Operations on *t*-structures and perverse coherent sheaves

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Introduction.

Suppose we have an equivalence $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$ between the derived categories of two abelian categories \mathcal{A} and \mathcal{B} . How this helps to study $\mathcal{D}(\mathcal{A})$? The virtue is that we can transport the standard *t*-structure from $\mathcal{D}(\mathcal{B})$ into $\mathcal{D}(\mathcal{A})$ via the equivalence and look how it interplays with the standard *t*-structure in $\mathcal{D}(\mathcal{A})$. In particular, we can try to produce new *t*-structures from these two.

In this paper we consider two binary operations on t-structures in a triangulated category, \mathcal{D} . More precisely, there is a natural partial order on the set of t-structures and the operations are just that of intersection and union as they are defined in the lattice theory. The basic problem is to prove the existence of intersection and union for two given t-structures.

We define *lower and upper consistent pairs* of t-structures. These are some elementary conditions which guarantee the existence of intersection or union for the pairs. Then we take a triple of t-structures and examine various conditions of consistency in pairs of t-structures in this triple which imply consistency in the pair that comprises one of them and the intersection (or union) t-structure of the other two. This gives some rules for iterating the operations of union and intersection.

The lattice of vector subspaces in a vector space does not satisfy distributivity law, but rather *modularity law*. The partially ordered set of t-structures is somewhat similar. We prove, in particular, some versions of modularity law under suitable conditions on consistency in pairs. We consider abstract partially ordered sets with two binary relations subject to axioms, which upper and low consistency obey. We call them *sets with consistencies*. We prove that such a set P allows a map into a universal *lattice* with consistencies, U(P).

A chain in a poset is a fully ordered set of elements. A theorem due to Birkhoff [4] claims that two chains in a modular lattice generate a distributive lattice. A pair of chains $\{a_i\}$ and $\{b_j\}$ is said to be consistent if all pairs (a_i, b_j) are both upper and low consistent. We prove a theorem that a consistent pair of chains in a set with consistencies generates a distributive lattice L.

Our application of the developed techniques is to perverse coherent sheaves on schemes of finite type over a field. We consider the derived category $\mathcal{D}(X) = \mathcal{D}^b_{\mathrm{coh}}(X)$ of coherent sheaves on such a scheme X. It has the standard t-structure. The category possesses Grothendieck-Serre duality functor $D: \mathcal{D}(X) \to \mathcal{D}(X)^{opp}$, defined for $\mathcal{F} \in \mathcal{D}(X)$ by

$$D\mathcal{F} = \mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet}),$$

where ω_X^{\cdot} stands for the dualizing complex on X. Since D is an anti-equivalence, it allows to define another, 'dual', t-structure on $\mathcal{D}(X)$ by transporting the standard t-structure via D. We consider two chains of t-structures: one is obtained by shifts of the standard t-structure and the other one by shifts of the dual t-structure. We prove that this pair of chains is consistent. Therefore, they generate a distributive lattice of t-structures by the above version of Birkhoff theorem. It is a set of perverse t-structures on coherent sheaves (those which are independent of stratification of X) considered independently by Deligne, Bezrukavnikov [3], Kashiwara [11], Gabber [9].

The hard part in constructing a *t*-structure with required properties is to prove existence of the adjoint to the embedding of what-is-to-be $\mathcal{D}^{\leq 0}$ into \mathcal{D} . The standard way to produce such a functor is by using limits in the category.

According to the proposal in paper [6], derived categories of coherent sheaves can play a crucial role in the Minimal Model Program of birational algebraic geometry. We expect that important ingredients of the Minimal Model Program, such as flips and flops, could be constructed and understood by means of transformation of t-structures (see [7] for 3-dimensional flops).

This problem requires considering triangulated categories of 'small' size, like those arising in 'realistic' algebraic geometry, such as the bounded derived categories of complexes of coherent sheaves on algebraic varieties as opposed to the unbounded derived categories of quasi-coherent sheaves. There is no suitable general properties on existence of limits in such categories. For that reason, standard constructions often lead us *a priori* beyond the 'small' category. For example, the derived functor of cohomology with support takes, in general, coherent complexes to quasi-coherent ones.

One of the main technical problems in studying algebraic geometry by means of triangulated categories can be formulated as to find appropriate finiteness properties on suitable subcategories in the derived categories of coherent sheaves and keep track of these properties under relevant transformations of these subcategories.

We hope that this paper shed light on the interplay of t-structures, thus giving approach to understand the finiteness conditions relevant to t-structures.

It would be interesting to compare the approach of this paper with Tom Bridgeland's work on producing stability conditions (hence a plenty of t-structures!) via deformation argument [8].

I am indebt to Michel Van-den-Bergh for useful discussions.

This work was done more than 15 years ago and was, for the first time, reported on Shafarevich's seminar at Steklov Institute in late 90's. It is my pleasure to devote this paper to I.R. Shafarevich on the occasion of his 90-th anniversary. The influence of Shafarevich on Russian algebraic geometry can hardly be overestimated.

1 Consistent systems of *t*-structures

1.1 The partially ordered set of *t*-structures

Let k be any field, \mathcal{D} a k-linear triangulated category with an auto-equivalence T, called translation (or shift) functor (see [15]).

For a subcategory \mathcal{C} in \mathcal{D} its *right* (resp. *left*) *orthogonal* \mathcal{C}^{\perp} ($^{\perp}\mathcal{C}$) is the strictly full subcategory in \mathcal{D} of objects X such that Hom(\mathcal{C}, X) = 0 (resp. Hom(X, \mathcal{C}) = 0).

We use the standard notation $X[k] := T^k X, k \in \mathbb{Z}$.

Recall (see [1]) that a *t*-structure in a triangulated category \mathcal{D} is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$, satisfying the following conditions:

- i) $T\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}, T^{-1}\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 1};$
- *ii*) Hom $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0;$
- *iii*) for any $X \in \mathcal{D}$ there exist $X_{-} \in \mathcal{D}^{\leq 0}$, $X_{+} \in \mathcal{D}^{\geq 1}$ and an exact triangle

$$X_{-} \to X \to X_{+} \to X_{-}[1] \tag{1}$$

The basic example of a *t*-structure comes up when \mathcal{D} is equivalent to the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of some abelian category \mathcal{A} . For this case $\mathcal{D}^{\geq 1}$ (resp. $\mathcal{D}^{\leq 0}$) comprises the complexes with cohomology in positive (resp. nonpositive) degree.

Categorically, the main virtue of the notion of t-structure is that in view of conditions (i) and (ii) the triangle (1) for a given X is unique up to unique isomorphism [1]. In other words, formulas

$$\tau_{\leq 0}(X) = X_{-}, \ \ \tau_{\geq 1}(X) = X_{+}$$

define the truncation functors $\tau_{\leq 0}: \mathcal{D} \to \mathcal{D}^{\leq 0}, \tau_{\geq 1}: \mathcal{D} \to \mathcal{D}^{\geq 1}$, which are easily seen to be respectively the right and the left adjoint to the inclusion functors $i_{\leq 0}: \mathcal{D}^{\leq 0} \to \mathcal{D}, i_{\geq 1}:$ $\mathcal{D}^{\geq 1} \to \mathcal{D}$. This suggest the following reformulation of the data defining a *t*-structure.

Lemma 1 To define a t-structure in a triangulated category \mathcal{D} is equivalent to one of the following data:

- **Data 1.** A strictly full subcategory $\mathcal{D}^{\leq 0}$ in \mathcal{D} , closed under the action of T and such that the inclusion functor $i_{\leq 0}: \mathcal{D}^{\leq 0} \to \mathcal{D}$ has a right adjoint $\tau_{\leq 0}$,
- **Data 2.** A strictly full subcategory $\mathcal{D}^{\geq 1}$ in \mathcal{D} , closed under the action of T^{-1} and such that the inclusion functor $i_{\geq 1}: \mathcal{D}^{\geq 1} \to \mathcal{D}$ has a left adjoint $\tau_{\geq 1}$.

Proof. As we have already constructed adjoint functors for a given *t*-structure, it remains to explain how to construct a *t*-structure from say data 1.

Put $\mathcal{D}^{\geq 1} = (\mathcal{D}^{\leq 0})^{\perp}$. As $T\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$, then obviously $T^{-1}\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 1}$.

For any $X \in \mathcal{D}$, the adjunction morphism $i_{\leq 0}\tau_{\leq 0}X \to X$ can be inserted in a triangle:

$$i_{\leq 0}\tau_{\leq 0}X \to X \to X_+ \to i_{\leq 0}\tau_{\leq 0}X[1].$$

 X_+ is easily seen to be in $\mathcal{D}^{\geq 1}$. Hence the triangle is of the form (1).

This simple observation is useful in constructing t-structures.

Remark 2 Note that in general neither inclusion nor truncation functors are exact. But one can easily see that if $A \to B \to C$ is an exact triangle with $A, C \in \mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$), then $B \in \mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$). It follows by 'rotation of triangle' that if A and B are in $\mathcal{D}^{\leq 0}$ (resp. $B, C \in \mathcal{D}^{\geq 1}$), then also $C \in \mathcal{D}^{\leq 0}$ (resp. $A \in \mathcal{D}^{\geq 1}$). Any *t*-structure generates two sequences of subcategories $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 1}[1-n]$. The subcategory $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ has a natural structure of abelian category (see [1]), it is called the *heart of t-structure*. The objects which lie in the heart (and also their shifts) are called *pure*.

When $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$, the heart of the standard *t*-structure comprises the complexes over \mathcal{A} with cohomology in degree zero. Thus, the heart is equivalent to the primary abelian category \mathcal{A} . We identify \mathcal{A} with this full subcategory in $\mathcal{D}^b(\mathcal{A})$.

If \mathcal{D} is triangulated, then the opposite category \mathcal{D}^{op} is also naturally triangulated. Namely, $T_{\mathcal{D}^{op}} = T_{\mathcal{D}}^{-1}$ and triangles in \mathcal{D}^{op} are defined by inverting arrows in triangles in \mathcal{D} . A *t*-structure \mathcal{T} in \mathcal{D} defines a *t*-structure \mathcal{T}^{op} in \mathcal{D}^{op} by exchanging the roles of $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$. The heart of \mathcal{T}^{op} is the opposite abelian category to the heart of \mathcal{T} .

