THE EYNARD–ORANTIN RECURSION FOR THE TOTAL ANCESTOR POTENTIAL

TODOR MILANOV

Abstract. It was proved recently that the correlation functions of a semi-simple cohomological field theory satisfy the so called Eynard–Orantin topological recursion. We prove that in the settings of singularity theory, the relations can be expressed in terms of periods integrals and the so called phase forms. In particular, we prove that the Eynard-Orantin recursion is equivalent to N copies of Virasoro constraints for the ancestor potential, which follow easily from the definition of the potential.

Contents

1. Introduction

The Eynard–Orantin recursion (see [5]) was discovered first for the correlation functions of certain matrix integrals. However, its applications go beyond the theory of matrix models. The recursion is turning into a powerful tool for computing the correlation functions in various quantum field theories. In particular, it provides an efficient algorithm for computing quite complicated invariants such as Gromov–Witten invariants and certain polynomial invariants in knot theory.

In order to set up the recursion one needs an analytic curve, called *spectral curve*, two holomorphic functions on it, and a certain symmetric 2-form satisfying some additional properties. At first, one might think that this is a serious restriction, so the applications would be only limited. The surprising fact however, is that the initial data can be determined from the 1-point and the 2-point correlation functions only (of a given quantum field theory) (see [7]). This feature makes the recursion quite universal. In particular, this observation was exploited in the paper [4], where the authors prove that the ancestor Gromov–Witten (GW for shortness) invariants of manifolds with a semi-simple quantum cohomology can be computed via the Eynard–Orantin recursion. Although our work appears after [4], the main observation namely, that one should study the n-point series (4) and that they should satisfy the Eynard–Orantin recursion with kernel given by formulas (36) and (37) was done independently.

The goal in this paper is to interpret the Eynard–Orantin recursion in terms of differential operator constraints for the total ancestor potential. In particular, this allows us to obtain a simple proof of the recursion relation. In particular, we prove that the correlation functions can be expressed in terms of period integrals and phase forms, which suggests that they should be compared to the correlation functions of the twisted Vertex algebra representation introduced in [2].

1.1. **Preliminary notation.** Let $f \in \mathcal{O}_{\mathbb{C}^{2l+1},0}$ be the germ of a holomorphic function with an isolated critical point at 0. We fix a miniversal deformation $F(t,x)$, $t \in B$ and a primitive form ω in the sense of K. Saito [13, 15], so that B inherits a Frobenius structure (see [11, 14]). In particular we have the following identifications (c.f. Section 2.1):

$$T^*B \cong TB \cong B \times T_0B \cong B \times H,$$

where H is the Jacobi algebra of f, the first isomorphism is given by the residue pairing, the second by the flat residue metric and the last one is the Kodaira–Spencer isomorphism

$$T_0B \cong H, \quad \partial/\partial t_i \mapsto \partial_{t_i}F\Big|_{t=0} \mod (f_{x_0},\dots,f_{x_{2l}}). \tag{1}$$

We need also the period integrals

$$I^{(k)}_\alpha(t,\lambda) = -d^B\,(2\pi)^{-l}\,\partial_\lambda^{k+l}\int_{\alpha_{t,\lambda}} d^{-1}\omega \;\in T_t^*B \cong H, \tag{2}$$

where α is a cycle from the vanishing cohomology, d^B is the de Rham differential on B, and $d^{-1}\omega$ is any $(n-1)$-form η such that $d\eta = \omega$. The periods are multivalued analytic functions on $B \times \mathbb{P}^1$ with poles along the so called *discriminant locus* (c.f. Section 2.2). We make use of the following series

$$\mathbf{f}^\alpha(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k)}_\alpha(t,\lambda)\,(-z)^k, \quad \phi^\alpha(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k+1)}_\alpha(t,\lambda)(-z)^k\,d\lambda.$$

Note that $\phi^\alpha(t,\lambda;z) = d^{\mathbb{P}^1}\mathbf{f}^\alpha(t,\lambda;z)$.

Let $B_{ss} \subset B$ be the subset of semi-simple points, i.e., points $t \in B$ such that the critical values of $F(t,\cdot)$ form a coordinate system in a neighborhood of t. For every $t \in B_{ss}$ Givental's higher-genus reconstruction formalism gives rise to ancestor correlation functions of the following form

$$\langle v_1\psi_1^{k_1},\dots,v_n\psi^{k_n}\rangle_{g,n}(t), \quad v_i \in H, \quad k_i \in \mathbb{Z}_+ (1 \le i \le n). \tag{3}$$

Apriory, each correlator depends analytically on $t \in B_{ss}$, but it might have poles along the divisor $B \setminus B_{ss}$. Given n vanishing cycles $\alpha_1, \ldots, \alpha_n$ and a generic point $t \in B$ we define the following n-point symmetric forms

$$\omega_{g,n}^{\alpha_1,\ldots,\alpha_n}(t; \lambda_1, \ldots, \lambda_n) = \Big\langle \phi_+^{\alpha_1}(t, \lambda_1, \psi_1), \ldots, \phi_+^{\alpha_n}(t, \lambda_n, \psi_n) \Big\rangle_{g,n}(t), \tag{4}$$

where the + means truncation of the terms in the series with negative powers of z. The functions (4) will be called *n-point series* of genus g. They should be interpreted formally via their Laurent series expansions at the singular points. They are the main object of our study and we expect that they have many remarkable properties yet to be discovered. Probably the first question to be addressed is whether the n-point functions are global objects, i.e., multivalued analytic functions with finite order poles on the *configuration space* $C_n(\mathbb{P}^1) = \mathcal{F}_n(\mathbb{P}^1)/\mathfrak{S}_n$, where

$$\mathcal{F}_n(\mathbb{P}^1) = \Big\{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{P}^1)^n \ : \ \lambda_i \neq \lambda_j \text{ for } i \neq j \Big\}$$

and $\mathfrak{S}_n$ is the symmetric group acting by permutation of the coordinates.

1.2. **Statement of the results.** Let us assume that $t \in B_{ss}$ is generic so that the function $F(t, \cdot)$ has $N := \dim_{\mathbb{C}} H$ pairwise different critical values $u_j(t)$. Let β_j be a cycle vanishing over $\lambda = u_j$. We introduce the following quadratic differential operator

$$Y_{t,\lambda}^{u_j} =: (\partial_\lambda \widehat{\mathbf{f}^{\beta_j}}(t, \lambda))^2 : \ + P_0^{\beta_j,\beta_j}(t, \lambda), \tag{5}$$

where : : is the normal ordering, the differential operator (c.f. Section 3.1)

$$\partial_\lambda \widehat{\mathbf{f}^{\beta_j}}(t, \lambda) = \partial_\lambda \widehat{\mathbf{f}_+^{\beta_j}}(t, \lambda) + \partial_\lambda \widehat{\mathbf{f}_-^{\beta_j}}(t, \lambda)$$

is defined by

$$\partial_\lambda \widehat{\mathbf{f}^{\beta_j}}(t, \lambda)_+ = \sum_{k=0}^{\infty} \sum_{i=1}^{N} (-1)^{k+1} \, (I_{\beta_j}^{(k+1)}(t, \lambda), v^i) \, \hbar^{1/2} \, \frac{\partial}{\partial q_k^i}\,, \tag{6}$$

$$\partial_\lambda \widehat{\mathbf{f}^{\beta_j}}(t, \lambda)_- = \sum_{k=0}^{\infty} \sum_{i=1}^{N} (I_{\beta_j}^{(-k)}(t, \lambda), v_i) \, \hbar^{-1/2} \, q_k^i\,, \tag{7}$$

where $\{v^i\}$ and $\{v_i\}$ are dual bases for H with respect to the residue pairing $(\,,)$. Finally, $P_0^{\beta_j,\beta_j}$ is the free term in the Laurent series expansion of the *propagator*

$$[\partial_\lambda \widehat{\mathbf{f}_+^{\beta_j}}(t, \mu), \partial_\lambda \widehat{\mathbf{f}_-^{\beta_j}}(t, \lambda)] = 2\,(\mu - \lambda)^{-2} + \sum_{k=0}^{\infty} P_k^{\beta_j,\beta_j}(t, \lambda)\,(\mu - \lambda)^k.$$

