DEMAZURE MODULES AND GRADED LIMITS OF MINIMAL AFFINIZATIONS

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Abstract. For a minimal affinization over a quantum loop algebra of type BC, we provide a character formula in terms of Demazure operators and multiplicities in terms of crystal bases. We also provide a simple formula for the limit of characters. These are achieved by verifying that its graded limit (a variant of a classical limit) is isomorphic to some multiple generalization of a Demazure module, and by determining the defining relations of the graded limit.

1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, and $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the associated loop algebra. The theory of finite-dimensional representations of the quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$ has been intensively studied from various viewpoints in recent years. For example, see the survey [CH10] and references therein.

In [Cha95], Chari introduced the notion of minimal affinizations. An affinization $\widehat{V}$ of a simple $U_q(\mathfrak{g})$-module V is by definition a simple $U_q(\mathbf{L}\mathfrak{g})$-module whose highest weight is equal to that of V. Two affinizations of V are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. Then one can define a partial ordering on the set of equivalence classes of affinizations of V, and minimal ones with respect to this ordering are called minimal affinizations. An almost complete classification of minimal affinizations was done by Chari and Pressley in [Cha95, CP95a, CP96a, CP96b], and in particular it was proved that, if $\mathfrak{g}$ is of type $ABCFG$, for every simple $U_q(\mathfrak{g})$-module its minimal affinization is unique.

Given a minimal affinization, one can consider its classical limit. By restricting it to the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ and taking a pull-back, a graded $\mathfrak{g}[t]$-module is obtained. In this article we call this the *graded limit*. Graded limits are quite important for the study of minimal affinizations since the $U_q(\mathfrak{g})$-module structure of a minimal affinization is completely determined by the $U(\mathfrak{g})$-module structure of its graded limit. This idea was applied in [Cha01, CM06b] to Kirillov-Reshetikhin modules, which are minimal affinizations whose highest weights are multiples of a fundamental weight.

The graded limits of general minimal affinizations were first studied by Moura in [Mou10]. In the article, he defined some two $\mathfrak{g}[t]$-modules using the graded limits of Kirillov-Reshetikhin modules, and conjectured in all types that the graded limit of a minimal affinization is isomorphic to them. This conjecture was proved in type A and partially in type BD in the article, and partially in type E_6 in [MP11].

In the present paper we study in more detail the graded limits of minimal affinizations in type ABC. These are the classical types in which minimal affinizations are unique. (Our main interest is in type BC since type A is well-known.)

To introduce our results, let us define some $\mathfrak{g}[t]$-modules. Denote by $\widehat{\mathfrak{g}}$ the non-twisted affine Lie algebra associated with $\mathfrak{g}$, and by $\widehat{\mathfrak{b}} \subseteq \widehat{\mathfrak{g}}$ the standard Borel subalgebra. Let $\xi_1, \dots, \xi_p \in \widehat{P}$ be a sequence of weights of $\widehat{\mathfrak{g}}$, and assume that each ξ_i belongs to the affine Weyl group orbit $\widehat{W}\Lambda^i$ of a dominant integral weight

$\Lambda^i \in \widehat{P}^+$. We define a $\widehat{\mathfrak{b}}$-module $D(\xi_1, \dots, \xi_p)$ by

$$D(\xi_1, \dots, \xi_p) = U(\widehat{\mathfrak{b}})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}) \subseteq \widehat{V}(\Lambda^1) \otimes \cdots \otimes \widehat{V}(\Lambda^p),$$

where $\widehat{V}(\Lambda)$ is the simple highest weight $\widehat{\mathfrak{g}}$-module with highest weight Λ, and v_ξ is an extremal weight vector with weight ξ. When $p = 1$, $D(\xi_1)$ is called a Demazure module. It is easily seen that, if each ξ_i is dominant with respect to $\mathfrak{g}$, then $D(w_\circ\xi_1, \dots, w_\circ\xi_p)$ is $\mathfrak{g}[t]$-stable, where $w_\circ$ is the longest element of the Weyl group of $\mathfrak{g}$.

Let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$, and assume that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ (here $\boldsymbol{\pi}$ denotes the ℓ-highest weight. See Subsection 3.2). By $L(\boldsymbol{\pi})$ we denote its graded limit. Our first main theorem is the following (Theorem 4.5).

Theorem 1.1. *As a $\mathfrak{g}[t]$-module, $L(\boldsymbol{\pi})$ is isomorphic to $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ with suitable $\mathfrak{g}$-dominant $\xi_1, \dots, \xi_n \in \widehat{P}$.*

When $\mathfrak{g}$ is of type AB and $\lambda = \sum_i m_i\varpi_i$ (ϖ_i are the fundamental weights of $\mathfrak{g}$), we set $\xi_i = m_i\varpi_i + m_i'\Lambda_0$, where $m_i' = \lceil m_i/2 \rceil$ if $\mathfrak{g}$ is of type B and $i = n$, and $m_i' = m_i$ otherwise. Here Λ_0 is the fundamental weight of $\widehat{\mathfrak{g}}$ associated with the additional index 0. In type C, we need to choose ξ_i's in a little more complicated way. For the detail see Subsection 4.2.

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a triangular decomposition of $\mathfrak{g}$, and denote by α_i and $\alpha_i^\vee$ the simple roots and coroots respectively. Our second main theorem gives the defining relations of $L(\boldsymbol{\pi})$ (Theorem 4.6).

Theorem 1.2. *The graded limit $L(\boldsymbol{\pi})$ is isomorphic to the cyclic $\mathfrak{g}[t]$-module generated by a nonzero vector v with relations*

$$\mathfrak{n}_+[t]v = 0, \ \ (h \otimes t^s)v = \delta_{s0}\langle h, \lambda\rangle v \text{ for } h \in \mathfrak{h}, s \geq 0, \ \ f_i^{\langle \alpha_i^\vee, \lambda\rangle + 1}v = 0 \text{ for } 1 \leq i \leq n,$$
$$t^2\mathfrak{n}_-[t]v = 0, \text{ and } (f_\alpha \otimes t)v = 0 \text{ for } \alpha \in \Delta_+^1,$$

where Δ_+^1 is a subset of the positive roots Δ_+ defined by

$$\Delta_+^1 = \Big\{\alpha \in \Delta_+ \ \Big| \ \alpha = \sum_{1 \leq i \leq n} n_i\alpha_i \text{ with } n_i \leq 1 \text{ for all } i\Big\}.$$

These theorems are motivated by the Moura's conjecture stated above. In fact when $\mathfrak{g}$ is of type B, the conjecture is proved from our theorems. More precisely, in type B the modules appearing in the theorems are isomorphic to the ones defined in [Mou10], and therefore the two theorems and his conjecture are equivalent. (They are not in type C, and the module $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ is essentially needed to formulate Theorem 1.1 in this type.) It should also be noted that Theorem 1.2, together with a result of [Her07], gives a proof to [CG11, Conjecture 1.13] in type B.

The module $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ can be constructed in another way as follows. For a $\widehat{\mathfrak{b}}$-submodule D of a $\widehat{\mathfrak{g}}$-module V and an index i, let F_iD be the $(\widehat{\mathfrak{b}} \oplus \mathbb{C}f_i)$-submodule of V generated by D. For an element $w \in \widehat{W}$ with reduced expression $w = s_{i_1} \cdots s_{i_p}$, set $F_wD = F_{i_1} \cdots F_{i_p}D$. One can naturally extend F_w to $w \in \widehat{W} \rtimes \Sigma$ (see Section 2.2), where Σ is the group of Dynkin diagram automorphisms. For each $1 \leq i \leq n$, let Λ^i be the dominant integral weight satisfying $\xi_i \in \widehat{W}\Lambda^i$. Then for suitable $w_1, \dots, w_n \in \widehat{W} \rtimes \Sigma$ (see Subsection 4.3), it follows that

$$\begin{aligned} &D(w_\circ\xi_1, \dots, w_\circ\xi_n) \qquad (1.1)\\ &\cong F_{w_\circ w_1}\Big(D(\Lambda^1) \otimes F_{w_2}\Big(D(\Lambda^2) \otimes \cdots \otimes F_{w_{n-1}}\Big(D(\Lambda^{n-1}) \otimes F_{w_n}D(\Lambda^n)\Big)\cdots\Big)\Big). \end{aligned}$$

The character of such a module is given by [LLM02] in terms of Demazure operators $\mathcal{D}_w$ (see Subsection 2.2). Since the character of $L_q(\boldsymbol{\pi})$ is equal to that of $L(\boldsymbol{\pi})$, we obtain the following character formula as a corollary of Theorem 1.1 (Corollary 4.10).

Corollary 1.3.

$$\operatorname{ch} L_q(\boldsymbol{\pi}) = \mathcal{D}_{w_\circ w_1}\Big(e^{\Lambda^1}\cdot\mathcal{D}_{w_2}\Big(e^{\Lambda^2}\cdots\mathcal{D}_{w_{n-1}}\Big(e^{\Lambda^{n-1}}\cdot\mathcal{D}_{w_n}\big(e^{\Lambda^n}\big)\Big)\cdots\Big)\Big)\Big|_{e^{\Lambda_0}=e^{\delta}=1},$$

where δ is the null root.

The right-hand side of (1.1) has a crystal analogue. Using this, we can express the $U_q(\mathfrak{g})$-module multiplicities of $L_q(\boldsymbol{\pi})$ as the number of some elements in a crystal basis (Corollary 4.11). We would like to emphasize that the crystal basis appearing here is (essentially) of *finite type* (see Remark 4.12).

On the other hand, we deduce from Theorem 1.2 the following formula for the limit of normalized characters (Corollary 4.13).

Corollary 1.4. *Let $J \subseteq \{1,\dots,n\}$, and $\lambda_1,\lambda_2,\dots$ be an infinite sequence of elements of P^+ such that $\lim_{k\to\infty}\langle\lambda_k,\alpha_i^\vee\rangle = \infty$ if $i\notin J$ and $\langle\lambda_k,\alpha_i^\vee\rangle = 0$ for all k otherwise. Assume that $L_q(\boldsymbol{\pi}_k)$ is a minimal affinization of $V_q(\lambda_k)$ for each k. Then $\lim_{k\to\infty} e^{-\lambda_k}\operatorname{ch} L_q(\boldsymbol{\pi}_k)$ exists, and*

$$\lim_{k\to\infty} e^{-\lambda_k}\operatorname{ch} L_q(\boldsymbol{\pi}_k) = \prod_{\alpha\in\Delta_+\setminus\Delta_+^J}\frac{1}{1-e^{-\alpha}}\cdot\prod_{\alpha\in\Delta_+\setminus\Delta_+^{1,J}}\frac{1}{1-e^{-\alpha}},$$

where $\Delta_+^J = \Delta_+\cap\Big(\sum_{i\in J}\mathbb{Z}\alpha_i\Big)$ and $\Delta_+^{1,J} = \{\alpha\in\Delta_+ \mid \alpha = \sum_i n_i\alpha_i$ with $n_i\leq 1$ if $i\notin J\}$.

This corollary, together with [MY12, Corollary 5.6], gives a proof to [loc. cit., Conjecture 6.3].

The theorems are established by showing one by one the existence of three surjective homomorphisms

$$D(w_\circ\xi_1,\dots,w_\circ\xi_n)\twoheadrightarrow M(\lambda),\quad M(\lambda)\twoheadrightarrow L(\boldsymbol{\pi}),\quad L(\boldsymbol{\pi})\twoheadrightarrow D(w_\circ\xi_1,\dots,w_\circ\xi_n),$$

where $M(\lambda)$ denotes the $\mathfrak{g}[t]$-module defined in Theorem 1.2. The key idea to verify the first one is to determine the defining relations of $D(w_\circ\xi_1,\dots,w_\circ\xi_n)$ inductively using the isomorphism (1.1). A main tool to prove the latter two is the theory of q-characters introduced by Frenkel and Reshetikhin [FR99]. A q-character is a generalization of a usual character which records the dimensions of generalized eigenspaces (i.e., ℓ-weight spaces) of a $U_q(\mathbf{L}\mathfrak{g})$-module with respect to the commutative subalgebra $U_q(\mathbf{L}\mathfrak{h})$.

In this article we concentrate only on the type ABC. However, (at least a part of) these results would hold in the other types. These will be studied in future publications.

It should be noted that a module similar to the right-hand side of (1.1) also appears in another study of graded limits. In [Nao12], it was proved that the fusion product of graded limits of Kirillov-Reshetikhin modules is isomorphic to such a module. This fact was essentially used to prove the $X = M$ conjecture in type AD.

The plan of this article is as follows. In Section 2, after fixing some notation we define modules $D(\xi_1,\dots,\xi_p)$ and study their properties. In Section 3, we review the theory of finite-dimensional representations of a quantum loop algebra. In Section 4, we state our main theorems and corollaries. Finally in Section 5, we establish our main theorems by showing the existence of three surjective homomorphisms. For this we need some results on q-characters, which are also recalled in this section.

Index of notation

We provide for the reader's convenience a brief index of the notation which is used repeatedly in this paper:

2.1: $C = (c_{ij})_{1\leq i,j\leq n}$, I, $d_1,\ldots,d_n$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, Δ, Δ_+, α_i, ϖ_i, P, P^+, Q, Q^+, W, $w_\circ$, e_α, f_α, $\alpha^\vee$ $(\alpha \in \Delta)$, $\mathfrak{n}_\pm$, $\mathfrak{g}_J$, $\widehat{\mathfrak{g}}$, K, d, $\widehat{\mathfrak{h}}$, $\widehat{\mathfrak{b}}$, $\widehat{\Delta}$, $\widehat{\Delta}_+$, $\widehat{\Delta}^{\mathrm{re}}$, $\widehat{\Delta}^{\mathrm{re}}_+$, δ, $\widehat{I}$, α_0, e_0, f_0, $\alpha^\vee$ $(\alpha\in\widehat{\Delta}^{\mathrm{re}})$, Λ_0, $\widehat{P}$, $\widehat{P}^+$, $\widehat{Q}$, $\widehat{Q}^+$, $\lambda\leq\nu$, $\widehat{W}$, Σ, $\widetilde{W}$, $\mathbf{L}\mathfrak{a}$, $\mathfrak{a}[t]$, $t^s\mathfrak{g}[t]$, $V(\lambda)$, $V(\lambda,a)$, $\mathrm{ch}_{\mathfrak{h}}V$.
2.2: $\widehat{V}(\Lambda)$, v_ξ, $D(\xi_1,\ldots,\xi_p)$, $\widehat{\mathfrak{p}}_i$, F_i, F_w, $\mathcal{D}_w$, v^τ.
3.1: q_i, $U_q(\mathbf{L}\mathfrak{g})$, $x^\pm_{i,r}$, $k^\pm_i$, $h_{i,m}$, $U_q(\mathbf{L}\mathfrak{n}_\pm)$, $U_q(\mathbf{L}\mathfrak{h})$, $U_q(\mathfrak{g})$, $U_q(\mathbf{L}\mathfrak{g}_J)$, $U_q(\mathbf{L}\mathfrak{h}_J)$.
3.2: $V_q(\lambda)$, P^+_q, $\boldsymbol{\varpi}_{i,a}$, P_q, wt, $V_{\boldsymbol{\rho}}$, $L_q(\boldsymbol{\pi})$, $v_{\boldsymbol{\pi}}$, $\boldsymbol{\pi}^*$, ${}^*\boldsymbol{\pi}$.
3.3: $\boldsymbol{\pi}^{(i)}_{m,a}$, $P_{q,J}$, $\boldsymbol{\rho}_J$.
3.4: $\mathbf{A}$, $\big(x^\pm_{i,r}\big)^{(k)}$, $U_{\mathbf{A}}(\mathbf{L}\mathfrak{g})$, $P^+_{\mathbf{A}}$, $L_{\mathbf{A}}(\boldsymbol{\pi})$, $\overline{L_q(\boldsymbol{\pi})}$.
4.1: $L(\boldsymbol{\pi})$, $\overline{v}_{\boldsymbol{\pi}}$.
4.2: $\xi_1,\ldots,\xi_n$, $i^\flat$, p_i, Δ^1_+.
4.3: $w_1,\ldots,w_n$, $w_{[r,t]}$, $\Lambda^1,\ldots,\Lambda^n$.
5.1: $M(\lambda)$, $\alpha_{p,q}$, v_M, D, v_D.
5.2: $\mathrm{wt}_\ell(V)$, ch_qV, $\boldsymbol{\alpha}_{i,a}$, $\boldsymbol{\nu}\leq\boldsymbol{\rho}$.

2. Lie algebras

2.1. **Notation and basics.** Let $C = (c_{ij})_{1\leq i,j\leq n}$ be a Cartan matrix of finite type, and set $I = \{1,\ldots,n\}$. Denote by $D = \mathrm{diag}(d_1,\ldots,d_n)$ the diagonal matrix such that DC is symmetric and the numbers $d_1,\ldots,d_n$ are coprime positive integers.

Let $\mathfrak{g}$ be the complex simple Lie algebra associated with C. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let Δ be the root system, and Δ_+ the set of positive roots. Denote by α_i $(i\in I)$ the simple roots and by ϖ_i $(i \in I)$ the fundamental weights, which are labeled as in [Kac90, Section 4.8]. For notational convenience, we set $\varpi_0 = 0$. Let P be the weight lattice, P^+ the set of dominant integral weights, Q the root lattice and $Q^+ = \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. Let W be the Weyl group and $w_\circ$ the longest element.

For each $\alpha\in\Delta$, denote by $\mathfrak{g}_\alpha$ the corresponding root space, and fix nonzero elements $e_\alpha\in\mathfrak{g}_\alpha$, $f_\alpha\in\mathfrak{g}_{-\alpha}$ and $\alpha^\vee\in\mathfrak{h}$ such that

$$[e_\alpha,f_\alpha]=\alpha^\vee,\quad [\alpha^\vee,e_\alpha]=2e_\alpha,\quad [\alpha^\vee,f_\alpha]=-2f_\alpha.$$

We also use the notation $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$ for $i\in I$. Set $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$. For a subset $J\subseteq I$, denote by $\mathfrak{g}_J$ the semisimple Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i \mid i\in J\}$.

Let $\theta\in\Delta_+$ be the highest root, and denote by $(\ ,\ )$ the unique non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ normalized so that $(\theta^\vee,\theta^\vee) = 2$. The restriction of this bilinear form on $\mathfrak{h}$ induces a linear isomorphism $\nu\colon \mathfrak{h}\to\mathfrak{h}^*$. By $(\ ,\ )$ we also denote the bilinear form on $\mathfrak{h}^*$ induced by ν^{-1}.

Let $\widehat{\mathfrak{g}}$ be the non-twisted affine Lie algebra associated with $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}d,$$

where K denotes the canonical central element and d is the degree operator. The Lie bracket of $\widehat{\mathfrak{g}}$ is given by

$$\begin{aligned}[x\otimes t^m&+a_1K+b_1d, y\otimes t^n+a_2K+b_2d]\\ &=[x,y]\otimes t^{m+n}+nb_1y\otimes t^n-mb_2x\otimes t^m+m\delta_{m,-n}(x,y)K.\end{aligned}$$

Naturally $\mathfrak{g}$ is regarded as a Lie subalgebra of $\widehat{\mathfrak{g}}$. A Cartan subalgebra $\widehat{\mathfrak{h}}$ and a Borel subalgebra $\widehat{\mathfrak{b}}$ are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \mathfrak{g} \otimes t\mathbb{C}[t].$$

Set $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus \mathfrak{g} \otimes t\mathbb{C}[t]$.

