Spectra of coset sigma models

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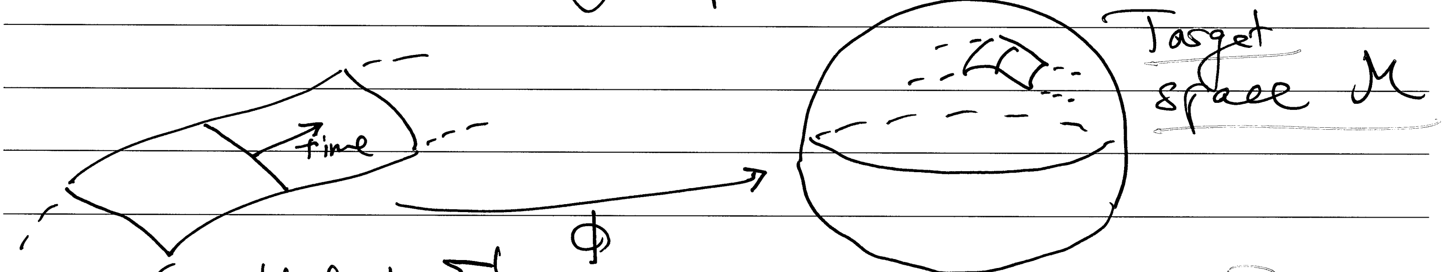
12.11.2238
1308.5981

1) General introduction

Sigma models describe the free movement of membranes of dimension d-1 in an embedding space



Worldsheet Σ of dimension d

$$\mathcal{L} \sim g_{ij}(\phi)\, d\phi^i \wedge * d\phi^j$$

(simplest case) (no B-field, etc...)

(Add Ref. on the other side)

- Non-renormalizable (pert.) for $d > 2$
- Nevertheless usefull ~~in~~ as ~~toy models for~~ effective theories for the study of
  - spontaneous breaking of chiral symmetry
  - ~~low energy limit of condensed matter models~~
  - study of asymptotic freedom
  - $N \to \infty$ limit (put the theory on S^N)
  - In 2d → study of the θ-term
- Condensed matter applications: limits of spin chains, disordered metals, superconductors ...
- String theory: sigma-models are the first ingredients.
  - require reparametrisation invariance → phys. states
  - string interactions → sum over worldsheets

(~~2d σ-models on Sym. spaces are classically integrable~~)

- Rel. between RG-flow, Ricci-flow, Perelman entropy and Zamolodchikov's c-theorem

Title: Symmetric Supercosets

2) I want to concentrate on the situation in which

- $d=2$
- $\mathcal{M}$ is a homogeneous supermanifold G/H

Motivation: ① disordered systems in condensed matter

Hamiltonian $H = H(\psi, \bar\psi, V)$ random pot., need to average over it

fermions

$$\text{Green's fkt } G(x_1, \dots, x_{2N}) = \frac{\int \mathcal{D}\bar\psi \mathcal{D}\psi \, \psi(x_1) \cdots \bar\psi(x_{2N}) \, e^{-i\bar\psi H(V)\psi}}{\int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-i\bar\psi H(V)\psi}}$$

The random potential appears in the num. and in the denominator

Trick: introduce bosonic fields $\beta, \bar\beta$

$$= \int \mathcal{D}\bar\psi \mathcal{D}\psi \, \psi(x_1) \cdots \bar\psi(x_{2N}) \, e^{-i\bar\psi H(V)\psi} \int \mathcal{D}\bar\beta \mathcal{D}\beta \, e^{-i\bar\beta H\beta}$$

$X := \begin{pmatrix} \beta \\ \psi \end{pmatrix}$ (target space susy field) ← X lives on a supermanifold

$$= \int \mathcal{D}\bar X \mathcal{D}X \, X^{a_1}(x_1) \cdots \bar X^{a_{2N}}(x_{2N}) \, e^{-i\bar X H(V) X}$$

the random pot. is only in the num.