The definition (5) is very natural from the point of view of vertex algebras (see [2]). It is the field that determines a representation of the Virasoro vertex operator algebra with central charge 1 on the *Fock space*

$$\mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \ldots]], \quad q_k = (q_k^1, \ldots, q_k^N), \quad \mathbb{C}_\hbar := \mathbb{C}(\!(\sqrt{\hbar})\!),$$

where $\mathbf{1}$ is the unity of the local algebra H. We may assume that $v_1 = \mathbf{1}$.

Let us denote by $\mathcal{A}_t(\hbar;\mathbf{q})$ the *total ancestor potential* of the singularity. By definition, it is a vector in the Fock space of the form (c.f. Section 3.2)

$$\exp\Big(\sum_{g,n=0}^{\infty}\frac{1}{n!}\Big\langle\mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_n)\Big\rangle_{g,n}(t)\,\hbar^{g-1}\Big),$$

where $\mathbf{t}(\psi) = \sum_{k,i} t_k^i\, v_i\psi^k$ and the relation between the set of formal variables $\{q_k^i\}$ and $\{t_k^i\}$ is given by the *dilaton shift*

$$t_k^i = \begin{cases} q_k^i & \text{if } (k,i)\neq(1,1),\\ q_1^1+1 & \text{otherwise.}\end{cases} \tag{8}$$

We define the following set of differential operators

$$L_{m-1,i} = \frac{1}{4}\sum_{j=1}^{N}\mathrm{Res}_{\lambda=u_j}\frac{(I^{(-m-1)}_{\beta_j}(t,\lambda),v_i)\,Y^{u_j}_{t,\lambda}}{(I^{(-1)}_{\beta_j}(t,\lambda),\mathbf{1})}\,d\lambda,\quad m\geq 0,\quad 1\leq i\leq N. \tag{9}$$

Note that although the periods are multi-valued analytic functions, the above expression is single valued with respect to the local monodromy around $\lambda = u_j$, so the residue is well defined. Our first result is the following .

Theorem 1.1. *The total ancestor potential satisfies the following constraints:*

$$L_{m-1,i}\,\mathcal{A}_t(\hbar;\mathbf{q}) = 0,\quad m\geq 0,\quad 1\leq i\leq N.$$

The proof follows easily from the definition of the ancestor potential and some known properties of the periods (2). More precisely one can prove that $Y^{u_j}_{t,\lambda}\,\mathcal{A}_t$ is regular near $\lambda = u_j$, so each residue vanishes. The main property of the construction (9) is that $L_{m-1,i}$ has only one term that involves the dilaton-shifted variable q_1^1 and this term is $q_1^1\partial/\partial q_m^i$. This fact allows us to interpret the differential operator constraints as a system of recursion relations. Our next result is the following.

Theorem 1.2. *The differential operator constraints determine a system of recursion relations that coincide with the local Eynard–Orantin recursion.*

We postpone the definition of the Eynard–Orantin recursion until Section 4. Following [4] we give the definition of the recursion only locally. It will be very important to find the global formulation (see [3]) as well, but for now this seems to be a very challenging problem, except may be for simple and simple elliptic singularities in singularity theory or the projective line and its orbifold versions in GW theory. The problems is that in the settings of singularity theory, or GW theory, the spectral curve has highly transcendental nature. Its description requires inverting the period map: very classical and very difficult problem. Our set up is slightly different from the standard conventions ([3, 4, 7, 5]), because we would like to work with multivalued correlation functions. The main idea is that whatever is the spectral curve Σ, we always have a projection $\Sigma \to \mathbb{P}^1$ which in general is an infinite sheet branched covering. Galois theory tells us that studying the field of meromorphic functions on Σ is the same as the field of multivalued meromorphic functions on $\mathbb{P}^1$ invariant under some monodromy (Galois) group. Our proposal is to formulate the Eynard–Orantin recursion for correlation functions on $\mathbb{P}^1$ that take values in some local system $\mathcal{L}$. The definition that we give in Section 4 is just a first attempt to set up this idea. Probably a better formulation is possible if one takes into account more examples not only the ones that come from GW theory.

Let us point out that although we work in the settings of singularity theory, the differential operators (9) can be defined for any semi-simple Frobenius manifold that has an Euler vector field. The period vectors should be introduced as the solutions to a system of differential equations (see Lemma 2.3) and it is easy to see that the proofs of Theorems 1.1 and 1.2 remain the same. In particular, we obtain the main result of [4]

Corollary 1.3. *The n-point series of a semi-simple cohomological field theory satisfy the local Eynard–Orantin recursion relations.*

1.3. **Acknowledgement.** I am thankful to M. Mulase for teaching me how the Eynard–Orantin recursion can be constructed in terms of 1- and 2-point functions. Also, I thank S. Shadrin for useful e-mail communication and clarifying some points from [4]. This work is supported by Grant-In-Aid and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

2. Frobenius structures in singularity theory

Let $f\colon (\mathbb{C}^{2l+1},0)\to(\mathbb{C},0)$ be the germ of a holomorphic function with an isolated critical point of multiplicity N. Denote by

$$H = \mathbb{C}[[x_0,\dots,x_{2l}]]/(\partial_{x_0}f,\dots,\partial_{x_{2l}}f)$$

the *local algebra* of the critical point; then $\dim H = N$.

Definition 2.1. *A* miniversal deformation *of f is a germ of a holomorphic function $F\colon(\mathbb{C}^N\times\mathbb{C}^{2l+1},0)\to(\mathbb{C},0)$ satisfying the following two properties:*

(1) F is a deformation of f, i.e., $F(0,x)=f(x)$.
(2) The partial derivatives $\partial F/\partial t^i$ $(1\le i\le N)$ project to a basis in the local algebra

$$\mathcal{O}_{\mathbb{C}^N,0}[[x_0,\dots,x_{2l}]]/\langle\partial_{x_0}F,\dots,\partial_{x_{2l}}F\rangle.$$

Here we denote by $t=(t^1,\dots,t^N)$ and $x=(x_0,\dots,x_{2l})$ the standard coordinates on $\mathbb{C}^N$ and $\mathbb{C}^{2l+1}$ respectively, and $\mathcal{O}_{\mathbb{C}^N,0}$ is the algebra of germs at 0 of holomorphic functions on $\mathbb{C}^N$.

