Stringy Differential Geometry and
\[ \mathcal{N} = 2 \quad D = 10 \]  Supersymmetric Double Field Theory

朴廷塽  Jeong-Hyuck Park  박정혁

Sogang University, Seoul

東京 2014年 2月 4日
Prologue

Without vector notation, Maxwell’s original equations consisted of eight (or twenty) formulas.

It was the rotational $\text{SO}(3)$ or Lorentz $\text{SO}(1, 3)$ symmetry that reorganized them into four or two compact equations.

This talk aims to show that

Type IIA & IIB supergravities may undergo an analogous ‘simplification’ and ‘unification’, restructured by ‘Stringy Differential Geometry’,

in the name of $\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory.
Without vector notation, Maxwell’s original equations consisted of eight (or twenty) formulas.

It was the rotational $SO(3)$ or Lorentz $SO(1, 3)$ symmetry that reorganized them into four or two compact equations.

This talk aims to show that

_Type IIA & IIB supergravities may undergo an analogous ‘simplification’ and ‘unification’,

restructured by ‘Stringy Differential Geometry’,

_in the name of $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory._
Without vector notation, Maxwell’s original equations consisted of eight (or twenty) formulas.

It was the rotational $\text{SO}(3)$ or Lorentz $\text{SO}(1, 3)$ symmetry that reorganized them into four or two compact equations.

This talk aims to show that Type IIA & IIB supergravities may undergo an analogous ‘simplification’ and ‘unification’, restructured by ‘Stringy Differential Geometry’, in the name of $\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory.
Without vector notation, Maxwell’s original equations consisted of eight (or twenty) formulas.

It was the rotational $\mathbf{SO}(3)$ or Lorentz $\mathbf{SO}(1, 3)$ symmetry that reorganized them into four or two compact equations.

This talk aims to show that Type IIA & IIB supergravities may undergo an analogous ‘simplification’ and ‘unification’, restructured by ‘Stringy Differential Geometry’, in the name of $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory.
Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  JHEP 1104:014 arXiv:1011.1324

- Double field formulation of Yang-Mills theory

- Stringy differential geometry, beyond Riemann

- Incorporation of fermions into double field theory
  JHEP 1111:025 arXiv:1109.2035

- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity

- Ramond-Ramond Cohomology and O(D,D) T-duality
  JHEP 1209:079 arXiv:1206.3478

- Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2 \ D = 10$
  Supersymmetric Double Field Theory

- Comments on double field theory and diffeomorphisms
  JHEP 1306:098 arXiv:1304.5946

- Covariant action for a string in doubled yet gauged spacetime
Related works on U-duality


- **M-theory and F-theory from a Duality Manifest Action**
  with Chris Blair and Emanuel Malek, to appear in JHEP arXiv:1311.5109
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

- Diffeomorphism: $\partial_\mu \mapsto \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu\nu} = 0$, $\Gamma^\lambda_{[\mu\nu]} = 0 \rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

- Curvature: $[\nabla_\mu, \nabla_\nu] \mapsto R_{\kappa\lambda\mu\nu} \mapsto R$

On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

$\nabla_\lambda g_{\mu\nu} = 0, \quad \Gamma^\lambda_{[\mu\nu]} = 0 \quad \longrightarrow \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$

On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

- **Diffeomorphism:** $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu\nu} = 0$, $\Gamma^\lambda_{[\mu\nu]} = 0 \rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

- **Curvature:** $[\nabla_\mu, \nabla_\nu] \rightarrow R^\kappa_{\lambda\mu\nu} \rightarrow R$

On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

- **Diffeomorphism:** $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu\nu} = 0, \ \Gamma^\lambda_{[\mu\nu]} = 0 \rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

- **Curvature:** $[\nabla_\mu, \nabla_\nu] \rightarrow R_{\kappa\lambda\mu\nu} \rightarrow R$

On the other hand, string theory puts $g_{\mu\nu}, B_{\mu\nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

- Diffeomorphism: $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu\nu} = 0, \Gamma^\lambda_{[\mu\nu]} = 0 \rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

- Curvature: $[\nabla_\mu, \nabla_\nu] \rightarrow R_{\kappa\lambda\mu\nu} \rightarrow R$

On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, \( g_{\mu\nu} \).

Diffeomorphism: \( \partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu} \)

\[ \nabla_{\lambda} g_{\mu\nu} = 0, \quad \Gamma_{[\mu\nu]}^\lambda = 0 \rightarrow \Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu}) \]

Curvature: \([\nabla_{\mu}, \nabla_{\nu}] \rightarrow R_{\kappa\lambda\mu\nu} \rightarrow R\]

On the other hand, string theory puts \( g_{\mu\nu}, B_{\mu\nu} \) and \( \phi \) on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
In Riemannian geometry, the fundamental object is the metric, $g_{\mu \nu}$.

- Diffeomorphism: $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu \nu} = 0$, $\Gamma^\lambda_{[\mu \nu]} = 0 \rightarrow \Gamma^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \rho} \left( \partial_\mu g_{\nu \rho} + \partial_\nu g_{\mu \rho} - \partial_\rho g_{\mu \nu} \right)$

- Curvature: $[\nabla_\mu, \nabla_\nu] \rightarrow R_{\kappa \lambda \mu \nu} \rightarrow R$

On the other hand, string theory puts $g_{\mu \nu}$, $B_{\mu \nu}$ and $\phi$ on an equal footing, as they form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
The low energy effective action of $g_{\mu\nu}$, $B_{\mu\nu}$, $\phi$ is well known in terms of Riemannian geometry

$$S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g} e^{-2\phi} \left( R_g + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right).$$

Diffeomorphism and $B$-field gauge symmetry are manifest,

$$x^\mu \rightarrow x^\mu + \delta x^\mu, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu.$$

Though not manifest, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. Buscher
The low energy effective action of $g_{\mu\nu}$, $B_{\mu\nu}$, $\phi$ is well known in terms of Riemannian geometry

$$S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g} e^{-2\phi} \left( R_g + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right).$$

Diffeomorphism and $B$-field gauge symmetry are manifest,

$$x^\mu \to x^\mu + \delta x^\mu, \quad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu.$$

Though not manifest, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. Buscher
The low energy effective action of $g_{\mu\nu}, B_{\mu\nu}, \phi$ is well known in terms of Riemannian geometry

$$S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g} e^{-2\phi} \left( R_g + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right).$$

Diffeomorphism and $B$-field gauge symmetry are manifest,

$$x^\mu \rightarrow x^\mu + \delta x^\mu, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu.$$

Though not manifest, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. Buscher
Redefine the dilaton,

\[ e^{-2d} = \sqrt{-g} e^{-2\phi} \]

Set a \((D + D) \times (D + D)\) symmetric matrix, \textbf{Duff}

\[
\mathcal{H}_{AB} = \begin{pmatrix}
g^{-1} & -g^{-1}B \\
Bg^{-1} & g - Bg^{-1}B
\end{pmatrix}
\]

Hereafter, \(A, B, \ldots\) : ‘doubled’ \((D + D)\)-dimensional vector indices, with \(D = 10\) for SUSY.
T-duality is realized by an $O(D, D)$ rotation, Tseytlin, Siegel

$$\mathcal{H}_{AB} \rightarrow M^C_A M^D_B \mathcal{H}_{CD}, \quad d \rightarrow d,$$

where

$$M \in O(D, D).$$
\( \Omega(D, D) \) metric,

\[
\mathcal{J}_{AB} := \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

freely raises or lowers the \((D + D)\)-dimensional vector indices.
Hull and Zwiebach, later with Hohm

$$S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d),$$

where

$$L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left(4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}.$$ 

Spacetime is formally doubled, $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, 2, \cdots, D+D$.

Yet, Double Field Theory (for NS-NS sector) is a $D$-dimensional theory written in terms of $(D + D)$-dimensional language, i.e. tensors.

All the fields MUST live on a $D$-dimensional null hyperplane or ‘section’, $\Sigma_D$. 

Supersymmetry Double Field Theory
Hull and Zwiebach, later with Hohm

\[ S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d), \]

where

\[ L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}. \]

Spacetime is formally doubled, \( y^A = (\tilde{x}_\mu, x^\nu) \), \( A = 1, 2, \cdots, D+D \).

Yet, Double Field Theory (for NS-NS sector) is a \( D \)-dimensional theory written in terms of \((D + D)\)-dimensional language, i.e. tensors.

All the fields MUST live on a \( D \)-dimensional null hyperplane or ‘section’, \( \Sigma_D \).
Hull and Zwiebach, later with Hohm

\[ S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d), \]

where

\[ L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}. \]

Spacetime is formally doubled, \( y^A = (\tilde{x}_\mu, x^\nu), A = 1, 2, \cdots, D+D. \)

Yet, Double Field Theory (for NS-NS sector) is a \( D \)-dimensional theory written in terms of \( (D+D) \)-dimensional language, i.e. tensors.

All the fields MUST live on a \( D \)-dimensional null hyperplane or ‘section’, \( \Sigma_D. \)
By stating DFT lives on a $D$-dimensional null hyperplane, we mean that, the $O(D, D)$ d’Alembert operator is trivial, acting on arbitrary fields as well as their products:

$$\partial_A \partial^A \Phi = 2 \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0 : \text{section condition}$$
What does $O(D, D)$ do in Double Field Theory?

- $O(D, D)$ rotates the $D$-dimensional null hyperplane where DFT lives.
- *A priori*, the $O(D, D)$ structure in DFT is a ‘meta-symmetry’ or ‘hidden symmetry’ rather than a Noether symmetry.
- Only after dimensional reductions,
  
  $$D = d + n \implies d,$$
  
  it can generate a Noether symmetry,

  $$O(n, n)$$

  which is a subgroup of $O(D, D)$ and ‘enhanced’ from $O(n)$. 
What does $O(D, D)$ do in Double Field Theory?

- $O(D, D)$ rotates the $D$-dimensional null hyperplane where DFT lives.

- *A priori*, the $O(D, D)$ structure in DFT is a ‘meta-symmetry’ or ‘hidden symmetry’ rather than a Noether symmetry.

- Only after dimensional reductions,

  $$D = d + n \implies d,$$

  it can generate a Noether symmetry,

  $$O(n, n)$$

  which is a subgroup of $O(D, D)$ and ‘enhanced’ from $O(n)$. 
What does $O(D, D)$ do in Double Field Theory?

- $O(D, D)$ rotates the $D$-dimensional null hyperplane where DFT lives.

- *A priori*, the $O(D, D)$ structure in DFT is a ‘meta-symmetry’ or ‘hidden symmetry’ rather than a Noether symmetry.

- Only after dimensional reductions,

$$ D = d + n \implies d , $$

it can generate a Noether symmetry,

$$ O(n, n) $$

which is a subgroup of $O(D, D)$ and ‘enhanced’ from $O(n)$. 
Closed string

\[ X_L(\sigma^+) = \frac{1}{2}(x + \tilde{x}) + \frac{1}{2}(p + w)\sigma^+ + \cdots, \]

\[ X_R(\sigma^-) = \frac{1}{2}(x - \tilde{x}) + \frac{1}{2}(p - w)\sigma^- + \cdots. \]

Under T-duality,

\[ X_L + X_R \rightarrow X_L - X_R, \]

such that

\[ (x, \tilde{x}, p, w) \rightarrow (\tilde{x}, x, w, p). \]

Level matching condition for the massless sector,

\[ p \cdot w \equiv 0 \iff \partial_A \partial_A' = 2 \frac{\partial^2}{\partial x^\mu \partial \tilde{x}_\mu} \equiv 0. \]
Closed string

\[ X_L(\sigma^+) = \frac{1}{2}(x + \tilde{x}) + \frac{1}{2}(p + w)\sigma^+ + \cdots , \]
\[ X_R(\sigma^-) = \frac{1}{2}(x - \tilde{x}) + \frac{1}{2}(p - w)\sigma^- + \cdots . \]

Under T-duality,

\[ X_L + X_R \rightarrow X_L - X_R , \]

such that

\[ (x, \tilde{x}, p, w) \rightarrow (\tilde{x}, x, w, p) . \]

Level matching condition for the massless sector,

\[ p \cdot w \equiv 0 \iff \partial_A \partial^A = 2 \frac{\partial^2}{\partial x^\mu \partial \tilde{x}_\mu} \equiv 0 . \]
Up to $O(D, D)$ rotation, we may further choose to set
\[ \frac{\partial}{\partial \bar{x}_\mu} \equiv 0. \]

Then DFT reduces to the effective action:
\[ S_{\text{DFT}} \Rightarrow S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g}e^{-2\phi} \left( R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right). \]
Up to $O(D, D)$ rotation, we may further choose to set
\[
\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0.
\]

Then DFT reduces to the effective action:
\[
S_{\text{DFT}} \Rightarrow S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g} e^{-2\phi} \left( R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right).
\]
Thus, in the DFT formulation of the effective action by Hull, Zwiebach & Hohm the $O(D,D)$ T-duality structure is manifest.

What about the diffeomorphism and the B-field gauge symmetry?
Thus, in the DFT formulation of the effective action by Hull, Zwiebach & Hohm the $O(D, D)$ T-duality structure is manifest.

What about the diffeomorphism and the B-field gauge symmetry?
Introducing a unifying \((D + D)\)-dimensional parameter,

\[ X^A = (\Lambda_\mu, \delta x^\nu) \]

it is possible to spell a unifying transformation rule, up to the section condition,

\[
\delta_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_C \mathcal{H}^C_{B]} + 2 \partial_{[B} X_C \mathcal{H}^C_{A]} ,
\]

\[
\delta_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) .
\]

In fact, these coincide with the generalized Lie derivative,

\[
\delta_X \mathcal{H}_{AB} = \hat{\mathcal{L}}_X \mathcal{H}_{AB} , \quad \delta_X (e^{-2d}) = \hat{\mathcal{L}}_X (e^{-2d}) = -2(\hat{\mathcal{L}}_X d) e^{-2d} .
\]
Introducing a unifying \((D + D)\)-dimensional parameter,

\[ X^A = (\Lambda_\mu, \delta x^\nu) \]

it is possible to spell a unifying transformation rule, up to the section condition,

\[ \delta_X \mathcal{H}_{AB} = X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A}X_{C]} \mathcal{H}^C_B + 2\partial_{[B}X_{C]} \mathcal{H}_A^C \]

\[ \delta_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) . \]

In fact, these coincide with the generalized Lie derivative,

\[ \delta_X \mathcal{H}_{AB} = \hat{L}_X \mathcal{H}_{AB} , \quad \delta_X (e^{-2d}) = \hat{L}_X (e^{-2d}) = -2(\hat{L}_X d) e^{-2d} . \]
**Definition** Siegel, Courant, Grana ...

\[ \hat{\mathcal{L}}_{X} T_{A_{1}...A_{n}} := X^{B} \partial_{B} T_{A_{1}...A_{n}} + \omega \partial_{B} X^{B} T_{A_{1}...A_{n}} + \sum_{i=1}^{n} (\partial_{A_{i}} X_{B} - \partial_{B} X_{A_{i}}) T_{A_{1}...A_{i-1}B} A_{i+1}...A_{n}. \]

*cf.* ordinary one in Riemannian geometry,

\[ \mathcal{L}_{X} T_{A_{1}...A_{n}} := X^{B} \partial_{B} T_{A_{1}...A_{n}} + \omega \partial_{B} X^{B} T_{A_{1}...A_{n}} + \sum_{i=1}^{n} \partial_{A_{i}} X^{B} T_{A_{1}...A_{i-1}BA_{i+1}...A_{n}}. \]

Commutator of the generalized Lie derivatives,

\[ [\hat{\mathcal{L}}_{X}, \hat{\mathcal{L}}_{Y}] \equiv \hat{\mathcal{L}}_{[X, Y]_{C}}, \]

where \([X, Y]_{C}\) denotes the C-bracket,

\[ [X, Y]^{A}_{C} := X^{B} \partial_{B} Y^{A} - Y^{B} \partial_{B} X^{A} + \frac{1}{2} Y^{B} \partial^{A} X_{B} - \frac{1}{2} X^{B} \partial^{A} Y_{B}. \]
Definition

Siegel, Courant, Grana ...

\[ \hat{L}_X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^{n} (\partial_A X_B - \partial_B X_A) T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

cf. ordinary one in Riemannian geometry,

\[ L_X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^{n} \partial_A X^B T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

Commutator of the generalized Lie derivatives,

\[ [\hat{L}_X, \hat{L}_Y] \equiv \hat{L}_{[X, Y]_C}, \]

where \([X, Y]_C\) denotes the C-bracket,

\[ [X, Y]_C^A := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B. \]
Definition  \[ \hat{\mathcal{L}} X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^n \left( \partial_{A_i} X_B - \partial_B X_{A_i} \right) T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

cf. ordinary one in Riemannian geometry,

\[ \mathcal{L} X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^n \partial_{A_i} X^B T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

Commutator of the generalized Lie derivatives,

\[ [\hat{\mathcal{L}} X, \hat{\mathcal{L}} Y] := \hat{\mathcal{L}} [X, Y], \]

where \([X, Y]_c\) denotes the C-bracket,

\[ [X, Y]_c^A := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B. \]
Diffeomorphism & \( B \)-field gauge symmetry

Brute force computation can show that

\[
\hat{L}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_C \mathcal{H}^C_{B]} + 2 \partial_{[B} X_C \mathcal{H}_{A]}^C,
\]

\[
\hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}),
\]

are symmetry of the action by Hull, Zwiebach & Hohm

\[
S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d),
\]

\[
L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}.
\]

This expression may be analogous to the case of writing the Riemannian scalar curvature, \( R \), in terms of the metric and its derivative.
Brute force computation can show that

\[ \hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A}X_{C]} \mathcal{H}^C_B + 2\partial_{[B}X_{C]} \mathcal{H}^C_A , \]

\[ \hat{\mathcal{L}}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}) , \]

are symmetry of the action by Hull, Zwiebach & Hohm

\[ S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d) , \]

\[ L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} . \]

This expression may be analogous to the case of writing the Riemannian scalar curvature, \( R \), in terms of the metric and its derivative.
Brute force computation can show that

\[ \hat{L}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_{C]} \mathcal{H}^C_B + 2 \partial_{[B} X_{C]} \mathcal{H}_{A C}, \]

\[ \hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}), \]

are symmetry of the action by Hull, Zwiebach & Hohm

\[ S_{\text{DFT}} = \int_{\Sigma^D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d), \]

\[ L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left( 4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}. \]

The underlying differential geometry is missing here.
Diffeomorphism & $B$-field gauge symmetry

Brute force computation can show that

$$\hat{L}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2 \partial_{[A} X_{C]} \mathcal{H}^C_B + 2 \partial_{[B} X_{C]} \mathcal{H}_A^C,$$

$$\hat{L}_X (e^{-2d}) \equiv \partial_A (X^A e^{-2d}),$$

are symmetry of the action by Hull, Zwiebach & Hohm

$$S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} L_{\text{DFT}}(\mathcal{H}, d),$$

$$L_{\text{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB} \left(4 \partial_A \partial_B d - 4 \partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right)$$

$$+ 4 \partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB}.$$

The underlying differential geometry is missing here.
Stringy differential geometry and Supersymmetric Double Field Theory (SDFT)

The remaining of this talk is structured to explain our works:

[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078, 1304.5946]

- Proposal of a underlying stringy differential geometry for DFT
- The full order construction of \( \mathcal{N} = 2 \ D = 10 \) SDFT
  - which ‘unifies’ IIA and IIB SUGRAs and ‘contains’ more.
- The reduction of SDFT to ordinary SUGRA via gauge fixing.
Stringy differential geometry and
Supersymmetric Double Field Theory (SDFT)

The remaining of this talk is structured to explain our works:
[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078, 1304.5946]

- Proposal of a underlying **stringy differential geometry for DFT**

- The *full order* construction of \( \mathcal{N} = 2 \ D = 10 \) SDFT

  which ‘unifies’ IIA and IIB SUGRAs and ‘contains’ *more*.