See Pruisken's model on $SU(1,1|2)/U(1|1)\times U(1|1)$ for the int. quantum Hall effect

- These are generically non-unitary models

② The AdS/CFT correspondence

$AdS_5 \to \dfrac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ $\qquad AdS_2 \to \dfrac{PSU(1,1|2)}{U(1)\times U(1)}$

$AdS_4 \to \dfrac{OSP(6|2,2)}{U(3)\times SO(3,1)}$

$AdS_3 \to PSU(1,1|2)$

Remarks:
- Many of these models are conformal →
- These models are non-unitary
- They are LCFTs ⇒ $L_0, \bar L_0$ are non-diagonalizable
- Classically integrable, maybe even quantum

Remarks (continued)

- (Some of these models exhibit interesting dualities)
- ex: it is conjectured that the sigma model on $S^{2N-1|2N}$ is dual (strong-weak duality) to the Gross-Neveu model of 2N+2 fermions and 2N ghost bosons.
  (simplest case: $N=0 \Rightarrow$ Massless Thirring, proven)

- ~~Berkovits' idea for weak-weak dualities?~~

$$\mathcal{L}_{GN} = \frac{1}{2\pi}\left(\Psi \cdot \bar{\partial}\Psi + \bar{\Psi}\cdot\partial\bar{\Psi} + g(\Psi\cdot\bar{\Psi})^2 \right)$$

$\boxed{R^2 = \frac{1-g}{1+g}}$ related to $S^{2N-1|2N}$

~~Optional~~

- Cohomological reduction:
  G supergroup, $Q \in \mathrm{Lie}\, H$, $Q^2=0 \Rightarrow$ $\boxed{\begin{array}{l}\mathrm{Lie}\, G' = \mathrm{Coh}_Q\, \mathrm{Lie}\, G \\ \mathrm{Lie}\, H' = \mathrm{Coh}_Q\, \mathrm{Lie}\, H\end{array}}$

$G'/H' \subset G/H$ is a subsector

3) Sigma Models on symmetric superspaces: definitions & gen. properties

- G is a (basic classical) supergroup with Lie superalgebra $\mathrm{Lie}\,G$
- $\sigma : G \to G$, automorphism with $\sigma^2 = \mathrm{id}$
- $H := \{g \in G : \sigma(g) = g\}$ subgroup with algebra $\mathrm{Lie}\,H$, $\mathrm{Lie}\,G = \mathrm{Lie}\,H \oplus m$

m is a rep. of H

- $G/H = G/\sim$, $g \sim gh \ \forall h \in H$
- $(\cdot,\cdot) : \mathrm{Lie}\,G \times \mathrm{Lie}\,G \to \mathbb{C}$, inv. non-deg. (basic) bil form
- $d\mu$: left-inv. measure on G/H

Usually: $\phi : \Sigma \to G/H$ $\quad \mathcal{L} \sim g_{ij}(\phi)\, \partial\phi^i \bar\partial\phi^j$ (Euclidean signature)

$i,j = 1, \dots, \dim m$

(cumbersome to work with)

Want to formulate things in a more covariant way

Reformulation: $i : G/H \to G$ local embedding

$gH \mapsto g$

Maurer-Cartan form $\boxed{w = g^{-1}dg}$ can be used to construct the vielbein on G/H

$$\phi^* i^* \omega = \left(t_\alpha w^\alpha{}_j(\phi) + t_i E^i{}_j\right) d\phi^j (z,\bar z) = J dz + \bar J d\bar z$$

where $(t_\alpha)_{\alpha=1}^{\dim \mathrm{Lie}\,H}$ basis of $\mathrm{Lie}\,H$, $(t_i)_{i=1}^{\dim m}$ basis of m

$P : \mathrm{Lie}\,G \to m$ projector

invariance

$$\boxed{\mathcal{L} = R^2 \,(PJ, P\bar J)}$$

; R: radius, i.e. prop. constant that def. the volume

$$= \left(R^2 (t_k, t_l)\, E^k{}_i(\phi)\, E^l{}_j(\phi)\, (-1)^{|l|(|k|+|i|)}\, \partial\phi^i \bar\partial\phi^j\right)$$

4) States of the theory : [The space of fields is spanned by products of worldsheet derivatives of vertex operators, viewed as functions on G/H]

- Simplest states : Tachyonic states (vertex operators) 1:1 correspondence to elements in $L^2(G/H)$

(If we put the theory on a cylinder, these states correspond to field conf that do not depend on the spacial direction. This becomes a 1-dim problem (q.m.) with a Hamiltonian that is just the Laplace operator of G/H. States with eigenvalues prop to Cas_Λ of G)

General philosophy. Better to work on G than G/H, Ex. functions on G/H are right H-inv on G $\Longrightarrow$

!