We fix a representative of the holomorphic germ F, which we denote again by F, with a domain X constructed as follows. Let
$$B_\rho^{2l+1} \subset \mathbb{C}^{2l+1}, \qquad B = B_\eta^N \subset \mathbb{C}^N, \qquad B_\delta^1 \subset \mathbb{C}$$
be balls with centers at 0 and radii ρ, η, and δ, respectively. We set
$$S = B \times B_\delta^1 \subset \mathbb{C}^N \times \mathbb{C}, \quad X = (B \times B_\rho^{2l+1}) \cap \phi^{-1}(S) \subset \mathbb{C}^N \times \mathbb{C}^{2l+1},$$
where
$$\phi\colon B \times B_\rho^{2l+1} \to B \times \mathbb{C}, \qquad (t, x) \mapsto (t, F(t, x)).$$
This map induces a map $\phi\colon X \to S$ and we denote by X_s or $X_{t,\lambda}$ the fiber
$$X_s = X_{t,\lambda} = \{(t, x) \in X \,|\, F(t, x) = \lambda\}, \qquad s = (t, \lambda) \in S.$$
The number ρ is chosen so small that for all r, $0 < r \leq \rho$, the fiber $X_{0,0}$ intersects transversely the boundary ∂B_r^{2l+1} of the ball with radius r. Then we choose the numbers η and δ small enough so that for all $s \in S$ the fiber X_s intersects transversely the boundary ∂B_ρ^{2l+1}. Finally, we can assume without loss of generality that the critical values of F are contained in a disk $B_{\delta_0}^1$ with radius $\delta_0 < 1 < \delta$.

Let Σ be the *discriminant* of the map ϕ, i.e., the set of all points $s \in S$ such that the fiber X_s is singular. Put
$$S' = S \setminus \Sigma \subset \mathbb{C}^N \times \mathbb{C}, \qquad X' = \phi^{-1}(S') \subset X \subset \mathbb{C}^N \times \mathbb{C}^{2l+1}.$$
Then the map $\phi\colon X' \to S'$ is a smooth fibration, called the *Milnor fibration*. In particular, all smooth fibers are diffeomorphic to $X_{0,1}$. The middle homology group of the smooth fiber, equipped with the bilinear form $(\cdot|\cdot)$ equal to $(-1)^l$ times the intersection form, is known as the *Milnor lattice* $Q = H_{2l}(X_{0,1}; \mathbb{Z})$.

For a generic point $s \in \Sigma$, the singularity of the fiber X_s is Morse. Thus, every choice of a path from $(0, 1)$ to s avoiding Σ leads to a group homomorphism $Q \to H_{2l}(X_s; \mathbb{Z})$. The kernel of this homomorphism is a free $\mathbb{Z}$-module of rank 1. A generator $\alpha \in Q$ of the kernel is called a *vanishing cycle* if $(\alpha|\alpha) = 2$.

2.1. **Frobenius structure.** Let $\mathcal{T}_B$ be the sheaf of holomorphic vector fields on B. Condition (2) in Definition 2.1 implies that the map
$$\partial/\partial t^i \mapsto \partial F/\partial t^i \quad \bmod \langle \partial_{x_0} F, \ldots, \partial_{x_{2l}} F \rangle \qquad (1 \leq i \leq N)$$
induces an isomorphism between $\mathcal{T}_B$ and $p_* \mathcal{O}_C$, where $p\colon X \to B$ is the natural projection $(t, x) \mapsto t$ and
$$\mathcal{O}_C := \mathcal{O}_X / \langle \partial_{x_0} F, \ldots, \partial_{x_{2l}} F \rangle$$
is the structure sheaf of the critical set of F. In particular, since $\mathcal{O}_C$ is an algebra, the sheaf $\mathcal{T}_B$ is equipped with an associative commutative multiplication, which will be denoted by $\bullet$. It induces a product $\bullet_t$ on the tangent space of every point $t \in B$. The class of the function F in $\mathcal{O}_C$ defines a vector field $E \in \mathcal{T}_B$, called the *Euler vector field*.

Given a holomorphic volume form ω on $(\mathbb{C}^{2l+1}, 0)$, possibly depending on $t \in B$, we can equip $p_*\mathcal{O}_C$ with the so-called *residue pairing*:

$$(\psi_1(t,x), \psi_2(t,x)) := \Big(\frac{1}{2\pi i}\Big)^{2l+1} \int_{\Gamma_\epsilon} \frac{\psi_1(t,y)\,\psi_2(t,y)}{\partial_{y_0}F \cdots \partial_{y_{2l}}F}\,\omega\,,$$

where $y = (y_0, \ldots, y_{2l})$ is a unimodular coordinate system for ω (i.e. $\omega = dy_0 \wedge \cdots \wedge dy_{2l}$) and the integration cycle Γ_ϵ is supported on $|\partial_{y_0}F| = \cdots = |\partial_{y_{2l}}F| = \epsilon$. Using that $\mathcal{T}_B \cong p_*\mathcal{O}_C$, we get a non-degenerate complex bilinear form $(\ ,\ )$ on $\mathcal{T}_B$, which we still call residue pairing.

For $t \in B$ and $z \in \mathbb{C}^*$, let $\mathcal{B}_{t,z}$ be a semi-infinite cycle in $\mathbb{C}^{2l+1}$ of the following type:

$$\mathcal{B}_{t,z} \in \lim_{\rho \to \infty} H_{2l+1}(\mathbb{C}^{2l+1}, \{\mathrm{Re}\, z^{-1}F(t,x) < -\rho\}; \mathbb{C}) \cong \mathbb{C}^N\,.$$

The above homology groups form a vector bundle on $B \times \mathbb{C}^*$ equipped naturally with a Gauss–Manin connection, and $\mathcal{B} = \mathcal{B}_{t,z}$ may be viewed as a flat section. According to K. Saito's theory of *primitive forms* [13, 15] there exists a form ω, called primitive, such that the oscillatory integrals (d^B is the de Rham differential on B)

$$J_{\mathcal{B}}(t,z) := (2\pi z)^{-l-\frac{1}{2}}\ (z d^B) \int_{\mathcal{B}_{t,z}} e^{z^{-1}F(t,x)}\omega \in \mathcal{T}_B^*$$

are horizontal sections for the following connection:

$$\nabla_{\partial/\partial t^i} = \nabla^{\mathrm{L.C.}}_{\partial/\partial t^i} - z^{-1}(\partial_{t^i} \bullet_t), \qquad 1 \le i \le N \tag{10}$$

$$\nabla_{\partial/\partial z} = \partial_z - z^{-1}\theta + z^{-2}E \bullet_t\ . \tag{11}$$

Here $\nabla^{\mathrm{L.C.}}$ is the Levi–Civita connection associated with the residue pairing and

$$\theta := \nabla^{\mathrm{L.C.}}E - \Big(1 - \frac{d}{2}\Big)\mathrm{Id},$$

where d is some complex number. In particular, this means that the residue pairing and the multiplication $\bullet$ form a *Frobenius structure* on B of conformal dimension d with identity 1 and Euler vector field E. For the definition of a Frobenius structure we refer to [6] .