- The reduction of SDFT to ordinary SUGRA *via* gauge fixing.
Stringy differential geometry and
Supersymmetric Double Field Theory (SDFT)

The remaining of this talk is structured to explain our works:
[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078, 1304.5946]

- Proposal of a underlying stringy differential geometry for DFT
- The full order construction of $\mathcal{N} = 2 \ D = 10$ SDFT

which ‘unifies’ IIA and IIB SUGRAs and ‘contains’ more.

- The reduction of SDFT to ordinary SUGRA via gauge fixing.
The remaining of this talk is structured to explain our works:
[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078, 1304.5946]

- Proposal of a underlying stringy differential geometry for DFT

- The full order construction of $\mathcal{N} = 2 \ D = 10$ SDFT

  which ‘unifies’ IIA and IIB SUGRAs and ‘contains’ more.

- The reduction of SDFT to ordinary SUGRA via gauge fixing.
Symmetries of $\mathcal{N} = 2 \ D = 10$ SDFT

- **$O(D, D)$ T-duality: Meta-symmetry**

- **Gauge symmetries**
  1. DFT-diffeomorphism (generalized Lie derivative)
  2. A *pair* of local Lorentz symmetries, $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$
  3. Local $\mathcal{N} = 2$ SUSY with 32 supercharges.

- All the bosonic symmetries will be realized manifestly and simultaneously.
- For this, it is crucial to have the right field variables.
- We shall postulate $O(D, D)$ covariant genuine DFT-field-variables, and NOT employ Riemannian variables such as metric, $B$-field, R-R $p$-forms.

Jeong-Hyuck Park  $\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
Symmetries of $\mathcal{N} = 2$ $D = 10$ SDFT

- **$O(D,D)$ T-duality**: Meta-symmetry

- **Gauge symmetries**

  1. DFT-diffeomorphism (generalized Lie derivative)

  2. A *pair* of local Lorentz symmetries, $\text{Spin}(1,D-1)_L \times \text{Spin}(D-1,1)_R$

  3. **Local $\mathcal{N} = 2$ SUSY** with 32 supercharges.

   - All the bosonic symmetries will be realized manifestly and simultaneously.

   - For this, it is crucial to have the right field variables.

   - We shall postulate $O(D,D)$ covariant genuine DFT-field-variables, and NOT employ Riemannian variables such as metric, $B$-field, R-R $p$-forms.
Symmetries of $\mathcal{N} = 2 \ D = 10$ SDFT

- **$O(D,D)$ T-duality: Meta-symmetry**

- **Gauge symmetries**
  - 1. DFT-diffeomorphism (generalized Lie derivative)
  - 2. A pair of local Lorentz symmetries, $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$
  - 3. Local $\mathcal{N} = 2$ SUSY with 32 supercharges.

- All the bosonic symmetries will be realized manifestly and simultaneously.

- For this, it is crucial to have the right field variables.

- We shall postulate $O(D,D)$ covariant genuine DFT-field-variables, and NOT employ Riemannian variables such as metric, $B$-field, R-R $p$-forms.
Symmetries of $\mathcal{N} = 2 \ D = 10$ SDFT

- **O(D, D) T-duality: Meta-symmetry**

- Gauge symmetries
  - 1. DFT-diffeomorphism (generalized Lie derivative)
  - 2. A pair of local Lorentz symmetries, $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$
  - 3. Local $\mathcal{N} = 2$ SUSY with 32 supercharges.

- All the bosonic symmetries will be realized manifestly and simultaneously.

- For this, it is crucial to have the right field variables.

- We shall postulate $\text{O}(D, D)$ covariant genuine DFT-field-variables, and NOT employ Riemannian variables such as metric, $B$-field, R-R $p$-forms.
All the fields are required to satisfy the section condition,

$$\partial_A \partial^A \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0,$$

which implies an invariance under a shift set by a ‘derivative-index-valued’ vector,

$$\Phi(x + \Delta) = \Phi(x) \quad \text{if} \quad \Delta^A = \varphi \partial^A \varphi' \quad \text{for arbitrary functions} \ \varphi \ \text{and} \ \varphi'.$$

The section condition implies, and in fact can be shown to be equivalent to, an equivalence relation for the coordinates,

$$x^A \sim x^A + \varphi \partial^A \varphi'$$

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in the coordinate space.

$$\Rightarrow$$ The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

**Spacetime is doubled yet gauged!** (further remarks to come at the end of this talk).
All the fields are required to satisfy the section condition,

$$\partial_A \partial^A \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0,$$

which implies an invariance under a shift set by a ‘derivative-index-valued’ vector,

$$\Phi(x + \Delta) = \Phi(x) \quad \text{if} \quad \Delta^A = \varphi \partial^A \varphi' \quad \text{for arbitrary functions} \ \varphi \ \text{and} \ \varphi'.$$

The section condition implies, and in fact can be shown to be equivalent to,

an equivalence relation for the coordinates,

$$x^A \sim x^A + \varphi \partial^A \varphi'$$

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in the coordinate space.

The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

Spacetime is doubled yet gauged! (further remarks to come at the end of this talk).
All the fields are required to satisfy the section condition,

\[ \partial_A \partial^A \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0, \]

which implies an invariance under a shift set by a ‘derivative-index-valued’ vector,

\[ \Phi(x + \Delta) = \Phi(x) \quad \text{if} \quad \Delta^A = \varphi \partial^A \varphi' \quad \text{for arbitrary functions} \ \varphi \ \text{and} \ \varphi'. \]

The section condition implies, and in fact can be shown to be equivalent to, an equivalence relation for the coordinates,

\[ x^A \sim x^A + \varphi \partial^A \varphi' \]

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in the coordinate space.

\[ \Rightarrow \text{The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’}. \]

Spacetime is doubled yet gauged! (further remarks to come at the end of this talk).
Spacetime is doubled yet gauged

All the fields are required to satisfy the section condition,

$$\partial_A \partial^A \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0,$$

which implies an invariance under a shift set by a ‘derivative-index-valued’ vector,

$$\Phi(x + \Delta) = \Phi(x) \quad \text{if} \quad \Delta^A = \varphi \partial^A \varphi'$$

for arbitrary functions $\varphi$ and $\varphi'$.

The section condition implies, and in fact can be shown to be equivalent to, an equivalence relation for the coordinates,

$$x^A \sim x^A + \varphi \partial^A \varphi'$$

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in the coordinate space.

The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

Spacetime is doubled yet gauged! (further remarks to come at the end of this talk).
Field contents of Type II SDFT

**Bosons**

- NS-NS sector
  - DFT-dilaton: \( d \)
  - DFT-vielbeins: \( V_{\alpha p}, \bar{V}_{\bar{\alpha} \bar{p}} \)
- R-R potential: \( C^{\alpha \bar{\alpha}} \)

**Fermions**

- DFT-dilatinos: \( \rho^\alpha, \rho'^{\bar{\alpha}} \)
- Gravitinos: \( \psi^\alpha_{\bar{p}}, \psi'^{\bar{\alpha}}_{\bar{p}} \)
Field contents of Type II SDFT

**Bosons**

- **NS-NS sector**
  - DFT-dilaton: $d$
  - DFT-vielbeins: $V_{Ap}$, $\bar{V}_{A\bar{p}}$

- R-R potential: $C^\alpha{}_{\bar{\alpha}}$

**Fermions**

- DFT-dilatinos: $\rho^\alpha$, $\rho^{\bar{\alpha}}$
- Gravitinos: $\psi_p^\alpha$, $\psi_p^{\bar{\alpha}}$
Field contents of Type II SDFT

- **Bosons**
  - **NS-NS sector**
    - DFT-dilaton: $d$
    - DFT-vielbeins: $V_{Ap}$, $\bar{V}_{A\bar{p}}$
  - R-R potential: $C^\alpha \bar{\alpha}$

- **Fermions**
  - DFT-dilatinos: $\rho^\alpha$, $\rho^{\bar{\alpha}}$
  - Gravitinos: $\psi^\alpha_p$, $\psi^{\bar{\alpha}}_p$
Field contents of Type II SDFT

**Bosons**

- **NS-NS sector**
  - DFT-dilaton: \( d \)
  - DFT-vielbeins: \( V_{Ap}, \bar{V}_{A\bar{p}} \)

- R-R potential: \( C^\alpha \bar{\alpha} \)

**Fermions**

- DFT-dilatinos: \( \rho^\alpha, \rho^{\prime \bar{\alpha}} \)
- Gravitinos: \( \psi^{\alpha}_{\bar{p}}, \psi^{\prime \bar{\alpha}}_{\bar{p}} \)
Field contents of Type II SDFT

Bosons

- **NS-NS sector**
  - DFT-dilaton: $d$
  - DFT-vielbeins: $V_{Ap}$, $\bar{V}_{A\bar{p}}$

- R-R potential: $C^\alpha \bar{\alpha}$

Fermions

- DFT-dilatinos: $\rho^\alpha$, $\rho^{i\bar{\alpha}}$
- Gravitinos: $\psi_\alpha$, $\psi_{i\bar{\alpha}}$
Field contents of Type II SDFT

- **Bosons**
  - **NS-NS sector**
    - DFT-dilaton: $d$
    - DFT-vielbeins: $V_{Ap}, \bar{V}_{A\bar{p}}$
  - R-R potential: $C^\alpha \bar{\alpha}$

- **Fermions**
  - DFT-dilatinos: $\rho^\alpha, \rho^{\prime \bar{\alpha}}$
  - Gravitinos: $\psi^{\alpha \bar{p}}, \psi^{\prime \bar{\alpha} p}$
Field contents of Type II SDFT

- **Bosons**
  - **NS-NS sector**
    - DFT-dilaton: $d$
    - DFT-vielbeins: $V_A$, $\bar{V}_{\bar{A}}$
  - **R-R potential:** $C^\alpha_{\bar{\alpha}}$

- **Fermions**
  - DFT-dilatinos: $\rho^\alpha$, $\rho^{\bar{\alpha}}$
  - Gravitinos: $\psi^\alpha_{\bar{p}}$, $\psi^{\bar{\alpha}}_p$

<table>
<thead>
<tr>
<th>Index</th>
<th>Representation</th>
<th>Metric (raising/lowering indices)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$, $B$, \cdots</td>
<td>$O(D, D)$ &amp; DFT-diffeom. vector</td>
<td>$J_{AB}$</td>
</tr>
<tr>
<td>$p$, $q$, \cdots</td>
<td>Spin$(1, D-1)_L$ vector</td>
<td>$\eta_{pq} = \text{diag}(- + + \cdots +)$</td>
</tr>
<tr>
<td>$\alpha$, $\beta$, \cdots</td>
<td>Spin$(1, D-1)_L$ spinor</td>
<td>$C^+<em>{\alpha\beta}$, $(\gamma^p)^T = C</em>+ \gamma^p C^{-1}_+$</td>
</tr>
<tr>
<td>$\bar{p}$, $\bar{q}$, \cdots</td>
<td>Spin$(D-1, 1)_R$ vector</td>
<td>$\bar{\eta}_{\bar{p}\bar{q}} = \text{diag}(+ - - \cdots -)$</td>
</tr>
<tr>
<td>$\bar{\alpha}$, $\bar{\beta}$, \cdots</td>
<td>Spin$(D-1, 1)_R$ spinor</td>
<td>$\bar{C}^+<em>{\bar{\alpha}\bar{\beta}}$, $(\bar{\gamma}^{\bar{p}})^T = \bar{C}</em>+ \bar{\gamma}^{\bar{p}} \bar{C}^{-1}_+$</td>
</tr>
</tbody>
</table>
Field contents of Type II SDFT

- **Bosons**
  - NS-NS sector
    - DFT-dilaton: $d$
    - DFT-vielbeins: $V_{A\rho}$, $\bar{V}_{A\bar{\rho}}$
  - R-R potential: $C^{\alpha \bar{\alpha}}$

- **Fermions**
  - DFT-dilatinos: $\rho^\alpha$, $\rho^{\dagger \bar{\alpha}}$
  - Gravitinos: $\psi^\alpha_{\bar{p}}$, $\psi^{\dagger \bar{\alpha}}_{p}$

R-R potential and Fermions carry NOT $(D + D)$-dimensional BUT undoubled $D$-dimensional indices.
Field contents of Type II SDFT

**Bosons**

- **NS-NS sector**
  - DFT-dilaton: \( d \)
  - DFT-vielbeins: \( V_{Ap}, \bar{V}_{A\bar{p}} \)

- **R-R potential:**
  \( C^\alpha \bar{\alpha} \)

**Fermions**

- DFT-dilatinos: \( \rho^\alpha, \rho^{\bar{\alpha}} \)
- Gravitinos: \( \psi^\alpha_p, \psi^{\bar{\alpha}}_{\bar{p}} \)

_A priori_, \( O(D, D) \) rotates only the \( O(D, D) \) vector indices (capital Roman), and the R-R sector and all the fermions are \( O(D, D) \) T-duality singlet.

The usual IIA ⇔ IIB exchange will follow only after fixing a gauge.
The DFT-dilaton gives rise to a scalar density with weight one, 
\[ e^{-2d}. \]

The DFT-vielbeins satisfy the **four defining properties:**

\[ V_A p V^A_q = \eta_{pq}, \quad \bar{V}_{A\bar{p}} \bar{V}^A_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}, \quad V_A p \bar{V}^A_{\bar{q}} = 0, \quad V_A p V_B p + \bar{V}_{A\bar{p}} \bar{V}_{B\bar{q}} = J_{AB}. \]

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

\[ \gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}}, \quad \gamma^{(D+1)} \rho = -c \rho, \]
\[ \bar{\gamma}^{(D+1)} \psi'_{\bar{p}} = c' \psi'_{\bar{p}}, \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho', \]

where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1 \).

Lastly for the R-R sector, we set the R-R potential, \( C^\alpha_{\bar{\alpha}} \), to be in the bi-fundamental spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,

\[ \gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C. \]
The DFT-dilaton gives rise to a scalar density with weight one,
\[ e^{-2d}. \]

The DFT-vielbeins satisfy the **four defining properties:**
\[ V_A p V^A q = \eta_{pq}, \quad \bar{V}_A \bar{p} \bar{V}^A q = \bar{\eta}_{\bar{p} \bar{q}}, \quad V_A p \bar{V}^A q = 0, \quad V_A p V_B p + \bar{V}_A \bar{p} \bar{V}^B \bar{p} = J_{AB}. \]

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,
\[ \gamma^{(D+1)} \psi \bar{p} = c \psi \bar{p}, \quad \gamma^{(D+1)} \rho = -c \rho, \]
\[ \bar{\gamma}^{(D+1)} \psi' \bar{p} = c' \psi' \bar{p}, \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho', \]
where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1. \)

Lastly for the R-R sector, we set the R-R potential, \( C^\alpha \bar{\alpha} \), to be in the **bi-fundamental** spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,
\[ \gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C. \]
The DFT-dilaton gives rise to a scalar density with weight one,
\[ e^{-2d} \].

The DFT-vielbeins satisfy the **four defining properties**:

\[ V_A p V^A q = \eta p q, \quad \bar{V}_{\bar{A} \bar{p}} \bar{V}^{\bar{A} \bar{q}} = \bar{\eta} \bar{p} \bar{q}, \quad V_A p \bar{V}^{\bar{A} \bar{q}} = 0, \quad V_A p V_B p + \bar{V}_{\bar{A} \bar{p}} \bar{V}_{\bar{B} \bar{p}} = \mathcal{J}_{AB}. \]

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

\[ \gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}}, \quad \gamma^{(D+1)} \rho = -c \rho, \]
\[ \tilde{\gamma}^{(D+1)} \psi_{\bar{p}} = c' \psi_{\bar{p}}, \quad \tilde{\gamma}^{(D+1)} \rho' = -c' \rho', \]

where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1 \).

Lastly for the R-R sector, we set the R-R potential, \( C^\alpha_{\bar{\alpha}} \), to be in the **bi-fundamental** spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,

\[ \gamma^{(D+1)} C \tilde{\gamma}^{(D+1)} = cc' C. \]
The DFT-dilaton gives rise to a scalar density with weight one,
\[ e^{-2d} \].