We often consider $\mathfrak{h}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\langle K, \lambda \rangle = \langle d, \lambda \rangle = 0$ for $\lambda \in \mathfrak{h}^*$. Let $\widehat{\Delta}$ be the root system of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}_+$ the set of positive roots, $\widehat{\Delta}^{\mathrm{re}}$ the set of real roots, and $\widehat{\Delta}^{\mathrm{re}}_+ = \widehat{\Delta}^{\mathrm{re}} \cap \widehat{\Delta}_+$. Denote by δ the indivisible imaginary root in $\widehat{\Delta}_+$. Set $\widehat{I} = I \sqcup \{0\}$, $\alpha_0 = \delta - \theta$, $e_0 = f_\theta \otimes t$ and $f_0 = e_\theta \otimes t^{-1}$. For $\alpha = \beta + s\delta \in \widehat{\Delta}^{\mathrm{re}}$ with $\beta \in \Delta$ and $s \in \mathbb{Z}$, define $\alpha^\vee \in \widehat{\mathfrak{h}}$ by

$$\alpha^\vee = \beta^\vee + \frac{2s}{(\alpha, \alpha)} K.$$

Denote by $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ the unique element satisfying $\langle K, \Lambda_0 \rangle = 1$ and $\langle \mathfrak{h}, \Lambda_0 \rangle = \langle d, \Lambda_0 \rangle = 0$, and define $\widehat{P}, \widehat{P}^+ \subseteq \widehat{\mathfrak{h}}^*$ by

$$\widehat{P} = P \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{C}\delta \quad \text{and} \quad \widehat{P}^+ = \big\{\lambda \in \widehat{P} \mid \langle \alpha_i^\vee, \lambda \rangle \geq 0 \text{ for all } i \in \widehat{I}\big\}.$$

Let $\widehat{Q} = \sum_{i \in \widehat{I}} \mathbb{Z}\alpha_i$ and $\widehat{Q}^+ = \sum_{i \in \widehat{I}} \mathbb{Z}_{\geq 0}\alpha_i$. For $\lambda, \mu \in \widehat{P}$, we write $\lambda \leq \mu$ if $\mu - \lambda \in \widehat{Q}^+$. Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$, and regard W naturally as a subgroup of $\widehat{W}$. Let $\ell \colon \widehat{W} \to \mathbb{Z}_{\geq 0}$ be the length function. Denote by Σ the group of Dynkin diagram automorphisms of $\widehat{\mathfrak{g}}$. A linear action of Σ on $\widehat{\mathfrak{h}}^*$ is defined by letting $\tau \in \Sigma$ act as follows:

$$\tau(\alpha_i) = \alpha_{\tau(i)} \ \text{ for } i \in \widehat{I}, \quad \tau(\Lambda_0) = \varpi_{\tau(0)} + \Lambda_0 - \frac{1}{2}\big(\varpi_{\tau(0)}, \varpi_{\tau(0)}\big)\delta.$$

Let $\widetilde{W}$ be the subgroup of $GL(\widehat{\mathfrak{h}}^*)$ generated by $\widehat{W}$ and Σ. Since $\tau s_i = s_{\tau(i)}\tau$ holds for $\tau \in \Sigma$ and $i \in \widehat{I}$, we have $\widetilde{W} = \widehat{W} \rtimes \Sigma$. We also define an action of Σ on $\widehat{\mathfrak{g}}$ by letting $\tau \in \Sigma$ act as a Lie algebra automorphism given by

$$\tau(e_i) = e_{\tau(i)}, \ \ \tau(\alpha_i^\vee) = \alpha_{\tau(i)}^\vee, \ \ \tau(f_i) = f_{\tau(i)} \ \text{ for } \ i \in \widehat{I} \ \text{ and } \ \tau(d) = d + \nu^{-1}\big(\varpi_{\tau(0)}\big).$$

The length function ℓ is extended on $\widetilde{W}$ by setting $\ell(w\tau) = \ell(w)$ for $w \in \widehat{W}, \tau \in \Sigma$.

Given a Lie algebra $\mathfrak{a}$, its *loop algebra* $\mathbf{L}\mathfrak{a}$ is defined by the tensor product $\mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ equipped with the Lie algebra structure given by $[x \otimes f, y \otimes g] = [x, y] \otimes fg$. Let $\mathfrak{a}[t]$ and $t^s\mathfrak{a}[t]$ for $s \in \mathbb{Z}_{>0}$ denote the Lie subalgebras $\mathfrak{a} \otimes \mathbb{C}[t]$ and $\mathfrak{a} \otimes t^s\mathbb{C}[t]$ respectively. The Lie algebra $\mathfrak{a}[t]$ is called the *current algebra* associated with $\mathfrak{a}$.

Denote by $V(\lambda)$ the simple $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. For $a \in \mathbb{C}^\times$, let $\mathrm{ev}_a \colon \mathbf{L}\mathfrak{g} \to \mathfrak{g}$ denote the *evaluation map* defined by $\mathrm{ev}_a(x \otimes f) = f(a)x$. Denote by $V(\lambda, a)$ the simple $\mathbf{L}\mathfrak{g}$-module defined by the pull-back of $V(\lambda)$ with respect to ev_a, which is called an *evaluation module*. An evaluation module for $\mathfrak{g}[t]$ is similarly defined, and also denoted by $V(\lambda, a)$ ($\lambda \in P^+, a \in \mathbb{C}$).

For a finite-dimensional semisimple $\mathfrak{h}$-module V, define the $\mathfrak{h}$-character $\mathrm{ch}_{\mathfrak{h}} V$ by

$$\mathrm{ch}_{\mathfrak{h}} V = \sum_{\lambda \in \mathfrak{h}^*} e^\lambda \dim V_\lambda \in \mathbb{Z}[\mathfrak{h}^*],$$

where $V_\lambda = \{v \in V \mid hv = \langle h, \lambda \rangle v \text{ for } h \in \mathfrak{h}\}$. For a finite-dimensional semisimple $\widehat{\mathfrak{h}}$-module $\widehat{V}$, the $\widehat{\mathfrak{h}}$-character $\mathrm{ch}_{\widehat{\mathfrak{h}}} \widehat{V} \in \mathbb{Z}[\widehat{\mathfrak{h}}^*]$ is defined similarly. We will omit the subscript $\mathfrak{h}$ or $\widehat{\mathfrak{h}}$ when they are obvious from the context.

2.2. **Demazure modules and generalizations.** For each $\xi \in \widehat{W}(\widehat{P}^+)$, we define a $\widehat{\mathfrak{b}}$-module $D(\xi)$ as follows: let Λ be the unique element of $\widehat{P}^+$ such that $\xi \in \widehat{W}\Lambda$, and denote by $\widehat{V}(\Lambda)$ the simple highest weight $\widehat{\mathfrak{g}}$-module with highest weight Λ. Let $v_\xi \in \widehat{V}(\Lambda)$ be an extremal weight vector with weight ξ, and set $D(\xi) = U(\widehat{\mathfrak{b}})v_\xi \subseteq \widehat{V}(\Lambda)$.

Definition 2.1. The $\widehat{\mathfrak{b}}$-module $D(\xi)$ is called a *Demazure module*.

Note that, for $i \in \widehat{I}$, $D(\xi)$ is f_i-stable if and only if $\langle \alpha_i^\vee, \xi\rangle \leq 0$. In this article we consider the following generalization of a Demazure module. Let $\xi_1, \dots, \xi_p$ be a sequence of elements of $\widehat{W}(\widehat{P}^+)$. For each $1 \leq j \leq p$, let Λ^j be the element of $\widehat{P}^+$ satisfying $\xi_j \in \widehat{W}\Lambda^j$, and define a $\widehat{\mathfrak{b}}$-submodule $D(\xi_1, \dots, \xi_p)$ of $\widehat{V}(\Lambda^1)\otimes\cdots\otimes\widehat{V}(\Lambda^p)$ by

$$D(\xi_1, \dots, \xi_p) = U(\widehat{\mathfrak{b}})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}).$$

If $\langle \alpha_i^\vee, \xi_j\rangle \leq 0$ holds for all $1 \leq j \leq p$, then $D(\xi_1, \dots, \xi_p)$ is f_i-stable.

Though it seems difficult to give characters of $D(\xi_1, \dots, \xi_p)$ in general, when the sequence $\xi_1, \dots, \xi_p$ has some special property, the character is given in terms of Demazure operators. To explain this, let us recall a result in [LLM02].

Denote by $\widehat{\mathfrak{p}}_i$ for $i \in \widehat{I}$ the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathbb{C}f_i \subseteq \widehat{\mathfrak{g}}$. For a $\widehat{\mathfrak{g}}$-module V, a $\widehat{\mathfrak{b}}$-submodule D of V and $i \in \widehat{I}$, let $F_iD = U(\widehat{\mathfrak{p}}_i)D \subseteq V$. For $w \in \widehat{W}$ with reduced expression $w = s_{i_1}\cdots s_{i_k}$, we set

$$F_wD = F_{i_1}\cdots F_{i_k}D.$$

Though the definition of F_w depends on the choice of a reduced expression, we will use this by abuse of notation (most of the modules F_wD in this article do not depend on the choices). For $i \in \widehat{I}$, define a linear operator $\mathcal{D}_i$ on $\mathbb{Z}[\widehat{P}]$ by

$$\mathcal{D}_i(f) = \frac{f - e^{-\alpha_i}\cdot s_i(f)}{1 - e^{-\alpha_i}},$$

where s_i acts on $\mathbb{Z}[\widehat{P}]$ by $s_i(e^\lambda) = e^{s_i(\lambda)}$. The operator $\mathcal{D}_i$ is called the *Demazure operator* associated with i. For $w \in \widehat{W}$ and its reduced expression $w = s_{i_1}\cdots s_{i_k}$, the operator $\mathcal{D}_w = \mathcal{D}_{i_1}\cdots\mathcal{D}_{i_k}$ is independent of the choice of the expression [Kum02]. The following theorem is a reformulation of [Theorem 5]MR1887117 for our setting (note that $D(\Lambda)$ is 1-dimensional if $\Lambda \in \widehat{P}^+$):

Theorem 2.2. *For sequences $\Lambda^1, \dots, \Lambda^p$ of elements of $\widehat{P}^+$ and $w_1, \dots, w_p$ of elements of $\widehat{W}$, we have*

$$\begin{aligned}\mathrm{ch}_{\widehat{\mathfrak{h}}} F_{w_1}\Big(D(\Lambda^1)\otimes F_{w_2}\Big(D(\Lambda^2)\otimes\cdots\otimes F_{w_{p-1}}\Big(D(\Lambda^{p-1})\otimes F_{w_p}D(\Lambda^p)\Big)\cdots\Big)\Big)\\ = \mathcal{D}_{w_1}\Big(e^{\Lambda_1}\cdot\mathcal{D}_{w_2}\Big(e^{\Lambda^2}\cdots\mathcal{D}_{w_{p-1}}\Big(e^{\Lambda_{p-1}}\cdot\mathcal{D}_{w_p}(e^{\Lambda_p})\Big)\cdots\Big)\Big).\end{aligned} \tag{2.1}$$

Remark 2.3. In [LLM02], the authors studied $\widehat{\mathfrak{b}}$-modules $V_{\mathbf{i},\mathbf{m}}$ called *generalized Demazure modules*. The $\widehat{\mathfrak{b}}$-module in the left-hand side of (2.1) is easily identified with a generalized Demazure module (see [loc. cit., Subsection 1.1], in which the authors explain how a Demazure module is identified with a generalized Demazure module). Under this identification, the above equality follows from [loc. cit., Theorem 5].

In some cases, we can construct $D(\xi_1, \dots, \xi_p)$ using F_w's. To see this, we need the following lemma.

Lemma 2.4. *Let $\xi_1, \dots, \xi_p$ be a sequence of elements of $\widehat{W}(\widehat{P}^+)$ and $i \in \widehat{I}$. If $\langle \alpha_i^\vee, \xi_j \rangle \geq 0$ holds for all $1 \leq j \leq p$, then we have*

$$F_i D(\xi_1, \dots, \xi_p) = D(s_i\xi_1, \dots, s_i\xi_p).$$

Proof. Let $\mathfrak{sl}_{2,i}$ be the Lie subalgebra of $\widehat{\mathfrak{g}}$ spanned by $\{e_i, \alpha_i^\vee, f_i\}$. Since $e_i v_{\xi_j} = 0$ and $f_i v_{s_i\xi_j} = 0$ hold for all j, we easily see that

$$U(\mathfrak{sl}_{2,i})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}) = U(\mathfrak{sl}_{2,i})(v_{s_i\xi_1} \otimes \cdots \otimes v_{s_i\xi_p}).$$

Since $D(s_i\xi_1, \dots, s_i\xi_p)$ is $\widehat{\mathfrak{p}}_i$-stable, this implies the assertion. □

Let $w_1, \dots, w_p$ be a sequence of elements of $\widehat{W}$, and denote by $w_{[r,t]}$ the element $w_r w_{r+1} \cdots w_t \in \widehat{W}$ for $1 \leq r \leq t \leq p$. We assume that $\ell(w_{[1,p]}) = \sum_{j=1}^p \ell(w_j)$. Then for every $1 \leq r \leq t \leq p$ and $\Lambda \in \widehat{P}^+$, if $w_r = s_{i_1} \cdots s_{i_{N(r)}}$ is a reduced expression, then

$$\langle \alpha_{i_u}^\vee, s_{i_{u+1}} \cdots s_{i_{N(r)}} w_{[r+1,t]}\Lambda \rangle \geq 0$$

holds for all $1 \leq u \leq N(r)$ since $\ell(w_{[r,t]}) = \sum_{j=r}^t \ell(w_j)$. Hence by applying Lemma 2.4 several times, the following proposition is proved.

Proposition 2.5. *Let $\Lambda^1, \dots, \Lambda^p$ be a sequence of elements of $\widehat{P}^+$, and $w_1, \dots, w_p$ a sequence of elements of $\widehat{W}$ such that $\ell(w_{[1,p]}) = \sum_{j=1}^n \ell(w_j)$. Then we have*

$$\begin{aligned} F_{w_1}\Big(D(\Lambda^1) \otimes F_{w_2}\Big(D(\Lambda^2) \otimes \cdots \otimes F_{w_{p-1}}\Big(D(\Lambda^{p-1}) \otimes F_{w_p}D(\Lambda^p)\Big)\cdots\Big)\Big) \\ = D\Big(w_{[1,1]}\Lambda^1, w_{[1,2]}\Lambda^2, \dots, w_{[1,p-1]}\Lambda^{p-1}, w_{[1,p]}\Lambda^p\Big). \qquad (2.2) \end{aligned}$$

In conclusion, if there is a sequence $w_1, \dots, w_p \in \widehat{W}$ satisfying $\xi_j = w_{[1,j]}\Lambda^j$ with $\Lambda^j \in \widehat{P}^+$ and $\ell(w_{[1,p]}) = \sum_{j=1}^p \ell(w_j)$, then the character of $D(\xi_1, \dots, \xi_p)$ is given by Theorem 2.2 and Proposition 2.5.

For later use, we need to generalize the above results for elements of $\widetilde{W}$. For $\tau \in \Sigma$ and a $\widehat{\mathfrak{g}}$-module V, let us denote by $F_\tau V$ the $\widehat{\mathfrak{g}}$-module $\{v^\tau \mid v \in V\}$ with $\tau(x)v^\tau = (xv)^\tau$ for $x \in \widehat{\mathfrak{g}}$ and $v \in V$. Note that if $v \in V_\xi$ $(\xi \in \widehat{P})$, then $v^\tau \in (F_\tau V)_{\tau\xi}$. For a $\widehat{\mathfrak{b}}$-submodule D of V, let $F_\tau D$ denote the $\widehat{\mathfrak{b}}$-submodule $\{v^\tau \mid v \in D\}$ of $F_\tau V$. By definition we have $F_\tau F_i = F_{\tau(i)}F_\tau$. Set $F_{w\tau} = F_w F_\tau$ for $w \in \widehat{W}$ and $\tau \in \Sigma$.

Lemma 2.6. *For a sequence $\xi_1, \dots, \xi_p$ of elements of $\widetilde{W}(\widehat{P}^+)$, we have*

$$F_\tau D(\xi_1, \dots, \xi_p) \cong D(\tau\xi_1, \dots, \tau\xi_p).$$

Proof. It is easy to see that $F_\tau \widehat{V}(\Lambda) \cong \widehat{V}(\tau\Lambda)$ for $\Lambda \in \widehat{P}^+$. Let $\Lambda^1, \dots, \Lambda^p \in \widehat{P}^+$ be the elements such that $\xi_j \in \widetilde{W}\Lambda^j$. We have

$$\begin{aligned} F_\tau\big(\widehat{V}(\Lambda^1) \otimes \cdots \otimes \widehat{V}(\Lambda^p)\big) &\cong F_\tau\widehat{V}(\Lambda^1) \otimes \cdots \otimes F_\tau\widehat{V}(\Lambda^p) \\ &\cong \widehat{V}(\tau\Lambda^1) \otimes \cdots \otimes \widehat{V}(\tau\Lambda^p), \end{aligned}$$

and this isomorphism maps $(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p})^\tau$ to a nonzero scalar multiple of $v_{\tau\xi_1} \otimes \cdots \otimes v_{\tau\xi_p}$. Hence the assertion follows. □

Now the following proposition is an easy generalization of Proposition 2.5 ($w_{[r,t]}$ are defined as above).

Proposition 2.7. *Let $\Lambda^1,\dots,\Lambda^p$ be a sequence of elements of $\widehat{P}^+$, and $w_1,\dots,w_p$ a sequence of elements of $\widetilde{W}$ such that $\ell(w_{[1,p]})=\sum_{j=1}^p \ell(w_j)$. Then we have*

$$F_{w_1}\Big(D(\Lambda^1)\otimes F_{w_2}\Big(D(\Lambda^2)\otimes\cdots\otimes F_{w_{p-1}}\Big(D(\Lambda^{p-1})\otimes F_{w_p}D(\Lambda^p)\Big)\cdots\Big)\Big)$$
$$\cong D\Big(w_{[1,1]}\Lambda^1, w_{[1,2]}\Lambda^2,\dots,w_{[1,p-1]}\Lambda^{p-1},w_{[1,p]}\Lambda^p\Big).$$

For $\tau\in\Sigma$, define a linear operator $\mathcal{D}_\tau$ on $\mathbb{Z}[\widehat{P}]$ by $\mathcal{D}_\tau(e^\lambda)=e^{\tau(\lambda)}$. Obviously $\mathrm{ch}\,F_\tau D=\mathcal{D}_\tau\mathrm{ch}\,D$ holds, and we have $\mathcal{D}_\tau\mathcal{D}_i=\mathcal{D}_{\tau(i)}\mathcal{D}_\tau$ [FL06, Lemma 4]. Set $\mathcal{D}_{w\tau}=\mathcal{D}_w\mathcal{D}_\tau$ for $w\in\widehat{W}$ and $\tau\in\Sigma$. Now the following corollary is obvious from Theorem 2.2.