Assume that a primitive form ω is chosen. Note that the flatness of the Gauss–Manin connection implies that the residue pairing is flat. Denote by $(\tau^1, \ldots, \tau^N)$ a coordinate system on B that is flat with respect to the residue metric, and write ∂_i for the vector field $\partial/\partial\tau^i$. We can further modify the flat coordinate system so that the Euler field is the sum of a constant and linear fields:

$$E = \sum_{i=1}^N (1 - d_i)\tau^i \partial_i + \sum_{i=1}^N \rho_i \partial_i\,.$$

The constant part represents the class of f in H, and the spectrum of degrees $d_1, \ldots, d_N$ ranges from 0 to d. Note that in the flat coordinates τ^i the operator θ (called sometimes the *Hodge grading operator*) assumes diagonal form:

$$\theta(\partial_i) = \Big(\frac{d}{2} - d_i\Big)\partial_i\,, \qquad 1 \le i \le N\,.$$

Finally, the vectors $v_i \in H$ appearing in formula (3) are the images of the flat vector fields ∂_i via the Kodaira–Spencer isomorphism (1).

2.2. **Period integrals.** Given a middle homology class $\alpha \in H_{2l}(X_{0,1};\mathbb{C})$, we denote by $\alpha_{t,\lambda}$ its parallel transport to the Milnor fiber $X_{t,\lambda}$. Let $d^{-1}\omega$ be any $2l$-form whose differential is ω. We can integrate $d^{-1}\omega$ over $\alpha_{t,\lambda}$ and obtain multivalued functions of λ and t ramified around the discriminant in S (over which the Milnor fibers become singular).

Definition 2.2. *To* $\alpha \in H_{2l}(X_{0,1};\mathbb{C})$*, we associate the* period vectors $I^{(k)}_\alpha(t,\lambda) \in H$ ($k \in \mathbb{Z}$) *defined by*

$$(I^{(k)}_\alpha(t,\lambda), \partial_i) := -(2\pi)^{-l}\partial_\lambda^{l+k}\partial_i \int_{\alpha_{t,\lambda}} d^{-1}\omega\,, \qquad 1 \le i \le N\,. \tag{12}$$

Note that this definition is consistent with the operation of stabilization of singularities. Namely, adding the squares of two new variables does not change the right-hand side, since it is offset by an extra differentiation $(2\pi)^{-1}\partial_\lambda$. In particular, this defines the period vector for a negative value of $k \ge -l$ with l as large as one wishes. Note that, by definition, we have

$$\partial_\lambda I^{(k)}_\alpha(t,\lambda) = I^{(k+1)}_\alpha(t,\lambda)\,, \qquad k \in \mathbb{Z}\,.$$

The following lemma is due to A. Givental [10].

Lemma 2.3. *The period vectors* (12) *satisfy the differential equations*

$$\partial_i I^{(k)} = -\partial_i \bullet_t (\partial_\lambda I^{(k)})\,, \qquad 1 \le i \le N\,, \tag{13}$$

$$(\lambda - E\bullet_t)\partial_\lambda I^{(k)} = \Big(\theta - k - \frac{1}{2}\Big) I^{(k)}\,. \tag{14}$$

Using equation (14), we analytically extend the period vectors to all $|\lambda| > \delta$. It follows from (13) that the period vectors have the symmetry

$$I^{(k)}_\alpha(t,\lambda) \;=\; I^{(k)}_\alpha(t - \lambda\mathbf{1}, 0)\,, \tag{15}$$

where $t \mapsto t - \lambda\mathbf{1}$ denotes the time-λ translation in the direction of the flat vector field $\mathbf{1}$ obtained from $1 \in H$. (The latter represents identity elements for all the products $\bullet_t$.)

2.3. **Stationary phase asymptotic.** Let $u_i(t)$ $(1 \leq i \leq N)$ be the critical values of $F(t,\cdot)$. For a generic t, they form a local coordinate system on B in which the Frobenius multiplication and the residue pairing are diagonal. Namely,

$$\partial/\partial u_i \bullet_t \partial/\partial u_j = \delta_{ij}\partial/\partial u_j\,, \quad \Big(\partial/\partial u_i, \partial/\partial u_j\Big) = \delta_{ij}/\Delta_i\,,$$

where Δ_i is the Hessian of F with respect to the volume form ω at the critical point corresponding to the critical value u_i. Therefore, the Frobenius structure is *semi-simple*.

We denote by Ψ_t the following linear isomorphism

$$\Psi_t\colon \mathbb{C}^N \to T_tB\,, \qquad e_i \mapsto \sqrt{\Delta_i}\partial/\partial u_i\,,$$

where $\{e_1, \ldots, e_N\}$ is the standard basis for $\mathbb{C}^N$. Let U_t be the diagonal matrix with entries $u_1(t), \ldots, u_N(t)$.

According to Givental [8], the system of differential equations (cf. (10), (11))

$$z\partial_i J(t,z) = \partial_i \bullet_t J(t,z)\,, \qquad 1 \leq i \leq N\,, \tag{16}$$

$$z\partial_z J(t,z) = (\theta - z^{-1}E\bullet_t)J(t,z) \tag{17}$$

has a unique formal asymptotic solution of the form $\Psi_t R_t(z) e^{U_t/z}$, where

$$R_t(z) = 1 + R_1(t)z + R_2(t)z^2 + \cdots\,,$$

and $R_k(t)$ are linear operators on $\mathbb{C}^N$ uniquely determined from the differential equations (16) and (17). Introduce the formal series

$$\mathbf{f}_\alpha(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k)}_\alpha(t,\lambda)\,(-z)^k\,. \tag{18}$$

Note that for A_1-singularity $F(t,x) = x^2/2 + t$ we have $u := u_1(t) = t$ and the series (18) takes the form

$$\mathbf{f}_{A_1}(t,\lambda;z) = \sum_{k\in\mathbb{Z}} I^{(k)}_{A_1}(u,\lambda)\,(-z)^k,$$

where

$$I^{(k)}_{A_1}(u,\lambda) = (-1)^k\,\frac{(2k-1)!!}{2^{k-1/2}}\,(\lambda-u)^{-k-1/2}, \quad k \geq 0$$

$$I^{(-k-1)}_{A_1}(u,\lambda) = 2\,\frac{2^{k+1/2}}{(2k+1)!!}\,(\lambda-u)^{k+1/2}, \quad k \geq 0.$$

The key lemma, which is due to Givental [10] is the following.

Lemma 2.4. *Let $t \in B$ be generic and β be a vanishing cycle vanishing over the point $(t, u_i(t)) \in \Sigma$. Then for all λ near $u_i := u_i(t)$, we have*

$$\mathbf{f}_\beta(t,\lambda;z) = \Psi_t R_t(z)\,e_i\,\mathbf{f}_{A_1}(u_i,\lambda;z)\,.$$

3. Symplectic loop space formalism

The goal of this section is to introduce Givental's quantization formalism (see [9]) and use it to define the higher genus potentials in singularity theory.

