of solution from this (or previous) subsection is related to the notion of solution from Sec 1.1 as follows. First of all we have $dz = \sqrt{V(x)}dx$, where $\sqrt{V(x)}$ is a nowhere vanishing holomorphic function on X. Set $\psi_0 = \psi^0$ and $\psi_1(s) = \sqrt{V(x_0)}\psi^1(s)$. We then see that the notion of solution of the Cauchy problem with the initial data ψ_0, ψ_1, as in Sec 1.1, coincides with the current notion of solution of the initial value problem given by the initial data ψ^0, ψ^1.

3.3.3 Formulation of the analytic continuation problem

Our analytic continuation problem is now as follows. Find a connected complex surface $\mathcal{S}$ along with a complex analytic local diffeomorphism $p_\mathcal{S}: \mathcal{S} \to X \times \mathbb{C}$ and a factorization $g = hp_\mathcal{S}$, where $h: S_\alpha \to \mathcal{S}$ is as in the previous subsection, satisfying: given any initial condition ψ as in (18), there should exist a solution to the initial value problem with the initial data ψ. By Claim 3.1, this solution is then unique.

3.4 Semi-orthogonal decomposition of $\mathcal{F}_0$

Our main tool in solving the analytic continuation problem is a certain semi-orthogonal decomposition theorem, to be now stated.

Let $\mathcal{F}_0 = Rg_!\mathbb{Z}_{S_\alpha}[-2]$; let $\mathcal{C}^X, {}^\perp\mathcal{C}^X$ be the same as in Sec. 2.10.1.

Theorem 3.2 *1) There exists a distinguished triangle*

$$\to \mathcal{F}_0 \xrightarrow{i_\Phi} \Phi \to \delta \xrightarrow{+1} \tag{20}$$

where $\Phi \in \mathcal{C}^X$ *and* $\delta \in {}^\perp\mathcal{C}^X$.

2) The object Φ *belongs to* $\mathbf{D}_{\geq 0}(X \times \mathbb{C})$ *(that is: the stalks of* Φ *at every point of* $X \times \mathbb{C}$ *have no negative cohomology).*

Remark. The distinguished rectangle (20) is called "left semi-orthogonal decomposition of $\mathcal{F}_0$". It is well known that such a triangle, if exists, is unique up-to a unique isomorphism.

We will devote the rest of this section by deducing a solution to the analytic continuation problem from this theorem.

3.4.1 Factorization of the initial condition

Since $g : S_\alpha \to X \times \mathbb{C}$ is locally a closed embedding of codimension 2, whose normal bundle is canonically trivialized, we have a natural transformation of functors

$$\kappa \ : \ g^{-1} \to g^![2]. \tag{21}$$

Since Sol is microsupported on Ω_X, one can easily check that Sol is non-characteristic with respect to g. Accoriding to [KS, Prop.5.4.13], κ induces an isomorphism $g^{-1}Sol \to g^!Sol[2]$. We now have an isomorphism

$$\Gamma(S_\alpha; g^{-1}Sol) = R^0\hom(\mathbb{Z}_{S_\alpha}; g^{-1}Sol) = R^0\hom(\mathbb{Z}_{S_\alpha}; g^!Sol[2]) = R^0\hom(Rg_!\mathbb{Z}_{S_\alpha}[-2]; Sol). \tag{22}$$

Let us denote the images of ψ under these identifications as follows:

$$\nu_\psi : \mathbb{Z}_{S_\alpha} \to g^{-1}Sol;$$

$$m'_\psi : \mathbb{Z}_{S_\alpha} \to g^!Sol[2];$$

$$m_\psi : g_!\mathbb{Z}_{S_\alpha}[-2] \to Sol.$$

Since $Sol \in \mathcal{C}$, the semi-orthogonal decomposition theorem 20 implies that m_ψ uniquely factors as

$$m_\psi : Rg_!\mathbb{Z}_{S_\alpha}[-2] \xrightarrow{i_\Phi} \Phi \xrightarrow{\psi'} Sol. \tag{23}$$

The map i_Φ defines, by the conjugacy, a map $\mathbf{I}' : \mathbb{Z}_{S_\alpha} \to g^!\Phi[2]$. Let also $\psi_1 : g^!\Phi[2] \to g^!Sol[2]$ be the map induced by ψ'. The equation (23) now implies the following factorization (by the conjugacy between $Rg_!$ and $g^!$):

$$m'_\psi : \mathbb{Z}_{S_\alpha} \xrightarrow{\mathbf{I}'} g^!\Phi[2] \xrightarrow{\psi_1} g^!Sol[2]. \tag{24}$$

Since $\Phi[2]$ is microsupported within Ω_X, the transformation κ, cf. (21), induces an isomorphism $\kappa_\Phi : g^{-1}\Phi \to g^!\Phi[2]$ so that we have a unique map $\mathbf{I} : \mathbb{Z}_{S_\alpha} \to g^{-1}\Phi$ such that $\mathbf{I}' = \kappa_\Phi \mathbf{I}$. Let $\tilde{\psi} : g^{-1}\Phi \to g^{-1}Sol$ be the map induced by ψ'. We can now rewrite (24) as follows:

$$\nu_\psi : \mathbb{Z}_{S_\alpha} \xrightarrow{\mathbf{I}} g^{-1}\Phi \xrightarrow{\tilde{\psi}} g^{-1}Sol. \tag{25}$$

3.4.2 Truncation

The second statement of the theorem implies that $\Phi_0 := \tau_{\leq 0}\Phi$ is a sheaf of abelian groups. The canonical map $c : \tau_{\leq 0}\Phi \to \Phi$ induces a map $c' : g^{-1}\Phi_0 \to g^{-1}\Phi$.

Let us show that

Proposition 3.3 *The map* $\mathbf{I}$ *factorizes uniquely through* c'.

PROOF.

We have a distinguished triangle

$$\xrightarrow{+1} g^{-1}\Phi_0 \xrightarrow{c'} g^{-1}\Phi \to g^{-1}\tau_{>0}\Phi \xrightarrow{+1},$$

which induces a long exact sequence

$$\cdots R^{-1}\hom(\mathbb{Z}_{S_\alpha}; g^{-1}\tau_{>0}\Phi) \to R^0\hom(\mathbb{Z}_{S_\alpha}; g^{-1}\Phi_0) \xrightarrow{*} R^0\hom(\mathbb{Z}_{S_\alpha}; g^{-1}\Phi) \to R^0\hom(\mathbb{Z}_{S_\alpha}; g^{-1}\tau_{>0}\Phi) \cdots .$$

where the arrow $*$ is given by the composition with c'. Since the functor g^{-1} is exact, $g^{-1}\tau_{>0}\Phi \in \mathbf{D}_{>0}(S_\alpha)$ so that $R^{\leq 0}\hom(\mathbb{Z}_{S_\alpha}; g^{-1}\tau_{>0}\Phi) = 0$, meaning that the map $*$ is an isomorhism. This implies the statement. $\square$

Denote by

$$\mathbf{I}_0 : \mathbb{Z}_{S_\alpha} \to g^{-1}\Phi_0 \tag{26}$$

the factorization map (unique by the above Proposition):

$$\mathbf{I} : \mathbb{Z}_{S_\alpha} \xrightarrow{\mathbf{I}_0} g^{-1}\Phi_0 \xrightarrow{c'} g^{-1}\Phi.$$

We can also factorize:

$$\nu_\psi : \mathbb{Z}_{S_\alpha} \xrightarrow{\mathbf{I}_0} g^{-1}\Phi_0 \xrightarrow{\tilde{\psi}\circ c'} g^{-1}Sol.$$

3.5 Etale space of Φ_0

3.5.1 Choice of a covering space Σ

Set $p_\Sigma : \Sigma \to X \times \mathbb{C}$ to be the etale space of Φ_0. Observe that the etale space of $g^{-1}\Phi_0$ is $S_\alpha \times_{X\times\mathbb{C}} \Sigma$. The etale space of $\mathbb{Z}_{S_\alpha}$ is $S_\alpha \times \mathbb{Z}$, so that we have a map

$$S_\alpha \times \mathbb{Z} \to S_\alpha \times_{X\times\mathbb{C}} \Sigma$$

over S_α, induced by the map $\mathbf{I}_0$. Let us restrict this map to $S_\alpha = S_\alpha \times 1$ and denote by h the through map

$$h : S_\alpha = S_\alpha \times 1 \to S_\alpha \times \mathbb{Z} \to S_\alpha \times_{X\times\mathbb{C}} \Sigma \to \Sigma. \tag{27}$$

By the definition of fibered product, we have $p_\Sigma h = g$.

Thus, $p_\Sigma : \Sigma \to X \times \mathbb{C}$ and $h : S_\alpha \to \Sigma$ yield a factorization of the map (17), as required by (19).

3.5.2 Solving the initial value problem

Let us show that the initial value problem $\psi \in \Gamma(S_\alpha; g^{-1}Sol)$ has a solution on Σ, in the sense of Sec. 3.3.1, where Σ is as in Sec.3.5.1.

We have a canonical map $\lambda : \mathbb{Z}_\Sigma \to p_\Sigma^{-1}\Phi_0$ which comes from the canonical section of $p_\Sigma^{-1}\Phi_0$: over a point of Σ corresponding to $((x,s), \varphi_{(x,s)} \in (\Phi_0)_{(x,s)})$, the stalk of this canonical section equals $\varphi_{(x,s)}$. Let us apply the functor h^{-1} and obtain a map

$$\mathbf{I}' : \mathbb{Z}_{S_\alpha} = h^{-1}\mathbb{Z}_\Sigma \to h^{-1}p_\Sigma^{-1}\Phi_0 = g^{-1}\Phi_0.$$

Lemma 3.4 *We have $\mathbf{I}' = \mathbf{I}$.*

PROOF It is easy to see that for each $s \in S_\alpha$, the map $\mathbf{I}'$ induces the same map on stalks as $\mathbf{I}$. $\square$

We have a composition $F_\psi : \mathbb{Z}_\Sigma \xrightarrow{\lambda} p_\Sigma^{-1}\Phi_0 \xrightarrow{\tilde{\psi}c'} p_\Sigma^{-1}Sol$. Let us prove that F_ψ is a solution to the initial value problem. Indeed, applying h^{-1} induces a map $\mathbb{Z}_{S_\alpha} \to g^{-1}Sol$ which, by virtue of Lemma, coincides with ν_ψ, which means that F_ψ is a solution.

3.5.3 Solving the analytic continuation problem

We replace Σ with its connected component $\mathcal{S}$ containing the image of h. It is clear that $\mathcal{S}$ is a solution to the analytic continuation problem as in Sec. 3.3.3.

3.6 Structure of the object Φ.

We construct the semi-orthogonal decomposition of $g_!\mathbb{Z}_{S_\alpha}[-2]$ via representing $g_!\mathbb{Z}_{S_\alpha}[-2]$ as a cone of some arrow $A \to B$, and then constructing the semi-orthogonal decompositions for A and B; these decompositions are then glued into the desired decomposition of $g_!\mathbb{Z}_{S_\alpha}[-2]$.

3.6.1 Decomposition of $\pi_!\mathbb{Z}_{S_\alpha} \in \mathbf{D}(\mathbb{C})$

Let $\pi_{S_\alpha} : S_\alpha \to \mathbb{C}$ be the projection. We are going to represent $\pi_{S_\alpha!}\mathbb{Z}_{S_\alpha}$ as a cone of a certain map. To this end let us introduce the following subsets of $\mathbb{C}$ (same as in Sec 2.1)

$$K \;=\; \{re^{i\varphi} \;:\; r \geq 0;\; -\alpha \leq \varphi \leq \alpha\};$$

$$\mathbf{r}_\alpha \;=\; \{re^{i\varphi} \;:\; r \geq 0;\; \varphi = \alpha\};$$

$$\mathbf{r}_{-\alpha} \;=\; \{re^{i\varphi} \;:\; r \geq 0;\; \varphi = -\alpha\}.$$

We have natural restriction maps

$$\mathbb{Z}_\mathbb{C} \stackrel{\rho_{\mathbb{C}K}}{\to} \mathbb{Z}_K \stackrel{\rho_{K\mathbf{r}_{\pm\alpha}}}{\to} \mathbb{Z}_{\mathbf{r}_{\pm\alpha}}$$

in $\mathbf{D}(\mathbb{C})$.

The identification $\mathbb{Z}_{S_\alpha} = \pi^!_{S_\alpha}\mathbb{Z}_\mathbb{C}$ induces, by conjugacy, a map

$$p_\mathbb{C} : \pi_{S_\alpha!}\mathbb{Z}_{S_\alpha} \to \mathbb{Z}_\mathbb{C}.$$

We are now up to defining a map $p_K : \pi_{S_\alpha!}\mathbb{Z}_{S_\alpha} \to \mathbb{Z}_K$. We have

$$\pi^{-1}_{S_\alpha}K = (0,\infty) \times (-\alpha;\alpha] \sqcup (0,\infty) \times [2\pi-\alpha; 2\pi+\alpha) =: K_1 \sqcup K_2.$$

Denote by $i_1 : K_1 \to S_\alpha$ the closed embedding. We have natural surjections of sheaves on S_α: $\iota_1 : \mathbb{Z}_{S_\alpha} \to i_{1!}\mathbb{Z}_{K_1}$ and $\iota_2 : \mathbb{Z}_{S_\alpha} \to i_{2!}\mathbb{Z}_{K_2}$.

The map π_{S_α} induces open embeddings $\pi_{S_\alpha}i_1 : K_1 \to K$ and $\pi_{S_\alpha}i_2 : K_2 \to K$. We have $\pi_{S_\alpha}(K_1) = K\backslash\mathbf{r}_\alpha$; $\pi_{S_\alpha}K_2 = K\backslash\mathbf{r}_{-\alpha}$. These open embeddings induce the following embeddings of sheaves on $\mathbb{C}$: $\pi_{S_\alpha!}i_{1!}\mathbb{Z}_{K_1} \to \mathbb{Z}_K$; $\pi_{S_\alpha!}i_{2!}\mathbb{Z}_{K_2} \to \mathbb{Z}_K$. Combining these maps with ι_1, ι_2, we get the following through map

$$p_K : \pi_{S_\alpha!}\mathbb{Z}_{S_\alpha} \stackrel{\iota_1}{\to} \pi_{S_\alpha!}i_{1!}\mathbb{Z}_{K_1} \to \mathbb{Z}_K.$$

One checks that $\rho_{K\mathbf{r}_\alpha} p_K = \rho_{C\mathbf{r}_\alpha} p_C$. Let us now construct the following sequence of maps

$$\begin{tikzcd}
 & & \mathbb{Z}_{\mathbb{C}} \arrow[rr, "\rho_{C\mathbf{r}_\alpha}"] & & \mathbb{Z}_{\mathbf{r}_\alpha} & \\
0 \arrow[r] & \pi_{S_\alpha !}\mathbb{Z}_{S_\alpha} \arrow[ur, "p_C"] \arrow[dr, "p_K"'] & \oplus & & \oplus \arrow[r] & 0 \\
 & & \mathbb{Z}_K \arrow[uurr, "-\rho_{K\mathbf{r}_\alpha}"] \arrow[rr, "\rho_{K\mathbf{r}_{-\alpha}}"] & & \mathbb{Z}_{\mathbf{r}_{-\alpha}} &
\end{tikzcd} \tag{28}$$

It is clear that the composition of every two consecutive maps is zero. In fact, this sequence is exact, which can be shown by proving exactness of the induced sequences on stalks for every point $z \in \mathbb{C}$.

Let $g' : \mathbb{C} \to X \times \mathbb{C}$ be given by $g'(s) = (\mathbf{x}_0, s)$ so that $g = g'\pi_{S_\alpha}$. Applying $g'_!$ to the exact sequence above yields the following exact sequence of sheaves:

$$\begin{tikzcd}
 & & \mathbb{Z}_{\mathbf{x}_0\times\mathbb{C}} \arrow[rr] & & \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_\alpha} & \\
0 \arrow[r] & g_!\mathbb{Z}_{S_\alpha} \arrow[ur, "g'_!(p_C)"] \arrow[dr, "g'_!(p_K)"'] & \oplus & & \oplus \arrow[r] & 0 \\
 & & \mathbb{Z}_{\mathbf{x}_0\times K} \arrow[uurr, "-g'_!(\rho_{K\mathbf{r}_\alpha})"] \arrow[rr, "g'_!(\rho_{K\mathbf{r}_{-\alpha}})"] & & \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_{-\alpha}} &
\end{tikzcd} \tag{29}$$

3.6.2 Semi-orthogonal decomposition for $\mathbb{Z}_{\mathbf{x}_0\times\mathbb{C}}, \mathbb{Z}_{\mathbf{x}_0\times K}, \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_{\pm\alpha}}$

Theorem 3.5 *There are objects $\Phi^{\mathbb{C}}$, Φ^K, $\Phi^{\mathbf{r}_\alpha}$, $\Phi^{\mathbf{r}_{-\alpha}}$ in the category of sheaves of abelian groups and maps in $\mathbf{D}^b(X \times \mathbb{C})$:*

$$\begin{aligned}
i_{\Phi^{\mathbb{C}}} &: \mathbb{Z}_{\mathbf{x}_0\times\mathbb{C}}[-2] \to \Phi^{\mathbb{C}} & \qquad i_{\Phi^K} &: \mathbb{Z}_{\mathbf{x}_0\times K}[-2] \to \Phi^K \\
i_{\Phi^{\mathbf{r}_\alpha}} &: \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_\alpha}[-2] \to \Phi^{\mathbf{r}_\alpha} & \qquad i_{\Phi^{\mathbf{r}_{-\alpha}}} &: \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_{-\alpha}}[-2] \to \Phi^{\mathbf{r}_{-\alpha}}
\end{aligned}$$

whose cones are in ${}^{\perp}\mathcal{C}$ and $\Phi^{\mathbb{C}}, \Phi^K, \Phi^{\mathbf{r}_\alpha}, \Phi^{\mathbf{r}_{-\alpha}} \in \mathcal{C}$.

Based on this theorem, let us construct a semi-orthogonal decomposition of $g_!\mathbb{Z}_{S_\alpha}$. Let us rewrite the sequence (29) as

$$0 \to g_!\mathbb{Z}_{S_\alpha} \xrightarrow{\iota} \mathcal{X} \xrightarrow{q} \mathcal{Y} \to 0,$$

where $\mathcal{X} = \mathbb{Z}_{\mathbf{x}_0\times\mathbb{C}} \oplus \mathbb{Z}_{\mathbf{x}_0\times K}$ and $\mathcal{Y} = \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_\alpha} \oplus \mathbb{Z}_{\mathbf{x}_0\times\mathbf{r}_{-\alpha}}$. By virtue of Theorem 3.5 we have semi-orthogonal decompositions of $\mathcal{X}$ and $\mathcal{Y}$

$$\to \xi \to \mathcal{X}[-2] \xrightarrow{P_{\mathcal{X}}} \mathcal{X}' \xrightarrow{+1}; \quad \eta \to \mathcal{Y}[-2] \xrightarrow{P_{\mathcal{Y}}} \mathcal{Y}' \xrightarrow{+1},$$

where $\mathcal{X}' = \Phi^{\mathbb{C}} \oplus \Phi^K \in \mathcal{C}$; $\mathcal{Y}' = \Phi^{\mathbf{r}_\alpha} \oplus \Phi^{\mathbf{r}_{-\alpha}} \in \mathcal{C}$; $\xi, \eta \in {}^{\perp}\mathcal{C}$. The map $P_{\mathcal{Y}}q : \mathcal{X}[-2] \to \mathcal{Y}'$, by the univerality of $\mathcal{X}'$, uniquely factors as

$$P_{\mathcal{Y}}q = q' P_{\mathcal{X}} \tag{30}$$

for some $q' : \mathcal{X}' \to \mathcal{Y}'$ so that we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}[-2] \arrow[r, "q"] \arrow[d, "P_{\mathcal{X}}"] & \mathcal{Y}[-2] \arrow[d, "P_{\mathcal{Y}}"] \\ \mathcal{X}' \arrow[r, "q'"] & \mathcal{Y}'. \end{tikzcd}$$

We have $g_! \mathbb{Z}_{S_\alpha}[-2] \cong \mathrm{Cone}q[-1]$. Set $\Phi := \mathrm{Cone}q'[-1]$. It is well known that the commutative diagram above implies existence of a map $i_\Phi : g_! \mathbb{Z}_{S_\alpha}[-2] \to \Phi'$ fitting into the following commutative diagram whose rows are distinguished triangles:

$$\begin{tikzcd} {} \arrow[r] & g_! \mathbb{Z}_{S_\alpha}[-2] \arrow[r] \arrow[d, "i_\Phi"] & \mathcal{X}[-2] \arrow[r, "q"] \arrow[d, "P_{\mathcal{X}}"] & \mathcal{Y}[-2] \arrow[r, "+1"] \arrow[d, "P_{\mathcal{Y}}"] & {} \\ {} \arrow[r] & \Phi \arrow[r] & \mathcal{X}' \arrow[r, "q'"] & \mathcal{Y}' \arrow[r, "+1"] & {} \end{tikzcd}$$

Furthermore, we have a distinguished triangle

$$\to \mathrm{Cone}(i_\Phi) \to \mathrm{Cone}P_{\mathcal{X}} \to \mathrm{Cone}P_{\mathcal{Y}} \xrightarrow{+1},$$

which implies that $\delta := \mathrm{Cone}(i_\Phi) \in {}^{\perp}\mathcal{C}$ satisfies all the conditions of Theorem 3.2.

We will now give an explicit description of the sheaves $\Phi^{\mathbb{C}}, \Phi^K, \Phi^{\mathbf{r}_{\pm\alpha}}$, as well as the maps $i_{\Phi^{\mathbb{C}}}, i_{\Phi^K}, i_{\Phi^{\mathbf{r}_{\pm\alpha}}}$ from Theorem 3.5. This theorem will be proven below.

3.6.3 $\Phi^{\mathbb{C}}$

We set $\Phi^{\mathbb{C}} = \mathbb{Z}_{\mathcal{X}\times\mathbb{C}}$. We have a codimension 2 embedding

$$i_{\mathbb{C},\mathbf{x}_0} : \mathbb{C} \to X \times \mathbb{C},$$

so that we have a natural map

$$\mathbb{Z}_{\mathbf{x}_0\times\mathbb{C}}[-2] \to \mathbb{Z}_{X\times\mathbb{C}},$$

and we assign $i_{\Phi^{\mathbb{C}}}$ to be this map.

3.7 Notation: convolution functor $\mathbf{D}(X \times \mathbb{C}) \times \mathbf{D}(\mathbb{C}) \to \mathbf{D}(X \times \mathbb{C})$

Define a convolution functor

$$* \ : \ \mathbf{D}(X \times \mathbb{C}) \times \mathbf{D}(\mathbb{C}) \ \to \ \mathbf{D}(X \times \mathbb{C}) \tag{31}$$

as follows. Let $\mathcal{F} \in \mathbf{D}(X \times \mathbb{C})$, $\Sigma \in \mathbf{D}(\mathbb{C})$. Let

$$a : X \times \mathbb{C} \times \mathbb{C} \to X \times \mathbb{C} \ : \ a(x, s_1, s_2) = (x, s_1 + s_2)$$

Set

$$\mathcal{F} * \Sigma \ = \ Ra_!(\mathcal{F} \boxtimes \Sigma).$$

3.8 Construction of Φ^K

3.8.1 Subdivision into α-strips

Let us split X into α-strips as in Sec. 2.7. We will freely use the notation from this section below.

We will define a sheaf Φ^K on $X \times \mathbb{C}$ via prescribing the following data.

1) For each α-strip P we will define a sheaf Φ^K_P on $P \times \mathbb{C}$. Recall that by α-strip we always mean a closed α-strip.

2) Let P_1, P_2 be intersecting closed α-strips so that $P_1 \cap P_2 = \ell \in \mathcal{L}^\alpha$. We will construct an isomorphism

$$\Gamma^{P_1 P_2}_{\Phi^K} : \Phi^K_{P_1}|_{\ell \times \mathbb{C}} \xrightarrow{\sim} \Phi^K_{P_2}|_{\ell \times \mathbb{C}},$$

where we assume $\Gamma^{P_2 P_1}_{\Phi^K} = (\Gamma^{P_1 P_2}_{\Phi^K})^{-1}$.

Since every triple of distinct closed α-strips has an empty intersection, the data 1),2) define a sheaf Φ^K unambiguously. More precisely, there exists a sheaf Φ^K endowed with the following structure:

— isomorphisms $j_P : \Phi^K|_{P \times \mathbb{C}} \xrightarrow{\sim} \Phi^K_P$ for every α-strip P satisfying: for every pair of intersecting strips P_1 and P_2, $P_1 \cap P_2 = \ell$, the following maps must conicide:

$$\Phi^K|_{\ell \times \mathbb{C}} \xrightarrow{j_{P_1}|_\ell} \Phi^K_{P_1}|_{\ell \times \mathbb{C}} \xrightarrow{\Gamma^{P_1 P_2}_{\Phi^K}} \Phi^K_{P_2}|_{\ell \times \mathbb{C}}$$

and

$$\Phi^K|_{\ell \times \mathbb{C}} \xrightarrow{j_{P_2}|_{\ell \times \mathbb{C}}} \Phi^K_{P_2}|_{\ell \times \mathbb{C}}.$$

The sheaf Φ^K is unique up-to a unique isomorphism compatible with all the structure maps j_P.

3.8.2 Words

We will use the notation from Sec. 2.7.1. Let $\mathbf{W}^\alpha$ be the set of words from the alphabet $\mathcal{L}^\alpha \cup \{L, R\}$ such that:

1) each word is non-empty and its rightmost letter in L or R

2) every word is either of the form

$$(\ell_n ... \ell_3 \ell_2 \ell_1 L) \tag{32}$$

where

$$\ell_1, \ell_3, \ell_5, ... \in \mathcal{L}^\alpha_{\text{right}}, \quad \ell_2, \ell_4, \ell_6, ... \in \mathcal{L}^\alpha_{\text{left}}$$

or

$$(\ell_n ... \ell_1 R) \tag{33}$$

where

$$\ell_1, \ell_3, ... \in \mathcal{L}^\alpha_{\text{left}}; \quad \ell_2, \ell_4, \ell_6, .. \in \mathcal{L}^\alpha_{\text{right}}$$

(alternating pattern).

Let $\mathbf{W}^\alpha = \mathbf{W}^\alpha_{\text{left}} \cup \mathbf{W}^\alpha_{\text{right}}$, where

$$\mathbf{W}^\alpha_{\text{left}} = \{(\ell_n ...) \ : \ \ell_n \in \mathcal{L}^\alpha_{\text{left}}\} \cup \{L\}; \qquad \mathbf{W}^\alpha_{\text{right}} = \{(\ell_n ...) \ : \ \ell_n \in \mathcal{L}^\alpha_{\text{right}}\} \cup \{R\}.$$

Let us stress that $\mathbf{W}^\alpha_{\text{left}}$ contains words both ending with L and words ending with R, and the same is true for $\mathbf{W}^\alpha_{\text{right}}$.

3.8.3 Sheaves S_ℓ, S_w on $\mathbb{C}$

Given a ray $\ell \in \mathcal{L}^\alpha_{\text{left}}$, let is define the following sheaf on $\mathbb{C}$:

$$S_\ell := \mathbb{Z}_{\{s \in +2\hat{c}(\ell)+K\}}, \tag{34}$$

Given a ray $\ell \in \mathcal{L}^\alpha_{\text{right}}$, we set

$$S_\ell := \mathbb{Z}_{\{s \in -2\hat{c}(\ell)+K\}}.$$

Set

$$S_L := \mathbb{Z}_{\{s \in z(\mathbf{x}_0)+K\}}; \quad S_R := \mathbb{Z}_{\{s \in -z(\mathbf{x}_0)+K\}}. \tag{35}$$

Let

$$S_w := S_{\ell_1} * S_{\ell_2} * ... * S_{\ell_n} * S_L, \quad \text{if } w := \ell_1 .. \ell_n L \in \mathbf{W}^\alpha,$$

$$S_w := S_{\ell_1} * S_{\ell_2} * ... * S_{\ell_n} * S_R, \quad \text{if } w := \ell_1 ... \ell_n R \in \mathbf{W}^\alpha,$$

where $*$ denotes the convolution functor $\mathbf{D}(\mathbb{C}) \times \mathbf{D}(\mathbb{C}) \to \mathbf{D}(\mathbb{C})$ in the sense of (31). It is clear that $S_w = \mathbb{Z}_{\hat{c}(w)+K}$, where we set:

$$\hat{c}(w) = z(\mathbf{x}_0) - 2\hat{c}(\ell_n) + 2\hat{c}(\ell_{n-1}) - \cdots + (-1)^n \hat{c}(\ell_1) \quad \text{if } w := \ell_1..\ell_n L; \tag{36}$$

$$\hat{c}(w) = -z(\mathbf{x}_0) + 2\hat{c}(\ell_n) - 2\hat{c}(\ell_{n-1}) + \cdots - (-1)^n \hat{c}(\ell_1) \quad \text{if } w := \ell_1..\ell_n R. \tag{37}$$

Let us further set

$$S_- := \oplus_{w \in \mathbf{W}^\alpha_{\mathrm{right}}} S_w; \qquad S_+ := \oplus_{w \in \mathbf{W}^\alpha_{\mathrm{left}}} S_w. \tag{38}$$

3.8.4 Definition of Φ_P^K

For any subset $U \subset X$, we define the following sheaf on $U \times \mathbb{C}$:

$$\Phi_U^K \;:=\; \Lambda_U^{K-} * S_- \;\oplus\; \Lambda_U^{K+} * S_+, \tag{39}$$

where $\Lambda_U^{K\pm} := \mathbb{Z}_{\{(x,s)|s \pm z(x) \in K\}}$ are the same as in Sec 2.11.

Set $\Phi_U^{K\pm} = \Lambda_U^{K\pm} * S_\pm$. In particular, we have defined sheaves $\Phi_P^{K\pm}$ for every α-strip P.

3.8.5 Constructuion of the identification $\Gamma_{\Phi^K}^{P_1P_2}$

We have identifications:

$$\Phi_{P_1}^K|_{\ell\times\mathbb{C}} = \Phi_{P_2}^K|_{\ell\times\mathbb{C}} = \Lambda_\ell^{K+} * S_+ \oplus \Lambda_\ell^{K-} * S_-.$$

Let us now construct the gluing maps

$$\Gamma_{\Phi^K}^{P_1P_2} : \Lambda_\ell^{K+} * S_+ \oplus \Lambda_\ell^{K-} * S_- \to \Lambda_\ell^{K+} * S_+ \oplus \Lambda_\ell^{K-} * S_-.$$

There are two cases.

Case A). Let $\ell \in \mathcal{L}^\alpha_{\mathrm{left}}$.

Assume that the z-image of P_2 is above the z-image of P_1 in the complex plane, fig. 4, a).

Let us define the following morphism of sheaves on $\ell \times \mathbb{C}$

$$\nu_\ell^K : \Lambda_\ell^{K-} \;\to S_\ell * \Lambda_\ell^{K+}, \tag{40}$$

or, more explicitly,

$$\nu_\ell^K : \mathbb{Z}_{\{z \in \hat{c}(\ell) - e^{i\alpha}.[0,\infty),\; s - z \in K\}} \to \mathbb{Z}_{\{s \in 2\hat{c}(\ell)+K\}} * \mathbb{Z}_{\{z \in \hat{c}(\ell) - e^{i\alpha}.[0,\infty), s+z \in K\}}. \tag{41}$$

We have $\mathbb{Z}_{\{s\in 2\hat{c}(\ell)+K\}} * \mathbb{Z}_{\{z\in\hat{c}(\ell)-e^{i\alpha}.[0,\infty),s+z\in K\}} = \mathbb{Z}_{\{z\in\hat{c}(\ell)-e^{i\alpha}.[0,\infty);s\in -z+2\hat{c}(\ell)+K\}}$. The map ν_ℓ^K is thus determined by a closed embedding

$$\{z\in\hat{c}(\ell)-e^{i\alpha}.[0,\infty); s\in -z+2\hat{c}(\ell)+K\}\subset\{z\in\hat{c}(\ell)-e^{i\alpha}.[0,\infty),\ s-z\in K\}.$$

Let us now define a map

$$N_\ell^K : \Lambda_\ell^{K-} * S_- \to \Lambda_\ell^{K+} * S_+.$$

as follows. We have $\Lambda_\ell^{K-} * S_- = \bigoplus_{w\in\mathbf{W}^\alpha_{\text{right}}} \Lambda_\ell^{K-} * S_w$.

We denote

$$N_\ell^w : \Lambda_\ell^{K-} * S_w \xrightarrow{\nu_\ell^K} \Lambda_\ell^{K+} * S_\ell * S_w = \Lambda_\ell^{K+} * S_{\ell w}. \tag{42}$$

Observe that $\ell w\in\mathbf{W}^\alpha_{\text{left}}$, so that $\Lambda_\ell^{K+} * S_{\ell w}$ is a direct summand of $\Lambda_\ell^{K+} * S_+$. We therefore can define N_ℓ^K as the direct sum of all N_ℓ^w, $w\in\mathbf{W}^\alpha_{\text{right}}$.

Let

$$\mathbf{N}_\ell^K : \Lambda_\ell^{K-} * S_- \oplus \Lambda_\ell^{K+} * S_+ \to \Lambda_\ell^{K-} * S_- \oplus \Lambda_\ell^{K+} * S_+$$

be the extension of N_ℓ^K whose all components are zero, except for $\Lambda_\ell^{K-} * S_- \to \Lambda_\ell^{K+} * S_+$ which equals N_ℓ^K.

We set

$$\Gamma^{P_1P_2}_{\Phi^K} := \mathrm{Id} + \mathbf{N}_\ell^K. \tag{43}$$

Finally, we set

$$\Gamma^{P_2P_1}_{\Phi^K} := (\Gamma^{P_1P_2}_{\Phi^K})^{-1} = \mathrm{Id} - \mathbf{N}_\ell^K.$$

Let us now rewrite the definition for the gluing maps in a more uniform way. Let P and P' be two neighboring strips such that $P\cap P'$ goes to the left. Let us define the sign

$$\vartheta(P,P') = 1 \text{ if } P' \text{ is above } P, \text{ and } \vartheta(P,P') = -1 \text{ if } P' \text{ is below } P. \tag{44}$$

We now have

$$\Gamma^{PP'}_{\Phi^K} := \mathrm{Id} + \vartheta(P,P')\mathbf{N}_\ell^K. \tag{45}$$

<u>Case B).</u> Let $\ell\in\mathcal{L}_{\text{right}}$, fig. 4,b). Assume first that P_2 is below P_1.

The formulas are similar to the case A but $+$ and $-$ get exchanged. We have a map

$$\nu_\ell^K : \Lambda_\ell^{K+} \to \Lambda_\ell^{K-} * S_\ell \tag{46}$$

which gives rise to a map

$$N_\ell^K\ :\ \Lambda_\ell^{K+} * S_+ \xrightarrow{\nu_\ell^K} \Lambda_\ell^{K-} * S_\ell * S_+\ \to\ \Lambda_\ell^{K-} * S_-. \tag{47}$$

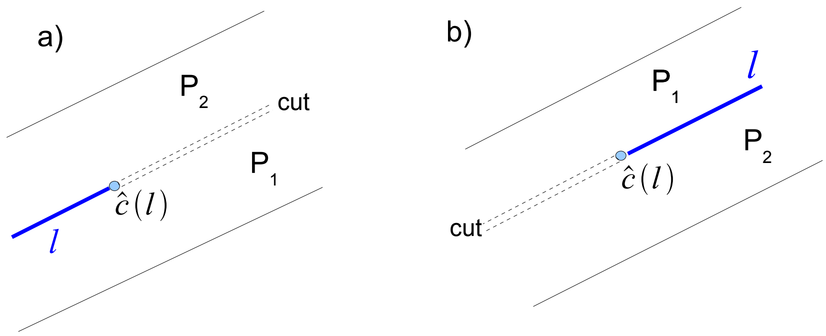

Figure 4: Notations in the construction of the sheaf Φ^K: a) $\ell \in \mathcal{L}^\alpha_{\text{left}}$, b) $\ell \in \mathcal{L}^\alpha_{\text{right}}$

Similar to above, we define a map

$$\mathbf{N}^K_\ell : \Lambda^{K+}_\ell * S_+ \oplus \Lambda^{K-}_\ell * S_- \to \Lambda^{K+}_\ell * S_+ \oplus \Lambda^{K-}_\ell * S_-$$

as the extension of N^K_ℓ whose all components are zero except for $\Lambda^{K+}_\ell * S_+ \to \Lambda^{K-}_\ell * S_-$ which is N^K_ℓ.

We set

$$\Gamma^{P_1P_2}_{\Phi^K} := \mathrm{Id} + \mathbf{N}^K_\ell; \tag{48}$$

$$\Gamma^{P_2P_1}_{\Phi^K} := (\Gamma^{P_1P_2}_{\Phi^K})^{-1} = \mathrm{Id} - \mathbf{N}^K_\ell.$$

Similarly to above, let us rewrite the definition as follows. Let P and P' be two neighboring strips such that $P \cap P'$ goes to the right. Let us define the sign

$$\vartheta(P, P') = 1 \text{ if } P' \text{ is below } P;\ \vartheta(P, P') = -1 \text{ if } P \text{ is below } P'. \tag{49}$$

We now have

$$\Gamma^{PP'}_{\Phi^K} := \mathrm{Id} + \vartheta(P, P')\mathbf{N}^K_\ell. \tag{50}$$

3.8.6 Description of the map $i_{\Phi^K} : \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to \Phi^K$

Let P_0 be the strip such that $\mathbf{x}_0 \in \mathrm{Int}P_0$.

By construction,

$$\Phi^K|_{\mathrm{Int}P_0 \times \mathbb{C}} = \Lambda^{K+}_{\mathrm{Int}P_0} * S_+ \oplus \Lambda^{K-}_{\mathrm{Int}P_0} * S_-.$$

The direct summand inclusions

$$S_L \to S_+ \ ; \quad S_R \to S_-$$

induce maps $\Lambda^{K+}_{\mathrm{Int}P_0} * S_L \to \Lambda^{K+}_{\mathrm{Int}P_0} * S_+$, $\Lambda^{K-}_{\mathrm{Int}P_0} * S_R \to \Lambda^{K-}_{\mathrm{Int}P_0} * S_-$.

We have the following closed embedding of codimension 2:

$$\left\{\begin{array}{c} x=\mathbf{x}_0 \\ s\in K\end{array}\right\} \hookrightarrow \left\{\begin{array}{c} x\in \mathrm{Int}P_0 \\ s\pm z(x)\in \pm z(\mathbf{x}_0)+K\end{array}\right\}.$$

We have the following maps in $\mathbf{D}(\mathrm{Int}P_0\times\mathbb{C})$:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z}_{\left\{\begin{array}{c} x\in \mathrm{Int}P_0 \\ s+z(x)\in z(\mathbf{x}_0)+K\end{array}\right\}} \arrow[r] & \Lambda^{K+}_{\mathrm{Int}P_0}*S_L & & \\
\mathbb{Z}_{\left\{\begin{array}{c} x=\mathbf{x}_0 \\ s\in K\end{array}\right\}}[-2] \arrow[ur] \arrow[dr] & \oplus & \oplus & \rightarrow & \Phi^K|_{\mathrm{Int}P_0\times\mathbb{C}} \\
 & \mathbb{Z}_{\left\{\begin{array}{c} x\in \mathrm{Int}P_0 \\ s-z(x)\in -z(\mathbf{x}_0)+K\end{array}\right\}} \arrow[r] & \Lambda^{K-}_{\mathrm{Int}P_0}*S_R & &
\end{tikzcd} \tag{51}$$

We thus have constructed a map

$$\mathbb{Z}_{\left\{\begin{array}{c} x=\mathbf{x}_0 \\ s\in K\end{array}\right\}}[-2] = \mathbb{Z}_{\mathbf{x}_0\times K}[-2]\to \Phi^K|_{\mathrm{Int}P_0\times\mathbb{C}} \tag{52}$$

As $\mathbb{Z}_{\mathbf{x}_0\times K}[-2]$ is supported on $\mathrm{Int}P_0$, our map extends canonically to a map $i_{\Phi^K}:\mathbb{Z}_{\mathbf{x}_0\times K}[-2]\to\Phi^K$ in $\mathbf{D}(X\times\mathbb{C})$.

3.9 Alternative construction of Φ^K via $-\alpha$-strips

It is clear that one can repeat all the steps of the previous section using $-\alpha$-strips instead of α strips. We denote the resulting sheaf Ψ^K; we also get an analogue of the map i_{Φ^K}, to be denoted by

$$i_{\Psi^K}:\mathbb{Z}_{\mathbf{x}_0\times K}[-2]\to\Psi^K. \tag{53}$$

By means of Ψ^K, we also get a semiorthogonal decomposition of $\mathbb{Z}_{\mathbf{x}_0\times K}[-2]$. This implies the existence of a unique isomorphism

$$I_{\Psi\Phi}:\Psi^K\to\Phi^K \tag{54}$$

satisfying $i_{\Phi^K}=I_{\Psi\Phi}i_{\Psi^K}$ (because of the unicity of semiorthogonal decomposition). We will now briefly go over the construction of Ψ^K.

3.9.1 Notation for $-\alpha$-strips

Let $\mathcal{L}^{-\alpha} = \mathcal{L}^{-\alpha}_{\text{left}} \cup \mathcal{L}^{-\alpha}_{\text{right}}$ be the set of all intersection rays of $-\alpha$-strips. $\mathcal{L}^{-\alpha}_{\text{left}}$ consists of the rays going to the left, $\mathcal{L}^{-\alpha}_{\text{right}}$ consists of the rays going to the right. Every ray $\ell \in \mathcal{L}^{-\alpha}_{\text{left}}$ (resp. $\ell \in \mathcal{L}^{-\alpha}_{\text{right}}$) is of the form $p_z(\ell) = \hat{c}(\ell) - (0,\infty)e^{-i\alpha}$; (resp. $p_z(\ell) = \hat{c}(\ell) + (0,\infty)e^{-i\alpha}$) for some $\hat{c}(\ell) \in \mathbb{C}$.