The DFT-vielbeins satisfy the **four defining properties**:

\[ V_{Ap} V^A_p = \eta_{\bar{p}q} \],
\[ \bar{V}_{A\bar{p}} \bar{V}^A_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}} \],
\[ V_{Ap} \bar{V}^A_{\bar{q}} = 0 \],
\[ V_{Ap} V_B^p + \bar{V}_{A\bar{p}} \bar{V}_{B\bar{q}} = \mathcal{J}_{AB} \].

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

\[ \gamma^{(D+1)}\psi_{\bar{p}} = c \psi_{\bar{p}} \],
\[ \gamma^{(D+1)}\rho = -c \rho \],
\[ \bar{\gamma}^{(D+1)}\psi_{\bar{p}} = c' \psi_{\bar{p}} \],
\[ \bar{\gamma}^{(D+1)}\rho' = -c' \rho' \],

where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1 \).

Lastly for the R-R sector, we set the R-R potential, \( C^\alpha_{\bar{\alpha}} \), to be in the bi-fundamental spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,

\[ \gamma^{(D+1)}C\bar{\gamma}^{(D+1)} = cc' C \].
The DFT-dilaton gives rise to a scalar density with weight one,
\[ e^{-2d}. \]

The DFT-vielbeins satisfy the **four defining properties**:

\[ V_{Ap} V^A_q = \eta_{pq}, \quad \bar{V}_{Ap} \bar{V}^A_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}, \quad V_{Ap} \bar{V}^A_{\bar{q}} = 0, \quad V_{Ap} V_B^p + \bar{V}_{Ap} \bar{V}^B_{\bar{p}} = J_{AB}. \]

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

\[
\gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}}, \quad \gamma^{(D+1)} \rho = -c \rho, \\
\bar{\gamma}^{(D+1)} \psi'_{\bar{p}} = c' \psi'_{\bar{p}}, \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho',
\]

where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1 \).

Lastly for the R-R sector, we set the R-R potential, \( C^{\alpha \bar{\alpha}} \), to be in the **bi-fundamental** spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,

\[
\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C.
\]
\[ \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \] chiralities:

\[
\gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}} , \quad \gamma^{(D+1)} \rho = -c \rho , \\
\bar{\gamma}^{(D+1)} \psi'_{\bar{p}} = c' \psi'_{\bar{p}} , \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho' , \\
\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C .
\]

\[ A \text{ priori} \] all the possible four different sign choices are equivalent up to \( \text{Pin}(1, D-1)_L \times \text{Pin}(D-1, 1)_R \) rotations.

That is to say, \( \mathcal{N} = 2 \ D = 10 \) SDFT is chiral with respect to both \( \text{Pin}(1, D-1)_L \) and \( \text{Pin}(D-1, 1)_R \), and the theory is unique, unlike IIA/IIB SUGRAs.

Hence, without loss of generality, we may safely set

\[ c = c' \equiv +1 . \]

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
\[ \textbf{Spin}(1, D-1)_L \times \textbf{Spin}(D-1, 1)_R \] chiralities:

\[
\gamma^{(D+1)} \psi^\rho = c \psi^\rho, \quad \gamma^{(D+1)} \rho = -c \rho,
\]

\[
\bar{\gamma}^{(D+1)} \psi^\rho' = c' \psi^\rho', \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho',
\]

\[
\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C.
\]

\textit{A priori} all the possible four different sign choices are equivalent up to \( \text{Pin}(1, D-1)_L \times \text{Pin}(D-1, 1)_R \) rotations.

That is to say, \( \mathcal{N} = 2 \) \( D = 10 \) SDFT is chiral with respect to both \( \text{Pin}(1, D-1)_L \) and \( \text{Pin}(D-1, 1)_R \), and the theory is unique, unlike IIA/IIB SUGRAs.

Hence, without loss of generality, we may safely set

\[
c \equiv c' \equiv +1.
\]

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
Spin\((1, D-1)_L \times \text{Spin}(D-1, 1)_R\) chiralities:

\[
\gamma^{(D+1)} \psi p = c \psi p, \quad \gamma^{(D+1)} \rho = -c \rho, \\
\tilde{\gamma}^{(D+1)} \psi' p = c' \psi' p, \quad \tilde{\gamma}^{(D+1)} \rho' = -c' \rho', \\
\gamma^{(D+1)} C \tilde{\gamma}^{(D+1)} = cc' C.
\]

\textit{A priori} all the possible four different sign choices are equivalent up to \(\text{Pin}(1, D-1)_L \times \text{Pin}(D-1, 1)_R\) rotations.

That is to say, \(\mathcal{N} = 2\) \(D = 10\) SDFT is chiral with respect to both \(\text{Pin}(1, D-1)_L\) and \(\text{Pin}(D-1, 1)_R\), and the theory is unique, unlike IIA/IIB SUGRAs.

Hence, without loss of generality, we may safely set

\[c \equiv c' \equiv +1.\]

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
\textbf{Spin}(1, D-1)_L \times \textbf{Spin}(D-1, 1)_R \text{ chiralities:}

\begin{align*}
\gamma^{(D+1)} \psi_p &= c \psi_p , & \gamma^{(D+1)} \rho &= -c \rho , \\
\bar{\gamma}^{(D+1)} \psi'_p &= c' \psi'_p , & \bar{\gamma}^{(D+1)} \rho' &= -c' \rho' , \\
\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} &= cc' C .
\end{align*}

\textit{A priori} all the possible four different sign choices are equivalent up to \textbf{Pin}(1, D-1)_L \times \textbf{Pin}(D-1, 1)_R \text{ rotations.}

That is to say, \( \mathcal{N} = 2 \; D = 10 \) SDFT is chiral with respect to both \textbf{Pin}(1, D-1)_L \text{ and } \textbf{Pin}(D-1, 1)_R, \text{ and the theory is unique, unlike IIA/IIB SUGRAs.}

Hence, without loss of generality, we may safely set

\( c \equiv c' \equiv +1 . \)

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
**Spin**(1, D−1)\(_L\) \(\times\) **Spin**(D−1, 1)\(_R\) chiralities:

\[
\gamma^{(D+1)} \psi_\bar{p} = c \psi_\bar{p}, \quad \gamma^{(D+1)} \rho = -c \rho,
\]

\[
\bar{\gamma}^{(D+1)} \psi_\bar{p}' = c' \psi_\bar{p}', \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho'.
\]

\[
\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C.
\]

**A priori** all the possible four different sign choices are equivalent up to **Pin**(1, D−1)\(_L\) \(\times\) **Pin**(D−1, 1)\(_R\) rotations.

That is to say, \(\mathcal{N} = 2\ D = 10\) SDFT is chiral with respect to both **Pin**(1, D−1)\(_L\) and **Pin**(D−1, 1)\(_R\), and the theory is unique, unlike IIA/IIB SUGRAs.

Hence, without loss of generality, we may safely set

\[
c \equiv c' \equiv +1.
\]

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
The DFT-vielbeins generate a pair of rank-two projectors,

\[ P_{AB} := V_A^p V_{Bp}, \quad P_A^B P_B^C = P_A^C, \quad \bar{P}_{AB} := \bar{V}_A^{\bar{p}} \bar{V}_{B\bar{p}}, \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C, \]

which are symmetric, orthogonal and complementary to each other,

\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A^B \bar{P}_B^C = 0, \quad P_A^B + \bar{P}_A^B = \delta_A^B. \]

It follows

\[ P_A^B V_{Bp} = V_{Ap}, \quad \bar{P}_A^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}, \quad \bar{P}_A^B V_{Bp} = 0, \quad P_A^B \bar{V}_{B\bar{p}} = 0. \]

Note also

\[ \mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}. \]

However, our emphasis lies on the ‘projectors’ rather than the “generalized metric".
The DFT-vielbeins generate a pair of rank-two projectors,

\[ P_{AB} := V_A^p V_{Bp}, \quad P_A^B P_B^C = P_A^C, \quad \bar{P}_{AB} := \bar{V}_A^{\bar{p}} \bar{V}_{B\bar{p}}, \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C, \]

which are symmetric, orthogonal and complementary to each other,

\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A^B \bar{P}_B^C = 0, \quad P_A^B + \bar{P}_A^B = \delta_A^B. \]

It follows

\[ P_A^B V_{Bp} = V_{Ap}, \quad \bar{P}_A^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}, \quad \bar{P}_A^B V_{Bp} = 0, \quad P_A^B \bar{V}_{B\bar{p}} = 0. \]

Note also

\[ \mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}. \]

However, our emphasis lies on the ‘projectors’ rather than the “generalized metric".
The DFT-vielbeins generate a pair of rank-two projectors,

\[ P_{AB} := V_A^p V_{Bp}, \quad P^B_A P^C_B = P^C_A, \quad \bar{P}_{AB} := \bar{V}_{A\bar{p}} \bar{V}_{B\bar{p}}, \quad \bar{P}^B_A \bar{P}^C_B = \bar{P}^C_A, \]

which are symmetric, orthogonal and complementary to each other,

\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P^B_A \bar{P}^C_B = 0, \quad P_A^B + \bar{P}_A^B = \delta_A^B. \]

It follows

\[ P_A^B V_{Bp} = V_A^p, \quad \bar{P}_A^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}, \quad \bar{P}_A^B V_{Bp} = 0, \quad P_A^B \bar{V}_{B\bar{p}} = 0. \]

Note also

\[ \mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}. \]

However, our emphasis lies on the ‘projectors’ rather than the “generalized metric". 
Further, we construct a pair of rank-six projectors,

$$\mathcal{P}_{CAB \, DEF} := P_C \, D \, P_{[A} \, [E \, P_{B} \, F]} + \frac{2}{D-1} \, P_{C[A} \, P_{B]} \, [E \, P_{F}] \, D \,, \quad \mathcal{P}_{CAB \, DEF} \, \mathcal{P}_{DEF \, GHI} = \mathcal{P}_{CAB \, GHI} \,,$$

$$\bar{\mathcal{P}}_{CAB \, DEF} := \bar{P}_C \, D \, \bar{P}_{[A} \, [E \, \bar{P}_{B} \, F]} + \frac{2}{D-1} \, \bar{P}_{C[A} \, \bar{P}_{B]} \, [E \, \bar{P}_{F}] \, D \,, \quad \bar{\mathcal{P}}_{CAB \, DEF} \, \bar{\mathcal{P}}_{DEF \, GHI} = \bar{\mathcal{P}}_{CAB \, GHI} \,,$$

which are symmetric and traceless,

$$\mathcal{P}_{CAB \, DEF} = \mathcal{P}_{DEF \, CAB} = \mathcal{P}_{C[AB] \, D[EF]} \,, \quad \bar{\mathcal{P}}_{CAB \, DEF} = \bar{\mathcal{P}}_{DEF \, CAB} = \bar{\mathcal{P}}_{C[AB] \, D[EF]} \,,$$

$$\mathcal{P}^A_{\, ABDEF} = 0 \,, \quad \mathcal{P}^{AB} \, \mathcal{P}_{ABCDEF} = 0 \,, \quad \bar{\mathcal{P}}^A_{\, ABDEF} = 0 \,, \quad \bar{\mathcal{P}}^{AB} \, \bar{\mathcal{P}}_{ABCDEF} = 0 \,.$$
Having all the ‘right’ field-variables prepared, we now discuss their derivatives or what we call, ‘semi-covariant derivative’.

The meaning of “semi-covariant” will be clarified later.
Having all the ‘right’ field-variables prepared, we now discuss their derivatives or what we call, ‘semi-covariant derivative’.

The meaning of “semi-covariant” will be clarified later.
Semi-covariant derivatives

For each gauge symmetry we assign a corresponding connection,

- $\Gamma_A$ for the DFT-diffeomorphism (generalized Lie derivative),
- $\Phi_A$ for the ‘unbarred’ local Lorentz symmetry, $\text{Spin}(1, D-1)_L$,
- $\bar{\Phi}_A$ for the ‘barred’ local Lorentz symmetry, $\text{Spin}(D-1, 1)_R$.

Combining all of them, we introduce the master ‘semi-covariant’ derivative

$$D_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A.$$
Semi-covariant derivatives

For each gauge symmetry we assign a corresponding connection,

- $\Gamma_A$ for the DFT-diffeomorphism (generalized Lie derivative),
- $\Phi_A$ for the ‘unbarred’ local Lorentz symmetry, $\text{Spin}(1, D-1)_L$,
- $\bar{\Phi}_A$ for the ‘barred’ local Lorentz symmetry, $\text{Spin}(D-1, 1)_R$.

Combining all of them, we introduce **master ‘semi-covariant’ derivative**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A.$$
It is also useful to set

\[ \nabla_A = \partial_A + \Gamma_A, \quad D_A = \partial_A + \Phi_A + \bar{\Phi}_A. \]

The former is the ‘semi-covariant’ derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

\[ \nabla_C T_{A_1 A_2 \ldots A_n} := \partial_C T_{A_1 A_2 \ldots A_n} - \omega \Gamma^B_{BC} T_{A_1 A_2 \ldots A_n} + \sum_{i=1}^n \Gamma_{CA_i}^B T_{A_1 \ldots \hat{A_i} \ldots A_{i-1} B A_{i+1} \ldots A_n}. \]

And the latter is the covariant derivative for the \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \) local Lorenz symmetries.
It is also useful to set
\[ \nabla A = \partial A + \Gamma_A, \quad D_A = \partial A + \Phi_A + \bar{\Phi}_A. \]

The former is the ‘semi-covariant’ derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),
\[ \nabla C T_{A_1 A_2 \cdots A_n} := \partial C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_C A_i^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n}. \]

And the latter is the covariant derivative for the $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ local Lorenz symmetries.
It is also useful to set

\[ \nabla_A = \partial_A + \Gamma_A, \quad D_A = \partial_A + \Phi_A + \bar{\Phi}_A. \]

The former is the ‘semi-covariant’ derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

\[ \nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}^B T_{A_1 \cdots A_{i-1} BA_{i+1} \cdots A_n}. \]

And the latter is the covariant derivative for the \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \) local\n
Lorenz symmetries.
By definition, the master derivative annihilates all the ‘constants’,

\[ \mathcal{D}_A \mathcal{J}_{BC} = \nabla_A \mathcal{J}_{BC} = \Gamma^{D}_{AB} \mathcal{J}_{DC} + \Gamma^{D}_{AC} \mathcal{J}_{BD} = 0, \]

\[ \mathcal{D}_A \eta_{pq} = D_A \eta_{pq} = \Phi_{Ap} r \eta_{rq} + \Phi_{Aq} r \eta_{pr} = 0, \]

\[ \mathcal{D}_A \bar{\eta}_{\bar{p}q} = D_A \bar{\eta}_{\bar{p}q} = \bar{\Phi}_{A\bar{p}} r \bar{\eta}_{\bar{r}q} + \bar{\Phi}_{A\bar{q}} r \bar{\eta}_{\bar{r}p} = 0, \]

\[ \mathcal{D}_A C_{+\alpha\beta} = D_A C_{+\alpha\beta} = \Phi_{A\alpha} \delta C_{+\delta\beta} + \Phi_{A\beta} \delta C_{+\alpha\delta} = 0, \]

\[ \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = D_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = \bar{\Phi}_{A\bar{\alpha}} \delta \bar{C}_{+\bar{\delta}\bar{\beta}} + \bar{\Phi}_{A\bar{\beta}} \delta \bar{C}_{+\bar{\alpha}\bar{\delta}} = 0, \]

including the gamma matrices,

\[ \mathcal{D}_A (\gamma^p)^{\alpha}_{\beta} = D_A (\gamma^p)^{\alpha}_{\beta} = \Phi_{A\alpha}^{\beta} q (\gamma^q)^{\alpha}_{\beta} + \Phi_{A\alpha}^{\beta} \delta (\gamma^p)^{\delta}_{\beta} - (\gamma^p)^{\alpha}_{\beta} \Phi_{A\delta}^{\beta} = 0, \]

\[ \mathcal{D}_A (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\beta}} = D_A (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\beta}} = \Phi_{A\bar{\alpha}}^{\bar{\beta}} q (\bar{\gamma}^\bar{q})^{\bar{\alpha}}_{\bar{\beta}} + \Phi_{A\bar{\alpha}}^{\bar{\beta}} \delta (\bar{\gamma}^\bar{p})^{\bar{\delta}}_{\bar{\beta}} - (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\beta}} \Phi_{A\bar{\delta}}^{\bar{\beta}} = 0. \]
It follows then that the connections are all anti-symmetric,

\[ \Gamma_{ABC} = -\Gamma_{ACB}, \]

\[ \Phi_{Apq} = -\Phi_{Aqp}, \quad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}, \]

\[ \Phi_{A\bar{p}\bar{q}} = -\Phi_{A\bar{q}\bar{p}}, \quad \Phi_{A\bar{\alpha}\bar{\beta}} = -\Phi_{A\bar{\beta}\bar{\alpha}}, \]

and as usual,

\[ \Phi_{A}^{\alpha \beta} = \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^{\alpha \beta}, \quad \Phi_{A}^{\bar{\alpha} \bar{\beta}} = \frac{1}{4} \Phi_{A\bar{p}\bar{q}} (\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha} \bar{\beta}}. \]
Further, the master derivative is compatible with the whole NS-NS sector,

\[ D_A d = \nabla_A d := -\frac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \frac{1}{2} \Gamma^B_{BA} = 0 , \]

\[ D_A V_{Bp} = \partial_A V_{Bp} + \Gamma^C_{AB} V_{Cp} + \Phi_{Apq} V_{Bq} = 0 , \]

\[ D_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma^C_{AB} \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}q} \bar{V}_{B\bar{q}} = 0 . \]

It follows that

\[ D_A P_{BC} = \nabla_A P_{BC} = 0 , \quad D_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0 , \]

and the connections are related to each other,

\[ \Gamma_{ABC} = V^\rho_B D_A V_{C\rho} + \bar{V}^\bar{\rho}_B D_A \bar{V}_{C\bar{\rho}} , \]

\[ \Phi_{Apq} = V^\rho_B \nabla_A V_{Bq} , \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^\bar{\rho}_B \nabla_A \bar{V}_{B\bar{q}} . \]
Further, the master derivative is compatible with the whole NS-NS sector,

\[ D_A d = \nabla_A d := -\frac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \frac{1}{2}\Gamma^B_{BA} = 0 , \]

\[ D_A V_{Bp} = \partial_A V_{Bp} + \Gamma^C_{AB} V_{Cp} + \Phi_{Apq} V_{Bq} = 0 , \]

\[ D_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma^C_{AB} \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}q} \bar{V}_{B\bar{q}} = 0 . \]

It follows that

\[ D_A P_{BC} = \nabla_A P_{BC} = 0 , \quad D_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0 , \]

and the connections are related to each other,

\[ \Gamma^C_{ABC} = V^D_{Bp} D_A V_{Cp} + \bar{V}^{D}_{B\bar{p}} D_A \bar{V}_{C\bar{p}} , \]

\[ \Phi_{Apq} = V^D_{Bp} \nabla_A V_{Bq} , \]

\[ \Phi_{A\bar{p}q} = \bar{V}^{D}_{B\bar{p}} \nabla_A \bar{V}_{B\bar{q}} . \]
The connections assume the following most general forms:

\[ \Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A^p V_B^q + \Delta_{C\bar{p}\bar{q}} \bar{V}_A^\bar{p} \bar{V}_B^\bar{q}, \]

\[ \Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}}. \]

Here

\[ \Gamma^0_{CAB} = 2 (P \partial_C \bar{P})_{[AB]} + 2 (\bar{P}_{[A}^D \bar{P}_{B]}^E - P_{[A}^D P_{B]}^E) \partial_D P_{EC} \]

\[ - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}^D + P_{C[A} P_{B]}^D) (\partial_D d + (P \partial^E P \bar{P})_{[ED]}), \]

and, with the corresponding derivative, \( \nabla^0_A = \partial_A + \Gamma^0_A, \)

\[ \Phi^0_{Apq} = V^B_p \nabla^0_A V_{Bq} = V^B_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B_p V^C_q, \]

\[ \bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B_{\bar{p}} \bar{V}^C_{\bar{q}}. \]
The connections assume the following most general forms:

\[ \Gamma_{CAB} = \Gamma_{CAB}^0 + \Delta_{Cpq} V_A^p V_B^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A^\bar{p} \bar{V}_B^\bar{q}, \]

\[ \Phi_{Apq} = \Phi_{Apq}^0 + \Delta_{Apq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}_{A\bar{p}\bar{q}}^0 + \bar{\Delta}_{A\bar{p}\bar{q}}. \]

Further, the extra pieces, \( \Delta_{Apq} \) and \( \bar{\Delta}_{A\bar{p}\bar{q}} \), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \( \mathcal{D}_A d = 0 \), must satisfy

\[ \Delta_{Apq} V_A^p = 0, \quad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0. \]

Otherwise they are arbitrary.