Corollary 2.8. *For sequences $\Lambda^1,\dots,\Lambda^p\in\widehat{P}^+$ and $w_1,\dots,w_p\in\widetilde{W}$, the equality* (2.1) *holds.*

3. Quantum loop algebras

3.1. **Definitions and basics.** Let $\mathbb{C}(q)$ denote the ring of rational functions in an indeterminate q. Set $q_i=q^{d_i}$ for $i\in I$, and

$$[l]_{q_i}=\frac{q_i^l-q_i^{-l}}{q_i-q_i^{-1}},\quad [s]_{q_i}!=[s]_{q_i}[s-1]_{q_i}\cdots[1]_{q_i},\quad \begin{bmatrix} s\\ s'\end{bmatrix}_{q_i}=\frac{[s]_{q_i}!}{[s-s']_{q_i}![s']_{q_i}!}$$

for $l\in\mathbb{Z}$ and $s,s'\in\mathbb{Z}_{\geq 0}$ with $s\geq s'$. The quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$ is the associative $\mathbb{C}(q)$-algebra with generators

$$x_{i,r}^{\pm}\ (i\in I, r\in\mathbb{Z}),\quad k_i^{\pm1}\ (i\in I),\quad h_{i,m}\ (i\in I, m\in\mathbb{Z}\setminus\{0\})$$

and the following relations ($i,j\in I, r,r'\in\mathbb{Z}, m,m'\in\mathbb{Z}\setminus\{0\}$):

$$k_ik_i^{-1}=k_i^{-1}k_i=1,\qquad [k_i,k_j]=[k_i,h_{j,m}]=[h_{i,m},h_{j,m'}]=0,$$
$$k_ix_{j,m}^{\pm}k_i^{-1}=q_i^{\pm c_{ij}}x_{j,m}^{\pm},\qquad [h_{i,m},x_{j,r}^{\pm}]=\pm\frac{1}{m}[mc_{ij}]_{q_i}x_{j,r+m}^{\pm},$$
$$[x_{i,r}^+,x_{j,r'}^-]=\delta_{ij}\frac{\phi_{i,r+r'}^+-\phi_{i,r+r'}^-}{q_i-q_i^{-1}},$$
$$x_{i,r+1}^{\pm}x_{j,r'}^{\pm}-q_i^{\pm c_{ij}}x_{j,r'}^{\pm}x_{i,r+1}^{\pm}=q_i^{\pm c_{ij}}x_{i,r}^{\pm}x_{j,r'+1}^{\pm}-x_{j,r'+1}^{\pm}x_{i,r}^{\pm},$$
$$\sum_{\sigma\in\mathfrak{S}_s}\sum_{k=0}^{s}(-1)^k\begin{bmatrix} s\\ k\end{bmatrix}_{q_i}x_{i,r_{\sigma(1)}}^{\pm}\cdots x_{i,r_{\sigma(k)}}^{\pm}x_{j,r'}^{\pm}x_{i,r_{\sigma(k+1)}}^{\pm}\cdots x_{i,r_{\sigma(s)}}^{\pm}=0\qquad\text{if } i\neq j$$

for all sequences of integers $r_1,\dots,r_s$, where $s=1-c_{ij}$, $\mathfrak{S}_s$ is the symmetric group on s letters, and $\phi_{i,r}^{\pm}$'s are determined by equating coefficients of powers of u in the formula

$$\sum_{r=0}^{\infty}\phi_{i,\pm r}^{\pm}u^{\pm r}=k_i^{\pm1}\exp\left(\pm(q_i-q_i^{-1})\sum_{r'=1}^{\infty}h_{i,\pm r'}u^{\pm r'}\right),$$

and $\phi_{i,\mp r}^{\pm}=0$ for $r>0$. The algebra $U_q(\mathbf{L}\mathfrak{g})$ is isomorphic to the quotient of the quantum affine algebra $U_q'(\widehat{\mathfrak{g}})$ by the ideal generated by a certain central element [Dri87, Bec94]. Denote by $U_q(\mathbf{L}\mathfrak{n}_{\pm})$ and $U_q(\mathbf{L}\mathfrak{h})$ the subalgebras of $U_q(\mathbf{L}\mathfrak{g})$ generated by $\{x_{i,r}^{\pm}\mid i\in I, r\in\mathbb{Z}\}$ and $\big\{k_i^{\pm1},h_{i,m}\mid i\in I, m\in\mathbb{Z}\setminus\{0\}\big\}$ respectively. Then we have

$$U_q(\mathbf{L}\mathfrak{g})=U_q(\mathbf{L}\mathfrak{n}_-)U_q(\mathbf{L}\mathfrak{h})U_q(\mathbf{L}\mathfrak{n}_+)\tag{3.1}$$

by the existence of a Poincaré-Birkhoff-Witt type basis [Bec94]. Denote by $U_q(\mathfrak{g})$ the subalgebra generated by $\{x_{i,0}^{\pm},k_i^{\pm1}\mid i\in I\}$ which is isomorphic to the quantized enveloping algebra associated with $\mathfrak{g}$. For a subset $J\subseteq I$, let $U_q(\mathbf{L}\mathfrak{g}_J)$ denote the subalgebra generated by $\{k_i^{\pm1},h_{i,r},x_{i,s}^{\pm}\mid i\in J, r\in\mathbb{Z}\setminus\{0\}, s\in\mathbb{Z}\}$. We also define

$U_q(\mathbf{L}\mathfrak{h}_J)$ in an obvious way. When $J = \{i\}$ for some $i \in I$, we simply write $U_q(\mathbf{L}\mathfrak{g}_i)$ and $U_q(\mathbf{L}\mathfrak{h}_i)$.

The algebra $U_q(\mathbf{L}\mathfrak{g})$ has a Hopf algebra structure [Lus93, CP94]. In particular if V and W are $U_q(\mathbf{L}\mathfrak{g})$-modules, then $V \otimes W$ and V^* are $U_q(\mathbf{L}\mathfrak{g})$-modules, and we have $(V \otimes W)^* \cong W^* \otimes V^*$.

3.2. **Finite-dimensional modules.** For a $U_q(\mathfrak{g})$-module V and $\lambda \in P$, set

$$V_\lambda = \big\{v \in V \mid k_i v = q_i^{\langle \alpha_i^\vee, \lambda\rangle} v \text{ for } i \in I\big\}.$$

We say V is *of type* 1 if V satisfies

$$V = \bigoplus_{\lambda \in P} V_\lambda.$$

For a finite-dimensional $U_q(\mathfrak{g})$-module V of type 1, define its character $\operatorname{ch} V$ by

$$\operatorname{ch} V = \sum_{\lambda \in P} e^\lambda \dim V_\lambda \in \mathbb{Z}[P].$$

The category of finite-dimensional $U_q(\mathfrak{g})$-modules of type 1 is semisimple. For $\lambda \in P^+$, let $V_q(\lambda)$ denote the $U_q(\mathfrak{g})$-module generated by a nonzero vector v_λ with relations

$$x_{i,0}^+ v_\lambda = 0, \quad k_i v_\lambda = q_i^{\langle \alpha_i^\vee, \lambda\rangle} v_\lambda, \quad \big(x_{i,0}^-\big)^{\langle \alpha_i^\vee, \lambda\rangle + 1} v_\lambda = 0 \quad \text{for } i \in I.$$

The module $V_q(\lambda)$ is simple, finite-dimensional and of type 1, and every simple finite-dimensional $U_q(\mathfrak{g})$-module of type 1 is isomorphic to some $V_q(\lambda)$. Moreover, we have $\operatorname{ch} V_q(\lambda) = \operatorname{ch} V(\lambda)$. For details of these results, see [CP94] for example.

Now we recall the basic results on finite-dimensional $U_q(\mathbf{L}\mathfrak{g})$-modules. Let P_q^+ denote the monoid (under coordinate-wise multiplication) of I-tuples of polynomials $\boldsymbol{\pi} = \big(\boldsymbol{\pi}_1(u), \dots, \boldsymbol{\pi}_n(u)\big)$ such that each $\boldsymbol{\pi}_i(u)$ is expressed as

$$\boldsymbol{\pi}_i(u) = (1 - a_1 u)(1 - a_2 u) \cdots (1 - a_k u)$$

for some $k \geq 0$ and $a_j \in \mathbb{C}(q)^\times$. In other words, P_q^+ is a free abelian monoid generated by $\{\boldsymbol{\varpi}_{i,a} \mid i \in I, a \in \mathbb{C}(q)^\times\}$ where

$$\big(\boldsymbol{\varpi}_{i,a}\big)_j(u) = \begin{cases} 1 - au & \text{if } j = i, \\ 1 & \text{otherwise.} \end{cases}$$

Denote by P_q the corresponding free abelian group, which is called the *ℓ-weight lattice.* We say $\boldsymbol{\rho} \in P_q$ is *dominant* if $\boldsymbol{\rho} \in P_q^+$. Define a homomorphism $\operatorname{wt}\colon P_q \to P$ by

$$\operatorname{wt}(\boldsymbol{\varpi}_{i,a}) = \varpi_i \quad \text{for all } a \in \mathbb{C}(q)^\times.$$

A nonzero vector v of a $U_q(\mathbf{L}\mathfrak{g})$-module V is said to be an *ℓ-weight vector* with ℓ-weight $\boldsymbol{\rho} \in P_q$ if

$$(\phi_{i,\pm r}^\pm - \gamma_{i,\pm r}^\pm)^N v = 0 \quad \text{for } N \gg 0$$

holds for all $i \in I$ and $r \in \mathbb{Z}_{\geq 0}$, where $\gamma_{i,\pm r}^\pm$ $(i \in I, r \in \mathbb{Z}_{\geq 0})$ are the rational functions in q determined by the formula

$$\sum_{r=0}^\infty \gamma_{i,r}^+ u^r = q_i^{\langle \alpha_i^\vee, \operatorname{wt}(\boldsymbol{\rho})\rangle} \frac{\boldsymbol{\rho}_i(q_i^{-1} u)}{\boldsymbol{\rho}_i(q_i u)} = \sum_{r=0}^\infty \gamma_{i,-r}^- u^{-r},$$

in the sense that the left- and right-hand sides are the Laurent expansions of the middle term about $u = 0$ and $u = \infty$, respectively. Denote by $V_{\boldsymbol{\rho}}$ the subspace consisting of ℓ-weight vectors with ℓ-weight $\boldsymbol{\rho}$. If $V = \bigoplus_{\boldsymbol{\rho} \in P_q} V_{\boldsymbol{\rho}}$ holds, we say V is an *ℓ-weight module*. For an ℓ-weight module V and $\mu \in P$, we have

$$V_\mu = \bigoplus_{\substack{\boldsymbol{\rho} \in P_q \\ \mathrm{wt}(\boldsymbol{\rho}) = \mu}} V_{\boldsymbol{\rho}},$$

since $\phi^+_{i,0} = k_i$ and $\gamma^+_{i,0} = q_i^{\langle \alpha_i^\vee, \mathrm{wt}(\boldsymbol{\rho}) \rangle}$. In particular, an ℓ-weight module is of type 1 as a $U_q(\mathfrak{g})$-module. We say a $U_q(\mathbf{L}\mathfrak{g})$-module V is *ℓ-highest weight* with ℓ-highest weight vector v and ℓ-highest weight $\boldsymbol{\pi} \in P_q^+$ if $v \in V_{\boldsymbol{\pi}}$, $U_q(\mathbf{L}\mathfrak{n}_+)v = 0$ and $U_q(\mathbf{L}\mathfrak{n}_-)v = V$ hold. A standard argument using (3.1) shows that for each $\boldsymbol{\pi} \in P_q^+$, there exists a unique simple ℓ-highest weight module $L_q(\boldsymbol{\pi})$ with ℓ-highest weight $\boldsymbol{\pi}$ (up to isomorphism). Denote an ℓ-highest weight vector of $L_q(\boldsymbol{\pi})$ by $v_{\boldsymbol{\pi}}$.

Theorem 3.1 ([CP95b]). *For every $\boldsymbol{\pi} \in P_q^+$, $L_q(\boldsymbol{\pi})$ is finite-dimensional and ℓ-weight. Moreover, every simple finite-dimensional ℓ-weight $U_q(\mathbf{L}\mathfrak{g})$-module is isomorphic to $L_q(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in P_q^+$.*

Remark 3.2. For every sequence of (not necessarily splitting) polynomials $\boldsymbol{\pi} = \big(\boldsymbol{\pi}_1(u), \dots, \boldsymbol{\pi}_n(u)\big)$ such that $\boldsymbol{\pi}_i(0) = 1$ ($i \in I$), the simple module $L_q(\boldsymbol{\pi})$ defined as above is finite-dimensional. For later use, however, it is more convenient to restrict our consideration only to $L_q(\boldsymbol{\pi})$ with $\boldsymbol{\pi} \in P_q^+$. In particular, this restriction makes it easier to apply the theory of q-characters. Since $\boldsymbol{\pi} \in P_q^+$ implies that $L_q(\boldsymbol{\pi})$ is ℓ-weight by [FM01, Theorem 4.1], the above theorem follows from [CP95b].

Let $i \mapsto \bar{i}$ be the bijection $I \to I$ determined by $\alpha_{\bar{i}} = -w_\circ(\alpha_i)$. We define endomorphisms $\boldsymbol{\pi} \mapsto \boldsymbol{\pi}^*$ and $\boldsymbol{\pi} \mapsto {}^*\boldsymbol{\pi}$ of the monoid P_q^+ by setting

$$\boldsymbol{\varpi}^*_{i,a} = \boldsymbol{\varpi}_{\bar{i}, aq^{-r^\vee h^\vee}}, \quad {}^*\boldsymbol{\varpi}_{i,a} = \boldsymbol{\varpi}_{\bar{i}, a^{-1}q^{-r^\vee h^\vee}}$$

respectively, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $r^\vee = \max\{c_{ij}c_{ji} \mid i \neq j\}$. By [Cha95, Proposition 1.6], there is a unique involution σ of $U_q(\mathbf{L}\mathfrak{g})$ such that

$$\sigma(x^\pm_{i,r}) = -x^\mp_{i,-r}, \quad \sigma(h_{i,m}) = -h_{i,-m}, \quad \sigma(k_i^{\pm 1}) = k_i^{\mp 1}, \quad \sigma(\phi^\pm_{i,r}) = \phi^\mp_{i,-r}$$

for $i \in I, r \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$, and σ is also a coalgebra anti-involution. For a $U_q(\mathbf{L}\mathfrak{g})$-module V, denote by σ^*V its pull-back with respect to σ. As a consequence of [FM01, Corollary 6.9], we have the following lemma.

Lemma 3.3. *For every $\boldsymbol{\pi} \in P_q^+$, we have*

$$L_q(\boldsymbol{\pi})^* \cong L_q(\boldsymbol{\pi}^*) \quad \text{and} \quad \sigma^* L_q(\boldsymbol{\pi}) \cong L_q({}^*\boldsymbol{\pi})$$

as $U_q(\mathbf{L}\mathfrak{g})$-modules.

3.3. **Minimal affinizations.** Here we recall the definition of minimal affinizations and their classification when the shape of the Dynkin diagram of $\mathfrak{g}$ is a straight line, i.e., $\mathfrak{g}$ is of type $ABCFG$.

Definition 3.4 ([Cha95]). Let $\lambda \in P^+$.
(i) A simple finite-dimensional $U_q(\mathbf{L}\mathfrak{g})$-module $L_q(\boldsymbol{\pi})$ is said to be an *affinization* of $V_q(\lambda)$ if $\mathrm{wt}(\boldsymbol{\pi}) = \lambda$.
(ii) Affinizations V and W of $V_q(\lambda)$ are said to be *equivalent* if they are isomorphic as $U_q(\mathfrak{g})$-modules. We denote by $[V]$ the equivalence class of V.

If V is an affinization of $V_q(\lambda)$, as a $U_q(\mathfrak{g})$-module we have

$$V \cong V_q(\lambda) \oplus \bigoplus_{\mu < \lambda} V_q(\mu)^{\oplus m_\mu(V)}$$

with some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. Let V and W be affinizations of $V_q(\lambda)$, and define $m_\mu(V), m_\mu(W)$ as above. We write $[V] \leq [W]$ if for all $\mu \in P^+$, either of the following holds:

(i) $m_\mu(V) \leq m_\mu(W)$, or
(ii) there exists some $\nu > \mu$ such that $m_\nu(V) < m_\nu(W)$.

Then $\leq$ defines a partial ordering on the set of equivalence classes of affinizations of $V_q(\lambda)$ [Cha95, Proposition 3.7].

Definition 3.5 ([Cha95])**.** We say an affinization V of $V_q(\lambda)$ is *minimal* if $[V]$ is minimal in the set of equivalence classes of affinizations of $V_q(\lambda)$ with respect to this ordering.

Given $i \in I$, $a \in \mathbb{C}(q)^\times$ and $m \in \mathbb{Z}_{\geq 0}$, define $\boldsymbol{\pi}_{m,a}^{(i)} \in P_q^+$ by

$$\boldsymbol{\pi}_{m,a}^{(i)} = \prod_{k=1}^{m} \boldsymbol{\varpi}_{i,aq_i^{m-2k+1}}.$$

Note that $\boldsymbol{\pi}_{0,a}^{(i)}$ is the unit element of the monoid P_q^+ for all $i \in I$ and $a \in \mathbb{C}(q)^\times$. For $\mathfrak{g}$ of type $ABCFG$, minimal affinizations are completely classified.

Theorem 3.6 ([Cha95, CP95a])**.** *Assume $\mathfrak{g}$ is of type $ABCFG$. For each $\lambda \in P^+$, there exists a unique minimal affinization of $V_q(\lambda)$ up to equivalence. Moreover for $\lambda = \sum_{i\in I} m_i\varpi_i$, $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ if and only if $\boldsymbol{\pi}$ is in the form $\prod_{i\in I} \boldsymbol{\pi}_{m_i,a_i}^{(i)}$ with $(a_i)_{i\in I}$ satisfying one of the following conditions:*

(I) *For all $1 \leq i < j \leq n$, $a_i/a_j = \prod_{i\leq k<j} c_k(\lambda)$,*
(II) *For all $1 \leq i < j \leq n$, $a_i/a_j = \prod_{i\leq k<j} c_k(\lambda)^{-1}$,*

where we set $c_k(\lambda) = q^{d_k m_k + d_{k+1} m_{k+1} + d_k - c_{k,k+1} - 1}$.

Remark 3.7. (i) Because of different normalizations in some definitions, the conditions of a_i's are rewritten in a slightly different way from the ones in [Cha95, CP95a]. (ii) The situation is more complicated in type DE because of the existence of a trivalent node, and the number of the equivalence classes of minimal affinizations of $V_q(\lambda)$ differs depending on λ. In this case, the classification has been achieved except for λ orthogonal to the trivalent node (see [CP96a, CP96b]). We omit the details since we do not consider this case in this article.

Definition 3.8. The simple modules $L_q(\boldsymbol{\pi}_{m,a}^{(i)})$, which are minimal affinizations of $V_q(m\varpi_i)$, are called *Kirillov-Reshetikhin modules.* Among them, the ones with $m = 1$ are called *fundamental modules.*

For a nonempty subset $J \subseteq I$ such that $\mathfrak{g}_J$ is simple, denote by $P_{q,J}$ the ℓ-weight lattice of $U_q(\mathbf{L}\mathfrak{g}_J)$, and define a map $P_q \ni \boldsymbol{\rho} \to \boldsymbol{\rho}_J \in P_{q,J}$ by letting $\boldsymbol{\rho}_J$ be the J-tuple $\big(\boldsymbol{\rho}_i(u)\big)_{i\in J}$.

Lemma 3.9 ([CP96b, Lemma 2.3])**.** *For every $\boldsymbol{\pi} \in P_q^+$, the $U_q(\mathbf{L}\mathfrak{g}_J)$-submodule of $L_q(\boldsymbol{\pi})$ generated by an ℓ-highest weight vector $v_{\boldsymbol{\pi}}$ is isomorphic to the simple $U_q(\mathbf{L}\mathfrak{g}_J)$-module with ℓ-highest weight $\boldsymbol{\pi}_J$.*

For $\mu = \sum_{i\in I} m_i\varpi_i \in P$, write $\mu_J = \sum_{i\in J} m_i\varpi_i$. From this lemma and Theorem 3.6, the following corollary is easily proved.