Let $\mathbf{W}^{-\alpha}, \mathbf{W}^{-\alpha}_{\text{left}}, \mathbf{W}^{-\alpha}_{\text{right}}$ be defined in the same way as $\mathbf{W}^{\alpha}, \mathbf{W}^{\alpha}_{\text{left}}, \mathbf{W}^{\alpha}_{\text{right}}$. ($\mathbf{W}^{-\alpha}_{\text{left}}$ consists of words of the form $w = \ell_n\ell_{n-1}...\ell_2\ell_1 L$ or $w = \ell_n...\ell_1 R$ where $\ell_n \in \mathcal{L}^{-\alpha}_{\text{left}}$ and we have an alternating pattern $\ell_{n-1} \in \mathcal{L}^{-\alpha}_{\text{right}}$, $\ell_{n-1} \in \mathcal{L}^{-\alpha}_{\text{left}}$,... ; if $\ell_1 \in \mathcal{L}^{-\alpha}_{\text{right}}$, then the right-most letter of w is L; if $\ell_1 \in \mathcal{L}^{-\alpha}_{\text{left}}$ then the right-most letter of w is R; we also add a one letter word L to $\mathbf{W}^{-\alpha}_{\text{left}}$.) Similarly to the previous section, we set

$$\tilde{\mathcal{S}}_\ell := \mathbb{Z}_{\{s:s\in 2\hat{c}(\ell)+K\}} \in \mathbf{D}(\mathbb{C}), \quad \ell \in \mathcal{L}^{-\alpha}_{\text{left}};$$

$$\tilde{\mathcal{S}}_\ell := \mathbb{Z}_{\{s:s\in -2\hat{c}(\ell)+K\}} \in \mathbf{D}(\mathbb{C}), \quad \ell \in \mathcal{L}^{-\alpha}_{\text{right}};$$

$$\tilde{\mathcal{S}}_L := \mathbb{Z}_{\{s:s\in z(\mathbf{x}_0)+K\}} \in \mathbf{D}(\mathbb{C});$$

$$\tilde{\mathcal{S}}_R := \mathbb{Z}_{\{s:s\in -z(\mathbf{x}_0)+K\}} \in \mathbf{D}(\mathbb{C}),$$

For $w \in \mathbf{W}^{-\alpha}$, $w = \ell_n...\ell_1(L \text{ or } R)$ set

$$\tilde{\mathcal{S}}_w = \tilde{\mathcal{S}}_{\ell_n} * \tilde{\mathcal{S}}_{\ell_{n-1}} * ... * \tilde{\mathcal{S}}_{\ell_1} * (\tilde{\mathcal{S}}_L \text{ or } \tilde{\mathcal{S}}_R).$$

Set

$$\tilde{S}_- := \oplus_{w\in \mathbf{W}^{-\alpha}_{\text{right}}} \tilde{\mathcal{S}}_w; \qquad \tilde{S}_+ := \oplus_{w\in \mathbf{W}^{-\alpha}_{\text{left}}} \tilde{\mathcal{S}}_w.$$

3.9.2 Sheaves Ψ^K_Π

Let $\Lambda^{K\pm}_U$ mean the same thing as in Sec.2.11. On every $(-\alpha)$-strip Π consider the sheaf on Π

$$\Psi^K_\Pi \;:=\; \Lambda^{K+}_\Pi * \tilde{\mathcal{S}}_+ \oplus \Lambda^{K-}_\Pi * \tilde{\mathcal{S}}_-.$$

3.9.3 Gluing maps

Let Π_1, Π_2 be neighboring strips, $\Pi_1 \cap \Pi_2 = \ell$.

Case A. If ℓ goes to the left, we denote by Π_1 the bottom strip, fig. 5, a).
We then define a map

$$\tilde{\nu}^K_\ell : \Lambda^{K-}_\ell \to \Lambda^{K+}_\ell * \tilde{S}_\ell$$

similar to ν^K_ℓ from the previous subsection. The maps $\tilde{\nu}^K_\ell$ induce maps

$$\tilde{N}^K_\ell : \Lambda^{K-}_\ell * \tilde{S}_+ \to \Lambda^{K+}_\ell * \tilde{S}_-$$

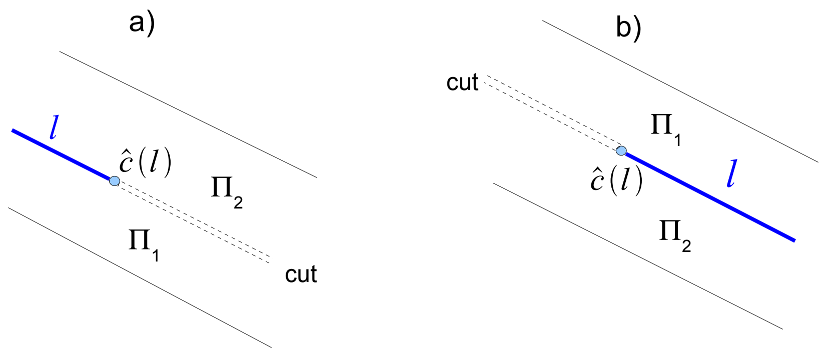

Figure 5: Notations in the construction of the sheaf Ψ^K: a) $\ell \in \mathcal{L}_{\text{left}}$, b) $\ell \in \mathcal{L}_{\text{right}}$

and

$$\tilde{\mathbf{N}}^K_\ell : \Lambda^{K+}_\ell * \tilde{S}_+ \oplus \Lambda^{K-}_\ell * \tilde{S}_- \to \Lambda^{K+}_\ell * \tilde{S}_+ \to \Lambda^{K-}_\ell * \tilde{S}_-,$$

in the same way as in Sec 3.8.5.

We now set

$$\Gamma^{\Pi_1\Pi_2}_{\Psi^K} := \mathrm{Id} + \tilde{\mathbf{N}}^K_\ell. \tag{55}$$

We set $\Gamma^{\Pi_2\Pi_1}_{\Psi^K} := (\Gamma^{\Pi_1\Pi_2}_{\Psi_K})^{-1} = \mathrm{Id} - \tilde{\mathbf{N}}^K_\ell$.

Similarly to the previous subsection, we can combine the definitions as follows. Let Π and Π' be intersecting $-\alpha$-strips whose intersection ray $\ell := \Pi \cap \Pi'$ goes to the left. Define a number $\vartheta(\Pi, \Pi') = 1$ if Π is below Π' and $\vartheta(\Pi, \Pi') = -1$ otherwise. We then have $\Gamma^{\Pi\Pi'}_{\Psi^K} = \mathrm{Id} + \vartheta(\Pi, \Pi')\mathbf{N}^K_\ell$.

<u>Case B.</u> Analogously, assume that $\ell = \Pi_1 \cap \Pi_2$ goes to the right and that Π_2 is below Π_2, fig. 5, b). Similar to above, we have a map

$$\tilde{\nu}^K_\ell : \Lambda^{K+}_\ell \to \Lambda^{K-}_\ell * \tilde{\mathcal{S}}_\ell, \tag{56}$$

which enables us to define maps

$$\tilde{N}^K_\ell : \Lambda^{K+}_\ell * \tilde{\mathcal{S}}_+ \to \Lambda^{K-}_\ell * \tilde{\mathcal{S}}_-;$$

$$\tilde{\mathbf{N}}^K_\ell : \Lambda^{K+}_\ell * \tilde{\mathcal{S}}_+ \oplus \Lambda^{K-}_\ell * \tilde{\mathcal{S}}_- \to \Lambda^{K+}_\ell * \tilde{\mathcal{S}}_+ \oplus \Lambda^{K-}_\ell * \tilde{\mathcal{S}}_-$$

in the same way as above. We set

$$\Gamma^{\Pi_1\Pi_2}_{\Psi^K} := \mathrm{Id} + \tilde{\mathbf{N}}^K_\ell; \tag{57}$$

$$\Gamma^{\Pi_2\Pi_1}_{\Psi^K} := (\Gamma^{\Pi_1\Pi_2}_{\Psi^K})^{-1} = \mathrm{Id} - \tilde{\mathbf{N}}^K_\ell \tag{58}$$

Finally, given two intersecting $-\alpha$-strips Π and Π' whose intersection ray ℓ goes to the right, we set $\vartheta(\Pi, \Pi') = 1$ if Π' is below Π and $\vartheta(\Pi, \Pi') = -1$ otherwise so that $\Gamma^{\Pi\Pi'}_{\Psi^K} = \mathrm{Id} + \vartheta(\Pi, \Pi')\tilde{\mathbf{N}}^K_\ell$.

The sheaf Ψ^K is obtained by gluing of the sheaves Ψ_Π along the boundary rays by means of the maps $\Gamma^{\Pi\Pi'}_{\Psi^K}$, similarly to Φ^K.

The map

$$i_{\Psi^K}:\mathbb{Z}_{\mathbf{x}_0\times K}[-2]\to\Psi^K, \tag{59}$$

same as in (53), is constructed similarly to i_{Φ^K}.

3.10 The map $I_{\Psi\Phi}$

We now pass to discussing the identification $I_{\Psi\Phi}:\Psi^K\to\Phi^K$ as in (54). Explicit formulas for the map $I_{\Psi\Phi}$ are complicated, see Sec. 7. Let us, however, formulate a result on this map, to be proven in Sec . 7.

Let P be an α-strip and Π be a $-\alpha$-strip. Suppose $P\cap\Pi\neq\emptyset$. We have identifications

$$\Phi^K|_{P\cap\Pi}=\Phi_P|_{P\cap\Pi}=\Lambda^{K+}_{P\cap\Pi}*S_+\oplus\Lambda^{K-}_{P\cap\Pi}*S_-;$$

$$\Psi^K|_{P\cap\Pi}=\Psi_\Pi|_{P\cap\Pi}=\Lambda^{K+}_{P\cap\Pi}*\tilde{S}_+\oplus\Lambda^{K-}_{P\cap\Pi}*\tilde{S}_-.$$

Set $i_{\Pi P}:=I_{\Psi\Phi}|_{P\cap\Pi}$. In view of the above identifications, we can rewrite:

$$i_{\Pi P}:\Lambda^{K+}_{P\cap\Pi}*\tilde{S}_+\oplus\Lambda^{K-}_{P\cap\Pi}*\tilde{S}_-\to\Lambda^{K+}_{P\cap\Pi}*S_+\oplus\Lambda^{K-}_{P\cap\Pi}*S_-.$$

We are now going to take advantage of direct sum decompositions of both parts of this map.

3.10.1 Decomposing $i_{\Pi P}$ into components

Let us now rewrite both sides of this map as follows.

For a$w\in\mathbf{W}^\alpha_{\text{left}}$ or $w\in\mathbf{W}^{-\alpha}_{\text{left}}$, we define $A(K,w)\subset(P\cap\Pi)\times\mathbb{C}$:

$$A(K,w):=\{(x,s)|s+p_z(x)\in\hat{c}(w)+K\},$$

where $\hat{c}w$ is as in (36), (37).

For $w\in\mathbf{W}^\alpha_{\text{right}}$, or $w\in\mathbf{W}^{-\alpha}_{\text{right}}$, we set

$$A(K,w):=\{(x,s)|s-p_z(x)\in-\hat{c}(w)+K\}.$$

We then have

$$\Lambda^{K+}_{P\cap\Pi}*S_+\oplus\Lambda^{K-}_{P\cap\Pi}*S_-=\bigoplus_{w\in\mathbf{W}^\alpha}\mathbb{Z}_{A(K,w)};$$

$$\Lambda^{K+}_{P\cap\Pi} * \tilde{S}_+ \oplus \Lambda^{K-}_{P\cap\Pi} * \tilde{S}_- = \bigoplus_{\tilde{w}\in\mathbf{W}^{-\alpha}} \mathbb{Z}_{A(K,\tilde{w})}.$$

Next,

$$\begin{aligned}\mathrm{Hom}(\bigoplus_{\tilde{w}\in\mathbf{W}^{-\alpha}} \mathbb{Z}_{A(K,\tilde{w})}; \bigoplus_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(K,w)}) &= \prod_{\tilde{w}\in\mathbf{W}^{-\alpha}} \mathrm{Hom}(\mathbb{Z}_{A(K,\tilde{w})}; \bigoplus_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(K,w)}) \\ &\hookrightarrow \prod_{\tilde{w}\in\mathbf{W}^{-\alpha}; w\in\mathbf{W}^{\alpha}} \mathrm{Hom}(\mathbb{Z}_{A(K,\tilde{w})}; \mathbb{Z}_{A(K,w)}). \end{aligned} \tag{60}$$

In Sec 7.1 we prove that $\mathrm{Hom}(\mathbb{Z}_{A(K,\tilde{w})}; \mathbb{Z}_{A(K,w)}) = 0$ unless $A(K,w) \subset A(K,\tilde{w})$, in which case $\mathrm{Hom}(\mathbb{Z}_{A(K,\tilde{w})}; \mathbb{Z}_{A(K,w)}) = \mathbb{Z}.e_{\tilde{w},w}$, where $e_{\tilde{w},w}$ is the homomorphism induced by the embedding $A(K,w) \subset A(K,\tilde{w})$. Elements of $\prod\limits_{\tilde{w}\in\mathbf{W}^{-\alpha}; w\in\mathbf{W}^{\alpha}} \mathrm{Hom}(\mathbb{Z}_{A(K,\tilde{w})}; \mathbb{Z}_{A(K,w)})$ are thus identified with infinite sums of the form

$$\sum_{\tilde{w},w} n_{\tilde{w}w} e_{\tilde{w}w}, \tag{61}$$

where $n_{\tilde{w}w} \in \mathbb{Z}$, and $A(K,w) \subset A(K,\tilde{w})$. By Prop.7.2, under the inclusion (60) the set $\hom(\bigoplus\limits_{\tilde{w}\in\mathbf{W}^{-\alpha}} \mathbb{Z}_{A(K,\tilde{w})}; \bigoplus\limits_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(K,w)})$ is identified with the set of all sums as in (61), satisfying

<u>for every point $y \in (P\cap\Pi)\times\mathbb{C}$ and every $\tilde{w} \in \mathbf{W}^{-\alpha}$, there are only finitely many $w \in \mathbf{W}^{\alpha}$ such that $n_{\tilde{w}w} \neq 0$ and $y \in A(K,w)$.</u>

3.10.2 Identification $\mathbf{W}^{-\alpha} \to \mathbf{W}^{\alpha}$.

Let us first define an identification $\mathbf{A} : \mathcal{L}^{-\alpha} \to \mathcal{L}^{\alpha}$. Let $\ell \in \mathcal{L}^{-\alpha}$. Suppose ℓ goes to the right. Let P be the leftmost strip among all α-strips that intersect ℓ. There are exactly two boundary rays of P, ℓ_l and ℓ_r such that $\hat{c}(\ell_l) = \hat{c}(\ell_r) = \hat{c}(\ell)$, ℓ_l goes to the left, and ℓ_r goes to the right. Let us assign $\mathbf{A}(\ell) = \ell_r$.

Similarly, if $\ell \in \mathcal{L}^{-\alpha}$, ℓ goes to the left, we consider the leftmost strip P among all α-strips that intersect ℓ. There are exactly two boundary rays of P, ℓ_l and ℓ_r such that

$$\hat{c}(\ell_l) = \hat{c}(\ell_r) = \hat{c}(\ell). \tag{62}$$

ℓ_l goes to the left, and ℓ_r goes to the right. Let us assign $\mathbf{A}(\ell) = \ell_l$. The map $\mathbf{A}$ extends in the obvious way to a map $\mathbf{A} : \mathbf{W}^{-\alpha} \to \mathbf{W}^{\alpha}$: a word $\ell_n \cdots \ell_1 L \in \mathbf{W}^{-\alpha}$ (resp. $\ell_n \cdots \ell_1 R \in \mathbf{W}^{-\alpha}$) is mapped into $\mathbf{A}(\ell_n)\cdots\mathbf{A}(\ell_1)L$ (resp. $\mathbf{A}(\ell_n)\cdots\mathbf{A}(\ell_1)R$). Because of (62), we have $A(K,\tilde{w}) = A(K,\mathbf{A}(\tilde{w}))$ for all $\tilde{w} \in \mathbf{W}^{-\alpha}$.

3.10.3 Formulation of the result

Let us write $i_{\Pi P}$ in the form (61):

$$i_{\Pi P} = \sum_{\tilde{w}\in\mathbf{W}^{-\alpha};w\in\mathbf{W}^{\alpha}} n_{\tilde{w}w} e_{\tilde{w}w}. \tag{63}$$

In order to formulate the result, let us introduce some notation. For $\tilde{w} \in \mathbf{W}^{-\alpha}$, $\tilde{w} = \ell_n \cdots \ell_1 L \in \mathbf{W}^{-\alpha}$ (resp. $\tilde{w} = \ell_n \cdots \ell_1 R \in \mathbf{W}^{-\alpha}$), set $|\tilde{w}| := n$, to be the length of $\tilde{w}$ (in particular $|L| = |R| = 0$).

Proposition 3.6 *1) We have $n_{\tilde{w}\mathbf{A}(\tilde{w})} = (-1)^{|\tilde{w}|}$;*

2) If $n_{\tilde{w}w} \neq 0$, then $A(K, w) \neq A(K, \tilde{w})$ (we have a strict embedding $A(K, w) \subset A(K, \tilde{w})$).

This proposition is proven in Sec 7.6.4.

3.11 Description of $\Phi^{\mathbf{r}_\alpha}$

We construct the sheaf $\Phi^{\mathbf{r}_\alpha}$ and a map $i_{\Phi^{\mathbf{r}_\alpha}}$ in a way very similar to the construction Φ^K, using the decomposition of X into α-strips and replacing K with $\mathbf{r}_\alpha$ everywhere. We then get sheaves

$$\Lambda_U^{\mathbf{r}_\alpha\pm} := \mathbb{Z}_{\{(x,s)|x\in U, s\in\mathbb{C}; s\pm x\in\mathbf{r}_\alpha\}}.$$

$$\Phi_P^{\mathbf{r}_\alpha} := \Lambda_P^{\mathbf{r}_\alpha+} * S_+ \oplus \Lambda^{\mathbf{r}_\alpha-} * S_-.$$

If ℓ goes to the left (resp. to the right) we still have a map

$$\nu_\ell^{\mathbf{r}_\alpha} : \Lambda^{\mathbf{r}_\alpha-} \to \Lambda^{\mathbf{r}_\alpha+} * S_\ell \text{ resp. } \nu_\ell^{\mathbf{r}_\alpha} : \Lambda^{\mathbf{r}_\alpha+} \to \Lambda^{\mathbf{r}_\alpha-} * S_\ell,$$

so that we can define the gluing maps $\Gamma_{\Phi^{\mathbf{r}_\alpha}}^{P_1P_2}$ similarly to $\Gamma_{\Phi^K}^{P_1P_2}$.

3.12 Description of $\Phi^{\mathbf{r}_{-\alpha}}$

In order to construct $\Phi^{\mathbf{r}_{-\alpha}}$ and $i_{\Phi^{\mathbf{r}_{-\alpha}}}$ we switch to $-\alpha$-strips (sticking to α-strips leads to a failure to define the maps $\nu_\ell^{\mathbf{r}_{-\alpha}}$). The construction is then similar to the construction of Ψ^K (just replace K with $\mathbf{r}_{-\alpha}$ everywhere).

3.13 Constructing the map (30)

Let us construct a map $\mathcal{Q}$, satisfying (30). It will be convenient for us to replace Φ^K with the isomorphic sheaf Ψ^K.

First, we will construct maps $q_{C\mathbf{r}_\alpha} : \Phi^C \to \Phi^{\mathbf{r}_\alpha}$; $q_{K\mathbf{r}_{\pm\alpha}} : \Psi^K \to \Psi^{\mathbf{r}_{\pm\alpha}}$ satisfying $i_{\Psi^K} = q_{C\mathbf{r}_\alpha} i_{\Phi^C}$; $i_{\Phi^{\mathbf{r}_{\pm\alpha}}} = q_{K\mathbf{r}_{\pm\alpha}} i_{\Psi^K}$.

We define $\mathcal{Q}$ as follows:

$$\mathcal{Q} : \begin{tikzcd} \Phi^C \arrow[r, "q_{C\mathbf{r}_\alpha}"] & \Phi^{\mathbf{r}_\alpha} \\ \oplus & \oplus \\ \Psi^K \arrow[ur, "q_{K\mathbf{r}_\alpha}"] \arrow[r, "q_{K\mathbf{r}_{-\alpha}}"] & \Phi^{\mathbf{r}_{-\alpha}}. \end{tikzcd} \tag{64}$$

The categorical definition of the maps in this diagram was discussed in section 3.6.

Let us now pass to constructing the above mentioned maps $q_{C\mathbf{r}_\alpha}$ and $q_{K\mathbf{r}_{\pm\alpha}}$.

3.13.1 The map $q_{C\mathbf{r}_\alpha}$

We have $\Phi^C = \mathbb{Z}_{X\times\mathbb{C}}$ so that

$$\hom(\Phi^C; \Phi^{\mathbf{r}_\alpha}) = \Gamma(X \times \mathbb{C}; \Phi^{\mathbf{r}_\alpha})$$

so that a map $q_{C\mathbf{r}_\alpha}$ can be defined by means of specifying a section $\mathbf{q} \in \Gamma(X \times \mathbb{C}; \Phi^{\mathbf{r}_\alpha})$. This can be done strip-wise: we can instead specify sections $\mathbf{q}_P \in \Gamma(P \times \mathbb{C}; \Phi_P^{\mathbf{r}_\alpha})$ which agree on intresections as follows. Let $P_1 \cap P_2 = \ell$. We then have restriction maps

$$|_{\ell\times\mathbb{C}} : \Gamma(P_i \times \mathbb{C}; \Phi_{P_i}^{\mathbf{r}_\alpha}) \to \Gamma(\ell \times \mathbb{C}; \Phi_\ell^{\mathbf{r}_\alpha}), \quad i = 1, 2.$$

We then should have

$$\mathbf{q}_{P_1}|_{\ell\times\mathbb{C}} = \mathbf{q}_{P_2}|_{\ell\times\mathbb{C}}. \tag{65}$$

It is clear that any collection of data $\mathbf{q}_P$, satisfying (65) for all pairs of neighboring strips, determines a section $\mathbf{q} \in \Gamma(X \times \mathbb{C}; \Phi^{\mathbf{r}_\alpha})$ in a unique way.

We have $\mathbb{Z} = \Gamma(\Pi \times \mathbb{C}; \Lambda_P^{\pm} * S_w)$ for all $w \in \mathbf{W}^\alpha$.

Let us take the direct sum of these identifications over all $w \in \mathbf{W}^\alpha$ so as to get a map

$$s_P : \mathbb{Z}[\mathbf{W}^\alpha] \to \Gamma(P \times \mathbb{C}; \Phi_P^{\mathbf{r}_\alpha}),$$

where $\mathbb{Z}[\mathbf{W}^\alpha]$ is the $\mathbb{Z}$-span of the set $\mathbf{W}^\alpha$. Similarly, we define

$$s_\ell : \mathbb{Z}[\mathbf{W}^\alpha] \to \Gamma(\ell \times \mathbb{C}; \Phi_\ell^{\mathbf{r}_\alpha}),$$

where ℓ is the intersection ray of a pair of neighboring α-strips . The maps s_Π, s_ℓ are inclusions; denote by $\Gamma'(P \times \mathbb{C}; \Phi_P^{\mathbf{r}_\alpha}), \Gamma'(\ell \times \mathbb{C}; \Phi_\ell^{\mathbf{r}_\alpha})$ the images of these inclusions. As easily follows from the definition of the gluing maps $\Gamma_{\Phi^{\mathbf{r}_\alpha}}^{\Pi_1\Pi_2}$, the restriction maps induce isomorphisms

$$|_{\ell\times\mathbb{C}} : \Gamma'(P \times \mathbb{C}; \Phi_P^{\mathbf{r}_\alpha}) \to \Gamma'(\ell \times \mathbb{C}; \Phi_\ell^{\mathbf{r}_\alpha}),$$

where ℓ is a boundary ray of P.

Since the graph formed by α-strips and their intersection rays is a tree, it follows that given an element $\mathbf{q}_{P_0} \in \Gamma'(P_0 \times \mathbb{C}; \Phi^{\mathbf{r}_\alpha}_{P_0})$, we have unique elements

$$\mathbf{q}_P \in \Gamma'(P \times \mathbb{C}; \Phi^{\mathbf{r}_\alpha}_P)$$

satisfying (65). We set $\mathbf{q}_{P_0} := s_{P_0}(L+R)$, where L, R are words of of length 1 in $\mathbf{W}^\alpha$ viewed as elements in $\mathbb{Z}[\mathbf{W}^\alpha]$. This way we get a section $\mathbf{q}$ and a map $q_{C\mathbf{r}_\alpha}$. It is clear that Condition $i_{\Phi^{\mathbf{r}_\alpha}} = q_{C\mathbf{r}_\alpha} i_{\Phi^C}$ is satisfied.

Denote by $\mathbf{e}_P \in \mathbb{Z}[\mathbf{W}^\alpha]$ a unique element such that $s_P(\mathbf{e}_P) = \mathbf{q}_P$. Denote by $W_P \subset \mathbf{W}^\alpha$ a finite subset such that

$$\mathbf{e}_P = \sum_{w \in W_P} \mathbf{e}_{Pw} w,$$

where $\mathbf{e}_{Pw} \in \mathbb{Z}\backslash 0$.

3.13.2 Map $q_{K\mathbf{r}_{-\alpha}} : \Psi^K \to \Phi^{\mathbf{r}_{-\alpha}}$

Let us define this map stripwise. For every $-\alpha$-strip Π we have a map $\Lambda^{K\pm}_\Pi \to \Lambda^{\mathbf{r}_{-\alpha}\pm}_\Pi$ induced by the embedding of the corresponding closed subsets of $\Pi \times \mathbb{C}$. Whence induced maps $\Lambda^{K\pm}_\Pi * \tilde{S}_w \to \Lambda^{\mathbf{r}_{-\alpha}\pm}_\Pi * \tilde{S}_w$. Taking a direct sum over all $w \in \mathbf{W}^\alpha$ yields a map

$$\Lambda^{K+}_\Pi * \tilde{S}_+ \oplus \Lambda^{K-}_\Pi * \tilde{S}_- \to \Lambda^{\mathbf{r}_{-\alpha}+}_\Pi * \tilde{S}_+ \oplus \Lambda^{\mathbf{r}_{-\alpha}-}_\Pi * \tilde{S}_-,$$

and we assign $q_{K\mathbf{r}_{-\alpha},\Pi} : \Psi^K_\Pi \to \Phi^{\mathbf{r}_{-\alpha}}_\Pi$ to be this map. It is clear that thus defined maps agree on all intersection rays, thereby defining the desired map $q_{K\mathbf{r}_{-\alpha}}$. The condition $i_{\Phi^{\mathbf{r}_{-\alpha}}} = q^{K\mathbf{r}_{-\alpha}} i_{\Psi^K}$ is clearly satisfied.

3.13.3 Map $q_{K\mathbf{r}_\alpha} : \Psi^K \to \Phi^{\mathbf{r}_\alpha}$

We first construct a map $q'_{K\mathbf{r}_{-\alpha}} : \Phi^K \to \Phi^{\mathbf{r}_\alpha}$ using α strip in the same way as we constructed $q_{K\mathbf{r}_{-\alpha}}$. We set

$$q_{K\mathbf{r}_\alpha} := q'_{K\mathbf{r}_\alpha} I_{\Psi\Phi}.$$

The condition $i_{\Phi^{\mathbf{r}_\alpha}} = q_{K\mathbf{r}_\alpha} i_{\Psi^K}$ is clearly satisfied.

3.13.4 Restriction of $\mathcal{Q}$ to a parallelogram

Let P and Π be a pair of intersecting α- and $(-\alpha)$-strips.

First, in view of identification $\mathbf{A}$, let us write w instead of $\mathbf{A}^{-1}w \in \mathbf{W}^{-\alpha}$. Next, for a $w \in \mathbf{W}^{\alpha}$ and a subset $\Delta \in \mathbb{C}$, let us define a subset $A(\Delta, w) \subset (P \cap \Pi) \times \mathbb{C}$ as follows. If $w \in \mathbf{W}^{\alpha}_{\text{left}}$ (resp., $w \in \mathbf{W}^{\alpha}_{\text{right}}$), we set $A(\Delta, w) = \{(x,s)|s + z(x) \in \hat{c}(w) + \Delta\}$ (resp., $A(\Delta, w) = \{(x,s)|s - z(x) \in \hat{c} + \Delta\}$. Set $A_0 := (\Pi \cap P) \times \mathbb{C}$. We then have identifications

$$\Phi^{C}_{\Pi\cap P} = \mathbb{Z}_{A_0};$$

$$\Psi^{K}_{\Pi\cap P} = \bigoplus_{w\in\mathbf{W}^{-\alpha}} \mathbb{Z}_{A(K,w)};$$

$$\Phi^{\mathbf{r}_\alpha}_{\Pi\cap P} = \bigoplus_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(\mathbf{r}_\alpha;w)};$$

$$\Phi^{\mathbf{r}_{-\alpha}}_{\Pi\cap P} = \bigoplus_{w\in\mathbf{W}^{-\alpha}} \mathbb{Z}_{A(\mathbf{r}_{-\alpha};w)}.$$

Let us now rewrite the maps from diagrams (64) in terms of these identifications.

3.13.5 The map $q_{C\mathbf{r}_\alpha}$.

Let $E^{C\mathbf{r}_\alpha}_w : \mathbb{Z}_{A_0} \to \mathbb{Z}_{A(\mathbf{r}_\alpha,w)}$ be the map induced by the closed embedding of the corresponding sets. According to Sec 3.13.1,

$$q_{C\mathbf{r}_\alpha} = \sum_{w\in W_P} \mathbf{e}_{Pw} E^{C\mathbf{r}_\alpha}_w. \tag{66}$$

3.13.6 The map $q_{K\mathbf{r}_{-\alpha}}$

It follows that the map

$$q_{K\mathbf{r}_{-\alpha}} : \bigoplus_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(K,w)} \to \bigoplus_{w\in\mathbf{W}^{\alpha}} \mathbb{Z}_{A(\mathbf{r}_{-\alpha},w)}$$

is a direct sum, over all $w \in \mathbf{W}^{\alpha}$, of the maps

$$\mathbb{Z}_{A(K,w)} \to \mathbb{Z}_{A(\mathbf{r}_{-\alpha},w)},$$

over all $w \in \mathbf{W}^{\alpha}$.

3.13.7 The map $q_{K\mathbf{r}_\alpha}$

Let $w, w' \in \mathbf{W}^\alpha$ be such that $A(K,w) \supset A(\mathbf{r}_\alpha; w')$. Let $E^{K\mathbf{r}_\alpha}_{ww'} : \mathbb{Z}_{A(K,w)} \to \mathbb{Z}_{A(\mathbf{r}_\alpha;w')}$ be the map induced by this embedding.

We then have

$$q_{K\mathbf{r}_\alpha} = \sum_{ww'} n^{K\mathbf{r}_\alpha}_{ww'} E^{K\mathbf{r}_{-\alpha}}_{ww'}.$$

Proposition 3.7 *1)* $n^{K\mathbf{r}_\alpha}_{ww} = (-1)^{|w|}$*;*
2) for every compact subset $L \in (P\cap\Pi)\times\mathbb{C}$ *and every* $w \in \mathbf{W}^\alpha$*, there are only finitely many* $w' \in \mathbf{W}^\alpha$ *such that* $n_{ww'} \neq 0$ *and* $L \cap A(\mathbf{r}_{-\alpha}; w') \neq \emptyset$*;*
3) If $n^{K\mathbf{r}_\alpha}_{ww'} \neq 0$*, then we have a strict embedding* $A(w', K) \subset A(w, K)$*.*

Proof. Parts 1) and 3) follow from Sec.3.13.3 and Prop. 3.6, part 2) follows from Prop.7.2. □

3.14 $\Sigma and \mathcal{S}$ are Hausdorff

Let us start with some general observations.

3.14.1 Generalities on étale spaces

Let F be a sheaf of abelian groups on a topological space Z. Call F rigid if its étale space is Hausdorff. The following facts are easy to check.

1) Let $U \subset X$ be a Hausdorff open subset. Then $\mathbb{Z}_U$ is rigid. Indeed, the corresponding étale space is $\mathbb{Z} \times U$.

2) Every sub-sheaf F_1 of a rigid sheaf F is rigid. Indeed, the étale space of F_1 is identified with a closed subspace of a Hausdorff étale space of F.

3) Let $0 \to A \to B \to C \to 0$ be an exact sequence of sheaves, where A, C are rigid. Then so is B. Indeed, Let $A' \to B' \xrightarrow{\pi} C'$ be the étale spaces of A, B, and C. Let $b_1, b_2 \in B'$. Suppose $\pi(b_1) \neq \pi(b_2)$; we then have separating neighborhoods $\pi(b_1) \in U_1$; $\pi(b_2) \in U_2$ so that $\pi^{-1}U_1, \pi^{-1}U_2$ separate b_1 and b_2. Let now $\pi(b_1) = \pi(b_2) = c$ but $b_1 \neq b_2$. Since π is a local homeomorhisms, there are neigborhoods W_i of b_i in B' such that W_i are projected homeomorhically into C'. By possible shrinking we may achieve that W_i project to the same open subset $U \in C'$; $c \in U$. so that we have homeomorphisms $\pi_i^{-1} : U \to W_i$. We then have a continuous map $\delta : U \to A'$, where $\delta(u) = \pi_2^{-1}u - \pi_1^{-1}u \in A_u \subset A'$. Since $b_1 \neq b_2$,$\delta(c) \neq 0$, so that we have a neighborhood $U' \subset U$ of c on which δ does not vanish. It now follows that the neighborhoods $\pi_i^{-1}U'$ do separate b_1 and b_2.

4) Let $i_n : F_n \to F_{n+1}$, $n \geq 0$ be a directed sequence of embeddings, where F_0 and all $F_{n+1}/i_n F_n$ are rigid. Then $F := \varinjlim_n F_n$ is also rigid. Indeed, 3) implies that all F_n are rigid. Let F'_n, F' be the étale spaces of F_n, F. We have induced maps $F'_n \to F'$; $F'_n \to F'_{n+1}$ which induce a map $\varinjlim F'_n \to F'$ which can be easily proven to be a homeomorphism. Since all the maps $F'_n \to F'_{n+1}$ are closed embeddings, it follows that F' is Hausdorff.

5) Let $p : Y \to X$ be a local homeomorphism, where Y is Hausdorff. Let $\emptyset \neq U \subset V \subset X$ be open sets, where V is connected. Suppose we are given a section $s : U \to Y$. There exist at most one way to extend s to V. Indeed, let $s_1, s_2 : V \to Y$ be extensions of s. Let us prove that the set $W := \{v \in V : s_1(v) \neq s_2(v)\}$ is open. Indeed, let $v \in W$. The points $s_1(v)$, $s_2(v)$ can be separated by neighborhoods $U_1, U_2 \subset Y$. Let $\mathcal{U} := {s_1}^{-1}U_1 \cap {s_2}^{-1}U_2$; $\mathcal{U}$ is a neighborhood of v. It now follows that $s_i(\mathcal{U}) \subset U_i$, therefore $s_i(\mathcal{U})$ do not intersect; we have thus found an open neighborhood $\mathcal{U} \subset W$ of v, hence W is open.

Let us now prove that $W' := \{v \in V : s_1(v) = s_2(v)\}$. It is clear that $s_i(U)$ are open subsets of Y, so that $W' = s_1(U) \cap s_2(U)$ is open.

Finally, $V = W \sqcup W'$ and $W' \neq \emptyset$. This implies $W = \emptyset$.

3.14.2 Reduction to rigidity on $\Pi \cap P$

Since $\mathcal{S} \subset \Sigma$ is a connected component, it suffices to prove that Σ is Hausdorff. The latter reduces to showing that $p_\Sigma^{-1}((P \cap \Pi) \times \mathbb{C})$ is Hausdorff for every pair of intersecting α-strip P and $-\alpha$-strip Π, which is equivalent to the rigidity of the sheaf $\Phi_0|_{(\Pi \cap P) \times \mathbb{C}}$, which is isomorphic to $\operatorname{Ker} \mathcal{Q}$.

3.14.3 Filtration on $\Phi_0|_{\Pi \cap P \times \mathbb{C}}$

Let us choose an arbitrary identification $\mathbb{Z}_{>0} \xrightarrow{\sim} W$; $n \mapsto w_n$. Define a filtration on $\mathcal{G} := \Phi^C \oplus \Psi^K|_{\Pi \cap P \times \mathbb{C}}$ by setting

$$\mathcal{G}^n := \Phi^C|_{\Pi \cap P \times \mathbb{C}} \oplus \mathbb{Z}_{A(K,w_1)} \oplus \cdots \oplus \mathbb{Z}_{A(K,w_n)}.$$

It is clear that

$$\Phi^C|_{\Pi \cap P \times \mathbb{C}} =: \mathcal{G}^0 \subset \mathcal{G}^1 \subset \cdots \mathcal{G}^n \subset \cdots \subset \mathcal{G}$$

is an exhaustive filtration. It is also clear that $\mathcal{G}^n \subset \mathcal{G}$ is a direct summand. Denote by $P_n^{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^n$ the projection.

Set

$$F_n \Phi_0 := \operatorname{Ker} \mathcal{Q}|_{\mathcal{G}^n}.$$

It follows that F is an exhaustive filtration of $\Phi_0|_{\Pi\cap P\times\mathbb{C}}$. By Sec. 3.14.1 2), it suffices to show that each sheaf F_n is rigid.

3.14.4 Sheaf $F_n' \supset F_n$

We have the following projection onto a direct summand

$$P_n : \Phi^{\mathbf{r}_\alpha}_{\Pi\cap P} \oplus \Phi^{\mathbf{r}_{-\alpha}}_{\Pi\cap P} \to \bigoplus_{m=1}^{n} \mathbb{Z}_{A(\mathbf{r}_\alpha;w_m)} \oplus \mathbb{Z}_{A(\mathbf{r}_{-\alpha};w_m)} =: \mathcal{L}_n.$$

Let $F_n' := \operatorname{Ker} P_n\mathcal{Q}|_{\mathcal{G}^n}$. We have: F_n is a sub-sheaf of F_n', so that it suffices to show that each F_n' is rigid.

3.14.5 Further filtrations on $\mathcal{G}^n, \mathcal{L}_n, F_n'$

Fix $n \in \mathbb{Z}_{>0}$. Let us re-label the words $w_1, w_2, \ldots, w_n$ to, say $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$, so that the following holds true:

<u>if $i > j$, then it is impossible that $A(K, \mathbf{w}_i)$ is a proper subset of $A(K, \mathbf{w}_j)$.</u>

Since we are dealing with only finitely many words, this is always possible. Let $j \leq n$. Set $\mathbf{F}^j\mathcal{G}^n := \mathbb{Z}(K, \mathbf{w}_1) \oplus \cdots \oplus \mathbb{Z}(K, \mathbf{w}_j) \subset \mathcal{G}^n$ Set $\mathbf{F}^j\mathcal{L}_n := \mathbb{Z}_{A(\ell_{\pm\alpha},\mathbf{w}_1)} \oplus \cdots \oplus \mathbb{Z}_{A(\ell_{\pm\alpha},\mathbf{w}_j)} \subset \mathcal{G}^n$. We also set $\mathbf{F}^{n+1}\mathcal{G}^n = \mathcal{G}^n$; $\mathbf{F}^{n+1}\mathcal{L}_n = \mathcal{L}_n$. Let $\mathbf{Gr}^j\mathcal{G}^n$; $\mathbf{Gr}^j\mathcal{L}_n$ be the associated graded quotients.

Proposition 3.7 and Sec. 3.13.6 imply that the map $P_n\mathcal{Q}$ preserves the filtration $\mathbf{F}$: $P_n\mathcal{Q} : \mathbf{F}^j\mathcal{G}^n \to \mathbf{F}^j\mathcal{L}_n$. Set $\mathbf{F}^jF_n' := \operatorname{Ker} P_n\mathcal{Q}|_{\mathbf{F}^j\mathcal{G}^n}$. It is clear that this way we get a filtration on F_n'. Let $\mathbf{Gr}^jF_n'$ be the associated graded quotients. Our problem now reduces to proving rigidity of $\mathbf{Gr}^jF_n'$ by Sec. 3.14.1, 3). Since $P_n\mathcal{Q}$ preserves $\mathbf{F}$, we have

$$\mathbf{Gr}^jF_n' \subset \operatorname{Ker} \mathbf{Gr}^jP_n\mathcal{Q} : \mathbf{Gr}^j\mathcal{G}^n \to \mathbf{Gr}^j\mathcal{L}_n.$$

By Sec 3.14.1 2), the problem reduces to showing rigidity of $\operatorname{Ker} \mathbf{Gr}^jP_n\mathcal{Q} : \mathbf{Gr}^j\mathcal{G}^n \to \mathbf{Gr}^j\mathcal{L}^n$.

3.14.6 Finishing the proof

Let $j \leq n$. We then have $\mathbf{Gr}^j\mathcal{G}^n = \mathbb{Z}_{A(K,\mathbf{w}_j)}$; $\mathbf{Gr}^j\mathcal{L}_n = \mathbb{Z}_{A(\mathbf{r}_\alpha;\mathbf{w}_j)} \oplus \mathbb{Z}_{A(\mathbf{r}_{-\alpha};\mathbf{w}_j)}$. By Sec. 3.13.6 and Proposition 3.7, we have:

$$\mathbf{Gr}^jP_n\mathcal{Q} = (-1)^{|\mathbf{w}_j|}E^{\mathbf{r}_\alpha}_{\mathbf{w}_j} \oplus E^{\mathbf{r}_{-\alpha}}_{\mathbf{w}_j},$$

where the morphisms

$$E^{\mathbf{r}_{\pm\alpha}}_{\mathbf{w}_j} : \mathbb{Z}_{A(K,\mathbf{w}_j)} \to \mathbb{Z}_{A(\mathbf{r}_{\pm\alpha};\mathbf{w}_j)}$$

are induced by the closed embeddings of the corresponding sets. It now follows that $\operatorname{Ker} \mathbf{Gr}^j P_n \mathcal{Q} = \mathbb{Z}_{A(\operatorname{Int}K;\mathbf{w}_j)}$, which is rigid by Sec. 3.14.1,1).

Let now $j = n+1$. We have $\mathbf{Gr}^{n+1}\mathcal{L}_n = 0$; $\mathbf{Gr}^{n+1}\mathcal{G}^n = \mathbb{Z}_{A_0}$, so that

$$\operatorname{Ker} \mathbf{Gr}^j P_n \mathcal{Q} = \mathbb{Z}_{A_0},$$

which is also rigid, as a sheaf on $(\Pi \cap P) \times \mathbb{C} = A_0$, by Sec. 3.14.1,1). This finishes the proof.