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

\[ \bar{\rho} \gamma_{pq} \psi_A, \quad \bar{\psi}_\bar{p} \gamma A \psi_\bar{q}, \quad \bar{\rho} \gamma_{Apq} \rho, \quad \bar{\psi}_\bar{p} \gamma_{Apq} \psi_\bar{p}, \]

where we set \( \psi_A = \bar{V}_A^\bar{p} \psi_{\bar{p}}, \gamma_A = V_A^p \gamma_p. \)
The connections assume the following most general forms:

\[ \Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A^p V_B^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_{A\bar{p}} \bar{V}_{B\bar{q}}, \]

\[ \Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}}. \]

Further, the extra pieces, \( \Delta_{Apq} \) and \( \bar{\Delta}_{A\bar{p}\bar{q}} \), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \( D_A d = 0 \), must satisfy

\[ \Delta_{Apq} V^A^p = 0, \quad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0. \]

Otherwise they are arbitrary.

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

\[ \rho_{\gamma pq} \psi_A, \quad \bar{\psi}_\rho \gamma A \psi \bar{q}, \quad \rho_{\gamma Apq} \rho, \quad \bar{\psi}_\rho \gamma Apq \psi \bar{p}, \]

where we set \( \psi_A = \bar{V}_A \bar{p} \psi_{\bar{p}}, \gamma_A = V_A^p \gamma_p \).
The connections assume the following most general forms:

\[
\Gamma_{CAB} = \Gamma^{0}_{CAB} + \Delta_{Cpq} V_A^p V_B^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A^{\bar{p}} \bar{V}_B^{\bar{q}},
\]

\[
\Phi_{Apq} = \Phi^{0}_{Apq} + \Delta_{Apq},
\]

\[
\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^{0}_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}}.
\]

Further, the extra pieces, \(\Delta_{Apq}\) and \(\bar{\Delta}_{A\bar{p}\bar{q}}\), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \(\mathcal{D}_A d = 0\), must satisfy

\[
\Delta_{Apq} V^A^p = 0, \quad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0.
\]

Otherwise they are arbitrary.

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

\[
\bar{\rho} \gamma_{pq} \psi_A, \quad \bar{\psi}_{\bar{p}} \gamma_A \psi_{\bar{q}}, \quad \bar{\rho} \gamma_{Apq} \rho, \quad \bar{\psi}_{\bar{p}} \gamma_{Apq} \psi_{\bar{p}},
\]

where we set \(\psi_A = \bar{V}_A^{\bar{p}} \psi_{\bar{p}}, \gamma_A = V_A^p \gamma_p\).
The ‘torsionless’ connection,

\[ \Gamma^0_{CAB} = 2 \left( P \partial_C P \bar{P} \right)_{[AB]} + 2 \left( \bar{P} [A^D P_B]^E - P [A^D P_B]^E \right) \partial_D P_{EC} \]

\[ - \frac{4}{D-1} \left( \bar{P} C[A \bar{P} B] D + P C[A P_B]^D \right) \left( \partial_D d + (P \partial^E P \bar{P})_{[ED]} \right), \]

further obeys

\[ \Gamma^0_{ABC} + \Gamma^0_{BCA} + \Gamma^0_{CAB} = 0, \]

and

\[ P_{CAB}^{DEF} \Gamma^0_{DEF} = 0, \quad \bar{P}_{CAB}^{DEF} \Gamma^0_{DEF} = 0. \]
In fact, the torsionless connection,

\[ \Gamma^0_{CAB} = 2 \left( P \partial_C P \bar{P} \right)_{[AB]} + 2 \left( \bar{P}_{[A}^D P_{B]}^E - P_{[A}^D P_{B]}^E \right) \partial_D P_{EC} \]

\[ - \frac{4}{D-1} \left( \bar{P}_{[A} C \bar{P}_{B]}^D + P_{[A} C P_{B]}^D \right) \left( \partial_D d + (P \partial^E P \bar{P})_{[ED]} \right), \]

is uniquely determined by requiring

\[ \nabla_A J_{BC} = 0 \quad \iff \quad \Gamma_{CAB} + \Gamma_{CBA} = 0, \]

\[ \nabla_A P_{BC} = 0, \]

\[ \nabla_A d = 0, \]

\[ \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0, \]

\[ (\mathcal{P} + \bar{\mathcal{P}})_{CAB}^D E F \Gamma_{DEF} = 0. \]
Having the two symmetric properties, $\Gamma_{A(BC)} = 0$, $\Gamma_{[ABC]} = 0$, we may safely replace $\partial_A$ by $\nabla^0_A = \partial_A + \Gamma^0_A$ in $\hat{\mathcal{L}}_X$ and also in $[X, Y]_C^A$,

$$\hat{\mathcal{L}}_X T_{A_1 \ldots A_n} = X^B \nabla^0_B T_{A_1 \ldots A_n} + \omega \nabla^0_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^n (\nabla^0_{A_i} X_B - \nabla^0_B X_{A_i}) T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n},$$

$$[X, Y]_C^A = X^B \nabla^0_B Y^A - Y^B \nabla^0_B X^A + \frac{1}{2} Y^B \nabla^0 A X_B - \frac{1}{2} X^B \nabla^0 A Y_B,$$

just like in Riemannian geometry.

In this way, $\Gamma^0_{ABC}$ is the DFT analogy of the Christoffel connection.

Precisely the same expression was later re-derived by Hohm & Zwiebach.
Having the two symmetric properties, $\Gamma_{A(BC)} = 0$, $\Gamma_{[ABC]} = 0$, we may safely replace $\partial_A$ by $\nabla^0_A = \partial_A + \Gamma^0_A$ in $\hat{\mathcal{L}}_X$ and also in $[X, Y]_C^A$,

$$\hat{\mathcal{L}}_X T_{A_1 \cdots A_n} = X^B \nabla^0_B T_{A_1 \cdots A_n} + \omega \nabla^0_B X^B T_{A_1 \cdots A_n} + \sum_{i=1}^n (\nabla^0_A X_B - \nabla^0_B X_A) T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n},$$

$$[X, Y]_C^A = X^B \nabla^0_B Y^A - Y^B \nabla^0_B X^A + \frac{1}{2} Y^B \nabla^0 A X_B - \frac{1}{2} X^B \nabla^0 A Y_B,$$

just like in Riemannian geometry.

In this way, $\Gamma^0_{A(BC)}$ is the **DFT analogy of the Christoffel connection**.

Precisely the same expression was later re-derived by Hohm & Zwiebach.
Having the two symmetric properties, \( \Gamma_{A(BC)} = 0, \Gamma_{[ABC]} = 0 \), we may safely replace \( \partial_A \) by \( \nabla^0_A = \partial_A + \Gamma^0_A \) in \( \hat{\mathcal{L}}_X \) and also in \([X,Y]^A_C\),

\[
\hat{\mathcal{L}}_X T_{A_1 \cdots A_n} = X^B \nabla^0_B T_{A_1 \cdots A_n} + \omega \nabla^0_B X^B T_{A_1 \cdots A_n} + \sum_{i=1}^{n} (\nabla^0_{A_i} X_B - \nabla^0_B X_{A_i}) T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n},
\]

\[
[X,Y]^A_C = X^B \nabla^0_B Y^A - Y^B \nabla^0_B X^A + \frac{1}{2} Y^B \nabla^0 A X_B - \frac{1}{2} X^B \nabla^0 A Y_B,
\]

just like in Riemannian geometry.

- In this way, \( \Gamma^0_{ABC} \) is the **DFT analogy of the Christoffel connection**.

- Precisely the same expression was later re-derived by Hohm & Zwiebach.
The usual curvatures for the three connections,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC} E \Gamma_{BED} - \Gamma_{BC} E \Gamma_{AED}, \]

\[ F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr} \Phi_{B'q} - \Phi_{Bpr} \Phi_{A'q}, \]

\[ \tilde{F}_{AB\tilde{p}\tilde{q}} = \partial_A \tilde{\Phi}_{B\tilde{p}\tilde{q}} - \partial_B \tilde{\Phi}_{A\tilde{p}\tilde{q}} + \tilde{\Phi}_{A\tilde{r}\tilde{r}} \tilde{\Phi}_{B'\tilde{q}} - \tilde{\Phi}_{B\tilde{r}\tilde{r}} \tilde{\Phi}_{A'\tilde{q}}, \]

are, from \([D_A, D_B] V_{Cp} = 0\) and \([D_A, D_B] \tilde{V}_{C\bar{p}} = 0\), related to each other,

\[ R_{ABCD} = F_{CDpq} V_A^p V_B^q + \tilde{F}_{CD\tilde{p}\tilde{q}} \tilde{V}_A^{\tilde{p}} \tilde{V}_B^{\tilde{q}}. \]

However, the crucial object in DFT turns out to be

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma^E_{AB} \Gamma_{ECD} \right), \]

which we name semi-covariant curvature.
Semi-covariant curvature

The usual curvatures for the three connections,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}, \]

\[ F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{A pq} + \Phi_{Apr} \Phi_{B^r q} - \Phi_{Bpr} \Phi_{A^r q}, \]

\[ \bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A \bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}} \bar{\Phi}_{B^r \bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}} \bar{\Phi}_{A^r \bar{q}}, \]

are, from \([\mathcal{D}_A, \mathcal{D}_B] V_{Cp} = 0\) and \([\mathcal{D}_A, \mathcal{D}_B] \bar{V}_{C\bar{p}} = 0\), related to each other,

\[ R_{ABCD} = F_{CDpq} V_A^p V_B^q + \bar{F}_{CD\bar{p}\bar{q}} \bar{V}_A^\bar{p} \bar{V}_B^\bar{q}. \]

However, the crucial object in DFT turns out to be

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma_{EAB}^E \Gamma_{ECD} \right), \]

which we name semi-covariant curvature.
Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,
  \[ S_{ABCD} = \frac{1}{2} \left( S_{[AB][CD]} + S_{[CD][AB]} \right), \]
  \[ S^0_{[ABC]D} = 0. \]

- Projection property,
  \[ P_I^A \bar{P}_J^B P^K_C \bar{P}^D_L S_{ABCD} \equiv 0. \]

- Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as
  \[ \delta S_{ABCD} = \mathcal{D}_{[A} \delta \Gamma_{B]CD} + \mathcal{D}_{[C} \delta \Gamma_{D]AB} - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^E_{CD} - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^E_{AB}, \]
  \[ \delta S^0_{ABCD} = \mathcal{D}_{[A} \delta \Gamma^0_{B]CD} + \mathcal{D}_{[C} \delta \Gamma^0_{D]AB}. \]
Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,

\[ S_{ABCD} = \frac{1}{2} \left( S_{[AB][CD]} + S_{[CD][AB]} \right), \]

\[ S^0_{[ABC]D} = 0. \]

- Projection property,

\[ P^A_P^B_P^C_P^D S_{ABCD} \equiv 0. \]

- Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as

\[ \delta S_{ABCD} = D_A \delta \Gamma_{B[CD]} + D_C \delta \Gamma_{D[AB]} - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^E_{CD} - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^E_{AB}, \]

\[ \delta S^0_{ABCD} = D_A \delta \Gamma_{B[CD]}^0 + D_C \delta \Gamma_{D[AB]}^0. \]
Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,

\[ S_{ABCD} = \frac{1}{2} \left( S_{[AB][CD]} + S_{[CD][AB]} \right), \]
\[ S^0_{[ABC]D} = 0. \]

- Projection property,

\[ P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} \equiv 0. \]

- Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as

\[ \delta S_{ABCD} = D_{[A} \delta \Gamma_{B]CD} + D_{[C} \delta \Gamma_{D]}AB - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^E \text{CD} - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^E \text{AB}, \]
\[ \delta S^0_{ABCD} = D_{[A} \delta \Gamma^0_{B]CD} + D_{[C} \delta \Gamma^0_{D]}AB. \]
Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,
  \[ S_{ABCD} = \frac{1}{2} (S_{[AB][CD]} + S_{[CD][AB]}) , \]
  \[ S^0_{[ABC]D} = 0 . \]

- Projection property,
  \[ P^A_I \bar{P}^B_J P^K_C \bar{P}^D_L S_{ABCD} \equiv 0 . \]

- Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as
  \[ \delta S_{ABCD} = \mathcal{D}_{[A} \delta \Gamma_{B]CD} + \mathcal{D}_{[C} \delta \Gamma_{D]AB} - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^E CD - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^E AB , \]
  \[ \delta S^0_{ABCD} = \mathcal{D}_{[A} \delta \Gamma^0_{B]CD} + \mathcal{D}_{[C} \delta \Gamma^0_{D]AB} . \]
Generically, under $\delta_X P_{AB} = \hat{L}_X P_{AB}$, $\delta_X d = \hat{L}_X d$ (DFT-diffeomorphism), the variation of $\nabla_C T_{A_1...A_n}$ contains an anomalous non-covariant part,

$$\delta_X (\nabla_C T_{A_1...A_n}) \equiv \hat{L}_X (\nabla_C T_{A_1...A_n}) + \sum_i 2(\mathcal{P} + \overline{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[DX_E]} T_{B...}.$$

Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{L}_X.$$

However, the characteristic property of our ‘semi-covariant’ derivative is that, combined with the projectors it can generate various fully covariant quantities, as listed below.
Generically, under $\delta_X P_{AB} = \hat{L}_X P_{AB}$, $\delta_X d = \hat{L}_X d$ (DFT-diffeomorphism), the variation of $\nabla_C T_{A_1 \ldots A_n}$ contains an anomalous non-covariant part,

$$\delta_X (\nabla_C T_{A_1 \ldots A_n}) \equiv \hat{L}_X (\nabla_C T_{A_1 \ldots A_n}) + \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[DX_E]} T_{B \ldots}.$$

Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{L}_X.$$

However, the characteristic property of our ‘semi-covariant’ derivative is that, combined with the projectors it can generate various fully covariant quantities, as listed below.
Generically, under $\delta_X P_{AB} = \hat{L}_X P_{AB}$, $\delta_X d = \hat{L}_X d$ (DFT-diffeomorphism), the variation of $\nabla_C T_{A_1\ldots A_n}$ contains an anomalous non-covariant part,

$$\delta_X (\nabla_C T_{A_1\ldots A_n}) \equiv \hat{L}_X (\nabla_C T_{A_1\ldots A_n}) + \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[DX]} T_{B\ldots}.$$

Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{L}_X.$$

However, the characteristic property of our ‘semi-covariant’ derivative is that, combined with the projectors it can generate various fully covariant quantities, as listed below.
For $O(D, D)$ tensors:

\begin{align*}
P_C^D \bar{P}_{A_1} B_1 \bar{P}_{A_2} B_2 \cdots \bar{P}_{A_n} B_n \nabla_D T_{B_1 B_2 \cdots B_n}, \\
\bar{P}_C^D P_{A_1} B_1 P_{A_2} B_2 \cdots P_{A_n} B_n \nabla_D T_{B_1 B_2 \cdots B_n},
\end{align*}

\begin{align*}
P^{AB} \bar{P}_{C_1} D_1 \bar{P}_{C_2} D_2 \cdots \bar{P}_{C_n} D_n \nabla_A T_{B D_1 D_2 \cdots D_n}, \\
\bar{P}^{AB} P_{C_1} D_1 P_{C_2} D_2 \cdots P_{C_n} D_n \nabla_A T_{B D_1 D_2 \cdots D_n}
\end{align*}

\{ Divergences, \\
\{ Laplacians. 