An important special case is a *t*-structure satisfying $T^{-1}\mathcal{D}^{\leq 0} \in \mathcal{D}^{\leq 0}$. It follows from the remark 2 that for such a *t*-structure both $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are triangulated subcategories. We call such a *t*-structure *translation invariant* or, simply, *T*-invariant.

A strictly full triangulated subcategory \mathcal{A} in \mathcal{D} which has the right (resp. left) adjoint to the inclusion $i: \mathcal{A} \to \mathcal{D}$ is called *right* (resp. *left*) *admissible* [5]. For a *T*-invariant *t*-structure, $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$) is right (resp. left) admissible. This gives a 1-1 correspondence between *T*-invariant *t*-structures and right (or left) admissible subcategories.

For a *T*-invariant *t*-structure all the subcategories $\mathcal{D}^{\leq k}$, $k \in \mathbb{Z}$, are equal (the same for $\mathcal{D}^{\geq k}$, $k \in \mathbb{Z}$). Hence the heart of a *T*-invariant *t*-structure is zero.

Given a triangulated category \mathcal{D} , consider the following partial order on the set of its *t*-structures (we ignore all possible set theoretical questions which arize from the fact that the objects of a category comprise a class not a set, as being irrelevant to our discussion).

For a pair $(\mathcal{T}_1, \mathcal{T}_2)$ of *t*-structures, we say $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if $\mathcal{D}_1^{\leq 0} \subseteq \mathcal{D}_2^{\leq 0}$. Then also $\mathcal{D}_2^{\geq 1} \subseteq \mathcal{D}_1^{\geq 1}$.

The minimal element **0** in this ordered set is the *t*-structure with $\mathcal{D}^{\leq 0} = 0$; the maximal one, **1**, has $\mathcal{D}^{\leq 0} = \mathcal{D}$.

Following the standard definition of union and intersection for partially ordered sets we introduce:

Definition The *(abstract) intersection* $\bigcap_{i \in I} \mathcal{T}_i$ of a set $\{\mathcal{T}_i\}_{i \in I}$ of *t*-structures in \mathcal{D} is the supremum of *t*-structures \mathcal{T} in \mathcal{D} such that $\mathcal{T} \subseteq \mathcal{T}_i$, for all $i \in I$. If it exists, it is unique. The *(abstract) union* $\bigcup_{i \in I} \mathcal{T}_i$ is defined dually.

We also denote the intersection and union operations by \cdot and + and call them *product* and *sum*. Indeed, these are just product and coproduct in the category related to the partially ordered set.

The intersection and the union of T-invariant t-structures, when exists, is T-invariant.

There is a natural 'naive' candidate for $\mathcal{D}_{\cap}^{\leq 0}$ (resp. $\mathcal{D}_{\cup}^{\geq 1}$) for the intersection (resp. union) *t*-structure of a set $\{\mathcal{T}_i\}$: $\mathcal{D}_{\cap}^{\leq 0} = \bigcap_{i \in I} \mathcal{D}_i^{\leq 0}$, (resp. $\mathcal{D}_{\cup}^{\geq 1} = \bigcap_{i \in I} \mathcal{D}_i^{\geq 1}$).

There is a duality functor on the category of partially ordered sets that inverts the partial order. It takes the union operation into the intersection and vice versa. We refer often to this duality in the present paper.

1.2 An example

The following example of geometric nature provides with a pair $(\mathcal{T}_1, \mathcal{T}_2)$ of *t*-structures such that $\mathcal{D}_1^{\leq 0} \cap \mathcal{D}_2^{\leq 0}$ does not have right adjoint. This shows that the 'naive' intersection does not necessarily coincides with the abstract one.

Here we assume k to be algebraically closed and of characteristic zero. Let $\mathcal{D} = \mathcal{D}^b_{\text{coh}}(\mathbb{P}^2)$ be the bounded derived category of coherent sheaves on the projective plane $\mathbb{P}^2 = \mathbb{P}(V)$, dim V = 3. We consider two translation invariant t-structures in \mathcal{D} . Define corresponding right admissible triangulated subcategories $\mathcal{D}_1^{\leq 0}$ and $\mathcal{D}_2^{\leq 0}$ by

$$\mathcal{D}_1^{\leq 0} = \langle \mathcal{O}(-2), \mathcal{O}(-1) \rangle, \qquad \mathcal{D}_2^{\leq 0} = \langle \mathcal{O}(1), \mathcal{O}(2) \rangle$$

i.e. the minimal strictly full triangulated subcategories generated by the indicated sheaves.

Lemma 3 The category $\mathcal{D}_{\cap} = \mathcal{D}_1^{\leq 0} \cap \mathcal{D}_2^{\leq 0}$ is not right admissible.

Proof. The category $\mathcal{D}_2^{\leq 0}$ is equivalent as a triangulated category to $\mathcal{D}^b(\text{mod}-A)$, the bounded derived category of representations of the path algebra of the quiver (see [5]):

$$\xrightarrow{V}$$

The equivalence can be given by the functor that assigns to a complex of representations $\stackrel{U^{\bullet}}{\cdot} \xrightarrow{W^{\bullet}}$ with the structure map $\varphi: U^{\bullet} \otimes V \to W^{\bullet}$, the complex of sheaves $U^{\bullet} \otimes \mathcal{O}(1) \to W^{\bullet} \otimes \mathcal{O}(2)$ in $\mathcal{D}_{2}^{\leq 0}$ with the differential induced by φ .

Since A is hereditary, any indecomposable object in $\mathcal{D}^b(\text{mod}-A)$ is pure, i.e. it is defined by a map $\varphi: U \otimes V \to W$, with U, W vector spaces (up to a common shift). The corresponding complex of sheaves in $\mathcal{D}_2^{\leq 0}$ has the form

$$U \otimes \mathcal{O}(1) \to W \otimes \mathcal{O}(2)$$
 (2)

with the components in degrees 0 and 1.

Let us describe indecomposables which are in \mathcal{D}_{\cap} . Note that an object $X \in \mathcal{D}$ is in $\mathcal{D}_1^{\leq 0}$ iff Ext' $(\mathcal{O}, X) = 0$. Applying this to (2) we get that the map

$$\widehat{\varphi}: U \otimes V^* \to W \otimes S^2 V^*$$

obtained from φ by partial dualization $\widetilde{\varphi}: U \to W \otimes V^*$, tensoring with the identity map in V^* and then by partial symmetrization, has to be an isomorphism. As dim V = 3, we see that for $X \in \mathcal{D}_{\cap} = \mathcal{D}_1^{\leq 0} \cap \mathcal{D}_2^{\leq 0}$ we have dim $U = 2 \dim W$.

If we take W to be 1-dimensional, then dim U = 2, hence the image of $\tilde{\varphi}$ annihilates a line in V. It follows that the image of $\hat{\varphi}$ consists of quadrics that vanish on this line, i. e. $\hat{\varphi}$ is not an isomorphism. Therefore, there is no indecomposable in \mathcal{D}_{\cap} with dim W = 1.

Yet, \mathcal{D}_{\cap} is not empty. Indeed, take any exceptional vector bundle on \mathbb{P}^2 , i.e. such that

$$\operatorname{Hom}(E, E) = k, \qquad \operatorname{Ext}^{i}(E, E) = 0, \quad \text{for } i \neq 0.$$
(3)

Consider the sheaf $F = \mathcal{E}nd_0E$ of traceless local endomorphisms of E. It is non-zero, if $\operatorname{rk} E > 1$. Then (3) implies $\operatorname{Ext}^{\bullet}(\mathcal{O}, F) = \operatorname{H}^{\bullet}(F) = 0$. Since $\mathcal{D}_1^{\leq 0} = \mathcal{O}^{\perp}$ in \mathcal{D} , then $F \in \mathcal{D}_1^{\leq 0}$. As $F^* = F$, then $\operatorname{Ext}^{\bullet}(F, \mathcal{O}) = \operatorname{H}^{\bullet}(F^*) = 0$. Since $\mathcal{D}_2^{\leq 0} = {}^{\perp}\mathcal{O}$, then $F \in \mathcal{D}_2^{\leq 0}$, i.e. $F \in \mathcal{D}_{\cap}$.

Now take $G = \mathcal{O}(-1)[1]$. If $F \in \mathcal{D}_{\cap}$ is of the form (2), then

$$\text{Hom}^{1}(F,G) = W^{*}, \quad \text{Hom}^{i}(F,G) = 0, \text{ for } i \neq 1.$$
 (4)

Suppose that \mathcal{D}_{\cap} is right admissible and τ is the right adjoint to the embedding functor $\mathcal{D}_{\cap} \hookrightarrow \mathcal{D}$. Put $G' = \tau G$. Then $G' \in \mathcal{D}_{\cap}$. If G' is of the form $U' \otimes \mathcal{O}(1) \to W' \otimes \mathcal{O}(2)$, while F is of the form (2), then Hom'(F, G') is the homology of the complex

$$U^* \otimes U' \oplus W^* \otimes W' \to U^* \otimes W' \otimes V^*, \tag{5}$$

with two non-trivial components in degrees 0 and 1. Since dim $U = 2 \dim W$ and dim $U' = 2 \dim W'$, the Euler characteristic of this complex is $-\dim W \cdot \dim W'$.

From the adjunction property and from (4) we conclude:

$$-\dim W = \chi(F,G) = \chi(F,G') = -\dim W \cdot \dim W'.$$

Therefore, $\dim W'$ must be equal to 1, which is, as we have seen, impossible.

Generally, G', being an object in \mathcal{D}_{\cap} , has a decomposition into a sum of objects of the form (2) and their translations. Any non-trivial translation would give a nontrivial contribution in $\operatorname{Hom}^{i}(F, G')$, for $i \neq 1$, because these Hom-groups are calculated by translations of the complex (5). But this contradicts the adjunction property for τ . Therefore, there is no chance for G' to exist. The lemma is proved.

One shows in the same way that the abstract intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is **0**.

1.3 Consistent pairs of *t*-structures and operations

Here we define lower and upper consistent pairs of t-structures. There are various possible ways for generalizing these notion, which we hope to address in an appropriate place.

We mark truncation and inclusion functors of t-structures with relevant superscripts.