How to generalize? : Instead of functions on G/H, we consider (right) H-vector bundles

Let W_λ be a repn of H (finite dim)

$$\Gamma_\lambda = \{ \mathcal{D} \in L_2(G) : \mathcal{D}(gh) = R_\lambda(h^{-1})\mathcal{D}(g) \ \forall h \in H \}$$

(These elements cannot be states in G/H because they transform non-trivially under H. We need to combine them with something in W_λ)

Under the left G-action

$$\Gamma_\lambda = \bigoplus_\Lambda n_{\Lambda\lambda} V_\Lambda$$

The only other element that we have are the currents and their derivatives

$$\boxed{j = PJ \,,\; \bar{j} = P\bar{J}} \qquad \left(\mathcal{L} = R^2 (j \bar{j}) \right)$$

EOM: $\partial_A \bar{j} + \bar{\partial}_{\bar{A}} j = 0$ Maurer-Cartan: $\partial_A \bar{j} - \bar{\partial}_A j = 0$ $\quad d\omega + \frac{1}{2}[\omega, \omega] = 0$

$\left(A = (1-P)J \,,\; \bar{A} = (1-P)\bar{J} \right)$

$\left(\partial_A \bar{j} = \partial \bar{j} + [A, \bar{j}] \,,\; \bar{\partial}_A j = \bar{\partial} j + [\bar{A}, j] \ldots \right)$

$\Rightarrow$ j cov. hol, $\bar{j}$ cov. anti-hol.

Def: $\boxed{j_n = \partial^{n_1} j \otimes \partial^{n_2} j \otimes \cdots \otimes \partial^{n_r} j}$

Let $p_\mu : m \otimes \cdots \otimes m \to W_\mu$ be a proj.
And $C_{\Lambda \mu \bar{\mu}} : W_\mu \otimes W_{\bar{\mu}} \to \mathbb{C}$

$$\boxed{\underbrace{\Phi_{(\Lambda, \lambda, \mu, \bar{\mu})}}_{=: \Lambda} := C_{\Lambda \mu \bar{\mu}} \Big(\underbrace{\mathcal{D}_{\Lambda \lambda}^{\,i(\phi)} \otimes p_\mu(j_n) \otimes p_{\bar{\mu}}(\bar{j}_{\bar{n}})}_{\text{transforms triv. under } H} \Big)}$$

(These are the most general states that we consider)

(This allows us to compute the partition functions of sigma models in the ∞-volume limit (semiclassical))

9\. In that limit • $\mathcal{D}_{\Lambda \lambda}$ has $h = \bar{h} = 0$

• j has $h = 1, \bar{h} = 0$, $\bar{j}$ has $h = 0, \bar{h} = 1$

• Derivatives increase h, resp. $\bar{h}$ by 1

5) Example: spectra of S^{N-1} models (N even)

① first count the number of states that one can build from the "currents" j and covariant derivatives

$$Z_j^{(y,q)} = \prod_{n=1}^{\infty} \frac{1}{\det(1 - y q^n)} \,, \quad Z_J = \ldots$$

② Count the number of bundles (compact)

$$Z_{L^2_2(SO(N))}(x,y) = \sum_\Lambda{}' \, \underset{\text{left}}{sd_\Lambda(x)} \, \underset{\text{right}}{sd_\Lambda(y)}$$

$\Lambda \leftarrow$ tensor

③ $$Z = \left(Z_{L^2_2(SO(N))}(x,y) \, Z_j(y,q) \, Z_J(y,\bar q) \right)^{SO(N-1)\text{-inv.}}$$

$\cancel{= \sum_{\Lambda \lambda}' sd_\Lambda(x)\, n_{\Lambda\lambda}\, sb_\lambda(y)}$

a) $$\sum_\Lambda{}' \, sd_\Lambda(x) \, \underset{\text{Schur}}{s_\Lambda(y)} = \frac{\prod_{i \leq j = 1}^{\infty} (1 - y_i y_j)}{\prod_{j=1}^{\infty} \det(1 - x y_j)}$$