3.15 Surjectivity of the projection $p_{\mathcal{S}} : \mathcal{S} \to X$.

In this subsection we will prove

Theorem 3.8 *The projection $p_{\mathcal{S}} : \mathcal{S} \to X$ is surjective.*

Proof of this theorem will occupy the rest of this subsection. We will construct an open subset $\mathcal{U} \subset \Sigma$ such that

1) $\mathcal{U}$ projects surjectively onto X;

2) $\mathcal{U}$ is connected;

3) $\mathcal{U} \cap h(S_\alpha) \neq \emptyset$, where $h : S_\alpha \to \Sigma$ is as in (27).

Conditions 2),3) imply that $\mathcal{U} \subset \mathcal{S}$, and Theorem follows.

Let us now construct $\mathcal{U}$ and verify 1)-3).

3.15.1 Constructing $\mathcal{U}$

We construct $\mathcal{U}$ stripwise. We will freely use the notation from Sec 3.13.1. Let P be an α-strip. Define a closed subset

$$A(P) := \bigcup_{w \in W_P} A(\mathbf{r}_\alpha, w) \subset P \times \mathbb{C} \subset X \times \mathbb{C}.$$

Let $\mathcal{U} := X \times \mathbb{C} \backslash \bigcup_P A(P)$, where the union is taken over the set of all α-strips P. Denote by $j_{\mathcal{U}}^X : \mathcal{U} \to X \times \mathbb{C}$ the open embedding.

Let us now embed $\mathcal{U}$ into Σ. We have a natural embedding $J_{\mathcal{U}} : \mathbb{Z}_{\mathcal{U}} \to \mathbb{Z}_{X \times \mathbb{C}} = \Phi^{\mathbb{C}}$. As follows from (66), we have $q_{C\mathbf{r}_a} J_{\mathcal{U}} = 0$, which implies that the map $J_{\mathcal{U}}$ factors through $\operatorname{Ker} q_{C\mathbf{r}_\alpha}$:

$$J_{\mathcal{U}} : \mathbb{Z}_{\mathcal{U}} \overset{J_{\mathcal{U}}^q}{\hookrightarrow} \operatorname{Ker} q_{C\mathbf{r}_\alpha} \to \Phi^C.$$

As follows from the diagram (64), we have a natural embedding $\iota_q : \operatorname{Ker} q_{C\mathbf{r}_\alpha} \hookrightarrow \operatorname{Ker} \mathcal{Q}$, and we set

$$J_{\mathcal{Q}} := \iota_q J^q_{\mathcal{U}}, \tag{67}$$

which is an injection $J_{\mathcal{Q}} : \mathbb{Z}_{\mathcal{U}} \hookrightarrow \operatorname{Ker} \mathcal{Q} = \Phi_0$.

The map $J_{\mathcal{Q}}$ induces an embedding of the étale spaces: $\mathcal{U} \times \mathbb{Z} \to \Sigma$. Let $j_{\mathcal{U}} : \mathcal{U} \to \Sigma$ be the restriction of this map onto $\mathcal{U} \times 1 \subset \mathcal{U} \times \mathbb{Z}$. This map is a local homeomorphism and an embedding, therefore, j is an open embedding. Let us identify $\mathcal{U}$ with $j_{\mathcal{U}}(\mathcal{U})$.

3.15.2 Verifying 1)

Checking 1): we see that the composition $p_\Sigma j_{\mathcal{U}}$ coincides with the composition $\mathcal{U} \overset{J^X_{\mathcal{U}}}{\to} X \times \mathbb{C} \to X$, where the rightmost arrow is the obvious projection. Let us check that this map is surjective. Indeed, let $x \in X$. There are at most two α-strips which contain x. We therefore have: $\mathcal{U} \cap x \times \mathbb{C}$ is obtained from $x \times \mathbb{C} = \mathbb{C}$ by removing a finite number of α-rays, which is non-empty.

3.15.3 Verifying 2)

As the sets W_P are finite, it easily follows that

— the sets $\mathcal{U}(P) := P \times \mathbb{C} \backslash A(P)$ are connected;

— if $P_1 \cap P_2 \neq \emptyset$, then $\mathcal{U}(P_1) \cap \mathcal{U}(P_2) \neq \emptyset$. This implies that $\mathcal{U}$ is connected.

The rest of the subsection is devoted by verifying 3).

3.15.4 Reformulation of 3)

Recall that the map $h : S_\alpha \to \Sigma$ is induced by the map $\mathbf{I}_0 : S_\alpha \to g^{-1}\Phi_0$, see (26). The injection $j_{\mathcal{U}} : \mathcal{U} \to \Sigma$ is induced by the map $J_{\mathcal{Q}} : \mathbb{Z}_{\mathcal{U}} \to \operatorname{Ker} \mathcal{Q} = \Phi_0$, see (67). Let $i_{\mathbf{x}_0} : \mathbb{C} \to X \times \mathbb{C}$ be the embedding $i_{\mathbf{x}_0}(s) = (\mathbf{x}_0, s)$. We have $g = i_{\mathbf{x}_0} \pi_{S_\alpha}$. Let us denote $\mathcal{U}_{\mathbf{x}_0} := i^{-1}_{\mathbf{x}_0}\mathcal{U}$. Observe that $\mathcal{U}_{\mathbf{x}_0}$ is obtained from $\mathbb{C}$ by removing a finite number of α-rays.

Lemma 3.9 *There exists a non-empty open subset $V \subset \mathcal{U}_{\mathbf{x}_0}$ such that:*

1) the map π_{S_α} induces a homeomorphism $\pi^{-1}_{S_\alpha}V \to V$, so that we have $\pi^{-1}_{S_\alpha}\mathbb{Z}_V = \mathbb{Z}_{\pi^{-1}_{S_\alpha}V}$;

2) the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}_{\pi^{-1}_{S_\alpha}V} \arrow[r, "j_{VS}"] \arrow[d, "j_{V\mathcal{U}}"] & \mathbb{Z}_{S_\alpha} \arrow[d, "\mathbf{I}_0"] \\
g^{-1}\mathbb{Z}_{\mathcal{U}} \arrow[r, "J_{\mathcal{Q}}"] & g^{-1}\Phi_0
\end{tikzcd}$$

where the arrow j_{VS} is induced by the open embedding $\pi_{S_\alpha}^{-1}V \subset S_\alpha$, and the arrow $j_{V\mathcal{U}}$ is the composition $\mathbb{Z}_{\pi_{S_\alpha}^{-1}V} = \pi_{S_\alpha}^{-1}\mathbb{Z}_V \xrightarrow{*} \pi_{S_\alpha}^{-1}\mathbb{Z}_{\mathcal{U}_{\mathbf{x}_0}} = g^{-1}\mathbb{Z}_{\mathcal{U}}$, where the arrow $*$ is induced by the open embedding $V \subset \mathcal{U}_{\mathbf{x}_0}$.

Let us first explain how Lemma implies 3). Indeed, it follows from Lemma that the embedding $h|_{\pi_{S_\alpha}^{-1}V} : \pi_{S_\alpha}^{-1}V \to \Sigma$ coincides with

$$\pi_{S_\alpha}^{-1}V \xrightarrow{\pi_{S_\alpha}} V \subset \mathcal{U}_{\mathbf{x}_0} \xrightarrow{i_{\mathbf{x}_0}} \mathcal{U} \xrightarrow{j_{\mathcal{U}}} \Sigma,$$

so that $h(S_\alpha) \cap j_{\mathcal{U}}(\mathcal{U}) \supset j_{\mathcal{U}}(i_{\mathbf{x}_0}V)$.

We will now prove Lemma.

3.15.5 Subset $W \subset S_\alpha$

Let $W := \pi_{S_\alpha}^{-1}(\mathbb{C}\backslash K) \subset S_\alpha$. Denote by $J_W : \mathbb{Z}_W \to \mathbb{Z}_{S_\alpha}$ the map induced by the open embedding $j_W : W \subset S_\alpha$. Let us consider the composition hj_W, which is induced by the map $\mathbf{I}_0 J_W : \mathbb{Z}_W \to g^{-1}\Phi_0$. Denote by $\pi : \Phi_0 \to \Phi^C \oplus \Phi^K$ the natural embedding (recall that $\Phi_0 = \operatorname{Ker} \mathcal{Q}$). Set $\pi_{0K} := \Pi_K \pi : \Phi_0 \to \Phi^K$, where $\Pi_K : \Phi^C \oplus \Phi^K \to \Phi^K$ is the projection.

Let us show

Lemma 3.10 *We have $(g^{-1}\pi_{0K})\mathbf{I}_0 J_W = 0$.*

Proof Indeed, the map π factors as

$$\Phi_0 \xrightarrow{\iota} \Phi = \operatorname{Cone}\mathcal{Q}[-1] \xrightarrow{P_\Phi} \Phi^C \oplus \Phi^K,$$

where the last arrow is the canonical map. Set $\pi_K := \Pi_K P_\Phi$. We have

$$(g^{-1}\pi_{0K})\mathbf{I}_0 = (g^{-1}\Pi_K)(g^{-1}\pi)\mathbf{I}_0 = (g^{-1}\Pi_K)(g^{-1}P_\Phi)g^{-1}\iota\mathbf{I}_0 = (g^{-1}\pi_K)\mathbf{I}.$$

Upon, the isomorphism $g^{-1}\Phi = g^!\Phi[2]$, the map $\mathbf{I}$ corresponds by the conjugacy to the map $i_\Phi : g_!\mathbb{Z}_{S_\alpha}[2] \to \Phi$. The map $(g^{-1}\pi_K)\mathbf{I}$ corresponds by the conjugacy to $\pi_K i_\Phi$. Denote by $\lambda : g_!\mathbb{Z}_W[-2] \to g_!\mathbb{Z}_{S_\alpha}[-2]$ the map induced by j_W. The problem now reduces to showing that $\pi_K i_\Phi \lambda = 0$.

By the construction of the map i_Φ, the map $\pi_K i_\Phi$ factors as $g_!\mathbb{Z}_{S_\alpha}[-2] \xrightarrow{p_K} \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \xrightarrow{i_{\Phi K}} \Phi^K$, where p_K is as in (28), so that $\pi_K i_\Phi \lambda = i_{\Phi^K} p_K \lambda$. It is easy to see that $p_K\lambda = 0$, which finishes the proof. $\square$

It now follows that the map $\mathbf{I}_0 J_W : \mathbb{Z}_W \to g^{-1}\Phi_0$ factors as

$$\mathbb{Z}_W \xrightarrow{\mathcal{J}_W} g^{-1} \operatorname{Ker} q_{C\mathbf{r}_\alpha} \to g^{-1}\Phi_0,$$

where the right arrow is induced by the obvious embedding $\operatorname{Ker} q_{C\mathbf{r}_\alpha} \subset \Phi_0$ coming from the definition $\Phi_0 = \operatorname{Ker} \mathcal{Q}$.

3.15.6 Finishing the proof

Recall that the map $J_{\mathcal{Q}} : \mathbb{Z}_{\mathcal{U}} \to \Phi_0$ factors as $J_{\mathcal{Q}} := \iota_q J^q_{\mathcal{U}}$, see (67).

Suppose that the susbet $V \subset \mathcal{U}$ from Lemma 3.9 satisfies: $\pi^{-1}_{S_\alpha} V \subset W$. The statement 2) of Lemma 3.9 now follows from the commutativity of the following diagram

$$\begin{tikzcd}
\mathbb{Z}_{\pi^{-1}_{S_\alpha} V} \arrow[r, "j_{VW}"] \arrow[d, "j_{V\mathcal{U}}"] & \mathbb{Z}_W \arrow[d, "\mathcal{J}_W"] \\
g^{-1}\mathbb{Z}_{\mathcal{U}} \arrow[r, "(J^q_{\mathcal{U}})'"] & g^{-1} \operatorname{Ker} q_{C\mathbf{r}_\alpha}
\end{tikzcd}$$

where $j_{V\mathcal{U}}$ is the composition $\mathbb{Z}_{\pi^{-1}_{S_\alpha} V} = \pi^{-1}_{S_\alpha}\mathbb{Z}_V \xrightarrow{*} \pi^{-1}_{S_\alpha}\mathbb{Z}_{\mathcal{U}_{\mathbf{x}_0}} = g^{-1}\mathbb{Z}_{\mathcal{U}}$, where the arrow $*$ is induced by the open embedding $V \subset \mathcal{U}_{\mathbf{x}_0}$; the map j_{VW} is induced by the open embedding $\pi^{-1}_{S_\alpha} V \subset W$. The map $(J^q_{\mathcal{U}})'$ is induced by $J^q_{\mathcal{U}}$.

We have an injection $\kappa : \operatorname{Ker} q_{C\mathbf{r}_\alpha} \to \Phi^C = \mathbb{Z}_{X\times\mathbb{C}}$ which induces an injection $\kappa' : g^{-1} \operatorname{Ker} q_{C\mathbf{r}_\alpha} \to g^{-1}\mathbb{Z}_{X\times\mathbb{C}}$. The commutativity of the above diagram is equivalent to the commutativity of

$$\begin{tikzcd}
\mathbb{Z}_{\pi^{-1}_{S_\alpha} V} \arrow[r, "j_{VW}"] \arrow[d, "j_{V\mathcal{U}}"] & \mathbb{Z}_W \arrow[d, "\kappa'\mathcal{J}_W"] \\
g^{-1}\mathbb{Z}_{\mathcal{U}} \arrow[r, "\kappa'(J^q_{\mathcal{U}})'"] & g^{-1}\mathbb{Z}_{X\times\mathbb{C}}
\end{tikzcd} \tag{68}$$

Let us now define

$$V := (\mathbb{C}\backslash K) \cap \mathcal{U}_{\mathbf{x}_0}.$$

Let us check that V satisfies all the conditions:

a) V is non-empty. The set $\mathcal{U}_{\mathbf{x}_0}$ is obtained by removing from $\mathbb{C}$ a finite number of α-rays, which implies non-emptyness of $(\mathbb{C}\backslash K) \cap \mathcal{U}_{\mathbf{x}_0}$.

b) $\pi^{-1}_{S_\alpha} V \subset W$ —this is clear.

c) $\pi_{S_\alpha} : \pi^{-1}_{S_\alpha} V \to V$ is a homeomorphism —clear.

d) Commutativity of (68). We have $g^{-1}\mathbb{Z}_{X\times\mathbb{C}} = \mathbb{Z}_{S_\alpha}$. It follows that the composition $\kappa'\mathcal{J}_W$ equals the map $\mathbb{Z}_W \to \mathbb{Z}_{S_\alpha}$ induced by the inclusion $W \subset S_\alpha$. Next, the map $\kappa J_{\mathcal{U}} : \mathbb{Z}_{\mathcal{U}} \to \mathbb{Z}_{X\times\mathbb{C}}$ is induced by the open embedding $j_{\mathcal{U}} : \mathcal{U} \to X \times \mathbb{C}$. The commutativity now follows. This finishes the proof.

3.16 Infinite continuation in the direction of K

We need some definitions

3.16.1 Parallelogram $\mathbf{U}$

Let $\mathbf{U} \subset \mathbb{C}$ be an open parallelogram with vertices A, B, C, and D, such that $\vec{AB}$ and $\vec{DC}$ are collinear to $e^{-i\alpha}$ and $\vec{BC}$ and $\vec{AD}$ are collinear to $e^{i\alpha}$.

3.16.2 Small sets

Let $\Gamma \subset \mathbb{C}$. Call Γ small if for every point $c \in \mathbb{C}$, the intersection $\Gamma \cap c - K$ is a finite set.

Claim 3.11 *Let $L \subset \mathbb{C}$ be a bounded subset. The set $\Gamma \cap (L - K)$ is then also finite.*

Proof. Assuming the contrary, let $\{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\} \in \Gamma \cap (L - K)$ so that $\gamma_i = c_i - z_i$, $z_i \in K$, $c_i \in L$. Since L is bounded, the sequence c_i has a convergent sub-sequence $c_{i_n} \to c$ for some $c \in \mathbb{C}$. Let $\varepsilon \in \mathrm{Int}K$. It follows, that $c_{i_n} \in c + \varepsilon - K$ for all n large enough, which contradicts to smallness of Γ $\square$

3.16.3 Theorem

Theorem 3.12 *Suppose we have a section σ of P_z:*

$$\begin{tikzcd} \mathcal{S}_z \arrow[r, "P_z"] & \mathbb{C} \\ & \mathbf{U} \arrow[ul, "\sigma"] \arrow[u] \end{tikzcd}$$

Then there exists a small subset $\Gamma \subset \mathbf{U} + K$ such that σ extends to $(\mathbf{U}+K)\backslash(\Gamma+\mathbf{r}_{-\alpha})$ and $(\Gamma+\mathbf{r}_{-\alpha}) \cap \mathbf{U} = \emptyset$.

Remark For every bounded set L there are only finitely many $\gamma \in \Gamma$ such that $(\gamma + \mathbf{r}_{-\alpha}) \cap L \neq \emptyset$, as follows from Claim 3.11.

Before proving this theorem, let us observe that it easily implies Theorem 1.1. Indeed, given $\underline{x} \in \mathbb{C}$, we see that $\mathcal{S}^{\underline{x}}$ is a disjoint union of all $\mathcal{S}_z$, where $p_X(z) = \underline{x}$, which reduces Theorem 1.1 to the current Theorem. The rest of this subsection is devoted to its proof.

3.16.4 Reformulation in terms of sheaves

By basic properties of an étale space of a sheaf, liftings σ as in Theorem, are in 1-to-1 correspondence with maps of sheaves $f_\sigma : \mathbb{Z}_{\mathbf{U}} \to \Phi_0|_{z\times\mathbb{C}}$.

For every $w \in \mathbf{W}^\alpha$ and a fixed $z \in X$, set $\mathcal{A}_z(K,w) = \mathcal{A}(K,w) \cap (z \times \mathbb{C}) \subset \mathbb{C}$, where $\mathcal{A}(K,w)$ are the same is in Sec 3.10.1 We define $\mathcal{A}_z(\mathbf{r}_\alpha, w)$, $\mathcal{A}_z(\mathbf{r}_{-\alpha}, w)$ in a similar way.

We then have the following maps:

$$f_\sigma : \mathbb{Z}_{\mathbf{U}} \xrightarrow{f} \begin{tikzcd} \mathbb{Z}_{\mathbb{C}} \arrow[r, "{q_0}^{C\mathbf{r}_\alpha}"] & \bigoplus_w \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha, w)} \\ \oplus & \\ \bigoplus_w \mathbb{Z}_{\mathcal{A}_z(K,w)} \arrow[ruu, "-{q_0}^{K\mathbf{r}_\alpha}"] \arrow[r, "{q_0}^{K\mathbf{r}_{-\alpha}}"] & \bigoplus_w \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha}, w)} \end{tikzcd}$$

where ${q_0}^{C\mathbf{r}_\alpha}$, ${q_0}^{K\mathbf{r}_\alpha}$, ${q_0}^{K\mathbf{r}_{-\alpha}}$ are the restrictions of the maps $q^{C\mathbf{r}_\alpha}$, $q^{K\mathbf{r}_\alpha}$, $q^{K\mathbf{r}_{-\alpha}}$ onto $\mathbf{x}_0 \times \mathbb{C}$. Let $\mathcal{Q}_{\mathbf{x}_0}$ be the restriction of the map $\mathcal{Q}$ onto $\mathbf{x}_0 \times \mathbb{C}$, so that $\mathcal{Q}_{\mathbf{x}_0}$ is the sum of ${q_0}^{C\mathbf{r}_\alpha}$, $-{q_0}^{K\mathbf{r}_\alpha}$, and ${q_0}^{K\mathbf{r}_{-\alpha}}$. We now have

$$\mathcal{Q} f_\sigma = 0. \tag{69}$$

3.16.5 Writing f_σ in terms of its components

We have components:

$$f_\sigma(w) \ : \ \mathbb{Z}_{\mathbf{U}} \to \mathbb{Z}_{\mathcal{A}_z(K,w)}$$

$$f_\sigma(0) \ : \ \mathbb{Z}_{\mathbf{U}} \to \mathbb{Z}_{\mathbb{C}}$$

we have (if $\mathbf{U} \cap \mathcal{A}_z(K,w) \neq \emptyset$):

$$\hom(\mathbb{Z}_{\mathbf{U}}; \mathbb{Z}_{\mathcal{A}_z(K,w)}) \ = \ \mathbb{Z} \cdot g_w$$

where

$$g_w \ : \ \mathbb{Z}_{\mathbf{U}} \to \mathbb{Z}_{\mathbf{U}\cap\mathcal{A}_z(K,w)} \to \mathbb{Z}_{\mathcal{A}_z(K,w)} \tag{70}$$

(the first arrow is induced by the closed embedding $\mathbf{U} \cap A(K,w) \subset \mathbf{U}$; the second arrow is an open embedding)
if $\mathbf{U} \cap \mathcal{A}_z(K,w) = \emptyset$, then $\hom(\mathbb{Z}_{\mathbf{U}}, \mathbb{Z}_{\mathcal{A}_z(K,w)}) = 0$.

So,

$$f_\sigma(w) = n_w \cdot g_w, \quad \text{where } n_w \in \mathbb{Z}, \tag{71}$$

and $f_\sigma(w) = 0$ if $\mathbf{U} \cap \mathcal{A}_z(K,w) = \emptyset$.

Analogously, $hom(\mathbb{Z}_{\mathbf{U}}, \mathbb{Z}_{\mathbb{C}}) = \mathbb{Z} \cdot g_0$, so

$$f(0) = n_0 \cdot g_0. \tag{72}$$

It also follows that:

Claim 3.13 *for every point $s \in \mathbf{U}$ there are only finitely many w such that $f(w) \neq 0$ and $s \in \mathcal{A}_z(K, w)$.*

Proof This follows from consideration of the induced map on stalks at w:

$$(f_\sigma)_s : (\mathbb{Z}_{\mathbf{U}})_s = \mathbb{Z} \to \bigoplus_{w: s \in \mathcal{A}_z(K,w)} \mathbb{Z} = (\bigoplus_{w \in \mathbf{W}^\alpha} \mathcal{A}_z(K, w))_s.$$

The image of this map must be contained in the direct sum of only finitely many copies of $\mathbb{Z}$, the statement now follows. □

3.16.6 Restriction to a sub-parallelogram V

Let $\mathbf{V} \subset \mathbf{U}$ be a parallelogram, $\mathbf{V} = AB'C'D'$, such that $B' \in (AB)$, $D' \in (AD)$ (so that $C' \in \mathbf{U}$).

The restriction

$$f_{\sigma,\mathbf{V}} := f_\sigma|_{\mathbf{V}} \;:\; \mathbb{Z}_{\mathbf{V}} \to \mathbb{Z}_{\mathbf{U}} \stackrel{f_\sigma}{\to} \mathbb{Z}_{\mathbb{C}} \oplus \bigoplus_w \mathbb{Z}_{\mathcal{A}_z(K,w)}$$

can thus be expressed as

$$f_{\sigma,\mathbf{V}} \;=\; \sum_{w \in \mathbf{W}^\alpha} n_w \cdot g_w|_{\mathbf{V}}.$$

Here $g_w|_{\mathbf{V}}$ is the following composition:

$$\mathbb{Z}_{\mathbf{V}} \to \mathbb{Z}_{\mathbf{U}} \stackrel{g_w}{\to} \mathbb{Z}_{\mathcal{A}_z(K,w)}$$

and g_w is the same as in (70).

Let $S \subset \mathbf{W}^\alpha$ consist of all w such that $n_w \neq 0$ and $g_w|_{\mathbf{V}} \neq 0$. We can now rewrite

$$f_{\sigma,\mathbf{V}} \;=\; \sum_{w \in S} n_w \cdot g_w|_{\mathbf{V}} \tag{73}$$

Observe that

$$g_w|_{\mathbf{V}} \neq 0 \text{ iff } \mathbf{V} \cap \mathcal{A}_z(K, w) \neq \emptyset. \tag{74}$$

Next, there are only finitely many w such that $f(w) \neq 0$ and $\mathcal{A}_z(K, w) \cap \mathbf{V} \neq \emptyset$. Indeed, $\mathcal{A}_z(K, w) \cap \mathbf{V} \neq \emptyset$ implies $C' \in \mathcal{A}_z(K, w)$, and we can set $z = C'$ in Claim 3.13. This shows that S is a finite set.

3.16.7 Proof of a weaker version of the Theorem

We are going to prove the following statement: <u>there exists a small set $\Gamma \subset \mathbf{V}+K$, such that $\sigma|_{\mathbf{V}\cap\mathcal{V}}$ extends to $\mathcal{V}$, where $\mathcal{V} := \mathbf{V}+K\backslash(\Gamma+K)$.</u>

Define the extensions $\mathbb{Z}_{\mathbf{V}+K} \stackrel{G_w}{\to} \mathbb{Z}_{\mathcal{A}_z(K,w)}$ as follows:

$$G_w \ : \ \mathbb{Z}_{\mathbf{V}+K} \stackrel{c}{\to} \mathbb{Z}_{(\mathbf{V}+K)\cap\mathcal{A}_z(K,w)} \to \mathbb{Z}_{\mathcal{A}_z(K,w)},$$

where the map c is the restriction onto a closed subset and the second map is induced by the embedding of an open subset).

Let $G_0 : \mathbb{Z}_{\mathbf{V}+K} \to \mathbb{Z}_{\mathbb{C}}$ be the map coming from the open embedding of the corresponding sets.

Let

$$F_{\sigma,\mathbf{V}} := n_0 G_0 + \sum_{w\in S} n_w G_w : \mathbb{Z}_{\mathbf{V}+K} \to \mathbb{Z}_{\mathbb{C}} \oplus \bigoplus_{w\in\mathbf{W}^\alpha} \mathbb{Z}_{\mathcal{A}_z(K,w)},$$

where the coefficients n_w, n_0 are the same as in (71), (72). Let $J_{\mathbf{V}} : \mathbb{Z}_{\mathbf{V}} \to \mathbb{Z}_{\mathbf{V}+K}$ be the map coming from the open embedding of the corresponding sets. We have:

$$f_{\sigma,\mathbf{V}} = F_{\sigma,\mathbf{V}} J_{\mathbf{V}}. \tag{75}$$

Let us now find a a subset $\mathcal{V} \subset \mathbf{V}+K$ such that $\mathcal{Q}\circ F_{\sigma,\mathbf{V}}|_{\mathcal{V}} = 0$. This vanishing along with (75) imply that $F_{\sigma,\mathbf{V}}$ determines an extension of $\sigma|_{\mathbf{V}}$ onto $\mathcal{V}$.

1) Consider the through map for some $w \in S$:

$$\begin{tikzcd}
 & \mathbb{Z}_{\mathbb{C}} \arrow[r] & \bigoplus_{w\in\mathbf{W}^\alpha} \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha;w)} & \\
\beta_w \ : \ \mathbb{Z}_{\mathbf{V}} \stackrel{f_{\sigma,\mathbf{V}}}{\to} & \oplus & \oplus & \stackrel{\rho_w}{\to} \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha},w)} \\
 & \bigoplus_{w\in\mathbf{W}^\alpha} \mathbb{Z}_{\mathcal{A}_z(K,w)} \arrow[r] \arrow[uur] & \bigoplus_{w\in\mathbf{W}^\alpha} \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha};w)} &
\end{tikzcd}$$

ρ_w is the projection onto a direct summand, and the middle map is $\mathcal{Q}_z$.

By (69), $\beta_w = 0$; on the other hand, $\beta_w = n_w \cdot h_w$, where

$$h_w \ : \ \mathbb{Z}_{\mathbf{V}} \stackrel{G_w}{\to} \mathbb{Z}_{\mathcal{A}_z(K,w)} \stackrel{restr}{\to} \mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha},w)}.$$

But $h_w = 0$ iff $\mathbf{V}\cap\mathcal{A}_z(\mathbf{r}_{-\alpha};w) = \emptyset$. So if $n_w \neq 0$, then

$$\mathbf{V}\cap\mathcal{A}_z(\mathbf{r}_{-\alpha};w) = \emptyset. \tag{76}$$

Since $w \in S$ and because of (74), we have

$$\mathbf{V}\cap\mathcal{A}_z(K;w) \neq \emptyset. \tag{77}$$

From (76) and (77) it follows that $(\mathbf{V}+K)\cap\mathcal{A}_z(\mathbf{r}_{-\alpha};w)=\emptyset$. Hence, we have

$$\rho_w\circ\mathcal{Q}\circ F_{\sigma,\mathbf{V}}:\mathbb{Z}_{(\mathbf{V}+K)}\to\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha},w)}=0. \tag{78}$$

Let us now consider the maps $\kappa\circ\mathcal{Q}\circ F_{\sigma,\mathbf{V}}$, where κ is the projection onto $\oplus_w\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha},w)}$ as shown in the following diagram:

$$\begin{tikzcd}
 & \mathbb{Z}_{\mathbb{C}} \arrow[r] & \bigoplus_{w\in\mathbf{W}^\alpha}\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha;w)} & \\
\kappa\circ\mathcal{Q}\circ F_{\sigma,\mathbf{V}}\ :\ \mathbb{Z}_{\mathbf{V}+K}\stackrel{F_{\sigma,\mathbf{V}}}{\to} & \oplus & \oplus & \stackrel{\kappa}{\to}\ \bigoplus\limits_{w\in\mathbf{W}^\alpha}\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha,w)} \\
 & \bigoplus_{w\in\mathbf{W}^\alpha}\mathbb{Z}_{\mathcal{A}_z(K,w)} \arrow[r] \arrow[uur] & \bigoplus_{w\in\mathbf{W}^\alpha}\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_{-\alpha};w)} &
\end{tikzcd}$$

Let $M_w:\mathbb{Z}_{\mathbb{C}}\to\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha;w)}$ be the components of the map ${q_0}^{C\mathbf{r}_\alpha}$. Let

$$\Delta\subset\mathbf{W}^\alpha=\{w'\ :\ \exists w\in S\ :\ N_{ww'}\neq 0 \text{ or } M_{w'}\neq 0\}.$$

Here S is as in (73), $N_{ww'}:=n_{\mathbf{A}^{-1}(w);w'}$, and $n_{\tilde{w};w'}$ are the same as in Prop. 3.6.

For each $w'\in\mathbf{W}^\alpha$ let us write

$$\mathcal{A}_z(K,w')=d_{w'}+K.$$

Set $\Gamma:=\{d_{w'}\ :\ w'\in\Delta\}\subset\mathbb{C}$. As S is finite (see end of section 3.16.6), for any $s\in\mathbb{C}$ there are only finitely many $w'\in\Delta\ :\ A(K,w')\ni s$. Equivalently there are only finitely many w' such that $d_{w'}\in s-K$ so that Γ is small .

Let

$$\pi_w:\bigoplus_{w'\in\mathbf{W}^\alpha}\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha,w')}\to\mathbb{Z}_{\mathcal{A}_z(\mathbf{r}_\alpha,w)}$$

be the projection. It follows that $\pi_w\kappa\mathcal{Q}F_{\sigma,\mathbf{V}}\neq 0$ only if $w\in\Delta$. Set $\mathcal{V}:=\mathbf{V}+K\backslash(\Gamma+K)$. It follows that $\pi_w\kappa\mathcal{Q}F_{\mathbf{V}}|_{\mathcal{V}}=0$, which implies $\kappa\mathcal{Q}F_{\sigma,\mathbf{V}}|_{\mathcal{V}}=0$. Taking into account (78), we conclude $\mathcal{Q}F_{\sigma,\mathbf{V}}|_{\mathcal{V}}=0$, i.e. $\sigma|_{\mathcal{V}\cap\mathbf{V}}$ extends onto $\mathcal{V}$, as we wanted.

3.16.8 Proof of the theorem for U

Denote by σ' the extension of $\sigma|_{\mathcal{V}\cap\mathbf{V}}$ onto $\mathcal{V}$. Observe that the set $\mathcal{V}\cap\mathbf{U}$ is connected and that $\mathcal{V}\cap\mathbf{V}\subset\mathcal{V}\cap\mathbf{U}$. Thus, σ and σ' are two extensions of $\sigma|_{\mathbf{V}\cap\mathcal{V}}$ onto $\mathcal{V}\cap\mathbf{U}$. By Sec 3.14.1 we have $\sigma|_{\mathcal{V}\cap\mathbf{U}}=\sigma'|_{\mathcal{V}\cap\mathbf{U}}$. Thus, σ extends to $\mathcal{V}\cup\mathbf{U}$ which is of the required type. $\square$

4 Orthogonality criterion – a simplified version

The goal of this section is to prove Theorem 4.1 below. This theorem will only be used in the next Section 5.

4.1 Formulation of the Theorem

Let X be a smooth manifold. We will work on a manifold $Y = X \times \mathbb{R} \times \mathbb{R}$. Let us refer to points of Y as $(x, s_1, s_2) \in X \times \mathbb{R} \times \mathbb{R}$. Let $P_1, P_2 : Y \to X \times \mathbb{R}$ be projections

$$P_i(x, s_1, s_2) = (x, s_i).$$

Let us refer to points of T^*Y as $(x, s_1, s_2, \omega, a_1 ds_1, a_2 ds_2)$, where $\omega \in T^*_x X$; $a_1 ds_1 \in T^*_{s_1}\mathbb{R}$; $a_2 ds_2 \in T^*_{s_2}\mathbb{R}$. Let $\Omega_Y \subset T^*Y$ be the closed subset consisting of all points $(x, s_1, s_2, \omega, a_1 ds_1, a_2 ds_2)$ where $a_1 = 0$ or $a_2 = 0$ (or both). Let $\mathcal{C}_Y \subset \mathbf{D}(Y)$ be the full subcategory consisting of all objects microsupported within Ω_Y. Let ${}^\perp\mathcal{C}_Y$ be the left orthogonal complement to $\mathcal{C}_Y$ (consisting of all $F \in \mathbf{D}(Y)$ such that $RHom(F, G) = 0$ for all $G \in \mathbf{D}(Y)$).

Theorem 4.1 *$F \in {}^\perp\mathcal{C}_Y$ iff $RP_{1!}F = RP_{2!}F = 0$.*

Let us start with proving that $F \in {}^\perp\mathcal{C}_Y$ implies $RP_{1!}F = RP_{2!}F = 0$. Indeed, given any $G \in \mathbf{D}(X \times \mathbb{R})$, we have

$$RHom(RP_{1!}F; G) = RHom(F, P_1^! G).$$

It is well known that every element $(x, s_1, s_2, \omega, a_1 ds_1 + a_2 ds_2) \in S.S.(p_1^! G)$ satisfies $a_2 = 0$, i.e. $P_1^! G \in \mathcal{C}_Y$ and

$$RHom(RP_{1!}F; G) = RHom(F, P_1^! G) = 0.$$

As G is arbitrary, we conclude $RP_{1!}F = 0$. One can prove the equality $RP_{2!}F = 0$ in a similar way.

The rest of this section will be devoted to proving the opposite implication:

Theorem 4.2 *Let $F \in \mathbf{D}(Y)$ satisfy $RP_{1!}F = RP_{2!}F = 0$. Let $G \in \mathcal{C}_Y$. Then $RHom(F, G) = 0$.*

We start with introducing the major tool, namely a version of Fourier-Sato transform.

4.2 Fourier-Sato Kernel

Let E be the dual real vector space to $\mathbb{R}^2$ so that we have a pairing $<,>: \mathbb{R}^2 \times E \to \mathbb{R}$. Let us use the standard coordinates s_1, s_2 on $\mathbb{R}^2$ and σ_1, σ_2 on E so that

$$< (s_1, s_2), (\sigma_1, \sigma_2) >= s_1\sigma_1 + s_2\sigma_2.$$

Let $Y_2 := X \times \mathbb{R}^2 \times \mathbb{R}^2$. Define projections $\pi_1, \pi_2 : Y_2 \to Y$:

$$\pi_1(x, s, s') = (x, s);$$

$$\pi_2(x,s,s')=(x,s'),$$

where $s=(s_1,s_2)\in\mathbb{R}^2$ and $s'=(s'_1,s'_2)\in\mathbb{R}^2$.

Let $K\subset Y_2\times E$ be the following closed subset

$$K=\{(y,s,s',\sigma)|\langle s-s',\sigma\rangle\geq 0\}.$$

Let us also define the projections

$$p_1\ :\ Y_2\times E\ \to\ Y_2\stackrel{\pi_1}{\to}Y;$$

$$p_2\ :\ Y_2\times E\ \stackrel{\pi_2\times \mathrm{id}_E}{\longrightarrow}\ Y\times E.$$

We then have the following functor: $\Psi:\mathbf{D}(Y)\to\mathbf{D}(Y\times E)$:

$$\Psi(F):=Rp_{2*}R\mathcal{H}om(\mathbb{Z}_K;p_1^!F)$$

which are modified versions of Fourier-Sato transform. Let us establish certain properties of these functors (similar to those of Fourier-Sato transform).

4.2.1 Properties of the modified Fourier-Sato transform.

Lemma 4.3 *Let $\pi_E:Y\times E\to Y$ be the projection. We then have a natural isomorphism*

$$F\to R\pi_{E*}\Psi(F)[2].$$

PROOF Let $p_E:Y_2\times E\to Y_2$ be the projection. We then have

$$R\pi_{E*}\Psi(F)\sim R\pi_{2*}R\mathcal{H}om(Rp_{E!}\mathbb{Z}_K;R\pi_1^!F). \tag{79}$$

(Indeed, one uses $p_1=\pi_1\circ p_E$, the adjunction formula for $p_{E!}$, and $\pi_E\circ p_2=\pi_E\circ\pi_2$.)

A simple computation shows that we have

$$Rp_{E!}\mathbb{Z}_K\cong\mathbb{Z}_\Delta[-2].$$

where $\Delta\subset Y_2$ is the diagonal, i.e. the set of all points of the form (x,s,s). The statement now follows. □

4.2.2 Singular support estimation

Let us define the following set

$$C := \{(\sigma_1, \sigma_2) | \sigma_1 = 0 \text{ or } \sigma_2 = 0\} \subset E. \tag{80}$$

Let $U := E \backslash C$.

Lemma 4.4 *Suppose $G \in \mathcal{C}_Y$. Then we have:*

$$SS(\Psi(G)) \cap T^*(Y \times U) \subset \{(x, s, \sigma, \omega, 0, bd\sigma)\} \subset T^*(Y \times U),$$

where $(x, s) \in X \times \mathbb{R}^2 = Y$; $\sigma \in U$; $\omega \in T_x^* X$; $bd\sigma \in T_\sigma^* U$.

Proof. First of all, by [KS, Prop.5.3.9],

$$SS(\mathbb{Z}_K) = \{((s, s', \sigma), \lambda d\langle s - s', \sigma\rangle) \ : \ \lambda\langle s - s', \sigma\rangle = 0, \ \lambda \geq 0, \ \langle s - s', \sigma\rangle \geq 0\}. \tag{81}$$

By [KS, proof of Prop.5.4.2], $S.S.p_1^! G$ is contained in the following subset of $T^*(Y_2 \times E)$:

$$(x, s, s', \sigma, \omega, ads, 0 \cdot ds', 0 \cdot d\sigma),$$

where $(x, s, \omega, ads) \in \Omega_Y$.

Let us now check that

$$S.S.p_1^! G \cap S.S.\mathbb{Z}_K \subset \{\text{zero section}\}. \tag{82}$$

Suppose we have an element η in this intersection which does not belong to the zero section. It should be of the form as in (81). Since $\eta \neq 0$, $\lambda > 0$ and $\langle s - s', \sigma\rangle = 0$. We have

$$\lambda d\langle s - s', \sigma\rangle = \lambda\langle s - s', d\sigma\rangle + \lambda\langle ds - ds', \sigma\rangle.$$

The ds' component of η is thus $-\lambda\langle ds', \sigma\rangle$. In order for $\eta \in SS\pi_1^! G$, this component must vanish, which implies $\sigma = 0$. Analogously, $d\sigma$-component of η must vanish as well, i.e. $s - s' = 0$. This implies that η is in the zero section, contradiction. This proves (82).

It now follows that

$$SSR\mathcal{H}om(\mathbb{Z}_K; p_1^! G) \subset SS(p_1^! G) - SS(\mathbb{Z}_K)$$

(where "$-$" means subtraction in each fiber of $T^*(Y_2 \times E)$), [KS, Cor.6.4.5]), i.e.

$$SSR\mathcal{H}om(\mathbb{Z}_K; p_1^! G) \ \subset \ \big\{(x, s, s', \sigma, \omega, ads - \lambda d\langle s - s', \sigma\rangle)\big\} \tag{83}$$

where

$$(x, s, \omega, ads) \in \Omega_Y \tag{84}$$

and s, s', σ, λ satisfy the same conditions as in (81).