Definition A pair of t-structures $(\mathcal{T}_1, \mathcal{T}_2)$ is called *lower consistent* if

$$au_{\leq 0}^1 \mathcal{D}_2^{\leq 0} \subset \mathcal{D}_2^{\leq 0},$$

and upper consistent if

$$au_{\geq 1}^2 \mathcal{D}_1^{\geq 1} \subset \mathcal{D}_1^{\geq 1}.$$

It is called *consistent* if it is simultaneously lower and upper consistent.

For consistent *t*-structures the intersection (union) exists and is 'naive'.

Proposition 4 Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a lower, respectively upper, consistent pair of t-structures. Then the formulas

$$\mathcal{D}_{\cap}^{\leq 0} = \mathcal{D}_{1}^{\leq 0} \cap \mathcal{D}_{2}^{\leq 0}, \qquad \mathcal{D}_{\cap}^{\geq 1} = (\mathcal{D}_{\cap}^{\leq 0})^{\perp}$$

with the truncation functor $\tau_{\leq 0}^{\cap} := \tau_{\leq 0}^{1} \tau_{\leq 0}^{2}$, respectively

$$\mathcal{D}_{\cup}^{\geq 1} = \mathcal{D}_{1}^{\geq 1} \cap \mathcal{D}_{2}^{\geq 1}, \qquad \mathcal{D}_{\cup}^{\leq 0} = {}^{\perp}(\mathcal{D}_{\cup}^{\geq 1})$$

with the truncation functor $\tau_{\geq 1}^{\cup} := \tau_{\geq 1}^2 \tau_{\geq 1}^1$, define a new t-structure in the category \mathcal{D} .

Proof. For a lower consistent pair $(\mathcal{T}_1, \mathcal{T}_2)$ of t-structures the category $\mathcal{D}_{\cap}^{\leq 0} = \mathcal{D}_1^{\leq 0} \cap \mathcal{D}_2^{\leq 0}$ forms the data 1 from lemma 1. Indeed, it is obviously preserved by the shift functor T. Moreover, the functor $\tau_{\leq 0}^{\cap} := \tau_{\leq 0}^{1} \tau_{\leq 0}^{2}$ has its image in $\mathcal{D}_{\cap}^{\leq 0}$ due to lower consistency of the pair. One can easily see that it is right adjoint to the inclusion functor $i_{\leq 0}^{\cap} : \mathcal{D}_{\cap}^{\leq 0} \to \mathcal{D}$. Similarly for an upper consistent pair.

The example from the preceding subsection yields a non-consistent pair of t-structures.

Now we shall specify conditions on a sequence of t-structures under which one can iterate the operations of intersection and union.

First, intersection defines an associative operation on ordered sequences with pairwise lower consistent elements, preserving lower consistency at intermediate steps.

Proposition 5 Let $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ be a triple of t-structures with $(\mathcal{T}_i, \mathcal{T}_i)$ being lower consistent for i < j. Then the pairs $(\mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{T}_3)$ and $(\mathcal{T}_1, \mathcal{T}_2 \cap \mathcal{T}_3)$ are lower consistent. Moreover, $(\mathcal{T}_1 \cap \mathcal{T}_2) \cap \mathcal{T}_3$ coincides with $\mathcal{T}_1 \cap (\mathcal{T}_2 \cap \mathcal{T}_3)$.

Proof. By proposition 4 the truncation functor for the *t*-structure $\mathcal{T}_1 \cap \mathcal{T}_2$ is the composite of $\tau_{\leq 0}^1$ and $\tau_{\leq 0}^2$. As both functors preserve $\mathcal{D}_3^{\leq 0}$, lower consistency of the pair $(\mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{T}_3)$ follows.

Further, $\mathcal{D}_{2\cap 3}^{\leq 0} = \mathcal{D}_{2}^{\leq 0} \cap \mathcal{D}_{3}^{\leq 0}$. Both categories $\mathcal{D}_{2}^{\leq 0}$ and $\mathcal{D}_{3}^{\leq 0}$ are preserved by $\tau_{\leq 0}^{1}$. Hence, $\tau_{\leq 0}^{1} \mathcal{D}_{2\cap 3}^{\leq 0} \subset \mathcal{D}_{2\cap 3}^{\leq 0}$, i.e. the second pair $(\mathcal{T}_{1}, \mathcal{T}_{2} \cap \mathcal{T}_{3})$ is lower consistent. Finally, for both *t*-structures $(\mathcal{T}_{1} \cap \mathcal{T}_{2}) \cap \mathcal{T}_{3}$ and $\mathcal{T}_{1} \cap (\mathcal{T}_{2} \cap \mathcal{T}_{3})$ the categories $\mathcal{D}^{\leq 0}$ coincide

with $\mathcal{D}_1^{\leq 0} \cap \mathcal{D}_2^{\leq 0} \cap \mathcal{D}_3^{\leq 0}$. This gives coincidence of these *t*-structures.

By the dual argument, one proves

Proposition 6 Let $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ be a triple of t-structures with $(\mathcal{T}_i, \mathcal{T}_i)$ being upper consistent for i < j. Then the pairs $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{T}_3)$ and $(\mathcal{T}_1, \mathcal{T}_2 \cup \mathcal{T}_3)$ are upper consistent. Moreover, $(\mathcal{T}_1 \cup \mathcal{T}_2) \cup \mathcal{T}_3$ and $\mathcal{T}_1 \cup (\mathcal{T}_2 \cup \mathcal{T}_3)$ coincide.

It follows that for any finite sequence $\{\mathcal{T}_i\}_{i \in [1,n]}$ of t-structures such that for all i < jthe pairs $(\mathcal{T}_i, \mathcal{T}_j)$ are lower (resp. upper) consistent, one can construct a new *t*-structure $\bigcap_{i \in [1,n]} \mathcal{T}_i$ (resp. $\bigcup_{i \in [1,n]} \mathcal{T}_i$) with $\mathcal{D}_{\cap}^{\leq 0} = \bigcap_{i \in [1,n]} \mathcal{D}_i^{\leq 0}$ (resp. $\mathcal{D}_{\cup}^{\geq 1} = \bigcap_{i \in [1,n]} \mathcal{D}_i^{\geq 1}$) and the truncation functor $\tau_{\leq 0}^{\cap} = \tau_{\leq 0}^1 \tau_{\leq 0}^2 \dots \tau_{\leq 0}^n$ (resp. $\tau_{\leq 0}^{\cup} = \tau_{\geq 1}^n \dots \tau_{\geq 1}^1$).

The next two propositions describe conditions under which one can alternate the intersection and union operations.

Proposition 7 Let $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ be a triple of t-structures, such that the pairs $(\mathcal{T}_1, \mathcal{T}_2)$ and $(\mathcal{T}_1, \mathcal{T}_3)$ are upper consistent and the pair $(\mathcal{T}_2, \mathcal{T}_3)$ is lower consistent. Then the pair $(\mathcal{T}_1, \mathcal{T}_2 \cap \mathcal{T}_3)$ is upper consistent.

Proof. We denote by $\tau_{\leq 0}^{\cap}$ and $\tau_{\geq 1}^{\cap}$ the truncation functors for the *t*-structure $\mathcal{T}_2 \cap \mathcal{T}_3$.

We know by proposition 4 that $\tau_{\leq 0}^{\cap} = \tau_{\leq 0}^2 \tau_{\leq 0}^3$. Then, for any $X \in \mathcal{D}$, by axioms of triangulated categories, the commutative triangle



can be embedded into the following 'octahedron' with all columns and rows being exact triangles (for convenience we draw it as a square):

We have to show that $\tau_{\geq 1}^{\cap} \mathcal{D}_1^{\geq 1} \subset \mathcal{D}_1^{\geq 1}$. Let $X \in \mathcal{D}_1^{\geq 1}$, then, since the pair $(\mathcal{T}_1, \mathcal{T}_3)$ is upper consistent, then $\tau_{\geq 1}^3 X \in \mathcal{D}_1^{\geq 1}$. From the middle row in (6) and remark 2 it follows that $\tau_{\leq 0}^3 X \in \mathcal{D}_1^{\geq 1}$. By upper consistency of the pair $(\mathcal{T}_1, \mathcal{T}_2)$ we deduce that $\tau_{\geq 1}^2 \tau_{\leq 0}^3 X \in \mathcal{D}_1^{\geq 1}$. Now from the upper row of (6) it follows that $\tau_{\geq 1}^{\cap} X \in \mathcal{D}_1^{\geq 1}$. This proves the proposition.

Thus, under conditions of proposition 7 one can construct a new *t*-structure $\mathcal{T}_1 \cup (\mathcal{T}_2 \cap \mathcal{T}_3)$.

By duality, one proves

Proposition 8 Let $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ be a triple of t-structures, such that the pairs $(\mathcal{T}_1, \mathcal{T}_3)$ and $(\mathcal{T}_2, \mathcal{T}_3)$ are lower consistent and the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is upper consistent. Then the pair $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{T}_3)$ is lower consistent.

Suppose we have a pair $(\mathcal{T}_1, \mathcal{T}_2)$ of t-structures with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. It is easily seen to be lower and upper consistent. Moreover, $(\mathcal{T}_2, \mathcal{T}_1)$ is also a lower and upper consistent pair. In particular, the pair $(\mathcal{T}_1, \mathcal{T}_1[k])$ is consistent for any $k \in \mathbb{Z}$. Note that truncation functors commute:

$$\tau_{\leq 0}^1 \tau_{\leq 0}^2 = \tau_{\leq 0}^1 = \tau_{\leq 0}^2 \tau_{\leq 0}^1, \quad \tau_{\geq 1}^1 \tau_{\geq 1}^2 = \tau_{\geq 1}^2 = \tau_{\geq 1}^2 \tau_{\geq 1}^1$$

We conclude this subsection by proving a lemma, used in the sequel, on commutation of truncation functors for an ordered pair of t-structures.

Lemma 9 For an ordered pair $(\mathcal{T}_1, \mathcal{T}_2)$ of t-structures with $\mathcal{T}_1 \subseteq \mathcal{T}_2$ the truncation functors commute:

$$\tau_{\le 0}^2 \tau_{\ge 1}^1 = \tau_{\ge 1}^1 \tau_{\le 0}^2$$

Proof. For any object $X \in \mathcal{D}$ we have an isomorphism: $\tau_{\geq 1}^2 \tau_{\geq 1}^1 X = \tau_{\geq 1}^2 X$.