Corollary 3.10. *Assume that $\mathfrak{g}$ is of type $ABCFG$. If $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$, then the $U_q(\mathbf{L}\mathfrak{g}_J)$-submodule of $L_q(\boldsymbol{\pi})$ generated by $v_{\boldsymbol{\pi}}$ is a minimal affinization of the simple $U_q(\mathfrak{g}_J)$-module with highest weight λ_J.*

In type A, the structure of minimal affinizations are much simpler than that of the other types.

Theorem 3.11 ([Jim86, Section 2], [CP96b, Theorem 3.1])**.** *Assume that $\mathfrak{g}$ is of type A. If $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$, then $L_q(\boldsymbol{\pi})$ is isomorphic to $V_q(\lambda)$ as a $U_q(\mathfrak{g})$-module.*

3.4. **Classical limits.** In this subsection we assume that $\mathfrak{g}$ is classical, since some of the results below (for example Proposition 3.13) have been proved under this assumption.

Let $\mathbf{A} = \mathbb{C}[q, q^{-1}] \subseteq \mathbb{C}(q)$. An $\mathbf{A}$-submodule L of a $\mathbb{C}(q)$-vector space V is called an $\mathbf{A}$-*lattice* if L is a free $\mathbf{A}$-module and $\mathbb{C}(q) \otimes_{\mathbf{A}} L = V$ holds.

Set $\big(x_{i,r}^{\pm}\big)^{(k)} = \big(x_{i,r}^{\pm}\big)^{k}/[k]_{q_i}!$ for $i \in I$, $r \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, and denote by $U_{\mathbf{A}}(\mathbf{L}\mathfrak{g})$ the $\mathbf{A}$-subalgebra of $U_q(\mathbf{L}\mathfrak{g})$ generated by $\big\{k_i^{\pm 1}, \big(x_{i,r}^{\pm}\big)^{(k)} \bigm| i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{>0}\big\}$. Define $U_{\mathbf{A}}(\mathfrak{g})$ in a similar way. Then $U_{\mathbf{A}}(\mathbf{L}\mathfrak{g})$ and $U_{\mathbf{A}}(\mathfrak{g})$ are $\mathbf{A}$-lattices of $U_q(\mathbf{L}\mathfrak{g})$ and $U_q(\mathfrak{g})$ respectively [Cha01, Lemma 2.1], [Lus93]. Set

$$U_1(\mathbf{L}\mathfrak{g}) = \mathbb{C} \otimes_{\mathbf{A}} U_{\mathbf{A}}(\mathbf{L}\mathfrak{g}) \ \text{ and } \ U_1(\mathfrak{g}) = \mathbb{C} \otimes_{\mathbf{A}} U_{\mathbf{A}}(\mathfrak{g}),$$

where $\mathbb{C}$ is regarded as an $\mathbf{A}$-module by letting q act by 1. As shown in the proof of [Cha01, Lemma 2.1], the $\mathbf{A}$-lattice of the quantum affine algebra $U_q'(\widehat{\mathfrak{g}})$ in [Lus93] is mapped onto $U_{\mathbf{A}}(\mathbf{L}\mathfrak{g})$ under the canonical projection. Hence the following proposition is proved from [Lus93], [CP94, Proposition 9.3.10].

Proposition 3.12. *The universal enveloping algebra $U(\mathbf{L}\mathfrak{g})$ is isomorphic to the quotient of $U_1(\mathbf{L}\mathfrak{g})$ by the ideal generated by $1 \otimes k_i - 1 \otimes 1$ $(i \in I)$. In particular if V is a $U_1(\mathbf{L}\mathfrak{g})$-module on which $1 \otimes k_i$'s act by 1, then V is an $\mathbf{L}\mathfrak{g}$-module. Similar statement also holds for $U(\mathfrak{g})$ and $U_1(\mathfrak{g})$.*

Following [CP01], we call $\boldsymbol{\pi} \in P_q^+$ *integral* if, for all $i \in I$, the polynomial $\boldsymbol{\pi}_i(u)$ has coefficients in $\mathbf{A}$ and the coefficient of the highest power of u lies in $\mathbb{C}^{\times} q^{\mathbb{Z}}$. Denote by $P_{\mathbf{A}}^+$ the set of integral elements in P_q^+. Let $\boldsymbol{\pi} \in P_{\mathbf{A}}^+$, and $L_{\mathbf{A}}(\boldsymbol{\pi}) = U_{\mathbf{A}}(\mathbf{L}\mathfrak{g}) v_{\boldsymbol{\pi}} \subseteq L_q(\boldsymbol{\pi})$.

Proposition 3.13 ([CP01, Cha01])**.**
(i) *$L_{\mathbf{A}}(\boldsymbol{\pi})$ is spanned by the vectors*

$$\big(x_{i_1,l_1}^{-}\big)^{(s_1)} \big(x_{i_2,l_2}^{-}\big)^{(s_2)} \cdots \big(x_{i_p,l_p}^{-}\big)^{(s_p)} v_{\boldsymbol{\pi}}$$

for $p \geq 0$, $i_j \in I$, $s_j \in \mathbb{Z}_{\geq 0}$ and $0 \leq l_j \leq N$ with sufficiently large N.
(ii) *$L_{\mathbf{A}}(\boldsymbol{\pi})$ is an $\mathbf{A}$-lattice of $L_q(\boldsymbol{\pi})$.*

Set

$$\overline{L_q(\boldsymbol{\pi})} = \mathbb{C} \otimes_{\mathbf{A}} L_{\mathbf{A}}(\boldsymbol{\pi}),$$

which is called the *classical limit* of $L_q(\boldsymbol{\pi})$. $\overline{L_q(\boldsymbol{\pi})}$ is an $\mathbf{L}\mathfrak{g}$-module by Proposition 3.12, and we have

$$\operatorname{ch} L_q(\boldsymbol{\pi}) = \operatorname{ch} \overline{L_q(\boldsymbol{\pi})}. \tag{3.2}$$

Since $\operatorname{ch} V_q(\mu) = \operatorname{ch} V(\mu)$ holds for all $\mu \in P^+$, we also have

$$[L_q(\boldsymbol{\pi}) : V_q(\mu)] = [\overline{L_q(\boldsymbol{\pi})} : V(\mu)] \tag{3.3}$$

for all $\mu \in P^+$, where the left- and right-hand sides are the multiplicities as a $U_q(\mathfrak{g})$-module and $\mathfrak{g}$-module respectively.

4. Main theorems and corollaries

In this section, we assume that $\mathfrak{g}$ is of type ABC.

4.1. **Graded limit.** Let $\lambda \in P^+$, and let $\boldsymbol{\pi} \in P_{\mathbf{A}}^+$ be such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ (such an element exists by Theorem 3.6). Denote by $\overline{\boldsymbol{\pi}} = \big(\overline{\boldsymbol{\pi}}_1(u), \dots, \overline{\boldsymbol{\pi}}_n(u)\big)$ the I-tuple of polynomials with coefficients in $\mathbb{C}$ obtained from $\boldsymbol{\pi}$ by evaluating q at 1. From Theorem 3.6, we easily see that there exists a unique nonzero complex number a satisfying

$$\overline{\boldsymbol{\pi}}_i(u) = (1 - au)^{\langle \alpha_i^\vee, \lambda \rangle} \quad \text{for all } i \in I.$$

Hence the following lemma is proved from [CP01, Lemma 4.7].

Lemma 4.1. *There exists a surjective* $\mathbf{L}\mathfrak{g}$*-module homomorphism from the classical limit* $\overline{L_q(\boldsymbol{\pi})}$ *to the evaluation module* $V(\lambda, a)$.

Define a Lie algebra automorphism $\tau_a \colon \mathfrak{g}[t] \to \mathfrak{g}[t]$ by

$$\tau_a\big(x \otimes f(t)\big) = x \otimes f(t - a) \quad \text{for } x \in \mathfrak{g}, f \in \mathbb{C}[t].$$

We consider $\overline{L_q(\boldsymbol{\pi})}$ as a $\mathfrak{g}[t]$-module by restriction, and define a $\mathfrak{g}[t]$-module $L(\boldsymbol{\pi})$ by the pull-back $\tau_a^* \overline{L_q(\boldsymbol{\pi})}$.

Definition 4.2. We call the $\mathfrak{g}[t]$-module $L(\boldsymbol{\pi})$ the *graded limit* of the minimal affinization $L_q(\boldsymbol{\pi})$.

In fact, $L(\boldsymbol{\pi})$ turns out to be a graded $\mathfrak{g}[t]$-module from our main theorems, which justify the name "graded limit". We see from Proposition 3.13 (i) and the construction that the vector $\overline{v}_{\boldsymbol{\pi}} = 1 \otimes v_{\boldsymbol{\pi}}$ generates $L(\boldsymbol{\pi})$ as a $\mathfrak{g}[t]$-module. Elementary properties of $L(\boldsymbol{\pi})$ are as follows.

Lemma 4.3. (i) *There exists a surjective* $\mathfrak{g}[t]$*-module homomorphism from* $L(\boldsymbol{\pi})$ *to* $V(\lambda, 0)$.
(ii) *The vector* $\overline{v}_{\boldsymbol{\pi}}$ *satisfies the relations*

$$\mathfrak{n}_+[t]\overline{v}_{\boldsymbol{\pi}} = 0, \quad (h \otimes t^s)\overline{v}_{\boldsymbol{\pi}} = \delta_{s0}\langle h, \lambda \rangle \overline{v}_{\boldsymbol{\pi}} \quad \textit{for } h \in \mathfrak{h}, s \geq 0.$$

(iii) *We have*

$$\operatorname{ch} L_q(\boldsymbol{\pi}) = \operatorname{ch} L(\boldsymbol{\pi}).$$

(iv) *For every* $\mu \in P^+$, *we have*

$$[L_q(\boldsymbol{\pi}) : V_q(\mu)] = [L(\boldsymbol{\pi}) : V(\mu)].$$

Proof. The assertion (i) follows from Lemma 4.1, and (ii) follows from the construction and (i). The assertions (iii) and (iv) are consequences of (3.2) and (3.3) since $\overline{L_q(\boldsymbol{\pi})} \cong L(\boldsymbol{\pi})$ as $\mathfrak{g}$-modules. □

The following is obvious from Theorem 3.11 and Lemma 4.3.

Corollary 4.4. *When* $\mathfrak{g}$ *is of type* A, *the graded limit* $L(\boldsymbol{\pi})$ *is isomorphic to the evaluation module* $V(\lambda, 0)$.

4.2. **Main theorems.** Throughout the rest of this section, we fix $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$ and $\boldsymbol{\pi} \in P_{\mathbf{A}}^+$ such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$.

In this subsection, we shall state our main theorems. (Although these are trivial in type A, we include this type for completeness.) Their proofs are given in the next section.

Let us define an n-tuple $\xi_1, \dots, \xi_n$ of elements of $\widehat{P}$ as follows. If $\mathfrak{g}$ is of type A_n, then $\xi_i = m_i(\varpi_i + \Lambda_0)$ for all $i \in I$. If $\mathfrak{g}$ is of type B_n, then

$$\xi_i = \begin{cases} m_i(\varpi_i + \Lambda_0) & \text{if } 1 \leq i \leq n-1, \\ m_n\varpi_n + \lceil m_n/2 \rceil \Lambda_0 & \text{if } i = n, \end{cases}$$

where $\lceil s \rceil = \min\{r \in \mathbb{Z} \mid r \geq s\}$. If $\mathfrak{g}$ is of type C_n, the definition is a little more complicated. Let $J = \{1 \leq i \leq n-1 \mid m_i > 0\}$, and define $i^\flat \in \widehat{I}$ for each $i \in I$ by

$$i^\flat = \begin{cases} \max\Big(\{0\} \sqcup \{j \in J \mid j < i\}\Big) & \text{if } i \in J, \\ i & \text{otherwise.} \end{cases}$$

Let $j_0 = \max\Big(\{0\} \sqcup J\Big)$, and define a sequence $p_0, p_1, \dots, p_n$ with $p_i \in \{0,1\}$ as follows: set $p_{j_0} = 0$, and define p_i for $i \in \Big(\{0\} \sqcup J\Big) \setminus \{j_0\}$ recursively by

$$p_{i^\flat} \equiv m_i + p_i \mod 2.$$

We put $p_i = 0$ for $i \in I \setminus J$. Now $\xi_1, \dots, \xi_n$ is defined by

$$\xi_i = p_{i^\flat}\varpi_{i^\flat} + (m_i - p_i)\varpi_i + \frac{d_i}{2}\big(m_i - p_i + p_{i^\flat}\big)\Lambda_0.$$

Note that $\sum_{i \in I} \xi_i \in \lambda + \mathbb{Z}_{>0}\Lambda_0$ holds in all cases. Our first main theorem is the following.

Theorem 4.5. *The graded limit $L(\boldsymbol{\pi})$ is isomorphic to $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ as a $\mathfrak{g}[t]$-module.*

In type A, $\xi_i \in \widehat{P}^+$ holds for all i. Thus the $\mathfrak{g}[t]$-module $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ is the submodule of $V(m_1\varpi_1, 0) \otimes \cdots \otimes V(m_n\varpi_n, 0)$ generated by the tensor product of highest weight vectors, which is isomorphic to $V(\lambda, 0)$. Hence the theorem follows from Corollary 4.4.

Let Δ_+^1 be a subset of Δ_+ defined by

$$\Delta_+^1 = \Big\{\alpha \in \Delta_+ \;\Big|\; \alpha = \sum_{i \in I} n_i\alpha_i \text{ with } n_i \leq 1 \text{ for all } i \in I\Big\}.$$

The second main theorem is the following.

Theorem 4.6. *The graded limit $L(\boldsymbol{\pi})$ is isomorphic to the cyclic $\mathfrak{g}[t]$-module generated by a nonzero vector v with relations*

$$\mathfrak{n}_+[t]v = 0, \quad (h \otimes t^s)v = \delta_{s0}\langle h, \lambda\rangle v \text{ for } h \in \mathfrak{h}, s \geq 0, \quad f_i^{m_i+1}v = 0 \text{ for } i \in I,$$
$$t^2\mathfrak{n}_-[t]v = 0 \text{ and } (f_\alpha \otimes t)v = 0 \text{ for } \alpha \in \Delta_+^1. \tag{4.1}$$

When $\mathfrak{g}$ is of type A, $\Delta_+^1 = \Delta_+$ holds and therefore the theorem is easily proved from Corollary 4.4.

Remark 4.7. This theorem implies that $L(\boldsymbol{\pi})$ is a projective object in a certain full-subcategory of the category of $\mathfrak{g} \otimes \big(\mathbb{C}[t]/t^2\mathbb{C}[t]\big)$-modules introduced in [CG11]. In particular this, together with [Her07, Theorem 6.1], gives a proof to [CG11, Conjecture 1.13] in type B.

Remark 4.8. As stated in the introduction, these two theorems are equivalent to [Mou10, Conjecture 3.20] in type AB.

4.3. **Corollaries.** Here we shall give several corollaries on minimal affinizations $L_q(\boldsymbol{\pi})$, which are obtained from the corresponding statements on $L(\boldsymbol{\pi})$ by applying Lemma 4.3 (iii), (iv).

First we apply the results in Subsection 2.2 to the module $D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ in Theorem 4.5. Let us define $w_i \in \widetilde{W}$ for each $1 \leq i \leq n$ as follows:
(i) If $\mathfrak{g}$ is of type A, then $w_i = \mathrm{id}$ for all $1 \leq i \leq n$.
(ii) If $\mathfrak{g}$ is of type B, then

$$w_i = s_{i-1}s_{i-2}\cdots s_1\tau,$$

where τ denotes the element of Σ which exchanges the nodes 0 and 1.
(iii) If $\mathfrak{g}$ is of type C, then

$$w_i = s_{i-1}s_{i-2}\cdots s_1 s_0.$$

For $1 \leq r \leq t \leq n$, denote by $w_{[r,t]}$ the product $w_r w_{r+1} \cdots w_t \in \widetilde{W}$. If $r > t$, we set $w_{[r,t]} = \mathrm{id}$.

Lemma 4.9. (i) (a) *When $\mathfrak{g}$ is of type B, we have*

$$w_{[1,i]}(\Lambda_0) \equiv \varpi_i + \delta_{i,n}\varpi_n + \Lambda_0 \quad \textit{mod } \mathbb{Q}\delta, \ \textit{and}$$
$$w_{[1,i]}(\varpi_n + \Lambda_0) = \varpi_n + \Lambda_0$$

for $1 \leq i \leq n$.
(b) *When $\mathfrak{g}$ is of type C, we have*

$$w_{[1,i]}(\varpi_j + \Lambda_0) \equiv \begin{cases} \varpi_{i-j} + \varpi_i + \Lambda_0 & (0 \leq j < i) \\ \varpi_j + \Lambda_0 & (i \leq j) \end{cases} \quad \textit{mod } \mathbb{Q}\delta$$

for $1 \leq i \leq n$.
(ii) *We have $\ell(w_{[1,n]}) = \sum_{i=1}^n \ell(w_i)$.*

Proof. The assertion (i) is proved by direct calculations. When $\mathfrak{g}$ is of type B, by applying the sequence

$$w_{[1,n]} = \tau(s_1\tau)\cdots(s_{n-1}\cdots s_1\tau)$$

to Λ_0, we see that each reflection s_j changes the weight by a positive multiple of α_j, which implies the assertion (ii). The proof for type C is the same. □

We also define a sequence $\Lambda^1, \ldots, \Lambda^n$ of elements of $\widehat{P}^+$ as follows:
(i) If $\mathfrak{g}$ is of type A, then $\Lambda^i = \xi_i$ for all i.
(ii) If $\mathfrak{g}$ is of type B, then

$$\Lambda^i = \begin{cases} m_i\Lambda_0 & \text{for } 1 \leq i \leq n-1, \\ \overline{m}_n\varpi_n + \lceil m_n/2 \rceil \Lambda_0 & \text{for } i = n, \end{cases}$$

where $\overline{m}_n = 0$ if m_n is even, and $\overline{m}_n = 1$ otherwise.
(iii) If $\mathfrak{g}$ is of type C, then

$$\Lambda^i = \begin{cases} p_{i^\flat}\varpi_{i-i^\flat} + \frac{1}{2}(m_i - p_i + p_{i^\flat})\Lambda_0 & \text{for } 1 \leq i \leq n-1, \\ m_n(\varpi_n + \Lambda_0)(= \xi_n) & \text{for } i = n. \end{cases}$$

We see from Lemma 4.9 (i) that

$$w_{[1,i]}\Lambda^i \equiv \xi_i \mod \mathbb{Q}\delta$$

for all $i \in I$. Hence by Proposition 2.7, we have a $\mathfrak{g}[t]$-module isomorphism

$$\begin{aligned} &D(\xi_1, \ldots, \xi_n) \qquad (4.2) \\ &\cong F_{w_1}\Big(D(\Lambda^1) \otimes F_{w_2}\Big(D(\Lambda^2) \otimes \cdots \otimes F_{w_{n-1}}\Big(D(\Lambda^{n-1}) \otimes F_{w_n} D(\Lambda^n)\Big)\cdots\Big)\Big). \end{aligned}$$

Then since the isomorphism

$$L(\boldsymbol{\pi}) \cong D(w_\circ\xi_1, \ldots, w_\circ\xi_n) \cong F_{w_\circ} D(\xi_1, \ldots, \xi_n)$$

follows from Theorem 4.5 and Lemma 2.4, we see that $L(\boldsymbol{\pi})$ is isomorphic to the $\mathfrak{g}[t]$-module

$$F_{w_\circ w_1}\Big(D(\Lambda^1) \otimes F_{w_2}\Big(D(\Lambda^2) \otimes \cdots \otimes F_{w_{n-1}}\Big(D(\Lambda^{n-1}) \otimes F_{w_n} D(\Lambda^n)\Big)\cdots\Big)\Big). \qquad (4.3)$$

Now the following character formula for $L_q(\boldsymbol{\pi})$ is obtained using Corollary 2.8 and Lemma 4.3 (iii).