$x = (x_1, \ldots,)$

b) $$\sum_\Lambda{}' \, s_\Lambda(x) \, s_\Lambda(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}$$

c) $s_\mu(x) \, s_\nu(x) = c^\lambda_{\mu\nu} \, s_\lambda(x)$

d) $\sum_{\mu\nu} c^\lambda_{\mu\nu} \, s_\mu(x) \, s_\nu(y) = s_\lambda(x, y)$

e) $N_{\lambda\mu\nu} = \sum_{\alpha\beta\gamma} c^\lambda_{\alpha\beta} \, c^\mu_{\beta\gamma} \, c^\nu_{\alpha\gamma}$

f) $n_{\Lambda\lambda} = \sum_{\ell=0}^{\infty} c^\Lambda_{\lambda(\ell)}$

write separatly

$X = \begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_N & & & \\ & & & x_1^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & x_N^{-1} \end{pmatrix}$

N arbitrary

$N_{\lambda\mu\nu} = \dim(W_\lambda \otimes W_\mu \otimes W_\nu)^{\text{inv}}$
$= \dim \mathrm{Hom}(W_\lambda \otimes W_\mu \otimes W_\nu, \mathbb{C})$

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$$Z^{free}_{inv} = \left(\sum_{\lambda\lambda\mu\bar\mu}' sd_\Lambda(x)\, n_{\Lambda\lambda}\, sd\lambda(y) \frac{1}{\left|\prod_{i\le j\ge 1}^{\infty}(1-q^{i+j})\right|^2} s_{\mu}(y) S_\mu(q) s_{\bar\mu}(y) S_{\bar\mu}(\bar q) \right)$$

$N\lambda\mu\bar\mu$

a), f), e)

$$= \sum_{\substack{\Lambda\lambda\mu\bar\mu\\ \alpha\beta\gamma\ell}} sd_\Lambda(x)\, C^\Lambda_{\lambda(\ell)} \left|\frac{1}{\prod_{i\le j\ge 1}^{\infty}(1-q^{i+j})}\right|^2 C^\lambda_{\alpha\beta}\, C^\mu_{\beta\gamma} C^{\bar\mu}_{\alpha\gamma}\, S_\mu(q) S_{\bar\mu}(\bar q)$$

c)

$$= \sum_{\substack{\Lambda\lambda\\ \alpha\beta\gamma\ell}} sd_\Lambda(x)\, C^\Lambda_{\lambda(\ell)} \left|\frac{1}{\prod_{i\le j\ge 1}^{\infty}(1-q^{i+j})}\right|^2 S_\alpha(q) S_\gamma(q) S_\beta(\bar q) S_\gamma(\bar q)\, C^\lambda_{\alpha\beta}$$

$$= \sum_{\Lambda\lambda\ell}' sd_\Lambda(x)\, C^\Lambda_{\lambda(\ell)} \left|\frac{1}{\prod_{i\le j\ge 1}^{\infty}(1-q^{i+j})}\right|^2 S_\lambda(q\bar q) \prod_{i,j\ge 1}^{\infty}(1-q^i\bar q^j)$$

d) b)

* Introduce a dummy variable u

$$= \lim_{u\to 1} \sum_{\Lambda\lambda\ell} \frac{\prod_{i,j\ge 1}^{\infty}(1-q^i\bar q^j)}{\left|\prod_{i,j\ge 1}^{\infty}(1-q^{i+j})\right|^2} sd_\Lambda(x)\, C^\Lambda_{\lambda(\ell)} \frac{u^\ell}{S_{(\ell)}(u)} S_\lambda(q\bar q)$$

$$= \lim_{u\to 1} \sum_\Lambda \frac{\prod_{i,j\ge 1}^{\infty}(1-q^i\bar q^j)}{\left|\prod_{i\le j\ge 1}^{\infty}(1-q^{i+j})\right|^2} sd_\Lambda(x)\, S_\Lambda(u, q, \bar q)$$

$$= \underbrace{\lim_{u\to 1} \frac{1-u^2}{\det(1-xu)}}_{\text{naively zero}} \prod_{n\ge 1}^{\infty} \frac{|1-q^n|^2}{|\det(1-xq^n)|^2}$$

b)

⇒ first expand denominator ⇒ gives $L^2(S^{N-1})$

optional, say it is an exercise for the reader

Remarks:

- Have an independent derivation ✓
- Extended to supersphere $S^{N-1|2M}$ $N=1$ worldsheet susy
- Strictly $R\to\infty$ limit!
  for the circle S^1 we know that we have winding modes ⇒ at $R\to\infty$ they get ∞-energy and decouple
- unclear modular invariance

6) Perturbation theory

We want to find the spectra also for finite values of R.

