INFINITE EXAMPLES OF CANCELLATIVE MONOIDS THAT DO NOT ALWAYS HAVE LEAST COMMON MULTIPLE.

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Abstract. We will study the presentations of fundamental groups of the complement of complexified real affine line arrangements that do not contain two parallel lines. By Yoshinaga's minimal presentation, we can give positive homogeneous presentations of the fundamental groups. We consider the associated monoids defined by the presentations. It turns out that, in some cases, left (resp. right) *least common multiple* does not always exist. Hence, the monoids are neither *Garside* nor *Artin.* Nevertheless, we will show that they carry certain particular elements similar to the *fundamental elements* in Artin monoids, and that, by improving the classical method in combinatorial group theory, they are *cancellative monoids.* As a result, we will show that the word problem can be solved and the center of them are determined.

1. Introduction

Early in $70's$ the braid groups are generalized to a wider class of groups, the fundamental groups of the regular orbit spaces of finite reflection groups ([B]), which are called either the *Artin group* ([B-S]) or the *generalized braid group* ([De]). In [B], E. Brieskorn gave a presentation of the fundamental groups by certain positive homogeneous relations, called Artin braid relations. The monoid defined by that presentation is called *Artin monoid* of finite type. In [B-S], by refering to the method in [G], they showed that the Artin monoid is *cancellative* (i.e. $axb = ayb$ implies $x = y$) and that, for any two elements in the monoid, left (resp. right) common multiples exist. Hence, due to the Öre's criterion, the Artin monoid of finite type injects in the corresponding Artin group. Furthermore, they showed that, for any two elements, left (resp. right) least common multiple exists (see [B-S] §4). By using this property, they defined a particular element Δ, the *fundamental element*, in the monoid. By using the injectivity and the existence of this element Δ, they solved the word and conjugacy problem in the Artin groups of finite type and determined the center of them.

After this work, in the late $90's$, the notion of Artin group (resp. Artin monoid) is generalized by French mathematicians ([D-P], [D]), which is called the *Garside group* (resp. *Garside monoid*). The Garside group is defined as the group of fractions of a Garside monoid. A Garside monoid is a finitely generated monoid that satisfies the following conditions: i) the monoid is cancellative; ii) *atomic* (i.e. the expressions of a given element have bounded lengths); iii) left (resp. right) least common multiples exist; iv) a *Garside element* exists. Hence, the Garside monoid trivially satisfies the Öre's criterion. We note that, under the assumption that the monoid is atomic and cancellative, an element Δ in the monoid is a Garside element if and only if Δ is a *fundamental element* (Proposition 2.2). Therefore, we prepare the condition iv)$'$ a fundamental element exists. For Garside group, the word problem can be solved. Moreover, the conjugacy problem can be solved ([P], [Ge]), by improving the method in [G] and [E-M]. Since the condition iii) is a strong assumption, some Zariski-van Kampen monoids do not satisfy the condition iii) ([B-M], [I1], [S-I]). Nevertheless, in [I1], for the monoid, called the type $\mathrm{B_{ii}}$, that does not satisfy only the condition iii), the author has solved the word problem and the conjugacy problem, and determined the center of it by showing the monoid injects in the corresponding group.

In this article, we will construct infinite examples that do not satisfy only the condition iii). To obtain the infinite examples, we will study the presentations of the fundamental groups of the complement of complexified real affine line arrangements that do not contain two parallel lines (§2). By Yoshinaga's minimal presentation, we can give positive homogeneous presentations of the fundamental groups. In Section 3, we will consider a special type of line arrangement. The line arrangement consists of $m+n+1$ real affine lines. We will compute the fundamental group of the complement of it's complexification by using Zariski-van Kampen method. The same presentation can be obtained by Yoshinaga's minimal presentation. It turns out that fundamental elements exist in the associated monoid defined by the presentation (Proposition 3.1). Moreover, we will show the cancellativity of it by improving the classical method in combinatorial group theory (for instance [G]) (Proposition 4.3). Due to Öre's criterion, the associated monoid injects in the corresponding group (Proposition 5.1). As a result, some decision problems in the group can be solved (Proposition 5.2, 5.4). We remark that the fundamental group is isomorphic to

$$\mathbb{Z} \times F_m \times F_n.$$

Hence, from a group theoretical point of view, we may say that this fundamental group is well-known.

Let us explain details of our motivation. Affine Zariski-van Kampen method is a convenient tool for computing fundamental groups of complements to affine plane curves (see [Ch], [T-S] for instance). It gives you the fundamental groups in terms of generators and relations. In some cases, the presentations of the fundamental groups are positive homogeneous presentations. Then, we can associate monoids defined by the presentations, which are called the Zariski-van Kampen monoids. When we deal with Zariski-van Kampen monoids, we often meet the situation in which the monoids satisfy the conditions ii) and iv)$'$ but do not satisfy the condition iii). We know that, for the groups corresponding to the monoids satisfying the conditions i), ii) and iv)$'$, the word problem can be solved (Lemma 2.3). As far as we know, for non-abelian monoids that do not satisfy the condition iii), it is difficult to show the cancellativity of them. Thus, we need to improve the technique to show the cancellativity. In Proposition 4.3, we partially succeed to improve the technique. In [I2], we have constructed examples to which only our method can be applied.

2. Positive Presentation

In this section, we first recall from [B-S] some basic definitions and notations. Secondly, for a positive finitely presented group

$$G = \langle L \,|\, R \rangle,$$

we associate a monoid defined by it. We will extend a basic notion in [B-S], *fundamental element*, for a positively presented atomic monoid. Lastly, by using a fundamental element Δ in the associated monoid, we will discuss the word problem in the group $G = \langle L \,|\, R\rangle$.

Let L be a finite set. Let $F(L)$ be the free group generated by L, and let L^* be the free monoid generated by L inside $F(L)$. We call the elements of $F(L)$ *words* and the elements of L^* *positive words*. The empty word ε is the identity element of L^*. If two words A, B are identical letter by letter, we write $A \equiv B$. Let $G = \langle L \,|\, R\rangle$ be a positive presented group (i.e. the set R of relations consists of those of the form $R_i = S_i$ where R_i and S_i are positive words), where R is the set of relations. We often denote the images of the letters and words under the quotient homomorphism

$$F(L) \longrightarrow G$$

by the same symbols and the equivalence relation on elements A and B in G is denoted by $A = B$.

Secondly, we recall some terminologies and concepts on a monoid M. An element $U \in M$ is said to *divide* $V \in M$ from the left (resp. right), and denoted by $U|_l V$ (resp. $U|_r V$), if there exists $W \in M$ such that $V = UW$ (resp. $V = WU$). We also say that V is *left-divisible* (resp.*right-divisible*) by U, or V is a *right-multiple* (resp.*left-divisible*) of U. We say that M *admits the left* (resp. *right*) *divisibility theory*, if for any two elements U, V in M, there always exists their left (resp. right) least common multiple, i.e. a left (resp. right) common multiple that divides any other left (resp. right) common multiple.

Lastly, we consider two operations on the set of subsets of a monoid M. For a subset J of M, we put

$$\mathrm{cm}_r(J) := \{u \in M \mid j \,|_l\, u,\ \forall j \in J\},$$
$$\min_r(J) := \{u \in J \mid \exists v \in J \text{ s.t. } v \,|_l\, u \Rightarrow v = u\},$$

and their composition by

$$\mathrm{mcm}_r(J) := \min_r(\mathrm{cm}_r(J)).$$

Next, we recall from [S-I], [I1] some terminologies and concepts on positive presented monoid. And we refer to some concepts from [D-P], [D].

Definition 2.1. *Let $G = \langle L \,|\, R\rangle$ be a positive finitely presented group, where L is the set of generators (called alphabet) and R is the set of relations. Then we associate a monoid $G^+ = \langle L \mid R\rangle_{mo}$ defined as the quotient of the free monoid L^* generated by L by the equivalence relation defined as follows:*

i) *two words U and V in L^* are called* elementarily equivalent *if either $U \equiv V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),*

ii) *two words U and V in L^* are called* equivalent, *denoted by $U \doteq V$, if there exists a sequence $U \equiv W_0, W_1, \ldots, W_n \equiv V$ of words in L^* for $n \in \mathbb{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \ldots, n$.*

1. We say that G^+ is atomic, *if there exists a map:*

$$\nu \ :\ G^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

such that i) $\nu(\alpha) = 0 \Longleftrightarrow \alpha = 1$ *and* ii) *an inequality:*

$$\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$$

is satisfied for any $\alpha, \beta \in G^+$. *If* $G^+ = \langle L \mid R \rangle_{mo}$ *is a positive homogeneously presented monoid (i.e. the set* R *of relations consists of those of the form* $R_i = S_i$ *where* R_i *and* S_i *are positive words of the same length), it is clear that* G^+ *is an atomic monoid. An element* $\alpha \neq 1$ *in* G^+ *is called an* atom *if it is indecomposable, namely,* $\alpha = \beta\gamma$ *implies* $\beta = 1$ *or* $\gamma = 1$.

2. We suppose that G^+ *satisfies the condition of atomic monoid. Here, we write the set of generators* L *by* $\{g_1, g_2, \ldots, g_m\}$. *If, for some positive word* $w(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m)$ *(i.e. a word that is written by the generators except* g_i*),* $g_i = w(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m)$ *is a relation of* R*, then we call the generator* g_i *a* dummy generator*. We note that, in the set* R*, a relation that has a form of* $g_i = w(g_1, \ldots, g_i, \ldots, g_m)$ *must be the form* $g_i = w(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m)$ *or a trivial form* $g_i = g_i$*, because we suppose here that* G^+ *is an atomic monoid. We denote by* L' *the set of all dummy generators of the monoid* G^+. *We put* $\widetilde{L} := L \setminus L'$. *We note that, if* G^+ *is an atomic monoid, the image of the set* $\widetilde{L}$ *in* G^+ *is equal to the set of all the atoms.*

3. We say that G^+ *is* cancellative*, if an equality* $AXB \doteqdot AYB$ *for* $A, B, X, Y \in G^+$ *implies* $X \doteqdot Y$.