Corollary 4.10.

$$\operatorname{ch} L_q(\boldsymbol{\pi}) = \mathcal{D}_{w_\circ w_1}\Big(e^{\Lambda^1}\cdot \mathcal{D}_{w_2}\Big(e^{\Lambda^2}\cdots \mathcal{D}_{w_{n-1}}\Big(e^{\Lambda^{n-1}}\cdot \mathcal{D}_{w_n}(e^{\Lambda^n})\Big)\cdots\Big)\Big)\Big|_{e^{\Lambda_0}=e^{\delta}=1}.$$

Next we give multiplicities of $U_q(\mathfrak{g})$-modules in $L_q(\boldsymbol{\pi})$ in terms of crystal bases. We refer to [HK02] for the basic theory of crystal bases. Let $B(\Lambda)$ be the crystal basis of $\widehat{V}(\Lambda)$ for $\Lambda \in \widehat{P}^+$, and u_Λ its highest weight element. For $\tau \in \Sigma$, let $b \mapsto b^\tau$ denote the bijection from $B(\Lambda)$ to $B(\tau\Lambda)$ satisfying

$$u_\Lambda^\tau = u_{\tau\Lambda} \text{ and } \tilde{f}_{\tau(i)}(b^\tau) = (\tilde{f}_i b)^\tau \text{ for } i \in \widehat{I},$$

where $\tilde{f}_i$ are the Kashiwara operators. Let $\Lambda_{(1)},\dots,\Lambda_{(p)}$ be an arbitrary sequence of elements of $\widehat{P}^+$, and T a subset of the crystal basis $B\big(\Lambda_{(1)}\big)\otimes\cdots\otimes B\big(\Lambda_{(p)}\big)$. We define a subset $\mathcal{F}_\tau T$ by

$$\mathcal{F}_\tau T = \big\{b_1^\tau \otimes\cdots\otimes b_p^\tau \bigm| b_1\otimes\cdots\otimes b_p \in T\big\} \subseteq B\big(\tau\Lambda_{(1)}\big)\otimes\cdots\otimes B\big(\tau\Lambda_{(p)}\big).$$

For $w \in \widehat{W}$ with reduced expression $w = s_{i_1}\cdots s_{i_k}$, we also define a subset $\mathcal{F}_w T$ by

$$\mathcal{F}_w T = \big\{\tilde{f}_{i_1}^{s_1}\tilde{f}_{i_2}^{s_2}\cdots\tilde{f}_{i_k}^{s_k} b \bigm| s_j \geq 0, b \in T\big\}\setminus\{0\} \subseteq B\big(\Lambda_{(1)}\big)\otimes\cdots\otimes B\big(\Lambda_{(p)}\big).$$

Set $\mathcal{F}_{w\tau} = \mathcal{F}_w\mathcal{F}_\tau$. Now let us define a subset Z' of a $U_q(\widehat{\mathfrak{g}})$-crystal basis by

$$Z' = \mathcal{F}_{w_\circ w_1}\Big(u_{\Lambda^1}\otimes \mathcal{F}_{w_2}\Big(u_{\Lambda^2}\otimes\cdots\otimes \mathcal{F}_{w_{n-1}}\Big(u_{\Lambda^{n-1}}\otimes \mathcal{F}_{w_n}(u_{\Lambda^n})\Big)\cdots\Big)\Big).$$

Since this is a crystal analogue of the module (4.3) (see [LLM02], in which $B(\Lambda)$ are realized using LS paths), there is one-to-one correspondence between the classically highest weight elements (i.e., elements annihilated by $\tilde{e}_i$ for $i \in I$) in Z' and the simple $\mathfrak{g}$-module components of $L(\boldsymbol{\pi})$. Note that $\mathcal{F}_{w_\circ}$ generates no new classically highest weight elements, which implies that the same statement also holds for

$$Z = \mathcal{F}_{w_1}\Big(u_{\Lambda^1}\otimes \mathcal{F}_{w_2}\Big(u_\Lambda^2\otimes\cdots\otimes \mathcal{F}_{w_{n-1}}\Big(u_{\Lambda^{n-1}}\otimes \mathcal{F}_{w_n}(u_{\Lambda^n})\Big)\cdots\Big)\Big)$$

instead of Z'. From this and Lemma 4.3 (iv), we have the following corollary.

Corollary 4.11. *For every $\mu \in P^+$, we have*

$$\big[L_q(\boldsymbol{\pi}) : V_q(\mu)\big] = \#\{b \in Z \mid \mathfrak{h}\text{-weight of } b \text{ is } \mu,\ \tilde{e}_i(b) = 0 \text{ for } i \in I\}.$$

Remark 4.12. Assume that $\mathfrak{g}$ is of type BC, and let $J = \{0,1,\dots,n-1\} \subseteq \widehat{I}$, $U_q(\widehat{\mathfrak{g}}_J)$ be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ whose set of simple roots are J, and W_J its Weyl group. Since w_i belongs to $W_J \rtimes \Sigma$ for all $i \in I$, we can regard Z as a subset of the crystal basis of a suitable $U_q(\widehat{\mathfrak{g}}_J)$-module. In view of this, Corollary 4.11 implies that the multiplicities of $L_q(\boldsymbol{\pi})$ are expressed in terms of $U_q(\widehat{\mathfrak{g}}_J)$-crystal bases, which is of *finite type* (D_n and C_n respectively).

Finally we give a formula for the limit of normalized characters of minimal affinizations. Let J be a subset of I, and set $\Delta_+^J = \Delta_+ \cap \Big(\sum_{i\in J}\mathbb{Z}_{\geq 0}\alpha_i\Big)$ and

$$\Delta_+^{1,J} = \{\alpha \in \Delta_+ \mid \alpha = \sum_{i\in I} n_i\alpha_i \text{ with } n_i \leq 1 \text{ if } i \notin J\}.$$

Assume that $\lambda_1,\lambda_2,\dots$ is an infinite sequence of elements of P^+ such that

$$\langle \alpha_i^\vee, \lambda_k\rangle = 0 \quad \text{for all } i \in J,\ k = 1,2,\dots,$$

and

$$\lim_{k\to\infty}\langle \alpha_i^\vee, \lambda_k\rangle = \infty \quad \text{for all } i \notin J.$$

Corollary 4.13. *Let $\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \dots$ be an infinite sequence of elements of $P_{\mathbf{A}}^+$ such that $L_q(\boldsymbol{\pi}_k)$ is a minimal affinization of $V_q(\lambda_k)$. Then $\lim_{k\to\infty} e^{-\lambda_k}\mathrm{ch}\, L_q(\boldsymbol{\pi}_k)$ exists, and*

$$\lim_{k\to\infty} e^{-\lambda_k}\mathrm{ch}\, L_q(\boldsymbol{\pi}_k) = \prod_{\alpha\in\Delta_+\setminus\Delta_+^J}\frac{1}{1-e^{-\alpha}}\cdot\prod_{\alpha\in\Delta_+\setminus\Delta_+^{1,J}}\frac{1}{1-e^{-\alpha}}. \tag{4.4}$$

Proof. By Lemma 4.3 (iii), it suffices to show that $\lim_{k\to\infty} e^{-\lambda_k}\mathrm{ch}_{\mathfrak{h}} L(\boldsymbol{\pi}_k)$ coincides with the right-hand side. Define a Lie subalgebra $\mathfrak{a}_J$ of $\mathfrak{g}[t]$ by

$$\mathfrak{a}_J = \mathfrak{n}_+[t]\oplus\mathfrak{h}[t]\oplus\bigoplus_{\alpha\in\Delta_+^J}\mathbb{C}f_\alpha\oplus\bigoplus_{\alpha\in\Delta_+^{1,J}}\mathbb{C}(f_\alpha\otimes t)\oplus t^2\mathfrak{n}_-[t].$$

For each λ_k, let $\mathbb{C}v_k$ be a 1-dimensional $\mathfrak{a}_J$-module defined by

$$hv_k = \langle h,\lambda_k\rangle v_k \ \text{ for } h\in\mathfrak{h}, \quad (\mathfrak{a}_J\cap\widehat{\mathfrak{n}}_+)v_k = f_\alpha v_k = 0 \ \text{ for } \alpha\in\Delta_+^J,$$

and define a $\mathfrak{g}[t]$-module M_k by

$$M_k = U(\mathfrak{g}[t])\otimes_{U(\mathfrak{a}_J)}\mathbb{C}v_k.$$

For all k, $e^{-\lambda_k}\mathrm{ch}_{\mathfrak{h}}\, M_k$ coincides with the right-hand side of (4.4). Note that M_k and $L(\boldsymbol{\pi}_k)$ have natural $\mathbb{Z}_{\geq 0}$-grading, which we normalize so that the degrees of v_k and $\overline{v}_{\boldsymbol{\pi}_k}$ are 0. We denote these gradings by superscripts. By Theorem 4.6, there exists a surjective homomorphism $\Phi_k\colon M_k\to L(\boldsymbol{\pi}_k)$, and $\ker\Phi_k$ is generated by $w_k^i = f_i^{\langle\alpha_i^\vee,\lambda_k\rangle+1}v_k$ for $i\notin J$. A standard calculation shows $\mathfrak{n}_+w_k^i = 0$, which implies $\ker\Phi_k = \sum_{i\notin J}U(\mathfrak{n}_-[t])U(t\mathfrak{h}[t])U(t\mathfrak{n}_+[t])w_k^i$. Hence if $\beta\in Q^+$ and $s\geq 0$ satisfies $(\ker\Phi_k)^s_{\lambda_k-\beta}\neq 0$, there exists i satisfying

$$-\beta+s\delta+(\langle\alpha_i^\vee,\lambda_k\rangle+1)\alpha_i\in w_\circ\widehat{Q}^+.$$

When β and s are fixed, this does not occur for sufficiently large k, which implies

$$\dim(M_k)^s_{\lambda_k-\beta} = \dim L(\boldsymbol{\pi}_k)^s_{\lambda_k-\beta} \quad \text{if } k\gg 0.$$

Since $(M_k)^s_{\lambda_k-\beta}=0$ except for finitely many s if β is fixed, the assertion follows from this. $\square$

5. Proof of main theorems

Throughout this section, we assume $\mathfrak{g}$ is of type BC unless specified otherwise, and fix $\lambda=\sum_{i\in I}m_i\varpi_i\in P^+$ and $\boldsymbol{\pi}\in P_{\mathbf{A}}^+$ such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$. We freely use the notation in Section 4, ξ_i, Λ^i, etc..

Let $M(\lambda)$ denote the $\mathfrak{g}[t]$-module defined in terms of generators and relations in Theorem 4.6. We shall verify one by one the existence of surjective homomorphisms

$$D(w_\circ\xi_1,\dots,w_\circ\xi_n)\twoheadrightarrow M(\lambda), \quad M(\lambda)\twoheadrightarrow L(\boldsymbol{\pi}), \quad L(\boldsymbol{\pi})\twoheadrightarrow D(w_\circ\xi_1,\dots,w_\circ\xi_n),$$

which implies Theorems 4.5 and 4.6 simultaneously. For the proof of the latter two, we need some results on q-characters. These are recalled in Subsection 5.2.

5.1. **Proof of $\boldsymbol{D(w_\circ\xi_1,\dots,w_\circ\xi_n)\twoheadrightarrow M(\lambda)}$.** Let us prepare several notation. For $1\leq p\leq q\leq n$, set $\alpha_{p,q}=\alpha_p+\alpha_{p+1}+\cdots+\alpha_q$. Then

$$\Delta_+ = \begin{cases}\{\alpha_{p,q}\mid p\leq q\}\sqcup\{\alpha_{p,n}+\alpha_{q,n}\mid p<q\} & \mathfrak{g}=B_n,\\ \{\alpha_{p,q}\mid p\leq q\}\sqcup\{\alpha_{p,n}+\alpha_{q,n-1}\mid p\leq q<n\} & \mathfrak{g}=C_n.\end{cases}$$

Set $\alpha_{p,q}=0$ if $p>q$. For $\alpha=\beta+s\delta\in\widehat{\Delta}^{\mathrm{re}}$ with $\beta\in\Delta$ and $s\in\mathbb{Z}$, we denote by x_α the vector $e_\beta\otimes t^s\in\widehat{\mathfrak{g}}$. Let v_M denote the generator of $M(\lambda)$ in the definition. Set $D=D(w_\circ\xi_1,\dots,w_\circ\xi_n)$, and $v_D=v_{\xi_1}\otimes\cdots\otimes v_{\xi_n}\in D$, which is also a generator as a $\mathfrak{g}[t]$-module. Recall that D is by definition a module over $\mathfrak{g}[t]\oplus\mathbb{C}K\oplus\mathbb{C}d$, and K and d act on v_D by some scalar multiplications. In this subsection, we also view

$M(\lambda)$ as a module over this Lie algebra by letting K and d act on v_M by the same multiplications.

For $\alpha \in \widehat{\Delta}^{\mathrm{re}}$, we define the nonnegative integer $\rho(\alpha)$ by

$$\rho(\alpha) = \sum_{k=1}^{n} \max\big\{0, -\langle \alpha^\vee, \xi_k\rangle\big\}.$$

Assume that $\alpha = \beta + s\delta \in \widehat{\Delta}^{\mathrm{re}}_+$. The following assertions are checked by direct calculations.

(i) We have $\rho(\beta + s\delta) = 0$ unless $-\beta \in \Delta_+ \setminus \Delta^1_+$ and $s = 1$.

(ii) If $\mathfrak{g}$ is of type B and $p < q$, we have

$$\rho\big(-(\alpha_{p,n} + \alpha_{q,n}) + \delta\big) = \sum_{k=q}^{n-1} m_k + \lfloor m_n/2 \rfloor.$$

(iii) If $\mathfrak{g}$ is of type C, we have for $p < q$ that

$$\rho\big(-(\alpha_{p,n} + \alpha_{q,n-1}) + \delta\big) \\ = \begin{cases} \sum_{k=q}^{n-1} m_k - 1 & \text{if } \sum_{k=q}^{n-1} m_k \in 2\mathbb{Z}_{\geq 0} + 1 \text{ and } m_k = 0 \text{ for all } p \leq k < q, \\ \sum_{k=q}^{n-1} m_k & \text{otherwise,} \end{cases}$$

and $\rho\big(-(\alpha_{q,n} + \alpha_{q,n-1}) + \delta\big) = \Big\lfloor \sum_{k=q}^{n-1} m_k/2 \Big\rfloor$.

The following proposition is essential in this subsection.

Proposition 5.1. (i) *If $\mathfrak{g}$ is of type B, we have*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v_D = U(\widehat{\mathfrak{n}}_+)\Big(\sum_{\alpha \in \widehat{\Delta}^{\mathrm{re}}_+} \mathbb{C} x_\alpha^{\rho(\alpha)+1} + t\mathfrak{h}[t]\Big).$$

(ii) *If $\mathfrak{g}$ is of type C, we have*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v_D = U(\widehat{\mathfrak{n}}_+)\Big(\sum_{\alpha \in \widehat{\Delta}^{\mathrm{re}}_+} \mathbb{C} x_\alpha^{\rho(\alpha)+1} + \sum_{\substack{(\alpha,\beta) \in S \\ 1 \leq k < \rho(\alpha)/2+1}} \mathbb{C} x_\alpha^{\rho(\alpha)-2k+1} x_\beta^k + t\mathfrak{h}[t]\Big),$$

where S is a subset of $\widehat{\Delta}^{\mathrm{re}}_+ \times \widehat{\Delta}^{\mathrm{re}}_+$ defined by

$$S = \Big\{\Big(-(\alpha_{p,n} + \alpha_{q,n-1}) + \delta, -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta\Big) \Bigm| 1 \leq p < q \leq n\Big\}.$$

Assuming Proposition 5.1 for a while, we shall prove $D \twoheadrightarrow M(\lambda)$. Set $D' = D(\xi_1, \dots, \xi_p) \subseteq D$, which is generated by v_D as a $\widehat{\mathfrak{b}}$-module.

Lemma 5.2. *There is a $\widehat{\mathfrak{b}}$-module homomorphism from D' to $M(\lambda)$ mapping v_D to v_M.*

Proof. Since the $\widehat{\mathfrak{h}}$-weights of v_D and v_M are same, it is enough to check that $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v_D$ annihilates v_M. First we shall show $x_\alpha^{\rho(\alpha)+1} v_M = 0$ when $\mathfrak{g} = B_n$ and $\alpha = -(\alpha_{p,n} + \alpha_{q,n}) + \delta$ with $p < q$. Set $\gamma = \alpha_{q,n}$. It is easily checked that

$$\Big\langle \gamma^\vee, \lambda + \big(\rho(\alpha)+1\big)\alpha \Big\rangle = \Big(2\sum_{k=q}^{n-1} m_k + m_n\Big) - 2\Big(\sum_{k=q}^{n-1} m_k + \lfloor m_n/2 \rfloor + 1\Big) < 0.$$

On the other hand, a direct calculation shows $x_\gamma x_\alpha^{\rho(\alpha)+1} v_M = 0$, which implies $x_\alpha^{\rho(\alpha)+1} v_M = 0$ as desired since $M(\lambda)$ is a finite-dimensional $\mathfrak{g}$-module. For $\mathfrak{g} = C_n$

and $\alpha = -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta$ with $q < n$, $x_\alpha^{\rho(\alpha)+1} v_M = 0$ is proved by the same argument with $\gamma = \alpha_{q,n-1}$.