Hence, by the octahedron axiom, we have the following diagram with exact rows and columns:

$$\begin{aligned}
\tau_{\leq 0}^{2} \tau_{\geq 1}^{1} X &\to \tau_{\geq 1}^{1} X \to \tau_{\geq 1}^{2} \tau_{\geq 1}^{1} X \\
\uparrow &\uparrow &\uparrow &\uparrow \\
\tau_{\leq 0}^{2} X \to X \to \tau_{\geq 1}^{2} X \\
\uparrow &\uparrow &\uparrow \\
\tau_{\leq 0}^{1} X \xrightarrow{\sim} \tau_{\leq 0}^{1} X \to 0
\end{aligned} \tag{7}$$

In the upper row, two terms $\tau_{\geq 1}^1 X$ and $\tau_{\geq 1}^2 \tau_{\geq 1}^1 X$ are in $\mathcal{D}_1^{\geq 1}$. Hence, the third term $\tau_{\leq 0}^2 \tau_{\geq 1}^1 X$ is in $\mathcal{D}_1^{\geq 1}$.

Then, in the left column, the lower term is in $\mathcal{D}_1^{\leq 0}$ and the upper one is in $\mathcal{D}_1^{\geq 1}$. Therefore, this column is the canonical decomposition of the middle term $\tau_{\leq 0}^2 X$ along the first *t*-structure. Hence, we have a canonical isomorphism: $\tau_{\leq 0}^2 \tau_{\geq 1}^1 X \simeq \tau_{\geq 1}^1 \tau_{\leq 0}^2 X$.

1.4 The standard postulates for lattices

The lattice theory allows to interpret particularly nice posets, called lattices, as abstract algebras with the union (sum) and intersection (product) operations that satisfy so-called standard postulates [4]. This interpretation is useful for manipulations with lattices, such as taking quotient.

Definition A *lattice* is a partially ordered set for which union and intersection of any pair of elements exist.

Recall the *standard postulates* known for the operations on a partially ordered set. We shall use now the $(\cdot, +)$ -notation with the usual priority of \cdot over +. In the following formulae, we suppose that both sides of equations are well-defined for elements $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ of a partially ordered set (when this is not *a priori* clear, as say for lattices):

$$\mathcal{T} \cdot \mathcal{T} = \mathcal{T}, \qquad \mathcal{T} + \mathcal{T} = \mathcal{T};$$
(8)

$$\mathcal{T}_1 \cdot \mathcal{T}_2 = \mathcal{T}_2 \cdot \mathcal{T}_1, \qquad \mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_2 + \mathcal{T}_1; \tag{9}$$

$$(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3), \qquad (T_1 + T_2) + T_3 = T_1 + (T_2 + T_3);$$
 (10)

$$\mathcal{T}_1 \cdot (\mathcal{T}_1 + \mathcal{T}_2) = \mathcal{T}_1 + \mathcal{T}_1 \cdot \mathcal{T}_2 = \mathcal{T}_1. \tag{11}$$

Proposition 10 (see [4]) Lattices are identified with abstract algebras with two binary operations \cdot and + satisfying standard postulates (8)-(11). Given such an algebra, the partial order on the set of its elements can be recovered from the rule:

$$x \le y \quad \text{iff} \quad x \cdot y = x.$$
 (12)

If 0 and/or 1 exist, they can be added as nullary operations.

An equivalence relation \sim in a lattice L is called *congruence* if $p \sim q$ implies $p+l \sim q+l$ and $p \cdot l \sim q \cdot l$ for any element $l \in L$. Given a number of pairs (equivalences) $p_i \sim q_i$, $i \in I$, we define the congruence generated by them as the minimal congruence containing all of them. It is the intersection of all congruences that contain $\{p_i \sim q_i\}_{i \in I}$. Intersection of arbitrary number of congruencies (as subsets in $L \times L$) is again a congruence.

For a congruence \sim in a lattice L, regarded as an abstract algebra, the quotient algebra $L' = L/\sim$ is well-defined. Being an abstract algebra of the same type (satisfying the same standard postulates), it is again a lattice.

Note that, for $x', y' \in L$, $x' \leq y'$ in L' iff there exist $x, y \in L$ such that $x \sim x', y \sim y'$ and $x \leq y$ in L. Indeed, if $x' \leq y'$ in L', then $x' \sim x'y'$ by (12). Once summed up with y', this gives: $x' + y' \sim x'y' + y' = y'$. Hence, we can take required x and y to be: x = x'y', y = x' + y'.

Unfortunately, I do not know any 'sufficiently large' class of *t*-structures in a triangulated category which constitute a lattice. It would be interesting to explore this problem.

1.5 Modularity laws

For any partially ordered set, the distributivity inequalities are verified (again under assumption that both sides are well defined):

$$(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 \geq \mathcal{T}_1 \cdot \mathcal{T}_3 + \mathcal{T}_2 \cdot \mathcal{T}_3, \tag{13}$$

$$\mathcal{T}_1 + \mathcal{T}_2 \cdot \mathcal{T}_3 \leq (\mathcal{T}_1 + \mathcal{T}_2) \cdot (\mathcal{T}_1 + \mathcal{T}_3).$$
(14)

The distributivity laws, i.e. when (13) and (14) are replaced by the equations with the same left and right hand sides, are not valid for *t*-structures.

Example. Let \mathcal{T}_1 and \mathcal{T}_2 be the *T*-invariant t-structures as in the example in subsection 1.2. Define a *T*-invariant *t*-structure \mathcal{T}_3 by

$$\mathcal{D}_3^{\leq 0} = \langle \mathcal{O}
angle$$

Then $\mathcal{T}_1 + \mathcal{T}_2 = \mathbf{1}$, because $\mathcal{D}_1^{\geq 1} \cap \mathcal{D}_2^{\geq 1} = \langle \mathcal{O}(-3) \rangle \cap \langle \mathcal{O} \rangle = 0$. Also $\mathcal{T}_1 \cdot \mathcal{T}_3 = \mathcal{T}_2 \cdot \mathcal{T}_3 = \mathbf{0}$, because $\mathcal{D}_1^{\leq 0} \cap \mathcal{D}_3^{\leq 0} = \mathcal{D}_2^{\leq 0} \cap \mathcal{D}_3^{\leq 0} = 0$. It follows that the distributivity law

$$(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 = \mathcal{T}_1 \cdot \mathcal{T}_3 + \mathcal{T}_2 \cdot \mathcal{T}_3$$

is violated, as the left hand side is T_3 , while the right hand side is **0**.

If $\mathcal{T}_1 \leq \mathcal{T}_3$, then (13) and (14) yield the following autodual inequality:

$$(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 \ge \mathcal{T}_1 + \mathcal{T}_2 \cdot \mathcal{T}_3. \tag{15}$$

When replaced by the equation, it is called *modularity law*.

We prove *under appropriate assumptions on consistency* several statements, which might be viewed as modularity laws for *t*-structures. The first theorem is autodual.

Theorem 11 (modularity law 1) Suppose we are given a triple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ of t-structures such that $\mathcal{T}_1 \leq \mathcal{T}_3$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is upper consistent and the pair $(\mathcal{T}_2, \mathcal{T}_3)$ is lower consistent. Then

- i) the pair $(T_1 + T_2, T_3)$ is lower consistent,
- ii) the pair $(\mathcal{T}_1, \mathcal{T}_2 \cdot \mathcal{T}_3)$ is upper consistent,
- *iii)* $(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 = \mathcal{T}_1 + \mathcal{T}_2 \cdot \mathcal{T}_3.$

Proof. Since $\mathcal{T}_1 \leq \mathcal{T}_3$, the pair $(\mathcal{T}_1, \mathcal{T}_3)$ is lower consistent. Hence, lower consistency of the pair $(\mathcal{T}_1 + \mathcal{T}_2, \mathcal{T}_3)$ follows from proposition 8. Similarly, the pair $(\mathcal{T}_1, \mathcal{T}_3)$ is upper consistent, hence $(\mathcal{T}_1, \mathcal{T}_2 \cdot \mathcal{T}_3)$ is upper consistent by proposition 7.

Thus, both sides of *iii*) are well-defined.

We have inequality (15). The opposite to (15) inequality is, by definition, equivalent to the inclusion of subcategories:

$$\mathcal{D}_{(1+2)3}^{\le 0} \subseteq \mathcal{D}_{1+2\cdot 3}^{\le 0}.$$
 (16)

This, in turn, is equivalent to the following statement:

(*) Let
$$X \in \mathcal{D}_{(1+2)3}^{\leq 0}$$
 and $Y \in \mathcal{D}_{1+2\cdot 3}^{\geq 1}$, then $\operatorname{Hom}(X, Y) = 0$.

Let us prove it. To this end, decompose X into an exact triangle:

$$\tau_{\le 0}^1 X \to X \to \tau_{\ge 1}^1 X. \tag{17}$$

As $\mathcal{T}_1 \subseteq (\mathcal{T}_1 + \mathcal{T}_2)\mathcal{T}_3$, the pair $(\mathcal{T}_1, (\mathcal{T}_1 + \mathcal{T}_2)\mathcal{T}_3)$ is lower consistent. Hence $\tau_{\leq 0}^1 X \in \mathcal{D}_{(1+2)3}^{\leq 0}$. Then, by remark 2, $\tau_{\geq 1}^1 X \in \mathcal{D}_{(1+2)3}^{\leq 0}$.

Now since $\mathcal{T}_1 \subseteq \mathcal{T}_1 + \mathcal{T}_2 \mathcal{T}_3$, then $\operatorname{Hom}(\mathcal{D}_1^{\leq 0}, \mathcal{D}_{1+2\cdot 3}^{\geq 1}) = 0$. It follows that $\operatorname{Hom}(\tau_{\leq 0}^1 X, Y) = 0$. Applying the functor $\operatorname{Hom}(\cdot, Y)$ to the triangle (17), we see that it is sufficient to show that $\operatorname{Hom}(\tau_{\geq 1}^1 X, Y) = 0$.