\}
Projector-aided, fully covariant derivatives

For Spin(1, D−1)$_L \times$ Spin(D−1, 1)$_R$ tensors:

\[
\begin{align*}
D_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n}, & \quad D_{\bar{p}} T_{q_1 q_2 \cdots q_n}, \\
D^p T_{p \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n}, & \quad D^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n}, \\
D_p D^p T_{q_1 \bar{q}_2 \cdots \bar{q}_n}, & \quad D_{\bar{p}} D^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n},
\end{align*}
\]

where we set

\[
D_p := V^A_p D_A, \quad D_{\bar{p}} := \bar{V}^A_{\bar{p}} D_A.
\]

These are the pull-back of the previous results using the DFT-vielbeins.
Dirac operators for fermions, \( \rho^\alpha, \psi^\alpha_p, \rho'^{\bar{\alpha}}, \psi'^{\bar{\alpha}} \):

\[
\begin{align*}
\gamma^p D_p \rho &= \gamma^A D_A \rho, \\
D_{\bar{p}} \rho &=, \\
\bar{\psi}^A \gamma_p (D_A \psi_q - \frac{1}{2} D_q \psi_A),
\end{align*}
\]

\[
\begin{align*}
\bar{\gamma}^\bar{p} D_{\bar{p}} \rho' &= \bar{\gamma}^A D_A \rho', \\
D_p \rho' &=, \\
\bar{\psi}'^A \bar{\gamma}_{\bar{p}} (D_A \psi'_q - \frac{1}{2} D_q \psi'_A).
\end{align*}
\]

Incorporation of fermions into DFT 1109.2035
For Spin\((1, D-1)_L \times \text{Spin}(D-1, 1)_R\) bi-fundamental spinorial fields, \(T^\alpha \bar{\beta}_\gamma\):

\[
\mathcal{D}_+ T := \gamma^A \mathcal{D}_A T + \gamma^{(D+1)} \mathcal{D}_A T \bar{\gamma}^A,
\]
\[
\mathcal{D}_- T := \gamma^A \mathcal{D}_A T - \gamma^{(D+1)} \mathcal{D}_A T \bar{\gamma}^A.
\]

Especially for the torsionless case, the corresponding operators are \text{nilpotent}:

\[
(\mathcal{D}_+^0)^2 T \equiv 0, \quad (\mathcal{D}_-^0)^2 T \equiv 0,
\]

and hence, they define \(O(D, D)\) covariant cohomology.

The field strength of the R-R potential, \(C^\alpha \bar{\alpha}_\gamma\), is then defined by

\[
\mathcal{F} := \mathcal{D}_+^0 C.
\]

Thanks to the nilpotency, the R-R gauge symmetry is simply realized

\[
\delta C = \mathcal{D}_+^0 \Delta \quad \Longrightarrow \quad \delta \mathcal{F} = \mathcal{D}_+^0 (\delta C) = (\mathcal{D}_+^0)^2 \Delta \equiv 0.
\]
For $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ bi-fundamental spinorial fields, $T^\alpha_{\bar{\beta}}$:

$$
\mathcal{D}_+ T := \gamma^A D_A T + \gamma^{(D+1)} D_A T \bar{\gamma}^A,
$$

$$
\mathcal{D}_- T := \gamma^A D_A T - \gamma^{(D+1)} D_A T \bar{\gamma}^A.
$$

Especially for the torsionless case, the corresponding operators are nilpotent

$$(\mathcal{D}_0^+)^2 T \equiv 0, \quad (\mathcal{D}_0^-)^2 T \equiv 0,$$

and hence, they define $O(D, D)$ covariant cohomology.

The field strength of the R-R potential, $C^\alpha_{\bar{\alpha}}$, is then defined by

$$
\mathcal{F} := \mathcal{D}_0^+ C.
$$

Thanks to the nilpotency, the R-R gauge symmetry is simply realized

$$
\delta C = \mathcal{D}_0^+ \Delta \quad \Rightarrow \quad \delta \mathcal{F} = \mathcal{D}_0^+ (\delta C) = (\mathcal{D}_0^+)^2 \Delta \equiv 0.
$$
**Projector-aided, fully covariant derivatives**

- **For Spin(1, D−1)\_L \times Spin(D−1, 1)\_R bi-fundamental spinorial fields, \( T^{\alpha \bar{\beta}} : \)**
  
  \[
  \mathcal{D}_+ T := \gamma^A D_A T + \gamma^{(D+1)} D_A T \gamma^A ,
  \]

  \[
  \mathcal{D}_- T := \gamma^A D_A T - \gamma^{(D+1)} D_A T \gamma^A .
  \]

- Especially for the torsionless case, the corresponding operators are **nilpotent**
  
  \[
  (\mathcal{D}_+^0)^2 T \equiv 0 , \quad (\mathcal{D}_-^0)^2 T \equiv 0 ,
  \]

  and hence, they define **O(D, D) covariant cohomology.**

- **The field strength of the R-R potential, \( C^{\alpha \bar{\alpha}} \), is then defined by**
  
  \[
  \mathcal{F} := \mathcal{D}_+^0 C .
  \]

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized
  
  \[
  \delta C = \mathcal{D}_+^0 \Delta \quad \implies \quad \delta \mathcal{F} = \mathcal{D}_+^0 (\delta C) = (\mathcal{D}_+^0)^2 \Delta \equiv 0 .
  \]
For Spin\((1, D-1)_L \times Spin(D-1, 1)_R\) bi-fundamental spinorial fields, \(T^\alpha{}_{\bar{\beta}}:\)

\[
\begin{align*}
D_+ T &= \gamma^A D_A T + \gamma^{(D+1)} D_A T \gamma^A, \\
D_- T &= \gamma^A D_A T - \gamma^{(D+1)} D_A T \gamma^A.
\end{align*}
\]

Especially for the torsionless case, the corresponding operators are nilpotent

\[(D^0_+)^2 T \equiv 0, \quad (D^0_-)^2 T \equiv 0,
\]

and hence, they define \(O(D, D)\) covariant cohomology.

The field strength of the R-R potential, \(C^\alpha{}_{\bar{\alpha}}\), is then defined by

\[
\mathcal{F} := D^0_+ C.
\]

Thanks to the nilpotency, the R-R gauge symmetry is simply realized

\[
\delta C = D^0_+ \Delta \quad \Rightarrow \quad \delta \mathcal{F} = D^0_+ (\delta C) = (D^0_+)^2 \Delta \equiv 0.
\]
Projector-aided, fully covariant curvatures

**Scalar curvature:**

\[(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} \cdot \]

**“Ricci” curvature:**

\[S_{p\bar{q}} + \frac{1}{2} D_{\bar{r}} \bar{\Delta}_{p\bar{q}} \bar{r} + \frac{1}{2} D_r \Delta_{\bar{p} \bar{q}} r,\]

where we set

\[S_{p\bar{q}} := V^A_p \bar{V}^B_{\bar{q}} S_{AB}, \quad S_{AB} = S_{ACB}^C.\]
**Scalar curvature:**

\[(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD}.\]

**“Ricci” curvature:**

\[S_{p\bar{q}} + \frac{1}{2} \bar{D}_\bar{r} \Delta_{p\bar{q}} \bar{r} + \frac{1}{2} D_r \Delta_{\bar{q}p} ^r,\]

where we set

\[S_{p\bar{q}} := V^A p \bar{V}^B_{\bar{q}} S_{AB}, \quad S_{AB} = S_{ACB}^C.\]
Combining all the results above, we are now ready to spell

**Type II i.e. \( \mathcal{N} = 2 \ D = 10 \) Supersymmetric Double Field Theory**
**Lagrangian**:

\[
\mathcal{L}_{\text{Type II}} = e^{-2d}\left[\frac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \frac{1}{2}\text{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_p\gamma_q\mathcal{F}\bar{\gamma}_p\psi'^q
\]

\[
+ i\frac{1}{2}\bar{\rho}\gamma^pD^*_p\rho - i\bar{\psi}_p\bar{\gamma}^qD^*_q\psi_p - i\frac{1}{2}\bar{\rho}'\bar{\gamma}_p\bar{D}^*_p\rho' + i\bar{\psi}'_p\bar{D}'^*_p\rho' + i\frac{1}{2}\bar{\psi}'_p\bar{\gamma}_q\bar{D}'^*_q\psi'^q\right].
\]

where $\bar{\mathcal{F}}\bar{\alpha}_\alpha$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_+^{-1}\mathcal{F}^T C_+$. 

As they are contracted with the DFT-vielbeins properly, 

**every term in the Lagrangian is fully covariant.**
Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

**Lagrangian**:

$$\mathcal{L}_{\text{Type II}} = e^{-2d} \left[ \frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(F \bar{F}) - i \bar{\rho} F \rho' + i \bar{\psi}_D \gamma_q F \bar{\gamma} \bar{\psi} \bar{\gamma} q \right]$$

$$+ i \frac{1}{2} \bar{\rho} \gamma^p D_{[p}^* \rho - i \bar{\psi}_D \bar{\gamma} q D_{q}^* \psi_{[p} - i \frac{1}{2} \bar{\rho}' \bar{\gamma} \bar{\gamma} D_{p}^* \rho' + i \bar{\psi}' D_{p}^* \rho' + i \frac{1}{2} \bar{\psi}' D_{[p} \bar{\gamma} D_{q}^* \psi_{[p} \right].$$

where $\bar{F}^{\alpha \alpha}$ denotes the charge conjugation, $\bar{F} := \bar{C}^{-1} F^T C_+$.  

As they are contracted with the DFT-vielbeins properly, 

**every term in the Lagrangian is fully covariant.**
Lagrangian:

\[
\mathcal{L}_{\text{Type II}} = e^{-2d} \left[ \frac{1}{8} (\mathcal{P}^{AB} \mathcal{P}^{CD} - \mathcal{P}^{AB} \mathcal{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \tilde{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi} \gamma_q \mathcal{F} \bar{\gamma} \bar{p} \psi' q + i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \bar{D}^*_p \rho - i \frac{1}{2} \bar{\psi} \gamma^q D^*_q \psi_p - i \frac{1}{2} \bar{\rho}' \bar{\gamma} \bar{p} D^*_p \rho' + i \bar{\psi}' \bar{D}^*_p \rho' + i \frac{1}{2} \bar{\psi}' \gamma^q D^*_q \psi'_p \right].
\]

Torsions: The semi-covariant curvature, \( S_{ABCD} \), is given by the connection,

\[
\Gamma_{ABC} = \Gamma^0_{ABC} + i \frac{1}{3} \bar{\rho} \gamma_{ABC} \rho - 2i \bar{\rho} \gamma_{BC} \psi_A - i \frac{1}{3} \bar{\psi} \bar{\gamma} \gamma_{ABC} \psi_p + 4i \bar{\psi}_B \gamma A \psi_C
\]

\[
+ i \frac{1}{3} \bar{\rho}' \bar{\gamma} \gamma_{ABC} \rho' - 2i \bar{\rho}' \bar{\gamma} \gamma_{BC} \psi'_A - i \frac{1}{3} \bar{\psi}' \bar{\gamma} \gamma_{ABC} \psi'_p + 4i \bar{\psi}'_B \bar{\gamma} A \psi'_C,
\]

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:

\( D^*_A \) for the unprimed fermions and \( D'^*_A \) for the primed fermions, set by

\[
\Gamma^*_A = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho} \gamma_{ABC} \rho + i \frac{5}{4} \bar{\rho} \gamma_{BC} \psi_A + i \frac{5}{24} \bar{\psi} \bar{\gamma} \gamma_{ABC} \psi_p - 2i \bar{\psi}_B \gamma A \psi_C + i \frac{5}{2} \bar{\rho}' \bar{\gamma} \gamma_{BC} \psi'_A,
\]

\[
\Gamma'^*_A = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho}' \bar{\gamma} \gamma_{ABC} \rho' + i \frac{5}{4} \bar{\rho}' \bar{\gamma} \gamma_{BC} \psi'_A + i \frac{5}{24} \bar{\psi}' \bar{\gamma} \gamma_{ABC} \psi'_p - 2i \bar{\psi}'_B \bar{\gamma} A \psi'_C + i \frac{5}{2} \bar{\rho} \gamma_{BC} \psi_A.
\]
**Lagrangian:**

\[
L_{\text{Type II}} = e^{-2d} \left[ \frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(F \tilde{F}) - i \bar{\rho} F \rho' + i \bar{\psi} \gamma_q F \gamma \bar{p} \psi' q \\
+ i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \bar{\bar{p}} \gamma^q D^*_q \psi \rho - i \frac{1}{2} \bar{\rho}' \gamma \bar{p} \gamma^q D^*_q \rho' + i \bar{\psi}' \bar{p} \gamma^q D^*_q \rho' + i \frac{1}{2} \bar{\psi}' \bar{p} \gamma \bar{q} \gamma^q D^*_q \rho' \right].
\]

**Torsions:** The semi-covariant curvature, \( S_{ABCD} \), is given by the connection,

\[
\Gamma_{ABC} = \Gamma^0_{ABC} + i \frac{1}{3} \bar{\rho} \gamma_{ABC} \rho - 2i \bar{\rho} \gamma_{BC} \psi_A - i \frac{1}{3} \bar{\psi} \bar{\bar{p}} \gamma_{ABC} \psi \rho + 4i \bar{\psi}_B \gamma A \psi_C \\
+ i \frac{1}{3} \bar{\rho}' \gamma_{ABC} \rho' - 2i \bar{\rho}' \gamma_{BC} \psi' A - i \frac{1}{3} \bar{\psi}' \bar{p} \gamma_{ABC} \psi' p + 4i \bar{\psi}'_B \gamma A \psi' C,
\]

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:

\( D^*_A \) for the unprimed fermions and \( D'^*_A \) for the primed fermions, set by

\[
\Gamma^*_A = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho} \gamma_{ABC} \rho + i \frac{5}{4} \bar{\rho} \gamma_{BC} \psi_A + i \frac{5}{24} \bar{\psi} \bar{\bar{p}} \gamma_{ABC} \psi \rho - 2i \bar{\psi}_B \gamma A \psi_C + i \frac{5}{2} \bar{\rho}' \gamma_{BC} \psi_A,
\]

\[
\Gamma'^*_A = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho}' \gamma_{ABC} \rho' + i \frac{5}{4} \bar{\rho}' \gamma_{BC} \psi' A + i \frac{5}{24} \bar{\psi}' \bar{p} \gamma_{ABC} \psi' p - 2i \bar{\psi}'_B \gamma A \psi' C + i \frac{5}{2} \bar{\rho} \gamma_{BC} \psi A.
\]
The $\mathcal{N} = 2$ supersymmetry transformation rules are

\[
\delta_{\epsilon} d = -i \frac{1}{2} (\bar{\epsilon} \rho + \bar{\epsilon'} \rho'), \\
\delta_{\epsilon} V_{Ap} = i \bar{V}_A \bar{q} (\bar{\epsilon'} \bar{\gamma q} \psi'_{\bar{p}} - \bar{\epsilon} \gamma p \psi_{\bar{q}}), \\
\delta_{\epsilon} \bar{V}_{A\bar{p}} = i V_A q (\bar{\epsilon} \gamma q \psi_{\bar{p}} - \bar{\epsilon'} \bar{\gamma p} \psi'_{\bar{q}}), \\
\delta_{\epsilon} C = i \frac{1}{2} (\gamma^p \epsilon \bar{\psi}'_{\bar{p}} - \epsilon \bar{\rho} - \psi_{\bar{p}} \bar{\epsilon}' \bar{\gamma} \bar{p} + \rho \bar{\epsilon}'), + C \delta_{\epsilon} d - \frac{1}{2} (\bar{V}_A q \delta_{\epsilon} V_{Ap}) \gamma^{(d+1)} \gamma^p C \bar{\gamma} q, \\
\delta_{\epsilon} \rho = -\gamma^p \hat{D}_p \epsilon + i \frac{1}{2} \gamma^p \epsilon \bar{\psi}'_{\bar{p}} \rho' - i \gamma^p \psi_{\bar{q}} \bar{\epsilon}' \bar{\gamma} q \psi', \\
\delta_{\epsilon} \rho' = -\bar{\gamma} \hat{D}'_{\bar{p}} \epsilon' + i \frac{1}{2} \bar{\gamma} \bar{\epsilon}' \bar{\psi}'_{\bar{p}} \rho - i \bar{\gamma} q \psi'_{\bar{p}} \bar{\epsilon} \gamma p q, \\
\delta_{\epsilon} \psi_{\bar{p}} = \hat{D}_p \epsilon + (\bar{\mathcal{F}} - i \frac{1}{2} \gamma^q \rho \bar{\psi}_q + i \frac{1}{2} \psi_{\bar{q}} \bar{\rho} \bar{\gamma} q) \bar{\gamma}_p \epsilon' + i \frac{1}{4} \epsilon \bar{\psi}_{\bar{p}} \rho + i \frac{1}{2} \psi_{\bar{p}} \bar{\epsilon} \rho, \\
\delta_{\epsilon} \psi'_{\bar{p}} = \hat{D}'_p \epsilon' + (\bar{\mathcal{F}} - i \frac{1}{2} \bar{\gamma} \rho' \bar{\psi} q + i \frac{1}{2} \psi'_{\bar{q}} \bar{\rho} \gamma q) \gamma p \epsilon + i \frac{1}{4} \epsilon' \bar{\psi}'_{\bar{p}} \rho' + i \frac{1}{2} \psi'_{\bar{p}} \bar{\epsilon}' \rho',
\]

where

\[
\hat{\Gamma}_{ABC} = \Gamma_{ABC} - i \frac{17}{48} \bar{\rho} \gamma_{ABC} \rho + i \frac{5}{2} \bar{\rho} \gamma_{BC} \psi_{\bar{A}} + i \frac{1}{4} \psi_{\bar{p}} \gamma_{ABC} \psi_{\bar{p}} - 3 i \bar{\psi}'_{\bar{B}} \bar{\gamma} A \psi'_{\bar{C}}, \\
\hat{\Gamma}'_{ABC} = \Gamma_{ABC} - i \frac{17}{48} \bar{\rho}' \gamma_{ABC} \rho' + i \frac{5}{2} \bar{\rho}' \gamma_{BC} \psi'_{\bar{A}} + i \frac{1}{4} \psi'_{\bar{p}} \gamma_{ABC} \psi'_{\bar{p}} - 3 i \bar{\psi}'_{\bar{B}} \bar{\gamma} A \psi'_{\bar{C}}.
\]
Lagrangian:

\[ \mathcal{L}_{\text{Type II}} = e^{-2d} \left[ \frac{1}{2} (P^{AB} P^{CD} - \tilde{P}^{AB} \tilde{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \tilde{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi}_{\bar{p}} \gamma q \mathcal{F} \bar{\gamma} \bar{\rho} \psi' q \\
+ i \frac{1}{2} \bar{\rho} \gamma p D^*_p \rho - i \bar{\psi}_{\bar{p}} \gamma q D^*_q \bar{\psi}_{\bar{p}} - i \frac{1}{2} \bar{\psi}_{\bar{p}} \gamma q D^*_q \psi_{\bar{p}} - i \frac{1}{2} \bar{\rho}' \bar{\gamma} \bar{p} D'^*_p \rho' + i \psi' p \bar{\gamma} \bar{q} D'^*_q \psi' p \right] . \]