4. The natural homomorphism $\pi : G^+ \to G$ *will be called the* localization homomorphism.

5. An element $\Delta \in G^+$ *is called a* Garside element *if the sets of left- and right-divisors of* Δ *coincide, generate* G^+*, and are finite in number.*

6. An element Δ *in an atomic monoid* G^+ *is called a* fundamental element *if there exists a permutation* σ_Δ *of* $\widetilde{L}$ *such that, for any* $s \in \widetilde{L}$*, there exists* $\Delta_s \in G^+$ *satisfying the following relation:*

$$\Delta \doteqdot s \cdot \Delta_s \doteqdot \Delta_s \cdot \sigma_\Delta(s).$$

We note that, if the monoid G^+ *is a cancellative monoid, there exists a unique permutation* σ_Δ *for a fundamental element* Δ. *We denote by* $\mathcal{F}(G^+)$ *the set of all fundamental elements of* G^+. *The order of an element* σ_Δ *in the permutation group* $\mathfrak{S}(\widetilde{L})$ *is denoted by* $\mathrm{ord}(\sigma_\Delta)$. *Note that* $\varepsilon \notin \mathcal{F}(G^+)$.

From the definitions, it follows that the notion of fundamental elements is equivalent to the notion of Garside elements.

Proposition 2.2. *Let* $G = \langle L \mid R \rangle$ *be a positively presented group, and let* $G^+ = \langle L \mid R \rangle_{mo}$ *be the associated monoid. Assume that the monoid* G^+ *is an atomic, cancellative monoid.*
Then, an element Δ *in* G^+ *is a fundamental element if and only if* Δ *is a Garside element.*

Proof. Assume that Δ is a fundamental element. We put $N := \mathrm{ord}(\sigma_\Delta)$. We decompose Δ into $U \cdot V$. We write U and V by $u_1 u_2 \cdots u_k$ and $v_1 v_2 \cdots v_\ell$ respectively ($u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_\ell \in \widetilde{L}$). Since the monoid G^+ is a cancellative monoid, by the definition of Δ we have

$$\begin{aligned}
& u_1 u_2 \cdots u_k \cdot v_1 v_2 \cdots v_\ell \doteqdot v_1 v_2 \cdots v_\ell \cdot \sigma_\Delta(u_1)\sigma_\Delta(u_2) \cdots \sigma_\Delta(u_k) \\
& \doteqdot \sigma_\Delta(u_1)\sigma_\Delta(u_2) \cdots \sigma_\Delta(u_k) \cdot \sigma_\Delta(v_1)\sigma_\Delta(v_2) \cdots \sigma_\Delta(v_\ell) \\
& \doteqdot \sigma_\Delta^{N-1}(u_1)\sigma_\Delta^{N-1}(u_2) \cdots \sigma_\Delta^{N-1}(u_k) \cdot \sigma_\Delta^{N-1}(v_1)\sigma_\Delta^{N-1}(v_2) \cdots \sigma_\Delta^{N-1}(v_\ell) \\
& \doteqdot \sigma_\Delta^{N-1}(v_1)\sigma_\Delta^{N-1}(v_2) \cdots \sigma_\Delta^{N-1}(v_\ell) \cdot u_1 u_2 \cdots u_k.
\end{aligned}$$

Hence, the element U is also a right divisor of Δ.

Next, we assume that Δ is a Garside element. We recall that the set $\widetilde{L}$ is equal to the set of all the atoms. Here, we write $\widetilde{L}$ by $\{s_1, s_2, \ldots, s_m\}$. Since Δ is a Garside element, for each $i \in \{1, 2, \ldots, m\}$, s_i devides Δ from the left. Thus, we can associate a quotient Δ_{s_i} (i.e. $\Delta \doteqdot s_i \cdot \Delta_{s_i}$ holds). Since the monoid G^+ is a cancellative monoid, we remark that the element Δ_{s_i} can be determined uniquely. We show the following Claim.

Claim. For arbitrary two atoms $s_i, s_j (i \neq j)$, Δ_{s_i} cannot be a substring of Δ_{s_j}.

Proof. We assume that there exist two words w_1 and w_2 such that Δ_{s_i} and Δ_{s_j} satisfy the following equation

$$\Delta_{s_i} \doteqdot w_1 \cdot \Delta_{s_j} \cdot w_2.$$

By substituting Δ_{s_i} by $w_1 \cdot \Delta_{s_j} \cdot w_2$, we have

$$\Delta \doteqdot s_j \cdot \Delta_{s_j} \doteqdot s_i \cdot \Delta_{s_i} \doteqdot s_i \cdot w_1 \cdot \Delta_{s_j} \cdot w_2. \tag{2.1}$$

We consider the following two cases.

Case 1: $w_2 = 1$

Due to the cancellativity, we have the following equation

$$s_j \doteqdot s_i \cdot w_1.$$

A contradiction.

Case 2: $w_2 \neq 1$

Since Δ is a Garside element, we say that, from (2.1), the element $s_i \cdot w_1 \cdot \Delta_{s_j}$ is also a right divisor. Hence, there exists a positive word $\widetilde{w_2} \neq 1$ such that

$$s_i \cdot w_1 \cdot \Delta_{s_j} \cdot w_2 \doteqdot \widetilde{w_2} \cdot s_i \cdot w_1 \cdot \Delta_{s_j}.$$

Thus, due to the cancellativity, we have

$$s_j \doteqdot \widetilde{w_2} \cdot s_i \cdot w_1.$$

A contradiction. $\square$

Since the monoid G^+ is a cancellative monoid, there exists a unique element A such that

$$\Delta \doteqdot s_i \cdot \Delta_{s_i} \doteqdot \Delta_{s_i} \cdot A. \tag{2.2}$$

We write A in the form $\alpha_1 \alpha_2 \cdots \alpha_k$ letter by letter ($\alpha_1, \alpha_2, \ldots, \alpha_k \in \widetilde{L}$). Assume that $k \geq 2$. Since Δ is a Garside element, we say that the element $\Delta_{s_i} \cdot \alpha_1 \cdots \alpha_{k-1}$ is also a right divisor. Hence, there exists a positive word $B \neq 1$ such that

$$\Delta \doteqdot B \cdot \Delta_{s_i} \cdot \alpha_1 \cdots \alpha_{k-1}.$$

Due to the Claim, we have a contradiction. Hence, we say that $k = 1$. From (2.2), there exists a unique permutation σ_Δ of $\widetilde{L}$ such that, for any $s \in \widetilde{L}$, the following relation holds:

$$\Delta \doteqdot s \cdot \Delta_s \doteqdot \Delta_s \cdot \sigma_\Delta(s).$$

$\square$

Lastly, we discuss the word problem in a positively presented group.

Lemma 2.3. *Let $G = \langle L \,|\, R\rangle$ be a positively presented group, and let $G^+ = \langle L \mid R\rangle_{mo}$ be the associated monoid. Assume that the monoid G^+ is an atomic, cancellative monoid and $\mathcal{F}(G^+) \neq \emptyset$. Then:*

(1) The localization homomorphism $\pi : G^+ \to G$ is injective.

(2) The word problem in G is solvable.

Proof. (1) Let $\Delta \in \mathcal{F}(G^+)$ be a fundamental element. We can easily show that, for any $U \in G^+$, there exists a sufficiently large integer ℓ such that U devides Δ^ℓ from the left and the right. Hence, we show that the monoid G^+ satisfies Öre's condition (see [C-P]). Therefore, the localization homomorphism π is injective.

(2) We put $\Lambda := \Delta^{\mathrm{ord}(\sigma_\Delta)}$, which belongs to the center $\mathcal{Z}(G^+)$ of the monoid G^+. For any two elements U, V in G, there exists a non-negative integer k in $\mathbb{Z}_{\geq 0}$ such that both $(\pi(\Lambda))^k U$ and $(\pi(\Lambda))^k V$ are equivalent to positive words. Since the localization homomorphism π is injective, there exists a unique element $U' \in G^+$ (resp. $V' \in G^+$) such that

$$\pi(U') = (\pi(\Lambda))^k U (\mathrm{resp.}\pi(\mathrm{V}') = (\pi(\Lambda))^{\mathrm{k}}\mathrm{V}).$$

Therefore, we can show that $U = V$ can be shown in G algorithmically if and only if $U' \doteqdot V'$ can be shown in G^+ algorithmically. Because the monoid G^+ is an atomic monoid, we can obtain algorithmically all the possible expressions of two words U' and V' in G^+ in a finite number of steps. Hence, by comparing two types of complete lists of all the possible expressions of words U' and V', we decide in a finite number of steps whether $U' \doteqdot V'$ or not. Consequently, the word problem in G can be solved.

□

Here is an important observation on the existence of fundamental elements in the monoid associated with the presentation of fundamental group of the complement of line arranement that is given by Yoshinaga's minimal presentation ([Y]). Let $\mathcal{A} = \{\ell_1, \ell_2, \ldots, \ell_N\}$ be a real line arrangement in $\mathbb{R}^2$ that does not contain two parallel lines and is equipped with an oriented generic flag $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2$. By Yoshinaga's minimal presentation, we give a positive homogeneous presentation of the fundamental group $\pi_1(M(\mathcal{A}))$. Here, we write the generator system by $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$. When we take the generic line $\mathcal{F}^1$ far away from all the intersection points, we can show that, in the finitely presented group, a cyclic defining relation $[\gamma_1, \gamma_2, \ldots, \gamma_N]$ holds. As a corollary, we have the following statement.