Now let us estimate

$$SSRp_{2*}R\mathcal{H}om(\mathbb{Z}_K; p_1^! G) = SS(\Psi(G)).$$

By [T08, Lemma 3.3], we have: if $(a')^0 d(s')^0 \neq 0$, then

$$(x^0, (s')^0, \sigma^0, \omega^0, (a')^0 d(s')^0 + b_0 d\sigma^0) \notin S.S.Rp_{2*}R\mathcal{H}om(\mathbb{Z}_K; p_1^! G)$$

as long as:

there exists ε such that $R\mathcal{H}om(\mathbb{Z}_K; p_1^! G)$ is nonsingular at all points $(x_\star, s_\star, s'_\star, \sigma_\star, \omega_\star, a_\star ds + a'_\star ds' + b_\star d\sigma)$, where

$$\begin{cases} |x_\star - x^0| < \varepsilon, & \text{any } s_\star \in \mathbb{R}^2, \quad |s'_\star - (s')^0| < \varepsilon, \quad |\sigma_\star - \sigma^0| < \varepsilon, \\ |\omega_\star - \omega^0| < \varepsilon, & |a_\star| < \varepsilon, \qquad |a'_\star - (a')^0| < \varepsilon, \quad |b_\star - b^0| < \varepsilon. \end{cases} \tag{85}$$

Thus, the proof of the lemma 4.4 reduces to the following statement:

Let $(x^0, (s')^0, \sigma^0, \omega^0, (a')^0 d(s')^0 + b_0 d\sigma^0) \in T^*(Y \times E)$ *satisfy:*
a) $\sigma^0 = (\sigma_1^0, \sigma_2^0)$ *is such that*

$$\sigma_1^0 \neq 0 \text{ and } \sigma_2^0 \neq 0; \tag{86}$$

b) $(a')^0 \neq 0$.
Then for some $\varepsilon > 0$ *there are no solution* $(x_\star, s_\star, s'_\star, \sigma_\star, \omega_\star, a_\star, a'_\star, b_\star)$ *of the inequalities (85) satisfying the conditions (coming from* (83) *)*

$$\begin{cases} x_\star = x, & s_\star = s, & s'_\star = s', & \sigma_\star = \sigma, \\ \omega_\star = \omega, & a_\star = a - \lambda\sigma, & a'_\star = \lambda\sigma, & b_\star = -\lambda(s - s'), \end{cases} \tag{87}$$

such that condition of (81) *and* (84) *hold.*

Eliminating the variables with $\star$ and conditions on x, ω, b, we must, for fixed 0-variables find ε making the following list of conditions inconsistent:

1. $|s' - (s')^0| < \varepsilon$
2. $|\sigma - \sigma^0| < \varepsilon$
3. $|a - \lambda\sigma| < \varepsilon$
4. $|\lambda\sigma - (a')^0| < \varepsilon$
5. $a_1 = 0$ or $a_2 = 0$

6. $\lambda \geq 0$

7. $\lambda \langle s - s', \sigma \rangle = 0$

8. $\langle s - s', \sigma \rangle \geq 0$

Indeed, suppose there is a solution to this system of inequalities such that $a_1 = 0$. Then by condition 3, $|\lambda \sigma_1| < \varepsilon$, i.e.

$$|\lambda| < \frac{\varepsilon}{|\sigma_1|} \tag{88}$$

By condition 2,

$$|\sigma| < |\sigma^0| + \varepsilon. \tag{89}$$

Combining condition 4 with (88) and (89), obtain

$$\varepsilon > |(a')^0 - \lambda \sigma| \geq |(a')^0| - \lambda \cdot (|\sigma^0| + \varepsilon) \geq |(a')^0| - \frac{\varepsilon}{|\sigma_1|}(|\sigma^0| + \varepsilon) \tag{90}$$

If we choose $\varepsilon > 0$ to satisfy (cf. condition a))

$$\varepsilon < \frac{1}{2} \min\{|\sigma_1^0|, |\sigma_2^0|\} \tag{91}$$

then (90) yields

$$\varepsilon > |(a')^0| - \frac{2\varepsilon}{|\sigma_1^0|}(|\sigma^0| + \varepsilon) \tag{92}$$

We have assumed $a_1 = 0$ above; if we assume $a_2 = 0$ (cf. condition 5), we get an analogous inequality. Choosing $\varepsilon > 0$ to satisfy (91) and to violate both (92) and its analog for $a_2 = 0$, finishes the proof.

□

4.2.3

Lemma 4.5 *Let $G \in Ob(\mathcal{C}_Y)$. Then $\Psi(G)|_{Y \times U} = 0$.*

Proof Let $q : Y \times U \to X \times U$ be the projection $q(x, s, \sigma) = (x, \sigma)$. We have a natural map

$$\iota : q^{-1} Rq_*(\Psi(G)|_{Y \times U}) \to \Psi(G)|_{Y \times U}$$

By virtue of lemma 4.4 and the fact that the fibers of q are diffeomorphic to $\mathbb{R}^2$, we see that ι is an isomorphism.

It now remains to show that $Rq_*(\Psi(G)|_{Y \times U}) = 0$.

Let $K_U := K \cap (Y_2 \times U)$. Let $q_1 : Y_2 \times U \to Y \times U$, $q_2 : Y \times U \to Y$, $q_3 : Y \times U \to X \times U$ be the projections

$$q_1(x, s, s', \sigma) = (x, s', \sigma);$$

$$q_2(x, s, \sigma) = (x, s);$$

$$q_3(x, s, \sigma) = (x, \sigma).$$

In this notation,

$$Rq_*(\Psi(G)|_{Y\times U}) = Rq_{3*} R\mathcal{H}om_{Y\times U}(Rq_{1!}\mathbb{Z}_{K_U}; q_2^! G).$$

Finally, we observe that $Rq_{1!}\mathbb{Z}_{K\times U} = 0$ (pointwise computation). □

4.2.4 Representation of G

Let $i_C : C \subset E$ be the closed embedding; here C is as in (80). Let $K_C := K \cap (Y_2 \times C)$. Let

$$p_1^C : Y_2 \times C \to Y_2 \xrightarrow{\pi_1} Y$$

and

$$p_2^C \ : \ Y_2 \times C \ \xrightarrow{\pi_2 \times \mathrm{id}_C} \ Y \times C.$$

Let $q^C : Y \times C \to Y$ be the projection. Let $G \in \mathcal{C}_Y$. It now follows from Lemma 4.5 that $\Psi(G) = (\mathrm{id}_Y \times i_C)_*(\mathrm{id}_Y \times i_C)^{-1}\Psi(G)$, which together with Lemma 4.3 yields a natural isomorphism

$$G \cong Rq_*^C Rp_{2*}^C R\mathcal{H}om_{Y_2\times C}(\mathbb{Z}_{K_C}; (p_1^C)^! G)[2].$$

So that we have an induced isomorphism

$$RHom(F, G) \cong RHom(F; Rq_*^C Rp_{2*}^C R\mathcal{H}om_{Y_2\times C}(\mathbb{Z}_{K_C}; (p_1^C)^! G))[2].$$

Let us rewrite the RHS.

First of all, set

$$\pi_2^C := q^C p_2^C \ : \ Y_2 \times C \ \to \ Y \ : \ (x, s, s', \sigma) \mapsto (x, s').$$

We then have

$$RHom(F; Rq_*^C Rp_{2*}^C R\mathcal{H}om_{Y_2\times C}(\mathbb{Z}_{K_C}; (p_1^C)^! G))$$

$$= RHom((\pi_2^C)^{-1}F; \mathcal{H}om(\mathbb{Z}_{K_C}; (p_1^C)^! G))$$

$$= RHom((\pi_2^C)^{-1}F \otimes \mathbb{Z}_{K_C}; (p_1^C)^! G)).$$

Next, we factor $p_1^C = q^C \pi_1^C$, where

$$\pi_1^C : Y_2 \times C \overset{\pi_1 \times \mathrm{id}_C}{\longrightarrow} Y \times C$$

so that we can continue

$$RHom((\pi_2^C)^{-1}F \otimes \mathbb{Z}_{K_C}; (p_1^C)^! G)) = RHom_{Y\times C}(R(\pi_1^C)_!((\pi_2^C)^{-1}F \otimes \mathbb{Z}_{K_C}); (q^C)^! G).$$

Let us show that $\mathbf{F} := R\pi_{1!}^C((\pi_2^C)^{-1}F \otimes \mathbb{Z}_{K_C}) = 0$ under assumtions on F from Theorem 4.2. Indeed, let $(a,0) \in C$, $a \neq 0$. Then, for any $F \in \mathbf{D}(Y)$, we have

$$RP_{1!}F \cong \mathbf{F}|_{Y\times(a,0)}.$$

Similarly,

$$RP_{2!}F \cong \mathbf{F}|_{Y\times(0,a)}.$$

Finally,

$$RP_{0!}F \cong \mathbf{F}|_{Y\times(0,0)},$$

where $P_0 : Y \times C \to Y$ is the projection. Since P_0 passes through P_1, all the restriction listed vanish under assumptions from Theorem 4.2. This concludes the proof.

5 Orthogonality criterion for a generalized strip

5.1 Conventions and notations

Let $\alpha \in (0, \pi/2)$ be an acute angle, same as in Sec.1.1.1.

Set $\mathbf{e} = e^{-i\alpha}$; $\mathbf{f} = e^{i\alpha}$ so that $\mathbf{e}, \mathbf{f}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$ and every complex number z can be uniquely written as $z = x\mathbf{e} + y\mathbf{f}$, $x, y \in \mathbb{R}$ so that we identify

$$\mathbb{C} \overset{\sim}{\to} \mathbb{R}^2 \tag{93}$$

using the coordinates (x, y).

Define a generalized strip which is a set of one of the following types:

<u>First type:</u>

$$\mathbf{S} = \{x\mathbf{e} + y\mathbf{f} \ : \ x > \gamma; \ y \in (A, B)\} \subset \mathbb{R}^2 = \mathbb{C}, \tag{94}$$

where $-\infty \leq \gamma < \infty$ and $-\infty \leq A < B \leq \infty$.

<u>Second type:</u>

$$\mathbf{S} = \{x\mathbf{e} + y\mathbf{f} \ : \ x < \gamma; \ y \in (A, B)\} \subset \mathbb{R}^2 = \mathbb{C}, \tag{95}$$

where $-\infty < \gamma \leq \infty$ and $-\infty \leq A < B \leq \infty$.

5.1.1 Convolution

Let M, N be smooth manifolds Define a convolution bi-functor

$$* \ : \ \mathbf{D}(M \times \mathbb{R}^2) \times \mathbf{D}(N \times \mathbb{R}^2) \to \mathbf{D}(M \times N \times \mathbb{R}^2)$$

as follows. Denote

$$A \ : \ M \times \mathbb{R}^2 \times N \times \mathbb{R}^2 \to M \times N \times \mathbb{R}^2 \ : \quad A(m, u, n, v) = (m, n, u + v) \tag{96}$$

We now define

$$F * S := RA_!(F \boxtimes^{\mathbb{L}} S).$$

5.1.2 The category $\mathcal{C}_{\mathbf{S}}$.

Let $\Omega_{\mathbf{S}} \subset T^*(\mathbf{S} \times \mathbb{R}^2)$ be a closed conic subset consisting of all points

$$(x_1, y_1, x_2, y_2, a_1 dx_1 + b_1 dy_1; a_2 dx_2 + b_2 dy_2)$$

where $(x_1, y_1) \in \mathbf{S}$ and $(a_1, b_1) = \pm(a_2, b_2)$.

In terms of the complex coordinate $z = x\mathbf{e} + y\mathbf{f}$ and the identification (93) we have:

$$\Omega_{\mathbf{S}} = \{(z, s, adz + bds | z \in \mathbf{S}, s \in \mathbb{C}, a = \pm b\}.$$

Let $\mathcal{C}_{\mathbf{S}} \subset \mathbf{D}(\mathbf{S} \times \mathbb{R}^2)$ be the full subcategory consisting of all objects microsupported within $\Omega_{\mathbf{S}}$.

5.1.3 Rays l_+ and l_-

Let

$$l_+ := \{(x, 0) | x \geq 0\} \subset \mathbb{R}^2 \ ; \quad l_- := \{(x, 0); x \leq 0\} \subset \mathbb{R}^2,$$

5.1.4 Projectors $P_\pm$

Let us define the following projectors $P_\pm \ : \mathbf{S} \times \mathbb{R}^2 \to \mathbb{R}^2$, where

$$P_\pm(x_1, y_1, x_2, y_2) = (x_1 \pm x_2; y_1 \pm y_2). \tag{97}$$

5.2 Formutation of the criterion

Introduce the following projections.

Our criterion is then as follows.

Proposition 5.1 *Consider constant sheaves $\mathbb{Z}_{l_\pm} \in \mathbf{D}(\mathbb{R}^2)$. Let $F \in \mathbf{D}(S \times \mathbb{R}^2)$ and suppose that one of the natural maps*

$$\mathbb{Z}_{l_+} * F \to \mathbb{Z}_0 * F = F \tag{98}$$

$$\mathbb{Z}_{l_-} * F \to \mathbb{Z}_0 * F = F; \tag{99}$$

is a quasi-isomorphism.

Suppose that both $RP_{+!}F = 0$ and $RP_{-!}F = 0$. Then $F \in {}^\perp\mathcal{C}$.

The rest of this section is devoted to proving this criterion under the assumption (98). The case (99) is treated in a fairly similar way and is omitted.

5.3 Fourier-Sato decomposition

Denote by E the dual vector space to $\mathbb{R}^2$. We have the standard identification $E = \mathbb{R}^2$. Let $\langle,\rangle$ be the standard pairing $E \times \mathbb{R}^2 \to \mathbb{R}$. Let $Z \subset E \times \mathbb{R}^2$; $Z = \{(\zeta, u) | \langle \zeta, u\rangle \geq 0\}$.

As was explained above, we have the convolution

$$* \ : \ \mathbf{D}(E \times \mathbb{R}^2) \times \mathbf{D}(\mathbf{S} \times \mathbb{R}^2) \to \mathbf{D}(E \times \mathbf{S} \times \mathbb{R}^2).$$

For $F \in \mathbf{D}(\mathbf{S} \times \mathbb{R}^2)$ set

$$\mathbb{F}(F) := \mathbb{Z}_Z * F \in \mathbf{D}(E \times \mathbf{S} \times \mathbb{R}^2), \tag{100}$$

where $\mathbb{Z}_Z \in \mathbf{D}(E \times \mathbb{R}^2)$ is the constant sheaf on Z. Notice that $\mathbb{F}(F)$ is an analog of (but is not directly equal to) the Fourier-Sato transform of [KS, Ch.3.7].

Lemma 5.2 *(Fourier-Sato decomposition of F) Consider the projection $q : E \times \mathbf{S} \times \mathbb{R}^2 \to S \times \mathbb{R}^2$. Then for any $F \in \mathbf{D}(\mathbf{S} \times \mathbb{R}^2)$, we have a natural isomorphism*

$$Rq_!\mathbb{F}(F)[2] \cong F.$$

PROOF. Let us introduce the following projections (where, say p_{24} means the projection onto the 2-nd and the 4-th factor):

$$\begin{tikzcd}
 & E\times\mathcal{S}\times\mathbb{R}^2\times\mathbb{R}^2 \arrow[dl, "p_{123}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{234}"] \arrow[drr, "p_{24}"] & & \\
E\times\mathcal{S}\times\mathbb{R}^2 \arrow[r, "q"'] & \mathcal{S}\times\mathbb{R}^2 & \mathcal{S}\times\mathbb{R}^2\times\mathbb{R}^2 \arrow[l, "\tilde{p}_{13}"] \arrow[r, "r"] & \mathcal{S}\times\mathbb{R}^2
\end{tikzcd}$$

Introduce the following closed subset

$$Z' = \{(\xi, z, x, y) \ : \ \langle \xi, x-y\rangle \geq 0\} \subset E\times\mathcal{S}\times\mathbb{R}^2\times\mathbb{R}^2.$$

We can now rewrite:

$$\mathbb{F}(F) = Rp_{123!}(\mathbb{Z}_{Z'}\otimes p_{24}^{-1}F),$$

hence

$$Rq_!\mathbb{F}(F) = R\tilde{p}_{13!}Rp_{234!}(\mathbb{Z}_{Z'}\otimes p_{24}^{-1}F) \ =$$

(projection formula [KS, Prop.2.5.13(ii)] is used)

$$= \ R\tilde{p}_{13!}(Rp_{234!}\mathbb{Z}_{Z'}\otimes r^{-1}F)$$

We have a natural isomorphism $Rp_{234!}\mathbb{Z}_{\Omega_{\mathbf{S}}} \cong \mathbb{Z}_{S\times\Delta}[-2]$, where $\Delta\subset\mathbb{R}^2\times\mathbb{R}^2$ is the diagonal. The result now follows. □

5.4 Transfer of the conditions $P_{\pm!}F = 0$ to $\mathbb{F}F$

Claim 5.3 *Let $F\in\mathbf{D}(\mathbf{S}\times\mathbb{R}^2)$ satisfy $RP_{\pm!}F = 0$. We then have $R(\mathrm{id}_E\times P_{\pm})_!\mathbb{F}(F) = 0$.*

PROOF. Let us pick a point $(\eta, s_0)\in E\times\mathbb{R}^2$ and show that, say, $R(\mathrm{id}_E\times P_+)_!\mathbb{F}(F)|_{(\eta,s_0)} = 0$. We have:

$$R(\mathrm{id}_E\times P_+)_!\mathbb{F}(F)|_{(\eta,s_0)} = R\Gamma_c(E\times\mathcal{S}\times\mathbb{R}^2; (\mathrm{id}_E\times P_+)^{-1}\mathbb{Z}_{(\eta,s_0)}\otimes^L\mathbb{F}(F))$$

$$= R\Gamma_c(E\times\mathcal{S}\times\mathbb{R}^2; \mathbb{Z}_{(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0)}\otimes RA_!(\mathbb{Z}_Z\boxtimes F))$$

$$\overset{[\mathrm{KS,\ Prop.2.5.13(ii)}]}{=} R\Gamma_c(E\times\mathbb{R}^2\times\mathcal{S}\times\mathbb{R}^2; \mathbb{Z}_{A^{-1}P_+^{-1}(\eta,s_0)}\otimes p_{12}^{-1}\mathbb{Z}_Z\otimes p_{34}^{-1}F), \tag{101}$$

where:

$$p_{12} : E\times\mathbb{R}^2\times\mathcal{S}\times\mathbb{R}^2\to E\times\mathbb{R}^2$$

is the projection onto the first two factors;

$$p_{34} : E\times\mathbb{R}^2\times\mathcal{S}\times\mathbb{R}^2\to\mathcal{S}\times\mathbb{R}^2$$

is the projection onto the last two factors; and finally,

$$A \;:\; E\times\mathbb{R}^2\times\mathcal{S}\times\mathbb{R}^2 \;\to\; E\times\mathcal{S}\times R^2 \;:\quad (\eta,s_1,z,s_2)\mapsto(\eta,z,s_1+s_2)$$

(as in (96)).

We have:

$$A^{-1}(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0)=\{(\eta,s_1,z,s_2)|s_1+s_2+z=s_0\}.$$

Note that

$$\mathbb{Z}_{A^{-1}(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0)}\otimes p_{12}^{-1}\mathbb{Z}_Z=\mathbb{Z}_{A^{-1}(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0)}\otimes\mathbb{Z}_{p_{12}^{-1}Z}=\mathbb{Z}_{(A^{-1}(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0))\cap p_{12}^{-1}Z}$$

and put

$$T:=(A^{-1}(\mathrm{id}_E\times P_+)^{-1}(\eta,s_0))\cap p_{12}^{-1}Z=\{(\eta,s_1,z,s_2)|s_1+z+s_2=s_0;\langle\eta,s_1\rangle\geq 0\}.$$

Denote by i the restriction of p_{34} to T:

$$i\;:\;T\to\mathcal{S}\times\mathbb{R}^2\;:\;T\ni(\eta,s_1,z,s_2)\mapsto(z,s_2).$$

We see that i is a closed embedding and that

$$i(T)=\{(z,s)|\langle\eta,s_0-s-z\rangle\geq 0\}=P_+^{-1}K,\quad K=\{w|\langle\eta,s_0-w\rangle\geq 0\}\subset\mathbb{R}^2,$$

where $P_+:\mathcal{S}\times\mathbb{R}^2\to\mathbb{R}^2$ is as in (97).

We thus can continue our computation from (101)

$$=R\Gamma_c(E\times\mathbb{R}^2\times\mathcal{S}\times\mathbb{R}^2;\mathbb{Z}_T\otimes p_{34}^{-1}F)$$

$$=R\Gamma_c(\mathcal{S}\times\mathbb{R}^2;(Rp_{34!}\mathbb{Z}_T)\otimes F)=R\Gamma_c(\mathcal{S}\times\mathbb{R}^2;\mathbb{Z}_{i(T)}\otimes F)=$$

$$=R\Gamma_c(\mathcal{S}\times\mathbb{R}^2;P_+^{-1}\mathbb{Z}_K\otimes F)=$$

$$\overset{\text{[KS, Prop.2.5.13(ii)]}}{=}R\Gamma_c(\mathbb{R}^2;\mathbb{Z}_K\otimes RP_{+!}F)=0.$$

The equality $RP_{-!}\mathbb{F}F=0$ can be proven in the same way. $\square$

5.5 Fourier-Sato decomposition for sheaves satisfying (98)

Define:

$$\Pi_+=\;\{(\xi,\eta)\in E|\xi>0\}\;\subset E. \tag{102}$$

Suppose (98) is the case. Then we have

$$\mathbb{F}(F)\overset{\sim}{\to}\mathbb{F}(\mathbb{Z}_{l_+}*F)\overset{\sim}{\to}(\mathbb{Z}_Z*\mathbb{Z}_{l_+})*F. \tag{103}$$

5.5.1 Computing $\mathbb{Z}_Z * \mathbb{Z}_{l_+}$

Lemma 5.4 *We have an isomorphism*

$$\mathbb{Z}_Z * \mathbb{Z}_{l_+} = \mathbb{Z}_{Z_+}. \tag{104}$$

Proof. The inclusion $\{0\} \hookrightarrow l_+$ induces a map

$$\mathbb{Z}_Z * \mathbb{Z}_{l_+} \to \mathbb{Z}_Z * \mathbb{Z}_0 = \mathbb{Z}_Z. \tag{105}$$

It suffices to prove the following two statements:

1) Let $x \in Z_+ \subset E \times \mathbb{R}^2$. The map

$$(\mathbb{Z}_Z * \mathbb{Z}_{l_+})_x \to (\mathbb{Z}_{Z_+})_x = \mathbb{Z}, \tag{106}$$

induced by (105), is an isomorphism.

2) Let $x \in (E \times \mathbb{R}^2)\backslash Z_+$. Then $(\mathbb{Z}_Z * \mathbb{Z}_{l_+})_x = 0$.

Let us now prove 1,2. First of all, for a point $x := (\zeta, v) \in E \times \mathbb{R}^2$, let us introduce a set

$$K_x = \{(\zeta, u_1, u_2) | (\zeta, u_1) \in Z; u_2 \in L_+; u_1 + u_2 = v\} \subset E \times \mathbb{R}^2 \times \mathbb{R}^2,$$

so that we have

$$(\mathbb{Z}_Z * \mathbb{Z}_{L_+})_x = R^\bullet\Gamma_c(K_x, \mathbb{Z}_{K_x}). \tag{107}$$

Let

$$L_x\{(zeta, u_1, u_2) | (\zeta, u_1) \in Z; u_2 = 0; u_1 + u_2 = v\} \subset E \times \mathbb{R}^2 \times \mathbb{R}^2$$

so that

$$(\mathbb{Z}_Z * \mathbb{Z}_0)_x = R^\bullet\Gamma_c(L_x, \mathbb{Z}_{L_x}). \tag{108}$$

We have $L_x \subset K_x$ is a closed subset. Under the identifications (107), (108), the map (106) corresponds to the restriction map

$$R^\bullet\Gamma_c(K_x, \mathbb{Z}_{K_x}) \to R^\bullet\Gamma_c(L_x, \mathbb{Z}_{L_x}).$$

Let $v = (v_1, v_2)$. We then have

$$K_x = \{((\xi, \eta), (x_1, v_2), (x_2, 0) | \xi x_1 + \eta y_1 \geq 0; x_2 \geq 0; x_1 + x_2 = v_1\}$$

The subset $L_x \subset K_x$ consists of all points with $x_2 = 0$.

The set K_x is identified with the set

$$K'_x := \{(x_1, y_1) \in \mathbb{R}^2 | \xi x_1 + \eta y_1 \geq 0; x_1 \leq v_1\}.$$

The set L_x gets identified with the subset L'_x of K'_x consisting of all points with $x_1 = v_1$.

Let us check 1). Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the second coordinate It suffices to check that the natural map

$$R\pi_! \mathbb{Z}_{K'_x} \to R\pi_! \mathbb{Z}_{L'_x}$$

(induced by the embedding $L'_x \subset K'_x$) is an isomorphism. We further reduce the statement so that it reads: the following induced map on stalks at every point $y \in \mathbb{R}$ is an isomorphism:

$$(R\pi_! \mathbb{Z}_{K'_x})_y \to (R\pi_! \mathbb{Z}_{L'_x})_y. \tag{109}$$

We have

$$(R\pi_! \mathbb{Z}_{K'_x})_y \cong R\Gamma_c(K'_{xy}; \mathbb{Z}_{K'_{xy}}); \tag{110}$$

$$(R\pi_! \mathbb{Z}_{L'_x})_y \cong R\Gamma_c(L'_{xy}; \mathbb{Z}_{L'_{xy}});$$

where

$$K'_{xy} = \{(x_1, y) \in \mathbb{R}^2 | \xi x_1 + \eta y \geq 0; x_1 \leq v_1\}; \tag{111}$$

$$L'_{xy} = \{(x_1, y) \in \mathbb{R}^2 | \xi x_1 + \eta y \geq 0; x_1 = v_1\}.$$

The map (109) corresponds to the natural map

$$R\Gamma_c(K'_{xy}; \mathbb{Z}_{K'_{xy}}) \to R\Gamma_c(L'_{xy}; \mathbb{Z}_{L'_{xy}}) \tag{112}$$

induced by the closed embedding $L'_{xy} \subset K'_{xy}$.

We have $\xi > 0$ (because $x \in \Pi_+ \times \mathbb{R}^2$), in which case either both L'_{xy} and K'_{xy} are empty sets, or K'_{xy} is a closed segment and L'_{xy} is its boundary point, which implies that (112) and hence (109) are isomorphisms.

Let us now check 2). We have $\xi \leq 0$. It suffices to check that $(R\pi_! \mathbb{Z}_{K_x})_y$=0 for all $y \in \mathbb{R}$. Using (110), we can equivalently rewrite this condition as follows:

$$R\Gamma(K'_{xy}; \mathbb{Z}_{K'_{xy}}) = 0.$$

As follows from (111), the condition $\xi \leq 0$ implies that K'_{xy} is homeomorphic to a closed ray, which implies the statement. $\Box$.

Combining (103) and (104), we immediately obtain:

Corollary 5.5 *Suppose $F \in \mathbf{D}(\mathcal{S} \times \mathbb{R}^2)$ satisfies* (98)*. Then*

$$\operatorname{supp} \mathbb{F}(F) \ \subset \Pi_+ \times \mathbf{S} \times \mathbb{R}^2. \tag{113}$$

Motivated by the corollary 5.5, set

$$\mathbb{F}'(F) := \mathbb{F}(F)|_{\Pi_+ \times \mathbf{S} \times \mathbb{R}^2} \in \mathbf{D}(\Pi_+ \times \mathbf{S} \times \mathbb{R}^2).$$

Introduce the following subset

$$Z_+ := Z \cap \Pi_+ \times \mathbb{R}^2 \subset \Pi_+ \times \mathbb{R}^2.$$

so that

$$\mathbb{F}'(F) = \mathbb{Z}_{Z_+} * F. \tag{114}$$

Let $\pi_+ : \Pi_+ \times \mathbf{S} \times \mathbb{R}^2 \to \mathbf{S} \times \mathbb{R}^2$ be the projection.

Lemma 5.2 and (113) imply the following isomorphism:

$$F[-2] \sim R\pi_{+!}\mathbb{F}'(F) = R\pi_{+!}(\mathbb{Z}_{Z_+} * F). \tag{115}$$

5.5.2 Further reformulation

Let us introduce a map

$$Q \ : \ \Pi_+ \to \mathbb{R} \ , \quad Q(\xi, \eta) = \eta/\xi.$$

Let also

$$q : \mathbb{R} \times \mathcal{S} \times \mathbb{R}^2 \to \mathcal{S} \times \mathbb{R}^2$$

be the projection. Finally, let us set

$$W := \{(a, (x, y))| x + ay \geq 0\} \subset \mathbb{R} \times \mathbb{R}^2.$$

There is a commutative diagram with a Cartesian square:

$$\begin{tikzcd}
Z_+ \times \mathbf{S} \times \mathbb{R}^2 \arrow[r, phantom, "\subset"] & \Pi_+ \times \mathbb{R}^2 \times \mathbf{S} \times \mathbb{R}^2 \arrow[rr, "Q \times \mathrm{id}_{\mathbb{R}^2 \times S \times \mathbb{R}^2}"] \arrow[d, "A"'] \arrow[drr, phantom, "\square"] & & \mathbb{R} \times \mathbb{R}^2 \times \mathbf{S} \times \mathbb{R}^2 \arrow[d, "A"] \arrow[r, phantom, "\supset"] & W \times \mathbf{S} \times \mathbb{R}^2 \\
& \Pi_+ \times \mathbf{S} \times \mathbb{R}^2 \arrow[rr, "Q \times \mathrm{id}_{S \times \mathbb{R}^2}"] \arrow[dr, "\pi_+"'] & & \mathbb{R} \times \mathbf{S} \times \mathbb{R}^2, \arrow[dl, "q"] & \\
& & \mathbf{S} \times \mathbb{R}^2 & &
\end{tikzcd} \tag{116}$$

The map A in this diagram is induced by the addition $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 5.6 *i) "$\mathbb{Z}_{Z_+} * F$ is constant along fibers of $Q \times \mathrm{id}_{S\times\mathbb{R}^2}$" in the sense that*

$$\mathbb{Z}_{Z_+} * F \cong (Q \times \mathrm{id}_{S\times\mathbb{R}^2})^{-1}(\mathbb{Z}_W * F); \tag{117}$$

ii) If F satisfies (98)*, then there is a quasi-isomorphism*

$$F \cong Rq_!(\mathbb{Z}_W * F)[1]. \tag{118}$$

PROOF From the definition of a constant sheaf as a pull-back of $\mathbb{Z}_{pt}$, we have $(Q\times\mathrm{id}_{\mathbb{R}^2})^{-1}\mathbb{Z}_{W\times\mathbf{S}\times\mathbb{R}^2} = \mathbb{Z}_{Z_+\times\mathbf{S}\times\mathbb{R}^2}$; and then, by the base change [KS, (2.5.6)] in the Cartesian square of (116), we obtain (117).

To prove (118), write

$$F \overset{(115)}{=} R\pi_{+!}(\mathbb{Z}_{Z_+} * F)[2] \overset{(117)}{=} R\pi_{+!}(Q \times \mathrm{id}_{S\times\mathbb{R}^2})^{-1}(\mathbb{Z}_W * F)[2] =$$

$$= R\pi_{+!}(Q\times\mathrm{id}_{S\times\mathbb{R}^2})^{-1}RA_!(\mathbb{Z}_W\boxtimes F)[2]{=}Rq_!R(Q\times\mathrm{id}_{S\times\mathbb{R}^2})_!(Q\times\mathrm{id}_{S\times\mathbb{R}^2})^{-1}RA_!(\mathbb{Z}_W\boxtimes F)[2] \overset{Q^{-1}=Q^![-1]}{=}$$

$$= Rq_!R(Q \times \mathrm{id}_{S\times\mathbb{R}^2})_!(Q \times \mathrm{id}_{S\times\mathbb{R}^2})^!(\mathbb{Z}_W * F)[1]{=}Rq_!(\mathbb{Z}_W * F)[1].$$

□

5.5.3 Rewriting the map (118)

Define a map $l : \mathbb{R} \times \mathbb{R}^2 \to R$, where R is another copy of $\mathbb{R}$, as follows: $l(a, x, y) := x + ay$.

Let

$$L : \mathbb{R} \times \mathbf{S} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbf{S} \times R;$$

be given by $L(a, z, u) = (a, z, l(a, u))$.

Let $W' \subset \mathbb{R} \times \mathbb{R}^2 \times R$ be given by

$$W' = \{(a, (x_1, y_1), t)|t - x - ay \geq 0\}.$$

Let

$$p_{\mathbf{S}} : \mathbb{R} \times \mathbf{S} \times \mathbb{R}^2 \times R \to \mathbb{R} \times \mathbb{R}^2 \times R;$$

$$p_{\mathbb{R}\times R} : \mathbb{R} \times \mathbf{S} \times \mathbb{R}^2 \times R \to \mathbf{S} \times \mathbb{R}^2;$$

and

$$p_{\mathbb{R}^2} : \mathbb{R} \times \mathbf{S} \times \mathbb{R}^2 \times R \to \mathbb{R} \times \mathbf{S} \times R$$

be projections.

We have the following cartesian diagram:

$$\begin{tikzcd}
 & & (a,u_1,z,u_2) \arrow[r, mapsto] & (a,z,u_2,\ell(a,u_1+u_2)) & & \\
 & & \Cap & \Cap & & \\
(a,u_1,z,u_2) \arrow[d, mapsto] & \in & \mathbb{R}\times\mathbb{R}^2\times\mathbf{S}\times\mathbb{R}^2 \arrow[r, "\tilde{L}"] \arrow[d, "A"'] \arrow[dr, phantom, "\square"] & \mathbb{R}\times\mathbf{S}\times\mathbb{R}^2\times R \arrow[d, "p_{\mathbb{R}^2}"] & \ni & (a,z,u,t) \arrow[d, mapsto] \\
(a,z,u_1+u_2) & \in & \mathbb{R}\times\mathbf{S}\times\mathbb{R}^2 \arrow[r, "L"] & \mathbb{R}\times\mathbf{S}\times R & \ni & (a,z,t) \\
 & & \Cup & \Cup & & \\
 & & (a,z,u) \arrow[r, mapsto] & (a,z,\ell(a,u)) & &
\end{tikzcd} \tag{119}$$

and $W \times \mathbb{R}^2_{u_2} \times \mathbf{S} = \tilde{L}^{-1}(W' \times \mathbf{S})$.

By the base change [KS, (2.5.6)] applied to the diagram (119), we have for all F satisfying (98):

$$\mathbb{Z}_W * F = L^{-1} R p_{\mathbb{R}^2 !}(p^{-1}_{\mathbb{R}\times R} F \otimes p^{-1}_{\mathbf{S}} \mathbb{Z}_{W'}). \tag{120}$$

Denote

$$\Phi_F := \mathbb{Z}_W * F := R p_{\mathbb{R}^2 !}(p^{-1}_{\mathbb{R}\times R} F \otimes p^{-1}_{\mathbf{S}} \mathbb{Z}_{W'}) \ \in\ \mathbf{D}(\mathbb{R} \times \mathbf{S} \times R).$$

5.5.4 Transferring Claim 5.3 to Φ_F

Let $P'_{\pm} : \mathbb{R} \times \mathbf{S} \times R \to \mathbb{R} \times R$ be given by

$$P'_{\pm}(a,(x,y),t) = (a, x + ay \pm t). \tag{121}$$

Lemma 5.7 *If $F \in \mathbf{D}(\mathbf{S} \times \mathbb{R}^2)$ satisfies both (98) and $RP_{+!}F = 0$ then*

$$RP'_{+!}(\Phi_F) = 0. \tag{122}$$

Analogously, if F satisfies both (99) and $RP_{-!}F = 0$, then $RP'_{-!}(\Phi_F) = 0$.

Proof of Lemma 5.7. Extend the diagram (119) as follows:

$$
\begin{tikzcd}
 & & \mathbb{R}\times\mathbb{R}^2\times\mathbf{S}\times\mathbb{R}^2 \arrow[r, "\tilde{L}"] \arrow[d, "A"] & \mathbb{R}\times\mathbf{S}\times\mathbb{R}^2\times\mathbb{R} \arrow[d, "p_{\mathbb{R}^2}"] \\
E\times\mathbf{S}\times\mathbb{R}^2 \arrow[d, "\mathrm{id}_E\times P_+"] & \Pi_+\times\mathbf{S}\times\mathbb{R}^2 \arrow[l, hook', "\iota\times\mathrm{id}"'] \arrow[r, "Q\times\mathrm{id}_{\mathbf{S}\times\mathbb{R}^2}"] \arrow[d, "\mathrm{id}_{\Pi_+}\times P_+"] & \mathbb{R}\times\mathbf{S}\times\mathbb{R}^2 \arrow[r, "L"] \arrow[d, "\mathrm{id}_{\mathbb{R}}\times P_+"] & \mathbb{R}\times\mathbf{S}\times R \arrow[d, "P'_+"] \\
E\times\mathbb{R}^2 & \Pi_+\times\mathbb{R}^2 \arrow[l, hook', "\iota\times\mathrm{id}"'] \arrow[r, "Q\times\mathrm{id}_{\mathbb{R}^2}"] & \mathbb{R}\times\mathbb{R}^2 \arrow[r, "L'"] & \mathbb{R}\times R \\
 & & \cup & \cup \\
 & & (a,w) \arrow[r, mapsto] & (a,\ell(a,w))
\end{tikzcd}
\tag{123}
$$

where $\iota:\Pi_+\hookrightarrow E$ is the open inclusion.

We have $Z_+ = Z\cap(\iota\times\mathrm{id}_{\mathbb{R}^2})\Pi_+$ and $\mathbb{Z}_{Z_+} = (i\times\mathrm{id}_{\mathbb{R}^2})^{-1}\mathbb{Z}_Z$. Thus by the base change [KS, (2.5.6)], $\mathbb{Z}_{Z_+} * F\in\mathbf{D}(\Pi_+\times\mathbf{S}\times\mathbb{R}^2$ is quasi-isomorphic to $(\iota\times\mathrm{id}_{\mathbf{S}\times\mathbb{R}^2})^{-1}(\mathbb{Z}_Z * F)$. Thus,

$$R(\mathrm{id}_{\Pi_+\times P_+}\times P_+)_!(\mathbb{Z}_{Z_+}*F) \overset{\text{[KS, (2.5.6)]}}{=} (\iota\times\mathrm{id}_{\mathbb{R}^2})^{-1}R(\mathrm{id}_E\times P_+)_!(\mathbb{Z}_Z*F) \overset{\text{Claim 5.3}}{=} 0.$$

But on the other hand,

$$\mathbb{F}(F) \overset{(114)}{=} \mathbb{Z}_{Z_+}*F \overset{(117)}{=} (Q\times\mathrm{id}_{S\times\mathbb{R}^2})^{-1}(\mathbb{Z}_W*F) \overset{(120)}{=} (Q\times\mathrm{id}_{S\times\mathbb{R}^2})^{-1}L^{-1}\Phi_F$$

hence

$$R(\mathrm{id}_{\Pi_+\times P_+}\times P_+)_!(Q\times\mathrm{id}_{S\times\mathbb{R}^2})^{-1}L^{-1}\Phi_F = 0,$$

or applying the base change [KS, (2.5.6)] to the middle and right bottom squares of (123), we have

$$(Q\times\mathrm{id}_{\mathbb{R}^2})^{-1}(L')^{-1}RP'_{+!}(\Phi_F) = 0.$$

Since both maps $(Q\times\mathrm{id}_{\mathbb{R}^2})$ and L' are locally trivial fibrations with a vector space as a fiber, we conclude that $RP'_{+!}\Phi_F = 0$. $\square$

5.6 Rewriting the condition of orthogonality to $\mathcal{C}$

Let F satisfy the conditions of Proposition 5.1 (assuming (98). Let $H\in\mathcal{C}_{\mathbf{S}}$, where $\mathcal{C}_{\mathbf{S}}$ is defined in section 5.1.2. Proposition 5.1 now reduces to proving that $RHom(F,H)=0$.

Let us investigate $RHom(F,H)$ using the representation (118) of F. We have:

$$
\begin{aligned}
R\,\mathrm{Hom}(F,H) &\overset{(118)}{=} R\,\mathrm{Hom}(Rq_!(\mathbb{Z}_W*F),H)[-1] \overset{(120)}{=} R\,\mathrm{Hom}(Rq_!L^{-1}(\Phi_F);H)[-1] \\
&= R\,\mathrm{Hom}_{\mathbb{R}\times\mathbf{S}\times R}(\Phi_F;RL_*q^!H)[-1].
\end{aligned}
\tag{124}
$$

Singular support estimate shows that

Proposition 5.8 *We have:*

$$S.S.RL_*q^!H \ \subset \Omega_{\mathcal{H}},$$

where

$$\Omega_{\mathcal{H}} := \bigcup_{\text{"+" and "-"}} \{(a, x_1, y_1, t, \mathbb{R}.(d(x_1 + ay_1) \pm dt) + \mathbb{R}.da)\} \tag{125}$$

and where $a \in \mathbb{R}$, $(x_1, y_1) \in \mathbf{S}$, $t \in R$.

PROOF Because q is a projection on a direct factor, by [KS, Prop.3.3.2(ii)] we have $S.S.q^!H = S.S.q^{-1}H$ which in turn can be, using [KS, Prop.5.4.13], estimated by (in the notation of that proposition) ${}^tq'(q_\pi^{-1}(S.S.(H)))$; thus

$$S.S.q^!H \ \subset \ \{a, z, u, \alpha da + \upsilon du \ : \ \zeta = \pm\upsilon\}.$$

By [KS, Prop.5.4.4],

$$S.S.RL_*q^!H \subset L_\pi({}^tL'^{-1}\{a, z, u, \alpha da + \zeta dz + \upsilon du \ : \ \zeta = \pm\upsilon\}).$$

We have

$$\begin{array}{rcl} T^*(\mathbb{R}_a \times \mathbf{S}_z \times \mathbb{R}^2_{u=(x,y)}) & \xleftarrow{{}^tL'} & \mathbb{R}_a \times \mathbf{S}_z \times \mathbb{R}^2_{u=(x,y)} \times_{(\mathbb{R}_a \times \mathbf{S}_z \times \mathbb{R}_t)} T^*(\mathbb{R}_a \times \mathbf{S}_z \times R_t) \\ (a, z, u, \alpha da + \zeta dz + \xi dx + \eta dy) & & (a, z, u, \alpha da + \zeta dz + \tau dt) \\ \upsilon = (\xi, \eta) & & t = \ell(a, u) \\ dx + ady + yda & \leftrightarrow & dt. \end{array}$$

Thus

$$S.S.RL_*q^!H \ \subset \ L_\pi(\{a, z, u, \alpha da + \zeta dz + \tau dt \ : \ \zeta = \pm\tau(1, a)\}) \ =$$

$$= \ \{a, z, t, \alpha da + \zeta dz + \tau dt \ : \ \zeta = \pm\tau(1, a)\}$$

which is equivalent to (125). $\square$

Thus, Proposition 5.1 follows from the following one:

Claim 5.9 *Let* $\Phi_F, \mathcal{H} \in \mathbf{D}(\mathbb{R} \times \mathbf{S} \times R)$ *satisfy:* $RP'_{\pm!}\Phi_F = 0$ *(where* $P'_\pm$ *are as in* (121)*);* $S.S.\mathcal{H} \subset \Omega_{\mathcal{H}}$*, where* $\Omega_{\mathcal{H}}$ *is as in* (125)*. Then we have:*

$$RHom(\Phi_F; \mathcal{H}) = 0.$$

5.7 Subdivision into 3 cases

We are going to subdivide the space $\mathbb{R} \times \mathbf{S} \times R$ with coordinates (a, z, u) into 3 parts according to the sign of a.