3.1. **Symplectic structure and quantization.** The space $\mathcal{H} := H(\!(z^{-1})\!)$ of formal Laurent series in z^{-1} with coefficients in H is equipped with the following *symplectic form*:

$$\Omega(\phi_1, \phi_2) := \mathrm{Res}_z\, (\phi_1(-z), \phi_2(z))\ , \qquad \phi_1, \phi_2 \in \mathcal{H}\,,$$

where, as before, $(,)$ denotes the residue pairing on H and the formal residue Res_z gives the coefficient in front of z^{-1}.

Let $\{\partial_i\}_{i=1}^N$ and $\{\partial^i\}_{i=1}^N$ be dual bases of H with respect to the residue pairing. Then

$$\Omega(\partial^i(-z)^{-k-1}, \partial_j z^l) = \delta_{ij}\delta_{kl}\,.$$

Hence, a Darboux coordinate system is provided by the linear functions q_k^i, $p_{k,i}$ on $\mathcal{H}$ given by:

$$q_k^i = \Omega(\partial^i(-z)^{-k-1}, \cdot)\,, \qquad p_{k,i} = \Omega(\cdot, \partial_i z^k)\,.$$

In other words,

$$\phi(z) = \sum_{k=0}^{\infty}\sum_{i=1}^{N} q_k^i(\phi)\partial_i z^k + \sum_{k=0}^{\infty}\sum_{i=1}^{N} p_{k,i}(\phi)\partial^i(-z)^{-k-1}\,, \qquad \phi \in \mathcal{H}\,.$$

The first of the above sums will be denoted $\phi(z)_+$ and the second $\phi(z)_-$.

The *quantization* of linear functions on $\mathcal{H}$ is given by the rules:

$$\widehat{q_k^i} = \hbar^{-1/2} q_k^i\,, \qquad \widehat{p_{k,i}} = \hbar^{1/2}\frac{\partial}{\partial q_k^i}\,.$$

Here and further, $\hbar$ is a formal variable. We will denote by $\mathbb{C}_\hbar$ the field $\mathbb{C}(\!(\hbar^{1/2})\!)$.

Every $\phi(z) \in \mathcal{H}$ gives rise to the linear function $\Omega(\phi, \cdot)$ on $\mathcal{H}$, so we can define the quantization $\widehat{\phi}$. Explicitly,

$$\widehat{\phi} = -\hbar^{1/2}\sum_{k=0}^{\infty}\sum_{i=1}^{N} q_k^i(\phi)\frac{\partial}{\partial q_k^i} + \hbar^{-1/2}\sum_{k=0}^{\infty}\sum_{i=1}^{N} p_{k,i}(\phi) q_k^i\,. \tag{19}$$

The above formula makes sense also for $\phi(z) \in H[[z, z^{-1}]]$ if we interpret $\widehat{\phi}$ as a formal differential operator in the variables q_k^i with coefficients in $\mathbb{C}_\hbar$.

Lemma 3.1. *For all $\phi_1, \phi_2 \in \mathcal{H}$, we have $[\widehat{\phi_1}, \widehat{\phi_2}] = \Omega(\phi_1, \phi_2)$.*

Proof. It is enough to check this for the basis vectors $\partial^i(-z)^{-k-1}$, $\partial_i z^k$, in which case it is true by definition. ◻

It is known that the operator series $\mathcal{R}_t(z) := \Psi_t R_t(z)\Psi_t^{-1}$ is a symplectic transformation. Moreover, it has the form $e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation. A linear operator $A(z)$ on $\mathcal{H} := H(\!(z^{-1})\!)$ is infinitesimal symplectic if and only if the map $\phi \in \mathcal{H} \mapsto A\phi \in \mathcal{H}$ is a Hamiltonian vector field with a Hamiltonian given by the quadratic function $h_A(\phi) = \frac{1}{2}\Omega(A\phi, \phi)$. By definition, the *quantization* of $e^{A(z)}$ is given by the differential operator $e^{\widehat{h_A}}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k,i}p_{l,j})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_k^i \partial q_l^j}\,, \quad (p_{k,i}q_l^j)\widehat{\ } = (q_l^j p_{k,i})\widehat{\ } = q_l^j\frac{\partial}{\partial q_k^i}\,, \quad (q_k^i q_l^j)\widehat{\ } = \frac{1}{\hbar}q_k^i q_l^j\,.$$

3.2. **The total ancestor potential.** Let us make the following convention. Given a vector

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} q_k z^k \in H[z]\,, \qquad q_k = \sum_{i=1}^{N} q_k^i \partial_i \in H\,,$$

its coefficients give rise to a vector sequence $q_0, q_1, \ldots$. By definition, a *formal function* on $H[z]$, defined in the formal neighborhood of a given point $c(z) \in H[z]$, is a formal power series in $q_0 - c_0, q_1 - c_1, \ldots$. Note that every operator acting on $H[z]$ continuously in the appropriate formal sense induces an operator acting on formal functions.

The *Witten–Kontsevich tau-function* is the following generating series:

$$\mathcal{D}_{\mathrm{pt}}(\hbar; Q(z)) = \exp\Big(\sum_{g,n} \frac{1}{n!}\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n}(Q(\psi_i) + \psi_i)\Big), \tag{20}$$

where $Q_0, Q_1, \ldots$ are formal variables, and ψ_i $(1 \le i \le n)$ are the first Chern classes of the cotangent line bundles on $\overline{\mathcal{M}}_{g,n}$ (see [16, 12]). It is interpreted as a formal function of $Q(z) = \sum_{k=0}^{\infty} Q_k z^k \in \mathbb{C}[z]$, defined in the formal neighborhood of $-z$. In other words, $\mathcal{D}_{\mathrm{pt}}$ is a formal power series in $Q_0, Q_1 + 1, Q_2, Q_3, \ldots$ with coefficients in $\mathbb{C}(\!(\hbar)\!)$.

Let $t \in B$ be a *semi-simple* point, so that the critical values $u_i(t)$ $(1 \le i \le N)$ of $F(t, \cdot)$ form a coordinate system. Recall also the flat coordinates $\tau = (\tau^1(t), \ldots, \tau^N(t))$ of t. The *total ancestor potential* of the singularity is defined as follows

$$\mathcal{A}_t(\hbar; \mathbf{q}(z)) = \widehat{\mathcal{R}}_t \prod_{i=1}^{N} \mathcal{D}_{\mathrm{pt}}(\hbar\Delta_i; {}^i\mathbf{q}(z)) \in \mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2 \ldots]], \tag{21}$$

where $\mathcal{R}_t(z) := \Psi_t R_t(z)\Psi_t^{-1}$ and

$${}^i\mathbf{q}(z) = \sum_{k=0}^{\infty}\sum_{j=1}^{N} q_k^j (\partial_j u_i) z^k\,.$$