Corollary 2.4. *An element $\Delta := \gamma_1\gamma_2\cdots\gamma_N$ in the associated monoid is a fundamental element.*

Due to the Lemma 2.3, if the cancellativity of the associated monoid is proved, we can solve the word problem in the presented group. Hence, to show the cancellativity of the associated monoids is important for an understanding of the corresponding fundamental groups. If the associated monoid is not a cancellative monoid (i.e. a relation $\alpha\beta \doteqdot \alpha\gamma$ holds but $\beta \doteqdot \gamma$ does not hold), we add the relation $\beta \doteqdot \gamma$ to the list of original defining relations. Then, we expect that the new monoid is a cancellative monoid. Even if the new monoid is not a cancellative monoid, by adding more new relations to the list each time, we expect that, in a finite number of steps, we can find a cancellative monoid. Contrary to our expectation, there are interesting examples, where the above process cannot finish in a finite number of

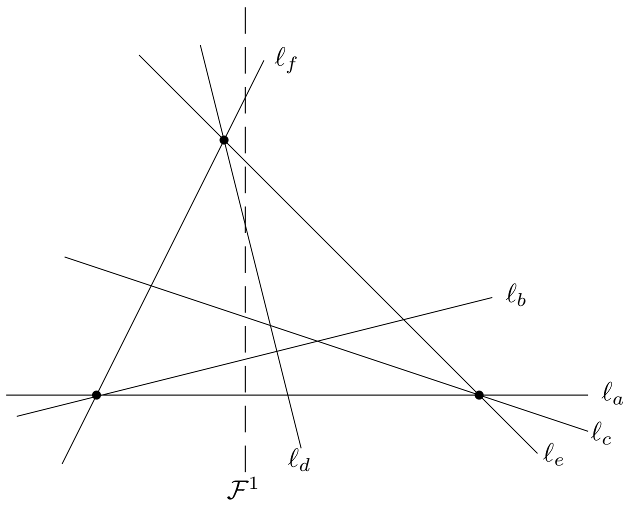

FIGURE 1. a line arrangement $\mathcal{A}_6$

steps.

Example 2.5. *Let $\mathcal{A}_6 = \{\ell_a, \ell_b, \ldots, \ell_f\}$ be the line arrangement that is written in Figure 1 and let $\mathcal{F}^1$ be a generic line. By Yoshinaga's minimal presentation, we give the following positive homogeneous presentation:*

$$\pi_1(M(\mathcal{A}_6)) \quad \cong \quad \left\langle a, b, c, d, e, f \;\middle|\; \begin{array}{l} abf = bfa = fab, ace = cea = eac, \\ def = efd = fde, ad = da, cd = dc, \\ bc = cb, bd = db, be = eb, cf = fc \end{array} \right\rangle .$$

For the above positive homogeneous presented group, we associate the monoid M_6. We show the following Claim.

Claim. *In the monoid M_6, $cdea^kf \underset{\cdot}{=} cea^kfd$ holds but $dea^kf \underset{\cdot}{=} ea^kfd$ does not hold ($k = 1, 2, \ldots$). Moreover, $bfe^kac \underset{\cdot}{=} fe^kabc$ holds but $bfe^ka \underset{\cdot}{=} fe^kab$ does not hold , and $cef^kab \underset{\cdot}{=} ef^kacb$ holds but $cef^ka \underset{\cdot}{=} ef^kac$ does not hold ($k = 1, 2, \ldots$).*

Proof. In the monoid M_6, we have

$$cdea^kf \underset{\cdot}{=} dcea^kf \underset{\cdot}{=} da^kcef \underset{\cdot}{=} a^kcdef \underset{\cdot}{=} a^kcefd \underset{\cdot}{=} cea^kfd.$$

However, we cannot show the relation $dea^kf \underset{\cdot}{=} ea^kfd$ by using only the above defining relations. □

Example 2.6. *Let $\mathcal{A}'_6 = \{\ell'_a, \ell'_b, \ldots, \ell'_f\}$ be the line arrangement that is written in Figure 2 and let $\mathcal{F}^1$ be a generic line. By Yoshinaga's minimal presentation, we give the following positive homogeneous presentation:*

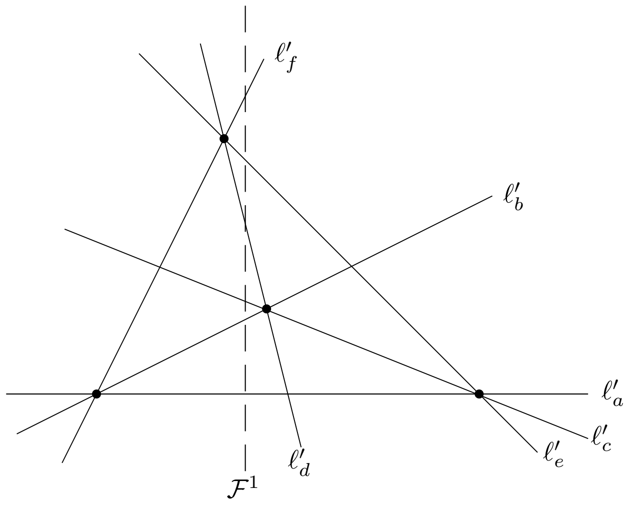

FIGURE 2. a line arrangement $\mathcal{A}'_6$

$$\pi_1(M(\mathcal{A}'_6)) \quad \cong \quad \left\langle a,b,c,d,e,f \;\middle|\; \begin{array}{l} abf = bfa = fab, bcd = cdb = dbc, \\ def = efd = fde, ad = da, cf = fc, \\ be = eb, abce = eabc, cdea = acde \end{array} \right\rangle .$$

For the above positive homogeneous presented group, we associate the monoid M'_6. We immediately show the following Claim.

Claim. *In the monoid M'_6, $dbcefa \underset{\cdot}{=} dbefac$ holds but $cefa \underset{\cdot}{=} efac$ does not hold.*

Proof. In the monoid M'_6, we have

$$\begin{aligned} & dbcefa \underset{\cdot}{=} bcdefa \underset{\cdot}{=} bcfdea \underset{\cdot}{=} bfcdea \underset{\cdot}{=} bfacde \underset{\cdot}{=} fabcde \\ & \underset{\cdot}{=} fadbce \underset{\cdot}{=} fdabce \underset{\cdot}{=} fdeabc \underset{\cdot}{=} defabc \underset{\cdot}{=} debfac \underset{\cdot}{=} dbefac. \end{aligned}$$

However, we cannot show the relation $cefa \underset{\cdot}{=} efac$ by using only the defining relations of the monoid M'_6. □

We add the relation $cefa \underset{\cdot}{=} efac$ to the list of original defining relations. We consider the associated monoid $\widetilde{M'_6}$. Then, we can find the following infinite new relations.

Claim. *In the monoid $\widetilde{M'_6}$, $acde^{k+1}abf \underset{\cdot}{=} de^kaabcef$ holds but $acde^{k+1}ab \underset{\cdot}{=} de^kaabce$ does not hold ($k = 1, 2, \ldots$). Moreover, $cefa^{k+1}cdb \underset{\cdot}{=} fa^kccdeab$ holds but $cefa^{k+1}cd \underset{\cdot}{=} fa^kccdea$ does not hold , and $eabc^{k+1}efd \underset{\cdot}{=} bc^keefacd$ holds but $eabc^{k+1}ef \underset{\cdot}{=} bc^keefac$ does not hold ($k = 1, 2, \ldots$).*

Proof. In the monoid $\widetilde{M'_6}$, we have

$$\begin{aligned} & de^kaabcef \underset{\cdot}{=} de^kaeabcf \underset{\cdot}{=} de^kaeabfc \underset{\cdot}{=} de^kaebfac \underset{\cdot}{=} de^kabefac \underset{\cdot}{=} de^kabcefa \\ & \underset{\cdot}{=} dabce^kefa \underset{\cdot}{=} adbce^kefa \underset{\cdot}{=} acdbe^kefa \underset{\cdot}{=} acde^kebfa \underset{\cdot}{=} acde^keabf. \end{aligned}$$

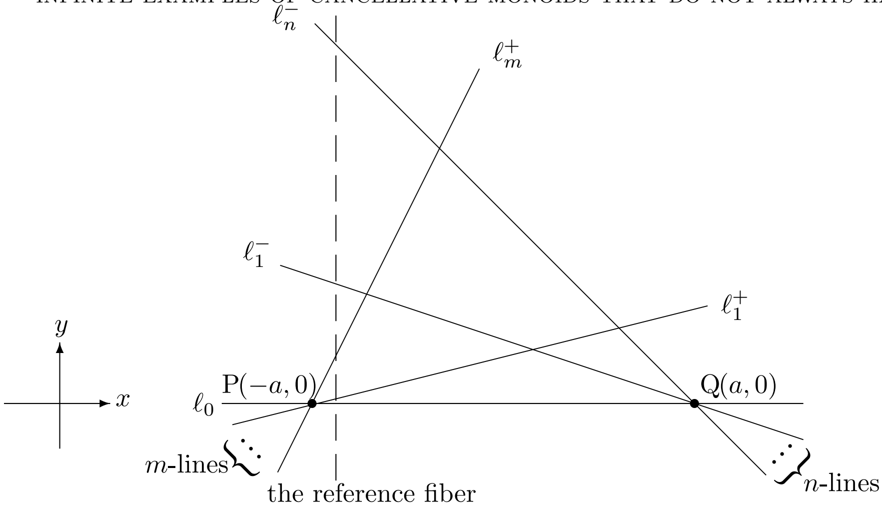

FIGURE 3. a line arrangement $\mathcal{A}_{m,n}$

However, we cannot show the relation $acde^{k+1}ab \doteq de^k aabce$ by using only the defining relations of the monoid $\widetilde{M_6'}$. $\square$

3. A Zariski-van Kampen presentation

In this section, we give a Zariski-van Kampen presentation of the fundamental groups of the complement of a certain complexified real affine line arrangement. We easily show that the same presentation can be obtained by Yoshinaga's minimal presentation. Next, for the presented group, we associate a monoid defined by it. And we show the existence of a fundamental element in it.