5.7.1 Subdivision of $\mathbb{R} \times \mathbf{S} \times R$

$$U_+ := (0, \infty) \times \mathbf{S} \times R \subset \mathbb{R} \times \mathbf{S} \times R$$

$$U_- := (-\infty, 0) \times \mathbf{S} \times R \subset \mathbb{R} \times \mathbf{S} \times R;$$

$$U_0 := 0 \times \mathbf{S} \times R \subset \mathbb{R} \times \mathbf{S} \times R.$$

Denote

$$j_\pm : U_\pm \to \mathbb{R} \times \mathbf{S} \times R$$

the corresponding open embeddings and by

$$i_0 : U_0 \to \mathbb{R} \times \mathbf{S} \times R$$

the corresponding closed embedding.

5.7.2 Subdivision of Φ_F

Set

$$\Phi_\pm := j_\pm^{-1} \Phi_F \;\in\; \mathbf{D}(U_\pm);$$

$$\Phi_0 := i_0^{-1} \Phi_F \;\in\; \mathbf{D}(U_0).$$

We have a distinguished triangle

$$\to j_{+!}\Phi_+ \oplus j_{-!}\Phi_- \to \Phi_F \to i_{0!}\Phi_0 \stackrel{+1}{\to} . \tag{126}$$

Let

$$P_\pm^{U_+} := P'_\pm j_+; \quad P_\pm^{U_-} := P'_\pm j_-; \quad P^{U_0} = P'_\pm i_0$$

be the restrictions of $P'_\pm$ from (121) onto U_+, U_-, and U_0. Base change theorem implies that

$$P_{\pm!}^{U_+} \Phi_+ = 0;$$

$$P_{\pm!}^{U_-} \Phi_- = 0;$$

$$P_{\pm!}^{U_0} \Phi_0 = 0.$$

5.7.3 Subdivision of $\mathcal{H}$

Let $\mathcal{H}_\pm \in \mathbf{D}(U_\pm)$;

$$\mathcal{H}_\pm := j_\pm^{-1}\mathcal{H}.$$

Let $\mathcal{H}_0 \in \mathbf{D}(U_0)$;

$$\mathcal{H}_0 := i_0^!\mathcal{H}.$$

Let us estimate the microsupports of these objects. Let

$$\Omega_{U_\pm} := \Omega_{\mathcal{H}} \cap T^*U_\pm \subset T^*U_\pm,$$

where we assume the embeddings $T^*U_\pm \subset T^*(\mathbb{R} \times \mathbf{S} \times R)$ induced by $j_\pm$.

It is immediate that $SS(\mathcal{H}_\pm) \subset \Omega_{U_\pm}$.

Let

$$\Omega_0 := \bigcup_{\text{“+” and “-”}} \{(x_1, y_1, t, \mathbb{R}.(dx_1 \pm dt)\} \subset T^*(\mathbf{S} \times R),$$

where, same as in (125), (x_1, y_1) are coordinates on $\mathbf{S}$, and t on R.

Corollary [KS] 6.4.4(ii) implies that

$$SS(\mathcal{H}_0) \subset \Omega_0.$$

5.7.4 Subdivision of Claim (5.9)

By virtue of the distinguished triangle in (126), Claim (5.9) gets split into showing the following vanishings:

$$RHom_{\mathbb{R}\times\mathbf{S}\times R}(j_{+!}\Phi_+; \mathcal{H}) = R\operatorname{Hom}_{U_+}(\Phi_+; \mathcal{H}_+) = 0;$$

$$RHom_{\mathbb{R}\times\mathbf{S}\times R}(j_{-!}\Phi_-; \mathcal{H}) = R\operatorname{Hom}_{U_-}(\Phi_-; \mathcal{H}_-) = 0;$$

$$RHom_{\mathbb{R}\times\mathbf{S}\times R}(i_0\Phi_+; \mathcal{H}) = R\operatorname{Hom}_{U_0}(\Phi_0; \mathcal{H}_0) = 0.$$

Our task now reduces to showing the following 3 statements:

Claim 5.10 *Let $\Phi_+, \mathcal{H}_+ \in \mathbf{D}(U_+)$. Suppose $RP_{\pm!}^{U_+}\Phi_+ = 0$ and $SS(\mathcal{H}_+) \subset \Omega_{U_+}$. Then*

$$RHom(\Phi_+, \mathcal{H}_+) = 0.$$

Claim 5.11 *Let $\Phi_-, \mathcal{H}_- \in \mathbf{D}(U_-)$. Suppose $RP_{\pm!}^{U_-}\Phi_+ = 0$ and $SS(\mathcal{H}_-) \subset \Omega_{U_-}$. Then*

$$RHom(\Phi_-, \mathcal{H}_-) = 0.$$

Claim 5.12 *Let $\Phi_0, \mathcal{H}_0 \in \mathbf{D}(U_0)$. Suppose $RP^{U_0}_{\pm!}\Phi_0 = 0$ and $SS(\mathcal{H}_0) \subset \Omega_{U_0}$. Then*

$$RHom(\Phi_0, \mathcal{H}_0) = 0.$$

5.7.5 Furhter reduction

Let $\diamondsuit$ be one of the symbols: $+, -$, or 0. Let $I_+ := (0, \infty)$; $I_- := (-\infty, 0)$; $I_0 := \{0\}$. Let

$$Q'_\diamondsuit : U_\diamondsuit \times \mathbf{S} \times R \to I_\diamondsuit \times \mathbb{R} \times R$$

be given by

$$Q'_\diamondsuit(a, (x, y), t) := (a, x + ay, t)$$

(in the case $\diamondsuit = 0$ we assume $a = 0$). Denote by $\mathbf{V}_\diamondsuit \subset \mathbb{R} \times \mathbb{R} \times R$ the image of $Q'_\diamondsuit$. Depending on $\mathbf{S}$, $\mathbf{V}_\diamondsuit$ can be of one of the following types:

1) For some linear function $f_\diamondsuit : I_\diamondsuit \to \mathbb{R}$,

$$\mathbf{V}_\diamondsuit = \{(a, v, t) | a \in I_\diamondsuit; v > f(a); \}.$$

In this case, set $\mathbf{U}_\diamondsuit := I_\diamondsuit \times (0, \infty) \times R$; set

$$Q_1 : U_\diamondsuit \to \mathbf{U}_\diamondsuit,$$

$$Q_1(a, (x, y), t) := (a, x + ay - f(a), t).$$

2) For some linear function $f_\diamondsuit : I_\diamondsuit \to \mathbb{R}$,

$$\mathbf{V}_\diamondsuit = \{(a, v, t) | a \in I_\diamondsuit; v < f(a)\}.$$

In this case, set $\mathbf{U}_\diamondsuit := I_\diamondsuit \times (-\infty, 0) \times R$; set

$$Q_1 : U_\diamondsuit \to \mathbf{U}_\diamondsuit;$$

$$Q_1(a, (x, y), t) := (a, x + ay - f(a), t).$$

3)

$$\mathbf{V}_\diamondsuit = I_\diamondsuit \times \mathbb{R} \times R.$$

In this case, set $\mathbf{U}_\diamondsuit := I_\diamondsuit \times (-\infty, \infty) \times \mathbb{R}$; set $Q_1 : U_\diamondsuit \to \mathbf{U}_\diamondsuit$,

$$Q_1(a, (x, y), t) := (a, x + ay, t).$$

It is easy to see that in each of the cases the map Q_1 is surjective; furthermore it is a smooth fibration with its typical fiber diffeomorphic to $\mathbb{R}$. We also see that the 1-forms from $\Omega_{U_\diamondsuit}$ vanish on fibers of Q_1, which implies that the natural map

$$\mathcal{H}_\diamondsuit \to Q_1^! RQ_{1!}\mathcal{H}_\diamondsuit$$

is an isomorphism.

Set

$$\mathcal{L}_\diamondsuit := RQ_{1!}\mathcal{H}_\diamondsuit \in \mathbf{D}(\mathbf{U}_\diamondsuit).$$

Define conic closed subsets $\Omega_{\mathbf{U}_\pm} \subset T^*\mathbf{U}_\pm$ as follows:

$$\Omega_{\mathbf{U}_\pm} := \bigcup_{\text{“+” and “-”}} \{(a, v, t, \mathbb{R}.(dv \pm dt) + \mathbb{R}.da\},$$

where $(a, v, t) \in \mathbf{U}_\pm \subset I_\pm \times \mathbb{R} \times R$. Define a conic closed subset $\Omega_{\mathbf{U}_0} \subset T^*\mathbf{U}_0$:

$$\Omega_{\mathbf{U}_\pm} := \bigcup_{\text{“+” and “-”}} \{(0, v, t, \mathbb{R}.(dv \pm dt)\}.$$

It is easy to see that

$$SS(\mathcal{L}_\diamondsuit) \subset \Omega_{\mathbf{U}_\diamondsuit}.$$

5.7.6

We have

$$RHom(\Phi_\diamondsuit; \mathcal{H}_\diamondsuit) = RHom(\Phi_\diamondsuit; Q_1^!\mathcal{L}_\diamondsuit) = RHom_{\mathbf{U}_\diamondsuit}(RQ_{1!}\Phi_\diamondsuit; \mathcal{L}_\diamondsuit).$$

Set $G_\diamondsuit := RQ_{1!}\Phi_\diamondsuit$. Let $P_\pm^{\mathbf{U}_\diamondsuit} : \mathbf{U}_\diamondsuit \to \mathbb{R}\times\mathbb{R}$ be the restrictions of the following maps $\mathbb{R}\times\mathbb{R}\times R \to \mathbb{R}\times\mathbb{R}$:

$$(a, v, t) \mapsto (a, v \pm t). \tag{127}$$

It now follows that

$$RP_{\pm!}^{\mathbf{U}_\diamondsuit} G_\diamondsuit = 0.$$

So, we can rewrite Claims 5.10—5.12 as follows.

Claim 5.13 *Let $G_\diamondsuit, \mathcal{L}_\diamondsuit \in \mathbf{D}(\mathbf{U}_\diamondsuit)$ satisfy:*

$$RP_{\pm!}^{\mathbf{U}_\diamondsuit} G_\diamondsuit = 0; \tag{128}$$

$SS(\mathcal{L}_\diamondsuit) \in \Omega_{\mathbf{U}_\diamondsuit}$. *Then $RHom(G_\diamondsuit; \mathcal{L}_\diamondsuit) = 0$.*

5.8 The case $\mathbf{U}_\diamondsuit = I_\diamondsuit \times (-\infty, \infty) \times R$

This case follows from Theorem 4.1 below. Below, we are going to consider the case $\mathbf{U}_\diamondsuit = I_\diamondsuit \times (0, \infty) \times \mathbb{R}$. The case $\mathbf{U}_\diamondsuit = I_\diamondsuit \times (-\infty, 0) \times \mathbb{R}$ is fairly similar.

5.9 Proof of Claim 5.13 for $\mathbf{U}_\diamondsuit = I_\diamondsuit \times (0, \infty) \times \mathbb{R}$

As above, our major tool is development of a certain representation of G.

5.9.1 Representation of G

Let $V_1 \subset I_\diamondsuit \times \mathbb{R} \times (0, \infty) \times \mathbb{R}$ be given by

$$V_1 = \{(a, u, v, t) |\ |t| < v\}. \tag{129}$$

Let $V := I_\diamondsuit \times \mathbb{R} \times (0, \infty) \times (0, \infty)$. We have an identification $J : V \to V_1$,

$$J(a, u, \xi_1, \xi_2) = (a, u, \frac{\xi_1 + \xi_2}{2}, \frac{\xi_1 - \xi_2}{2}). \tag{130}$$

Let $\mathbf{I}_1 : V_1 \to I_\diamondsuit \times (0, \infty) \times \mathbb{R}$ be given by

$$\mathbf{I}_1(a, u, v, t) = (a, v, u + t). \tag{131}$$

Let $\mathbf{I} = \mathbf{I}_1 J$:

$$\mathbf{I}(a, u, \xi_1, \xi_2) = (a, \frac{\xi_1 + \xi_2}{2}, u + \frac{\xi_1 - \xi_2}{2}),$$

so that $\xi_1 = v + t$; $\xi_2 = v - t$.

Let $q_1, q_2 : V \to I_\diamondsuit \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$q_i(a, u, \xi_1, \xi_2) = (a, u, \xi_i), \quad i = 1, 2. \tag{132}$$

Let us summarize our notation in the following diagram (a wavy line indicates that a sheaf is defined over the given space):

$$\begin{tikzcd}
 & & (a,u,v,t) \arrow[r, mapsto] & (a,v,u+t) & \\
 & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
X \times \mathbb{R} \times (\mathbb{R}_{>0} \times \mathbb{R}) & \supset & V_1 = \{(a,u,v,t) \,:\, |t| < v\} \arrow[r, "\mathbf{I}_1"] & I_\diamondsuit \times \mathbb{R}_{>0} \times \mathbb{R} \arrow[r, squiggly, dash] & G \\
 & H \arrow[r, squiggly, dash] & V = I_\diamondsuit \times \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \arrow[u, "J"'] \arrow[ur, "\mathbf{I}"'] \arrow[r, "q_i"'] & I_\diamondsuit \times \mathbb{R} \times \mathbb{R}_{>0}. & \\
 & & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \\
 & & (a,u,\xi_1,\xi_2) \arrow[r, mapsto] & (a,u,\xi_i) &
\end{tikzcd}$$

Claim 5.14 *Suppose that an object* $G \in \mathbf{D}(I_\diamondsuit \times (0,\infty) \times \mathbb{R})$ *satisfies (128) both with the sign "+" and with the sign "-". There exists an object* $H \in \mathbf{D}(V)$ *such that*
1) both $Rq_{1!}H \sim 0$ *and* $Rq_{2!}H \sim 0$*;*
(2)$R\mathbf{I}_!H \sim G$.

Remark. Observe that (128) reads as follows: $RP^1_{\pm !}G = 0$, where

$$P^1_\pm : I_\diamondsuit \times (0,\infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \quad : \quad P^1_\pm(a,v,t) = (a, v \pm t), \tag{133}$$

same as in (127).

Proof of this Claim will occupy the next subsection

5.10 Proof of Claim 5.14

5.10.1 Functors r_1 and r_2 and their properties

For $F \in \mathbf{D}(I_\diamondsuit \times \mathbb{R} \times (0,\infty) \times (0,\infty))$ we have natural maps (coming from the adjunction)

$$F \to q_1^! Rq_{1!}F; \quad F \to q_2^! Rq_{2!}F. \tag{134}$$

Let $r_1(F), r_2(F)$ be the cones of these maps so that we have natural maps (in the conventions of [KS, Ch.1.4])

$$r_1(F) \to F[1] \tag{135}$$

$$r_2(F) \to F[1]. \tag{136}$$

We therefore have a composition map

$$r_1 r_2 F \to F[2]. \tag{137}$$

Lemma 5.15 *We have* $Rq_{1!}r_1r_2 = Rq_{2!}r_1r_2 = 0$.

Proof First of all we observe that

$$Rq_{1!}r_1 \sim 0, \quad Rq_{2!}r_2 \sim 0. \tag{138}$$

Indeed, the question boils down to showing that $Rq_{1!}$ applied to (134) yields a quasi-isomorphism $Rq_{1!}F \xrightarrow{\sim} Rq_{1!}q_1^! Rq_{1!}F$.

There is a natural transformation of endofunctors on $\mathbf{D}(I_\diamondsuit \times \mathbb{R} \times (0,\infty))$: $\varepsilon : Rq_{1!}q_1^! \to \mathrm{Id}$ (since $Rq_{1!}$ is left adjoint to $q_1^!$). Since q_1 is a projection along $(0,\infty)$, it is well known that ε is an isomorphism of functors. By [MacLane, Ch.IV.1, Th.1(ii)], there is a diagram

$$\begin{tikzcd} Rq_{1!}F \arrow[r] \arrow[dr, "\mathrm{id}"'] & Rq_{1!}q_1^! Rq_{1!}F \arrow[d] \\ & Rq_{1!}F \end{tikzcd}$$

in which the vertical arrow is induced by ε, which implies that the vertical arrow is an isomorphism, hence, so is the horizontal arrow. This finishes proof of (138).

Secondly, we have a natural quasi-isomorphism

$$r_1 r_2 \sim r_2 r_1. \tag{139}$$

Indeed, let us represent q_1, q_2 as convolution with kernels. Let A, B, C be smooth manifolds. We have the convolution bifunctor $\circ : \mathbf{D}(A \times B) \times \mathbf{D}(B \times C) \to \mathbf{D}(A \times C)$ defined by

$$F \circ G = R\pi_{AC!}(\pi_{AB}^! F \otimes \pi_{BC}^! G). \tag{140}$$

Let $A = \mathbb{R}$, $B_1 = B_2 = (0, \infty)$, $C = \mathbf{pt}$ so that F is a sheaf on $A \times B_1 \times B_2$, $q_1 : A_1 \times B_1 \times B_2 \to A \times B_1 \times C$ is the projection along B_2.

We have $Rq_{1!}F = F \circ \mathbb{Z}_{B_2 \times C}$.

Set $q_1^\diamondsuit G \cong q_1^{-1}G[1] = G \circ \mathbb{Z}_{C \times B_2}[1]$.

Let us construct an isomorphism (natural in F and G)

$$RHom(Rq_{1!}F; G) \xrightarrow{\sim} RHom(F; q_1^\diamondsuit G).$$

Fix one of the two maps $I : \Delta_! \mathbb{Z}_{B_2} \to \mathbb{Z}_{B_2 \times B_2}[1]$ such that the induced map $RP_! \Delta_! \mathbb{Z}_{B_2} \to RP_! \mathbb{Z}_{B_2 \times B_2}[1]$ is an isomorphism, where $P : B_2 \times B_2 \to B_2$ is the projection along the second factor. We have an induced map

$$\alpha : F \xrightarrow{\cong} F \circ \Delta_! \mathbb{Z}_{B_2} \xrightarrow{I} F \circ \mathbb{Z}_{B_2 \times B_2}[1] \xrightarrow{\cong} q_1^\diamondsuit Rq_{1!}F$$

It follows that this map induces an isomorphism

$$Rq_{1!}F \to Rq_{1!}q_1^\diamondsuit Rq_{1!}F. \tag{141}$$

The induced map

$$RHom(Rq_{1!}F; G) \to RHom(q_1^\diamondsuit Rq_{1!}F; q_1^\diamondsuit G) \xrightarrow{-\circ\alpha} RHom(F; q_1^\diamondsuit G) \tag{142}$$

is an isomorphism for all F, G. Indeed, the right arrow is an isomorphism because of (141). The left arrow is an isomorphism because we have an isomorphism of functors $q_1^\diamondsuit G = G \boxtimes \mathbb{Z}[1]$ and the statement now follows from the Kuenneth formula.

Thus we have constructed an adjunction between the functors $q_1^{\diamondsuit}$ and $Rq_{1!}$ in the sense of [MacLane, Ch.IV.1]. In case $G = Rq_{1!}F$, the map (142) sends $\mathrm{id}_{Rq_{1!}F}$ to $q_1^!(\mathrm{id}_{Rq_{1!}F}) \circ \alpha = \alpha$, therefore α is the universal arrow associated to the adjunction (142) in the sense of [MacLane, Ch.IV.1, p.81]; by the uniqueness of an adjoint functor, see [MacLane, Cor.1, Ch.IV.1, p.85] and its proof, this means that α coincides with the "standard" adjunction map (coming from [KS, Ch.3.1]) up to some natural autoequivalence of the functor $q_1^! Rq_{1!}$. This means that we have a canonical isomorphism of functors $q_1^{\diamondsuit} \cong q_1^!$ so that we won't make difference between $q_1^{\diamondsuit}$ and $q_1^!$ We have

$$q_1^! Rq_{1!}F = F \circ (\mathbb{Z}_{B_2 \times C} \circ \mathbb{Z}_{C \times B_2})[1] = F \circ \mathbb{Z}_{B_2 \times B_2}[1]. \tag{143}$$

The above consideration shows that $r_1 F = \mathrm{Cone}\alpha \simeq F \circ \mathcal{L}_1$, where $\mathcal{L}_1 := \mathrm{Cone}(I : \Delta_! \mathbb{Z}_{B_2} \to \mathbb{Z}_{B_2 \times B_2}[1])$.

Analogously, $r_2 F \simeq F \circ \mathcal{L}_2$, where $\mathcal{L}_2 := \mathrm{Cone}(I : \Delta_! \mathbb{Z}_{B_1} \to \mathbb{Z}_{B_1 \times B_1}[1])$.

Therefore,

$$r_1 r_2 F \simeq F \circ [\mathcal{L}_1 \boxtimes \mathcal{L}_2] \simeq r_2 r_1 F,$$

as we wanted.

We now have: $Rq_{1!} r_1 r_2 = 0$ because of (138) and

$$Rq_{2!} r_1 r_2 \overset{(139)}{=} Rq_{2!} r_2 r_1 \overset{(138)}{=} 0. \tag{144}$$

This accomplishes proof of Lemma. □

5.10.2 Construction of the object H and proof of the Claim 5.14 1)

We set $\Phi = \mathbf{I}^! G$ and $H := r_1 r_2(\Phi)$. Lemma 5.15 says that $Rq_{1!} H \sim 0$ and $Rq_{2!} H \sim 0$, which proves part 1) of the Claim 5.14.

5.10.3 Reduction of part 2) of the Claim 5.14

Let us deduce part 2) of the Claim 5.14 from the following statement.

We have a map

$$\iota_H : H = r_1 r_2 \Phi \to \Phi[2],$$

where the right arrow is defined in (137). Let us apply the functor $R\mathbf{I}_!$ to ι_H so as to get a map

$$R\mathbf{I}_! H \to R\mathbf{I}_! \Phi[2] \tag{145}$$

Claim 5.16 *The map (145) is an isomorhpism.*

This Claim implies part 2) of the Claim 5.14. Indeed, we can rewrite (145) as follows.

$$R\mathbf{I}_!H \to R\mathbf{I}_!\Phi[2] = R\mathbf{I}_!\mathbf{I}^!G[2] \xrightarrow{\sim} G[2],$$

where the rightmost arrow is an isomorphism because $\mathbf{I}$ is a smooth fibration with fibers diffeomorphic to $\mathbb{R}^1$.

We now pass to proving Claim 5.16.

5.10.4 Subdivision into 3 cases

The map (145) factors as

$$R\mathbf{I}_!r_1r_2(\Phi) \overset{(135)}{\to} R\mathbf{I}_!r_2(\Phi)[1] \overset{(136)}{\to} R\mathbf{I}_{I!}\Phi[2].$$

As $\mathbf{I}^!G = \Phi$ and by [KS, Prop.1.4.4.(TR3)], the cone of the right arrow is isomorphic to $R\mathbf{I}_!q_2^!Rq_{2!}\mathbf{I}^!G[2]$. Analogously, the cone of the left arrow is $R\mathbf{I}_!q_1^!Rq_{1!}r_2\Phi[1]$ which, by definition of r_2, is the cone of the natural arrow

$$R\mathbf{I}_!q_1^!Rq_{1!}\mathbf{I}^!G \to R\mathbf{I}_!q_1^!Rq_{1!}Rq_2^!Rq_{2!}\mathbf{I}^!G.$$

Thus, isomorphicity of (145) is implied by the following three vanishing statements:

1) $R\mathbf{I}_!q_2^!Rq_{2!}\mathbf{I}^!G \sim 0$

2)$R\mathbf{I}_!q_1^!Rq_{1!}\mathbf{I}^!G \sim 0$;

3)$R\mathbf{I}_!q_1^!Rq_{1!}q_2^!Rq_{2!}\mathbf{I}^!G \sim 0$.

5.10.5 Proof of the 1-st and the 2-nd vanishing

Let $V_2 := I_\diamondsuit \times \mathbb{R} \times (0,\infty)^4$. Let $\pi_1, \pi_2 : V_2$ be given by

$$\pi_1(a, v, \xi_1, \xi_2, \xi_1', \xi_2') = (a, v, \xi_1, \xi_2)$$

and

$$\pi_2(a, v, \xi_1, \xi_2, \xi_1', \xi_2') = (a, v, \xi_1', \xi_2')$$

Let $L_2, \subset V_2$ be a closed subset of the form:

$$L_2 := \{(a, v, \xi_1, \xi_2, \xi_1', \xi_2')|\xi_2 = \xi_2'\};$$

Lemma 5.17 *For any $F \in \mathbf{D}(V)$ we have*

$$q_2^! Rq_{2!}F = R\pi_{2!}(\mathbb{Z}_{L_i} \otimes \pi_2{}^{-1}F).$$

PROOF Similar to proof of (143). $\square$

Let $X_2 := I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R})$. Let $\pi_1^X, \pi_2^X : X_2 \to I_\diamondsuit \times (0,\infty) \times \mathbb{R}$ be the projections along the 3rd and the 2nd factors respectively. Define closed subsets $L_\pm \subset X_2$:

$$L_\pm = \{(a,(s_1,t_1),(s_2,t_2)) \in I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R}) \ : \ s_1 \pm t_1 = s_2 \pm t_2\}$$

Lemma 5.18 *For any $F \in \mathbf{D}(I_\pm \times (0,\infty) \times \mathbb{R})$,*

$$(P_-^1)^{-1} RP_{-!}^1 F = R\pi_{1!}^X(\mathbb{Z}_{L_-} \otimes \pi_2^{X^{-1}} F),$$

where the map P_-^1 was defined in (133).

PROOF. The proof is analogous to the proof of lemma 5.17. $\square$

We now have

$$\begin{aligned} R\mathbf{I}_! q_2^! Rq_{2!}\mathbf{I}^! G[-2] &\sim R\mathbf{I}_! q_2^{-1} Rq_{2!}\mathbf{I}^{-1} G \\ &\sim R\pi'_{1!}(\mathbb{Z}_{L_2} \otimes (\pi'_2)^{-1} G), \end{aligned} \tag{146}$$

where $\pi'_i = \mathbf{I}\pi_i : V_2 \to I_\diamondsuit \times (0,\infty) \times \mathbb{R}$, as easily follows from Lemma 5.17.

Let us define the following map

$$J_2 : I_\diamondsuit \times \mathbb{R} \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R}) \to I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R}) = X_2$$

as follows:

$$J_2(a,v,(s_1,t_1),(s_2,t_2)) = (a, s_1, v+t_1, s_2, v+t_2).$$

Let us also define a map (which is a closed embedding)

$$K_2 : V_2 \to I_\diamondsuit \times \mathbb{R} \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R})$$

as follows:

$$K_2(a,v,\xi_1,\xi_2,\xi'_1,\xi'_2) := (a, v, \frac{\xi_1+\xi_2}{2}; \frac{\xi_1-\xi_2}{2}, \frac{\xi'_1+\xi'_2}{2}; \frac{\xi'_1-\xi'_2}{2}).$$

It follows that $\pi'_1 = \pi_1^X J_2 K_2$; $\pi'_2 = \pi_2^X J_2 K_2$

We can now rewrite (146) as follows:

$$R\mathbf{I}_!q_2^!Rq_{2!}\mathbf{I}^!G[-2] \sim R\mathbf{I}_!q_2^{-1}Rq_{2!}\mathbf{I}^{-1}G$$
$$\sim R\pi_{1!}^X((RJ_{2!}RK_{2!}\mathbb{Z}_{L_2}) \otimes (\pi_2^X)^{-1}G), \quad (147)$$

Let

$$L_2' \subset I_\diamondsuit \times \mathbb{R} \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R})$$

be a closed subset consisting of all points $(a, v, s_1, t_1, s_2, t_2)$ with $s_1 - t_1 = s_2 - t_2$.

It is easy to see that $K_2(L_2) \subset L_2'$ is an open embedding. Indeed, $K_2(L_2)$ consists of all points $(a, v, s_1, t_1, s_2, t_2)$ with $s_1 - t_1 = s_2 - t_2$, $s_1 > |t_1|$, $s_2 > |t_2|$.

Therefore, we have a map $RK_{2!}\mathbb{Z}_{L_2} \to \mathbb{Z}_{L_2'}$ which induces a map

$$R\pi_{1!}^X((RJ_{2!}RK_{2!}\mathbb{Z}_{L_2}) \otimes (\pi_2^X)^{-1}G) \to R\pi_{1!}^X((RJ_{2!}\mathbb{Z}_{L_2'}) \otimes (\pi_2^X)^{-1}G). \quad (148)$$

The cone of this arrow equals

$$R\pi_{1!}^X(M\otimes^{\mathbb{L}}(\pi_2^X)^{-1}G),$$

where

$$M \sim RJ_{2!}\mathbb{Z}_N,$$

and $N = L_2' \backslash K(L_2)$. Let us now show by a pointwise computation that $M \sim 0$. Indeed, let $\alpha := (a, \sigma_1, \tau_1, \sigma_2, \tau_2) \in X_2)$ be a point. Let us consider $H^\bullet(M_\alpha) = H_c^\bullet(J_2^{-1}\alpha; \mathbb{Z})$.

If $\sigma_1 - \tau_1 \neq \sigma_2 - \tau_2$, then $J_2^{-1}\alpha = \emptyset$. If $\sigma_1 - \tau_1 = \sigma_2 - \tau_2 = h$, then $J_2^{-1}\alpha$ gets identified with the set of all $v \in \mathbb{R}$ satisfying: $\underline{\text{either } \sigma_1 \leq |\tau_1 - v| \text{ or } \sigma_2 \leq |\tau_2 - v|}$. Let us denote this set by $Y_\alpha \subset \mathbb{R}$. It follows that Y_α consists of all points v satisfying: $h + v \leq 0$ or $h + v \geq 2\sigma$, where σ is the maximum of σ_1 and σ_2. In other words, Y_α is a disjoint union of two closed rays so that $H_c^\bullet(Y_\alpha, \mathbb{Z}) = 0$. This shows that $M \sim 0$.

The map (148) is therefore a quasiisomorphism. In view of (146), the first vanishing will be shown once we prove that

$$R\pi_{1!}^X((RJ_{2!}\mathbb{Z}_{L_2'}) \otimes (\pi_2^X) - 1G) \sim 0. \quad (149)$$

But $RJ_{2!}\mathbb{Z}_{L_2'} = \mathbb{Z}_{L_-^1}[-1]$, and hence the l.h.s. equals $(P_-^1)^{-1}RP_{-!}^1G[-1]$ which is zero by (128).

The second vanishing is shown analogously.

<u>Proof of the third vanishing</u> Define the following subset

$$L \subset I_\diamondsuit \times \mathbb{R} \times ((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R})) :$$

$$L = \{(a, v, s_1, t_1, s_2, t_2) | (a, v, s_1, t_1), (a, v, s_2, t_2) \in V\};$$

Similar to the proof of the 1-st vanishing, one shows that

$$R\mathbf{I}_! q_1^! Rq_{1!} q_2^! Rq_{2!} \mathbf{I}^! G[-3] \sim R\pi_{1!}^X ((RJ_{2!}\mathbb{Z}_L) \otimes (\pi_2^X)^{-1} G),$$

where

$$J_2 : I_\diamondsuit \times \mathbb{R} \times ((0, \infty) \times \mathbb{R}) \times ((0, \infty) \times \mathbb{R})) \to X_2$$

and

$$\pi_1^X, \pi_2^X : I_\diamondsuit \times \mathbb{R} \times ((0, \infty) \times \mathbb{R}) \times ((0, \infty) \times \mathbb{R})) \to I_\diamondsuit \times (0, \infty) \times \mathbb{R}$$

are the same as in the proof of the 1-st vanishing.

Observe that

$$J_2(L) = \{(a, (s_1, t_1), (s_2, t_2))| \ |t_1 - t_2| < s_1 + s_2\}.$$

the projecion $L \to J_2(L)$ is a smooth fibration whose fibers are diffeomorphic to $\mathbb{R}^1$; we now see that

$$RJ_{2!}\mathbb{Z}_L \sim \mathbb{Z}_{J_2(L)}[-1] \in \mathbf{D}(X_2).$$

We therefore need to show that

$$R\pi_{1!}^X (\mathbb{Z}_{J_2(L)} \otimes (\pi_2^X)^{-1} G) \sim 0$$

The complement to $J_2(L)$ in X_2 consists of two components

$$X_2 \backslash J_2(L) = M_+ \sqcup M_-,$$

where

$$M_+ = \{\{(x, (s_1, t_1), (s_2, t_2))| \ t_1 - t_2 \geq s_1 + s_2\}$$

and

$$M_- = \{\{(x, (s_1, t_1), (s_2, t_2))| \ t_1 - t_2 \leq -s_1 - s_2\}$$

We thus have a distinguished triangle

$$\to R\pi_{1!}(\mathbb{Z}_{J_2(L)} \otimes \pi_2^{-1} F) \to R\pi_{1!}(\mathbb{Z}_{X_2} \otimes \pi_2^{-1} G) \to R\pi_{1!}(\mathbb{Z}_{M_+} \otimes \pi_2^{-1} G) \oplus R\pi_{1!}(\mathbb{Z}_{M_-} \otimes \pi_2^{-1} G) \to$$

which comes from a short exact sequence

$$0 \to \mathbb{Z}_{J_2(L)} \to \mathbb{Z}_{X_2} \to \mathbb{Z}_{M_+} \oplus \mathbb{Z}_{M_-} \to 0.$$

The second term of this triangle is quasi-isomorphic to

$$\pi^{-1} R\pi_! G,$$

where $\pi : I_\diamondsuit \times (0,\infty) \times \mathbb{R} \to I_\diamondsuit$ is the projection. It follows that $R\pi_! G \sim 0$ because π passes through P^1_+ (as well as P^2_-) from (128).

We thus need to show that $R\pi^X_{1!}(\mathbb{Z}_{M_\pm} \otimes (\pi^X_2)^{-1} G) \sim 0$.

Introduce the following subsets $N_\pm \subset I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times \mathbb{R}$:

$$N_+ = \{(a,(s_1,t_1),y)|\ t_1 \geq s_1 + y\}$$

and

$$N_- = \{(a,(s_1,t_1),y)|\ t_1 \leq -s_1 - y\}.$$

Let $q_1 : I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times \mathbb{R} \to (0,\infty) \times \mathbb{R}$ and $q_2 : I_\diamondsuit \times ((0,\infty) \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be projections. We then have

$$R\pi^X_{1!}(\mathbb{Z}_{M_\pm} \otimes (\pi^X_2)^{-1} G) \sim Rq_{1!}(\mathbb{Z}_{N_\pm} \otimes q_2^{-1} RP^1_{\pm !} G) \sim 0$$

because $RP^1_{\pm !} G = 0$ by (128).

This completes the proof of the 3rd vanishing as well as the proof of Claim 5.14

5.11 Finishing proof of Claim 5.13

Let $I_\diamondsuit \times \mathbb{R}_{>0} \times \mathbb{R}$, the target of the map $\mathbf{I}_1$ from (131), have coordinates (a, v, η).

Let $G, H, \mathbf{I}$ be as in Claim 5.14 and let H' be a sheaf on $I_\diamondsuit \times \mathbb{R}_{>0} \times R$ microsupported on the set

$$\bigcup_{\text{“+” and “-”}} (a, v, \eta, \mathbb{R}.d(v \pm \eta) + \mathbb{R}.da). \tag{150}$$

We then have

$$RHom(G, H') \sim RHom(R\mathbf{I}_! H, H') \sim RHom(H, \mathbf{I}^! H').$$

By [KS, Prop.5.4.5(i)], it follows from (150) that

$$S.S.(\mathbf{I}^! H') \ \subset\ \{(a, u, \xi_1, \xi_2, bda + wdu + \tau_1 d\xi_1 + \tau_2 d\xi_2\ :\ \tau_1 = 0 \text{ or } \tau_2 = 0\}. \tag{151}$$

Set $A' = H$, $B' = \mathbf{I}^! H'$.

Let also $q_1, q_2 : I_\diamondsuit \times \mathbb{R} \times (0,\infty) \times (0,\infty) \to I_\diamondsuit \times \mathbb{R} \times (0,\infty)$ be projections as in (132): $q_i(a, u, \xi_1, \xi_2) = (a, u, \xi_i)$.

We then have $Rq_{i!} A' = 0$, $i = 1, 2$, by Claim 5.14,1), and we have the estimate (151) for B'.

Let us identify diffeomorphically $\mathbb{R} \to (0, \infty)$. Under this identification, we have two sheaves A, B on $Y \times \mathbb{R} \times \mathbb{R}$, where $Y = I_\diamondsuit \times \mathbb{R}$, such that

1) $Rp_{1!}A = Rp_{2!}A \sim 0$, where $p_1, p_2 : Y \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are projections;

2) B is microsupported on the set of points $(y, u_1, u_2, \omega + v_1 du_1 + v_2 du_2)$, where $\omega \in T^*_y Y$ $u_1, u_2 \in \mathbb{R}$; $v_1 = 0$ or $v_2 = 0$ (or both).

By Theorem 4.1, $RHom(A, B) = 0$, which finishes the proof of Claim 5.13, as well as Proposition 5.1.

6 Proof of Theorem 3.5

In section 3.6 -3.13, we have constructed objects $\Phi^K, \Phi^{\mathbf{r}_\alpha}, \Phi^{\mathbf{r}_{-\alpha}}$, as well as maps $i_{\Phi^K} : \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to \Phi^K$, $i_{\Phi^{\mathbf{r}_\alpha}} : \mathbb{Z}_{\mathbf{x}_0 \times \mathbf{r}_\alpha}[-2] \to \Phi^{\mathbf{r}_\alpha}$, and $i_{\Phi^{\mathbf{r}_{-\alpha}}} : \mathbb{Z}_{\mathbf{x}_0 \times \mathbf{r}_{-\alpha}}[-2] \to \Phi^{\mathbf{r}_{-\alpha}}$. In order to finish the proof of Theorem 3.5, it now remains to prove:

1) Each of the objects $\Phi^K, \Phi^{\mathbf{r}_\alpha}, \Phi^{\mathbf{r}_{-\alpha}}$ belongs to $\mathcal{C}$, to be done in Sec 6.1.

2) Cones of the maps $i_{\Phi^K}, i_{\Phi^{\mathbf{r}_\alpha}}, i_{\Phi^{\mathbf{r}_{-\alpha}}}$ are in ${}^\perp\mathcal{C}$, to be done in Sec 6.2

We only consider the case of Φ^K (and the map i_{Φ^K}), because the arguments for the remaning cases are very similar.

Proof of 2) is based on the orthogonality criterion of the previous section (Proposition 5.1).

6.1 Proof of $\Phi^K \in \mathcal{C}$.

Consider open subsets $\Sigma_\ell \subset X$, where Σ_ℓ is the union of two neighboring open strips $\mathrm{Int}P_1$, $\mathrm{Int}P_2$ and their common boundary ray ℓ. It is clear that Σ_ℓ form an open covering of X.

Let us consider the restriction estimate $\Phi^K|_{\Sigma_\ell \times \mathbb{C}}$. It suffices to show that

$$SS(\Phi^K|_{\Sigma_\ell \times \mathbb{C}}) \subset \Omega_X \cap T^*(\Sigma_\ell \times \mathbb{C})$$

for each element Σ_ℓ of the open covering. Let us fix the notation: let $\Sigma_\ell = \mathrm{Int}P_1 \sqcup \mathrm{Int}P_2 \sqcup \ell$; let $P'_i := \mathrm{Int}P_i \sqcup \ell$, $i = 1, 2$, be the closure of P_i in Σ_ℓ. Set for brevity

$$F := \Phi^K|_{\Sigma_\ell \times \mathbb{C}}.$$

Finally, we introduce the following sheaf on $\Sigma_\ell \times \mathbb{C}$:

$$\Lambda^{K\pm}_{\Sigma_\ell} := \mathbb{Z}_{\{z \in \Sigma_\ell \,:\, s \pm z \in K\}}.$$

Let us now suppose for definiteness that ℓ goes to the left. As follows from the construction of Φ^K in Sec 3.8.4,3.8.5, we have identifications ($i = 1, 2$):

$$F|_{P_i' \times \mathbb{C}} = (\Lambda_{\Sigma_\ell}^{K+} * S_+ \oplus \Lambda_{\Sigma_\ell}^{K-} * S_-)|_{P_i' \times \mathbb{C}}.$$

as well as a gluing map (43):

$$\Gamma_{\Phi^K}^{P_1 P_2} : (\Lambda_{\Sigma_\ell}^{K+} * S_+ \oplus \Lambda_{\Sigma_\ell}^{K-} * S_-)|_{\ell \times \mathbb{C}} \to (\Lambda_{\Sigma_\ell}^{K+} * S_+ \oplus \Lambda_{\Sigma_\ell}^{K-} * S_-)|_{\ell \times \mathbb{C}}$$

When restricted onto $\Lambda_{\Sigma_\ell}^{K+} * S_+|_{\ell \times \mathbb{C}}$, this map becomes the identity. This readily implies that we have an embedding

$$\Lambda_{\Sigma_\ell}^{K+} * S_+ \hookrightarrow F,$$

whose restriction onto each P_i' is just the identical embedding onto the direct summand. We can construct a surjection $F \to \Lambda_{\Sigma_\ell}^{K-} * S_-$ in a similar way. All together, we get a short exact sequence

$$0 \to \Lambda_{\Sigma_\ell}^{K+} * S_+ \to F \to \Lambda_{\Sigma_\ell}^{K-} * S_- \to 0,$$

The marginal terms of this sequence do clearly have their singular support inside $\Omega_X \cap T^*(\Sigma_\ell \times \mathbb{C})$, cf.(7), hence so does the middle term F. This finishes the proof.

6.2 Proof of orthogonality

In this subsection, we prove that the cone of the map i_{Φ^K} is in ${}^\perp\mathcal{C}$. We will exhibit an increasing exhaustive filtration F of Φ^K such that the map i_Φ factors through $F^0\Phi^K$. Our statement then reduces to showing that $Cone(\mathcal{F}_0 \to F^0\Phi^K)$, as well as all successive quotients of $F^{i+1}\Phi^K/F^i\Phi^K$, $i \geq 0$, belong to ${}^\perp\mathcal{C}$.

6.2.1 Regular sequences

Notation 6.1 Let $\lambda_n\lambda_{n-1}\cdots\lambda_1$ be a nonempty sequence of bounday α-rays.

Call this sequence <u>regular</u> if for each $k \geq 1$ the rays λ_k and λ_{k+1} are different and belong to the closure of a (unique) α-strip P_k, fig.6. We also assume that P_0 is the initial strip (i.e. $\mathbf{x}_0 \in P_0$.

Note that, in general, a ray can occur in a regular sequence several times.