3.3. **Proof of Theorem 1.1.** Using Lemma 2.4, it is easy to see that the ratio

$$(I_{\beta_j}^{(-m-1)}(t,\lambda), v_i)/(I_{\beta_j}^{(-1)}(t,\lambda), \mathbf{1})$$

is analytic in a neighborhood of $\lambda = u_j$ for all $m \geq 0$. Furthermore, $Y_{t,\lambda}^{u_j}\widehat{\mathcal{R}}_t = \widehat{\mathcal{R}}_t\, Y_{u_j}^{A_1}$, where $Y_{u_j}^{A_1}$ is the differential operator (5) on the variables $Q_k^j := {}^jq_k/\sqrt{\Delta_j}$ $(k \geq 0)$, defined for the A_1-singularity $F = x^2/2 + u_j$ (see Lemma 6.12 in [2]). Using that the Virasoro operators in the Virasoro constraints for the point coincide with the polar part of $Y_{u_j}^{A_1}$ (see Section 8.3 in [2]), we get that $Y_{t,\lambda}^{u_j}\mathcal{A}_t(\hbar;\mathbf{q})$ is a formal series in $\mathbf{q}$ whose coefficients are analytic at $\lambda = u_j$. This implies that the residue at $\lambda = u_j$ vanishes, which completes the proof. □

3.4. **The Virasoro recursions.** Let us compute the coefficient in front of q_1^1 in $L_{m-1,i}$. The contribution from the j-th residue is computed as follows. We put $\beta = \beta_j$ to avoid cumbersome notation. By definition the differential operator $Y_{t,\lambda}^{u_j}$ is a sum of quadratic expressions of the following 3 types:

$$(-1)^{k'+k''}\,(I_\beta^{(k'+1)}(t,\lambda), v^a)\,(I_\beta^{(k''+1)}(t,\lambda), v^b)\,\hbar\partial_{q_{k'}^a}\partial_{q_{k''}^b}\,,$$

$$2(-1)^{k''+1}\,(I_\beta^{(-k')}(t,\lambda), v_a)\,(I_\beta^{(k''+1)}(t,\lambda), v^b)\,q_{k'}^a\partial_{q_{k''}^b}\,, \tag{22}$$

and

$$(I_\beta^{(-k')}(t,\lambda), v_a)\,(I_\beta^{(-k'')}(t,\lambda), v_b)\,\hbar^{-1}q_{k'}^a\,q_{k''}^b\,,$$

where the sum is over all $k', k'' \geq 0$ and $a, b = 1, 2, ..., N$. The only contribution could come from the terms (22). The coefficient in front of $q_1^1\partial_{q_k^b}$ is

$$\frac{1}{2}\,(-1)^{k+1}\,\mathrm{Res}_{\lambda=u_j}\,(I_\beta^{(-m-1)}(t,\lambda), v_i)\,(I_\beta^{(k+1)}(t,\lambda), v^b). \tag{23}$$

Lemma 3.2. *The following identity holds*

$$\sum_{j=1}^{N}\mathrm{Res}_{\lambda=u_j}\,(I_\beta^{(k')}(t,\lambda), v_a)\,(I_\beta^{(k'')}(t,\lambda), v^b)\,d\lambda = 2(-1)^{k'}\delta_{a,b}\delta_{k'+k'',0},$$

for all $k', k'' \in \mathbb{Z}$ and $a, b = 1, 2, \ldots, N$.

Proof. According to Lemma 2.4 we have

$$I_\beta^{(k)}(t,\lambda) = \sum_{l=0}^{\infty} R_l\,(-\partial_\lambda)^{-l}\,I_{A_1}^{(k)}(u_j,\lambda)e_j.$$

Using this identity we find that the j-th term in the above sum is

$$\sum_{l',l''=0}^{\infty}({}^TR_{l'}v_a, e_j)({}^TR_{l''}v^b, e_j)(-1)^{l'+l''}\mathrm{Res}_{\lambda=u_j}\,I_{A_1}^{(k'+l')}(u_j,\lambda)I_{A_1}^{(k''+l'')}(u_j,\lambda)d\lambda. \tag{24}$$

The above residue is non-zero only if $k'+l' = -k''-l''$. In the latter case using integration by parts we find that the residue is

$$(-1)^{k'+l'}\mathrm{Res}_{\lambda=u_j}\, I^{(0)}_{A_1}(u_j,\lambda) I^{(0)}_{A_1}(u_j,\lambda)d\lambda = 2(-1)^{k'+l'}.$$

The sum (24) becomes

$$2(-1)^{k'}\sum_{l',l''=0}^{\infty}({}^TR_{l'}v_a,e_j)({}^TR_{l''}v^b,e_j)(-1)^{l''}\delta_{l'+l'',-k'-k''}$$

If we sum over all $j=1,2,\ldots,N$, since $\{e_j\}$ is an orthonormal basis of H, we get

$$2(-1)^{k'}\sum_{l',l''=0}^{\infty}({}^TR_{l'}v_a,{}^TR_{l''}v^b)(-1)^{l''}\delta_{l'+l'',-k'-k''}.$$

Using the symplectic condition $R(z)^TR(-z)=1$ we see that the only non-zero contribution in the above sum comes from the terms with $l'=l''=0$, which completes the proof. □

The above Lemma implies that the coefficient (23) is non-zero only if $k=m$ and $b=i$ and in the latter case the coefficient is 1. In order to obtain a recursion relation for the correlators (3) we replace $q_1^1 = t_1^1 - 1$ and compare the genus g degree n (with respect to $\mathbf{t}$) terms in the identity

$$\sum_{m=0}^{\infty}(-1)^{m+1}\,(I_\beta^{(m+1)}(t,\mu),v^a)L_{m-1,a}\,\mathcal{A}_t(\hbar;\mathbf{q}) = 0. \tag{25}$$

Note that if we ignore the dilaton shift, then $(Y^{u_j}_{t,\lambda}\mathcal{A}_t d\lambda\cdot d\lambda)/\mathcal{A}_t$ (here $\cdot$ is the *symmetric* product of differential forms) is a sum of terms of five different types. The first two are

$$\frac{\hbar^{g-1}}{n!}\Big\langle\phi_+^{\beta_j}(t,\lambda;\psi_1),\phi_+^{\beta_j}(t,\lambda;\psi_2),\mathbf{t},\ldots,\mathbf{t}\Big\rangle_{g-1,n+2}, \tag{26}$$

and

$$\sum_{\substack{g'+g''=g\\ n'+n''=n}}\frac{\hbar^{g-1}}{(n')!(n'')!}\Big\langle\phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\ldots,\mathbf{t}\Big\rangle_{g',n'+1}\Big\langle\phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\ldots,\mathbf{t}\Big\rangle_{g'',n''+1}. \tag{27}$$

The other three types are

$$P_0^{\beta_j,\beta_j}(t,\lambda), \tag{28}$$

$$\frac{2\hbar^{g-1}}{n!}\Omega(\phi_-^{\beta_j}(t,\lambda;z),\mathbf{t}(z))\Big\langle\phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\ldots,\mathbf{t}\Big\rangle_{g,n}, \tag{29}$$

and

$$\hbar^{-1}\Omega(\phi_-^{\beta_j}(t,\lambda;z),\mathbf{t}(z))\,\Omega(\phi_-^{\beta_j}(t,\lambda;z),\mathbf{t}(z)). \tag{30}$$

Let us point out that the ancestor correlators (3) are *tame* (see [10]), which by definition means that they vanish if $k_1+\cdots+k_n > 3g-3+n$. In particular, the ancestor potential does not have non-zero correlators in the genus-0 unstable range

$(g,n) = (0,0),(0,1)$, and $(0,2)$. However, motivated by the above formulas, it is convenient to extend the definition in the unstable range as well by setting

$$\left\langle \phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t}\right\rangle_{0,2} := \Omega(\phi_-^{\beta_j}(t,\lambda;z),\mathbf{t}(z))$$
$$\left\langle \phi_+^{\beta_j}(t,\lambda;\psi_1),\phi_+^{\beta_j}(t,\lambda;\psi_1)\right\rangle_{0,2} := P_0^{\beta_j,\beta_j}(t,\lambda)$$

and keeping the remaining unstable genus-0 correlators 0. If we allow such unstable correlators; then the terms (29) and (30) become the unstable part of the sum (27), while (28) becomes the unstable correlator in the set (26).