Let $\mathcal{A}_{m,n} = \{\ell_0, \ell_1^+, \ldots, \ell_m^+, \ell_1^-, \ldots, \ell_n^-\}$ be a real line arrangement in $\mathbb{R}^2$ with coordinates (x, y) (see Figure 2). The line ℓ_0 denotes the horizontal line in Figure 2. And we fix two points $P(-a, 0)$ and $Q(a, 0)$ $(a > 0)$ on the line ℓ_0. For $i \in \{1, \ldots, m\}$, the line ℓ_i^+ denotes the line that passes through the point P and has a positive slope k_i^+ $(0 < k_1^+ < \cdots < k_m^+)$. And, for $i \in \{1, \ldots, n\}$, the line ℓ_i^- denotes the line that passes through the point Q and has a negative slope k_i^- $(0 > k_1^- > \cdots > k_n^-)$. All the multiple points (i.e. points where more than two lines are intersected) are two points P and Q. We consider its complexification $\mathcal{A}_{m,n}^{\mathbb{C}} = \{\ell_0 \otimes \mathbb{C}, \ell_1^+ \otimes \mathbb{C}, \ldots, \ell_m^+ \otimes \mathbb{C}, \ell_1^- \otimes \mathbb{C}, \ldots, \ell_n^- \otimes \mathbb{C}\}$. We set

$$M(\mathcal{A}_{m,n}) = \mathbb{C}^2 - ((\ell_0 \otimes \mathbb{C}) \cup (\bigcup_{i=1}^{m} \ell_i^+ \otimes \mathbb{C}) \cup (\bigcup_{i=1}^{n} \ell_i^- \otimes \mathbb{C})).$$

By using the Zariski-van Kampen method (see [Ch], [T-S] for instance), we give a presentation of the fundamental group of the complement of the line arrangement $\mathcal{A}_{m,n}^{\mathbb{C}}$. We specify the technical data that are used in the computation. The dotted line in the Figure 3 denotes the reference fiber. And we have taken a generator system naturally in the reference fiber (see Figure 4). The presentation is the following:

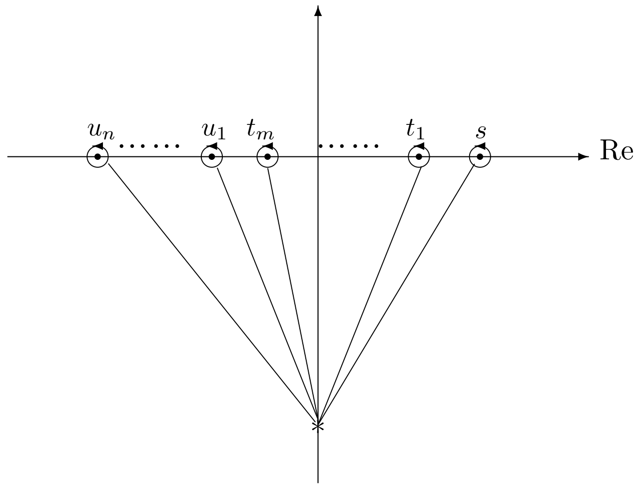

FIGURE 4. a generator system

$$\pi_1(M(\mathcal{A}_{m,n})) \quad \cong \quad \left\langle s, t_1, \ldots, t_m, u_1, \ldots, u_n \,\middle|\, \begin{array}{l} [s, t_1, \ldots, t_m], [s, u_1, \ldots, u_n], \\ [t_i, u_j] \, (i = 1, \ldots, m, j = 1, \ldots, n) \end{array} \right\rangle,$$

whereasymbol $[x_{i_1}, x_{i_2}, \ldots, x_{i_k}]$ denotes the cyclic relations:

$$x_{i_1} x_{i_2} \cdots x_{i_k} = x_{i_2} \cdots x_{i_k} x_{i_1} = x_{i_k} x_{i_1} \cdots x_{i_{k-1}}.$$

We have a remark on the group $\pi_1(M(\mathcal{A}_{m,n}))$.

Remark 1. *Let* $\{\overline{\ell_0}, \overline{\ell_1^+}, \ldots, \overline{\ell_m^+}, \overline{\ell_1^-}, \ldots, \overline{\ell_n^-}\}$ *be projectivization of the line arrangement* $\{\ell_0, \ell_1^+, \ldots, \ell_m^+, \ell_1^-, \ldots, \ell_n^-\}$*. We add the line at infinity* ℓ_∞ *to the list. After carrying out a projective transformation of* $\{\overline{\ell_0}, \overline{\ell_1^+}, \ldots, \overline{\ell_m^+}, \overline{\ell_1^-}, \ldots, \overline{\ell_n^-}, \ell_\infty\}$ *that transforms the line* $\overline{\ell_0}$ *to the position of the line at infinity, we consider an affinization of the arrangement. We write it by* $\widetilde{\mathcal{A}_{m,n}} = \{\widetilde{\ell_1^+}, \ldots, \widetilde{\ell_m^+}, \widetilde{\ell_1^-}, \ldots, \widetilde{\ell_n^-}, \widetilde{\ell_\infty}\}$*. We say that*

$$\pi_1(M(\mathcal{A}_{m,n})) \cong \pi_1(M(\widetilde{\mathcal{A}_{m,n}})).$$

By the theorem of Oka and Sakamoto ([O-S]*), we show the following*

$$\pi_1(M(\widetilde{\mathcal{A}_{m,n}})) \cong \mathbb{Z} \times F_m \times F_n.$$

From a group theoretical point of view, this group is understood well.

In this paper, we denote this presented group by $G_{m,n}$. For the presented group $G_{m,n}$, we associate the monoid $G^+_{m,n}$.

Next, we show the existence of a fundamental element in the monoid $G^+_{m,n}$.

Proposition 3.1. *An element* $\Delta := s \cdot t_1 \cdots t_m \cdot u_1 \cdots u_n$ *in the monoid* $G^+_{m,n}$ *is a fundamental element.*

Proof. By using the defining relations repeatedly, we show the cyclic relations $[s, t_1, \ldots, t_m, u_1, \ldots, u_n]$. Hence, we show that $\Delta = s \cdot t_1 \cdots t_m \cdot u_1 \cdots u_n \in \mathcal{F}(G^+_{m,n})$. □

4. Cancellativity of the monoid $G^+_{m,n}$

In this section, we prove the cancellativity of the monoid $G^+_{m,n}$.

Before continuing further, we prepare notation. We put

$$\Delta_1 := s\cdot t_1\cdots t_m,\ \Delta_2 := s\cdot u_1\cdots u_n,$$

$$I_1 := \{1,\dots,m\},\ I_2 := \{1,\dots,n\},$$

$$L_0 := \{\, s,t_1,\dots,t_m,u_1,\dots,u_n \,\}, L_1 := \{\, t_1,\dots,t_m \,\},\ L_2 := \{\, u_1,\dots,u_n \,\},$$

$$F_1^+ := F^+(\underline{t}),\ F_2^+ := F^+(\underline{u}),$$

$$F^+_{1,\mathrm{rm}} := \{w(\underline{t}) \in F_1^+ \mid (t_1\cdots t_m) \not|_r w(\underline{t})\},$$

$$F^+_{2,\mathrm{rm}} := \{w(\underline{u}) \in F_2^+ \mid (u_1\cdots u_n) \not|_r w(\underline{u})\},$$

$$F^+_{1,\mathrm{cons}} := \{w \in F_1^+ \mid \exists i_0, j_0 \in I_1 (i_0 \le j_0) \text{ s.t.}\, w = t_{i_0}t_{i_0+1}\cdots t_{j_0}\},$$

$$F^+_{2,\mathrm{cons}} := \{w \in F_2^+ \mid \exists i_0, j_0 \in I_2 (i_0 \le j_0) \text{ s.t.}\, w = u_{i_0}u_{i_0+1}\cdots u_{j_0}\}.$$

For arbitrary element $w(\underline{t})$ in F_1^+ and $w(\underline{u})$ in F_2^+, we put

$$\mathrm{Div}_1(w(\underline{t})) := \{w \in F^+_{1,\mathrm{cons}} \mid w \,|_r\, w(\underline{t})\},$$

$$\mathrm{Div}_2(w(\underline{u})) := \{w \in F^+_{2,\mathrm{cons}} \mid w \,|_r\, w(\underline{u})\}.$$

We remark that there exists a unique element $w_{0,1}$ in $\mathrm{Div}_1(w(\underline{t}))$ (resp. $w_{0,2}$ in $\mathrm{Div}_2(w(\underline{u}))$) such that $w_1 \,|_r\, w_{0,1}$ for any element w_1 in $\mathrm{Div}_1(w(\underline{t}))$ (resp. $w_2 \,|_r\, w_{0,2}$ for any element w_2 in $\mathrm{Div}_2(w(\underline{u}))$). We put

$$\mathrm{C}(w(\underline{t})) := w_{0,1}, \mathrm{C}(w(\underline{u})) := w_{0,2}.$$

In view of the defining relations of $G^+_{m,n}$, there exists an element $w'(\underline{t})$ in F_1^+ (resp. $w'(\underline{u})$ in F_2^+) such that we have a decomposition $w(\underline{t}) \equiv w'(\underline{t})\mathrm{C}(w(\underline{t}))$ (resp. $w(\underline{u}) \equiv w'(\underline{u})\mathrm{C}(w(\underline{u}))$) in $G^+_{m,n}$. We put

$$\mathrm{R}(w(\underline{t})) := w'(\underline{t}), \mathrm{R}(w(\underline{u})) := w'(\underline{u}).$$

For arbitrary element $w(\underline{t})$ in $F^+_{1,\mathrm{cons}}$ (resp. $w(\underline{u})$ in $F^+_{2,\mathrm{cons}}$), we say that, in the monoid $G^+_{m,n}$, $w(\underline{t}) \,|_r\, \Delta_1$ (resp. $w(\underline{u}) \,|_r\, \Delta_2$). Since the quotient can be uniquely determined respectively, we denote it by $\Delta_{1,w(\underline{t})}$ (resp. $\Delta_{2,w(\underline{u})}$).

Theorem 4.1. *The monoid $G^+_{m,n}$ is a cancellative monoid.*

Proof. First, we remark on the following.