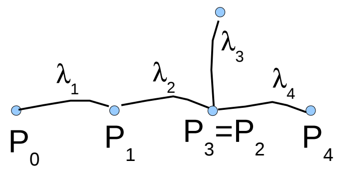

Figure 6**:** A regular sequence – Notation 6.1.

6.2.2 Admissible rays

We will freely use the notation from Sec. 3.8, such as $\mathcal{L}^\alpha$, W, $\Lambda^{K\pm}$.

Let $w \in \mathbf{W}^\alpha$ be of the form $\ell_m\ell_{m-1}\cdots\ell_1\{L \text{ or } R\}$ and let $\ell \in \mathcal{L}^\alpha$ be a boundary α-ray. We call ℓ $\underline{\lambda, w\text{-admissible}}$, if there exists a k such that $\ell = \lambda_k$ and and $\ell_m\ell_{m-1}\cdots\ell_1$ is a subsequence of $\lambda_k\lambda_{k-1}\cdots\lambda_1$ (i.e. there is an increasing sequence $\kappa_1 < ... < \kappa_m$ such that $\ell_1 = \lambda_{\kappa_1}$, ..., $\ell_m = \lambda_{\kappa_m}$).

Remark 6.2 Let $w = l_m l_{m-1}\cdots(L \text{ or } R)$. If $\ell_m = \ell$, then this condition is equivalent to $\ell_m\ell_{m-1}\cdots\ell_1$ being a subsequence of λ; it $\ell_m \neq \ell$, then the condition is equivalent to $\ell\ell_m\ell_{m-1}\cdots\ell_1$ being a subsequence of λ.

6.2.3 Subset $P_{\lambda,w}$

Let P be an α-strip. We define an open subset $P_{\lambda,w} \subset P$ as follows.

1) if every boundary ray of P is not λ, w-admissible, then we set $P_{\lambda,w} := \emptyset$.

2) otherwise (there are λ, w-admissible boundary rays of P) we define $P_{\lambda,w}$ as the union of $\mathrm{Int}P$ with all λ, w-admissible boundary rays of P.

6.2.4 Subsheaves $\Lambda^{K\pm}_{P,\lambda,w}$

Let $j := j^P_{\lambda,w} : P_{\lambda,w} \times \mathbb{C} \to P \times \mathbb{C}$ be the open embedding.

As in Sec.2.11, let $\Lambda^{K\pm}_P = \mathbb{Z}_{\{(z,s):\ z\in P,\ s\pm z\in K\}}$.

Accordingly, we can define subsheaves

$$\Lambda^{K\pm}_{P,\lambda,w} := j_! j^! \Lambda^{K\pm}_P \subset \Lambda^{K\pm}_P \in \mathbf{D}(P \times \mathbb{C}).$$

Observe that $\Lambda^{K\pm}_{P,\lambda,w} = 0$ if P has no λ, w-admissible boundary rays.

6.2.5 Subsheaves $\Phi_P^{K,\lambda} \subset \Phi_P^K$

We have an identification

$$\Phi^K|_P = \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{right}}} S_w * \Lambda_P^{K-} \oplus \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{left}}} S_w * \Lambda_P^{K+}.$$

For each regular sequence λ (where λ stands for $\lambda_n \lambda_{n-1} \ldots \lambda_1$), let us construct a sub-sheaf $\Phi^{K,\lambda} \subset \Phi^K$ as follows. Set

$$\Phi_P^{K,\lambda} := \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{right}}} S_w * \Lambda_{P,\lambda,w}^{K-} \oplus \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{left}}} S_w * \Lambda_{P,\lambda,w}^{K+} \tag{152}$$

We have an obvious embedding

$$\Phi_P^{K,\lambda} \to \Phi_P^K.$$

6.2.6 Sheaves $\Phi_P^{K,\lambda}$ match on the intersections

Let P and P' be two intersecting α-strips; let $\ell = P \cap P'$. We then have two sub-sheaves of Φ_ℓ^K, namely $\Phi_P^{K,\lambda}|_{\ell \times \mathbb{C}}$ and $\Phi_{P'}^{K,\lambda}|_{\ell \times \mathbb{C}}$. Let us check that these two subsheaves do in fact coincide:

Claim 6.3

$$\Phi_P^{K,\lambda}|_{\ell \times \mathbb{C}} = \Phi_{P'}^{K,\lambda}|_{\ell \times \mathbb{C}}$$

Proof Let $w \in \mathbf{W}^\alpha$. Consider the following sheaf: $\Lambda_{P,w}^{\pm} := \Lambda_{P,\lambda,w}^{K\pm}|_{\ell \times \mathbb{C}}$. By definition, $\Lambda_{P,w}^{\pm} = 0$ unless ℓ is λ, w-admissible, in which case $\Lambda_{P,w}^{\pm} = \Lambda^{K\pm}|_\ell$.

Let $\mathbf{W}(\ell, \lambda) \subset \mathbf{W}^\alpha$ be the subset consisting of all w, where ℓ is λ, w-admissible. Let $\mathbf{W}(\ell, \lambda) = \mathbf{W}(\ell, \lambda)_{\mathrm{left}} \sqcup S(\ell, \lambda)_{\mathrm{right}}$, where $\mathbf{W}(\ell, \lambda)_{\mathrm{left}} = \mathbf{W}(\ell, \lambda) \cap \mathbf{W}^\alpha_{\mathrm{left}}$; $\mathbf{W}(\ell, \lambda)_{\mathrm{right}} = \mathbf{W}(\ell, \lambda) \cap \mathbf{W}^\alpha_{\mathrm{right}}$.

It now follows that $\Phi_P^{K,\lambda}|_{\ell \times \mathbb{C}}$, as a subsheaf of $\Phi_P^K|_{\ell \times \mathbb{C}} = \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{left}}} S_w * \Lambda_\ell^{K+} \oplus \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{right}}} S_w * \Lambda_\ell^{K-}$, coincides with the following its direct summand:

$$\Phi_P^{K,\lambda}|_{\ell \times \mathbb{C}} = \Phi(\ell, \lambda) := \bigoplus_{w \in \mathbf{W}(\ell,\lambda)_{\mathrm{left}}} S_w * \Lambda_\ell^{K+} \oplus \bigoplus_{w \in \mathbf{W}(\ell,\lambda)_{\mathrm{right}}} S_w * \Lambda_\ell^{K-}.$$

Analogously, we have an equality

$$\Phi_{P'}^{K,\lambda}|_{\ell \times \mathbb{C}} = \Phi(\ell, \lambda)$$

of subsheaves of

$$\bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{left}}} S_w * \Lambda_\ell^{K+} \oplus \bigoplus_{w \in \mathbf{W}^\alpha_{\mathrm{right}}} S_w * \Lambda_\ell^{K-} = \Phi_{P'}^K|_{\ell \times \mathbb{C}}.$$

It now suffices to check that the sub-sheaf $\Phi(\ell,\lambda)$ is preserved by the gluing map $\Gamma^{PP'}_{\Phi^K}$ from Sec 3.8.5. By definition of $\Gamma^{PP'}_{\Phi^K}$, it suffices to check: let $w \in \mathbf{W}(\ell,\lambda)$ and suppose $\ell w \in \mathbf{W}^\alpha$ (meaning that the leftmost ray of the word w goes in the opposite direction to ℓ); then $\ell w \in \mathbf{W}(\ell,\lambda)$. Indeed, $w \in \mathbf{W}(\ell,\lambda)$, $\ell w \in \mathbf{W}^\alpha$ is equivalent to ℓw being a sub-sequence of λ, which is the same as $\ell w \in \mathbf{W}(\ell,\lambda)$. $\square$

This Claim implies that there is a unique sub-sheaf $\Phi^{K,\lambda} \subset \Phi^K$ such that $\Phi^{K,\lambda}_P = \Phi^{K,\lambda}|_{P\times\mathbb{C}}$ for all α-strips P.

6.2.7 Definition of a filtration on Φ^K

Notation 6.4 Choose and fix an infinite regular sequence

$$\ldots\lambda_n\lambda_{n-1}\ldots\lambda_2\lambda_1 \tag{153}$$

such that

—every ray occurs in this sequence infinitely many times;

—the ray λ_1 is adjacent to the α-strip $\mathbf{P}_0$ containing $\mathbf{x}_0$.

Denote by $\lambda^{(n)}$ the subsequence $\lambda_n\lambda_{n-1}\ldots\lambda_2\lambda_1$.

Set $F^n\Phi^K := \Phi^{K,\lambda^{(n)}}$. Let us check

Claim 6.5 *We have $F^n\Phi^K \subset F^{n+1}\Phi^K$.*

Proof. It suffices to check that $F^n\Phi^K|_{P\times\mathbb{C}} \subset F^{n+1}\Phi^K|_{P\times\mathbb{C}}$ for every strip P (as sub-sheaves of Φ^K_P). It suffices to check that $P(\lambda^{(n)},w) \subset P(\Lambda^{(n+1)},w)$ for all w, which follows from: if a ray ℓ is $\lambda^{(n)},w$-admissible, then ℓ is $\lambda^{(n+1)},w$-admissible. This follows from the definition of λ,w-admissibility. $\square$

Claim 6.6 *Subsheaves $F^n\Phi^K$ form an exhaustive filtration of Φ^K.*

Proof. It suffices to check that $\bigcup F^n\Phi^K|_{P\times\mathbb{C}} = \Phi^K_P$. This is implied by: for every $w \in \mathbf{W}^\alpha$ and every boundary ray ℓ of P, there exists an $n>0$ such that $\ell \in P_{\lambda^{(n)},w}$, equivalently: ℓ is $\lambda^{(n)},w$-admissible. Let us prove this statement. By the construction of λ, every finite sequence of rays, is a subsequence of $\lambda^{(n)}$ for n large enough (because every ray occurs in the sequence $\{\lambda_i\}_{i=1}^\infty$ infinitely many times). Let $w=\ell_m\cdots\ell_1(L \text{ or } R)$, then the sequence $\ell\ell_m\cdots\ell_1$ (if $\ell\neq\ell_m$) or $\ell_m\cdots\ell_1$ is a subsequence of $\lambda^{(n)}$ for some n, meaning that ℓ is λ,w-admisssible. $\square$

6.2.8 Computing $F^1\Phi^K$

In this subsection, P_* denotes the strip adjacent to λ_1 and different from P_0. We assume that λ_1 goes to the right and that P_0 is above P_* (all other cases are treated in a similar way).

Let us give an explicit description of $F^1\Phi^K$. First of all, a ray ℓ is $\lambda^{(1)}, w$-admissible iff $\ell = \lambda_1$ and w is one of the following $L, R, \lambda_1 L$. Therefore, $P_{\lambda^{(1)},w} \neq \emptyset$ iff: P contains λ_1, that is $P = P_0$ or $P = P_*$, and w is one of $L, R, \lambda_1 L$. In each of this cases $P_{\lambda^{(1)},w} = \mathrm{Int}P \cup \lambda_1$.

Thus, $F^1\Phi^K$ is supported on $\Sigma := \mathrm{Int}P_0 \cap \lambda_1 \cap \mathrm{Int}P_*$. Let $P_0' = \mathrm{Int}P_0 \cup \lambda_1$; $P_*' = \mathrm{Int}P_* \cup \lambda_1$. We have

$$F^1\Phi^K|_{P_*' \times \mathbb{C}} = A_* \oplus B_*;$$

$$F^1\Phi^K|_{P_0' \times \mathbb{C}} = A_0 \oplus B_0,$$

where $A_* = S_R * \Lambda^{K-}_{P_*'}$; $A_0 = S_R * \Lambda^{K-}_{P_0'}$; $B_* = S_L * \Lambda^{K+}_{P_*'} \oplus S_{\lambda_1 L} * \Lambda^{K-}_{P_*'}$; $B_0 = S_L * \Lambda^{K+}_{P_0'} \oplus S_{\lambda_1 L} * \Lambda^{K-}_{P_0'}$ The gluing map $\Gamma^{P_0 P_*}_{\Phi_K}$ maps $A_0|_{\lambda_1 \times \mathbb{C}}$ into $A_*|_{\lambda_1 \times \mathbb{C}}$ and $B_0|_{\lambda_1 \times \mathbb{C}}$ into $B_*|_{\lambda_1 \times \mathbb{C}}$, therefore, the sheaves A_* and A_0 get glued into a sheaf A on Σ, and B_* and B_0 into a sheaf B so that $F^1\Phi^K = A \oplus B$. One also sees that $A = S_R * \Lambda^{K-}_{\Sigma}$. Let $j : \mathrm{Int}P_0 \to \Sigma$ be the open embedding.

6.2.9 The map i_Ψ factorizes through $F^1\Phi^K$

Keeping the assumptions of the previous subsection, let us now construct the factorization of the map $i_\Psi : \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to \Phi^K$ through $F^1\Phi^K$. The cases when λ_1 goes to the left of P_* is above P_0 are treated in a similar way.

Let $j : \mathrm{Int}P_0 \times \mathbb{C} \to X \times \mathbb{C}$ be the open embedding. By definition, i_Ψ factors as

$$\mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to j_!(S_L * \Lambda^{K+}_{\mathrm{Int}P_0} \oplus S_R * \Lambda^{K-}_{\mathrm{Int}P_0}) \to \Phi^K, \tag{154}$$

where the first arrow is induced by the following maps in $\mathbf{D}(\mathrm{Int}P_0 \times \mathbb{C})$:

$$\iota_L : \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to \mathbb{Z}_{\{(z,s)|z \in \mathrm{Int}P_0, s+z \in \mathbf{x}_0 + K\}} = S_L * \Lambda^{K+}_{\mathrm{Int}_{P_0}};$$

$$\iota_R : \mathbb{Z}_{\mathbf{x}_0 \times K}[-2] \to \mathbb{Z}_{\{(z,s)|z \in \mathrm{Int}P_0, s-z \in -\mathbf{x}_0 + K\}} = S_R * \Lambda^{K-}_{\mathrm{Int}_{P_0}},$$

which are induced by the closed codimension 2 embeddings of the corresponding sets.

The right arrow in (154) factors through $F^1\Phi^K$ as follows. Let as decompose $j = j_1 j_0$, where $j_0 : \mathrm{Int}P_0 \times \mathbb{C} \to \Sigma \times \mathbb{C}$ and $j_1 : \Sigma \times \mathbb{C} \to X \times \mathbb{C}$ are the open embeddings. We have natural maps $i_A : j_{0!}(S_L * \Lambda^{K+}_{\mathrm{Int}P_0}) \to A$; $i_B : j_{0!}(S_R * \Lambda^{K-}_{\mathrm{Int}P_0} \to B$. Whence a map

$$i_A \oplus i_B : j_{0!}(S_L * \Lambda^{K+}_{\mathrm{Int}P_0} \oplus S_R * \Lambda^{K-}_{\mathrm{Int}P_0}) \to A \oplus B = F^1\Phi^K|_{\Sigma \times \mathbb{C}}.$$

The right arrow in (154) is then obtained by applying $j_{1!}$ to $i_A \oplus i_B$. For future references, let us consider $\mathrm{Cone}(\mathbb{Z}_{\mathbf{x}_0\times K}[-2] \to F^1\Phi^K)$, which is supported on $\Sigma \times \mathbb{C}$. We now see that

$$\mathrm{Cone}(\mathbb{Z}_{\mathbf{x}_0\times K}[-2] \to F^1\Phi^K)|_{\Sigma\times\mathbb{C}}$$

is isomorphic to the Cone of the following composition map in $\mathbf{D}(\Sigma \times \mathbb{C})$:

$$\mathbb{Z}_{\mathbf{x}_0\times K}[-2] \to j_{0!}(S_L * \Lambda^{K+}_{\mathrm{Int}P_0} \oplus S_R * \Lambda^{K-}_{\mathrm{Int}P_0}) \to A \oplus B, \tag{155}$$

where the right arrow is $i_A \oplus i_B$, and the left arrow is induced by $\iota_L \oplus \iota_R$.

6.2.10 Computing successive quotients of the filtration

. Let us compute the quotients $\mathcal{G}^n := F^n\Phi^K/F^{n-1}\Phi^K$, $n \geq 2$. Our computation will result in decompositions (158), (159)

For that purpose, we choose an α strip P and compute the restriction $\mathcal{G}^n_P := F^n\Phi^K/F^{n-1}\Phi^K|_P$.

Set

$$P(n,w) := P_{\Lambda^n,w} \backslash P_{\Lambda^{n-1},w} \subset P.$$

$P(n,w)$ is a locally closed subset of P so that we can define the following sheaves on $P \times \mathbb{C}$:

$$\Lambda^{K\pm}_{P(n,w)} = \mathbb{Z}_{\{(z,s)|z\in P(n,w); s\pm z\in K\}}.$$

We have an identification

$$\mathcal{G}^n_P = \bigoplus_{w\in\mathbf{W}^\alpha_{\mathrm{left}}} S_w * \Lambda^{K+}_{P(n,w)} \oplus \bigoplus_{w\in\mathbf{W}^\alpha_{\mathrm{right}}} S_w * \Lambda^{K-}_{P(n,w)}.$$

Let us now describe the sets $P(n,w)$. Below, for a $w \in \mathbf{W}^\alpha$, we set $\mathrm{trim}(w)$ to be the word w with its rigthmost letter (L or R) removed.

<u>Step 1</u> Consider all the situations when $\mathrm{Int}P \subset P(n,w)$

This occurs iff $\mathrm{Int}P$ is part of $P_{\lambda^{(n)},w}$ but not $P_{\lambda^{(n-1)},w}$. This is equivalent to the following:

Condition I: n is the minimal number satisfying:

(1) λ_n is a boundary ray of P;

(2) $\mathrm{trim}(w)$ is a subsequence of $\lambda^{(n)}$.

Let us reformulate these conditions. Introduce the following notation. For a word w set $M(w)$ to be the minimal number such that $\mathrm{trim}(w)$ is a subsequence of $\lambda^{(M(w))}$. For a word w , $w \neq \{R\}, \{L\}$, we also write $w = lw'$, where l is the leftmost ray of w.

Let us split our consideration into two cases:

A) $l = \lambda_n$, (meaning that $\mathrm{trim}(w)$ is non-empty);

B)$\mathrm{trim}(w)$ is empty or $l \neq \lambda_n$.

Case A). The combination Condition I+Case A) is equivalent to the following combination:

A) (i.e. $l = \lambda_n$), and

A1) $M(w) = n$, and

A2) λ_n is a boundary ray of P.

It follows that <u>given a boundary ray r of P different from λ_n, such an r is not $\lambda^{(n)}, w$-admissible</u>: the admissibility would mean that the word rw is a subsequence of $\lambda^{(n)}$ (see remark 6.2)); since $r \neq \lambda_n$, rw is also a subsequence of $\lambda^{(n-1)}$, which implies $M(w) < n$, contradiction.

Thus, in this case we have $P(n, w) = \mathrm{Int}P \cup \lambda_n$.

Case B)

Let us give an equivalent reformulation of the combination.

Lemma 6.7 *Condition I and case B). It is equivalent to the following combination:*

B) and

B1) λ_n is a boundary strip of P, and

B2) $M(\lambda_n w) = n$, and

B3) If $\mathrm{trim}(w)$ is non-empty, then l is not a boundary ray of P, and, finally,

B4) $M(rw) \geq n$ for any boundary ray r of P.

Proof. Let us first derive B1)-B4) from Condition I and B):

B1) is just the condition (1);

B2): (2) and B) imply $M(\lambda_n w) \leq n$. If $M(\lambda_n w) < n$, then n is not the minimal number satisfying (1) and (2);

Violation of B3) implies that $n - 1$ satisfies (1) and (2) — contradiction.
Violaton of B4) implies that $M(rw) < n$; since the number $M(rw)$ satisfies (1) and (2), we have a contradiction.

Let us now derive Condition I from B) and B1)-B4).

B1,B2 imply that n satisfies (1) and (2). Suppose n is not minimal, i.e there exists $p < n$ such that λ_p is a boundary ray of P and $M(w) \leq p$. B3 implies that λ_p is different from the leftmost ray of w. Therefore, $M(\lambda_p w) \leq p$, which is prohibited by B4. □

Let us now introduce a one more condition $B5$.

Let P_{n-1} be (a unique) α-strip which is adjacent to both λ_n and λ_{n-1}. Let P_* be the other α-strip adjacent to λ_n.

The condition B5 is as follows:

B5)$P = P_*$.

Let us prove that

Lemma 6.8 *Combination Condition I+ B is equivalent to the combination $B, B2, B5$.*

Proof. Let us first prove that B,B1-B4 imply B5. Since λ_n is a boundary ray of P, the only alternative to B5 is $P = P_{n-1}$. Then λ_{n-1} is a boundary ray of P and $M(\lambda_{n-1}w) \leq n-1$ which contradicts to B4.

Let us prove that $\underline{B, B2, B5 \text{ imply } B1, B3, B4}$.

B1: By B5 $P_* = P$, and λ_n is a boundary ray of P_*;

B3,B4: B2 implies that for all $p \in [M(w); n-1]$, $\lambda_p \neq \lambda_n$. This implies that $\underline{P_* \text{ is not adjacent to any of } \lambda_p \text{ with } p \in [M(w); n-1]}$ Indeed, suppose P_* is adjacent to such a λ_p. Consider the graph Γ whose vertices are strips and and whose edges are rays. We have two non-intersecting paths between P_{n-1} and P_*: one of them is λ_n, we also have a path between P_{n-1} and P_* in the connected graph composed of the edges $\lambda_{n-1}\lambda_{n-2}, \cdots, \lambda_p$, which contradicts to Γ being a tree.

The just proven statement implies B3 and

B4') $M(rw) > n$ for every boundary ray of $P = P_*$ which differs from $\lambda^{(n)}$.

Finally, B2) and B4') imply B4), which finishes the proof. □

Finally, we conclude from B4', that in the situation Condition 1+B we have:

$$P(n, w) = \mathrm{Int}P \sqcup \lambda_n.$$

<u>Step 2</u> Let us now examine the case (call it case C) when $P(n, w)$ is a non-empty union of boundary rays of P. Since $P_{\lambda^{(n-1)},w} \subset P_{\lambda^{(n)},w}$, this is equivalent to $P_{\lambda^{(n-1)},w}$ being a proper (in particular, non-empty) subset of $P_{\lambda^{(n)},w}$. As follows from definitions, this is equivalent to:

i') there is a $\lambda^{(n-1)}, w$-admissible ray of P;

ii') There exists a boundary ray r of P such that r is $\lambda^{(n)}, w$-admissible, but not $\lambda^{(n-1)}, w$-admissible.

By Remark 6.2, the condition i') is equivalent to:

i") there exists a boundary ray r of P such that either r is the leftmost ray of w and $M(w) \leq n-1$, or r is not the leftmost ray of w and $M(rw) \leq n-1$.

In any case, i') implies that $M(w) \leq n-1$.

Also by Remark 6.2, the condition ii') is equivalent to the following one

ii") There exists a boundary ray r of P such that either

a) r is not the leftmost ray of w and $M(rw) = n$;

or

b) r is the leftmost ray of w and $M(w) = n$.

The case b) contradicts to i'), which implies $M(w) \leq n-1$.

The condition a) implies $r = \lambda_n$ and hence λ_n is one and the only ray in $P_{\lambda^{(n)},w}$.

We thus can reformulate:

The case C occurs iff

i') holds and

ii-α) λ_n is a boundary ray of P;

ii-β)λ_n is not the leftmost ray of w;

ii-γ) $M(\lambda_n w) = n$.

In the case C we have $P(n, w) = \lambda^n$.

From ii-γ we conclude that

$$\lambda_p \neq \lambda_n \quad \text{for all } p \in [M(w); n-1]. \tag{156}$$

The condition i' is equivalent to

$$\exists p \in [M(w), n-1] \quad : \; \lambda_p \; \text{ is adjacent to } \; \bar{\varsigma}. \tag{157}$$

Let us show that $P = P_{n-1}$:

Indeed, by ii-α, the only alternative is $P = P_*$. In this case, analogously to the proof of B5$\Rightarrow$B4, the property (156) implies that P_* is not adjacent to any of λ_p with $p \in [M(w); n-1]$, and that contradicts (157).

Thus, we have the following condition which is equivalent to i' and ii' (the proof of the converse is trivial):

C1) $P = P_{n-1}$; λ_n is not the leftmost ray of w and $M(\lambda_n w) = n$.

In this case $P(n,w) = \lambda_n$.

Let us summarize our findings. Introduce the following notation. Let $\mathbf{W}^{\alpha}_{n,\text{left}}$ be the set of all words w in $\mathbf{W}^{\alpha}_{\text{left}}$ such that the leftmost ray of w is not λ_n and $M(\lambda_n w) = n$. Let $\mathbf{W}^{\alpha}_{n,\text{right}}$ be the similar thing.

We then have the following three cases when the set $P(n,w)$ is non-empty:

– Conditions $A, A1, A2$ is satisfied. Equivalently, the following conditions are the case:

a1) $P = P_{n-1}$ or $P = P_*$;

a2) $w = \lambda_n u$, where $u \in \mathbf{W}^{\alpha}_{n,\text{left}}$ if $\lambda_n \in \mathcal{L}_{\text{right}}$, and $u \in \mathbf{W}^{\alpha}_{n,\text{right}}$ if $\lambda_n \in \mathcal{L}_{\text{left}}$.

In this situation $P(n,w) = \text{Int}P \cup \lambda_n$.

— B,B2,B5 are satisfied. Equivalently: $P = P_*$; $w \in \mathbf{W}^{\alpha}_{n,\text{left}}$ if $\lambda_n \in \mathcal{L}_{\text{right}}$, and $w \in \mathbf{W}^{\alpha}_{n,\text{right}}$ if $\lambda_n \in \mathcal{L}_{\text{left}}$. Then $P(n,w) = \text{Int}P_* \cup \lambda_n$.

— C1 is satisfied. Equivalently:

b1) $P = P_{n-1}$;

b2) $w \in \mathbf{W}^{\alpha}_{n,\text{left}}$ if $\lambda_n \in \mathcal{L}_{\text{right}}$, and $w \in \mathbf{W}^{\alpha}_{n,\text{right}}$ if $\lambda_n \in \mathcal{L}_{\text{left}}$.

In this situation, we have $P(n,w) = \lambda_n$.

6.2.11 Description of $\mathcal{G}_n$

In particular, we see that the sheaf $\mathcal{G}_n = F^n\Phi^K/F^{n-1}\Phi^K$ is supported on the union $\text{Int}P_{n-1} \cap \lambda_n \cap \text{Int}P_*$.

Let $P'_* := \text{Int}P_* \cup \lambda_n$. We will now describe the restriction of $\mathcal{G}_n$ onto P'_*.

Suppose that $\lambda_n \in \mathcal{L}_{\text{left}}$. We then have

$$\mathcal{G}_n|_{P'_*\times\mathbb{C}} = \bigoplus_{w\in\mathbf{W}^{\alpha}_{n,\text{right}}} (S_w * \Lambda^{K-}_{P'_*} \oplus S_{\lambda_n w} * \Lambda^{K+}_{P'_*}) \oplus \bigoplus_{w\in\mathbf{W}^{\alpha}_{n,\text{left}}} S_w * \Lambda^{K+}_{P'_*}.$$

For $w \in \mathbf{W}^{\alpha}_{n,\text{right}}$, we denote

$$B^{P'_*}_w := S_w * \Lambda^{K-}_{P'_*} \oplus S_{\lambda_n w} * \Lambda^{K+}_{P'_*};$$

for $w \in \mathbf{W}^{\alpha}_{n,\text{left}}$, we set

$$A^{P'_*}_w := S_w * \Lambda^{K+}_{P'_*}.$$

so that we can rewrite

$$\mathcal{G}_n = \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}} B_w^{P'_*} \oplus \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}} A_w^{P'_*}.$$

In the case $\lambda_n \in \mathcal{L}_{\mathrm{right}}$, change all signs and all orientations: we have

$$\mathcal{G}_n = \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}} B_w^{P'_*} \oplus \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}} A_w^{P'_*},$$

where for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}$, we denote

$$B_w^{P'_*} := S_w * \Lambda^{K+}_{P'_*} \oplus S_{\lambda_n w} * \Lambda^{K-}_{P'_*};$$

for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}$, we set

$$A_w^{P'_*} := S_w * \Lambda^{K-}_{P'_*}.$$

(2) Let P'_{n-1} be the union of the interior of P_{n-1} and λ_n.

We then have in the case $\lambda_n \in \mathcal{L}_{\mathrm{left}}$:

$$\mathcal{G}_n|_{P'_{n-1} \times \mathbb{C}} = \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}} B_w^{P'_{n-1}} \oplus \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}} A_w^{P'_{n-1}},$$

where for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}$ we set

$$B_w^{P'_{n-1}} := S_w * \Lambda^{K-}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K+}_{P'_{n-1}};$$

for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}$ we set

$$A_w^{P'_{n-1}} := S_w * \Lambda^{K+}_{\lambda_n}.$$

If $\lambda_n \in \mathcal{L}_{\mathrm{right}}$, then one has to change all the directions and all the signs:

$$\mathcal{G}_n|_{P'_{n-1} \times \mathbb{C}} = \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}} B_w^{P'_{n-1}} \oplus \bigoplus_{w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}} A_w^{P'_{n-1}},$$

where for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{left}}$ we set

$$B_w^{P'_{n-1}} := S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{P'_{n-1}};$$

for $w \in \mathbf{W}^{\alpha}_{n,\mathrm{right}}$ we set

$$A_w^{P'_{n-1}} := S_w * \Lambda^{K-}_{\lambda_n}.$$

Analyzing the gluing maps, we see that

$$A_w^{P'_*}|_{\lambda_n\times\mathbb{C}} = A_w^{P'_{n-1}}|_{\lambda_n\times\mathbb{C}}$$

as sub-sheaves of $\mathcal{G}_n|_{\lambda_n\times\mathbb{C}}$ and similarly for B_w. Therefore, we have well defined sub-sheaves A_w, B_w of $\mathcal{G}_n$: A_w is defined by the conditions:

$$A_w|_{P'_*\times\mathbb{C}} = A_w^{P'_*};$$

$$A_w|_{P'_{n-1}\times\mathbb{C}} = A_w^{P'_{n-1}},$$

and similarly for B_w.

Let us stress that $B_w|_{\mathrm{Int}P_{n-1}\cup\lambda_n\cup\mathrm{Int}P_n}$ is *not* isomorphic to the direct sum of $S_w * \Lambda^{K+}_{\mathrm{Int}P_{n-1}\cup\lambda_n\cup\mathrm{Int}P_*}$ and $S_{\lambda_n w} * \Lambda^{K-}_{\mathrm{Int}P_{n-1}\cup\lambda_n\cup\mathrm{Int}P_*}$

We have in the case $\lambda_n \in \mathcal{L}_{\mathrm{left}}$:

$$\mathcal{G}_n = \bigoplus_{w\in\mathbf{W}^\alpha_{n,\mathrm{right}}} B_w \oplus \bigoplus_{w\in\mathbf{W}^\alpha_{n,\mathrm{left}}} A_w; \tag{158}$$

if $\lambda_n \in \mathcal{L}_{\mathrm{left}}$, then we have:

$$\mathcal{G}_n = \bigoplus_{w\in\mathbf{W}^\alpha_{n,\mathrm{left}}} B_w \oplus \bigoplus_{w\in\mathbf{W}^\alpha_{n,\mathrm{right}}} A_w. \tag{159}$$

6.2.12 Reduction of the orthogonality property

As was explained in Sec 6.2.9, the map map i_{Φ^K} factors as $\mathbb{Z}_{\{z=\mathbf{x}_0,s\in K\}}[-2] \to F^1\Phi^K \to \Phi^K$.

It therefore suffices to prove that A_w, B_w belong to ${}^\perp\mathcal{C}^\Sigma$, where $\Sigma = \mathrm{Int}P_{n-1}\cup\lambda_n\cup\mathrm{Int}P_*$ and that and $\mathrm{Cone}(\mathbb{Z}_{\{z=\mathbf{x}_0,s\in K\}}[-2] \to F^1\Phi^K) \in {}^\perp\mathcal{C}^X$. As was explained in Sec 6.2.8, the sheaf $F^1\Phi^K$ is supported on $\Sigma' := \mathrm{Int}P_0 \cap \lambda_1 \cap \int P_*$, so that it suffices to show that

$$\mathrm{Cone}(\mathbb{Z}_{\{z=\mathbf{x}_0,s\in K\}}[-2] \to F^1\Phi^K)|_{\Sigma'\times\mathbb{C}} \in {}^\perp\mathcal{C}^{\Sigma'}$$

We do it in the rest of the section.

6.2.13 Conventions

Suppose that the ray λ_n is directed to the right so that $\lambda_n = \hat{c}(\lambda_n) + \mathbb{R}_{>0}.e^{i\alpha}$; the case of the opposite direction is similar.

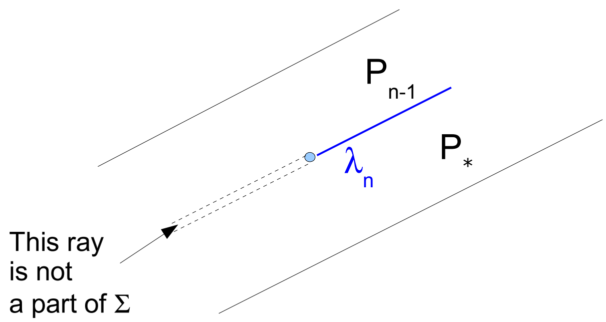

Figure 7

Assume the situation is as on figure 7, namely, we assume that P_{n-1} is above λ_n and P_* is below λ_n. The argument for the opposite situation is similar.

Define

$$U := \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} \in \Sigma \ : \ x, y \in \mathbb{R} \text{ and } x > 0\};$$

$$V := \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} \in \Sigma \ : \ x, y \in \mathbb{R} \text{ and } x \leq 0\}.$$

6.2.14 Orthogonality of A_w

Because of the assumptions above, we have $w \in \mathbf{W}^{\alpha}_{\text{right}}$ and

$$A_w = S_w * \Lambda^{K-}_{P'_*},$$

where

$$\Lambda^{K-}_{P'_*} = \mathbb{Z}_{\{(z,s): z \in P'_*; s - z \in K\}}.$$

We have a short exact sequence:

$$0 \to S_w * \Lambda^{K-}_{U \cap P'_*} \to A_w \to S_w * \Lambda^{K-}_{V \cap P'_*} \to 0, \tag{160}$$

where $\Lambda^{K\pm}_U := \mathbb{Z}_{(s,z)|z \in U; s \pm z \in K\}}$ and similarly for $\Lambda^{K\pm}_{V \cap P'_*}$.

(Note that in the case $\lambda_n \in \mathcal{L}_{\text{left}}$ we need to consider a sequence analogous to (160) with Λ^{K-} instead of Λ^{K+}.)

The problem is thus reduced to proving that

$$S_w * \Lambda^{K-}_{U \cap P'_*}, \quad S_w * \Lambda^{K-}_{V \cap P'_*} \quad \in \quad {}^{\perp}\mathcal{C}^{\Sigma}. \tag{161}$$

Now let us use the following consideration: if $j : U \times \mathbb{C} \to \Sigma \times \mathbb{C}$ is an open inclusion and if $F \in {}^{\perp}\mathcal{C}^U$, then $j_! F \in {}^{\perp}\mathcal{C}^{\Sigma}$ bacause $RHom(j_!F;G) \cong RHom(F;G|_{U\times\mathbb{C}})$. In application to the situation at hand, this allows us to reduce (161) to proving

$$S_w * \Lambda^-_{U\cap P'_*}|_U \in {}^{\perp}\mathcal{C}^U \tag{162}$$

and

$$S_w * \Lambda^-_V|_{P_*} \in {}^{\perp}\mathcal{C}^{P_*} \tag{163}$$

which we are going to do using Proposition 5.1.

PROOF OF (162). Denote $F := S_w * \Lambda^-_{U\cap P'_*}|_U$. We have $F = \mathbb{Z}_S$, where $S = \{(z,s) : z \in U \cap P'_*, s - z \in \hat{c}(w) + K\}$.

Next, $U = \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} | x > 0; y \in I\}$, where I is a generalized open interval containing 0, so that U is a generalized strip and we can apply Proposition 5.1.

We have $U \cap P'_* = \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} | x > 0; y \geq 0; y \in I\}$.

Let us now check that F satisfies all the assumptions of Prop. 5.1, which will show that $F \in {}^{\perp}\mathcal{C}^U$.

Namely, we need to show: a) the map $\mathbb{Z}_{\mathbf{r}_\alpha} * F \to \mathbb{Z}_{\{0\}} * F = F$, induced by the embedding $0 \in \mathbf{r}_\alpha$, is an isomorphism,

b) $RP_{+!}F = 0$;

c) $RP_{-!}F = 0$.

Proof of a) is easy: the word w contains at least one letter, hence S_w is a convolution of ≥ 1 sheaves of the type $\mathbb{Z}_{\{s \in a+K\}}$, $a \in \mathbb{C}$. But the map $\beta : \mathbb{Z}_{\mathbf{r}_\alpha} * \mathbb{Z}_{\{s\in a+K\}} \xrightarrow{\cong} \mathbb{Z}_0 * \mathbb{Z}_{\{s\in a+K\}}$, induced by the inclusion $0 \in \mathbf{r}_\alpha$, is an isomorphism.

Proof of b) It suffices to check that $(RP_{+!}F)_t = 0$ for every point $t \in \mathbb{C}$. We have $(R^\bullet P_{+!}F)_t = H^\bullet_c(P_+^{-1}t \cap S; \mathbb{Z})$. Denote $W_t := P_+^{-1}t \cap S$. The space W_t consists of all points (z,s), where $z \in U \cap P'_*$; $s + z \in K$; $s - z = t$. Since $s = z + t$, we can exclude s: the space W_t gets identified with a closed subset $W'_t \subset U$ consisting of all points $z \in U \cap P'_*$ such that $2z + t \in \hat{c}(w) + K$. Let us write $\hat{c}(w) - t - 2\hat{c}(\lambda_n) = 2(x_0 e^{i\alpha} + y_0 e^{-i\alpha})$. We then see that W'_t consists of all points $\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha}$, where $x > 0; y \geq 0; y \in I; x \geq x_0; y \geq y_0$. It is now easy to see that for all x_0, y_0, we have $H^\bullet_c(W_t, \mathbb{Z}) = 0$.

Proof of c) Similar to above, we need to show that $H^\bullet_c(V_t; \mathbb{Z}) = 0$, where $V_t = P_-^{-1}t \cap S$, for all $t \in \mathbb{C}$. If $t \notin \hat{c}(w) + K$, $V_t = \emptyset$. Otherwise, V_t gets identified with $U \cap P'_*$ i.e. the set of all points $(x,y) : x > 0; y \geq 0; y \in I$. The statement now follows.

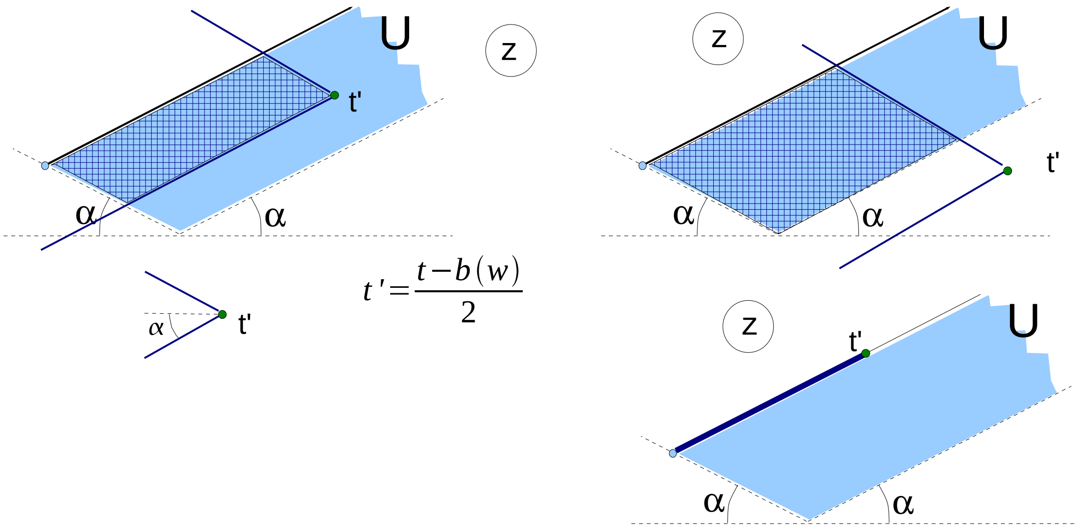

Figure 8: Proof of (162), part b).

Proof of (163). Set $G_1 := S_w * \Lambda^-_{V\cap P'_*}$. We have

$$V \cap P'_* = \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} | x \leq 0; y \in I; y > 0\}.$$

In particular, $V \cap P'_* \subset \mathrm{Int}P_*$. Similar to above, it suffices to show that $G := G_1|_{\mathrm{Int}P_*\times\mathbb{C}} \in \mathcal{C}^{\mathrm{Int}P_*}$. Since $\mathrm{Int}P_*$ is a generalized strip, we can apply Proposition 5.1. Let us check the assumptions of this Proposition.

We have $G = \mathbb{Z}_T$, where $T \subset \mathrm{Int}P_* \times \mathbb{C}$ consists of all points (z, s), where $z = \hat{(}\lambda_n) + xe^{i\alpha} + ye^{-i\alpha}$; $x \leq 0$; $y < 0$; $y \in I$; $s - z \in \hat{c}(w) + K$.

a) We see that the natural map $\mathbb{Z}_{\mathbf{r}_\alpha} * G \to \mathbb{Z}_0 * G = G$ is clearly an isomorphism.

b)$RP_{+!}G|_t = 0$ for all t. This is equivalent to $H^\bullet_c(W'_t, \mathbb{Z}) = 0$, where $W'_t = P_+^{-1}t \cap T$. Similar to above, the set W'_t gets identified with the set of all (x, y), where $x \leq 0$; $y < 0$; $y \in I$; $x \geq x_0; y \geq y_0$ for some numbers x_0, y_0, the statement follows.

c) We need to chech that $H^\bullet_c(V'_t, \mathbb{Z}) = 0$, where $V'_t = P_-^{-1}t \cap T$. We see that $V'_t = \emptyset$ for all $t \notin \hat{c}(w) + K$. Otherwise, V'_t gets identified with T.

6.2.15 Orthogonality of B_w

Let U, V be the same subsets of Σ as above. We see that $\Sigma\backslash U = V = V_1 \sqcup V_2$, where $V_1 \subset \mathrm{Int}P_*$, $V_2 \subset \mathrm{Int}P_{n-1}$.