Issue: Conformal-invariance. Not all sigma models are conf. invariant.
At 1-loop a necessary and sufficient condition is that G be Ricci-flat $\Leftrightarrow$ the Killing form on G vanishes. $\longrightarrow$

$\boxed{2\,\mathrm{Cas}^m_H = \mathrm{Cas}^{adj}_G}$

$\mu \frac{\partial g_{ij}}{\partial \mu} = \frac{R_{ij}}{2} \Rightarrow \mu \frac{dR^2}{d\mu} = 2\,\mathrm{Cas}^m_H$

For groups this implies that G is flat.
However, this is not true for supergroups.

Known examples: . $S^{2N-1|2N} = \frac{OSp(2N+2|2N)}{OSp(2N+1|2N)}$) real super grassmanians

higher loops

. complex supergrassmanians $\longrightarrow$

:

We concentrate on ~~...~~ conformal examples at first order in $\frac{1}{R^2}$ (we also did non-conf + $N=1$ susy)

Background field method: $\boxed{g = g_0 e^{i\phi}}$ $\quad \phi \in m$

1-loop: (① expand the action
② expand the sections on the bundles
③ expand the currents) don't write

① $S = \frac{R^2}{2}\int \frac{d^2z}{4}(j\bar{j}) = \frac{R^2}{2}\int \frac{d^2z}{4}\Big[\underbrace{(\partial\phi, \bar\partial\phi)}_{\text{bunch of free bosons + sympl. fermions}} + \underbrace{\tfrac{1}{3}([\phi,\partial\phi],[\phi,\bar\partial\phi])}_{=:\Omega} + \dots\Big]$

$\langle \phi(u,\bar u) \otimes \phi(v,\bar v)\rangle_0 = -\frac{t_i \otimes t^i}{R^2} \log\left|\frac{u-v}{\varepsilon}\right|^2$

$\uparrow$ u-v reg.

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ref: Pilch & Schellekens, Formulae for the Eigenvalues of the Laplacian on tensor harmonics on sym. coset spaces
J. Math. Phys. 25: 3455, 1984

(2) $D_{\Lambda\lambda}(g) \mapsto D_{\Lambda\lambda}(g_0) + i L_\Lambda(\mathrm{Ad}_{g_0}\phi(z,\bar z)) \cdot D_{\Lambda\lambda}(g_0) + \dots$

(3) $j \to i\partial\phi + \dots \quad {}_1 J_n^0 = i\partial^{n_1+1}\phi \otimes \dots \otimes i\partial^{n_r+1}\phi \quad n_1 \geq \dots \geq n_r \geq 0$

$$\Phi_{\vec\Lambda}(\Lambda,\lambda,\mu,\bar\mu)(z,\bar z|g_0) = d_{\lambda\mu\bar\mu}\left[D_{\Lambda\lambda}(g_0) + i L_\Lambda(\mathrm{Ad}_{g_0}\phi) D_{\Lambda\lambda}(g_0)\right] \otimes j^0_{\vec n}(z) \otimes \bar j^0_{\vec{\bar n}}(\bar z)$$

$d_{\lambda\mu\bar\mu} \leftarrow C_{\lambda\mu\bar\mu}(1\otimes\rho_\mu\otimes 1)(1\otimes 1\otimes\rho_{\bar\mu})$

We compute the two point correlation functions

Reminder: $\langle A(z) B(w)\rangle = \dfrac{\text{const}}{(z-w)^{2h}(\bar z-\bar w)^{2\bar h}}$ $\quad h = h_0 + \delta h$, $\bar h = \bar h_0 + \delta\bar h$ $\quad \left[\delta h = \delta\bar h\right]$ what was local remains local

+ non-log. contributions $\quad = \dfrac{\text{const}}{(z-w)^{2h_0}(\bar z-\bar w)^{2\bar h_0}}\left(1 + 2\delta h \log\left|\dfrac{1}{z-w}\right|^2\right)$

$$\Rightarrow \langle \phi_\Lambda(u)\otimes\phi_\Sigma(v)\rangle_1 = \langle 2\delta h \cdot \Phi_\Lambda(u)\otimes\phi_\Sigma(v)\rangle_0 \log\left|\frac{1}{u-v}\right|^2$$

$\uparrow$ 1-loop $\quad + \dots$

Corrections not prop. to zero-order term don't contribute to anom. dim.