Proposition 4.2. *The left cancellativity on $G^+_{m,n}$ implies the right cancellativity.*

Proof. Consider a map $\varphi : G^+_{m,n} \to G^+_{m,n}$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where σ is a permutation $\binom{s\ t_1\ \cdots\ t_m\ u_1\ \cdots\ u_n}{s\ t_m\ \cdots\ t_1\ u_n\ \cdots\ u_1}$ and $rev(W)$ is the reverse of the word $W = x_1x_2\cdots x_k$ (x_i is a letter) given by the word $x_k\cdots x_2x_1$. In view of the defining relation of $G^+_{m,n}$, φ is well-defined and is an anti-isomorphism. If $\beta\alpha \doteq \gamma\alpha$, then $\varphi(\beta\alpha) \doteq \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) \doteq \varphi(\alpha)\varphi(\gamma)$. Using the left cancellativity, we obtain $\varphi(\beta) \doteq \varphi(\gamma)$ and, hence, $\beta \doteq \gamma$.

□

The following is sufficient to show the left cancellativity on $G^+_{m,n}$.

Proposition 4.3. *Let X and Y be positive words in $G^+_{m,n}$ of length $r \in \mathbb{Z}_{\geq 0}$ and let $Y^{(h)}$ be a positive word in $G^+_{m,n}$ of length $h \in \{0, \ldots, r\}$.*
(i) *If $vX \doteqdot vY$ for some $v \in L_0$, then $X \doteqdot Y$.*
(ii) *If $t_iX \doteqdot u_jY$ $(t_i \in L_1, u_j \in L_2)$, then $X \doteqdot u_jZ$, $Y \doteqdot t_iZ$ for some positive word Z.*
(iii) *If $sX \doteqdot w(\underline{t})Y^{(h)}$ for some positive word $w(\underline{t})$ of length $r-h+1$ in F_1^+, then $X \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w(\underline{t})) \cdot Z$, $Y^{(h)} \doteqdot \Delta_{1,\mathrm{C}(w(\underline{t}))} \cdot Z$ for some positive word Z.*
(iv) *If $sX \doteqdot w(\underline{u})Y^{(h)}$ for some positive word $w(\underline{u})$ of length $r-h+1$ in F_2^+, then $X \doteqdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z$, $Y^{(h)} \doteqdot \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z$ for some positive word Z.*
(v) *If $t_iX \doteqdot w(\underline{t})Y^{(h)}$ for some t_i in L_1 and some positive word $w(\underline{t})$ of length $r-h+1$ in F_1^+ that satisfies $t_i \not|_l\, w(\underline{t})$, then there exists $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that $X \doteqdot w(\underline{u}) \cdot \Delta_{1,t_i} \cdot \mathrm{R}(w(\underline{t})) \cdot Z$, $Y^{(h)} \doteqdot w(\underline{u}) \cdot \Delta_{1,\mathrm{C}(w(\underline{t}))} \cdot Z$ for some positive word Z.*
(vi) *If $u_iX \doteqdot w(\underline{u})Y^{(h)}$ for some u_i in L_2 and some positive word $w(\underline{u})$ of length $r-h+1$ in F_2^+ that satisfies $u_i \not|_l\, w(\underline{u})$, then there exists $w(\underline{t})$ in $F^+_{1,\mathrm{rm}}$ such that $X \doteqdot w(\underline{t}) \cdot \Delta_{2,u_i} \cdot \mathrm{R}(w(\underline{u})) \cdot Z$, $Y^{(h)} \doteqdot w(\underline{t}) \cdot \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z$ for some positive word Z.*

Proof. We will show the general theorem, by refering to the double induction (see [G], [B-S], [S-I] for instance). The theorem for positive words X, Y of word-length r and $Y^{(h)}$ of word-length $h \in \{0, \ldots, r\}$ will be refered to as $\mathrm{H}_{r,h}$. For arbitrary $h \in \{0, \ldots, r\}$, it is easy to show that, for $r = 0, 1$, $\mathrm{H}_{r,h}$ is true. If a positive word U_1 is transformed into U_2 by using t single applications of the defining relations of $G^+_{m,n}$, then the whole transformation will be said to be of *chain-length* t. For induction hypothesis, we assume
(A) $\mathrm{H}_{s,h}$ is true for $0 \leq h \leq s \leq r$ for transformations of all chain-lengths,
and
(B) $\mathrm{H}_{r+1,h}$ is true for $0 \leq h \leq r+1$ for all chain-lengths $\leq t$.
We will show the theorem $\mathrm{H}_{r+1,h}$ for chain-lengths $t+1$. For the sake of simplicity, we devide the proof into two steps.
Step 1. $\mathrm{H}_{r+1,h}$ for $h = r+1$
Let X, Y' be of word-length $r+1$, and let

$$v_1X \doteqdot v_2W_2 \doteqdot \cdots \doteqdot v_{t+1}W_{t+1} \doteqdot v_{t+2}Y'$$

be a sequence of single transformations of $t+1$ steps, where $v_1, \ldots, v_{t+2} \in L_0$ and $W_2, \ldots, W_{t+1}$ are positive words of length $r+1$. By the assumption $t > 1$, there exists an index $\tau \in \{2, \ldots, t+1\}$ such that we can decompose the sequence into two steps

$$v_1X \doteqdot v_\tau W_\tau \doteqdot v_{t+2}Y',$$

in which each step satisfies the induction hypothesis (B).

If there exists τ such that v_τ is equal to either to v_1 or v_{t+2}, then by induction hypothesis, W_τ is equivalent either to X or to Y'. Hence, we obtain the statement for the $v_1X \doteqdot v_{t+2}Y'$. Thus, we assume from now on $v_\tau \neq v_1, v_{t+2}$ for $1 < \tau \leq t+1$.

Suppose $v_1 = v_{t+2}$. If there exists τ such that $(v_1 = v_{t+2}, v_\tau) \neq (t_i, t_j), (u_i, u_j)$, then each of the equivalences says the existence of $\alpha, \beta \in L_0$ and words Z_1, Z_2 such that $X \doteqdot \alpha Z_1$, $W_\tau \doteqdot \beta Z_1 \doteqdot \beta Z_2$ and $Y' \doteqdot \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \doteqdot \beta Z_2$, we get $Z_1 \doteqdot Z_2$. Hence, we obtain the statement

$X \doteq \alpha Z_1 \doteq \alpha Z_2 \doteq Y'$. Thus, we exclude these cases from our considerations. Next, we consider the case $(v_1 = v_{t+2}, v_\tau) = (t_i, t_j)$. However, because of the above consideration, we have only the case $v_2, \ldots, v_{t+1} \in L_1$. Hence, we consider the case $\tau = 1$, namely

$$t_i X \doteq t_j W_1 \doteq t_i Y'.$$

Applying the induction hypothesis (B) to each step, we say that there exist words Z_3, Z_4 and $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that

$$X \doteq \Delta_{1,t_i} \cdot Z_3,\ W_1 \doteq \Delta_{1,t_j} \cdot Z_3,$$

$$W_1 \doteq w(\underline{u}) \cdot \Delta_{1,t_j} \cdot Z_4,\ Y' \doteq w(\underline{u}) \cdot \Delta_{1,t_i} \cdot Z_4.$$

Moreover, we say that

$$\Delta_{1,t_j} \cdot Z_3 \doteq w(\underline{u}) \cdot \Delta_{1,t_j} \cdot Z_4. \cdots (*)$$

By induction hypothesis, we have

$$s \cdot t_1 \cdots t_{j-1} \cdot Z_3 \doteq w(\underline{u}) \cdot s \cdot t_1 \cdots t_{j-1} \cdot Z_4.$$

We consider the case $w(\underline{u}) \neq \varepsilon$. Applying the induction hypothesis to this equation, we say that there exists a word Z_5 such that

$$t_1 \cdots t_{j-1} \cdot Z_3 \doteq \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_5,$$

$$s \cdot t_1 \cdots t_{j-1} \cdot Z_4 \doteq \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z_5. \tag{4.1}$$

Moreover, we say that there exists a word Z_6 such that

$$Z_3 \doteq \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_6,$$

$$Z_5 \doteq t_1 \cdots t_{j-1} \cdot Z_6. \tag{4.2}$$

Applying (4.2) to the equation (4.1), we have

$$s \cdot t_1 \cdots t_{j-1} \cdot Z_4 \doteq \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot t_1 \cdots t_{j-1} \cdot Z_6. \tag{4.3}$$

We consider the following two cases.

Case 1: $\mathrm{C}(w(\underline{u})) \doteq u_a \cdots u_n$ for some integer $a \geq 2$

From (4.3), we obtain $Z_4 \doteq u_1 \cdots u_{a-1} \cdot Z_6$. Then, we have

$$X \doteq \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_6 \doteq \mathrm{R}(w(\underline{u})) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot Z_6,$$

$$Y' \doteq w(\underline{u}) \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{a-1} \cdot Z_6 \doteq \mathrm{R}(w(\underline{u})) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot Z_6.$$

Case 2: $\mathrm{C}(w(\underline{u})) \doteq u_a \cdots u_b$ for some integers a, b $(2 \leq a \leq b < n)$

We consider the equation

$$s \cdot t_1 \cdots t_{j-1} \cdot Z_4 \doteq u_{b+1} \cdots u_n \cdot s \cdot u_1 \cdots u_{a-1} \cdot t_1 \cdots t_{j-1} \cdot Z_6.$$

By applying the induction hypothesis to this equation, we say that there exists a word Z_7 such that

$$t_1 \cdots t_{j-1} \cdot Z_4 \doteq \Delta_{2,s} \cdot Z_7,$$

$$s \cdot u_1 \cdots u_{a-1} \cdot t_1 \cdots t_{j-1} \cdot Z_6 \doteq s \cdot u_1 \cdots u_b \cdot Z_7. \tag{4.4}$$

Moreover, we say that there exists a word Z_8 such that

$$Z_4 \doteq \Delta_{2,s} \cdot Z_8,\ Z_7 \doteq t_1 \cdots t_{j-1} \cdot Z_8. \tag{4.5}$$

Applying (4.5) to the equation (4.4), we have

$$Z_6 \doteq u_a \cdots u_b \cdot Z_8.$$

Then, we have

$$X \doteq \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot u_a \cdots u_b \cdot Z_8 \doteq \mathrm{R}(w(\underline{u})) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot \mathrm{C}(w(\underline{u})) \cdot Z_8,$$

$$Y' \doteq w(\underline{u}) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot Z_8 \doteq \mathrm{R}(w(\underline{u})) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot \mathrm{C}(w(\underline{u})) \cdot Z_8.$$

In the case of $(v_1 = v_{t+2}, v_\tau) = (u_i, u_j)$, we can prove the statement in a similar manner.