The above discussion and the fact that the dilaton shift changes $L_{m-1,a}$ simply by an additional differentiation $-\partial/\partial t_m^a$ yields the following identities:

$$\left\langle \phi_+^{\beta}(t,\mu;\psi_1),\mathbf{t},\dots,\mathbf{t}\right\rangle_{g,n+1} = \tag{31}$$

$$\frac{1}{4}\sum_{j=1}^{N}\mathrm{Res}_{\lambda=u_j}\frac{[\widehat{\phi^{\beta}}_+(t,\mu),\widehat{\mathbf{f}^{\beta_j}}_-(t,\lambda)]}{(I_{\beta_j}^{(-1)}(t,\lambda),\mathbf{1})\,d\lambda}\times\Bigg(\left\langle \phi_+^{\beta_j}(t,\lambda;\psi_1),\phi_+^{\beta_j}(t,\lambda;\psi_2),\mathbf{t},\dots,\mathbf{t}\right\rangle_{g-1,n+2}+ \tag{32}$$

$$\sum_{\substack{g'+g''=g\\ n'+n''=n}}\binom{n}{n'}\left\langle \phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\dots,\mathbf{t}\right\rangle_{g',n'+1}\left\langle \phi_+^{\beta_j}(t,\lambda;\psi_1),\mathbf{t},\dots,\mathbf{t}\right\rangle_{g'',n''+1}\Bigg), \tag{33}$$

where (g,n) is assumed to be in the stable range, i.e., $2g-2+n>0$, we are allowing unstable correlators on the RHS, and we suppressed the dependence of the correlators on $t\in B_{ss}$. Note that the RHS involves differential forms that should be treated formally with respect to the symmetric product $\cdot$ of differential forms.

$$\frac{d\mu\cdot d\lambda\cdot d\lambda}{d\lambda} = d\mu\cdot d\lambda = d\lambda\cdot d\mu.$$

The residue contracts $d\lambda$, so at the end the RHS involves only $d\mu$. Let us point out that the tameness condition is crucial, because it implies that for the correlator insertion of the type $\sum_m I_\beta^{(m+1)}(t,\lambda)\,(-\psi)^m$, only finitely many terms contribute. In particular, although the infinite sum of differential operators in (25) does not make sense in general, our argument goes through since on each step only finitely many terms of the sum (25) contribute.

4. The Eynard–Orantin recursion

The initial data for setting up the local Eynard–Orantin recursion is a complex line $\mathbb{C}$ with N marked points $u_1,\dots,u_N$ and a certain set of 1- and 2-forms defined only locally. More precisely, for each i we have a multi-valued holomorphic 1-form

$\omega^i(\lambda) = P^i(\lambda)\, d\lambda$ defined in some disk-neighborhood D_i of u_i, s.t.,

$$P^i(\lambda) := \sum_{k=0}^{\infty} P^i_k\, (\lambda - u_i)^{k+1/2}\,.$$

For each pair (i, j) we have a symmetric 2-form $\omega^{ij}(\mu, \lambda) := P^{ij}(\mu, \lambda)\, d\lambda \cdot d\mu$ on $D_i \times D_j$ obeying the symmetry $(i, \mu) \leftrightarrow (j, \lambda)$ and such that the function

$$(\mu - u_i)^{1/2}(\lambda - u_j)^{1/2}P^{ij}(\mu, \lambda) = (\mu - u_i)^{1/2}(\lambda - u_j)^{1/2}P^{ji}(\lambda, \mu)$$

is analytic on $D_i \times D_j$ except for a pole (with no residues) of order 2 along the diagonal in the case when $i = j$. In the latter case, we assume that the differentials are normalized in such a way that the Laurent series expansion with respect to μ in the annulus $0 < |\lambda - \mu| < |\lambda - u_i|$ has the form

$$P^{ii}(\mu, \lambda) = \frac{2}{(\mu - \lambda)^2} + \sum_{k=0}^{\infty} P^{ii}_k(\lambda)\, (\mu - \lambda)^k. \tag{34}$$

Note that for each fixed $k \geq 0$, the functions $P^{ii}_k(\lambda)$ are holomorphic on the punctured disk $D^*_i := D_i \setminus \{u_i\}$ with a finite order pole at $\lambda = u_i$.

Let us denote by $\mathcal{L}_i$ the local system in a neighborhood of D_i defined by the multi-valued function $(\lambda - u_i)^{1/2}$. We define a system of symmetric multi-valued analytic differential forms $\omega^{\alpha_1,\ldots,\alpha_n}_{g,n}(\lambda_1, \ldots, \lambda_n)$ for $\alpha_k \in \mathcal{L}_{i_k}$ and $\lambda_k \in D^*_{i_k}$ that are compatible with the (local) monodromy action on the local systems, i.e., the analytic continuation along a small loop around $\lambda_k = u_{i_k}$ transforms the differential form into $\omega^{\alpha_1,\ldots,\sigma(\alpha_k),\ldots,\alpha_n}_{g,n}(\lambda_1, \ldots, \lambda_n)$, where σ is the corresponding monodromy action on $\mathcal{L}_{i_k}$.

Let $\alpha \in \mathcal{L}_i, \beta \in \mathcal{L}_j$ be any sections; then the base of the recursion is the following

$$\omega^{\alpha}_{0,1}(\lambda) = 0,$$

$$\omega^{\alpha,\beta}_{0,2}(\mu, \lambda) = \begin{cases} P^{ij}(\mu, \lambda)\, d\mu \cdot d\lambda & \text{if } (i, \mu) \neq (j, \lambda), \\ P^{ii}_0(\lambda)\, d\lambda \cdot d\lambda & \text{if } (i, \mu) = (j, \lambda). \end{cases}$$

The *kernel* of the recursion is the following ratio of 1 forms:

$$K^{\alpha,\beta}(\mu, \lambda) = \frac{1}{2}\, \frac{\oint_{C_\lambda} P^{ij}(\mu, \lambda')\, d\lambda'}{P^j(\lambda)}\, \frac{d\mu}{d\lambda}, \tag{35}$$

where we fix $\mu \in D_i \setminus \{u_i\}$ and select a simple loop C_λ in D_j based at λ that goes around u_j. Then the recursion takes the form

$$\omega^{\alpha_0,\alpha_1,\ldots,\alpha_n}_{g,n+1}(\lambda_0, \lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{N} \mathrm{Res}_{\lambda=u_j}\, K^{\alpha_0,\beta_j}(\lambda_0, \lambda) \times$$

$$\left(\omega^{\beta_j,\beta_j,\alpha_1,\ldots,\alpha_n}_{g-1,n+2}(\lambda,\lambda,\lambda_1,\ldots,\lambda_n) + \sum_{\substack{g'+g''=g \\ I'\sqcup I''=\{1,\ldots,n\}}} \omega^{\beta_j,\alpha_{I'}}_{g',n'+1}(\lambda,\lambda_{I'})\,\omega^{\beta_j,\alpha_{I''}}_{g'',n''+1}(\lambda,\lambda_{I''}) \right),$$

where we are assuming that $2g-2+n>0$, $\beta_j\in\mathcal{L}_j$, the sum in the big brackets is over all splittings, n' and n'' are the number of elements respectively in I' and I'', and for a subset $I\subset\{1,2,\ldots,n\}$ we adopt the standard multi-index notation $x_I=(x_{i_1},\ldots,x_{i_k})$. Note that although the functions are multivalued, the local monodromy about $\lambda=u_j$ leaves the expression invariant, so the residue is well defined.