Therefore, we have reduced (*) to the case when $X \in \mathcal{D}_{(1+2)3}^{\leq 0} \cap \mathcal{D}_{1}^{\geq 1}$, $Y \in \mathcal{D}_{1+2\cdot 3}^{\geq 1}$. Now let us decompose Y with respect to the third t-structure:

$$\tau_{\le 0}^3 Y \to Y \to \tau_{\ge 1}^3 Y. \tag{18}$$

Since $\mathcal{T}_{1+2\cdot3} \subseteq \mathcal{T}_3$, the pair $(\mathcal{T}_{1+2\cdot3}, \mathcal{T}_3)$ is upper consistent. Hence $\tau_{\geq 1}^3 Y \in \mathcal{D}_{1+2\cdot3}^{\geq 1}$. Then, again by remark 2, $\tau_{\leq 0}^3 Y \in \mathcal{D}_{1+2\cdot3}^{\geq 1}$. Since $\mathcal{T}_{(1+2)3} \subseteq \mathcal{T}_3$, then $\operatorname{Hom}(\mathcal{D}_{(1+2)3}^{\leq 0}, \mathcal{D}_3^{\geq 1}) = 0$. It follows that $\operatorname{Hom}(X, \tau_{\geq 1}^3 Y) = 0$. Applying $\operatorname{Hom}(X, \cdot)$ to (18), we see that it is sufficient to show that $\operatorname{Hom}(X, \tau_{\leq 0}^3 Y) = 0$.

Therefore, we have reduced (*) to the statement:

(**) if
$$X \in \mathcal{D}_{(1+2)3}^{\leq 0} \cap \mathcal{D}_1^{\geq 1}$$
 and $Y \in \mathcal{D}_3^{\leq 0} \cap \mathcal{D}_{1+2\cdot 3}^{\geq 1}$, then $\operatorname{Hom}(X, Y) = 0$.

Let us show that

- a) $\mathcal{D}_{(1+2)3}^{\leq 0} \cap \mathcal{D}_1^{\geq 1} \subset \mathcal{D}_2^{\leq 0}$,
- b) $\mathcal{D}_3^{\leq 0} \cap \mathcal{D}_{1+2\cdot 3}^{\geq 1} \subset \mathcal{D}_2^{\geq 1}$.

Recall that, by proposition 4, since the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is upper consistent, $\tau_{\geq 1}^{1+2} = \tau_{\geq 1}^2 \tau_{\geq 1}^1$. Therefore, if $X \in \mathcal{D}_1^{\geq 1}$, then $\tau_{\geq 1}^{1+2}X = \tau_{\geq 1}^2 X$. If, in addition, $X \in \mathcal{D}_{(1+2)3}^{\leq 0}$, then $\tau_{\geq 1}^2 X = \tau_{\geq 1}^{1+2} X = 0$, in view of $\mathcal{T}_{(1+2)3} \subseteq \mathcal{T}_{1+2}$. This proves a).

Dually, by proposition 4, since the pair $(\mathcal{T}_2, \mathcal{T}_3)$ is lower consistent, $\tau_{\leq 0}^{2\cdot 3} = \tau_{\leq 0}^2 \tau_{\leq 0}^3$. Hence for $Y \in \mathcal{D}_3^{\leq 0}$, $\tau_{\leq 0}^{2\cdot 3}Y = \tau_{\leq 0}^2Y$. If, in addition, $Y \in \mathcal{D}_{1+2\cdot 3}^{\geq 1}$, then $\tau_{\leq 0}^2Y = \tau_{\leq 0}^{2\cdot 3}Y = 0$, in view of $\mathcal{T}_{2\cdot 3} \subseteq \mathcal{T}_{1+2\cdot 3}$. This proves b).

Since a) and b) obviously imply (**), this finishes the proof of the theorem.

We shall also need another incarnation of the modularity law.

Theorem 12 (modularity law 2) Suppose we are given a triple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ of t-structures such that $\mathcal{T}_1 \leq \mathcal{T}_3$, the pair $(\mathcal{T}_1, \mathcal{T}_2)$ is upper consistent and the pair $(\mathcal{T}_3, \mathcal{T}_2)$ is lower consistent. Then

- i) the pair $(\mathcal{T}_1, \mathcal{T}_2 \cdot \mathcal{T}_3)$ is upper consistent,
- ii) the pair $(\mathcal{T}_3, \mathcal{T}_1 + \mathcal{T}_2)$ is lower consistent,

iii) $(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 = \mathcal{T}_1 + \mathcal{T}_2 \cdot \mathcal{T}_3.$

Note that in the contrary to the theorem 11, lower consistency of the pair $(\mathcal{T}_3, \mathcal{T}_1 + \mathcal{T}_2)$ is not a formal consequence of the propositions from the previous subsection.

Proof. As $\mathcal{T}_1 \leq \mathcal{T}_3$, the pair $(\mathcal{T}_1, \mathcal{T}_3)$ is upper consistent. Hence, we can apply proposition 7 to the triple $(\mathcal{T}_1, \mathcal{T}_3, \mathcal{T}_2)$ to get upper consistency of the pair $(\mathcal{T}_1, \mathcal{T}_2 \cdot \mathcal{T}_3)$. This proves i) and the fact that $\mathcal{T}_{1+2\cdot 3}$ exists.

We know by proposition 4 that $\tau_{\leq 0}^{2\cdot 3} = \tau_{\leq 0}^3 \tau_{\leq 0}^2$.

Then, for any $X \in \mathcal{D}$, by the octahedron axiom, we have the diagram with exact rows and columns:

$$\begin{aligned} \tau_{\geq 1}^{3} \tau_{\leq 0}^{2} X &\to \tau_{\geq 1}^{2 \cdot 3} X \to \tau_{\geq 1}^{2} X \\ \uparrow &\uparrow &\uparrow &\uparrow \\ \tau_{\leq 0}^{2} X \to X \to \tau_{\geq 1}^{2} X \\ \uparrow &\uparrow &\uparrow \\ \tau_{\leq 0}^{3} \tau_{\leq 0}^{2} X \xrightarrow{\sim} \tau_{\leq 0}^{3} \tau_{\leq 0}^{2} X \to 0 \end{aligned}$$

$$(19)$$

By proposition 4, $\tau_{\geq 1}^{1+2} = \tau_{\geq 1}^2 \tau_{\geq 1}^1$ and $\tau_{\geq 1}^{1+2\cdot 3} = \tau_{\geq 1}^{2\cdot 3} \tau_{\geq 1}^1$. Hence, substitution $X = \tau_{\geq 1}^1 Y$ in the upper row of the octahedron yields:

$$\tau_{\geq 1}^{3} \tau_{\leq 0}^{2} \tau_{\geq 1}^{1} Y \to \tau_{\geq 1}^{2 \cdot 3 + 1} Y \to \tau_{\geq 1}^{1 + 2} Y.$$

Now, substitution $Y = \tau_{\leq 0}^{1+2} Z$ gives a natural isomorphism:

$$\tau_{\geq 1}^{3} \tau_{\leq 0}^{2} \tau_{\geq 1}^{1} \tau_{\leq 0}^{1+2} Z \xrightarrow{\sim} \tau_{\geq 1}^{1+2 \cdot 3} \tau_{\leq 0}^{1+2} Z.$$

It follows that, for any $Z \in \mathcal{D}$,

$$\tau_{\leq 0}^{3} \tau_{\geq 1}^{1+2 \cdot 3} \tau_{\leq 0}^{1+2} Z = 0.$$

Observe an obvious inequality: $\mathcal{T}_{1+2\cdot 3} \leq \mathcal{T}_3$. Hence, by lemma 9, $\tau_{\leq 0}^3$ and $\tau_{\geq 1}^{1+2\cdot 3}$ commute. This implies:

$$\tau_{\geq 1}^{1+2\cdot 3}\tau_{\leq 0}^{3}\tau_{\leq 0}^{1+2} = 0.$$

Equivalently, the image of $\tau_{\leq 0}^{3} \tau_{\leq 0}^{1+2}$ is in $\mathcal{D}_{1+2\cdot 3}^{\leq 0}$. On the other hand, $\mathcal{D}_{1+2\cdot 3}^{\leq 0} \subset \mathcal{D}_{1+2}^{\leq 0} \cap \mathcal{D}_{3}^{\leq 0}$. Therefore, the image of $\tau_{\leq 0}^{3} \tau_{\leq 0}^{1+2}$ is in $\mathcal{D}_{1+2}^{\leq 0} \cap \mathcal{D}_{3}^{\leq 0}$. Then $\tau_{\leq 0}^{3} \mathcal{D}_{1+2}^{\leq 0} \subset \mathcal{D}_{1+2}^{\leq 0}$, hence lower consistency of $(\mathcal{T}_{3}, \mathcal{T}_{1} + \mathcal{T}_{2})$. By proposition 4, functor $\tau_{\leq 0}^{3} \tau_{\leq 0}^{1+2}$ is the truncation functor for $\mathcal{T}_{(1+2)\cdot 3}$. Its image is $\mathcal{D}_{(1+2)\cdot 3}^{\leq 0} = \mathcal{D}_{1+2}^{\leq 0} \cap \mathcal{D}_{3}^{\leq 0}$. It was shown that the image is in $\mathcal{D}_{1+2\cdot 3}^{\leq 0}$. Together with the opposite inclusion (15) this yields *iii*).

The dual statement to theorem 12 is

Theorem 13 (modularity law 2') Suppose we are given a triple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ of t-structures such that $\mathcal{T}_1 \leq \mathcal{T}_3$, pair $(\mathcal{T}_2, \mathcal{T}_1)$ is upper consistent and pair $(\mathcal{T}_2, \mathcal{T}_3)$ is lower consistent. Then

- i) pair $(\mathcal{T}_2 \cdot \mathcal{T}_3, \mathcal{T}_1)$ is upper consistent,
- *ii)* pair $(\mathcal{T}_1 + \mathcal{T}_2, \mathcal{T}_3)$ is lower consistent,
- *iii*) $(\mathcal{T}_1 + \mathcal{T}_2) \cdot \mathcal{T}_3 = \mathcal{T}_1 + \mathcal{T}_2 \cdot \mathcal{T}_3$.

2 Sets with consistencies and pairs of chains

For the sake of clear and logical exposition, we formalize the properties of consistency proved in the previous subsections and present a convenient pictorial description. Then we prove that a consistent pair of chains generates a distributive lattice.