Suppose $v_1 \neq v_{t+2}$. We consider the following three cases.

Case 1: $(v_1, v_{t+2}) = (t_i, t_k), (u_i, u_k)$

We consider the case $(v_1, v_{t+2}) = (t_i, t_k)$. Then, we can easily show the case $v_\tau = s, u_j$. Thus, we have only the case $v_2, \ldots, v_{t+1} \in L_1$. Hence, we consider the case $\tau = 1$, namely

$$t_i X \doteq t_j W_1 \doteq t_k Y'.$$

Applying the induction hypothesis to each step, we say that there exist words Z_1, Z_2 and $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that

$$X \doteq \Delta_{1,t_i} \cdot Z_1,\ W_1 \doteq \Delta_{1,t_j} \cdot Z_1,$$

$$W_1 \doteq w(\underline{u}) \cdot \Delta_{1,t_j} \cdot Z_2,\ Y' \doteq w(\underline{u}) \cdot \Delta_{1,t_k} \cdot Z_2.$$

Thus, we say that $\Delta_{1,t_j} \cdot Z_1 \doteq w(\underline{u}) \cdot \Delta_{1,t_j} \cdot Z_2$. Since this equation has the same form as the equation $(*)$, we can find the solution in a similar way. Hence, we verify the statement in the case $(v_1, v_{t+2}) = (t_i, t_k)$. In the same way, we verify the statement in the case $(v_1, v_{t+2}) = (u_i, u_k)$.

Case 2: $(v_1, v_{t+2}) = (s, t_j), (s, u_j)$

We consider the case $(v_1, v_{t+2}) = (s, t_j)$. If $v_{t+1} = t_i$, then, by applying the induction hypothesis, we easily show the statement. Thus, we consider the case $(v_1, v_{t+1}, v_{t+2}) = (s, u_i, t_j)$, namely

$$sX \doteq u_i W_{t+1} \doteq t_j Y'.$$

Applying the induction hypothesis to each step, we say that there exist words Z_1 and Z_2 such that

$$X \doteq \Delta_{2,s} \cdot Z_1,\ W_{t+1} \doteq \Delta_{2,u_i} \cdot Z_1,$$

$$W_{t+1} \doteq t_j \cdot Z_2,\ Y' \doteq u_i \cdot Z_2.$$

Thus, we say that $\Delta_{2,u_i} \cdot Z_1 \doteq t_j \cdot Z_2$. By applying the induction hypothesis, there exists a word Z_3 such that

$$Z_2 \doteq u_{i+1} \cdots u_n \cdot Z_3,\ s \cdot u_1 \cdots u_{i-1} \cdot Z_1 \doteq t_j \cdot Z_3. \tag{4.6}$$

By applying the induction hypothesis to the equation (4.6), we say that there exists a word Z_4 such that

$$u_1 \cdots u_{i-1} \cdot Z_1 \doteq \Delta_{1,s} \cdot Z_4,\ Z_3 \doteq \Delta_{1,t_j} \cdot Z_4.$$

Moreover, we say that there exists a word Z_5 such that

$$Z_1 \doteq \Delta_{1,s} \cdot Z_5,\ Z_4 \doteq u_1 \cdots u_{i-1} \cdot Z_5.$$

Thus, we have

$$X \doteq \Delta_{2,s} \cdot \Delta_{1,s} \cdot Z_5 \doteq \Delta_{1,s} \cdot \Delta_{2,s} \cdot Z_5,$$

$$Y' \doteq u_i \cdots u_n \cdot \Delta_{1,t_j} \cdot u_1 \cdots u_{i-1} \cdot Z_5 \doteq \Delta_{1,t_j} \cdot \Delta_{2,s} \cdot Z_5.$$

We verify the statement in the case $(v_1, v_{t+2}) = (s, u_j)$ in a similar manner.

Case 3: $(v_1, v_{t+2}) = (t_i, u_j)$
First, we assume that there exists an index τ such that v_τ is equal to s. Then, we consider the case $(v_1, v_\tau, v_{t+2}) = (t_i, s, u_j)$, namely

$$t_i X \doteqdot sW_\tau \doteqdot u_j Y'.$$

Applying the induction hypothesis to each step, we say that there exist words Z_1 and Z_2 such that

$$X \doteqdot \Delta_{1,t_i} \cdot Z_1,\ W_\tau \doteqdot \Delta_{1,s} \cdot Z_1,$$
$$W_\tau \doteqdot \Delta_{2,s} \cdot Z_2,\ Y' \doteqdot \Delta_{2,u_j} \cdot Z_2.$$

Moreover, we say that

$$\Delta_{1,s} \cdot Z_1 \doteqdot \Delta_{2,s} \cdot Z_2.$$

Applying the induction hypothesis to this equation, we say that there exists a word Z_3 such that

$$Z_1 \doteqdot \Delta_{2,s} \cdot Z_3,\ Z_2 \doteqdot \Delta_{1,s} \cdot Z_3.$$

Thus, we have

$$X \doteqdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot Z_3 \doteqdot u_j \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot Z_3,$$
$$Y' \doteqdot \Delta_{2,u_j} \cdot \Delta_{1,s} \cdot Z_3 \doteqdot t_i \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot Z_3.$$

Thus, in the consideration of Case 3, we assume from now on $v_\tau \neq s$ for $1 < \tau \leq t+1$. We consider the following three cases.

Case 3 − 1: $(v_1, v_2, v_{t+2}) = (t_i, t_k, u_j)$
We consider the case

$$t_i X \doteqdot t_k W_2 \doteqdot u_j Y'.$$

Applying the induction hypothesis to each step, we say that there exist words Z_1 and Z_2 such that

$$X \doteqdot \Delta_{1,t_i} \cdot Z_1,\ W_2 \doteqdot \Delta_{1,t_k} \cdot Z_1,$$
$$W_2 \doteqdot u_j \cdot Z_2,\ Y' \doteqdot t_k \cdot Z_2.$$

Moreover, we obtain an equation $\Delta_{1,t_k} \cdot Z_1 \doteqdot u_j \cdot Z_2$. Then, there exists a word Z_3 such that

$$Z_2 \doteqdot t_{k+1} \cdots t_m \cdot Z_3,\ s \cdot t_1 \cdots t_{k-1} \cdot Z_1 \doteqdot u_j \cdot Z_3.$$

By the induction hypothesis, we say that there exists a word Z_4

$$t_1 \cdots t_{k-1} \cdot Z_1 \doteqdot \Delta_{2,s} \cdot Z_4,\ Z_3 \doteqdot \Delta_{2,u_j} \cdot Z_4.$$

Moreover, we say that there exists a word Z_5 such that

$$Z_1 \doteqdot \Delta_{2,s} \cdot Z_5,\ Z_4 \doteqdot t_1 \cdots t_{k-1} \cdot Z_5.$$

Thus, we have

$$X \doteqdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot Z_5 \doteqdot u_j \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot Z_5,$$
$$Y' \doteqdot t_k \cdot t_{k+1} \cdots t_m \cdot \Delta_{2,u_j} \cdot t_1 \cdots t_{k-1} \cdot Z_5 \doteqdot t_i \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot Z_5.$$

Case 3 − 2: $(v_1, v_{t+1}, v_{t+2}) = (t_i, u_k, u_j)$
In the same way as the Case 3 − 1, we verify the statement in this case.

Case 3 − 3: $(v_1, v_2, v_{t+1}, v_{t+2}) = (t_i, u_{j_1}, t_{i_1}, u_j)$
We consider the case

$$t_i X \doteqdot t_{i_1} W_{t+1} \doteqdot u_j Y'.$$

Applying the induction hypothesis to each step, we say that there exist words Z_1, Z_2 and $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that

$$X \doteqdot w(\underline{u}) \cdot \Delta_{1,t_i} \cdot Z_1,\ W_{t+1} \doteqdot w(\underline{u}) \cdot \Delta_{1,t_{i_1}} \cdot Z_1,$$

$$W_{t+1} \doteqdot u_j \cdot Z_2,\ Y' \doteqdot t_{i_1} \cdot Z_2.$$

Moreover, we say that $w(\underline{u}) \cdot \Delta_{1,t_{i_1}} \cdot Z_1 \doteqdot u_j \cdot Z_2$. And we say that there exists a word Z_3 such that

$$Z_2 \doteqdot t_{i_1+1} \cdots t_m \cdot Z_3,\ w(\underline{u}) \cdot s \cdot t_1 \cdots t_{i_1-1} \cdot Z_1 \doteqdot u_j \cdot Z_3.$$

Here, we consider the case $u_j \not|_l w(\underline{u})$. By applying the induction hypothesis, we say that there exist a word Z_4 and $w(\underline{t})$ in $F^+_{1,\mathrm{rm}}$ such that

$$s \cdot t_1 \cdots t_{i_1-1} \cdot Z_1 \doteqdot w(\underline{t}) \cdot \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z_4,\ Z_3 \doteqdot w(\underline{t}) \cdot \Delta_{2,u_j} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_4.$$

By applying the induction hypothesis to this equation, we say that there exists a word Z_5 such that

$$t_1 \cdots t_{i_1-1} \cdot Z_1 \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w(\underline{t})) \cdot Z_5,\ \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z_4 \doteqdot \Delta_{1,\mathrm{C}(w(\underline{t}))} \cdot Z_5. \tag{4.7}$$