For any locally closed subset $C \subset \Sigma$ we set $B_C := B_w \otimes \mathbb{Z}_{C\times\mathbb{C}_s} \in \mathbf{D}(\Sigma \times \mathbb{C}_s)$. We then have a distinguished triangle

$$\xrightarrow{+1} B_{V_1} \oplus B_{V_2} \to B_w \to B_U \xrightarrow{+1} .$$

Similarly to section 6.2.14, it suffices to prove that

$$B'_U := B_U|_{U\times\mathbb{C}} \in {}^{\perp}\mathcal{C}^U; \tag{164}$$

$$B_{V_1}|_{\overset{\circ}{P}_*\times\mathbb{C}} \in {}^{\perp}\mathcal{C}^{\mathrm{Int}P_*}, \tag{165}$$

$$B_{V_2}|_{\overset{\circ}{P}_{n-1}\times\mathbb{C}} \in {}^{\perp}\mathcal{C}^{\mathrm{Int}P_{n-1}}, \tag{166}$$

It is clear that U, V_1,and V_2 are generalized strips so that we can apply Prop. 5.1.

Proof of (164) Let $\mathbf{P}_1 := U \cap P_{n-1}$; $\mathbf{P}_2 := U \cap P_*$ so that $\mathbf{P}_1, \mathbf{P}_2 \subset U$ are closed subsets and $\mathbf{P}_1 \cap \mathbf{P}_2 = \lambda_n$.

As above, we have

$$U = \{\hat{c}(\lambda_n) + xe^{i\alpha} + ye^{-i\alpha} | x > 0; y \in I\},$$

where $I \subset \mathbb{R}$ is a generalized open interval containing 0. The subset $\mathbf{P}_1$ is given by $y \geq 0$, and $\mathbf{P}_2$ by $y \leq 0$.

We have identifications

$$B_1 := B'_U|_{\mathbf{P}_1\times\mathbb{C}} = S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_1};$$

$$B_2 := B'_U|_{\mathbf{P}_2\times\mathbb{C}} = S_w * \Lambda^{K+}_{\mathbf{P}_2} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_2}.$$

Whence induced identifications

$$B_1|_{\lambda_n\times\mathbb{C}} = S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n} \tag{167}$$

$$B_2|_{\lambda_n\times\mathbb{C}} = S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n} \tag{168}$$

The gluing map

$$B_1|_{\lambda_n\times\mathbb{C}} \to B_2|_{\lambda_n\times\mathbb{C}}$$

is induced by $\Gamma^{P_{n-1}P_*}_{\Phi^K}$ and equals

$$\Gamma = \mathrm{Id} + n \in \mathrm{End}(S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n}),$$

where the only non-zero component of n is

$$n^{+-} : S_w * \Lambda^{K+}_{\lambda_n} \to S_w * S_{\lambda_n} * \Lambda^{K-}_{\lambda_n} = S_{\lambda_n w}\Lambda^{K-}_{\lambda_n}$$

is defined by means of the map $\nu^K_{\lambda_n}$ from (46).

Let $i_k : \mathbf{P}_k \to U$, $k = 1,2$ and $i_0 : \lambda_n \to U$ be closed embeddings. Denote by $\iota_1 : i_{1!}B_1 \to i_{0!}(S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n})$ the natural isomorphism coming from the identification (167). Similarly, we have a map $\iota_2 : i_{2!}B_1 \to i_{0!}(S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n})$, coming from (168). We can rewrite the above consideration in terms of the following short exact sequence of sheaves of abelian groups

$$0 \to B'_U \to i_{1!}B_1 \oplus i_{2!}B_2 \to i_{0!}(S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n}) \to 0. \tag{169}$$

Where the left arrow is induced by the direct sum of the obvious restriction maps and the right arrow is $-\Gamma\iota_1 \oplus \iota_2$. Let us denote the components of this map

$$-\mathrm{Id} : i_{0!}S_w * \Lambda^{K+}_{\lambda_n} \to i_{0!}S_w * \Lambda^{K+}_{\lambda_n};$$

$$-\nu : i_{0!}S_w * \Lambda^{K+}_{\lambda_n} \to i_{0!}S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n};$$

$$-r_1 : i_{1!}S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_1} \to i_{0!}S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n};$$

$$r_2^+ : i_{2!}S_w * \Lambda^{K+}_{\mathbf{P}_2} \to i_{0!}S_w * \Lambda^{K+}_{\lambda_n};$$

$$r_2^- : i_{2!}S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_2} \to i_{0!}S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n}.$$

Consider the complex B'' composed of the 2 last terms of the sequence (169), which is quasi-isomorphic to B'_U. This complex has a filtration by the following subcomplexes:

F^1B'' is as follows:

$$i_{0!}S_w * \Lambda^{K+}_{\lambda_n} \xrightarrow{-\nu} i_{0!}S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n} \to 0;$$

F^2B'' is as follows:

$$i_{0!}S_w * \Lambda^{K+}_{\lambda_n} \oplus i_{2!}S_w * \Lambda^{K+}_{\mathbf{P}_2} \to i_{0!}(S_w * \Lambda^{K+}_{\lambda_n} \oplus S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n}) \to 0$$

We finally set $F^3B'' = B''$. The associated graded quotients are as follows: F^2/F^1 equals $\mathrm{Cone} r_2^+[-1]$, which is quasi-isomorphic to $S_w * \Lambda^{K+}_{\mathrm{Int}P_2}$.

F^3/F^2 equals

$$i_{1!}S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_1} \oplus i_{2!}S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_2}.$$

We will need a one more exact sequence. We have subsheaves (direct summands)

$$S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_1} \subset B_1; \quad S_{\lambda_n w} * \Lambda^{K-}_{\mathbf{P}_2} \subset B_2.$$

Since the map Γ induces identity on $S_{\lambda_n w} * \Lambda^{K-}_{\lambda_n}$, the two subsheaves glue into a subsheaf $S_{\lambda_n w} * \Lambda^{K-}_U \subset B'_U$. It is clear that we have a short exact sequence:

$$0 \to S_{\lambda_n w} * \Lambda^{K-}_U \to B'_U \to i_{2!}S_w * \Lambda^{K+}_{P_2} \to 0. \tag{170}$$

Let us now check the conditions of Prop 5.1. The isomorphicity of the map $\mathbb{Z}_{\mathbf{r}_\alpha} * B'_U \to B'_U$ can be checked directly.

Let us now show that $RP_{+!}B'_U = 0$. Because of the exact sequence (170), it suffices to prove that $RP_{+!}S_w * \Lambda^{K+}_{P_2} = 0$ and $RP_{+!}S_{\lambda_n w} * \Lambda^{K-}_U = 0$. This can be checked pointwise in a way similar to the previous subsection.

Let us now check that $RP_{-!}B'_U = 0$. It suffices to show that $RP_{-!}$, when applied to all associated graded quotients of the filtration F on B'', produces zero. The latter can be done pointwise in a way similar to the previous sections.

Proof of (165), (166) is very similar to the previous subsection.

6.2.16 Orthogonality of Cone($\mathbb{Z}_{z=\mathbf{x}_0,s\in K}[-2] \to F^1\Phi^K$)

The aim of this subsection is to prove that

$$\mathrm{Cone}(\mathbb{Z}_{z=\mathbf{x}_0,s\in K}[-2] \to F^1\Phi^K) \in {}^\perp\mathcal{C}^{\Sigma'}. \tag{171}$$

We will freely use the notation and the results from Sec 6.2.8,6.2.9. As was mentioned above, $\mathrm{Cone}(\mathbb{Z}_{z=\mathbf{x}_0,s\in K}[-2] \to F^1\Phi^K)$ is supported on $\Sigma \times \mathbb{C}$, where $\Sigma = \mathrm{Int}P_0 \cup \lambda_1 \cup \mathrm{Int}P_*$. The restriction $\mathrm{Cone}(\mathbb{Z}_{z=\mathbf{x}_0,s\in K}[-2] \to F^1\Phi^K)|_{\Sigma\times\mathbb{C}}$ is isomorphic to the Cone of the composition arrow in (155). Denote the cone of the left arrow in (155) by Γ_1 and the cone of the right arrow by Δ. Observe that $\Gamma_1 = j_{0!}\Gamma$, where $\Gamma = \mathrm{Cone}(\iota_L \oplus \iota_R)$; $\Gamma \in D(\mathrm{Int}P_0 \times \mathbb{C})$. The problem now reduces to showing that $\Gamma \in {}^\perp\mathcal{C}^{\mathrm{Int}P_0}$ and $\Delta \in {}^\perp\mathcal{C}^{\Sigma}$.

Denote $A_L := \mathrm{Coker}\, i_A$; $B_R := \mathrm{Coker}\, i_B$. Observe that A_L is of the form A_w with $w = L$, and B_R is of the form B_w with $w = R$, where A_w, B_w are as defined in Sec 6.2.11. It is also clear that $\Delta \cong A_L \oplus B_R$. As follows from the previous two subsections, $A_L, B_R \in {}^\perp\mathcal{C}^{\Sigma}$, hence, same is true for Δ. Let us now show that $\Gamma \in {}^\perp\mathcal{C}^{\mathrm{Int}P_0}$.

By Prop.5.1, it suffices to check statements a),b),c) below:

a) $\Gamma * \mathbb{Z}_{\{s\in e^{i\alpha}\mathbb{R}_{\geq 0}\}} \to \Gamma$ is an isomorphism: it suffices to check that a similar map applied to each of $\mathbb{Z}_{\mathbf{x}_0\times K}[-2]$, $S_L * \Lambda^{K+}_{\mathrm{Int}P_0}$, and $S_R * \Lambda^{K-}_{\mathrm{Int}P_0}$ is an isomorphism, which is straightforward.

b) $RP_{+!}\Gamma = 0$. It is enough to check $RP_{+!}\mathcal{G}_k = 0$, $k = 1, 2$, where

$$\mathcal{G}_1 = S_R * \Lambda^-_{\overset{\circ}{P}_0} = \mathbb{Z}_{\{(z,s):z\in\overset{\circ}{P}_0,\, s-z\in -\mathbf{x}_0+K\}},$$

$\mathcal{G}_2 = Cone(\mathbb{Z}_{\mathbf{x}_0\times K}[-2] \to S_L * \Lambda^+_{\overset{\circ}{P}_0}$ and where

$$S_L * \Lambda^+_{\overset{\circ}{P}_0} = \mathbb{Z}_{\{(z,s):z\in\overset{\circ}{P}_0,\, s+z\in \mathbf{x}_0+K\}}.$$

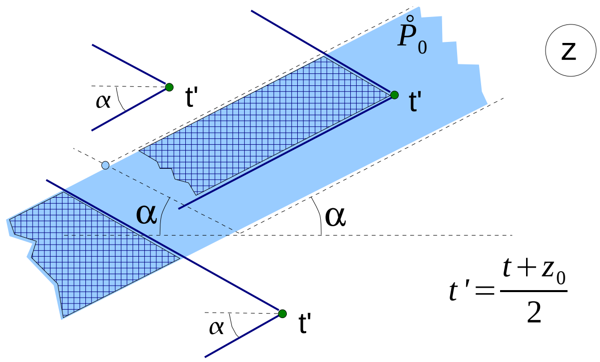

Figure 9**:** Proof of (171), Step b-i)

b-i) $RP_{+!}\mathcal{G}_1 = 0$. Indeed, by the base change, let us pass to the fiber of P_+ over $t \in \mathbb{C}$ and calculate $R\Gamma_c(\mathbb{Z}_{W_1})$ where $W_1 = \{(z,s) \in \mathbb{C} :\ z \in \overset{\circ}{P}_0,\ s - z \in -\mathbf{x}_0 + K\ z + s = t\}$. Eliminating s makes $W_1 = \{z \in \mathbb{C} : z \in \overset{\circ}{P}_0,\ z \in \frac{t+\mathbf{x}_0}{2} - K\}$. For different values of t this set is sketched on fig. 9. Thus, W_1 is either empty or homeomorphic to a closed half-plane, so the result follows.

b-ii) $RP_{+!}\mathcal{G}_2 = 0$. Indeed, by the base change, let us pass to the fiber of P_+ over $t \in \mathbb{C}$ and calculate $R\Gamma_c(\mathbb{Z}_{W_2'})[-2] \to R\Gamma_c(\mathbb{Z}_{W_2})$, where $W_2' = \{(z,s) \in \mathbb{C} :\ z = \mathbf{x}_0,\ s \in K\ z + s = t\}$, $W_2 = \{(z,s) \in \mathbb{C} : z \in \overset{\circ}{P}_0,\ s + z \in \mathbf{x}_0 + K\ z + s = t\}$. Eliminating s makes

$$\begin{array}{lll} \text{if } t - \mathbf{x}_0 \in K : & W_2' = \{\mathbf{x}_0\} & W_2 = \{z \in \mathbb{C} : z \in \overset{\circ}{P}_0\} \\ \text{otherwise:} & W_2' = \emptyset & W_2 = \emptyset \end{array}$$

and the map $R\Gamma_c(\mathbb{Z}_{W_2'})[-2] \to R\Gamma_c(\mathbb{Z}_{W_2})$ is the obvious quasi-isomorphism.

c) $RP_{-!}\Gamma = 0$. This can be shown similarly to $RP_{+!}\Gamma = 0$.

7 Identification of Φ^K and Ψ^K

We are going to construct an identification as in (54). Namely, we will construct a map

$$I_{\Psi\Phi} : \Psi^K \to \Phi^K$$

such that

$$i_\Phi = I_{\Psi\Phi} i_\Psi, \tag{172}$$

where $i_\Phi : \mathcal{F}_0^K \to \Phi^K$ is the map (52) and $i_\Psi : \mathcal{F}_0^K \to \Psi^K$ is the map (59).

The goal of this section is to give an explicit desciption of $I_{\Psi\Phi}$. This can be done as follows. Let P be a closed α-strip. Let Π be a closed $(-\alpha)$-strip such that $P \cap \Pi \neq \emptyset$. We then have identifications

$$\iota_{\Phi P}|_{(\Pi\cap P)\times\mathbb{C}} : \Lambda^+ * S_+ \oplus \Lambda^- * S_-|_{(\Pi\cap P)\times\mathbb{C}} = (\Phi^K|_{P\times\mathbb{C}})|_{(\Pi\cap P)\times\mathbb{C}} = \Phi^K|_{(\Pi\cap P)\times\mathbb{C}}$$

$$\iota_{\Psi\Pi}|_{(\Pi\cap P)\times\mathbb{C}} : \Lambda^+ * S_+ \oplus \Lambda^- * S_-|_{(\Pi\cap P)\times\mathbb{C}} = (\Psi^K|_{\Pi\times\mathbb{C}})|_{(\Pi\cap P)\times\mathbb{C}} = \Psi^K|_{(\Pi\cap P)\times\mathbb{C}}$$

meaning that the restriction $I_{\Psi\Phi}|_{(\Pi\cap P)\times\mathbb{C}}$ can be rendered as an automorphism $J_{\Pi P}$ of $\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{(\Pi\cap P)\times\mathbb{C}}$ in the abelian category of sheaves on $(\Pi\cap P)\times\mathbb{C}$, so that we have:

$$I_{\Psi\Phi}|_{(\Pi\cap P)\times\mathbb{C}} = \iota_{\Phi P}|_{(\Pi\cap P)\times\mathbb{C}} J_{\Pi P} \iota^{-1}_{\Psi\Pi}|_{(\Pi\cap P)\times\mathbb{C}}. \tag{173}$$

We are now motivated for the next subsection.

7.1 Endomorphisms of $\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{(P\cap\Pi)\times\mathbb{C}}$

We will do the study in a slightly more general context. Let Y be a locally closed connected subset of $\mathbb{C}$. For a $c\in\mathbb{C}$, set

$$A^{\pm}_c := \{(x,s)|s\pm x\in c+K\}\subset Y\times\mathbb{C}.$$

Let $W^{\pm}$ be sets; set $W := W^+\sqcup W^-$. Let $\mathbf{c}_W : W\to\mathbb{C}$ be a function. Let $w\in W_+$. Set $A_w := A^+_{\mathbf{c}(w)}$. For $w\in W_-$ we set $A_w := A^-_{\mathbf{c}(w)}$. Define the following sheaves on $Y\times\mathbb{C}$:

$$S_W := \bigoplus_{w\in W}\mathbb{Z}_{A_w}$$

Let $\mathbf{c}_i : W_i \to \mathbb{C}$; $W_i = W_i^+\sqcup W_i^-$, $i=1,2$; $\mathbf{c}_{W_i} : W_i\to\mathbb{C}$; and let us study a group $Hom_{Y\times\mathbb{C}}(S_{W_1}; S_{W_2})$.

We have

$$Hom_{Y\times\mathbb{C}}(S_{W_1}; S_{W_2}) \xrightarrow{\sim} \prod_{w_1\in W_1} Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; S_{W_2}) \tag{174}$$

Let us focus on $Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; S_{W_2})$. We have an embedding $S_{W_2}\hookrightarrow\prod_{w_2\in W_2}\mathbb{Z}_{A_{w_2}}$ which induces an embedding

$$\begin{aligned}\iota : Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; S_{W_2}) &\hookrightarrow Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; \prod_{w_2\in W_2}\mathbb{Z}_{A_{w_2}})\\ &= \prod_{w_2\in W_2} Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; \mathbb{Z}_{A_{w_2}}).\end{aligned} \tag{175}$$

Let us now compute

$$Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; \mathbb{Z}_{A_{w_2}}) = H^0(A_{w_2}; A_{w_2}\backslash A_{w_1}).$$

We have a homeomorphism $A_{w_2}\cong Y\times K$ so that A_{w_2} is connected and $H^0(A_{w_2}; A_{w_2}\backslash A_{w_1})$ is zero unless $A_{w_2}\backslash A_{w_1}$ is empty, in which case it equals $\mathbb{Z}$. In other words, we have an isomorphism $\varepsilon_{w_1w_2} : \mathbb{Z}\xrightarrow{\sim} Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; \mathbb{Z}_{A_{w_2}})$ if $A_{w_2}\subset A_{w_1}$; otherwise, $Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}}; \mathbb{Z}_{A_{w_2}}) = 0$. Set $e_{w_1w_2} := \varepsilon_{w_1w_2}(1)$.

Every element

$$\nu \in \prod_{w_2 \in W_2} Hom_{Y\times\mathbb{C}}(\mathbb{Z}_{A_{w_1}};\mathbb{Z}_{A_{w_2}})$$

can be uniquely written as

$$\sum_{w_2} \nu_{w_1w_2} e_{w_1w_2},$$

where the sum is taken over all w_2 such that $A_{w_2} \subset A_{w_1}$ and $\nu_{w_1w_2}$ are arbitrary integers.

Claim 7.1 *The element ν lies in the image of (175) iff for every compact subset $L \subset A_{w_1}$:*

$$\text{there are only finitely many } w_2 \text{ such that } \nu_{w_2w_1} = 0 \text{ and } A_{w_2} \cap L \neq 0. \tag{176}$$

PROOF We will use the following notation. For every $w \in W_1$ or $w \in W_2$, let us denote by $\mathbf{1}_w \in \Gamma(Y\times\mathbb{C};\mathbb{Z}_{A_w})$ the canonical section, such that for every $y \in Y \times \mathbb{C}$, the stalk $(\mathbf{1}_w)_y$ generates the group $(\mathbb{Z}_{A_w})_y$, which is equal to $\mathbb{Z}$ if $y \in A_w$ and to zero otherwise.

We have

$$\nu(\mathbf{1}_{w_1}) = \sum_{w_2\in W_2} n_{w_2w_1}\mathbf{1}_{w_2} \in \Gamma(Y\times\mathbb{C};\prod_{w_2\in w_2}\mathbb{Z}_{A_{w_2}}).$$

Let us now suppose that ν lies in the image of (175). This implies that the restriction $\nu(\mathbf{1}_{w_1})|_L \in \Gamma(L;\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}})$. Since L is compact, we have an isomorphism

$$\bigoplus_{w_2\in W_2}\Gamma(L;\mathbb{Z}_{A_{w_2}}) \to \Gamma(L;\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}}).$$

Given a section $\sigma \in \Gamma(L;\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}})$, denote by $\sigma_{w_2} \in \Gamma(L;\mathbb{Z}_{A_{w_2}})$ the corresponding component of σ. We have: $\sigma_{w_2} = 0$ for almost all $w_2 \in W_2$. We have $\nu(\mathbf{1}_{w_1})_{w_2} = n_{w_2w_1}\mathbf{1}_{w_2}|_L$. The element on the RHS does not vanish iff $n_{w_2w_1} \neq 0$ and $L \cap A_{w_2} \neq \emptyset$, which implies the statement.

Let us now assume that there only are finitely many $w_2 \in W_2$ such that $n_{w_2w_1} \neq 0$ and $L \cap A_{w_2} \neq \emptyset$. It suffices to show that

$$\nu(\mathbf{1}_{w_1}) \in \Gamma(Y\times\mathbb{C};\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}}) \subset \Gamma(Y\times\mathbb{C};\prod_{w_2\in W_2}\mathbb{Z}_{A_{w_2}}).$$

Let us choose an open covering of $Y\times\mathbb{C}$ by precompact sets U_a (i.e. the closure of each U_a in $Y\times\mathbb{C}$ must be compact). It suffices to show that $\nu(\mathbf{1}_{w_1}) \in \Gamma(U_a;\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}})$ for each U_a. Let L_a be the closure of U_a; it then suffices to show that $\nu(\mathbf{1}_{w_1}) \in \Gamma(L_a;\bigoplus_{w_2\in W_2}\mathbb{Z}_{A_{w_2}})$, In fact, $\nu(\mathbf{1}_{w_1}) \in \Gamma(L_a;\prod_{w_2\in W_2'}\mathbb{Z}_{A_{w_2}})$, where W_2' consists of all w_2 satisfying $n_{w_2w_1} \neq 0$, $A_{w_2} \cap L_a \neq 0$, which is finite, whence the statement. $\square$.

As follows from the proof of the Claim, ν belongs to the image of (175) iff the condition (176) is satisfied for a family of compact sets L_a whose interiors cover $X \times \mathbb{C}$.

Proposition 7.2 *Elements from $Hom_{X\times\mathbb{C}}(S_{W_1};S_{W_2})$ are in 1-to-1 correspondence with the sums*

$$\sum_{w_1\in W_1,w_2\in W_2,A_{w_2}\subset A_{w_1}} n_{w_1w_2}e_{w_1w_2},$$

satisfying:

there exists a family of compact subsets $L_a\subset X\times\mathbb{C}$ such that the sets $\mathrm{Int}L_a$ cover $X\times\mathbb{C}$, and: given a $w_1\in W_1$ and any L_a, there are only finitely many $w_2\in W_2$ such that $n_{w_1w_2}\neq 0$ and $L_a\cap A_{w_2}\neq\emptyset$.

7.1.1 Filtration on $Hom_{Y\times\mathbb{C}}(S_{W_1};S_{W_2})$

Let $\varepsilon\in K$. Let $T_\varepsilon: Y\times\mathbb{C}\to Y\times\mathbb{C}$ be the shift $(x,s)\mapsto(x,s+\varepsilon)$. We have $T_\varepsilon(A_c)\subset A_c$, for every $\varepsilon\in K$, whence an induced map

$$\tau_\varepsilon:\mathbb{Z}_{A_c}\to T_{\varepsilon!}\mathbb{Z}_{A_c}=\mathbb{Z}_{T_\varepsilon(A_c)}.$$

These maps give rise to a map

$$\tau_\varepsilon: S_{W_1}\to T_{\varepsilon!}S_{W_1}.$$

It is easy to see that $T_{\varepsilon!}S_{W_1}=S_{W_1'}$, where $W_1'=W_1$ and $\mathbf{c}_{W_1'}=\mathbf{c}_{W_1}+\varepsilon$, so that Proposition 7.2 applies to $T_{\varepsilon!}S_{W_1}$.

We say that $f\in F^\varepsilon Hom_{X\times\mathbb{C}}(S_{W_1};S_{W_2})$ if f factors as $f=g\tau_\varepsilon$ for some $g:T_{\varepsilon!}S_{W_1}\to S_{W_2}$. Using Proposition 7.2, one can check that such a g is unique, if exists.

We write $f\equiv f' \mod F^\varepsilon$ if $f-f'\in F^\varepsilon Hom(S_{W_1},S_{W_2})$.

We also write $f\equiv f'$ if $f\equiv f' \mod F^\varepsilon$ for some $\varepsilon\in\mathrm{Int}K$.

Let us prove that the filtration F is complete in the following sense. Let $f_n\in Hom(S_{W_1};S_{W_2})$ be a sequence of homomorphisms. Let us call f_n <u>a Cauchy</u> sequence if:

$$\forall\varepsilon\in K\ \exists N(\varepsilon):\forall n,m\geq N(\varepsilon): f_n\equiv f_m \mod F^\varepsilon.$$

We say that <u>f_n converges to f</u> if

$$\forall\varepsilon\in K\ \exists N(\varepsilon):\forall n\geq N(\varepsilon): f\equiv f_n \mod F^\varepsilon.$$

Claim 7.3 *Every Cauchy sequence f_n converges to a unique limit f.*

Let us first construct f. Decompose $f_n=\sum_{w_1,w_2\in W}(f_n)_{w_1w_2}e_{w_1w_2}$. Let $y\in X\times\mathbb{C}$ and let $n,m\geq N(\varepsilon)$. Since f_n-f_m passes through τ_ε, we deduce that $(f_n)_{w_1w_2}-(f_m)_{w_1w_2}\neq 0$ only if $A_{w_2}\subset T_\varepsilon A_{w_1}$. For every w_1,w_2 there exists $\varepsilon_{w_1w_2}$ such that this condition is violated, meaning that for $n,m\geq N(\varepsilon_{w_1w_2})$, $(f_n)_{w_1w_2}=(f_m)_{w_1w_2}=:f_{w_1w_2}$.

The data $f_{w_1w_2}$ define a homomorphism f by virtue of Proposition 7.2. If f' is another limit, it follows that $f-f'\equiv F^\varepsilon$ for all ε which implies $f_{w_1w_2}=f'_{w_1w_2}$ for all w_1,w_2, that is $f=f'$. $\square$

In particular, let $\gamma\in \mathrm{End}(A_W)$, $\gamma=Id+n$ and assume that for some $k>0$, $n^k\in F^\varepsilon$ for some $\varepsilon\in \mathrm{Int}K$,then γ is invertible, and we can set $\gamma^{-1}=Id-n+n^2-n^3+...$ (the sequence of partial sums of this series is Cauchy).

We conclude with several Lemmas for the future use.

7.1.2 Lemma on composition

As above, let P be an α-strip and let Π be a $-\alpha$-strip. Let $Y=\Pi\cap P$ and supose Y is a bounded subset of $\mathbb{C}$, so that the closure of Y is a parallelogram; let us denote its vertices $ABCD$ so that AC is one of the two diagonals and $\vec{AC}\in K$. It then follows that the closure of $P\cap\Pi$ equals $A+K\cap C-K$. Denote $\varepsilon:=\vec{AC}$.

Lemma 7.4 *Let $W_1^-=W_2^+=\emptyset$. And let $f:S_{W_1}\to S_{W_2}$ and $g:S_{W_2}\to S_{W_1}$. Then $gf\equiv 0 \mod F^{2\varepsilon}$ and $fg\equiv 0 \mod F^{2\varepsilon}$.*

Proof.

Let $f_{w_1w_2}e_{w_1w_2}$, $g_{w_2w_1}e_{w_2w_1}$ be components of f,g.

Let us consider the compositions $f_{w_1w_2}e_{w_1w_2}g_{w_2'w_1}e_{w_2'w_1}$ In order for this composition to be non-zero, there should be

$$A_{w_2}\subset A_{w_1}\subset A_{w_2'}.$$

Or, for every $z\in P\cap\Pi$ and $s\in\mathbb{C}$ we should have the following implications:

$$s-z\in \mathbf{c}_{W_2}(w_2)+K\Rightarrow s+z\in \mathbf{c}_{W_1}(w_1)+K\Rightarrow s-z\in \mathbf{c}_{W_2}(w_2')+K.$$

Set $\varsigma:=s-z-\mathbf{c}_{w_2}$. The first implication then reads as:

$$\varsigma\in K\Rightarrow \varsigma+2z+\mathbf{c}_{W_2}(w_2)-\mathbf{c}_{W_1}(w_1)\in K$$

or, equivalently, $2A+\mathbf{c}_{W_2}(w_2)-\mathbf{c}_{W_1}(w_1)\in K$. Similarly, the second implication can be rewritten as $-2C+\mathbf{c}_{W_1}(w_1)-\mathbf{c}_{W_2}(w_2')\in K$. Adding the two conditions yields $-2\varepsilon+\mathbf{c}_{W_2}(w_2)-\mathbf{c}_{W_2}(w_2')\in K$; $\mathbf{c}_{W_2}-\mathbf{c}_{W_2}(w_2')\in 2\varepsilon+K$. This implies that

$$f_{w_1w_2}e_{w_1w_2}g_{w_2'w_1}e_{w_2'w_1}:\mathbb{Z}_{A_{w_2'}}\to \mathbb{Z}_{A_{w_2}}$$

passes through $\tau_{2\varepsilon} : \mathbb{Z}_{A_{w_2'}} \to T_{2\varepsilon !}\mathbb{Z}_{A_{w_2'}}$, which implies the statement for fg. Proof for gf is similar. □.

Let us keep the assumption $W_1 = W_1^+$, $W_2 = W_2^-$ and consider now the case when $X = \Pi \cap P$ is not bounded. Then at least one of the following is true:

i) there is no $A \in \mathbb{C}$ such that $X \subset A + K$;

ii) there is no $C \in \mathbb{C}$ such that $X \subset C - K$.

Lemma 7.5 *Let us keep the same notation as in the previous Lemma. In the case i) we have $Hom(S_{W_1}; S_{W_2}) = 0$. In the case ii) we have $Hom(S_{W_2}; S_{W_1}) = 0$.*

Proof. In Case i), given $w_1 \in W_1$ and $w_2 \in W_2$, it is impossible that $A_{w_2} \subset A_{w_1}$, And similarly for the Case ii). □

7.1.3 Lemma on extension

We keep the same assumptions on W_1, W_2, namely,

$$W_1 = W_1^+, \quad W_2 = W_2^-.$$

Let Y be a locally closed non-empty connected subset of $\mathbb{C}$. Let $Y + K$ (resp. $Y - K$) be the arithmetic sum (resp. difference) of Y and K. Let Y_+, Y_- be connected locally closed subsets satisfying $Y \subset Y_+ \subset Y + K$; $Y \subset Y_- \subset Y - K$. Let Z be an arbitrary connected locally closed subset $\mathbb{C}$ containing Y.

Lemma 7.6 *1) The restriction maps*

$$Hom_{Y_+}(S_{W_1^+}; S_{W_2^-}) \to Hom_Y(S_{W_1^+}; S_{W_2^-});$$
$$Hom_{Y_-}(S_{W_2^-}; S_{W_1^+}) \to Hom_Y(S_{W_2^-}; S_{W_1^+})$$

are isomorphisms;

2) the restriction maps

$$Hom_Z(S_{W_1^+}; S_{W_2^+}) \to Hom_Y(S_{W_1^+}; S_{W_2^+});$$
$$Hom_Z(S_{W_2^-}; S_{W_1^-}) \to Hom_Y(S_{W_2^-}; S_{W_1^-})$$

are isomorphisms.

Proof. 1) Follows from Proposition 7.2: the inclusion $A_{w_2} \subset A_{w_1}$, $w_i \in W_i$ occurs on $Y_+ \times \mathbb{C}$ iff it occurs on $Y \times \mathbb{C}$, and similar for the inclusion $A_{w_1} \subset A_{w_2}$ on $Y_- \times \mathbb{C}$.

2) Follows from Proposition 7.2 in a similar way.

□

7.1.4 Decomposition Lemma

Let now $Y := \ell := c + (0,\infty).e^{i\alpha}$ be a ray which goes to the right. Let $a \in \mathbb{C}$. We have natural maps $\lambda_a^+ : \mathbb{Z}_{A_a^+} \to \mathbb{Z}_{A^-_{-2c+a}}$; $\lambda_a^- : \mathbb{Z}_{A_a^-} \to \mathbb{Z}_{A^+_{2c+a}}$; coming from the inclusions of the corresponding sets.

Lemma 7.7 *Let $f : \mathbb{Z}_{A_a^+} \to S_{W_2}$, $g : \mathbb{Z}_{A_a^-} \to S_{W_1}$ be a map of sheaves. Then f and g can be uniquely factored as $f = f'\lambda_a^+$; $g = g'\lambda_a^-$.*

Proof. Let $w \in W_2$. A simple analysis shows that $A_a^+ \subset A_w$ is equivalent to $A^-_{-2c+a} \supset A_w$. Proposition 7.2 now implies the factorization of f. The factorization of g can be proven similarly. $\square$

7.2 Restriction $\Phi^K|_\Pi$

As above, let Π be a closed $(-\alpha)$-strip.

The goal of this subsection is to construct an isomorphism

$$\phi_\Pi : (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)|_{\Pi\times\mathbb{C}} \xrightarrow{\sim} \Phi^K|_{\Pi\times\mathbb{C}}. \tag{177}$$

Denote by

$$\phi_\Pi^\pm : \Lambda^{K\pm} * S_\pm|_{\Pi\times\mathbb{C}} \to \Phi^K|_{\Pi\times\mathbb{C}}$$

the components.

7.2.1 Notation

Let us number all α-strips that intersect Π as $P_1, P_2, ..., P_n$ (there are only finitely many such stripes, Sec 2.3.2) as shown on the picture 10 so that we number the strips from the left to the right. The strips P_1 and P_n are necessarily half planes.

7.2.2 Prescription of $\phi_\Pi^+|_{(\Pi\cap P_1)\times\mathbb{C}}$

We have an identification

$$\Phi^K|_{\Pi\cap P_1} = (\Phi^K|_{P_1})|_{(\Pi\cap P_1)\times\mathbb{C}} = (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)|_{(\Pi\cap P_1)\times\mathbb{C}}.$$

This identification gives rise to a map (embedding onto a direct summand):

$$\Lambda^{K+} * S_+ \to \Phi^K|_{(\Pi\cap P_1)\times\mathbb{C}}.$$

We assign $\phi_\Pi^+|_{(\Pi\cap P_1)\times\mathbb{C}}$ to be this map.

Remark The reason why we do not define $\phi_\Pi^-|_{(\Pi\cap P_1\times\mathbb{C})}$ in a similar way, is that there is no way to extend it to the whole Π.

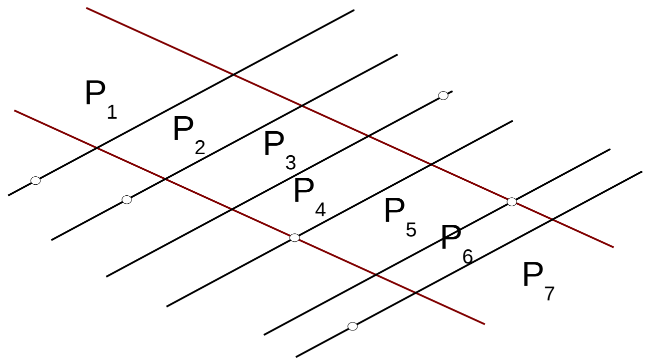

Figure 10

7.2.3 Extension to Π

For a subset $A \subset \mathbb{C}$, set $\underline{A} := (\Pi \cap A) \times \mathbb{C} \subset \Pi \times \mathbb{C}$.

Let us define ϕ^+_Π by constructing maps

$$j_k^+ : \ \Lambda^{K+} * \mathcal{S}_+|_{\underline{P_k}} \to \Phi|_{\underline{P_k}},$$

which agree on the intersections:

$$j_{k+1}^+|_{\underline{P_k \cap P_{k+1}}} = j_k^+|_{\underline{P_k \cap P_{k+1}}}. \tag{178}$$

We have identifications

$$\iota_k : \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\underline{P_k}} \to (\Phi^K|_{P_k \times \mathbb{C}})|_{\underline{P_k}} = \Phi|_{\underline{P_k}} \tag{179}$$

coming from the gluing construction of Φ_K.

We have

$$\iota_k|_{\underline{P_k \cap P_{k+1}}} = \iota_{k+1}|_{\underline{P_k \cap P_{k+1}}} \circ \Gamma^{P_k P_{k+1}}_{\Phi^K},$$

where $\Gamma^{P_k P_{k+1}}_{\Phi^K}$ is as in (43).

We can now prescribe j_k^+ in the following form: $j_k^+ = \iota_k \circ i_k^+$ where

$$i_k^+ : \Lambda^{K+} * S_+|_{\underline{P_k}} \to (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)|_{\underline{P_k}}.$$

The agreement conditions (178) now read as:

$$i_{k+1}^+|_{\underline{P_k \cap P_{k+1}}} = \Gamma^{P_k P_{k+1}}_{\Phi^K} i_k^+|_{\underline{P_k \cap P_{k+1}}} \tag{180}$$

The assignment from the previous subsection means that i_1^+ is the identity embedding onto the direct summand. Let us construct the remaining maps i_k inductively. Suppose i_k has been already defined.

According to Claim (7.6), the map $\Gamma^{P_kP_{k+1}}_{\Phi^K} i^+_k|_{\underline{P_k\cap P_{k+1}}}$ extends uniquely to $\underline{P_{k+1}}$ by Claim 7.6 (this is where the choice of + sign is crucial). We assign i^+_{k+1} to be this map. It is clear that thus defined map i^+_{k+1} satisfies (180) so that the maps j^+_{k+1} give rise to a well defined map ϕ^+_Π, as we wanted.

Let us denote by $i^{++}_k : \Lambda^{K+} * S_+|_{\underline{P_k}} \to \Lambda^{K+} * S_+|_{\underline{P_k}}$; $i^{+-}_k : \Lambda^{K+} * S_+|_{\underline{P_k}} \to \Lambda^{K-} * S_-|_{\underline{P_k}}$; the components of the map i^+_k.

7.3 Estimate

Denote by ε_k the diagonal vector of the parallellogram $P_k \cap \Pi$ such that $\varepsilon_k \in \mathrm{Int}K$ (there is a unique such a diagonal vector). Let $\varepsilon_\Pi \in \mathrm{Int}K$ be a vector such that $\varepsilon_k \in \varepsilon + K$ for all k.

The following Claim can be now proved by a direct computation.

Claim 7.8 *1)* $i^{++}_k \equiv 1 \mod F^{\varepsilon_\Pi}$ *for all* $k = 1, ..., n$.
2) Let $\mathcal{R}_\Pi \subset \{1, 2, ..., n-1\}$ *consist of all* k *s.th.* $P_k \cap P_{k+1}$ *goes to the right. We then have a transform*

$$\Gamma^{P_kP_{k+1}}_{+-} \;:\; \Lambda^{K+} * \mathcal{S}_+|_{\underline{P_k\cap P_{k+1}}} \to \Lambda^{K-} * \mathcal{S}_-|_{\underline{P_k\cap P_{k+1}}},$$

where $\Gamma^{P_kP_{k+1}}_{+-}$ *is the corresponding component of* $\Gamma^{P_kP_{k+1}}_{\Phi^K}$*., which extends uniquely to* $\underline{P_{k+1} \cup ... \cup P_n}$*.* $\Gamma^{P_kP_{k+1}}_{+-}$ *is the same as* N^K_ℓ*, where* $\ell = P_k \cap P_{k+1}$ *from (47).*

We then have:

$$i^{+-}_k \equiv -\sum_{k' \in \mathcal{R}_\Pi;\; k'<k} \Gamma^{P_{k'}P_{k'+1}}_{+-} \mod F^{\varepsilon_\Pi}. \tag{181}$$

7.3.1 Construction of ϕ^-_Π

The map ϕ^-_Π is constructed in a fairly similar way (the major difference is that we need to start the construction from $\underline{P_n}$ and then continue to the left until we reach $\underline{P_1}$.

Similar to above, we define ϕ^-_Π in terms of the restrictions to $\underline{P_k}$:

$$\phi^-_\Pi|_{\underline{P_k}} = \iota_k \circ i^-_k,$$

where ι_k is the same as above, see (179), and

$$i^-_k : \Lambda^{K-} * S_-|_{\underline{P_k}} \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\underline{P_k}}.$$

We have the following analogue of Claim 7.8.

Claim 7.9 *Let $\varepsilon_\Pi \in Int\ K$ be as in Claim 7.8. We have 1) $i_k^{--} \equiv 1 \mod F^{\varepsilon_\Pi}$ for all $k = 1, ..., n$. 2) Let $\mathcal{L}_\Pi \subset \{1, 2, ..., n-1\}$ consist of all k s.th. $P_k \cap P_{k-1}$ goes to the left. We then have transform*

$$\Gamma_{-+}^{P_{k-1}P_k} \;:\; \Lambda^{K-} * \mathcal{S}_-|_{\underline{P_k \cap P_{k-1}}} \to \Lambda^{K+} * \mathcal{S}_+|_{\underline{P_k \cap P_{k-1}}}$$

which extends uniquely to $\underline{P_{k-1} \cup ... \cup P_1}$. We then have:

$$i_k^{-+} \equiv -\sum_{k' \in R_\Pi;\ k' > k} \Gamma_{-+}^{P_{k'-1}P_{k'}} \mod F^{\varepsilon_\Pi}.$$

7.3.2 The map ϕ_Π is an isomorphism

Now that we have constructed the maps $\phi_\Pi|_{\underline{P_k}}$ from (177), let us prove that they are isomorphisms. We can write

$$\phi_\Pi|_{\underline{P_k}} = \iota_k \circ i_{\Pi P_k}, \tag{182}$$

where $i_{\Pi P_k}$ is an endomorpism of $\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\underline{P_k}}$ whose components $i_k^{\pm\pm}$ have been constructed above. We will abbreviate $i_{\Pi P_k} = i_k$. The problem reduces to showing invertibility of i_k.

Let us use the matrix notation

$$i_k = \begin{pmatrix} i_k^{++} & i_k^{-+} \\ & \\ i_k^{+-} & i_k^{--} \end{pmatrix} \in \mathrm{End} \left(\begin{array}{c} \Lambda^{K+} * S_+ \\ \oplus \\ \Lambda^{K-} * S_- \end{array} \Bigg|_{\underline{P_k}} \right).$$

We have

$$\begin{pmatrix} i_k^{++} & i_k^{-+} \\ i_k^{+-} & i_k^{--} \end{pmatrix} \equiv \begin{pmatrix} 1 & i_k^{-+} \\ i_k^{+-} & 1 \end{pmatrix}, \tag{183}$$

as follows from Claims 7.8 and 7.9.