The Lagrangian is pseudo: It is necessary to impose a self-duality of the R-R field strength by hand,

\[ \tilde{\mathcal{F}}_- := \left( 1 - \gamma^{(D+1)} \right) \left( \mathcal{F} - i \frac{1}{2} \rho \rho' + i \frac{1}{2} \gamma^p \psi_{\bar{p}} \bar{\psi}_{\bar{q}} \bar{\gamma} \bar{q} \right) = 0 . \]
Lagrangian:

\[ \mathcal{L}_{\text{Type II}} = e^{-2d} \left[ \frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \tilde{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi}_p \gamma_q \mathcal{F} \tilde{\gamma} \bar{\rho} \psi' q 
+ i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \bar{D}^*_p \rho - i \frac{1}{2} \bar{\psi} \bar{\gamma} q D^*_q \psi \bar{p} - i \frac{1}{2} \bar{\rho'} \tilde{\gamma} \bar{D}^*_p \rho' + i \bar{\psi}' \bar{D}^*_p \rho' + i \frac{1}{2} \bar{\psi}' \bar{\gamma} q D^*_q \psi' p \right] . \]

The Lagrangian is pseudo: It is necessary to impose a self-duality of the R-R field strength by hand,

\[ \tilde{\mathcal{F}}_- := \left( 1 - \gamma^{(D+1)} \right) \left( \mathcal{F} - i \frac{1}{2} \rho \rho' + i \frac{1}{2} \gamma^p \psi_q \bar{\psi}_p \tilde{\gamma} \bar{q} \right) \equiv 0 . \]
Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as
\[
\delta_\epsilon \mathcal{L}_{\text{Type II}} \simeq -\frac{1}{8} e^{-2d} \bar{V}^A \bar{q} \delta_\epsilon V_A \text{Tr} \left( \gamma^p \tilde{\mathcal{F}}_- \tilde{q} \bar{\mathcal{F}}_- \right),
\]
where
\[
\tilde{\mathcal{F}}_- := \left( 1 - \gamma^{(D+1)} \right) \left( \mathcal{F} - i \frac{1}{2} \rho \bar{\rho}' + i \frac{1}{2} \gamma^p \psi \bar{\gamma} \psi' \bar{p} \right).
\]
This verifies, to the full order in fermions, the supersymmetric invariance of the action, modulo the self-duality.

For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,
\[
\delta_\epsilon \tilde{\mathcal{F}}_- = -i \left( \tilde{D}_p \rho + \gamma^p \tilde{D}_p \psi \bar{p} - \gamma^p \mathcal{F} \bar{\gamma} \psi' \bar{p} \right) \bar{\varepsilon}' \bar{\gamma} \bar{p} - i \gamma^p \left( \tilde{D}_p \rho' + \tilde{D}_p \psi' \bar{\gamma} \bar{p} - \bar{\psi} \gamma_p \mathcal{F} \bar{\gamma} \bar{p} \right).
\]
Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_\varepsilon \mathcal{L}_{\text{Type II}} \simeq -\frac{1}{8} e^{-2d} V^A \bar{q} \delta_\varepsilon V_A \text{Tr} \left( \gamma^p \tilde{F}_- \bar{\gamma} \tilde{q} \tilde{F}_- \right),$$

where

$$\tilde{F}_- := \left( 1 - \gamma^{(D+1)} \right) \left( F - i \frac{1}{2} \bar{\rho} \bar{\rho}' + i \frac{1}{2} \gamma^p \psi \bar{q} \gamma^p \bar{\psi} \bar{\gamma} \right).$$

This verifies, to the full order in fermions, the supersymmetric invariance of the action, modulo the self-duality.

For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_\varepsilon \tilde{F}_- = -i \left( \tilde{D}_p \rho + \gamma^p \tilde{D}_p \bar{\psi} \bar{p} - \gamma^p F \bar{\gamma} \bar{p} \bar{\psi} \right) \bar{\varepsilon}' \bar{\gamma} \bar{p} - i \gamma^p \varepsilon \left( \tilde{D}_p \bar{\rho}' + \tilde{D}_p \bar{\psi}' \bar{\gamma} \bar{p} - \bar{\psi} \gamma_p F \bar{\gamma} \right).$$
Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{\bar{p}q} + \text{Tr}(\gamma_p \mathcal{F} \bar{\gamma}_q \bar{F}) + i \bar{\rho}_\gamma p \bar{D}_q \rho + 2i \bar{\psi}_q \bar{D}_p \rho - i \bar{\psi}^p \gamma_p \bar{D}_q \psi_p + i \bar{\rho}'_\gamma q \bar{D}_p \rho' + 2i \bar{\psi}'_p \bar{D}_q \rho' - i \bar{\psi}' q \bar{\gamma}_q \bar{D}_p \psi_q' = 0. \]

This is DFT-generalization of Einstein equation.

- **DFT-dilaton:**

\[ \mathcal{L}_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**

\[ \mathcal{D}^0_-(\mathcal{F} - i \rho \bar{\rho}' + i \gamma' \psi_s \bar{\psi}' \bar{\gamma}^s) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( \mathcal{D}^0_+ \),

\[ \mathcal{D}^0_-(\mathcal{F} - i \rho \bar{\rho}' + i \gamma' \psi_s \bar{\psi}' \bar{\gamma}^s) = \mathcal{D}^0_-(\gamma^{(D+1)}\mathcal{F}) = -\gamma^{(D+1)} \mathcal{D}^0_+ \mathcal{F} = -\gamma^{(D+1)}(\mathcal{D}^0_+)^2 c = 0. \]

- **The 1.5 formalism** works: The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{\rho \bar{q}} + \text{Tr}(\gamma_p \mathcal{F} \tilde{\gamma}_q \tilde{\mathcal{F}}) + i \bar{\rho} \gamma_p \tilde{D}_q \rho + 2 i \bar{\psi}_q \tilde{D}_p \rho - i \bar{\psi} \gamma_p \tilde{D}_q \psi_p + i \bar{\rho}' \tilde{\gamma}_q \tilde{D}_p \rho' + 2 i \bar{\psi}'_p \tilde{D}_q \rho' - i \bar{\psi}' q \tilde{\gamma}_q \tilde{D}_p \psi_q = 0. \]

This is **DFT-generalization of Einstein equation.**

- **DFT-dilaton:**

\[ \mathcal{L}_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**

\[ \mathcal{D}_-^0 \left( \mathcal{F} - i \rho \rho' + i \gamma' \psi_{\bar{s}} \bar{\psi}'_{\bar{s}} \bar{\gamma}_{\bar{s}} \right) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( \mathcal{D}_+^0, \)

\[ \mathcal{D}_-^0 \left( \mathcal{F} - i \rho \rho' + i \gamma' \psi_{\bar{s}} \bar{\psi}'_{\bar{s}} \bar{\gamma}_{\bar{s}} \right) = \mathcal{D}_-^0 \left( \gamma^{(D+1)} \mathcal{F} \right) = -\gamma^{(D+1)} \mathcal{D}_+^0 \mathcal{F} = -\gamma^{(D+1)} (\mathcal{D}_+^0)^2 \mathcal{C} = 0. \]

- **The 1.5 formalism** works: The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{p\bar{q}} + \text{Tr}(\gamma_p \tilde{F} \gamma_{\bar{q}} \tilde{F}) + i \bar{\rho} \gamma_p \tilde{D}_{\bar{q}} \rho + 2i \bar{\psi}_{\bar{q}} \tilde{D}_\rho \psi + i \bar{\rho}' \gamma_{\bar{q}} \tilde{D}_{\rho'} \psi + 2i \bar{\psi}' \rho \tilde{D}_{\bar{q}} \rho' - i \bar{\psi}' \gamma_{\bar{q}} \tilde{D}_\rho \psi' = 0. \]

This is **DFT-generalization of Einstein equation.**

- **DFT-dilaton:**

\[ \mathcal{L}_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**

\[ D_0^- \left( \mathcal{F} - i \rho \bar{\rho}' + i \gamma' \bar{\psi} \bar{\psi}' \bar{\gamma} \right) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( D_0^+ \),

\[ D_0^- \left( \mathcal{F} - i \rho \bar{\rho}' + i \gamma' \bar{\psi} \bar{\psi}' \bar{\gamma} \right) = D_0^+ \left( \gamma^{(D+1)} \mathcal{F} \right) = -\gamma^{(D+1)} D_0^+ \mathcal{F} = -\gamma^{(D+1)} (D_0^+) 2 \mathcal{C} = 0. \]

- **The 1.5 formalism works:** The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Bosons

- DFT-vielbein:

\[ S_{\rho \bar{q}} + \text{Tr}(\gamma_{\rho} F \bar{\gamma}_{\bar{q}} \tilde{F}) + i \bar{\rho} \gamma_{\rho} \tilde{D}_{\rho} \rho + 2i \bar{\psi}_{\bar{q}} \tilde{D}_{\rho} \psi_{\bar{p}} + i \bar{\rho}' \bar{\gamma}_{\bar{q}} \tilde{D}_{\rho} \rho' + 2i \bar{\psi}' p \tilde{D}_{\rho} \rho' - i \bar{\psi}' q \bar{\gamma}_{\bar{q}} \tilde{D}_{\rho} \psi_q = 0. \]

This is DFT-generalization of Einstein equation.

- DFT-dilaton:

\[ L_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- R-R potential:

\[ D_0^-(F - i \rho \bar{\rho}' + i \gamma' \psi_{\bar{s}} \bar{\psi}' \bar{\gamma} \bar{\bar{s}}) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( D_0^+ \),

\[ D_0^-(F - i \rho \bar{\rho}' + i \gamma' \psi_{\bar{s}} \bar{\psi}' \bar{\gamma} \bar{\bar{s}}) = D_0^-(\gamma^{(D+1)} F) = -\gamma^{(D+1)} D_0^+ F = -\gamma^{(D+1)} (D_0^0)^2 C = 0. \]

The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta L_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Bosons

- **DFT-vielbein:**
  \[ S_{\rho\bar{q}} + \text{Tr}(\gamma_{p} F \bar{\gamma}_{q} \bar{F}) + i \bar{\rho} \gamma_{p} \bar{D}_{q} \rho + 2i \bar{\psi}_{q} \bar{D}_{p} \rho - i \bar{\psi}^{\rho} \gamma_{p} \bar{D}_{q} \psi_{p} + i \bar{\psi}^{p} \bar{D}_{q} \rho' + 2i \bar{\psi}'_{p} \bar{D}_{q} \rho' - i \bar{\psi}'^{q} \bar{\gamma}_{q} \bar{D}_{p} \psi_{q} = 0. \]

  This is **DFT-generalization of Einstein equation.**

- **DFT-dilaton:**
  \[ \mathcal{L}_{\text{Type II}} = 0. \]

  Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**
  \[ D_{0}^{-} \left( F - i \rho \bar{\rho}' + i \gamma' \psi_{r} \bar{\psi}'_{s} \bar{\gamma}'_{s} \right) = 0, \]
  which is automatically met by the self-duality, together with the nilpotency of \( D_{0}^{+}, \)
  \[ D_{0}^{-} \left( F - i \rho \bar{\rho}' + i \gamma' \psi_{r} \bar{\psi}'_{s} \bar{\gamma}'_{s} \right) = D_{0}^{-} \left( \gamma^{(D+1)} F \right) = -\gamma^{(D+1)} D_{0}^{+} F = -\gamma^{(D+1)} (D_{0}^{+})^{2} \mathcal{C} = 0. \]

- The **1.5 formalism** works: The variation of the Lagrangian induced by that of the connection is trivial, \( \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \)
Equations of Motion for Fermions

\[ \gamma^p \tilde{D}_p \rho - \tilde{D}_p \tilde{\psi} \rho - \mathcal{F} \rho' = 0, \quad \tilde{\gamma}^\rho \tilde{D}_\rho \rho' - \tilde{D}_\rho \psi' \rho - \tilde{\mathcal{F}} \rho = 0. \]

Gravitinos,

\[ \tilde{D}_\rho \rho + \gamma^p \tilde{D}_p \tilde{\psi} \rho - \gamma^p \mathcal{F} \tilde{\gamma}_p \psi' \rho = 0, \quad \tilde{D}_p \rho' + \tilde{\gamma}^p \tilde{D}_p \psi' \rho - \tilde{\gamma}^p \tilde{\mathcal{F}} \gamma_p \psi_\rho = 0. \]
Equations of Motion for Fermions

- DFT-dilationos,

\[ \gamma^\rho \tilde{D}_\rho \rho - \tilde{D}_\rho \psi^\bar{\rho} - F \rho' = 0, \quad \bar{\gamma}^{\bar{\rho}} \tilde{D}_{\bar{\rho}} \rho' - \tilde{D}_\rho \psi'^\rho - F \rho = 0. \]

- Gravitinos,

\[ \tilde{D}_\rho \rho + \gamma^\rho \tilde{D}_\rho \psi^\bar{\rho} - \gamma^\rho F \bar{\gamma}_{\bar{\rho}} \psi' p = 0, \quad \tilde{D}_{\bar{\rho}} \rho' + \bar{\gamma}^{\bar{\rho}} \tilde{D}_{\bar{\rho}} \psi'^\rho - \bar{\gamma}^{\bar{\rho}} F \gamma_p \psi_{\bar{p}} = 0. \]
Turning off the primed fermions and the R-R sector truncates the \( \mathcal{N} = 2 D = 10 \) SDFT to \( \mathcal{N} = 1 D = 10 \) SDFT,

\[
\mathcal{L}_{\mathcal{N}=1} = e^{-2d} \left[ \frac{1}{8} \left( P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD} \right) S_{ACBD} + i \frac{1}{2} \bar{\rho} \gamma^A D_A^* \rho - i \bar{\psi}^A D_A^* \psi - i \frac{1}{2} \bar{\psi}^B \gamma^A D_A^* \psi_B \right].
\]

\( \mathcal{N} = 1 \) Local SUSY:

\[
\begin{align*}
\delta_\varepsilon d &= -i \frac{1}{2} \bar{\varepsilon} \rho, \\
\delta_\varepsilon V_{Ap} &= -i \bar{\varepsilon} \gamma^p \psi_A, \\
\delta_\varepsilon \bar{V}_{A\bar{p}} &= i \bar{\varepsilon} \gamma_A \psi_{\bar{p}}, \\
\delta_\varepsilon \rho &= -\gamma^A \hat{D}_A \varepsilon, \\
\delta_\varepsilon \psi_{\bar{p}} &= \bar{V}^A_{\bar{p}} \hat{D}_A \varepsilon - i \frac{1}{4} (\bar{\rho} \psi_{\bar{p}}) \varepsilon + i \frac{1}{2} (\bar{\varepsilon} \rho) \psi_{\bar{p}}.
\end{align*}
\]
Commutator of supersymmetry reads

\[ [\delta \epsilon_1, \delta \epsilon_2] \equiv \hat{\mathcal{L}} \chi_3 + \delta \epsilon_3 + \delta_{\text{so}(1,9)_L} + \delta_{\text{so}(9,1)_R} + \delta_{\text{trivial}}. \]

where

\[ \chi^A_3 = i \bar{\epsilon}_1 \gamma^A \epsilon_2, \quad \epsilon_3 = i^{1/2} \left[ (\bar{\epsilon}_1 \gamma^p \epsilon_2) \gamma_p \rho + (\bar{\rho} \epsilon_2) \epsilon_1 - (\bar{\rho} \epsilon_1) \epsilon_2 \right], \quad \text{etc.} \]

and \( \delta_{\text{trivial}} \) corresponds to the fermionic equations of motion.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the ‘unification’,
- choose a diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$,
- and reduce SDFT to SUGRAs.

However, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) backgrounds.
As stressed before, one of the characteristic features in our construction of $\mathcal{N} = 2 \ D = 10$ SDFT is the usage of the $\text{O}(D, D)$ covariant, genuine DFT-field-variables. However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, i.e. zehnbeins and $B$-field. Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{A\rho} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_\rho{}^\mu \\ (B + e)_\nu\rho \end{pmatrix}, \quad \tilde{V}_{A\bar{\rho}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\tilde{e}^{-1})_{\bar{\rho}}{}^\mu \\ (B + \tilde{e})_{\nu\bar{\rho}} \end{pmatrix}.$$

Here $e_\mu{}^p$ and $\tilde{e}_\nu{}^{\bar{\rho}}$ are two copies of the $D$-dimensional vielbein corresponding to the same spacetime metric,

$$e_\mu{}^p e_\nu{}^q \eta_{pq} = -\tilde{e}_\mu{}^{\bar{\rho}} \tilde{e}_\nu{}^{\bar{q}} \tilde{\eta}_{\bar{\rho}\bar{q}} = g_{\mu\nu},$$

and further, $B_{\mu\rho} = B_{\mu\nu} (e^{-1})_\nu{}^\rho, \ B_{\mu\bar{\rho}} = B_{\mu\nu} (\tilde{e}^{-1})_{\bar{\rho}}{}^\nu$. 

---

Jeong-Hyuck Park

$\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
As stressed before, one of the characteristic features in our construction of $\mathcal{N} = 2 \ D = 10$ SDFT is the usage of the $O(D, D)$ covariant, genuine DFT-field-variables.