We can find a general solution of the equation (4.7)

$$Z_4 \doteqdot \Delta_{1,s} \cdot \mathrm{C}(w(\underline{u})) \cdot Z_6,\ Z_5 \doteqdot \Delta_{2,s} \cdot \mathrm{C}(w(\underline{t})) \cdot Z_6.$$

Thus, we have

$$\begin{aligned}
&X \doteqdot w(\underline{u}) \cdot \Delta_{1,t_i} \cdot \Delta_{2,s} \cdot t_{i_1} \cdots t_m \cdot w(\underline{t}) \cdot Z_6 \\
&\doteqdot u_j \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot w(\underline{u}) \cdot t_{i_1} \cdots t_m \cdot w(\underline{t}) \cdot Z_6, \\
&Y' \doteqdot t_{i_1} \cdot t_{i_1+1} \cdots t_m \cdot w(\underline{t}) \cdot \Delta_{2,u_j} \cdot \mathrm{R}(w(\underline{u})) \cdot \Delta_{1,s} \cdot \mathrm{C}(w(\underline{u})) \cdot Z_6 \\
&\doteqdot t_i \cdot u_{j+1} \cdots u_n \cdot \Delta_{1,t_i} \cdot u_1 \cdots u_{j-1} \cdot w(\underline{u}) \cdot t_{i_1} \cdots t_m \cdot w(\underline{t}) \cdot Z_6.
\end{aligned}$$

Step 2. $\mathrm{H}_{r+1,h}$ for $0 \le h \le r+1$

We put $h' := r+1-h$. We will show the general theorem $\mathrm{H}_{r+1,h}$ by induction on h'. The case $h' = 0$ is proved in Step 1. We assume $h' = 0, \ldots, r-h$. Let X be of word-length $r+1$, and let $Y^{(h)}$ be of word-length h. We consider a sequence of single transformations of $t+1$ steps

$$v_1 X \doteqdot \cdots \doteqdot V \cdot Y^{(h)}, \tag{4.8}$$

where $v_1 \in L_0$ and V is a positive word of length $r-h+2$. To show the theorem $\mathrm{H}_{r+1,h}$ (i.e. $h' = r-h+1$) for chain-lengths $t+1$, we discuss the cases $(v_1, V) = (s, w(\underline{t})), (s, w(\underline{u})), (t_i, w(\underline{t})), (u_i, w(\underline{u}))$.

Case 1: $(v_1, V) = (s, w(\underline{t})), (s, w(\underline{u}))$.

First, we discuss the case $(v_1, V) = (s, w(\underline{t}))$ and decompose $w(\underline{t})$ into $w_1(\underline{t}) \cdot t_a$ (i.e. $w(\underline{t}) \equiv w_1(\underline{t}) \cdot t_a$). We consider the case

$$sX \doteqdot \cdots \doteqdot w_1(\underline{t}) \cdot t_a \cdot Y^{(h)}.$$

Applying the induction hypothesis, we say that there exists a word Z_1 such that

$$X \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t})) \cdot Z_1,\ t_a \cdot Y^{(h)} \doteqdot \Delta_{1,\mathrm{C}(w_1(\underline{t}))} \cdot Z_1. \cdots (**)$$

We consider the following two cases.

Case $1-1$: $\mathrm{C}(w_1(\underline{t})) \doteqdot t_b \cdots t_m$ for some integer $b \ge 2$

The equation is the following

$$t_a \cdot Y^{(h)} \doteqdot s \cdot t_1 \cdots t_{b-1} \cdot Z_1.$$

Applying the induction hypothesis, we say that there exists a word Z_2 such that

$$Y^{(h)} \doteqdot \Delta_{1,t_a} \cdot Z_2,\ Z_1 \doteqdot t_b \cdots t_m \cdot Z_2.$$

Thus, we have

$$X \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t})) \cdot t_b \cdots t_m \cdot Z_2 \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t}) \cdot t_a) \cdot Z_2,$$

$$Y^{(h)} \mathrel{\underset{\cdot}{=}} \Delta_{1,t_a} \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{1,\mathrm{C}(w_1(\underline{t})\cdot t_a)} \cdot Z_2.$$

Case $1-2$: $\mathrm{C}(w_1(\underline{t})) \mathrel{\underset{\cdot}{=}} t_b \cdots t_c$ for some integers b, c $(2 \leq b \leq c < m)$
The equation is the following

$$t_a \cdot Y^{(h)} \mathrel{\underset{\cdot}{=}} t_{c+1} \cdots t_m \cdot s \cdot t_1 \cdots t_{b-1} \cdot Z_1.$$

We discuss the case $t_a \neq t_{c+1}$. Applying the induction hypothesis, we say that there exist a word Z_2 and $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that

$$Y^{(h)} \mathrel{\underset{\cdot}{=}} w(\underline{u}) \cdot \Delta_{1,t_a} \cdot Z_2,\ s \cdot t_1 \cdots t_{b-1} \cdot Z_1 \mathrel{\underset{\cdot}{=}} w(\underline{u}) \cdot s \cdot t_1 \cdots t_c \cdot Z_2.$$

Moreover, we say that there exists a word Z_3 such that

$$t_1 \cdots t_{b-1} \cdot Z_1 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_3,\ s \cdot t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot Z_3.$$

We say that there exists a word Z_4 such that

$$Z_1 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_4,\ Z_3 \mathrel{\underset{\cdot}{=}} t_1 \cdots t_{b-1} \cdot Z_4.$$

Then, we have $s \cdot t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,\mathrm{C}(w(\underline{u}))} \cdot t_1 \cdots t_{b-1} \cdot Z_4$.

Case $1-2-1$: $\mathrm{C}(w(\underline{u})) \mathrel{\underset{\cdot}{=}} u_d \cdots u_n$ for some integer $d \geq 2$
The equation is the following

$$s \cdot t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} s \cdot u_1 \cdots u_{d-1} \cdot t_1 \cdots t_{b-1} \cdot Z_4.$$

Moreover, we say

$$t_b \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} u_1 \cdots u_{d-1} \cdot Z_4.$$

By the induction hypothesis, we say that there exists a word Z_5 such that

$$Z_2 \mathrel{\underset{\cdot}{=}} u_1 \cdots u_{d-1} \cdot Z_5,\ Z_4 \mathrel{\underset{\cdot}{=}} t_b \cdots t_c \cdot Z_5.$$

Thus, we have

$X \mathrel{\underset{\cdot}{=}} \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t})) \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot t_b \cdots t_c \cdot Z_5 \mathrel{\underset{\cdot}{=}} \Delta_{1,s} \cdot w_1(\underline{t}) \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_5$
$\mathrel{\underset{\cdot}{=}} \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t}) \cdot t_a) \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_5,$
$Y^{(h)} \mathrel{\underset{\cdot}{=}} w(\underline{u}) \cdot \Delta_{1,t_a} \cdot u_1 \cdots u_{d-1} \cdot Z_5 \mathrel{\underset{\cdot}{=}} \Delta_{1,t_a} \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_5,$
$\mathrel{\underset{\cdot}{=}} \Delta_{1,\mathrm{C}(w_1(\underline{t})\cdot t_a)} \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot Z_5.$

Case $1-2-2$: $\mathrm{C}(w(\underline{u})) \mathrel{\underset{\cdot}{=}} u_e \cdots u_f$ for some integers e, f $(2 \leq e \leq f < n)$
The equation is the following

$$s \cdot t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} u_{f+1} \cdots u_n \cdot s \cdot u_1 \cdots u_{e-1} \cdot t_1 \cdots t_{b-1} \cdot Z_4.$$

By the induction hypothesis, we say that there exists a word Z_5 such that

$$t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot Z_5,\ s \cdot u_1 \cdots u_{e-1} \cdot t_1 \cdots t_{b-1} \cdot Z_4 \mathrel{\underset{\cdot}{=}} s \cdot u_1 \cdots u_f \cdot Z_5.$$

Moreover, we say that $t_1 \cdots t_{b-1} \cdot Z_4 \mathrel{\underset{\cdot}{=}} u_e \cdots u_f \cdot Z_5$. By the induction hypothesis, there exists a word Z_6 such that

$$Z_4 \mathrel{\underset{\cdot}{=}} u_e \cdots u_f \cdot Z_6,\ Z_5 \mathrel{\underset{\cdot}{=}} t_1 \cdots t_{b-1} \cdot Z_6.$$

Hence, we say $t_1 \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot t_1 \cdots t_{b-1} \cdot Z_6$. By the induction hypothesis, we show

$$t_b \cdots t_c \cdot Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot Z_6.$$

We say that there exists a word Z_7 such that

$$Z_2 \mathrel{\underset{\cdot}{=}} \Delta_{2,s} \cdot Z_7,\ Z_6 \mathrel{\underset{\cdot}{=}} t_b \cdots t_c \cdot Z_7.$$

Thus, we have

$X \mathrel{\underset{\cdot}{=}} \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t})) \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot u_e \cdots u_f \cdot t_b \cdots t_c \cdot Z_7$

$\doteqdot \Delta_{1,s} \cdot w_1(\underline{t}) \cdot \Delta_{2,s} \cdot \mathrm{R}(w(\underline{u})) \cdot u_e \cdots u_f \cdot Z_7 \doteqdot \Delta_{1,s} \cdot \mathrm{R}(w_1(\underline{t}) \cdot t_a) \cdot \Delta_{2,s} \cdot w(\underline{u}) \cdot Z_7,$
$Y^{(h)} \doteqdot w(\underline{u}) \cdot \Delta_{1,t_a} \cdot \Delta_{2,s} \cdot Z_7 \doteqdot \Delta_{1,\mathrm{C}(w_1(\underline{t})\cdot t_a)} \cdot \Delta_{2,s} \cdot w(\underline{u}) \cdot Z_7.$
We can verify the statement in the case $(v_1, V) = (s, w(\underline{u}))$ in a similar manner.