Optional

$$\langle \phi_{\vec\Lambda}(u,\bar u)\otimes\phi_{\vec\Sigma}(v,\bar v)\rangle = \int_{G/H} d\mu(g_0H)\langle \phi_{\vec\Lambda}(u\bar u|g_0)\otimes\Phi_\Sigma(v\bar v|g_0)e^{-\frac{i}{\pi}S_{int}}\rangle$$

0) $\displaystyle\int_{G/H} d\mu(g_0H)\left(d_{\lambda\mu\bar\mu}\otimes d_{\xi\eta\bar\eta}\right)\langle D_{\Lambda\lambda}(g_0)\otimes j_n(u)\otimes\bar j_{\bar n}(\bar u)\otimes D_{\Sigma\xi}(v|g_0) \otimes j_{n'}(v)\otimes\bar j_{\bar n'}(\bar v)\rangle_0$

1) $\displaystyle\int_{G/H} d\mu(g_0H)\left\{(d_{\lambda\mu\bar\mu}\otimes d_{\xi\eta\bar\eta})\left[-\int_{C_\varepsilon}\frac{d^2z}{\pi}\langle D_{\Lambda\lambda}(g_0)\otimes j^0_n(u)\otimes\bar j^0_{\bar n}(\bar u)\otimes D_{\Sigma\xi}(g_0)\otimes j_{n'}(v)\otimes\bar j_{\bar n'}(\bar v)\,\mathcal{S}(z\bar z)\rangle\right.\right.$

$\left.\left. - \beta\langle :L_\Lambda(\mathrm{Ad}_{g_0}\phi(u,\bar u)) D_{\Lambda\lambda}(g_0)\otimes j^0_n(u)\otimes\bar j^{+}_{\bar n}(\bar u)\otimes D_{\Sigma\xi}(g_0)\otimes j^0_{n'}\otimes\bar j^{+0}_{\bar n'}(\bar v)\rangle\right]\right\}$

Mixex

0) write it down explicitly

1) 1-term: insertion of the perturbing field Ω 1a)

acts on currents

2-term: covariant derivatives on the sections 1b)

acts on $J^{\circ}\otimes\bar{J}^{\circ}$ acts on $\bar{J}^{\circ}$ acts on $D\lambda$ (i.e. sections)

$$1a) \Rightarrow \frac{1}{R^2}\left(\mathrm{Cas}^{D}_{H} - \mathrm{Cas}^{R}_{H} - \mathrm{Cas}^{L}_{H}\right)$$

acts on J°

$$1b) \Rightarrow \frac{1}{R^2}\left(\mathrm{Cas}_{G} - \mathrm{Cas}^{D}_{H}\right)$$ acts on $D\lambda$

$$\langle \phi_{\vec{\Lambda}} \otimes \phi_{\vec{\Xi}} \rangle = 2\,\frac{1}{2R^2}\,\langle \underbrace{\left(\mathrm{Cas}^{\Lambda}_{G} - \mathrm{Cas}^{M}_{H} - \mathrm{Cas}^{\bar{\mu}}_{H}\right)}_{\text{operator anomalous dimensions.}} \Phi_{\vec{\Lambda}} \otimes \Phi_{\Xi} \rangle_0$$

- We can plug this formula in the partition functions! (They are perfectly adapted for it!)

Also:
- non-conformal case $(J\bar{J})$ is not marginal anymore $+(r+\bar{r})\,\mathrm{Cas}^{m}_{H}$ ✓
- $N=1$ ~~su~~ worldsheet susy ✓

7) Outlook: future research.
1. $\mathbb{Z}_{2N}$ cosets
2. string theory $\Rightarrow$ pure spinors $AdS_2\times S^2$ case
3. Modular invariance, "winding"
4. higher-loops (difficult)