Lemma 7.4 implies that

$$\begin{pmatrix} 0 & i_k^{-+} \\ i_k^{+-} & 0 \end{pmatrix}^2 = \begin{pmatrix} i_k^{-+} \circ i_k^{+-} & 0 \\ 0 & i_k^{+-} \circ i_k^{-+} \end{pmatrix} \equiv 0.$$

It now follows that $X := \begin{pmatrix} i_k^{++} & i_k^{-+} \\ i_k^{+-} & i_k^{--} \end{pmatrix}$ is invertible (Sec 7.1.1).

We can multiply (183) by X^{-1} so as to get:

$$i_k X^{-1} \equiv \mathrm{Id},$$

which implies that $i_k X^{-1}$ and, thereby, i_k is invertible. Furthermore, we get:

$$i_k^{-1} \equiv \begin{pmatrix} 1 & -i_k^{-+} \\ -i_k^{+-} & 1 \end{pmatrix} \tag{184}$$

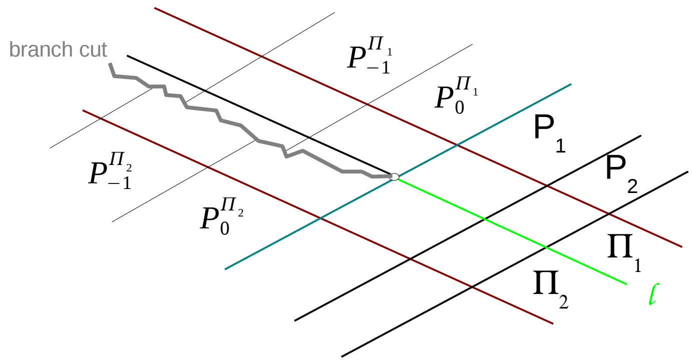

Figure 11

7.4 The maps ϕ_{Π_1}, ϕ_{Π_2} for a pair neighboring strips Π_1 and Π_2

Consider now the neighboring strips Π_1 and Π_2 and let $\ell = \Pi_1 \cap \Pi_2$. Let us find the relation between $\Phi^{\pm}_{\Pi_1}|_\ell$ and $\Phi^{\pm}_{\Pi_2}|_\ell$. Suppose ℓ goes to the right, fig. 11.

We have a canonical isomorphism

$$H_{\Pi_1\Pi_2} : (\Phi|_{\Pi_1\times\mathbb{C}})|_\ell \simeq (\Phi|_{\Pi_2\times\mathbb{C}})|_\ell.$$

Using the isomorphisms $\phi_{\Pi_1}, \phi_{\Pi_2}$ as in (177), we get an isomorphism

$$\tilde{A}_{\Pi_1\Pi_2} := \phi^{-1}_{\Pi_2}|_{\ell\times\mathbb{C}} \circ H_{\Pi_1\Pi_2} \circ \phi_{\Pi_1}|_{\ell\times\mathbb{C}} :$$

$$\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\ell\times\mathbb{C}} \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\ell\times\mathbb{C}}. \tag{185}$$

Let $P_1, P_2, ..., P_n$ be all α-strips which intersect ℓ, fig.11. We then have commutative diagrams

$$\begin{tikzcd}
\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-\big|_{\ell\cap P_k} \arrow[rr, "\tilde{A}_{\Pi_1\Pi_2}"] \arrow[dr, "i_{\Pi P_k}|_\ell"'] & & \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-\big|_{\ell\cap P_k} \arrow[dl, "i_{\Pi_2 P_k}|_\ell"] \\
& \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-\big|_{\ell\cap P_k} &
\end{tikzcd}$$

which implies that

$$\tilde{A}_{\Pi_1\Pi_2}|_{\ell\cap P_k} \;=\; (i_{\Pi_2 P_k}|_{\ell\cap P_k})^{-1} \circ i_{\Pi_1 P_k}|_{\ell\cap P_k}.$$

These formulas determine $\tilde{A}_{\Pi_1\Pi_2}$. Let us compute:

$$i_{\Pi_2 P_k} \circ \tilde{A}_{\Pi_1\Pi_2}|_{\ell\cap P_k} = i_{\Pi_1 P_k}|_{\ell\cap P_k}$$

$$\begin{pmatrix} 1 & i^{-+}_{\Pi_2 P_k} \\ i^{+-}_{\Pi_2 P_k} & 1 \end{pmatrix} \circ \tilde{A}_{\Pi_1\Pi_2}|_{\ell\cap P_k} \equiv \begin{pmatrix} 1 & i^{-+}_{\Pi_1 P_k} \\ i^{+-}_{\Pi_1 P_k} & 1 \end{pmatrix}.$$

Formula (183) yields

$$\begin{pmatrix} 1 & i^{-+}_{\Pi_2 P_k} \\ i^{+-}_{\Pi_2 P_k} & 1 \end{pmatrix}^{-1} \equiv \begin{pmatrix} 1 & -i^{-+}_{\Pi_2 P_k} \\ -i^{+-}_{\Pi_2 P_k} & 1 \end{pmatrix} .$$

Therefore,

$$\tilde{A}_{\Pi_1\Pi_2}|_{\ell\cap P_k} \equiv \begin{pmatrix} 1 & -i^{-+}_{\Pi_2 P_k} \\ -i^{+-}_{\Pi_2 P_k} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & i^{-+}_{\Pi_1 P_k} \\ i^{+-}_{\Pi_1 P_k} & 1 \end{pmatrix} \equiv$$

$$\equiv \left.\begin{pmatrix} 1 & i^{-+}_{\Pi_1 P_k} - i^{-+}_{\Pi_2 P_k} \\ i^{+-}_{\Pi_1 P_k} - i^{+-}_{\Pi_2 P_k} & 1 \end{pmatrix}\right|_{\ell\cap P_k} \tag{186}$$

because $i^{+-}_{\Pi_2 P_k} \circ i^{-+}_{\Pi_1 P_k} \equiv 0$ and $i^{-+}_{\Pi_2 P_k} \circ i^{+-}_{\Pi_1 P_k} \equiv 0$ by Lemma 7.4

Let us, cf. fig.11, number all the α-strips that meet Π_1 or Π_2:

$$P^{\Pi_1}_{-m_1}, P^{\Pi_1}_{-m_1+1}, ..., P^{\Pi_1}_{0}, P_1, P_2, ..., P_n;$$

$$P^{\Pi_2}_{-m_2}, P^{\Pi_2}_{-m_2+1}, ..., P^{\Pi_2}_{0}, P_1, P_2, ..., P_n.$$

Let us also set $P^{\Pi_1}_1 := P^{\Pi_2}_1 := P_1$. Lemma 7.8 yields,

$$i^{+-}_{\Pi_1 P_k} \equiv -\sum_{l<k}{}' \Gamma^{P_l P_{l+1}} - \sum_{m\le 0}{}' \Gamma^{P^{\Pi_1}_{m+1} P^{\Pi_1}_{m}};$$

$$i^{+-}_{\Pi_1 P_k} \equiv -\sum_{l<k}{}' \Gamma^{P_l P_{l+1}} - \sum_{m\le 0}{}' \Gamma^{P^{\Pi_2}_{m} P^{\Pi_2}_{m+1}},$$

where only those terms are included into the sums, for which the intersection ray of the corresponding α-strips goes to the right. Hence,

$$i^{+-}_{\Pi_1 P_k} - i^{+-}_{\Pi_2 P_k} \equiv \sum_{m\le 0}{}' \Gamma^{P^{\Pi_2}_{m} P^{\Pi_2}_{m+1}} - \sum_{m\le 0}{}' \Gamma^{P^{\Pi_1}_{m} P^{\Pi_1}_{m+1}}.$$

Let $\ell := \Pi_1 \cap \Pi_2$ be of the form $\{\hat{c}(\ell) + re^{-i\alpha} \;\; r > 0\}$.

It now follows that

$$i^{+-}_{\Pi_1 P_k} - i^{+-}_{\Pi_2 P_k}|_{\ell\cap P_k} \equiv -\Gamma^{P^{\Pi_1}_0 P_1}_{+-} . \tag{187}$$

Thus:

$$\tilde{A}_{\Pi_1\Pi_2}|_{\ell\cap P_k} \equiv \begin{pmatrix} 1 & * \\ -\Gamma^{P^{\Pi_1}_0 P_1} & 1 \end{pmatrix} .$$

This means that the same is true for $\tilde{A}_{\Pi_1\Pi_2}|_{\ell}$.

Let us write $\tilde{A}_{\Pi_1\Pi_2}$ in the matrix form.

$$\tilde{A}_{\Pi_1\Pi_2} = \begin{pmatrix} \tilde{A}^{++}_{\Pi_1\Pi_2} & \tilde{A}^{-+}_{\Pi_1\Pi_2} \\ & \\ \tilde{A}^{+-}_{\Pi_1\Pi_2} & \tilde{A}^{--}_{\Pi_1\Pi_2} \end{pmatrix} : \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell} \to \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell} .$$

Lemma 7.5 implies that $\tilde{A}^{-+}_{\Pi_1\Pi_2} = 0$. Indeed, the corresponding map is defined on an unbounded set $\Pi_1 \cap \Pi_2$; since the intersection ray goes to the right, we are under the conditions of the case *i)* of that Lemma.

Let us summarize our findings.

Claim 7.10 *Let Π_1, Π_2 be neighboring strips and $\ell = \Pi_1 \cap \Pi_2$ goes to the right. Assume that Π_1 is above Π_2. Then*
1) the map

$$\tilde{A}_{\Pi_1\Pi_2} : \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell} \to \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell}$$

is of the form

$$\tilde{A}_{\Pi_1\Pi_2} = \begin{pmatrix} \tilde{A}^{++}_{\Pi_1\Pi_2} & 0 \\ \tilde{A}^{+-}_{\Pi_1\Pi_2} & \tilde{A}^{--}_{\Pi_1\Pi_2} \end{pmatrix};$$

2) $\tilde{A}^{++}_{\Pi_1\Pi_2} \equiv Id$; $\tilde{A}^{--}_{\Pi_1\Pi_2} \equiv Id$; $\tilde{A}^{+-}_{\Pi_1\Pi_2} \equiv -\Gamma^{P_0^{\Pi_1} P_1}_{-+}$; *where P_1 is the leftmost α-strip that meets both Π_1 and Π_2 and $P_0^{\Pi_1}$ is the rightmost α-strip that meets Π_1 but not Π_2.*

Similar result holds true in the case when the intersection ray $\Pi_1 \cap \Pi_2$ goes to the left (proof is omitted).

Claim 7.11 *Let Π_1, Π_2 be neighboring strips and $\ell = \Pi_1 \cap \Pi_2$ goes to the left. Assume that Π_1 is below Π_2. Then*
1) the map

$$\tilde{A}_{\Pi_1\Pi_2} : \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell} \to \left.\begin{array}{c} \Lambda^{K+} * \mathcal{S}_+ \\ \oplus \\ \Lambda^{K-} * \mathcal{S}_- \end{array}\right|_{\ell}$$

is of the form

$$\tilde{A}_{\Pi_1\Pi_2} = \begin{pmatrix} \tilde{A}^{++}_{\Pi_1\Pi_2} & \tilde{A}^{-+}_{\Pi_1\Pi_2} \\ 0 & \tilde{A}^{--}_{\Pi_1\Pi_2} \end{pmatrix};$$

2) $\tilde{A}^{++}_{\Pi_1\Pi_2} \equiv Id$*;* $\tilde{A}^{--}_{\Pi_1\Pi_2} \equiv Id$*;* $\tilde{A}^{-+}_{\Pi_1\Pi_2} \equiv -\Gamma^{P_0^{\Pi_1}P_1}_{-+}$ *where* P_1 *is the rightmost* α*-strip that meets both* Π_1 *nad* Π_2 *and* $P_0^{\Pi_1}$ *is the leftmost* α*-strip that meets* Π_1 *but not* Π_2*.*

7.4.1 Identifications

Let $\ell = \Pi_1 \cap \Pi_2$, $\ell \in \mathcal{L}^{-\alpha}$.

In the notation of section 3.10.2, we can identify $\tilde{S}_\ell \xrightarrow{\sim} S_{\mathbf{A}^{-1}(\ell)}$; $B_w : \tilde{S}_w \xrightarrow{\sim} S_{\mathbf{A}^{-1}(w)}$ for every $w \in \tilde{W}$. For a word $w = \ell_n \cdots \ell_1 L$ or $w = \ell_n \cdots \ell_1 R$, set $|w| := n$ (we set $|L| = |R| = 0$).

Let $C_w := (-1)^{|w|} B_w : \tilde{S}_w \to S_{\mathbf{A}^{-1}(w)}$.

Let us define identifications

$$\mathbf{B}_\pm, \mathbf{C}_\pm : \tilde{S}_\pm \to S_\pm \tag{188}$$

where

$$\mathbf{B}_\pm|_{\tilde{S}_w} = B_w; \quad \mathbf{C}_\pm|_{\tilde{S}_w} = C_w.$$

We can conclude from 2)s of Claims 7.10, 7.11 that

$$\tilde{A}_{\Pi_1\Pi_2} \equiv \mathbf{C}^{-1}\Gamma^{\Pi_1\Pi_2}_{\Psi^K}\mathbf{C}, \tag{189}$$

where $\Gamma^{\Pi_1\Pi_2}_{\Psi^K}$ is as in (57).

7.5 The isomorphism $I_{\Psi\Phi} : \Psi^K \to \Phi^K$

Using the above developed results, we will construct a map $I_{\Psi\Phi} : \Psi^K \to \Phi^K$ which satisfies (172) (recall that such a map is unique). Equivalently, for each $(-\alpha)$-strip Π, let us specify maps

$$I_{\Psi\Phi,\Pi} : \Psi^K|_{\Pi\times\mathbb{C}} \to \Phi^K|_{\Pi\times\mathbb{C}}$$

which agree on itersections: if $\Pi_1 \cap \Pi_2 = \ell \neq \emptyset$, then we should have:

$$I_{\Psi\Phi,\Pi_1}|_{\ell\times\mathbb{C}} = I_{\Psi\Phi,\Pi_2}|_{\ell\times\mathbb{C}}. \tag{190}$$

Let us now reformulate condition (172).

Let $\mathbf{P}_0$ be an α strip and $\mathbf{\Pi}_0$ be a $-\alpha$-strip such that $\mathbf{x}_0 \in \mathbf{P}_0 \cap \mathbf{\Pi}_0$ (these strips are unique).

Let

$$i^0_\Phi : \mathcal{F}_0 \to \Phi^K|_{(\mathbf{\Pi}_0\cap\mathbf{P}_0)\times\mathbb{C}};$$

$$i^0_\Psi : \mathcal{F}_0 \to \Psi^K|_{(\mathbf{\Pi}_0\cap\mathbf{P}_0)\times\mathbb{C}}$$

be the restrictions of i_Φ, i_Ψ. Since $\mathcal{F}_0^K$ is supported on $(\mathbf{\Pi}_0 \cap \mathbf{P}_0) \times \mathbb{C}$, the condition (172) is equivalent to:

$$I_{\Psi\Phi}|_{(\mathbf{\Pi}_0 \cap \mathbf{P}_0)\times\mathbb{C}} i^0_\Psi = i^0_\Phi. \tag{191}$$

We have identifications

$$\tilde{\iota}_\Pi : \Lambda^{K+} * \tilde{S}_+ \oplus \Lambda^{K-} * \tilde{S}_-|_{\Pi\times\mathbb{C}} \to \Psi^K|_{\Pi\times\mathbb{C}}$$

$$\phi_\Pi : \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\Pi\times\mathbb{C}} \to \Phi^K|_{\Pi\times\mathbb{C}}.$$

Here $\tilde{\iota}_\Pi$ is defined similarly to (179) but for $\tilde{S}_\pm$, Ψ^K and $(-\alpha)$-strips instead of $S_\pm$, Φ^K and α-strips; and ϕ_Π is as in (177).

One can now equivalently look for $I_{\Psi\Phi,\Pi}$ in the form:

$$I_{\Psi\Phi,\Pi} = \phi_\Pi U_\Pi \tilde{\iota}_\Pi^{-1}, \tag{192}$$

where

$$U_\Pi : \Lambda^{K+} * \tilde{S}_+ \oplus \Lambda^{K-} * \tilde{S}_-|_{\Pi\times\mathbb{C}} \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\Pi\times\mathbb{C}}$$

is to be calculated.

Since Π satisfies both i) and ii) in Lemma 7.5, we have

$$Hom_{\Pi\times\mathbb{C}}(\Lambda^{K\pm} * \tilde{S}_\pm; \Lambda^{K\mp} * \tilde{S}_\mp) = 0.$$

Thus, we must have:

$$U_\Pi(\Lambda^{K\pm} * \tilde{S}_\pm) \subset \Lambda^{K\pm} * S_\pm. \tag{193}$$

Using (185) and (56), we rewrite the gluing condition (190) as follows:

$$U_{\Pi_2}|_{\ell\times\mathbb{C}} = \tilde{A}_{\Pi_1\Pi_2} U_{\Pi_1}|_{\ell\times\mathbb{C}} \Gamma^{\Pi_2\Pi_1}_{\Psi^K}. \tag{194}$$

Let us now rewrite the condition (191) (from now on all our maps are restricted onto $(\mathbf{\Pi}_0 \cap \mathbf{P}_0) \times \mathbb{C}$, unless otherwise specified). Let

$$\nu : \mathcal{F}_0^K \to \Lambda^{K+} * S_L \oplus \Lambda^{K-} * S_R$$

be the map given by the left arrow in (51). Let $\nu^+ : \mathcal{F}_0^K \to \Lambda^{K+} * S_L$; $\nu^- : \mathcal{F}_0^K \to \Lambda^{K-} * S_L$ be the components of ν.

We have the following obvious embeddings:

$$I_L : \Lambda^{K+} * S_L \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-; \quad I_R : \Lambda^{K-} * S_R \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-;$$

$$\tilde{I}_L : \Lambda^{K+} * S_L \to \Lambda^{K+} * \tilde{S}_+ \oplus \Lambda^{K-} * \tilde{S}_-; \quad \tilde{I}_R : \Lambda^{K-} * S_R \to \Lambda^{K+} * \tilde{S}_+ \oplus \Lambda^{K-} * \tilde{S}_-.$$

The formula (182) can now be rewritten as

$$\phi_{\mathbf{\Pi}_0} = \iota_{\mathbf{P}_0} i_{\mathbf{\Pi}_0 \mathbf{P}_0}.$$

We, therefore, can split

$$i^0_\Phi = \iota_{\mathbf{P}_0}(I_L \oplus I_R)\nu = \phi_{\mathbf{\Pi}_0} i^{-1}_{\mathbf{\Pi}_0 \mathbf{P}_0}(I_L \oplus I_R)\nu. \tag{195}$$

Next, we have

$$i^0_\Psi = \tilde{\iota}_{\mathbf{\Pi}_0}(\tilde{I}_L \oplus \tilde{I}_R)\nu.$$

Combining (192) and (195), we have

$$I_{\Psi\Phi,\mathbf{\Pi}_0} i^0_\Psi = \phi^{-1}_{\mathbf{\Pi}_0} U_{\mathbf{\Pi}_0}(\tilde{I}_L \oplus \tilde{I}_R)\nu;$$

so that the condition (191) is equivalent to the condition

$$U_{\mathbf{\Pi}_0}(\tilde{I}_L \oplus \tilde{I}_R)\nu = i^{-1}_{\mathbf{\Pi}_0 \mathbf{P}_0}(I_L \oplus I_R)\nu \ \text{ as maps } \ \mathcal{F}^K_0 \ \to \ \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\mathbf{\Pi}_0 \times \mathbb{C}} \ . \tag{196}$$

Denote

$$i^{-1}_{\mathbf{\Pi}_0 \mathbf{P}_0}(I_L \oplus I_R)\nu =: \mathbf{I}_0.$$

Let us make this condition (196) more specific.

Lemma 7.12 *Let $\mathcal{J} : \mathcal{F}^K_0 \to (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)[2]$ be an arbitrary map in $\mathbf{D}((\Pi_0 \cap P_0) \times \mathbb{C})$. There exist unique maps*

$$\mathcal{J}^+ : \Lambda^{K+} * S_L \to \Lambda^{K+} * S_+;$$

$$\mathcal{J}^- : \Lambda^{K-} * S_R \to \Lambda^{K-} * S_-$$

such that

$$\mathcal{J} = (\mathcal{J}^+ \oplus \mathcal{J}^-)\nu.$$

Proof We have identifications:

$$\beta \ : \ RHom_{\mathbb{C}}(\mathbb{Z}_K; i^{-1}_{\mathbf{x}_0}(\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} RHom_{\mathbb{C}}(\mathbb{Z}_K; i^!_{\mathbf{x}_0}(\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)[2]) \xrightarrow{\sim} RHom(\mathcal{F}^K_0; (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)[2]),$$

where $i_{\mathbf{x}_0} : \mathbb{C} \to (\Pi_0 \cap P_0) \times \mathbb{C}$ is the inclusion $s \mapsto (\mathbf{x}_0, s)$. Consider two more identification

$$\alpha^+ \ : \ RHom(\Lambda^{K+} * S_L; \Lambda^{K+} * S_+) \xrightarrow{\sim} RHom(i^{-1}_{\mathbf{x}_0}\Lambda^{K+} * S_L; i^{-1}_{\mathbf{x}_0}\Lambda^{K+} * S_+) = RHom(\mathbb{Z}_K; i^{-1}_{\mathbf{x}_0}\Lambda^{K+} * S_+);$$

$$\alpha^- \ : \ RHom(\Lambda^{K-} * S_R; \Lambda^{K-} * S_R) \xrightarrow{\sim} RHom(i_{\mathbf{x}_0}^{-1}\Lambda^{K-} * S_L; i_{\mathbf{x}_0}^{-1}\Lambda^{K-} * S_-) = RHom(\mathbb{Z}_K; i_{\mathbf{x}_0}^{-1}\Lambda^{K-} * S_-);$$

and let $\alpha = \alpha^+ \oplus \alpha^-$. Then we have a chain of identifications

$$RHom(\Lambda^{K+} * S_L; \Lambda^{K+} * S_+) \oplus RHom(\Lambda^{K-} * S_R; \Lambda^{K-} * S_-)$$

$$\xrightarrow{\alpha} RHom_{\mathbb{C}}(\mathbb{Z}_K; i_{\mathbf{x}_0}^{-1}(\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-))$$

$$\xrightarrow{\beta} RHom(\mathcal{F}_0^K; (\Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)[2]).$$

Let

$$\gamma : RHom(\Lambda^{K+} * S_L; \Lambda^{K+} * S_+) \oplus RHom(\Lambda^{K-} * S_R; \Lambda^{K-} * S_-) \to RHom(\mathcal{F}_0^K; \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-)[2])$$

be given by the pre-composition with ν. One can check that $\gamma\beta = \alpha$ so that γ is an isomorphism. The statement now follows. $\square$.

Let $\mathbf{I}_0^{\pm}$ denote the maps obtained from $\mathbf{I}_0$ by means of lemma 7.12. Observe that the maps $\mathbf{I}_0^{\pm}$ uniquely extend from $(\mathbf{\Pi}_0 \cap \mathbf{P}_0) \times \mathbb{C}$ onto $\mathbf{\Pi}_0 \times \mathbb{C}$. Denote the resulting extensions by the symbol $\mathbf{I}^{\pm} : \Lambda^{K\pm} * S_{L/R}|_{\mathbf{\Pi}_0 \times \mathbb{C}} \to \Lambda^{K+} * S_+ \oplus \Lambda^{K-} * S_-|_{\mathbf{\Pi}_0 \times \mathbb{C}}$.

Rewrite the condition (196) in the form:

$$U_{\Pi_0}(\tilde{I}_L \oplus \tilde{I}_R)\nu = (\mathbf{I}_0^+ \oplus \mathbf{I}_0^-)\nu.$$

It now follows that the condition (196) (and thus also (172)) will be satisfied iff

$$U_{\mathbf{\Pi}_0}|_{\Lambda^{K+} * S_L} = \mathbf{I}^+; \ U_{\mathbf{\Pi}_0}|_{\Lambda^{K-} * S_R} = \mathbf{I}^-. \tag{197}$$

Indeed, the implicaton (197) $\Rightarrow$ (196) is obvious, and (196) $\Rightarrow$ (197) follows from (193).

7.5.1 Estimate

Let us prove the following estimates:

Claim 7.13

$$\mathbf{I}^+ \equiv I_L; \quad \mathbf{I}^- \equiv I_R. \tag{198}$$

Let us bring the current notation into correspondence with that in Lemmas 7.8,7.9. Set $\Pi := \mathbf{\Pi}_0$. Let us denote all the α-strips intersecting Π by $P_0, P_1, \ldots, P_n$ in the order from the left to the right, in the same way as in Lemmas 7.8, 7.9. Suppose that $\mathbf{P}_0 = P_k$ so that $i_{\mathbf{\Pi}_0 \mathbf{P}_0} = i_k$ in the notation of Lemmas 7.8,7.9.

Let us now write $i_{\mathbf{\Pi}_0\mathbf{P}_0}^{-1} = i_k^{-1} = \mathrm{Id} + a_0$, where a_0 is an endomorhipsm of $\Lambda^{K+} * \tilde{S}_+ \oplus \Lambda^{K-} * \tilde{S}_-$. Let $\mathbf{a} := a_0(I_L \oplus I_R)\nu$. Our statement now reads as $\mathbf{a}^+ \equiv 0$; $\mathbf{a}^- \equiv 0$.

According to (184), we have

$$a_0 \equiv \begin{pmatrix} 0 & -i_k^{-+} \\ -i_k^{+-} & 0 \end{pmatrix}$$

so that

$$\mathbf{a} = -(i_k^{+-} I_L \oplus i_k^{-+} I_R)\nu. \tag{199}$$

Let us now examine the map $i_k^{+-} I_L \nu$. We have

$$i_k^{+-} I_L : \Lambda^{K+} * S_L|_{\mathbf{\Pi}_0 \cap \mathbf{P}_0 \times \mathbb{C}} \to \Lambda^{K-} * S_-|_{\mathbf{\Pi}_0 \cap \mathbf{P}_0 \times \mathbb{C}} = \bigoplus_{w \in \mathbf{W}^\alpha_{\text{right}}} \mathbb{Z}_{A(K,w)},$$

where, as in (36), (37), $A(K,w) := \{(z,s) | s - z \in K + \hat{c}(w)\} \subset (\mathbf{\Pi}_0 \cap \mathbf{P}_0) \times \mathbb{C}$.

As above, let $\mathbf{W}'_{\text{right}} \subset \mathbf{W}^\alpha_{\text{right}}$ consists of all w such that $A(K,w) \subset A(K,L)$, where

$$A(K,L) = \{(z,s) | s + z - \mathbf{x}_0 \in K\} \subset (\mathbf{\Pi}_0 \cap \mathbf{P}_0) \times \mathbb{C}.$$

Let $E_w : \mathbb{Z}_{A(K,L)} \to \mathbb{Z}_{A(K,w)}$ be the corresponding map of sheaves. We then have

$$i^{+-} I_L = \sum_{w \in \mathbf{W}'_{\text{right}}} n_w E_w,$$

where for each $(z,s) \in A(K,L)$ there are only finitely many w such that $n_w \neq 0$ and $(z,s) \in A(K,w)$.

Let A be a unique vertex of the parallelogram $\mathbf{\Pi}_0 \cap \mathbf{P}_0$ such that $\mathbf{\Pi}_0 \cap \mathbf{P}_0 \subset A + K$. The condition $A(K,w) \subset A(K,L)$ is then equivalent to $2A - \mathbf{x}_0 + \hat{c}(w) \in K$, or $\hat{c}(w) + \mathbf{x}_0 = -2(A - \mathbf{x}_0) + \varepsilon_w$ where $\varepsilon_w \in K$. Observe that $\mathbf{x}_0 - A \in \mathrm{Int}K$ because $\mathbf{x}_0 \in \mathrm{Int}\mathbf{\Pi}_0 \cap \mathbf{P}_0$. It now follows that for each $w \in \mathbf{W}'_{\text{right}}$, the map $E_w \nu^+ : \mathcal{F}_0 \to \mathbb{Z}_{A(K,w)}$ factors as

$$\mathcal{F}_0 \xrightarrow{\nu^-} \Lambda^- * S_R = \mathbb{Z}_{A(K,R)} \to \mathbb{Z}_{\{(z,s)|s-z+\mathbf{x}_0+2(A-\mathbf{x}_0) \in K\}} \xrightarrow{F_w} \mathbb{Z}_{\{(z,s)|s-z+\mathbf{x}_0+2(A-\mathbf{x}_0)-\varepsilon_w \in K\}} = \mathbb{Z}_{A(K,w)},$$

where all the arrows except the leftmost one are induced by the closed embeddings of the corresponding closed sets. It is easy to check that the sum $\sum n_w F_w$ gives rise to a well-defined map

$$J : \mathbb{Z}_{\{(z,s)|s-z+\mathbf{x}_0+2(A-\mathbf{x}_0) \in K\}} \to \bigoplus_{w \in \mathbf{W}^\alpha_{\text{right}}} A(K,w)$$

Let $\delta := 2(A - \mathbf{x}_0)$. We have $bZ_{\{(z,s)|s-z+\mathbf{x}_0+2(A-\mathbf{x}_0) \in K} = T_{\delta *}\mathbb{Z}_{A(K,R)}$. Let $\tau_\delta : \mathbb{Z}_{A(K,R)} \to T_{\delta *}\mathbb{Z}_{A(K,R)}$ be the map induced by the closed embedding of the corresponding closed sets. We then have a factorization

$$i_k^{+-} I_L \nu = J \tau_\delta \nu^-,$$

which implies that $(i_k^{+-} I_L \nu)^+ = J\tau_\delta \equiv 0$. Similarly, one can check that $(i_k^{-+} I_R \nu)^- \equiv 0$, which, by virtue of (199), that $\mathbf{a} = 0$. $\square$

7.6 Inductive construction of the maps U_Π.

We will now construct the maps U_Π satisfying (194) and (197). Taking into account (193), it is possible to construct U_Π in terms of its components

$$U^w_\Pi : \Lambda^{K+} * \tilde{S}_w \to \Lambda^{K+} * S_+, \text{ for all } w \in \mathbf{W}^{-\alpha}_{\text{left}};$$

$$U^w_\Pi : \Lambda^{K-} * \tilde{S}_w \to \Lambda^{K-} * S_-, \text{ for all } w \in \mathbf{W}^{-\alpha}_{\text{right}}.$$

7.6.1 Rewriting the gluing condition

Let us rewrite the conditions (194).

Case 1: ℓ goes to the left and $w \in \mathbf{W}^{\alpha}_{\text{left}}$ (set $\pm = +$ on both sides of (200)) or ℓ goes to the right and $w \in \mathbf{W}^{\alpha}_{\text{right}}$ (set $\pm = -$ on both sides of (200)) Let us rewrite (194):

$$U^w_{\Pi_2}|_{\ell\times\mathbb{C}} = \tilde{A}_{\Pi_1\Pi_2} U^w_{\Pi_1}|_{\ell\times\mathbb{C}} : \Lambda^{K\pm} * S_w|_\ell \to \Lambda^{K\pm} * S_\pm|_{\ell\times\mathbb{C}}. \tag{200}$$

Every map as on the RHS extends uniquely to Π_2 (Lemma 7.6)

so that we can equivalently rewrite

$$U^w_{\Pi_2} = (\Gamma^{\Pi_1\Pi_2}_{\Psi^K} U^w_{\Pi_1}|_\ell)_{\text{ext}}, \tag{201}$$

where ext means the extension onto Π_2.

Case 2:

$$\ell \text{ goes to the left and } w \in \mathbf{W}^{\alpha}_{\text{right}} \text{ (set } \pm = -) \text{ or } \ell \text{ goes to the right and } w \in \mathbf{W}^{\alpha}_{\text{left}} \text{ (set } \pm = +): \tag{202}$$

$$U^w_{\Pi_2}|_{\ell\times\mathbb{C}} = \Gamma^{\Pi_1\Pi_2}_{\Psi^K}(U^w_{\Pi_1}|_{\ell\times\mathbb{C}} \oplus \vartheta(\Pi_2,\Pi_1) U^{\ell w}_{\Pi_1}|_{\ell\times\mathbb{C}} N^w_\ell),$$

where $N^w_\ell : \Lambda^-_\ell * S_w \to \Lambda^+_\ell * S_{\ell w}$ is as in (42).

Recall that $\tilde{A}^{\mp\pm}_{\Pi_1\Pi_2} = 0$ by Claims 7.10, 7.11, so that we can rewrite the RHS as (using notation from Sec 3.8.5)

$$\tilde{A}^{\pm\pm}_{\Pi_1\Pi_2} U^w_{\Pi_1}|_{\ell\times\mathbb{C}} + (\tilde{A}^{\pm\mp}_{\Pi_1\Pi_2} U^w_{\Pi_1}|_{\ell\times\mathbb{C}} + \tilde{A}^{\mp\mp}_{\Pi_1\Pi_2} \vartheta(\Pi_2,\Pi_1) U^{\ell w}_{\Pi_1}|_{\ell\times\mathbb{C}} N^w_\ell).$$

So that we have (by separating $+$ and $-$ components):

$$U^w_{\Pi_2}|_{\ell\times\mathbb{C}} = \tilde{A}^{\pm\pm} U^w_{\Pi_1}|_{\ell\times\mathbb{C}}. \tag{203}$$

$$\tilde{A}^{\pm\mp}_{\Pi_1\Pi_2} U^w_{\Pi_1}|_{\ell\times\mathbb{C}} + \tilde{A}^{\mp\mp}_{\Pi_1\Pi_2} \vartheta(\Pi_2,\Pi_1) U^{\ell w}_{\Pi_1}|_{\ell\times\mathbb{C}} N^w_\ell) = 0. \tag{204}$$

As above, (203) can be equivalently rewritten in the same way as (201).

Let us rewrite (204):

$$U^{\ell w}_{\Pi_1}|_{\ell\times\mathbb{C}} N^w_\ell = -\vartheta(\Pi_2,\Pi_1)\tilde{A}^{\mp\mp}_{\Pi_2\Pi_1}\tilde{A}^{\pm\mp}_{\Pi_1\Pi_2}U^w_{\Pi_1}|_{\ell\times\mathbb{C}}.$$

Given a map $\mathbb{K} : \Lambda^{K\pm} * S_w|_\ell \to \Lambda^{K\mp} * S_\mp|_\ell$, one can uniquely factor it as

$$\mathbb{K} = \mathbb{K}' N^w_\ell,$$

where $\mathbb{K}' : \Lambda^{K\mp} * S_{lw}|_\ell \to \Lambda^{K\mp} * S_\mp|_\ell$ (Sec 7.1.3) which extends uniquely to a map

$$\mathbb{K}'_{\text{ext}} : \Lambda^{K\mp} * S_{lw}|_{\Pi_2} \to \Lambda^{K\mp} * S_\mp|_{\Pi_2}$$

by Lemma 7.6. In view of this remark, we finally write

$$U^{\ell w}_{\Pi_1} = \left(-\vartheta(\Pi_2,\Pi_1)\tilde{A}^{\mp\mp}_{\Pi_2\Pi_1}\tilde{A}^{\pm\mp}_{\Pi_1\Pi_2}U^w_{\Pi_1}|_\ell\right)_{\text{ext}}. \tag{205}$$

Let us summarize. Gluing conditions (194) can be equivalently formulated as follows:

For every pair of neighboring strips Π_1, Π_2, $\ell = \Pi_1 \cap \Pi_2$, we have (201). In the case (202) we also have (205).

Condition (201) implies that

$$U^w_{\Pi_2}|_\ell \equiv U^w_{\Pi_1}|_\ell. \tag{206}$$

7.6.2 Constructing U^w_Π

Let us proceed by the induction in the length of w. In the case $\Pi = \Pi_0$ and $w = L$ or $w = R$, $U^w_{\Pi_0}$ is determined by (197).

Given an arbitrary strip Π, there is a unique sequence

$$\Pi_0, \Pi_1, \ldots, \Pi_n = \Pi \tag{207}$$

where all Π_i are different and $\Pi_i \cap \Pi_{i+1} \neq \emptyset$ (because the graph formed by the strips is a tree). Formulas (201) (applied for all pairs Π_i, Π_{i+1}) determine U^L_Π, U^R_Π for all Π.

Suppose that U^w_Π for all words w of length $\leq N$. Let $w = \ell w'$ be a word of length $N+1$ (so that the length of w' is N). Let $\ell = \Pi_1 \cap \Pi_2$. The formulas (205) determine $U^w_{\Pi_1}$. Given an arbitrary strip Π we can join it with Π_1 by a path and define U^w_Π using (201) in the same way as above.

7.6.3 Estimate

We are going to prove the following estimate. Let Π be a strip. Consider a map $\mathbf{C} = \mathbf{C}_+ \sqcup \mathbf{C}_-$, cf. (188). We will prove

Claim 7.14

$$U_\Pi^w \equiv \mathbf{C} I_w = (-1)^{|w|} I_w.$$

Proof. Let us use induction in $|w|$. If $w = L$ or $w = R$ and Π is arbitrary, the estimate follows from (206). Suppose that the estimate is the case for all w with $|w| \leq N$. Let now $|w'| = N + 1$ and $w' = lw$, $|w| = N$. Let $\ell = \Pi_1 \cap \Pi_2$.

Combining 205 and the inductive assumption, we have:

$$\begin{aligned}
\mathbf{C}^{-1} U_{\Pi_1}^{\ell w} &\equiv \left(-\vartheta(\Pi_2, \Pi_1)\mathbf{C}^{-1} \tilde{A}^{\mp\mp}_{\Pi_2\Pi_1} \tilde{A}^{\pm\mp}_{\Pi_1\Pi_2} \mathbf{C} I_w|_\ell\right)_{\text{ext}} \left(-\vartheta(\Pi_2, \Pi_1)\mathbf{C}^{-1} \tilde{A}_{\Pi_1\Pi_2} \mathbf{C} I_w|_\ell\right)_{\text{ext}} \\
&\overset{\text{Claims 7.10,7.11}}{\equiv} \left(-\vartheta(\Pi_2, \Pi_1)\mathbf{C}^{-1} \tilde{A}^{\pm\mp}_{\Pi_1\Pi_2} \mathbf{C} I_w|_\ell\right)_{\text{ext}} \left(-\vartheta(\Pi_2, \Pi_1)\mathbf{C}^{-1} \tilde{A}_{\Pi_1\Pi_2} \mathbf{C} I_w|_\ell\right)_{\text{ext}} \\
&\overset{(189)}{\equiv} (-\vartheta(\Pi_2, \Pi_1)\tilde{\Gamma}^w_{\Pi_1\Pi_2}|_\ell)_{\text{ext}} \\
&\equiv (N^w_\ell)|_{\text{ext}} = I_{\ell w},
\end{aligned}$$

and (206) allows us to extend this equality to other strips. $\square$

7.6.4 Proof of Proposition (3.6)

Let us first find an expression for the maps $J_{\Pi P}$ as in (173). We have

$$I_{\Psi\Phi,\Pi}|_{\Pi\cap P\times\mathbb{C}} \overset{(192)}{=} \phi_\Pi|_{\Pi\cap P\times\mathbb{C}} U_\Pi|_{\Pi\cap P\times\mathbb{C}} \tilde{\iota}_\Pi^{-1}|_{\Pi\cap P\times\mathbb{C}} \overset{(182)}{=} \iota_{\Phi P}|_{\Pi\cap P\times\mathbb{C}} i_{\Pi P} U_\Pi|_{\Pi\cap P\times\mathbb{C}} \tilde{\iota}_{\Psi\Pi}^{-1}|_{\Pi\cap P\times\mathbb{C}}. \quad (208)$$

Comparison with (173) yields:

$$J_{\Pi P} = i_{\Pi P} U_\Pi|_{\Pi\cap P}.$$

We then have (for every $w \in \mathbf{W}^\alpha$)

$$J_{\Pi P} I_w \equiv i_{\Pi P} I_w (-1)^{|w|},$$

by Claim 7.14.

Let us write

$$i_{\Pi P} I_w : \mathbb{Z}_{A(K,w)} \to \bigoplus_{w'\in\mathbf{W}^\alpha} \mathbb{Z}_{A(K,w')}$$

as

$$i_{\Pi P} I_w = \sum_{w' \in \mathbf{W}'} m^{\Pi P}_{ww'} e_{ww'},$$

where the sum is taken over all w' such that $A(K, w') \subset A(K, w)$ and $e_{ww'} : \mathbb{Z}_{A(K,w)} \to \mathbb{Z}_{A(K,w')}$ is induced by this embedding. We are to show that $m^{\Pi P}_{ww'} \neq 0$ implies that $A(K, w) \neq A(K, w')$. Assume, on the contrary that $A(K, w) = A(K, w')$ for $w, w' \in \mathbf{W}^\alpha$. Since $P \cap \Pi \neq \emptyset$, this is only possible when $w, w' \in \mathbf{W}^\alpha_{\text{right}}$ or $w, w' \in \mathbf{W}^\alpha_{\text{left}}$. Suppose $w, w' \in \mathbf{W}^\alpha_{\text{right}}$. Lemma 7.8 then implies that either $w' = w$, or $\hat{c}(w') - \hat{c}(w) \in \text{Int}K$, i.e. $w \neq w'$, as we wanted. The case $w, w' \in \mathbf{W}^\alpha_{\text{left}}$ is treated in the same way by means of Lemma 7.9. $\square$

References

[G09] A.Getmanenko, *Shatalov-Sternin's construction of complex WKB solutions and the assosicated Riemann surface.* `arXiv:0907.2934`

[KS] Kashiwara, Schapira. *Sheaves on manifolds.*

[MacLane] S. Mac Lane, *Categories for the working mathematicians.* 2nd Edition.

[Sch] P.Schapira, *Microdifferential systems in the complex domain.*

[ShSt] B.Yu. Sternin, V.E. Shatalov, *Borel-Laplace transform and asymptotic theory. Introduction to resurgent analysis.* CRC Press, Boca Raton, FL, 1996.

[V83] A.Voros, *Return of the quatric oscillator. The complex WKB method.* Ann. Inst. H.Poincaré Phys. Théor. 39 (1983)

[T08] D.Tamarkin, *Microlocal condition for non-displaceablility.* `arXiv:0809.1584v1`