Case 2: $(v_1, V) = (t_i, w(\underline{t})), (u_i, w(\underline{u}))$.
First, we discuss the case $(v_1, V) = (t_i, w(\underline{t}))$. And we decompose $w(\underline{t})$ into $w_1(\underline{t})\cdot t_a$ (i.e. $w(\underline{t}) \equiv w_1(\underline{t}) \cdot t_a$). We consider the case

$$t_i X \doteqdot \cdots \doteqdot w_1(\underline{t}) \cdot t_a \cdot Y^{(h)}.$$

Applying the induction hypothesis, we say that there exist a word Z_1 and $w(\underline{u})$ in $F^+_{2,\mathrm{rm}}$ such that

$$X \doteqdot w(\underline{u}) \cdot \Delta_{1,t_i} \cdot \mathrm{R}(w_1(\underline{t})) \cdot Z_1,\ t_a Y^{(h)} \doteqdot w(\underline{u}) \cdot \Delta_{1,\mathrm{C}(w_1(\underline{t}))} \cdot Z_1.$$

By the induction hypothesis, we say that $w(\underline{u}) \,|_l\, Y^{(h)}$. Hence, we write $Y^{(h)} \doteqdot w(\underline{u}) \cdot \widetilde{Y}^{(h)}$. Then, we consider an equation

$$t_a \widetilde{Y}^{(h)} \doteqdot \Delta_{1,\mathrm{C}(w_1(\underline{t}))} \cdot Z_1.$$

Since this equation has the same form as the equation $(**)$, we can find the solution in a similar way. Hence, we verify the statement in the case $(v_1, V) = (t_i, w(\underline{t}))$. In the same way, we verify the statement in the case $(v_1, V) = (u_i, w(\underline{u}))$. □

This completes the proof of Theorem 4.1. □

Remark 2. *The sufficient criterion for the cancellativity given in* [D-P] *is not satisfied in the monoid* $G^+_{m,n}$.

5. Some decision problems on the group $G_{m,n}$

In this section, we will solve the word problem in the group $G_{m,n}$ and determine the center of it, by showing the monoid $G^+_{m,n}$ injects in the group $G_{m,n}$.

Since the existence of a fundamental element and the cancellativity of the monoid $G^+_{m,n}$ have be shown, we show the following by applying the Lemma 2.3 to this case.

Proposition 5.1. *The localization homomorphism* $\pi : G^+_{m,n} \to G_{m,n}$ *is injective.*

Proposition 5.2. *The word problem in the group* $G_{m,n}$ *can be solved.*

Thanks to Theorem 4.1, we can also show the following proposition.

Proposition 5.3. *The monoid* $G^+_{m,n}$ *does not always have least common multiples.*

Proof. Due to the Theorem 4.1, we say, for example,

$$\mathrm{mcm}_r(\{t_1, t_2\}) = \{w(\underline{u}) \cdot \Delta_1 \mid w(\underline{u}) \in F^+_{2,\mathrm{rm}}\}.$$

□

As a consequence of Proposition 5.3, the monoid $G^+_{m,n}$ is neither Garside nor Artin monoid. We have an important remark on the monoid $G^+_{m,n}$.

Remark 3. *For each letter v in L_0, both sides of the defining relations of $G^+_{m,n}$ contain the same number of the letter v. For arbitrary word W in $G^+_{m,n}$, the number of the letter v in W ought to be preserved in the process of rewriting W.*

Proposition 5.4. *The center $\mathcal{Z}(G_{m,n})$ is isomorphic to $\mathbb{Z}$ and generated by Δ.*

Proof. First, we prove the following two Claims.

Claim 1. $\delta \in \mathcal{Z}(G^+_{m,n}) \setminus \{\varepsilon\} \Rightarrow \Delta|_l\, \delta$.

Proof. Thanks to the Theorem 4.1, it is easy to show that δ contains at least one letter except for the letter s. Hence, there exist a non-negative integer k in $\mathbb{Z}_{\geq 0}$ and a letter v in $L_1 \cup L_2$ such that

$$\delta \doteq s^k \cdot v \cdot d$$

for some positive word d. Since δ belongs to the center, an equation $s \cdot \delta \doteq \delta \cdot s$ holds. By using the cancellativity of the monoid $G^+_{m,n}$, we have

$$s \cdot v \cdot d \doteq v \cdot d \cdot s.$$

By the Theorem 4.1, we easily show that

$$\Delta_{1,s} \,|_l\, \delta \;\text{ or }\; \Delta_{2,s} \,|_l\, \delta.$$

Without loss of generality, we assume that $\Delta_{1,s} \,|_l\, \delta$. Hence, there exists a positive word δ_1 such that

$$\delta \doteq \Delta_{1,s} \cdot \delta_1.$$

We easily show that $\delta_1 \neq \varepsilon$ and δ_1 contains at least one letter except for the letter s. Due to the cancellativity, we have $s \cdot \delta_1 \doteq \delta_1 \cdot s$. In the same way, we can show

$$\Delta_{1,s} \,|_l\, \delta_1 \;\text{ or }\; \Delta_{2,s} \,|_l\, \delta_1.$$

From the Theorem 4.1, we say that δ cannot be a power of $\Delta_{1,s}$. Hence, there exists a positive integer j such that $\Delta^j_{1,s} \cdot \Delta_{2,s} \,|_l\, \delta$. Then, there exists a positive word δ_2 such that

$$\delta \doteq \Delta^j_{1,s} \cdot \Delta_{2,s} \cdot \delta_2.$$

We devide δ_2 by $\Delta_{1,s}$ and $\Delta_{2,s}$ as much as we can. Namely, there exist positive integers j_1, j_2 and a positive word δ_3 such that

$$\delta \doteq \Delta^{j_1}_{1,s} \cdot \Delta^{j_2}_{2,s} \cdot \delta_3 \text{ and } \Delta_{1,s}, \Delta_{2,s} \not|_l\, \delta_3.$$

We easily show that $\delta_3 \neq \varepsilon$. If $s \not|_l\, \delta_3$, then , from the above consideration, we show that

$$\Delta_{1,s} \,|_l\, \delta_3 \;\text{ or }\; \Delta_{2,s} \,|_l\, \delta_3.$$

A contradiction. Hence, we say that $s \,|_l\, \delta_3$. Thus, we have

$$\Delta|_l\, \delta.$$

□

Claim 2. The center $\mathcal{Z}(G^+_{m,n})$ is isomorphic to an infinite cyclic monoid and generated by Δ.

Proof. We take an element δ in $\mathcal{Z}(G^+_{m,n})\setminus\{\varepsilon\}$. By applying the Claim 1 repeatedly, we say that there exists a positive integer j such that

$$\delta \doteqdot \Delta^j.$$

□

Next, for an arbitrary element V in $\mathcal{Z}(G_{m,n})$, there exists a non-negative integer k in $\mathbb{Z}_{\geq 0}$ such that $\Delta^k \cdot V$ is equivalent to a positive word. Since the localization homomorphism π is injective, there exists a unique element V' in $G^+_{m,n}$ such that $\pi(V') = \Delta^k \cdot V$. The element V' belongs to the center $\mathcal{Z}(G^+_{m,n})$. Due to the Claim 2, we show that there exists a positive integer k' such that

$$\Delta^{k'} = \Delta^k \cdot V.$$

Hence, we have $V = \Delta^{k'-k}$. □

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References

[B] E. Brieskorn: Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe, *Inventiones Math.* **12** (1971) 57-61.

[B-M] D. Bessis and J. Michel: Explicit presentations for exceptional braid groups, *Experimental Math.* **13** (2004) No.3, 257-266.

[B-S] E. Brieskorn and K. Saito: Artin-Gruppen und Coxeter-Gruppen, *Inventiones Math.* **17** (1972) 245-271, English translation by C. Coleman, R. Corran, J. Crisp, D. Easdown, R. Howlett, D. Jackson and A. Ram at the University of Sydney, 1996.

[Ch] D. Cheniot: Une démonstration du théorème de Zariski sur les sectionshyperplanes d'une hypersurface projective et du théorème de van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane, *Compositio Math.* **27** (1973) 141-158.

[C-P] A.H.Clifford and G.B. Preston: The algebraic theory of semigroups, Mathematical Surveys and Monographs **7** (American Mathematical Society, Providence, RI,1961).

[D] P. Dehornoy: Groupes de Garside, Ann. Scient. Éc. Norm. Sup. (4) 35 (2002), no. 2, 267-306.

[De] P. Deligne: Les immeubles des tresses généralizé, *Inventiones Math.* **17** (1972) 273-302.

[D-P] P. Dehornoy and L. Paris: Gaussian groups and Garside groups, two generalization of Artin groups, Proc. London Math. Soc.(3) **79** (1999) 569-604.

[E-M] E. ElRifai and H. Morton: Algorithms for positive braids, Quart. J. Math. Oxford, **45** (1994), 479-497.

[G] F.A. Garside: The braid groups and other groups, Quart. J. Math. Oxford, **20** (1969), 235-254.

[Ge] V. Gebhardt: A new approach to the conjugacy problem in Garside groups, Journal of Algebra 292, No.1 (2005), 282-302.

[I1] T. Ishibe: On the monoid in the fundamental group of type $\mathrm{B_{ii}}$, Hiroshima Mathematical Journal Vol.42 No.1(March) 2012.

[I2] T. Ishibe: The skew growth function $N_{M,deg}(t)$ for the monoid of type $\mathrm{B_{ii}}$ and others, in preparation.

[O-S] M. Oka and K. Sakamoto: Product theorem of the fundamental group of a reducible curve, J. Math. Soc. Japan **30**(4) (1978), 599-602.

[P] M. Picantin: The conjugacy problem in small Gaussian groups, Comm. Algebra**29**(3) (2001), 1021-1039.

[S-I] K. Saito and T. Ishibe: Monoids in the fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$, Journal of Algebra 344 (2011), 137-160.

[T-S] H. Tokunaga and I. Shimada: Algebraic curves and Singularities (Part I: Fundamental Groups and Singularities), Kyoritsu, 2001, published in Japanese.
[Y] M. Yoshinaga: Hyperplane arrangements and Lefschetz hyperplane section theorem, Kodai Math. J. **30** (2) (2007) 157-194.

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