2.1 Sets with consistencies

By a set with consistencies we call a partially ordered set containing $\mathbf{0}$ and $\mathbf{1}$ with two binary (non-reflexive) relations, called lower and upper consistencies, such that for any lower, respectively upper, consistent pair the abstract intersection, respectively union, exists and satisfies the following axioms (*prime* means the dual axiom):

SC1. if $a \leq b$, then both (a, b) and (b, a) are lower and upper consistent,

SC2. implication of lower consistency under intersection as in proposition 5,

SC2'. implication of upper consistency under union as in proposition 6,

SC3. implication of upper consistency under intersection as in proposition 7,

SC3'. implication of lower consistency under union as in proposition 8,

SC4. modularity law as in theorem 11,

SC5. modularity law as in theorem 12,

SC5'. modularity law as in theorem 13.

In this formalism, the words 'of t-structures' must be removed from all the quoted statements. In the sequel, we shall sometimes directly refer to the corresponding statements, meaning the above axioms for general sets with consistencies.

Note that the standard postulates (8),(9),(10),(11) are automatically satisfied for sets with consistencies. It follows from SC1 that **0** and **1** constitute a lower and upper consistent pair with any element.

Contrary to a lattice, a set with consistencies is not an abstract algebra.

2.2 Diagrams

It is instructive to use the following pictorial description for consistencies and operations with them.

Suppose we are given a set with consistencies V. Consider a graph $\Gamma(V)$ with vertices labelled by elements of V.

We draw a full arrow:

 $a \longrightarrow b$

if (a, b) is a lower consistent pair in V, and a dashed arrow

 $a \dashrightarrow b$

if (a, b) is an upper consistent pair.

Now the operations of union and intersection are interpreted as contractions of full and dashed arrows. For the intersection, this is:

$$a \longrightarrow b \implies a \cdot b$$

For the union, this is:

$$a \dashrightarrow b \implies a+b$$

The duality on t-structures described in subsection 1.1 exchanges full arrows with dashed ones and reverses their directions.

We shall actually consider subgraphs of $\Gamma(V)$ which are significant to ensure iterated contractions.

Axioms SC2, SC2', SC3, SC3' of the definition of a set with consistencies provides us with patterns which allow to connect by arrows a new vertex obtained as result of contractions of an arrow with other (old) vertices of the graph. Axiom SC2 (proposition 5) reads as possibility of contractions of two edges in a triangle with full arrows:



Axiom SC2' reads as the same picture with dashed arrows. Axiom SC3 (proposition 7) reads:







Note that we don't have in general the implications which are obtained from the last two pictures by changing full arrows for dashed ones and vice versa (while preserving their directions).

Sometimes it is useful to put the vertex marked with the intersection (resp. union) *t*-structure at the middle point of the corresponding full (resp. dashed) arrow, if it exists.

2.3 The universal lattice with consistencies

A morphism of sets with consistencies is a morphism of the corresponding posets which takes any upper (lower) consistent pair to an upper (lower) consistent pair. A set with consistencies which is a lattice is called a *lattice with consistencies*. Morphisms of lattices with consistencies are defined in the obvious way.

A morphism $\phi : P \to L$, where P is a poset and L a lattice, is an order preserving map such that $\phi(x+y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$, whenever x + y or xy exists.

Consider a poset P on which two arbitrary binary relations are fixed. We are interested in morphisms $P \to L$ of P into a lattice L with consistencies, such that the images of any pair of elements in P which is in the first, respectively, the second, relation, is lower, respectively upper, consistent in L. Such a morphism $P \to U$ is called *universal* if for any other morphism $P \to L$ of this sort, there exists a unique morphism of lattices $U \to L$ with consistencies which makes commutative the following diagram (of morphisms):



In this case, we call U = U(P) the universal lattice with consistencies generated by P.

When consistency relations are omitted in the above definitions, the notion of the universal lattice L(P) generated by a poset P takes place. The universal lattice L(P) classically known to exist [4]. Indeed, remember that the notion of a lattice is equivalent to that of abstract algebra satisfying the standard postulates. Thus, we can consider a free abstract algebra F(P), generated by P as a disjoint set, and take its quotient by the congruence generated by relations x + y = "x + y" and $x \cdot y = "x \cdot y"$ for all x and y for which the union "x + y" (resp. intersection " $x \cdot y$ ") in poset P exists (cf. 1.4). This quotient is L(P).

A lattice is said to be *distributive* if for any x, y, z:

$$x(y+z) = xy + xz,$$
$$x + yz = (x+y)(x+z).$$

These postulates are mutually dual and each one implies the other.

We can similarly define the universal distributive lattice D(P) generated by poset P. D(P) is the quotient of L(P) by the congruence generated by equivalences $(x + y)z \sim xz + yz$ for all $x, y, z \in L(P)$.

Proposition 14 Let P be a partially ordered set with two binary relations. Then a universal lattice with consistencies generated by P exists.

Proof. Consider the universal lattice L(P) generated by the poset P. Endow L(P) with two binary relations, called respectively upper and lower consistency: first, demand the images in L(P) of all pairs in P which belong to the first (resp. second) given binary relation to be upper (resp. lower) consistent; second, extend the two binary relations of

consistency by the minimal set of pairs that satisfy axioms $SC1, \ldots, SC3'$ and those parts of axioms SC5 and SC5' which imply consistencies (recall that not all of them formally follow from the other axioms).

Note that this procedure does not make L(P) a set with consistencies, because the modularity equations of axioms SC4, SC5, SC5' are not satisfied yet.

To meet this constraints, consider the quotient lattice $L_1(P)$ of L(P) by the congruence generated by equivalences $(t_1+t_2)t_3 \sim t_1+t_2t_3$, for all the triples $(t_1,t_2,t_3) \in L(P)$ which meet conditions of one of the modularity laws SC4, SC5, SC5'. Endow $L_1(P)$ with consistency relations inherited from L(P). The axiom SC1 obviously follows from the fact (proved in 1.4) that $x' \leq y'$ in $L_1(P)$ iff there exists $x, y \in L(P)$ such that $x \sim x'$, $y \sim y'$ and $x \leq y$ in L(P).

But it is not so good in what concerns the other axioms. For this reason, subject $L_1(P)$ (with its consistency relations) to the same procedure as we did with L(P), i.e. extend appropriately the binary relations and take the similar quotient, to get $L_2(P)$ from $L_1(P)$. By iterating this process, we get a sequence of lattices $L_i(P)$ together with maps $P \to L_i(P)$, compatible with the quotient maps. Then $U(P) = \operatorname{colim} L_i(P)$ with two colimit binary relations is the universal lattice with consistencies.

Indeed, consider any morphism $P \to L$, where L is a lattice with consistencies and the morphism takes the pairs in L from the first, respectively, the second binary relations on P into lower, respectively, upper consistent pairs in L. The universality of $P \to L(P)$ implies existence of the lattice homomorphism $L(P) \to L$, which makes the corresponding diagram commute. This homomorphism clearly descends to a homomorphism $L_i(P) \to L$, which, when passing to the limit, gives a homomorphism $U(P) \to L$. It is easy to see that this homomorphism is unique.

We can consider a distributive lattice as a lattice with consistencies for which all pairs of elements are both upper and lower consistent. Since distributivity implies modularity, we indeed get a correctly defined lattice with consistencies. A canonical homomorphism

$$\psi: U(P) \to D(P)$$

follows from the universality of U(P). It is clearly an epimorphism, because congruencies used to construct U(P) as a quotient of the lattice L(P) in proposition 14 are among those which were used to construct D(P) as the quotient of L(P).

2.4 Consistent pairs of chains

A chain in a partially ordered set is an ordered sequence of elements. A pair of chains $a_1 \geq \ldots \geq a_n$ and $b_1 \leq \ldots \leq b_m$ in a set with consistencies such that (a_i, b_j) is upper and lower consistent for any pair (i, j), is called *consistent pair of chains*. Our purpose in this subsection will be to prove that a consistent pair of chains generates (under operations of union and intersection) a distributive lattice.

There is a theorem due to Birkhoff [4] that two chains in a modular lattice generate a distributive lattice. When working with sets with consistencies we have two new difficulties against the hypothesis of Birkhoff theorem: first, intersection and union do not *a priori* exist, second, the modularity laws are verified only under some consistency conditions. The notion of sum and intersection of any number of elements in a partially ordered set is intrinsically defined. Say, the sum of a given set S of elements is the element which is greater than any element from S but less than any other element with this property. It may not exist. Recall the following trivial observation, which slightly strengthens associativity postulates (10).

Lemma 15 Let (a, b, c) be a triple of elements in a partially ordered set. If a + b exists, then existence of one side of the following equation implies existence of the other and the equation itself:

$$(a+b) + c = a + b + c. (20)$$

In the dual statement, + is replaced by \cdot .

Proof. Obvious.

Remark. Note that in the constructions of this subsection, this lemma is the only tool applied to deduce existence of union, respectively intersection, for pairs of elements ((a + b, c) here) which do not a priori constitute an upper, respectively lower, consistent pair. The typical situation is the following. Suppose we have a triple of elements (say *t*-structures) (a_1, a_2, a_3) with $a_i \rightarrow a_j$, for i < j. Then, by proposition 8 and the lemma, $(a_1 + a_2) + a_3 = a_1 + a_2 + a_3$ exists. Hence, again by the lemma, $(a_1 + a_3) + a_2$ exists, though neither $(a_1 + a_3, a_2)$ nor $(a_2, a_1 + a_3)$ is a priori lower consistent.

For the rest of the subsection, we assume to be given a consistent pair of chains $a_1 \geq \ldots \geq a_n$ and $b_1 \leq \ldots \leq b_m$. Denote:

$$u_{ij} = a_i b_j, \quad v_{ij} = a_i + b_j.$$

Lemma 16 Let $j \leq l$. Then, the pair (u_{kl}, u_{ij}) is upper consistent. Dually, the pair (v_{ij}, v_{kl}) is lower consistent for $i \leq k$.

Proof. First, we may suppose that $i \leq k$, because, if i > k, then $u_{ij} \leq u_{kl}$ and upper consistency for (u_{kl}, u_{ij}) follows.

Applying SC3 to (a_k, a_i, b_j) , we obtain upper consistency for the pair $(a_k, a_i b_j)$.

Hence, SC5' (the modularity law 2') is applicable to the triple (a_ib_j, a_k, b_l) . It follows, the pair $(u_{kl}, u_{ij}) = (a_kb_l, a_ib_j)$ is upper consistent. The rest follows from the duality.