Next we shall show $x_\alpha^{\rho(\alpha)-2k+1} x_\beta^k v_M = 0$ for $\mathfrak{g} = C_n$,

$$\alpha = -(\alpha_{p,n} + \alpha_{q,n-1}) + \delta, \quad \beta = -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta \ \text{ with } p < q,$$

and $0 \leq k < \rho(\alpha)/2 + 1$. If $\rho(\alpha) = \sum_{k=q}^{n-1} m_k$, this is proved by the same argument as above with $\gamma = \alpha_{q,n-1}$. So we may assume that $\sum_{k=q}^{n-1} m_k$ is odd and $m_k = 0$ for $p \leq k < q$ (in this case $\rho(\alpha) = \sum_{k=q}^{n-1} m_k - 1 \in 2\mathbb{Z}_{\geq 0}$). The assertion is proved by the descending induction on k. Set $\gamma = \alpha - \beta = -\alpha_{p,q-1}$. Since $\rho(\beta) = \rho(\alpha)/2$ and $x_\gamma v_M = 0$, we have

$$0 = x_\gamma x_\beta^{\rho(\alpha)/2+1} v_M \in \mathbb{C}^\times x_\alpha x_\beta^{\rho(\alpha)/2} v_M.$$

Hence the case $k = \rho(\alpha)/2$ is proved. Assume $k < \rho(\alpha)/2$. A direct calculation using $x_\gamma v_M = 0$ shows

$$\begin{aligned} x_\gamma^2 x_{\beta+\gamma}^{\rho(\alpha)-2k-1} x_\beta^{k+2} v_M =& a_1 x_{\beta+2\gamma}^2 x_{\beta+\gamma}^{\rho(\alpha)-2k-3} x_\beta^{k+2} v_M \\ &+ a_2 x_{\beta+2\gamma} x_{\beta+\gamma}^{\rho(\alpha)-2k-1} x_\beta^{k+1} v_M + a_3 x_{\beta+\gamma}^{\rho(\alpha)-2k+1} x_\beta^k v_M \end{aligned}$$

for some $a_1, a_2, a_3 \in \mathbb{C}$ with $a_3 \neq 0$ (let $x_{\beta+\gamma}^l = 0$ if $l \leq 0$). By the induction hypothesis, this implies $x_\alpha^{\rho(\alpha)-2k+1} x_\beta^k v_M = 0$ as desired. The other relations are trivially checked, and the lemma is proved. $\square$

Let $w_\circ = s_{i_1} \cdots s_{i_{r-1}} s_{i_r}$ be a reduced expression of $w_\circ$, and set $w_\circ^k = s_{i_k} \cdots s_{i_r}$. We define $D^k = F_{w_\circ^k} D'$ for $1 \leq k \leq r+1$. Note that $D^{r+1} = D'$, and $D^1 = D$ follows from Lemma 2.4. In the following, we shall verify by the descending induction on k that there exists a nonzero $\widehat{\mathfrak{b}}$-module homomorphism from D^k to $M(\lambda)$. This for $k = r+1$ is just Lemma 5.2. Assume $k \leq r$, and consider a $\widehat{\mathfrak{p}}_{i_k}$-module $U(\widehat{\mathfrak{p}}_{i_k}) \otimes_{U(\widehat{\mathfrak{b}})} D^{k+1}$. This has a unique maximal finite-dimensional $\widehat{\mathfrak{p}}_{i_k}$-quotient [Jos85], which we denote by $\widetilde{D}^{k+1}$. Note that, if N is a finite-dimensional $\widehat{\mathfrak{p}}_{i_k}$-module, every $\widehat{\mathfrak{b}}$-module homomorphism $D^{k+1} \to N$ uniquely extends to a $\widehat{\mathfrak{p}}_{i_k}$-module homomorphism $\widetilde{D}^{k+1} \to N$ by definition. By the induction hypothesis, there is a nonzero $\widehat{\mathfrak{b}}$-module homomorphism $D^{k+1} \to M(\lambda)$, which extends to a $\widehat{\mathfrak{p}}_{i_k}$-module homomorphism $\widetilde{D}^{k+1} \to M(\lambda)$. Hence it suffices to show that $\widetilde{D}^{k+1} \cong D^k$. The identity on D^{k+1} extends to a homomorphism

$$\widetilde{D}^{k+1} \to F_{i_k} D^{k+1} = U(\widehat{\mathfrak{p}}_{i_k}) D^{k+1} = D^k,$$

and this is obviously surjective. On the other hand, [Jos85, Lemmas 2.6, 2.8(i)] and Theorem 2.2 imply

$$\mathrm{ch}_{\widehat{\mathfrak{h}}} \widetilde{D}^{k+1} = \mathcal{D}_{i_k} \mathrm{ch}_{\widehat{\mathfrak{h}}} D^{k+1} = \mathrm{ch}_{\widehat{\mathfrak{h}}} F_{i_k} D^{k+1} = \mathrm{ch}_{\widehat{\mathfrak{h}}} D^k,$$

and therefore $\widetilde{D}^{k+1} \cong D^k$ holds as desired.

We have verified that there is a nonzero $\widehat{\mathfrak{b}}$-module homomorphism from D to $M(\lambda)$. Note that D and $M(\lambda)$ are generated by the 1-dimensional weight spaces $D_{w_\circ \lambda}$ and $M(\lambda)_{w_\circ \lambda}$ respectively, and these spaces are annihilated by $\mathfrak{n}_-$. From this, we easily see that the homomorphism is surjective, and extends to one of $\mathfrak{g}[t]$-modules. Hence $D \twoheadrightarrow M(\lambda)$ is proved.

It remains to show Proposition 5.1. For $1 \leq j \leq n$ and $0 \leq i \leq j$, let

$$v(i,j) = v_{w_i \Lambda^j} \otimes v_{w_i w_{j+1} \Lambda^{j+1}} \otimes \cdots \otimes v_{w_i w_{[j+1,n-1]} \Lambda^{n-1}} \otimes v_{w_i w_{[j+1,n]} \Lambda^n},$$

which is a generator of the $\widehat{\mathfrak{b}}$-module

$$D(i,j) = F_{w_i} \Big(D(\Lambda^j) \otimes F_{w_{j+1}} \Big(D(\Lambda^{j+1}) \otimes \cdots \otimes F_{w_{n-1}} \Big(D(\Lambda^{n-1}) \otimes F_{w_n} D(\Lambda^n) \Big) \cdots \Big) \Big)$$

by Proposition 2.7 (here we set $w_0 = \mathrm{id}$). Note that D' is isomorphic to $D(1,1)$ by (4.2), and this isomorphism maps v_D to $v(1,1)$. Hence it suffices to determine the annihilator of $v(1,1)$. In the following, we shall achieve this by determining the annihilators of $v(i,j)$'s inductively.

We prepare two lemmas. For $i \in \widehat{I}$, define a Lie subalgebra $\widehat{\mathfrak{n}}_i$ of $\widehat{\mathfrak{n}}_+$ by $\widehat{\mathfrak{n}}_i = \bigoplus_{\alpha\in\Delta_+^{\mathrm{re}}\setminus\{\alpha_i\}} \mathbb{C}x_\alpha \oplus t\mathfrak{h}[t]$. Note that $\widehat{\mathfrak{n}}_+ = \mathbb{C}e_i \oplus \widehat{\mathfrak{n}}_i$.

Lemma 5.3. *Let $i \in \widehat{I}$, $\xi \in \widehat{P}$ such that $\langle \alpha_i^\vee, \xi\rangle \geq 0$, V be an integrable $\widehat{\mathfrak{g}}$-module, and T a finite-dimensional $\widehat{\mathfrak{b}}$-submodule of V. Assume the following.*
(i) *T is generated by a weight vector $v \in T_\xi$ satisfying $e_i v = 0$.*
(ii) *There is an $ad(e_i)$-invariant left $U(\widehat{\mathfrak{n}}_i)$-ideal $\mathcal{I}$ such that*

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v = U(\widehat{\mathfrak{n}}_+)e_i + U(\widehat{\mathfrak{n}}_+)\mathcal{I}.$$

(iii) *We have $\mathrm{ch}_{\widehat{\mathfrak{h}}} F_iT = \mathcal{D}_i \mathrm{ch}_{\widehat{\mathfrak{h}}} T$.*
Let $v' = f_i^{\langle\alpha_i^\vee,\xi\rangle} v \in F_iT$. Then we have

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v' = U(\widehat{\mathfrak{n}}_+)e_i^{\langle\alpha_i^\vee,\xi\rangle+1} + U(\widehat{\mathfrak{n}}_+)r_i(\mathcal{I}),$$

where r_i denotes the algebra automorphism of $U(\widehat{\mathfrak{g}})$ corresponding to s_i.

Proof. The following proof is essentially same as a part of the proof of [Jos85, Theorem 3.4].

By the $\mathrm{ad}(e_i)$-invariance of $\mathcal{I}$, it follows that

$$U(\widehat{\mathfrak{n}}_+)\mathcal{I} = \mathbb{C}[e_i]\mathcal{I} \subseteq \mathcal{I} + U(\widehat{\mathfrak{n}}_+)e_i.$$

Hence (ii) implies $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_i)} v = \mathcal{I}$, for $U(\widehat{\mathfrak{n}}_+) = U(\widehat{\mathfrak{n}}_+)e_i \oplus U(\widehat{\mathfrak{n}}_i)$. By [Kac90, Lemma 3.8], there is a $\widehat{\mathfrak{p}}_i$-module automorphism r_i' on F_iT satisfying $r_i'(v) \in \mathbb{C}^\times v'$ and $(r_i')^{-1}xr_i' = r_i(x)$ for $x \in \widehat{\mathfrak{p}}_i$. Hence by applying r_i to $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_i)} v = \mathcal{I}$, we have $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_i)} v' = r_i(\mathcal{I})$. Now, since (iii) implies

$$\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v' = U(\widehat{\mathfrak{n}}_+)e_i^{\langle\alpha_i^\vee,\xi\rangle+1} + U(\widehat{\mathfrak{n}}_+)\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_i)} v'$$

by [Jos85, Proposition 3.2], the assertion is proved. □

For $1 \leq j \leq n$ and $0 \leq i \leq j$, define $\rho_{i,j} : \widehat{\Delta}^{\mathrm{re}} \to \mathbb{Z}_{\geq 0}$ by

$$\rho_{i,j}(\alpha) = \sum_{k=j}^{n} \max\Big\{0, -\langle \alpha^\vee, w_i w_{[j+1,k]}\Lambda^k\rangle\Big\},$$

and put $\widehat{\Delta}_+^{\mathrm{re}}(i,j) = \{\alpha \in \widehat{\Delta}_+^{\mathrm{re}} \mid \rho_{i,j}(\alpha) > 0\}$. When $j < n$, we have

$$\rho_{0,j}(\alpha) = \rho_{j+1,j+1}(\alpha) + \max\{0, -\langle\alpha^\vee, \Lambda^j\rangle\} = \rho_{j+1,j+1}(\alpha) \quad \text{for } \alpha \in \widehat{\Delta}_+^{\mathrm{re}}, \tag{5.1}$$

which implies $\widehat{\Delta}_+^{\mathrm{re}}(0,j) = \widehat{\Delta}_+^{\mathrm{re}}(j+1,j+1)$.

Lemma 5.4. *Assume $1 \leq i \leq j \leq n$.*
(i) *If $\mathfrak{g} = B_n$, then*

$$\begin{aligned}\widehat{\Delta}_+^{\mathrm{re}}(i,j) \subseteq &\Big\{\alpha_{p,i-1} \;\Big|\; 1 \leq p < i\Big\} \sqcup \Big\{\alpha_{p,q} \;\Big|\; 1 \leq p \leq j \leq q < n,\ p \neq i\Big\} \\ &\sqcup \Big\{-(\alpha_{i,n} + \alpha_{q,n}) + \delta \;\Big|\; j < q \leq n\Big\} \sqcup \Big\{-(\alpha_{p,n} + \alpha_{q,n}) + \delta \;\Big|\; j < p < q \leq n\Big\}.\end{aligned}$$

(ii) *If $\mathfrak{g} = C_n$, then*

$$\begin{aligned}\widehat{\Delta}_+^{\mathrm{re}}(i,j) \subseteq &\Big\{\alpha_{p,i-1} \;\Big|\; 1 \leq p < i\Big\} \sqcup \Big\{\alpha_{p,q} \;\Big|\; 1 \leq p \leq j \leq q < n,\ p \neq i\Big\} \\ &\sqcup \Big\{-(\alpha_{i,n} + \alpha_{q,n-1}) + \delta \;\Big|\; q = i \text{ or } j < q \leq n\Big\} \\ &\sqcup \Big\{-(\alpha_{p,n} + \alpha_{q,n-1}) + \delta \;\Big|\; j < p \leq q \leq n\Big\}.\end{aligned}$$

Proof. We prove (i) only (the proof of (ii) is similar). Note that the following two containments hold:

$$\widehat{\Delta}_+^{\mathrm{re}}(i+1,j) \subseteq s_i\Big(\widehat{\Delta}_+^{\mathrm{re}}(i,j)\Big) \sqcup \{\alpha_i\} \quad \text{for } i<j, \text{ and} \tag{5.2}$$

$$\widehat{\Delta}_+^{\mathrm{re}}(1,j) \subseteq \tau\Big(\widehat{\Delta}_+^{\mathrm{re}}(j+1,j+1)\Big) \quad \text{for } j<n. \tag{5.3}$$

In fact, (5.2) holds since $\rho_{i+1,j}(\alpha) = \rho_{i,j}(s_i\alpha)$, and (5.3) holds since

$$\rho_{1,j}(\alpha) = \rho_{0,j}(\tau\alpha) = \rho_{j+1,j+1}(\tau\alpha) \quad \text{for } \alpha \in \widehat{\Delta}_+^{\mathrm{re}}$$

by (5.1). Then the assertion can be proved inductively from $\widehat{\Delta}_+^{\mathrm{re}}(0,n) = \emptyset$ using these containments. $\square$

Now let us begin the proof of Proposition 5.1. First assume that $\mathfrak{g}$ is of type B_n. We verify the assertion

$$(\mathrm{B}_{i,j}) \qquad \mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v(i,j) = U(\widehat{\mathfrak{n}}_+)\Big(\sum_{\alpha\in\Delta_+^{\mathrm{re}}} \mathbb{C}x_\alpha^{\rho_{i,j}(\alpha)+1} + t\mathfrak{h}[t]\Big)$$

by the induction on (i,j), which with $i=j=1$ implies the proposition. $(\mathrm{B}_{0,n})$ is obvious since $D(0,n) = D(\Lambda^n)$ is a trivial $\widehat{\mathfrak{n}}_+$-module and $\rho_{0,n}(\alpha) = 0$ for all $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$. We easily see that $(\mathrm{B}_{j+1,j+1})$ implies $(\mathrm{B}_{0,j})$ from (5.1) and $v(0,j) = v_{\Lambda^j} \otimes v(j+1,j+1)$, and it is also easy to check that $(\mathrm{B}_{0,j})$ implies $(\mathrm{B}_{1,j})$ since $v(1,j) = v(0,j)^\tau$ and $\rho_{1,j}(\alpha) = \rho_{0,j}(\tau\alpha)$. It remains to show that $(\mathrm{B}_{i,j})$ with $0<i<j$ implies $(\mathrm{B}_{i+1,j})$. For this, it suffices to show the $\mathrm{ad}(e_i)$-invariance of the left $U(\widehat{\mathfrak{n}}_i)$-ideal

$$\mathcal{I}_{i,j} = U(\widehat{\mathfrak{n}}_i)\Big(\sum_{\alpha\in\widehat{\Delta}_+^{\mathrm{re}}\setminus\{\alpha_i\}} \mathbb{C}x_\alpha^{\rho_{i,j}(\alpha)+1} + t\mathfrak{h}[t]\Big)$$

by Lemma 5.3. Note that, if $\beta \in \widehat{\Delta}_+^{\mathrm{re}}$ is in the form $\beta = l\alpha + \alpha_i$ with some $\alpha \in \widehat{\Delta}_+$ and $l \in \mathbb{Z}_{>0}$, then $\rho_{i,j}(\beta) = 0$ holds. In fact, the condition implies $\beta \in \widehat{\Delta}_+^{\mathrm{re}} + \delta$, or

$$\begin{aligned}\beta \in \big\{\alpha_{p,i} \bigm| p<i\big\} &\sqcup \big\{\alpha_{i,q} \bigm| q>i\big\} \sqcup \big\{\alpha_{i,n}+\alpha_{q,n} \bigm| q\neq i\big\} \sqcup \big\{-\alpha_{p,i-1}+\delta \bigm| p<i\big\} \\ &\sqcup \big\{-\alpha_{i+1,q}+\delta \bigm| q>i\big\} \sqcup \Big\{-\big(\alpha_{i+1,n}+\alpha_{q,n}\big)+\delta \Bigm| q\neq i,i+1\Big\},\end{aligned}$$

and hence $\rho_{i,j}(\beta) = 0$ holds from Lemma 5.4 and $0<i<j$. Then the $\mathrm{ad}(e_i)$-invariance of $\mathcal{I}_{i,j}$ is immediately follows from this, as desired.

Next assume that $\mathfrak{g} = C_n$, and define a subset S_j of $\widehat{\Delta}_+^{\mathrm{re}} \times \widehat{\Delta}_+^{\mathrm{re}}$ for $1 \le j \le n$ by

$$S_j = \bigg\{\Big(-(\alpha_{p,n}+\alpha_{q,n-1})+\delta, -(\alpha_{q,n}+\alpha_{q,n-1})+\delta\Big) \biggm| j \le p < q \le n\bigg\}.$$

We verify the assertion $(\mathrm{C}_{i,j})$: $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v(i,j) = \mathcal{J}_{i,j}$ by the induction on (i,j), where $\mathcal{J}_{i,j}$ is a $U(\widehat{\mathfrak{n}}_+)$-ideal defined by

$$\mathcal{J}_{i,j} = U(\widehat{\mathfrak{n}}_+)\Big(\sum_{\alpha\in\Delta_+^{\mathrm{re}}} \mathbb{C}x_\alpha^{\rho_{i,j}(\alpha)+1} + \sum_{\substack{(\alpha,\beta)\in s_is_{i+1}\cdots s_{j-1}(S_j)\\ 1\le k<\rho_{i,j}(\alpha)/2+1}} \mathbb{C}x_\alpha^{\rho_{i,j}(\alpha)-2k+1}x_\beta^k + t\mathfrak{h}[t]\Big).$$

Here we set $w(S_j) = \big\{(w\alpha, w\beta) \bigm| (\alpha,\beta)\in S_j\big\}$ for $w \in W$. Note that $(\mathrm{C}_{1,1})$ implies the proposition, and $(\mathrm{C}_{0,n})$ is obvious.

Let us show that $(\mathrm{C}_{j+1,j+1})$ implies $(\mathrm{C}_{0,j})$. Since $v(0,j) = v_{\Lambda^j} \otimes v(j+1,j+1)$, we have $\mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v(0,j) = \mathrm{Ann}_{U(\widehat{\mathfrak{n}}_+)} v(j+1,j+1)$, and hence it suffices to show that $\mathcal{J}_{0,j} = \mathcal{J}_{j+1,j+1}$. We have $\rho_{0,j}(\alpha) = \rho_{j+1,j+1}(\alpha)$ for $\alpha \in \widehat{\Delta}_+^{\mathrm{re}}$ by (5.1), and direct calculation shows

$$s_0 s_1 \cdots s_{j-1} S_j = S_{j+1} \sqcup \left\{ \Big(\alpha_{1,q-1}, -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta\Big) \,\Big|\, j < q \leq n \right\}. \tag{5.4}$$

Hence it is enough to check

$$x_\alpha^{\rho_{j+1,j+1}(\alpha)-2k+1} x_\beta^k \in \mathcal{J}_{j+1,j+1} \tag{5.5}$$

for $\alpha = \alpha_{1,q-1}$, $\beta = -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta$ with $j < q$ and $1 \leq k < \rho_{j+1,j+1}(\alpha)/2 + 1$. For every $l_1 \in \mathbb{Z}_{\geq 2}$ and $l_2 \in \mathbb{Z}_{\geq 0}$, it is directly checked from $\beta = \alpha_0 + 2\alpha$ that

$$e_0 x_\alpha^{l_1} x_\beta^{l_2} = x_\alpha^{l_1} x_\beta^{l_2} e_0 + a_1 x_\alpha^{l_1-1} x_\beta^{l_2} x_{\alpha_0+\alpha} + a_2 x_\alpha^{l_1-2} x_\beta^{l_2+1} \tag{5.6}$$

with $a_1, a_2 \in \mathbb{C}^\times$. Then since e_0 and $x_{\alpha_0+\alpha}$ belong to $\mathcal{J}_{j+1,j+1}$, (5.5) is inductively proved from $x_\alpha^{\rho_{j+1,j+1}(\alpha)+1} \in \mathcal{J}_{j+1,j+1}$.