However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, \textit{i.e.} zehnbeins and $B$-field.

Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{A\rho} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} (e^{-1})_{\rho}^{\mu} \\ (B + e)_{\nu \rho} \end{pmatrix} \right), \quad \tilde{V}_{A\bar{\rho}} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} (\tilde{e}^{-1})_{\bar{\rho}}^{\mu} \\ (B + \tilde{e})_{\nu \bar{\rho}} \end{pmatrix} \right).$$

Here $e_{\mu}^p$ and $\tilde{e}_{\nu}^{\bar{p}}$ are two copies of the $D$-dimensional vielbein corresponding to the same spacetime metric,

$$e_{\mu}^p e_{\nu}^q \eta_{pq} = -\tilde{e}_{\mu}^{\bar{p}} \tilde{e}_{\nu}^{\bar{q}} \eta_{\bar{p} \bar{q}} = g_{\mu \nu},$$

and further, $B_{\mu \rho} = B_{\mu \nu} (e^{-1})_{\rho}^{\nu}$, $B_{\mu \bar{\rho}} = B_{\mu \nu} (\tilde{e}^{-1})_{\bar{\rho}}^{\nu}$. 

As stressed before, one of the characteristic features in our construction of $\mathcal{N} = 2 \ D = 10$ SDFT is the usage of the $O(D,D)$ covariant, genuine DFT-field-variables.

However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, i.e. zehnbeins and $B$-field.

Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{\alpha \rho} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{\rho}^{\mu} \\ (B + e)_{\nu \rho} \end{pmatrix}, \quad \bar{V}_{\alpha \bar{\rho}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{\rho}}^{\mu} \\ (B + \bar{e})_{\nu \bar{\rho}} \end{pmatrix}.$$

Here $e_{\mu}^{\rho}$ and $\bar{e}_{\nu}^{\bar{\rho}}$ are two copies of the $D$-dimensional vielbein corresponding to the same spacetime metric,

$$e_{\mu}^{\rho} e_{\nu}^{\sigma} \eta_{\rho \sigma} = -\bar{e}_{\mu}^{\bar{\rho}} \bar{e}_{\nu}^{\bar{\sigma}} \bar{\eta}_{\bar{\rho} \bar{\sigma}} = g_{\mu \nu},$$

and further, $B_{\mu \rho} = B_{\mu \nu} (e^{-1})_{\nu}^{\rho}$, $B_{\mu \bar{\rho}} = B_{\mu \nu} (\bar{e}^{-1})_{\bar{\rho}}^{\nu}$. 

---

**Jeong-Hyuck Park**  
$\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
Instead, we may choose an alternative parametrization,

\[
V_A^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^{p \nu} \end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu \bar{p}} \\ (\tilde{e}^{-1})^{\bar{p} \nu} \end{pmatrix},
\]

where \(\beta^{\mu p} = \beta^{\mu \nu} (\tilde{e}^{-1})^{p \nu}, \beta^{\mu \bar{p}} = \beta^{\mu \nu} (\tilde{e}^{-1})^{p \nu},\) and \(\tilde{e}^\mu_p, \tilde{e}^\mu_{\bar{p}}\) correspond to a pair of T-dual vielbeins for winding modes,

\[
\tilde{e}^\mu_p \tilde{e}^\nu_q \eta^{pq} = -\tilde{e}^\mu_{\bar{p}} \tilde{e}^\nu_{\bar{q}} \tilde{\eta}^{\bar{p} \bar{q}} = (g - B g^{-1} B)^{-1} \mu \nu .
\]

Note that in the T-dual winding mode sector, the \(D\)-dimensional curved spacetime indices are all upside-down: \(\tilde{x}_\mu, \tilde{e}^\mu_p, \tilde{e}^\mu_{\bar{p}}, \beta^{\mu \nu}\) (cf. \(x^\mu, e^\mu_p, e^\mu_{\bar{p}}, B_{\mu \nu}\)).
Instead, we may choose an alternative parametrization,

\[
V_A^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^{p \nu} \end{pmatrix}, \quad \tilde{V}_A^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^{p \nu} \end{pmatrix},
\]

where \( \beta^{\mu p} = \beta^{\mu \nu} (\tilde{e}^{-1})^{\nu p} \), \( \beta^{\mu \bar{p}} = \beta^{\mu \nu} (\tilde{e}^{-1})^{\nu \bar{p}} \), and \( \tilde{e}^{\mu p} \), \( \tilde{e}^{\mu \bar{p}} \) correspond to a pair of T-dual vielbeins for winding modes,

\[
\tilde{e}^{\mu p} \tilde{e}^{\nu q} \eta^{pq} = - \tilde{e}^{\mu \bar{p}} \tilde{e}^{\nu \bar{q}} \eta^{\bar{p} \bar{q}} = (g - B \eta^{-1} B)^{-1} \eta^{\mu \nu}.
\]

Note that in the T-dual winding mode sector, the \( D \)-dimensional curved spacetime indices are all upside-down: \( \tilde{x}_\mu \), \( \tilde{e}^{\mu p} \), \( \tilde{e}^{\mu \bar{p}} \), \( \beta^{\mu \nu} \) (cf. \( x^\mu \), \( e^{\mu p} \), \( e^{\mu \bar{p}} \), \( B^{\mu \nu} \)).
Two parametrizations:

\[ V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{\rho \mu} \\ (B + e)_{\nu \rho} \end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{\rho} \mu} \\ (B + \bar{e})_{\nu \bar{\rho}} \end{pmatrix} \]

versus

\[ V_{A_\rho} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu \rho} \\ (\bar{e}^{-1})_{\rho \nu} \end{pmatrix}, \quad \bar{V}_{A\bar{\rho}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu \bar{\rho}} \\ (\bar{e}^{-1})_{\bar{\rho} \nu} \end{pmatrix}. \]

In connection to the section condition, \( \partial^A \partial_A \equiv 0 \), the former matches well with the choice, \( \partial_{\partial x^{\mu}} \equiv 0 \), while the latter is natural when \( \partial_{\partial x^{\mu}} \equiv 0 \).

Yet if we consider dimensional reductions from \( D \) to lower dimensions, there is no longer preferred parametrization \( \Rightarrow \) "Non-geometry", other parametrizations Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun et al. (München)
Two parametrizations:

\[ V_{A^p} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})\rho^\mu \\ (B + e)_{\nu p} \end{pmatrix}, \quad \bar{V}_{A^\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})\bar{p}^\mu \\ (B + \bar{e})_{\nu \bar{p}} \end{pmatrix} \]

versus

\[ V_{A^\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu p} \\ (\bar{e}^{-1})^\rho_\nu \end{pmatrix}, \quad \bar{V}_{A^{\bar{\mu}}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu \bar{p}} \\ (\bar{e}^{-1})^{\rho_\nu} \end{pmatrix}. \]

In connection to the section condition, \( \partial^A \partial_A \equiv 0 \), the former matches well with the choice, \( \frac{\partial}{\partial x^\mu} \equiv 0 \), while the latter is natural when \( \frac{\partial}{\partial x^\mu} \equiv 0 \).

Yet if we consider dimensional reductions from \( D \) to lower dimensions, there is no longer preferred parametrization \( \implies \) “Non-geometry”, other parametrizations

Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun et al. (München)
Two parametrizations:

\[
V_{A\rho} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(e^{-1})_{\rho\mu} \\
(B + e)_{\nu \rho}
\end{pmatrix}, \quad \tilde{V}_{A\bar{\rho}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\tilde{e}^{-1})_{\bar{\rho}\mu} \\
(B + \tilde{e})_{\nu \bar{\rho}}
\end{pmatrix}
\]

versus

\[
V_{A\rho} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \tilde{e})^{\mu \rho} \\
(\tilde{e}^{-1})^p_{\rho}
\end{pmatrix}, \quad \tilde{V}_{A\bar{\rho}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \tilde{e})^{\mu \bar{\rho}} \\
(\tilde{e}^{-1})^p_{\bar{\rho}}
\end{pmatrix}.
\]

In connection to the section condition, \(\partial^A \partial_A \equiv 0\), the former matches well with the choice, \(\frac{\partial}{\partial \tilde{x}^\mu} \equiv 0\), while the latter is natural when \(\frac{\partial}{\partial x^\mu} \equiv 0\).

Yet if we consider dimensional reductions from \(D\) to lower dimensions, there is no longer preferred parametrization \(\implies\) “Non-geometry”, other parametrizations

Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun et al. (München)
Two parametrizations:

\[ V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{\rho}{}^{\mu} \\ (B + e)_{\nu p} \end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{\rho}}{}^{\mu} \\ (B + \bar{e})_{\nu \bar{p}} \end{pmatrix} \]

versus

\[ V_{A p} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu p} \\ (\bar{e}^{-1})_{\rho}{}^{\nu} \end{pmatrix}, \quad \bar{V}_{A \bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})^{\mu p} \\ (\bar{e}^{-1})_{\bar{\rho}}{}^{\nu} \end{pmatrix}. \]

However, let me emphasize that to maintain the clear \( O(D, D) \) covariant structure, it is necessary to work with the parametrization-independent, and \( O(D, D) \) covariant, DFT-vielbeins, \( V_{Ap}, \bar{V}_{A\bar{p}} \), rather than the Riemannian variables, \( e_{\mu}{}^{p}, B_{\mu \nu} \).

Furthermore, ‘degenerate’ cases are also allowed which lead to genuinely non-Riemannian ‘metric-less’ backgrounds \( \implies \) New type of string theory backgrounds 1307.8377
Two parametrizations:

\[
V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(e^{-1})^\rho_\mu \\
(B + e)_{\nu p}
\end{pmatrix},
\quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\bar{e}^{-1})_{\bar{p} \mu} \\
(B + \bar{e})_{\nu \bar{p}}
\end{pmatrix}
\]

versus

\[
V_{A^p} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \bar{e})^\mu_\rho \\
(\bar{e}^{-1})^\rho_\nu
\end{pmatrix},
\quad \bar{V}_{A^\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \bar{e})_{\mu \rho} \\
(\bar{e}^{-1})_{\rho \nu}
\end{pmatrix}.
\]

However, let me emphasize that to maintain the clear $O(D,D)$ covariant structure, it is necessary to work with the parametrization-independent, and $O(D,D)$ covariant, DFT-vielbeins, $V_{Ap}$, $\bar{V}_{A\bar{p}}$, rather than the Riemannian variables, $e_{\mu p}$, $B_{\mu \nu}$.

Furthermore, ‘degenerate’ cases are also allowed which lead to genuinely non-Riemannian ‘metric-less’ backgrounds $\implies$ New type of string theory backgrounds 1307.8377
Two parametrizations:

\[
V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(e^{-1})_{\rho}{}^{\mu} \\
(B + e)_{\nu \rho}
\end{pmatrix}, \quad
\tilde{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\
(B + \bar{e})_{\nu \bar{p}}
\end{pmatrix}
\]

versus

\[
V_{A}{}^{p} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \bar{e})^{\mu p} \\
(\bar{e}^{-1})_{p}{}^{\nu}
\end{pmatrix}, \quad
\tilde{V}_{A}{}^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(\beta + \bar{e})_{\mu p} \\
(\bar{e}^{-1})_{p}{}^{\nu}
\end{pmatrix}.
\]

However, let me emphasize that to maintain the clear \(O(D, D)\) covariant structure, it is necessary to work with the parametrization-independent, and \(O(D, D)\) covariant, DFT-vielbeins, \(V_{Ap}, \tilde{V}_{A\bar{p}}\), rather than the Riemannian variables, \(e_{\mu}{}^{p}, B_{\mu \nu}\).

Furthermore, ‘degenerate’ cases are also allowed which lead to genuinely non-Riemannian ‘metric-less’ backgrounds \(\implies\) New type of string theory backgrounds 1307.8377
From now on, let us restrict ourselves to the former parametrization and impose \( \frac{\partial}{\partial x_\mu} \equiv 0 \).

This reduces (S)DFT to generalized geometry

\[ \text{Hitchin; Grana, Minasian, Petrini, Waldram} \]

For example, the \( O(D,D) \) covariant Dirac operators become

\[ \sqrt{2} \gamma^A \mathcal{D}_A \rho = \gamma^m \left( \partial_m \rho + \frac{1}{4} \omega_{mnp} \gamma^{np} \rho + \frac{1}{24} H_{mnp} \gamma^{np} \rho - \partial_m \phi \rho \right), \]

\[ \sqrt{2} \gamma^A \mathcal{D}_A \psi \bar{p} = \gamma^m \left( \partial_m \psi \bar{p} + \frac{1}{4} \omega_{mnp} \gamma^{np} \psi \bar{p} + \bar{\omega}_{mp \bar{q}} \psi \bar{q} + \frac{1}{24} H_{mnp} \gamma^{np} \psi \bar{p} + \frac{1}{2} H_{mp \bar{q}} \psi \bar{q} - \partial_m \phi \psi \bar{p} \right), \]

\[ \sqrt{2} \bar{\mathcal{V}}^A \bar{p} \mathcal{D}_A \rho = \partial_{\bar{p}} \rho + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \rho + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \rho, \]

\[ \sqrt{2} \mathcal{D}_A \psi^A = \partial_\bar{p} \psi_\bar{p} + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \psi_\bar{p} + \bar{\omega}_{\bar{p} \bar{q} \bar{r}} \psi_\bar{q} + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \psi_\bar{p} - 2 \partial_\bar{p} \phi \psi_\bar{p}. \]

\( \omega_\mu \pm \frac{1}{2} H_\mu \) and \( \omega_\mu \pm \frac{1}{6} H_\mu \) naturally appear as spin connections. Liu, Minasian
From now on, let us restrict ourselves to the former parametrization and impose $\frac{\partial}{\partial x_\mu} \equiv 0$.

This reduces (S)DFT to generalized geometry

Hitchin; Grana, Minasian, Petrini, Waldram

For example, the $O(D, D)$ covariant Dirac operators become

$$\sqrt{2} \gamma^A D_A \rho \equiv \gamma^m \left( \partial_m \rho + \frac{1}{4} \omega_{mnp} \gamma^{np} \rho + \frac{1}{24} H_{mnp} \gamma^{np} \rho - \partial_m \phi \rho \right),$$

$$\sqrt{2} \gamma^A D_A \psi_\bar{p} \equiv \gamma^m \left( \partial_m \psi_{\bar{p}} + \frac{1}{4} \omega_{mpq} \gamma^{pq} \psi_{\bar{p}} + \frac{1}{24} H_{mpq} \gamma^{pq} \psi_{\bar{p}} + \frac{1}{2} H_{mpq} \gamma^{pq} \psi_{\bar{p}} - \partial_m \phi \psi_{\bar{p}} \right),$$

$$\sqrt{2} \bar{V}_A^p D_A \rho \equiv \partial_p \rho + \frac{1}{4} \omega_{pqr} \gamma^{qr} \rho + \frac{1}{8} H_{pqr} \gamma^{qr} \rho,$$

$$\sqrt{2} D_A \psi^A \equiv \partial_\bar{p} \psi_{\bar{p}} + \frac{1}{4} \omega_{\bar{pq}} \gamma^{\bar{q}} \psi_{\bar{p}} + \omega_{\bar{p}q} \gamma^{q} \psi_{\bar{p}} + \frac{1}{8} H_{pqr} \gamma^{qr} \psi_{\bar{p}} - 2 \partial_\bar{p} \phi \psi_{\bar{p}}.$$

$\omega_\mu \pm \frac{1}{2} H_\mu$ and $\omega_\mu \pm \frac{1}{6} H_\mu$ naturally appear as spin connections. Liu, Minasian
From now on, let us restrict ourselves to the former parametrization and impose $\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0$.

This reduces (S)DFT to generalized geometry

Hitchin; Grana, Minasian, Petrini, Waldram

For example, the $\mathbb{O}(D,D)$ covariant Dirac operators become

$$\sqrt{2} \gamma^A D_A \rho \equiv \gamma^m \left( \partial_m \rho + \frac{\omega_{mnp}}{4} \gamma^{np} \rho + \frac{1}{24} H_{mn} \gamma^{np} \rho - \partial_m \phi \rho \right),$$

$$\sqrt{2} \gamma^A D_A \psi_p \equiv \gamma^m \left( \partial_m \psi_p + \frac{\omega_{mnp}}{4} \gamma^{np} \psi_p + \bar{\omega}_{mpq} \psi_q + \frac{1}{24} H_{mp} \gamma^{np} \psi_p + \frac{1}{2} H_{mpq} \psi_q - \partial_m \phi \psi_p \right),$$

$$\sqrt{2} \bar{V}^A \bar{D}_A \rho \equiv \partial_{\bar{p}} \rho + \frac{\omega_{\bar{p}qr}}{4} \gamma^{qr} \rho + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \rho,$$

$$\sqrt{2} D_A \psi^A \equiv \partial_{\bar{p}} \psi_{\bar{p}} + \frac{\omega_{\bar{p}qr}}{4} \gamma^{qr} \psi_{\bar{p}} + \bar{\omega}_{\bar{p}q} \psi_q + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \psi_{\bar{p}} - 2 \partial_{\bar{p}} \phi \psi_{\bar{p}}.$$

$\omega_\mu \pm \frac{1}{2} H_\mu$ and $\omega_\mu \pm \frac{1}{6} H_\mu$ naturally appear as spin connections. Liu, Minasian
From now on, let us restrict ourselves to the former parametrization and impose $\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0$.