Recall that a decomposition $x = x_1 + \ldots + x_n$ is called *irreducible*, if $x \neq x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n$, for all *i*.

Proposition 17 Any finite sum of elements u_{ij} exists and has a decomposition of the form:

 $u = u_{i_1 j_1} + \ldots + u_{i_k j_k}, \text{ with } i_1 < \ldots < i_k; \ j_1 < \ldots < j_k.$ (21)

Dually, any finite product of v_{ij} exists and has the form

$$v = v_{i_1 j_1} \cdot \ldots \cdot v_{i_k j_k}, \text{ with } i_1 < \ldots < i_k; \ j_1 < \ldots < j_k.$$
 (22)

Proof. First, any sum of the form (21) exists. Indeed, summation can be done step by step from the left to the right in view of lemma 16 and SC2'.

The rest of the proof is done by induction on number of summands u_{ij} .

Let us add to a sum u of the form (21) some element u_{ij} . In fact, it is sufficient to prove only existence of

$$u' = u + u_{ij}.\tag{23}$$

Indeed, if it exists, then we can reduce, if necessary, the decomposition of u' to get an irreducible one. Any irreducible decomposition is automatically of the form (21), because otherwise there would exist summands u_{pq} and u_{rs} with $u_{pq} \leq u_{rs}$.

If $i \leq i_t$ and $j \geq j_t$ for some t, then $u_{ij} \geq u_{i_t j_t}$. Similarly, if $i \geq i_t$ and $j \leq j_t$ for some t, then $u_{ij} \leq u_{i_t j_t}$. In both cases, one of two elements in the sum for u', either u_{ij} or $u_{i_t j_t}$ can be absorbed by the rule (11). Then, in view of lemma 15 and the induction hypothesis the sum u' exists.

The alternative to the above inequalities is the possibility to order all summands in the decomposition (23) of u' to be of the form (21). It has already been shown that any such sum exists.

A chain is called *extended* if it contains **0** and **1** as extreme elements.

Proposition 18 Let $a_1 \ge \ldots \ge a_n$ and $b_1 \le \ldots \le b_m$ be a consistent pair of extended chains. Choose two ordered sets of indices $I = \{i_1 \le \ldots \le i_k\}, i_l \in [1, n], and J = \{j_1 \le \ldots \le j_k\}, j_l \in [1, m]$. Denote:

$$r_{IJ} = a_{i_1}(b_{j_1} + a_{i_2}) \dots (b_{j_{k-1}} + a_{i_k})b_{j_k},$$
$$s_{IJ} = a_{i_1}b_{j_1} + a_{i_2}b_{j_2} + \dots + a_{i_k}b_{j_k}.$$

Then:

- i) $r_{IJ} = s_{IJ}$,
- ii) $a_i \longrightarrow s_{IJ} \longrightarrow b_i$, for all $(i, j) \in [1, n] \times [1, m]$.

Remark. If we extend the chains of $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ by **0**, then the formula for r_{IJ} is simply the product of suitable v_{ij} 's. We allow the indices for $\{a_i\}$'s and $\{b_j\}$'s coincide for the reason of simplifying the write-down of the proof by induction.

Proof. Both r_{IJ} and s_{IJ} exist due to proposition 17.

We shall prove both statements of the proposition by simultaneous induction on k = |I| = |J|. For k = 1, $r_{IJ} = a_{i_1}b_{j_1} = s_{IJ}$ and *ii*) follows from *SC*2 and *SC*3'. Now, take the case |I| = |J| = k - 1 for granted.

Denote $I' = I \setminus i_k$, $J' = J \setminus j_k$. Evidently, $s_{I'J'} \leq b_{j_k}$. By the induction hypothesis $a_{ij} \dashrightarrow s_{I'J'}$. Then, we can apply SC5' (theorem 13) to the triple $(s_{I'J'}, a_{i_k}, b_{j_k})$ to get

$$(r_{I'J'} + a_{i_k})b_{j_k} = (s_{I'J'} + a_{i_k})b_{j_k} = s_{I'J'} + a_{i_k}b_{i_k} = s_{IJ}.$$
(24)

Let us show that the left hand side is r_{IJ} . Decompose $r_{I'J'} = qb_{j_{k-1}}$, where

$$q = a_{i_1}(b_{j_1} + a_{i_2})\dots(b_{j_{k-2}} + a_{i_{k-1}})$$

Since the second chain is extended, the last element b_m in it is **1**. Then, $q = s_{I'J''}$, where $J'' = \{j_1, \ldots, j_{k-2}, m\}$. Hence, applying the induction hypothesis to the subsets I' and J'', we have: $q \longrightarrow b_{j_{k-1}}$. Since, in addition, $a_{i_k} \leq q$ and $a_{i_k} \dashrightarrow b_{j_{k-1}}$, we can apply SC5 to the triple $(a_{i_k}, b_{j_{k-1}}, q)$:

$$r_{I'J'} + a_{i_k} = qb_{j_{k-1}} + a_{i_k} = q(b_{j_{k-1}} + a_{i_k}).$$

It follows that the left hand side of (24) is r_{IJ} . Thus, we have proven *i*).

In view of $a_{i_k} \rightarrow s_{I'J'}$ and the induction hypothesis, we can apply axioms SC2' and SC3' to get:

$$a_i \dashrightarrow (s_{I'J'} + a_{i_k}) \longrightarrow b_j$$

for all (i, j). Then, in view of $s_{I'J'} + a_{i_k} \longrightarrow b_{j_k}$ we can apply axioms SC2 and SC3 to get:

$$a_i \dashrightarrow (s_{I'J'} + a_{i_k})b_{j_k} \longrightarrow b_j.$$

By (24) the middle entry coincides with s_{IJ} . This proves *ii*) and completes the proof of the proposition.

Theorem 19 A consistent pair of chains in a set with consistencies generates in this set a distributive lattice with all elements being of the form (21).

Proof. By adding, if necessary, **0** and **1** as extreme elements of the chains, we may assume chains to be extended. Then all a_i and b_j are among u_{ij} . By proposition 17, any sum of u_{ij} exists and is of the form (21).

Let us show that any product of elements of the form (21) exists and is of the same form. By proposition 18, any sum of the form (21) can be recast as a product of v_{ij} (in the formula for r_{IJ} elements a_{i_1} and b_{j_k} can be interpreted as v_{i_1m} and v_{1j_k} respectively). Products of these elements exist again by proposition 17 and are of the form (22), which in turn, by proposition 18, can be recast as a sum of the form (21). This proves that the set of elements of the form (21) is a lattice, L. It is the lattice generated by the pair of chains.

Consider the poset $P = \sigma \cup \tau$, which is a disjoint union of two chains $\sigma = \{a_1 \geq \ldots \geq a_n\}$ and $\tau = \{b_1 \leq \ldots \leq b_m\}$, as an abstract ordered set. Let D(P) be the universal distributive lattice generated by P.

Endow P with two coinciding binary relations, which consist of all pairs (a_i, b_j) . Let U(P) be the universal lattice with consistencies generated by P, which exists by proposition 14, and $\psi : U(P) \to D(P)$ the canonical epimorphism. Let us show that it is an isomorphism.

Since U(P) is the quotient of the universal lattice L(P), it has only elements of the form (21). Hence, to show that ψ is an isomorphism it is sufficient to present a distributive lattice generated by two chains such that all elements of the form (21) are taken by ψ into distinct elements.

Such a lattice can be realized as a sublattice of subsets in the finite set of integer nodes in the rectangle $\{(x, y) \in [1, m] \times [-1, -n]\}$. Elements a_i are presented by the subsets $\{y \leq -i\}$ and b_j by the subsets $\{x \leq j\}$. Any element of the lattice is the set of integer points under a ladder descending from the point (1, -1) to the point (m, -n). Being a sublattice of the lattice of subsets, it is distributive and images under ψ of distinct elements of the form (21) are clearly distinct. Hence ψ is injective. Therefore, U(P) is distributive and, by universality property, L is distributive too.

Remark. An element x in a lattice is called *indecomposable* if x = a + b implies x = a or x = b, and *decomposable* otherwise. By another theorem of Birkhoff [4], any element in a finite distributive lattice can be uniquely decomposed into an irreducible sum of indecomposable elements. Clearly, all indecomposable elements are among u_{ij} . It is easy to see that an element u_{ij} is decomposable in L iff $u_{ij} = u_{i+1,j} + u_{i,j-1}$.

3 Perverse coherent sheaves

3.1 Grothendieck duality and the dual *t*-structure

Let k be a field of characteristic zero. In this section we assume, for simplicity, all schemes to be of finite type over k. For such a scheme X we consider the bounded derived category $\mathcal{D}(X) := \mathcal{D}^b_{\mathrm{coh}}(X)$ of coherent sheaves. We shall construct some perverse t-structures in $\mathcal{D}(X)$ using the machinery developed in the preceding section.

Let \mathcal{T} be the standard *t*-structure in $\mathcal{D}(X)$ with $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) consisting of complexes of coherent sheaves on X with trivial positive (resp. negative) degree cohomology.

By $\mathcal{H}^{i}(\mathcal{F})$ we denote the *i*-th cohomology sheaf of a complex $\mathcal{F} \in \mathcal{D}(X)$. \mathcal{O}_{X} is the structure sheaf of rings on X. We write f_{*} , $\mathcal{H}om$, etc. for derived push-forward, derived local Hom functor and other functors between derived categories. For a functor $\Phi : \mathcal{D}(X) \to \mathcal{D}(Y)$ we use notation $\mathbb{R}^{k}\Phi := \mathcal{H}^{k} \cdot \Phi$ for the corresponding cohomological functors $\mathcal{D}(X) \to \operatorname{Coh}(Y)$. Here Φ is usually the derived functor of a left exact functor $\operatorname{Coh}(X) \to \operatorname{Coh}(Y)$. If Φ is the derived one of a right exact functor $\operatorname{Coh}(X) \to \operatorname{Coh}(Y)$, then, in compliance with the tradition, we write $\mathbb{L}^{k}\Phi := \mathcal{H}^{-k} \cdot \Phi$ for the cohomology functors.