Finally, let us show that $(\mathrm{C}_{i,j})$ with $0 \leq i < j$ implies $(\mathrm{C}_{i+1,j})$. For this, it suffices to show the $\mathrm{ad}(e_i)$-invariance of the left $U(\widehat{\mathfrak{n}}_i)$-ideal

$$\mathcal{I}_{i,j} = U(\widehat{\mathfrak{n}}_i)\Bigg(\sum_{\alpha \in \Delta_+^{\mathrm{re}} \setminus \{\alpha_i\}} \mathbb{C} x_\alpha^{\rho_{i,j}(\alpha)+1} + \sum_{\substack{(\alpha,\beta) \in s_i s_{i+1} \cdots s_{j-1}(S_j) \\ 1 \leq k < \rho_{i,j}(\alpha)/2+1}} \mathbb{C} x_\alpha^{\rho_{i,j}(\alpha)-2k+1} x_\beta^k + t\mathfrak{h}[t] \Bigg)$$

by Lemma 5.3. When $0 < i < j$, it is checked similarly as above that, if $\beta \in \widehat{\Delta}_+^{\mathrm{re}}$ is in the form $\beta = l\alpha + \alpha_i$ with $\alpha \in \widehat{\Delta}_+$ and $l \in \mathbb{Z}_{>0}$, then $\rho_{i,j}(\beta) = 0$ holds. Since $[x_\alpha, x_\beta] = 0$ if $(\alpha, \beta) \in s_i \cdots s_{j-1} S_j$, the $\mathrm{ad}(e_i)$-invariance of $\mathcal{I}_{i,j}$ follows from this, as desired. Assume $i = 0$. Using $\widehat{\Delta}_+^{\mathrm{re}}(0,j) = \widehat{\Delta}_+^{\mathrm{re}}(j+1,j+1)$, it is similarly checked that, if $\beta \in \widehat{\Delta}_+^{\mathrm{re}}$ is in the form $\beta = l\alpha + \alpha_0$ with $\alpha \in \widehat{\Delta}_+$ and $l \in \mathbb{Z}_{>0}$, then we have $\rho_{0,j}(\beta) = 0$, or

$$l = 2, \quad \alpha = \alpha_{1,q-1} \quad \text{and} \quad \beta = -(\alpha_{q,n} + \alpha_{q,n-1}) + \delta$$

for some $j < q \leq n$. In the latter case, we see from (5.6) that

$$U(\widehat{\mathfrak{n}}_0)\Bigg(\sum_{0 \leq k < \rho_{0,j}(\alpha)/2+1} \mathbb{C} x_\alpha^{\rho_{0,j}(\alpha)-2k+1} x_\beta^k + x_{\alpha_0+\alpha} \Bigg)$$

is $\mathrm{ad}(e_0)$-invariant. Now the $\mathrm{ad}(e_0)$-invariance of $\mathcal{I}_{0,j}$ is easily proved from (5.4). The proof is complete.

5.2. **q-characters.** Here we recall some results on q-characters, which are necessary in Subsections 5.3 and 5.4. For a finite-dimensional ℓ-weight module V, define its ℓ-weight set $\mathrm{wt}_\ell V$ and *q-character* $\mathrm{ch}_q V$ by

$$\mathrm{wt}_\ell V = \{\boldsymbol{\rho} \in P_q \mid V_{\boldsymbol{\rho}} \neq 0\} \quad \text{and} \quad \mathrm{ch}_q V = \sum_{\boldsymbol{\rho} \in P_q} (\dim V_{\boldsymbol{\rho}}) \boldsymbol{\rho} \in \mathbb{Z}[P_q]$$

respectively. For finite-dimensional ℓ-weight modules V_1, V_2, we have $\mathrm{ch}_q V_1 \otimes V_2 = \mathrm{ch}_q V_1 \cdot \mathrm{ch}_q V_2$ [FR99]. For $i \in I$ and $a \in \mathbb{C}(q)^\times$, define $\boldsymbol{\alpha}_{i,a} \in P_q$ by

$$\boldsymbol{\alpha}_{i,a} = \boldsymbol{\pi}_{2,a}^{(i)} \prod_{j \neq i} \left(\boldsymbol{\pi}_{-c_{j,i},a}^{(j)} \right)^{-1}.$$

Let Q_q^+ denote the monoid generated by $\{\boldsymbol{\alpha}_{i,a} \mid i \in I, a \in \mathbb{C}(q)^\times\}$ and Q_q the corresponding abelian group. We write $\boldsymbol{\rho} \leq \boldsymbol{\nu}$ for $\boldsymbol{\rho}, \boldsymbol{\nu} \in P_q$ if $\boldsymbol{\nu}\boldsymbol{\rho}^{-1} \in Q_q^+$ holds.

Proposition 5.5 ([FM01, Theorem 4.1])**.** *For every $\boldsymbol{\rho} \in P_q^+$, $\boldsymbol{\nu} \in \mathrm{wt}_\ell L_q(\boldsymbol{\rho})$ implies $\boldsymbol{\nu} \leq \boldsymbol{\rho}$.*

The following proposition is proved from the study of $U_q(\mathbf{L}\mathfrak{sl}_2)$-modules in [CP91, CP95b, FR99].

Proposition 5.6. *Assume that $\mathfrak{g} = \mathfrak{sl}_2$. Then the following statements hold, where we omit the index i.*
(i)
$$\mathrm{ch}_q L_q(\boldsymbol{\pi}_{m,a}) = \boldsymbol{\pi}_{m,a} \sum_{0\le k\le m} \prod_{0\le j\le k-1} \boldsymbol{\alpha}^{-1}_{aq^{m-2j}}.$$
(ii) *If V is an ℓ-highest weight module with ℓ-highest weight $\boldsymbol{\pi}_{m,a}$, then we have*
$$\mathrm{ch}_q L_q(\boldsymbol{\pi}_{m,a}) \le \mathrm{ch}_q V \le \prod_{1\le j\le m} \mathrm{ch}_q L_q(\boldsymbol{\varpi}_{aq^{m-2j+1}}) = \boldsymbol{\pi}_{m,a} \prod_{1\le j\le m} (1 + \boldsymbol{\alpha}^{-1}_{aq^{m-2j+2}}),$$
where the inequality $f \le g$ means $g - f \in \mathbb{Z}_{\ge 0}[P_q]$. In particular, the dimension of each ℓ-weight space of V is at most 1.

Recall the map $P_q \ni \boldsymbol{\rho} \mapsto \boldsymbol{\rho}_J \in P_{q,J}$ for a subset $J \subseteq I$ defined in Subsection 3.3. The following proposition is an easy consequence of results in [FM01, Section 3].

Proposition 5.7. *Let V be a finite-dimensional ℓ-weight module, and $J \subseteq I$ a subset such that $\mathfrak{g}_J$ is simple. For an ℓ-weight vector $v \in V_{\boldsymbol{\rho}}$, let $W = U_q(\mathbf{L}\mathfrak{g}_J)v$. Assume that a vector $w \in W$ is an ℓ-weight with respect to $U_q(\mathbf{L}\mathfrak{h}_J)$, and its ℓ-weight is*
$$\boldsymbol{\rho}_J \prod_{i\in J, a\in\mathbb{C}(q)^\times} (\boldsymbol{\alpha}_{i,a})_J^{v(i,a)} \in P_{q,J}$$
with some integers $v(i,a)$. Then w is also ℓ-weight with respect to $U_q(\mathbf{L}\mathfrak{h})$, and its ℓ-weight is $\boldsymbol{\rho}\prod \boldsymbol{\alpha}_{i,a}^{v(i,a)}$.

Let $j \in I$. We say $\boldsymbol{\rho} = \prod_{i\in I, a\in\mathbb{C}(q)^\times} \boldsymbol{\varpi}_{i,a}^{u(i,a)} \in P_q$ is *j-dominant* if $u(j,a) \ge 0$ holds for all $a \in \mathbb{C}(q)^\times$. The following proposition was established by Hernandez.

Proposition 5.8 ([Her08, Lemma 5.6])**.** *Let $\boldsymbol{\rho} \in P_q^+$, and $\boldsymbol{\nu} \in \mathrm{wt}_\ell L_q(\boldsymbol{\rho}) \setminus \{\boldsymbol{\rho}\}$. Then there exist some $j \in I$ and $\boldsymbol{\nu}' \in \mathrm{wt}_\ell L_q(\boldsymbol{\rho})$ such that*

(i) *$\boldsymbol{\nu}'$ is j-dominant,*
(ii) *$\boldsymbol{\nu}' \in \boldsymbol{\nu} \prod_{a\in\mathbb{C}(q)^\times} \boldsymbol{\alpha}_{j,a}^{\mathbb{Z}_{\ge 0}}$,*
(iii) $\Big(U_q(\mathbf{L}\mathfrak{g}_j) L_q(\boldsymbol{\rho})_{\boldsymbol{\nu}}\Big) \cap L_q(\boldsymbol{\rho})_{\boldsymbol{\nu}'} \ne 0.$

Definition 5.9 ([FM01])**.** Let $\boldsymbol{\rho} \in P_q$ and assume that there is some $a \in \mathbb{C}^\times$ such that $\boldsymbol{\rho} = \prod_{i\in I, k\in\mathbb{Z}} \boldsymbol{\varpi}_{i,aq^k}^{u(i,k)}$ with some $u(i,k) \in \mathbb{Z}$. We say $\boldsymbol{\rho}$ is *right-negative* if $k_{\max} = \max\{k \in \mathbb{Z} \mid u(i,k) \ne 0 \text{ for some } i \in I\}$ satisfies $u(i,k_{\max}) \le 0$ for all $i \in I$.

Note that if $\boldsymbol{\rho}$ is right-negative, $\boldsymbol{\rho}$ does not belong to P_q^+. We easily see that $\boldsymbol{\alpha}_{i,a}^{-1}$ are right-negative.

Lemma 5.10 ([FM01])**.** (i) *If $\boldsymbol{\rho}$ is right-negative and $\boldsymbol{\nu} \le \boldsymbol{\rho}$, then $\boldsymbol{\nu}$ is also right-negative.*
(ii) *If $\boldsymbol{\rho} \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{i,a}) \setminus \{\boldsymbol{\varpi}_{i,a}\}$, then $\boldsymbol{\rho} \le \boldsymbol{\varpi}_{i,a}\boldsymbol{\alpha}^{-1}_{i,aq_i}$. In particular $\boldsymbol{\rho}$ is right-negative by* (i).

5.3. **Proof of $\boldsymbol{M(\lambda) \twoheadrightarrow L(\pi)}$.** Let $v_{\boldsymbol{\pi}} \in L_q(\boldsymbol{\pi})$ be an ℓ-highest weight vector. For $1 \le i \le j \le n$ and $p \in \mathbb{Z}$, define $v_p(i,j) \in L_q(\boldsymbol{\pi})$ by
$$v_p(i,j) = \begin{cases} x_{i,p}^- x_{i+1,0}^- x_{i+2,0}^- \cdots x_{j,0}^- v_{\boldsymbol{\pi}} & \text{if } \boldsymbol{\pi} \text{ satisfies (I)},\\ x_{j,p}^- x_{j-1,0}^- x_{j-2,0}^- \cdots x_{i,0}^- v_{\boldsymbol{\pi}} & \text{if } \boldsymbol{\pi} \text{ satisfies (II)},\end{cases}$$
where (I) and (II) are the conditions in Theorem 3.6. Let $\{a_i\}_{i\in I}$ be the sequence of rational functions in Theorem 3.6 associated with $\boldsymbol{\pi}$. The following lemma is crucial in this subsection.

Lemma 5.11. *Let $1 \leq i \leq j \leq n$.*
(i) *For all $p \in \mathbb{Z}$, $v_p(i,j)$ is a scalar multiple of $v_0(i,j)$.*
(ii) *The vector $v_0(i,j)$ is a simultaneous $U_q(\mathbf{L}\mathfrak{h})$-eigenvector, and its ℓ-weight is $\boldsymbol{\pi}\prod_{i\leq k\leq j}\boldsymbol{\alpha}^{-1}_{k,a_kq_k^{m_k}}$.*

Let us assume this lemma for a moment. To prove $M(\lambda) \twoheadrightarrow L(\boldsymbol{\pi})$, we need to check that the vector $\overline{v}_{\boldsymbol{\pi}} = 1 \otimes v_{\boldsymbol{\pi}} \in L(\boldsymbol{\pi})$ satisfies the defining relations of $M(\lambda)$. Using the commutativity $[x^-_{k,r}, x^-_{l,s}] = 0$ for $|k-l| \geq 2$, we easily see that Proposition 5.11 (i) implies

$$[x^-_{i,1},[x^-_{i+1,0},\dots,[x^-_{j-1,0},x^-_{j,0}]\dots]]v_{\boldsymbol{\pi}} \in V_q(\lambda) \subseteq L_q(\boldsymbol{\pi}) \quad \text{if } \boldsymbol{\pi} \text{ satisfies (I), and}$$
$$[x^-_{j,1},[x^-_{j-1,0},\dots,[x^-_{i+1,0},x^-_{i,0}]\dots]]v_{\boldsymbol{\pi}} \in V_q(\lambda) \subseteq L_q(\boldsymbol{\pi}) \quad \text{if } \boldsymbol{\pi} \text{ satisfies (II)}$$

for all $1 \leq i \leq j \leq n$. Here $V_q(\lambda)$ denotes (by abuse of notation) the $U_q(\mathfrak{g})$-submodule of $L_q(\boldsymbol{\pi})$ generated by $v_{\boldsymbol{\pi}}$. By the definition of $L(\boldsymbol{\pi})$, this implies

$$(f_\alpha \otimes t)\overline{v}_{\boldsymbol{\pi}} \in V(\lambda) \subseteq L(\boldsymbol{\pi}) \quad \text{if } \alpha = \alpha_i + \dots + \alpha_j \in \Delta^1_+.$$

Since the restriction of the surjection $L(\boldsymbol{\pi}) \twoheadrightarrow V(\lambda,0)$ in Lemma 4.3 (i) on $V(\lambda)$ is an isomorphism, this implies $(f_\alpha \otimes t)\overline{v}_{\boldsymbol{\pi}} = 0$ for all $\alpha \in \Delta^1_+$. The other relations are proved from this relation or follow from Lemma 4.3. The assertion is proved.

The following lemma is shown in [CP95a, Lemma 3.6].

Lemma 5.12. *Assume $\mathfrak{g}$ is of type ABC. Let $i \in I$, $\mu \in P^+$ such that $\langle \alpha_j^\vee, \mu\rangle = 0$ for $i < j < n$, and $\boldsymbol{\rho} \in P_q^+$ such that $L_q(\boldsymbol{\rho})$ is a minimal affinization of $V_q(\mu)$. Define $v_p(i,j) \in L_q(\boldsymbol{\rho})$ similarly as above. Then for all $i \in I$ and $p \in \mathbb{Z}$, the vector $v_p(i,n)$ is a scalar multiple of $v_0(i,n)$.*

We verify Lemma 5.11 by the induction on $j-i$, assuming $\boldsymbol{\pi}$ satisfies condition (I). (The proof for condition (II) is similar). In view of Corollary 3.10, the case $i=j$ follows from Proposition 5.6. Let $i \leq j-1$. Since $m_j = 0$ implies $v_p(i,j) = 0$ for $p \in \mathbb{Z}$, we may assume $m_j > 0$. Set $\boldsymbol{\nu} = \boldsymbol{\pi}\prod_{i+1\leq k\leq j}\boldsymbol{\alpha}^{-1}_{k,a_kq_k^{m_k}}$, and let $W = U_q(\mathbf{L}\mathfrak{g}_i)v_0(i+1,j) \subseteq L_q(\boldsymbol{\pi})$. By the induction hypothesis, W is an ℓ-highest weight $U_q(\mathbf{L}\mathfrak{g}_i)$-module, and its ℓ-highest weight with respect to $U_q(\mathbf{L}\mathfrak{h}_i)$ is

$$\boldsymbol{\nu}_i(u) = \boldsymbol{\pi}_i(u)\cdot\Big(\boldsymbol{\alpha}^{-1}_{i+1,a_{i+1}q^{m_{i+1}}}\Big)_i(u) = \prod_{1\leq k\leq m_i - c_{i+1,i}}(1 - a_iq_i^{m_i-2k+1}u).$$

Hence by Proposition 5.6 (ii), each ℓ-weight space of W is 1-dimensional. Let us assume that the assertion (i) of Lemma 5.11 does not hold, which implies that the dimension of the weight space $W_{\lambda-\sum_{i\leq k\leq j}\alpha_k}$ is at least 2. Hence by Proposition 5.7, we can take $Y_s \in \bigoplus_p \mathbb{C}(q)x^-_{i,p}$ $(s=1,2)$ such that $0 \neq Y_sv_0(i+1,j) \in L_q(\boldsymbol{\pi})_{\boldsymbol{\nu}\boldsymbol{\alpha}^{-1}_{i,b_s}}$ for some $b_s \in a_iq_i^{\mathbb{Z}}$ with $b_1 \neq b_2$. Let $l = \min\{i < l' < j \mid m_{l'} > 0\}$, which exists by Lemma 5.12 and Corollary 3.10. We easily see that $\boldsymbol{\nu}\boldsymbol{\alpha}^{-1}_{i,b_s}$ is not $(l+1)$-dominant, and hence there exists some $p_s \in \mathbb{Z}$ such that $x^+_{l+1,p_s}Y_sv_0(i+1,j) \neq 0$. From this and the induction hypothesis, we see that $Y_sx^-_{i+1,0}\cdots x^-_{l,0}v_0(l+2,j)$ is a nonzero ℓ-weight vector, and its ℓ-weight is

$$\boldsymbol{\nu}\boldsymbol{\alpha}^{-1}_{i,b_s}\boldsymbol{\alpha}_{l+1,a_{l+1}q^{m_{l+1}}} = \boldsymbol{\pi}\boldsymbol{\alpha}^{-1}_{i,b_s}\prod_{i+1\leq k\leq j, k\neq l+1}\boldsymbol{\alpha}^{-1}_{k,a_kq^{m_k}}.$$

by Proposition 5.5. By repeating this argument we finally see that $Y_sv_0(i+1,l)$ is a nonzero ℓ-weight vector with ℓ-weight $\boldsymbol{\pi}\boldsymbol{\alpha}^{-1}_{i,b_s}\prod_{i+1\leq k\leq l}\boldsymbol{\alpha}^{-1}_{k,a_kq^{m_k}}$ for $s=1,2$. Since $b_1 \neq b_2$, this contradicts with Lemma 5.12, and the assertion (i) is proved. Now the assertion (ii) is easily proved from (i) and the induction hypothesis. The proof is complete.

Remark 5.13. In type B, $M(\lambda) \twoheadrightarrow L(\boldsymbol{\pi})$ is also proved in [Mou10, Proposition 3.22] using the Frenkel-Mukhin algorithm.

5.4. **Proof of $L(\boldsymbol{\pi}) \twoheadrightarrow D(w_\circ\xi_1, \dots, w_\circ\xi_n)$.** We begin with the proof of the following lemma.