4.1. **Proof of Theorem 1.2.** In the settings of singularity theory, for a generic $t\in B_{ss}$ we let the marked points be the critical values $u_i=u_i(t)$. The choice of a section of the local system $\mathcal{L}_i$ is the same as choosing a vanishing cycle over $\lambda=u_i$. Let $\omega^{\beta_1,\ldots,\beta_n}_{g,n}(\lambda_1,\ldots,\lambda_n)$ be the n-point series (4).

In order to prove that these forms satisfy the Eynard–Orantin recursion, it is enough to notice that

$$\frac{1}{\sqrt{\hbar}}\,[\widehat{\phi^\beta}_+(t,\lambda),\mathbf{t}(\psi)]=\phi^\beta_+(t,\lambda;\psi).$$

Applying this formula n times to (31)–(33) with $\beta=\beta_1,\ldots,\beta_n$ we obtain the Eynard–Orantin recursion with

$$\omega^{ij}(\mu,\lambda)=[\widehat{\phi^\alpha}_+(t,\mu),\widehat{\phi^\beta}_-(t,\lambda)], \tag{36}$$

and

$$P^j(\lambda)=4(I^{(-1)}_\beta(t,\lambda),\mathbf{1}), \tag{37}$$

where α,β are vanishing cycles vanishing respectively over $\mu=u_i$ and $\lambda=u_j$. Note that

$$\oint_{C_\lambda} P^{ij}(\mu,\lambda')d\lambda'\cdot d\mu = 2\,[\widehat{\phi^\alpha}_+(t,\mu),\widehat{\mathbf{f}^\beta}_-(t,\lambda)]$$

so the kernel is given by formula (35).

In the opposite direction, in order to prove that the Eynard–Orantin recursion implies the Virasoro constraints, it is enough to notice that according to Lemma 3.2 we have the following identity

$$\mathbf{t}(\psi)=\frac{1}{2}\sum_{j=1}^N \mathrm{Res}_{\lambda=u_j}\,\Omega(\mathbf{f}^{\beta_j}_-(t,\lambda;z),\mathbf{t}(z))\,\phi^{\beta_j}(t,\lambda;\psi)\,. \quad \square$$

Finally, let us point out that using Lemma 2.4 one can express the Laurent series expansions of $\omega^{ij}(\mu,\lambda)$ and $\omega^j(\lambda)$ in terms of the symplectic operator series $\mathcal{R}$. The answer is the following. Let $V_{kl} \in \mathrm{End}(H)$ be defined via

$$\sum_{k,l=0}^{\infty} V_{kl} w^k z^l = \frac{1 - {}^T\mathcal{R}(-w)\mathcal{R}(-z)}{z+w},$$

then $(\mu - u_i)^{1/2}(\lambda - u_j)^{1/2} P^{ij}(\mu, \lambda)$ has the following Taylor's series expansion

$$\frac{\delta_{ij}}{(\mu-\lambda)^2}(\mu - u_i + \lambda - u_j) + \sum_{k,l=0}^{\infty} 2^{k+l+1}(e_i, V_{kl} e_j)\frac{(\mu-u_i)^k}{(2k-1)!!}\,\frac{(\lambda-u_j)^l}{(2l-1)!!}$$

Note that if $i = j$ and we fix λ near u_i; then the Laurent series expansion of $P^{ij}(\mu, \lambda)$ about $\mu = \lambda$ does take the form (34). The Taylor's series expansion of $P^j(\lambda)$ is

$$8 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+1/2}}{(2k+1)!!}\,(\mathcal{R}_k\, e_j, \mathbf{1})\,(\lambda - u_j)^{k+1/2}.$$

Up to an appropriate normalization of the correlation functions, our answer agrees with the formulas in [4].

References

[1] V. Arnold, S. Gusein-Zade, and A. Varchenko. *Singularities of Differentiable maps.* Vol. II. Monodromy and Asymptotics of Integrals. Boston, MA: Birkhäuser Boston, 1988. viii+492 pp

[2] B. Bakalov and T. Milanov *W-constraints for the total descendant potential of a simple singularity.* Preprint (2012); http://arxiv.org/abs/1203.3414

[3] V. Bouchard and B. Eynard *Think globally, compute locally.* Preprint (2012); http://arxiv.org/abs/1211.2302

[4] P. Dunnin–Barkowski, N. Orantin, S. Shadrin, and L. Spitz. *Identification of the Givental formula with the spectral curve topological recursion procedure.* Preprint (2012); http://arxiv.org/abs/1211.4021

[5] B. Eynard and N. Orantin *Invariants of algebraic curves and topological expansion.* Comm. in Number Theory and Physics **1**(2007): 347–552

[6] B. Dubrovin, *Geometry of 2D topological field theories*. In: "Integrable systems and quantum groups" (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[7] O. Dumitrescu, M. Mulase, B. Safnuk, and A. Sorkin. *The spectral curve of the Eynard-Orantin recursion via the Laplace transform.* Preprint (2012); http://arxiv.org/abs/1202.1159

[8] A. Givental. *Semisimple Frobenius structures at higher genus.* Internat. Math. Res. Notices 2001, no. 23, 1265-1286

[9] A. Givental. *Gromov–Witten invariants and quantization of quadratic Hamiltonians*. Mosc. Math. J. **1** (2001), 551–568

[10] A. Givental. *A_{n-1} singularities and nKdV Hierarchies.* Mosc. Math. J. 3.2(2003): 475–505

[11] C. Hertling. *Frobenius Manifolds and Moduli Spaces for Singularities.* Cambridge Tracts in Mathematics, 151. Cambridge University Press, Cambridge, 2002. x+270 pp

[12] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. **147** (1992), 1–23.

[13] K. Saito. *On Periods of Primitive Integrals, I.* Preprint RIMS(1982)
[14] K. Saito and A. Takahashi. *From primitive forms to Frobenius manifolds.* From Hodge theory to integrability and TQFT tt*-geometry, 31-48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008
[15] M. Saito, *On the structure of Brieskorn lattice*. Ann. Inst. Fourier **39** (1989), 27–72.
[16] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*. In: "Surveys in differential geometry," 243–310, Lehigh Univ., Bethlehem, PA, 1991.

Kavli IPMU (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
E-mail address: todor.milanov@ipmu.jp