Note that for a morphism $f : X \to Y$ there is the twisted inverse image functor $f^! : \mathcal{D}^+_{qcoh}(Y) \to \mathcal{D}^+_{qcoh}(X)$ between the bounded below derived categories of quasi-coherent sheaves (see [14] and [12]). It is defined by:

$$f^!(-) = (-) \otimes f^!(\mathcal{O}_Y),$$

where $f^!(\mathcal{O}_Y)$ is defined by applying the right adjoint functor for f_* to the structure sheaf \mathcal{O}_Y . When f is proper and of finite Tor-dimension, then $f^!$ coincides with the right adjoint to f_* and $f^!$ takes $\mathcal{D}(Y)$ to $\mathcal{D}(X)$. In the particular case when f is a closed embedding, this property gives an obvious construction for $f^!$ at least locally over the base. When f is smooth of relative dimension $n, f^!$ is defined by twisting with the relative canonical class $\omega_{X/Y}$:

$$f^!(-) = (-) \otimes \omega_{X/Y}[n].$$

If f is projective, it can be decomposed into f = pi, where i is a closed embedding and p is smooth. Then f' = p'i'. Note also that f is automatically of finite Tor-dimension if Y is a smooth variety.

Also, for a proper f, there is a duality isomorphism for local $\mathcal{H}om$'s:

$$f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) = \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}), \tag{25}$$

natural in $\mathcal{F} \in \mathcal{D}^+_{qcoh}(X), \, \mathcal{G} \in \mathcal{D}^+_{qcoh}(Y).$

Denote by ω_X^{\cdot} the dualizing complex on X. By definition $\omega_X^{\cdot} = \pi_X^! \mathcal{O}_{\text{pt}}$, where $\pi_X : X \to \text{pt}$ is the projection to a point pt. For a smooth variety X of dimension $n, \, \omega_X^{\cdot} = \omega_X[n]$, where $\omega_X = \Omega_X^n$ is the canonical sheaf of differential forms of the highest degree. For general $X, \, \omega_X^{\cdot}$ is known to be in $\mathcal{D}(X)$.

For a closed embedding $i: X \to Y$ of X into a smooth Y we have:

$$\omega_X^{\bullet} = i^! \omega_Y^{\bullet} = i^! \omega_Y [\dim Y]. \tag{26}$$

The category $\mathcal{D}(X)$ possesses a contravariant involutive exact functor $D = D_X$. For an $\mathcal{F} \in \mathcal{D}(X)$, it is defined by:

$$D\mathcal{F} = D_X \mathcal{F} = \mathcal{H}om_X(\mathcal{F}, \omega_X^{\bullet}).$$
⁽²⁷⁾

D is compatible with triangulated structures, i.e. DT is naturally isomorphic to $T^{-1}D$, and D takes an exact triangle $A \to B \to C$ to the exact triangle $DC \to DB \to DA$.

If $j: U \to X$ is an open embedding, then the dualizing complex over U is obtained by restricting of the one on X:

$$\omega_U^{\bullet} = j^* \omega_X^{\bullet}. \tag{28}$$

It follows that D is local, meaning that, given an open embedding as above, we have:

$$j^* D_X = D_U j^*. (29)$$

We use notation $D^{(i)}\mathcal{F} := \mathcal{H}^i(D\mathcal{F})$ for the cohomology sheaves of $D\mathcal{F}$. D preserves $\mathcal{D}(X)$ and $D \cdot D$ is naturally isomorphic to the identity functor ([10]).

Using D, we can define another t-structure $\tilde{\mathcal{T}} = (\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 1})$ on $\mathcal{D}(X)$ by requiring:

$$\mathcal{F} \in \tilde{\mathcal{D}}^{\leq 0} \iff D\mathcal{F} \in \mathcal{D}^{\geq 0},\tag{30}$$

$$\mathcal{F} \in \tilde{\mathcal{D}}^{\geq 0} \iff D\mathcal{F} \in \mathcal{D}^{\leq 0}.$$
(31)

As D is an anti-equivalence, $\tilde{\mathcal{T}}$ is a *t*-structure. We call it the *dual t-structure*. Note that the dualizing complex is a pure object for the dual *t*-structure.

3.2 A consistent pair of chains and perverse sheaves

We denote by $\tau_{\leq k}, \tau_{\geq k}$ (resp. $\tilde{\tau}_{\leq k}, \tilde{\tau}_{\geq k}$) the truncation functors for \mathcal{T} (resp. $\tilde{\mathcal{T}}$).

The definition of $\tilde{\mathcal{T}}$ implies:

$$\tilde{\tau}_{\leq r} = D\tau_{\geq -r}D, \quad \tilde{\tau}_{\geq r} = D\tau_{\leq -r}D. \tag{32}$$

Proposition 20 Let X be a scheme of finite type over k. Suppose that $G \in \mathcal{D}^{\geq 0}(X)$, then, for any $r \in \mathbb{Z}$, $D(\tau_{\leq r}DG) \in \mathcal{D}^{\geq 0}(X)$.

Proof. In view of (29), the statement of the proposition is local. Therefore, we may assume that X is affine and (being of finite type) embeddable into a smooth variety over k. Fix such a closed embedding $i: X \to Y$ of X into a smooth variety Y of dimension l. In view of (25) and (26), for any $G \in \mathcal{D}(X)$, we have:

$$i_*DG = i_*\mathcal{H}om_X(G,\omega_X^{\bullet}) = i_*\mathcal{H}om_X(G,i^!\omega_Y^{\bullet}) = \mathcal{H}om_Y(i_*G,\omega_Y[l]).$$

Since i_* is an exact functor with respect to the standard t-structure, then

$$i_*D^{(k)}G = \mathbb{R}^k i_*DG = \mathbb{R}^k \mathcal{H}om_Y(i_*G, \omega_Y[l]).$$
(33)

Let $G \in \mathcal{D}^{\geq 0}(X)$. Then $\mathbb{R}^k i_* G$ are zero for k < 0. Consider the spectral sequence with $E_2^{mj} = \mathcal{E}xt_Y^{m+l}(\mathbb{R}^{-j}i_*G,\omega_Y)$ that converges to $\mathbb{R}^{m+j}\mathcal{H}om_Y(i_*G,\omega_Y[l]).$

Since ω_Y is locally free, then, for any coherent sheaf \mathcal{F} on Y, $\mathcal{E}xt^k_Y(\mathcal{F},\omega_Y)$ has support of codimension $\geq k$ in Y (cf. [13], p.142). By convention, the support of the zero sheaf, which is the empty set, is of infinite codimension.

As $E_2^{mj} = 0$, for j > 0, it follows from the spectral sequence that the support of $\mathbb{R}^{s}\mathcal{H}om_{Y}(i_{*}G,\omega_{Y}[l])$ is of codimension at least s+l. Now, formula (33) implies that the same restriction on codimension of support is verified for $i_*D^{(s)}G$.

As the cohomology sheaves of the truncation $\tau_{\leq r}i_*DG$ either coincide with those of i_*DG or equal zero, the codimension of the support of $\mathcal{H}^s(\tau_{\leq r}i_*DG)$ in Y is at least s+lfor any $r \in \mathbb{Z}$.

Now consider the spectral sequence with $E_2^{js} = D^j \mathcal{H}^{-s}(\tau_{\leq r} DG)$, which converges to the cohomology sheaves $\mathcal{H}^{j+s}(D(\tau_{\leq r}DG))$.

By (33) and in view of exactness of i_* we obtain:

$$i_*D^j\mathcal{H}^{-s}(\tau_{\leq r}DG) = \mathbb{R}^j\mathcal{H}om_Y(i_*\mathcal{H}^{-s}(\tau_{\leq r}DG), \omega_Y[l]) = \mathcal{E}xt_Y^{j+l}(\mathcal{H}^{-s}(\tau_{\leq r}i_*DG), \omega_Y).$$

Using the fact that $\mathcal{E}xt_V^k(\mathcal{F},\mathcal{E}) = 0$ for any locally free sheaf \mathcal{E} and for any coherent sheaf \mathcal{F} with codimension of the support greater than k, we find that the right hand side is trivial for j < -s.

Then the spectral sequence implies

$$i_*\mathcal{H}^j(D(\tau_{< r}DG)) = 0.$$

The statement of the proposition follows.

Now consider the set of t-structures in $\mathcal{D}(X)$ as a set with consistencies where consistency binary relations are defined as in section 1.3. The sequences $\mathcal{T}[r]$ and $\mathcal{T}[r]$ can be viewed as chains in this partially ordered set. An ordered pair of t-structures is said to be *T*-consistent if the pair of chains obtained from these two t-structures by applying translation functor is consistent.

Theorem 21 The pair $(\mathcal{T}, \tilde{\mathcal{T}})$ is a *T*-consistent pair of *t*-structures.

Proof. We have to show that, for any $r \in \mathbb{Z}$, the pair $(\mathcal{T}[r], \tilde{\mathcal{T}})$ is lower and upper consistent.

If, for $F \in \tilde{\mathcal{D}}^{\leq 0}$, we denote G = DF, then $G \in \mathcal{D}^{\geq 0}$. By the proposition $D\tau_{\leq r}DG = D\tau_{\leq r}F$ belongs to $\mathcal{D}^{\geq 0}$. Hence $\tau_{\leq r}\tilde{\mathcal{D}}^{\leq 0} \subset \tilde{\mathcal{D}}^{\leq 0}$, which is the condition of lower consistency. Further, in view of (32), we have:

$$\tilde{\tau}_{>r}\mathcal{D}^{\geq 0} = D\tau_{<-r}D\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0},$$

which yields the upper consistency.

Due to theorem 21, pair of chains $\mathcal{T}[r]$ and $\mathcal{T}[r]$ is consistent. Hence, by theorem 19 we obtain a distributive lattice of *t*-structures. The *t*-structures that occur in this lattice are in fact coherent versions of perverse *t*-structures (cf. [3]).

Now recall that for *t*-structures there is the naive intersection (resp. union) defined by intersecting the subcategories $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$).

Proposition 22 Any (multiple) intersection and union of t-structures in the obtained lattice is naive.

Proof. Intersection or union in consistent pairs is manifestly naive. As follows from the remark after lemma 15, the situation described in that lemma is the only place where we should check naivety of the union and intersection. So the fact comes with the following reformulation of lemma 15 for *t*-structures: if a + b exists and is naive, then the existence and naivety for one side of the equation (20) implies existence and naivety for the other side and the equation itself. This statement is again obvious.

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