Lemma 5.14. *For every $i \in I$, $m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}(q)^\times$,*

$$L\big(\boldsymbol{\pi}^{(i)}_{m,a}\big) \cong D\big(-m\varpi_i + \lceil d_i m/2 \rceil \Lambda_0\big).$$

Proof. If $d_i m/2 \in \mathbb{Z}$, the assertion follows from [CM06b, Proposition 5.1.3] (see also [FL07, Theorem 4]). Hence we may assume $d_i m/2 \notin \mathbb{Z}$, which is equivalent to that α_i is short and $m = 2k+1$ for some $k \in \mathbb{Z}_{\geq 0}$. Then there is an injective homomorphism

$$L(\boldsymbol{\pi}^{(i)}_{m,a}) \hookrightarrow D(-2\varpi_i + \Lambda_0)^{\otimes k} \otimes D(-\varpi_i + \Lambda_0)$$

by [CM06b, Theorem 2.2], which implies

$$L(\boldsymbol{\pi}^{(i)}_{m,a}) \cong D(\underbrace{-2\varpi_i + \Lambda_0, \dots, -2\varpi_i + \Lambda_0}_{k}, -\varpi_i + \Lambda_0).$$

Since $w_\circ w_{[1,i]}(\Lambda_0) = -2\varpi_i + \Lambda_0$ and $w_\circ w_{[1,i]}(\varpi_i + \Lambda_0) = -\varpi_i + \Lambda_0$ hold by Lemma 4.9 (i), we have

$$\begin{aligned} D(\underbrace{-2\varpi_i+\Lambda_0, \dots, -2\varpi_i+\Lambda_0}_{k}, -\varpi_i+\Lambda_0) &\cong F_{w_\circ w_{[1,i]}}\Big(D(\Lambda_0)^{\otimes k} \otimes D(\varpi_i + \Lambda_0)\Big) \\ &\cong D\big(-m\varpi_i + (k+1)\Lambda_0\big) \end{aligned}$$

by Proposition 2.7. The assertion is proved. ☐

Hence in type B, $L(\boldsymbol{\pi}) \twoheadrightarrow D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ follows from [Mou10, Proposition 3.21]. (Note that this proposition does not imply our assertion in type C.)

In the rest of this subsection, we assume that $\mathfrak{g}$ is of type C. For the proof of the assertion in this type, we need the following lemma.

Lemma 5.15. *Let $1 \leq r < s \leq n-1$, and assume that $\boldsymbol{\rho}$ is an element of $P^+_{\boldsymbol{A}}$ such that $L_q(\boldsymbol{\rho})$ is a minimal affinization of $V_q(\varpi_r + \varpi_s)$. Then we have*

$$L(\boldsymbol{\rho}) \cong D(-\varpi_r - \varpi_s + \Lambda_0).$$

Assuming this lemma for a while, we shall prove $L(\boldsymbol{\pi}) \twoheadrightarrow D(w_\circ\xi_1, \dots, w_\circ\xi_n)$ for $\boldsymbol{\pi}$ satisfying condition (I). (The proof for condition (II) is similar.) The proof is carried out in a similar line as that of [Mou10, Proposition 3.21]. First we recall the following theorem, which is obtained by taking the dual of [Cha02, Theorem 5.1] and using Lemma 3.3.

Theorem 5.16. *Let $i_1, \dots, i_p \in I$, $b_1, \dots, b_p \in \mathbb{C}(q)^\times$, $l_1, \dots, l_p \in \mathbb{Z}_{\geq 0}$, and assume that*

$$b_r q_{i_r}^{-l_r} \notin q^{\mathbb{Z}_{>0}} b_s q_{i_s}^{-l_s} \quad \textit{for all } r < s. \tag{5.7}$$

Then the submodule of $L_q\big(\boldsymbol{\pi}^{(i_1)}_{l_1,b_1}\big) \otimes \cdots \otimes L_q\big(\boldsymbol{\pi}^{(i_p)}_{l_p,b_p}\big)$ generated by the tensor product of ℓ-highest weight vectors is isomorphic to $L_q\Big(\prod_{k=1}^p \boldsymbol{\pi}^{(i_k)}_{l_k,b_k}\Big)$.

The following is an easy consequence of this theorem.

Corollary 5.17. *Assume that $i_1, \dots, i_p \in I$, $b_1, \dots, b_p \in \mathbb{C}(q)^\times$, and $l_1, \dots, l_p \in \mathbb{Z}_{\geq 0}$ satisfy (5.7). Then for any sequence $0 = k_0 < k_1 < \ldots < k_{r-1} < k_r = p$, the submodule of $L_q\Big(\prod_{k=1}^{k_1} \boldsymbol{\pi}^{(i_k)}_{l_k,b_k}\Big) \otimes \cdots \otimes L_q\Big(\prod_{k=k_{r-1}+1}^{p} \boldsymbol{\pi}^{(i_k)}_{l_k,b_k}\Big)$ generated by the tensor product of ℓ-highest weight vectors is isomorphic to $L_q\Big(\prod_{k=1}^p \boldsymbol{\pi}^{(i_k)}_{l_k,b_k}\Big)$.*

For each $i \in I$, define $\boldsymbol{\pi}^{(i)} \in P_q^+$ by

$$\boldsymbol{\pi}^{(i)} = \begin{cases} \boldsymbol{\pi}^{(i)}_{m_i - p_i, a_i q^{p_i}} & \text{if } i^\flat = 0, \\ \boldsymbol{\pi}^{(i^\flat)}_{p_{i^\flat}, a_{i^\flat} q^{-m_{i^\flat}+1}} \boldsymbol{\pi}^{(i)}_{m_i - p_i, a_i q^{p_i}} & \text{otherwise.} \end{cases}$$

By Corollary 5.17, we have an injective $U_q(\mathbf{L}\mathfrak{g})$-module homomorphism

$$L_q(\boldsymbol{\pi}) \to L_q\big(\boldsymbol{\pi}^{(n)}\big) \otimes \cdots \otimes L_q\big(\boldsymbol{\pi}^{(2)}\big) \otimes L_q\big(\boldsymbol{\pi}^{(1)}\big),$$

which induces a $U_{\mathbf{A}}(\mathbf{L}\mathfrak{g})$-module homomorphism

$$L_{\mathbf{A}}(\boldsymbol{\pi}) \to L_{\mathbf{A}}\big(\boldsymbol{\pi}^{(n)}\big) \otimes \cdots \otimes L_{\mathbf{A}}\big(\boldsymbol{\pi}^{(2)}\big) \otimes L_{\mathbf{A}}\big(\boldsymbol{\pi}^{(1)}\big).$$

By applying $\mathbb{C}\otimes_{\mathbf{A}}$ and taking the pull-back, we have a $\mathfrak{g}[t]$-module homomorphism $L\big(\boldsymbol{\pi}\big) \to \bigotimes_{i=n}^{1} L\big(\boldsymbol{\pi}^{(i)}\big)$ mapping $\overline{v}_{\boldsymbol{\pi}}$ to $\overline{v}_{\boldsymbol{\pi}^{(n)}} \otimes \cdots \otimes \overline{v}_{\boldsymbol{\pi}^{(1)}}$. Since $D(w_\circ \xi_n, \ldots, w_\circ \xi_1) \cong D(w_\circ \xi_1, \ldots, w_\circ \xi_n)$, to complete the proof it suffices to show for each $1 \leq i \leq n$ the existence of a surjective homomorphism $L\big(\boldsymbol{\pi}^{(i)}\big) \twoheadrightarrow D(w_\circ \xi_i)$. If $i^\flat = 0$ or $p_{i^\flat} = 0$, this follows from Lemma 5.14. Assume that $p_{i^\flat} = 1$, and let

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}^{(i)}_{m_i - p_i - 1, a_i q^{p_i - 1}}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\varpi}_{i^\flat, a_{i^\flat} q^{-m_{i^\flat}+1}} \boldsymbol{\varpi}_{i, a_i q^{m_i - 1}}.$$

We have an inclusion $L_q\big(\boldsymbol{\pi}^{(i)}\big) \hookrightarrow L_q(\boldsymbol{\pi}_1) \otimes L_q(\boldsymbol{\pi}_2)$ by Corollary 5.17, and then using the same argument as above, we obtain a $\mathfrak{g}[t]$-module homomorphism $L\big(\boldsymbol{\pi}^{(i)}\big) \to L\big(\boldsymbol{\pi}_1\big) \otimes L(\boldsymbol{\pi}_2)$. By Lemmas 5.14 and 5.15, this induces a surjective homomorphism

$$L\big(\boldsymbol{\pi}^{(i)}\big) \twoheadrightarrow D\Big(-(m_i - p_i - 1)\varpi_i + \frac{1}{2}(m_i - p_i - 1)\Lambda_0, -\varpi_{i^\flat} - \varpi_i + \Lambda_0\Big).$$

By Proposition 2.7 and Lemma 4.9, we see that the right-hand side is isomorphic to

$$F_{w_\circ w_{[1,i]}}\left(D\Big(\frac{1}{2}(m_i - p_i - 1)\Lambda_0\Big) \otimes D\big(\varpi_{i - i^\flat} + \Lambda_0\big)\right) \cong D(w_\circ \xi_i),$$

and hence the assertion is proved.

It remains to show Lemma 5.15. Fix $1 \leq r < s \leq n-1$.

Lemma 5.18. *As $\mathfrak{g}$-modules,*

$$D(-\varpi_r - \varpi_s + \Lambda_0) \cong \bigoplus_{k=0}^{r} V(\varpi_{r-k} + \varpi_{s-k}).$$

Proof. Let $\widehat{J} = \{0, 1, \ldots, s-1\} \subseteq \widehat{I}$ and $J = \{1, \ldots, s-1\} \subseteq I$, and define $\widehat{\mathfrak{g}}_{\widehat{J}}$ by the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $\{e_i, f_i \mid i \in \widehat{J}\}$ and $\widehat{\mathfrak{h}}$. We also define $\mathfrak{g}_J \subseteq \mathfrak{g}$ similarly. Note that we have

$$D(-\varpi_r - \varpi_s + \Lambda_0) = F_{w_\circ w_{[1,s]}} D(\varpi_{s-r} + \Lambda_0).$$

Let $w_\circ^J$ be the longest element of the Weyl group of $\mathfrak{g}_J$. Then $F_{w_\circ^J w_{[1,s]}} D(\varpi_{s-r} + \Lambda_0)$ is a simple $\widehat{\mathfrak{g}}_{\widehat{J}}$-module with highest weight $\varpi_{s-r} + \Lambda_0$, and therefore we have

$$F_{w_\circ^J w_{[1,s]}} D(\varpi_{s-r} + \Lambda_0) \cong \bigoplus_{k=0}^{r} V_J(\varpi_{r-k} + \varpi_{s-k})$$

as $\mathfrak{g}_J$-modules, where $V_J(\nu)$ denotes the simple highest weight $\mathfrak{g}_J$-module with highest weight ν. Since $V_J(\nu)$ are Demazure modules for $\mathfrak{g}$, $F_{w_\circ w_\circ^J} V_J(\nu) = V(\nu)$ holds. Hence the assertion is proved. $\square$

We also need the following lemma.

Lemma 5.19. *There exists an exact sequence*

$$0 \to L_q(\boldsymbol{\varpi}_{r,a}\boldsymbol{\varpi}_{s,aq^{s-r+2}}) \to L_q(\boldsymbol{\varpi}_{r,a}) \otimes L_q(\boldsymbol{\varpi}_{s,aq^{s-r+2}}) \to L_q(\boldsymbol{\varpi}_{r-1,aq}\boldsymbol{\varpi}_{s+1,aq^{s-r+1}}).$$

Assuming this for a moment, we shall complete the proof of Lemma 5.15. Take $\boldsymbol{\rho}$ as in Lemma 5.15. By the results in Subsections 5.1 and 5.3, we have

$$D(-\varpi_r - \varpi_s + \Lambda_0) \twoheadrightarrow M(\varpi_r + \varpi_s) \twoheadrightarrow L(\boldsymbol{\rho}).$$

Hence it suffices to show that $\dim L(\boldsymbol{\rho}) \geq \dim D(-\varpi_r - \varpi_s + \Lambda_0)$. Recall that every fundamental module $L_q(\boldsymbol{\varpi}_{i,a})$ is simple as a $U_q(\mathfrak{g})$-module (in type C), and it follows for $1 \leq i < j \leq n$ that

$$V_q(\varpi_i) \otimes V_q(\varpi_j) \cong \begin{cases} \bigoplus_{k=0}^{i} V_q(\varpi_{i-k} + \varpi_{j-k}) \oplus \bigoplus_{k=1}^{i} V_q(\varpi_{i-k} + \varpi_{n-|n-j-k|}) & \text{if } j \leq n-1, \\ \bigoplus_{k=0}^{i} V_q(\varpi_{i-k} + \varpi_{n-k}) & \text{if } j = n. \end{cases} \tag{5.8}$$

By Theorem 3.6, $\boldsymbol{\rho} = \boldsymbol{\varpi}_{r,a}\boldsymbol{\varpi}_{s,aq^{\varepsilon(s-r+2)}}$ for some $a \in \mathbb{C}(q)^\times$ and $\varepsilon \in \{\pm 1\}$. Let us assume $\varepsilon = +1$ first. Using

$$L_q(\boldsymbol{\varpi}_{r-1,aq}\boldsymbol{\varpi}_{s+1,aq^{s-r+1}}) \hookrightarrow L_q(\boldsymbol{\varpi}_{r-1,aq}) \otimes L_q(\boldsymbol{\varpi}_{s+1,aq^{s-r+1}}),$$

we see from Lemma 5.19 and (5.8) that $L_q(\boldsymbol{\rho})$ contains $V_q(\varpi_{r-k}+\varpi_{s-k})$ $(0 \leq k \leq r)$ as simple $U_q(\mathfrak{g})$-components. Hence $\dim L(\boldsymbol{\rho}) = \dim L_q(\boldsymbol{\rho}) \geq \dim D(-\varpi_r - \varpi_s + \Lambda_0)$ holds by Lemma 5.18, as desired. Since $\dim L_q(\boldsymbol{\rho}) = \dim L_q({}^*\boldsymbol{\rho})$ holds by Lemma 3.3, the case $\varepsilon = -1$ is also proved.

Now let us prove Lemma 5.19. It suffices to show that

$$\mathrm{wt}_\ell\Big(L_q(\boldsymbol{\varpi}_{r,a}) \otimes L_q(\boldsymbol{\varpi}_{s,aq^{s-r+2}})\Big) \cap P_q^+ = \big\{\boldsymbol{\varpi}_{r,a}\boldsymbol{\varpi}_{s,aq^{s-r+2}}, \boldsymbol{\varpi}_{r-1,aq}\boldsymbol{\varpi}_{s+1,aq^{s-r+1}}\big\}$$

and each dominant ℓ-weight space is 1-dimensional. Assume that $\boldsymbol{\rho}_1 \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{r,a})$ and $\boldsymbol{\rho}_2 \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{s,aq^{s-r+2}})$ satisfy $\boldsymbol{\rho}_1\boldsymbol{\rho}_2 \in P_q^+$. If $\boldsymbol{\rho}_2 \neq \boldsymbol{\varpi}_{s,aq^{s-r+2}}$, it follows that

$$\boldsymbol{\rho}_1\boldsymbol{\rho}_2 \leq \boldsymbol{\varpi}_{r,a}\boldsymbol{\varpi}_{s,aq^{s-r+2}}\boldsymbol{\alpha}_{s,aq^{s-r+3}}^{-1}$$

by Lemma 5.10 (ii), and therefore $\boldsymbol{\rho}_1\boldsymbol{\rho}_2$ is right-negative by Lemma 5.10 (i). Hence we have $\boldsymbol{\rho}_2 = \boldsymbol{\varpi}_{s,aq^{s-r+2}}$.

We need to show one more lemma. For $\boldsymbol{\nu} \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{r,a})$, define $u_i(\boldsymbol{\nu}) \in \mathbb{Z}_{\geq 0}$ for $i \in I$ by $\varpi_r - \mathrm{wt}(\boldsymbol{\nu}) = \sum_{i \in I} u_i(\boldsymbol{\nu})\alpha_i$. Let $u(\boldsymbol{\nu}) = \sum_{i \in I} u_i(\boldsymbol{\nu}) \in \mathbb{Z}_{\geq 0}$.

Lemma 5.20. *Let $r \leq k \leq n$, and assume that $\boldsymbol{\nu} \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{r,a})$ satisfies $u_k(\boldsymbol{\nu}) > 0$ and $u_l(\boldsymbol{\nu}) = 0$ for $l > k$. Then $\boldsymbol{\nu} \leq \boldsymbol{\varpi}_{r,a}\boldsymbol{\alpha}_{k,aq^{p(k)}}^{-1}$ holds, where we set $p(k) = k - r + 1 + \delta_{kn}$.*

Proof. We prove this by the induction on k. The case $k = r$ follows from Lemma 5.10 (ii). Let $k > r$, and assume that there is an element $\boldsymbol{\nu}$ such that $u_k(\boldsymbol{\nu}) > 0$, $u_l(\boldsymbol{\nu}) = 0$ for $l > k$, and $\boldsymbol{\nu} \not\leq \boldsymbol{\varpi}_{r,a}\boldsymbol{\alpha}_{k,aq^{p(k)}}^{-1}$. We may assume that $u(\boldsymbol{\nu})$ is minimal among such elements. By Proposition 5.8, there exists $j \in I$ and $\boldsymbol{\nu}' \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{r,a})$ satisfying the conditions (i)–(iii). If $u_k(\boldsymbol{\nu}') > 0$, then $\boldsymbol{\nu}'$ also satisfies the assumption of $\boldsymbol{\nu}$, which contradicts the minimality of $u(\boldsymbol{\nu})$. Hence $j = k$ and $u_k(\boldsymbol{\nu}') = 0$ follow. We easily see that $u_{k-1}(\boldsymbol{\nu}) > 0$, and therefore $u_{k-1}(\boldsymbol{\nu}') > 0$ also holds. Hence by the induction hypothesis, $\boldsymbol{\nu}' \leq \boldsymbol{\varpi}_{r,a}\boldsymbol{\alpha}_{k-1,aq^{k-r}}^{-1}$ holds. On the other hand, we see from the weight set of $L_q(\boldsymbol{\varpi}_{r,a}) \cong V_q(\varpi_r)$ that $u_{k-1}(\boldsymbol{\nu}') = 1$, which implies

$$\boldsymbol{\nu}' \in \boldsymbol{\varpi}_{r,a}\boldsymbol{\alpha}_{k-1,aq^{k-r}}^{-1} \prod_{l<k-1, b \in \mathbb{C}(q)^\times} \boldsymbol{\alpha}_{l,b}^{\mathbb{Z}_{\leq 0}}.$$

Thus we have $\boldsymbol{\nu}'_k(u) = 1 - aq^{k-r}$. Then using Propositions 5.6 and 5.7, we see from the condition (iii) that $\boldsymbol{\nu} = \boldsymbol{\nu}'\boldsymbol{\alpha}^{-1}_{k,aq^{k-r}q_k} \leq \boldsymbol{\varpi}_{r,a}\boldsymbol{\alpha}^{-1}_{k,aq^{p(k)}}$, which is a contradiction. The assertion is proved. □

This lemma implies that, if $\boldsymbol{\nu} \in \mathrm{wt}_\ell L_q(\boldsymbol{\varpi}_{r,a})$ satisfies $u_k(\boldsymbol{\nu}) > 0$ for some $k > s$, then $\boldsymbol{\nu}\boldsymbol{\varpi}_{s,aq^{s-r+2}}$ is right-minimal. Hence $u_k(\boldsymbol{\rho}_1) = 0$ holds for all $k > s$. Since $\boldsymbol{\rho}_1\boldsymbol{\varpi}_{s,aq^{s-r+2}} \in P_q^+$ implies $\mathrm{wt}(\boldsymbol{\rho}_1) + \varpi_s \in P^+$, this implies

$$\mathrm{wt}(\boldsymbol{\rho}_1) \in \{\varpi_r, \varpi_{r-1} - \varpi_s + \varpi_{s+1}\}.$$

Then we see from [CM06a, Theorem 2.7] that

$$\boldsymbol{\rho}_1 \in \{\boldsymbol{\varpi}_{r,a}, \boldsymbol{\varpi}_{r-1,aq}\boldsymbol{\varpi}^{-1}_{s,aq^{s-r+2}}\boldsymbol{\varpi}_{s+1,aq^{s-r+1}}\}$$

and $\dim L_q(\boldsymbol{\varpi}_{r,a})_{\boldsymbol{\rho}_1} = 1$. Now the assertion is obvious. The proof is complete.

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