This reduces (S)DFT to generalized geometry

Hitchin; Grana, Minasian, Petrini, Waldram

For example, the $O(D, D)$ covariant Dirac operators become

$$\sqrt{2} \gamma^A D_A \rho \equiv \gamma^m \left( \partial_m \rho + \frac{1}{4} \omega_{mnp} \gamma^{np} \rho + \frac{1}{24} H_{mnp} \gamma^{np} \rho - \partial_m \phi \rho \right),$$

$$\sqrt{2} \gamma^A D_A \psi_p \equiv \gamma^m \left( \partial_m \psi_p + \frac{1}{4} \omega_{mnp} \gamma^{np} \psi_p + \tilde{\omega}_{m\bar{p}q} \psi_{\bar{q}} + \frac{1}{24} H_{mnp} \gamma^{np} \psi_{\bar{p}} + \frac{1}{2} H_{m\bar{p}q} \psi_{\bar{q}} - \partial_m \phi \psi_p \right),$$

$$\sqrt{2} \tilde{V}^A \bar{D}_A \rho \equiv \partial_{\bar{p}} \rho + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \rho + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \rho,$$

$$\sqrt{2} D_A \psi^A \equiv \partial_{\bar{p}} \psi_{\bar{p}} + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \psi_{\bar{p}} + \omega_{\bar{p}q\bar{q}} \psi_{\bar{q}} + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \psi_{\bar{p}} - 2 \partial_{\bar{p}} \phi \psi_{\bar{p}} .$$

$\omega_\mu \pm \frac{1}{2} H_\mu$ and $\omega_\mu \pm \frac{1}{6} H_\mu$ naturally appear as spin connections. Liu, Minasian
Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

\[(e^{-1} \bar{e})_p \bar{\partial} (e^{-1} \bar{e})_q \bar{\partial} \eta_{p\bar{q}} = -\eta_{p\bar{q}}.\]

Further, there is a spinorial representation of this Lorentz rotation,

\[S e \gamma_{\bar{p}} S e^{-1} = \gamma^{(D+1)} \gamma_p (e^{-1} \bar{e})_p \bar{\partial},\]

such that

\[S e \gamma^{(D+1)} S e^{-1} = -\det(e^{-1} \bar{e}) \gamma^{(D+1)}.\]
Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

\[(e^{-1} \bar{e})_{p}^{\bar{p}} (e^{-1} \bar{e})_{q}^{\bar{q}} \eta_{\bar{p} \bar{q}} = -\eta_{pq}.\]

Further, there is a spinorial representation of this Lorentz rotation,

\[S_{e} \gamma^{\bar{p}} S_{e}^{-1} = \gamma^{(D+1)} \gamma^{p} (e^{-1} \bar{e})_{p}^{\bar{p}},\]

such that

\[S_{e} \gamma^{(D+1)} S_{e}^{-1} = -\det(e^{-1} \bar{e}) \gamma^{(D+1)}.\]
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are then classified into two groups,

\[
\text{cc}' \det(e^{-1}\bar{e}) = +1 \quad : \quad \text{type IIA},
\]

\[
\text{cc}' \det(e^{-1}\bar{e}) = -1 \quad : \quad \text{type IIB}.
\]

This identification with the ordinary IIA/IIB SUGRAs can be established, if we ‘fix’ the two zehnbeins equal to each other,

\[
e_{\mu}^{\rho} \equiv \bar{e}_{\mu}^{\bar{\rho}},
\]

using a $\text{Pin}(D-1,1)_R$ local Lorentz rotation which may or may not flip the $\text{Pin}(D-1,1)_R$ chirality,

\[
c' \rightarrow \det(e^{-1}\bar{e})c'.
\]

Namely, the $\text{Pin}(D-1,1)_R$ chirality changes iff $\det(e^{-1}\bar{e}) = -1$.  

Jeong-Hyuck Park  
$\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are then classified into two groups,

\[
cc' \det(e^{-1}\bar{e}) = +1 : \text{type IIA},
\]
\[
cc' \det(e^{-1}\bar{e}) = -1 : \text{type IIB}.
\]

This identification with the ordinary IIA/IIB SUGRAs can be established, if we ‘fix’ the two zehnbeins equal to each other,

\[
e_\mu^\rho \equiv \bar{e}_\mu^{\bar{\rho}},
\]

using a $\text{Pin}(D-1, 1)_R$ local Lorentz rotation which may or may not flip the $\text{Pin}(D-1, 1)_R$ chirality,

\[
c' \longrightarrow \det(e^{-1}\bar{e})c'.
\]

Namely, the $\text{Pin}(D-1, 1)_R$ chirality changes iff $\det(e^{-1}\bar{e}) = -1$. 

Jeong-Hyuck Park
Unification of type IIA and IIB SUGRAs

- The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are classified into two groups,

  $$\mathbf{c c'} \det(e^{-1} \bar{e}) = +1 : \text{type IIA},$$

  $$\mathbf{c c'} \det(e^{-1} \bar{e}) = -1 : \text{type IIB}.$$

- That is to say, formulated in terms of the genuine DFT-field variables, i.e. $V_{A\rho}$, $\bar{V}_{A\bar{\rho}}$, $C^\alpha \bar{\alpha}$, etc. the $\mathcal{N} = 2 \ D = 10$ SDFT is a chiral theory with respect to the pair of local Lorentz groups. The possible four chirality choices are all equivalent and hence the theory is **unique**. We may safely put $\mathbf{c} \equiv \mathbf{c'} \equiv +1$ without loss of generality.

- However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

- Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \ D = 10$ SDFT of fixed chirality e.g. $\mathbf{c} \equiv \mathbf{c'} \equiv +1$.

- In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs.
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are classified into two groups,

$$cc' \det(e^{-1} \bar{e}) = +1 \quad : \quad \text{type IIA},$$

$$cc' \det(e^{-1} \bar{e}) = -1 \quad : \quad \text{type IIB}.$$

That is to say, formulated in terms of the genuine DFT-field variables, i.e. $V_A$, $\bar{V}_{\bar{A}}$, $C^\alpha_{\bar{\alpha}}$, etc. the $\mathcal{N} = 2 \ D = 10$ SDFT is a chiral theory with respect to the pair of local Lorentz groups. The possible four chirality choices are all equivalent and hence the theory is \textit{unique}. We may safely put $c \equiv c' \equiv +1$ without loss of generality.

However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \ D = 10$ SDFT of fixed chirality e.g. $c \equiv c' \equiv +1$.

In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs.
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are classified into two groups,

$$cc' \ det(e^{-1} \bar{e}) = +1 \quad : \quad \text{type IIA},$$

$$cc' \ det(e^{-1} \bar{e}) = -1 \quad : \quad \text{type IIB}.$$  

That is to say, formulated in terms of the genuine DFT-field variables, i.e. $V_A, \bar{V}_{\bar{A}}, C^\alpha \bar{\alpha}$, etc. the $\mathcal{N} = 2 \ D = 10$ SDFT is a chiral theory with respect to the pair of local Lorentz groups. The possible four chirality choices are all equivalent and hence the theory is \textit{unique}. We may safely put $c \equiv c' \equiv +1$ without loss of generality.

However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \ D = 10$ SDFT of fixed chirality e.g. $c \equiv c' \equiv +1$.

In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs.
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are classified into two groups,

$$\text{cc}' \det(e^{-1}\bar{e}) = +1 : \text{type IIA},$$

$$\text{cc}' \det(e^{-1}\bar{e}) = -1 : \text{type IIB}.$$

That is to say, formulated in terms of the genuine DFT-field variables, i.e. $V_A \bar{p}$, $\bar{V}_{\bar{A}p}$, $C^{\alpha} \bar{\alpha}$, etc. the $\mathcal{N} = 2 \ D = 10$ SDFT is a chiral theory with respect to the pair of local Lorentz groups. The possible four chirality choices are all equivalent and hence the theory is \textit{unique}. We may safely put $c \equiv c' \equiv +1$ without loss of generality.

However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \ D = 10$ SDFT of fixed chirality e.g. $c \equiv c' \equiv +1$.

In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs.
The $\mathcal{N} = 2 \ D = 10$ SDFT solutions are classified into two groups,

$$cc' \ det(e^{-1} \bar{e}) = +1 \ : \ \text{type IIA},$$

$$cc' \ det(e^{-1} \bar{e}) = -1 \ : \ \text{type IIB}.$$

That is to say, formulated in terms of the genuine DFT-field variables, i.e. $V_{Ap}$, $\bar{V}_{A\bar{p}}$, $C^{\alpha \bar{\alpha}}$, etc. the $\mathcal{N} = 2 \ D = 10$ SDFT is a chiral theory with respect to the pair of local Lorentz groups. The possible four chirality choices are all equivalent and hence the theory is unique. We may safely put $c \equiv c' \equiv +1$ without loss of generality.

However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \ D = 10$ SDFT of fixed chirality e.g. $c \equiv c' \equiv +1$.

In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs.
Setting the diagonal gauge,

\[ e_{\mu}^p \equiv \bar{e}_{\mu} \bar{p} \]

with \( \eta_{pq} = -\bar{\eta}_{\bar{p} \bar{q}} \), \( \bar{\gamma}^\bar{p} = \gamma^{(D+1)} \gamma^p \), \( \bar{\gamma}^{(D+1)} = -\gamma^{(D+1)} \), breaks the local Lorentz symmetry,

\[ \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \implies \text{Spin}(1, D-1)_D. \]

And it reduces SDFT to SUGRA:

- \( \mathcal{N} = 2 \; D = 10 \) SDFT \implies 10D Type II democratic SUGRA
  Bergshoeff, et al.; Coimbra, Strickland-Constable, Waldram

- \( \mathcal{N} = 1 \; D = 10 \) SDFT \implies 10D minimal SUGRA
  Chamseddine; Bergshoeff et al.
Setting the diagonal gauge,
\[ e_{\mu}^{\rho} \equiv \bar{e}_{\mu}^{\bar{\rho}} \]
with \( \eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}} \), \( \bar{\gamma}^{\bar{p}} = \gamma^{(D+1)} \gamma^{\rho} \), \( \bar{\gamma}^{(D+1)} = -\gamma^{(D+1)} \), breaks the local Lorentz symmetry,
\[ \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \implies \text{Spin}(1, D-1)_D. \]

And it reduces SDFT to SUGRA:
\[ \mathcal{N} = 2 \ D = 10 \ \text{SDFT} \implies 10D \text{ Type II democratic SUGRA} \]
Bergshoeff, et al.; Coimbra, Strickland-Constable, Waldram

\[ \mathcal{N} = 1 \ D = 10 \ \text{SDFT} \implies 10D \text{ minimal SUGRA} \]
Chamseddine; Bergshoeff et al.
To the full order in fermions, $\mathcal{N} = 1$ SDFT reduces to 10D minimal SUGRA:

$$
\mathcal{L}_{10D} = \det e \times e^{-2\phi} \left[ R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda \mu \nu} H^{\lambda \mu \nu}
+ i 2 \sqrt{2} \bar{\rho} \gamma^m [\partial_m \rho + \frac{1}{4} (\omega + \frac{1}{6} H)_{mnp} \gamma^{np} \rho] - i 4 \sqrt{2} \bar{\psi}^p [\partial_p \rho + \frac{1}{4} (\omega + \frac{1}{2} H)_{pq} \gamma^{qr} \rho]
- i 2 \sqrt{2} \bar{\psi}^p \gamma^m [\partial_m \psi_p + \frac{1}{4} (\omega + \frac{1}{6} H) \gamma^{np} \psi_p + \omega_{mpq} \psi^q - \frac{1}{2} H_{mpq} \psi^q]
+ \frac{1}{24} (\bar{\psi}^q \gamma_{mnp} \psi_q)(\bar{\psi}^r \gamma^{mnp} \psi_r) - \frac{1}{48} (\bar{\psi}^q \gamma_{mnp} \psi_q)(\bar{\rho} \gamma^{mnp} \rho) \right].
$$

$$
\delta_\epsilon \phi = i \frac{1}{2} \bar{\xi} (\rho + \gamma^a \psi_a), \quad \delta_\epsilon \mathbf{e}_\mu^a = i \bar{\xi} \gamma^a \psi_\mu, \quad \delta_\epsilon B_{\mu \nu} = -2 i \bar{\xi} \gamma_{[\mu} \psi_{\nu]},
$$

$$
\delta_\epsilon \rho = - \frac{1}{\sqrt{2}} \gamma^a [\partial_a \epsilon + \frac{1}{4} (\omega + \frac{1}{6} H)_{abc} \gamma^{bc} \epsilon - \partial_a \phi \epsilon]
+ i \frac{1}{48} (\bar{\psi}^d \gamma_{abc} \psi_d) \gamma^{abc} \epsilon + i \frac{1}{192} (\bar{\rho} \gamma_{abc} \rho) \gamma^{abc} \epsilon + i \frac{1}{2} (\bar{\epsilon} \gamma_{[a} \psi_{b]}) \gamma^{ab} \rho,
$$

$$
\delta_\epsilon \psi_a = \frac{1}{\sqrt{2}} [\partial_a \epsilon + \frac{1}{4} (\omega + \frac{1}{2} H)_{abc} \gamma^{bc} \epsilon]
- i \frac{1}{2} (\bar{\rho} \epsilon) \psi_a - i \frac{1}{4} (\bar{\rho} \psi_a) \epsilon + i \frac{1}{8} (\bar{\rho} \gamma_{bc} \psi_a) \gamma^{bc} \epsilon + i \frac{1}{2} (\bar{\epsilon} \gamma_{[b} \psi_{c]}) \gamma^{bc} \psi_a.
$$
After the diagonal gauge fixing, we may parameterize the R-R potential as

\[ C \equiv \left( \frac{1}{2} \right)^{\frac{D+2}{4}} \sum' \frac{1}{p!} C_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p} \]

and obtain the field strength,

\[ \mathcal{F} := \mathcal{D}_+^0 C \equiv \left( \frac{1}{2} \right)^{\frac{D}{4}} \sum' \frac{1}{(p+1)!} \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}} \]

where \( \sum'_p \) denotes the odd \( p \) sum for Type IIA and even \( p \) sum for Type IIB, and

\[ \mathcal{F}_{a_1 a_2 \cdots a_p} = p \left( D_{[a_1} C_{a_2 \cdots a_p]} - \partial_{[a_1} \phi C_{a_2 \cdots a_p]} \right) + \frac{p!}{3!(p-3)!} H_{[a_1 a_2 a_3} C_{a_4 \cdots a_p]} \]

The pair of nilpotent differential operators, \( \mathcal{D}_+^0 \) and \( \mathcal{D}_-^0 \), reduce to a ‘twisted K-theory’ exterior derivative and its dual, after the diagonal gauge fixing,

\[ \mathcal{D}_+^0 \quad \implies \quad d + (H - d\phi) \wedge \]

\[ \mathcal{D}_-^0 \quad \implies \quad * [ d + (H - d\phi) \wedge ] * \]
After the diagonal gauge fixing, we may parameterize the R-R potential as

\[ C \equiv \left( \frac{1}{2} \right)^{\frac{D+2}{4}} \sum' \frac{1}{p!} C_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p} \]

and obtain the field strength,

\[ F := D_+^0 C \equiv \left( \frac{1}{2} \right)^{\frac{D}{4}} \sum' \frac{1}{(p+1)!} F_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}} \]

where \( \sum'_p \) denotes the odd \( p \) sum for Type IIA and even \( p \) sum for Type IIB, and

\[ F_{a_1 a_2 \cdots a_p} = \epsilon \left( D_{[a_1} C_{a_2 \cdots a_p]} - \partial_{[a_1} \phi C_{a_2 \cdots a_p]} \right) + \frac{p!}{3!(p-3)!} H_{[a_1 a_2 a_3} C_{a_4 \cdots a_p]} \]

The pair of nilpotent differential operators, \( D_+^0 \) and \( D_-^0 \), reduce to a ‘twisted K-theory’ exterior derivative and its dual, after the diagonal gauge fixing,

\[ D_+^0 \quad \Rightarrow \quad d + (H - d\phi) \wedge \]

\[ D_-^0 \quad \Rightarrow \quad * \left[ d + (H - d\phi) \wedge \right] * \]
In this way, \textbf{ordinary SUGRA} $\equiv$ \textbf{gauge-fixed SDFT},

$$\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \implies \text{Spin}(1, D-1)_D.$$
The diagonal gauge, $e_\mu^p \equiv \bar{e}_\mu \bar{p}$, is incompatible with the vectorial $O(D, D)$ transformation rule of the DFT-vielbein.

In order to preserve the diagonal gauge, it is necessary to modify the $O(D, D)$ transformation rule.
The diagonal gauge, $e_\mu^p \equiv \tilde{e}_\mu{}^{\bar{p}}$, is incompatible with the vectorial $O(D, D)$ transformation rule of the DFT-vielbein.

In order to preserve the diagonal gauge, it is necessary to modify the $O(D, D)$ transformation rule.
The $O(D, D)$ rotation must accompany a compensating $\text{Pin}(D-1, 1)_R$ local Lorentz rotation, $\bar{L}_q \bar{p}$, $S_L \bar{\alpha} \bar{\beta}$, which we can construct explicitly,

$$
\bar{L} = \bar{e}^{-1} \left[ a^t - (g + B)b^t \right] \left[ a^t + (g - B)b^t \right]^{-1} \bar{e}, \quad \bar{\gamma} \bar{q} \bar{L}_q \bar{p} = S_L^{-1} \bar{\gamma} \bar{p} S_L,
$$

where $a$ and $b$ are parameters of a given $O(D, D)$ group element,

$$
M_A^B = \begin{pmatrix}
  a^{\mu \nu} & b^{\mu \sigma} \\
  c_{\rho \nu} & d^{\rho \sigma}
\end{pmatrix}.
$$
Modified $O(D, D)$ Transformation Rule After The Diagonal Gauge Fixing

\[
\begin{align*}
  d & \rightarrow d \\
  V_A^\rho & \rightarrow M_A^B V_B^\rho \\
  \bar{V}_A^{\bar{\rho}} & \rightarrow M_A^B \bar{V}_B^{\bar{\rho}} \bar{L}_{\bar{\rho} p} \\
  C^\alpha \bar{\alpha}, \quad \mathcal{F}^{\alpha} \bar{\alpha} & \rightarrow C^{\alpha} \bar{\beta}(S_L^{-1})^{\bar{\beta}} \bar{\alpha}, \quad \mathcal{F}^{\alpha} \bar{\beta}(S_L^{-1})^{\bar{\beta}} \bar{\alpha} \\
  \rho^\alpha & \rightarrow \rho^\alpha \\
  \rho'^{\bar{\alpha}} & \rightarrow (S_L)^{\bar{\alpha}} \bar{\beta} \rho'^{\bar{\beta}} \\
  \psi^\alpha \bar{p} & \rightarrow (\bar{L}^{-1})_{\bar{p} q} \psi^\alpha_q \\
  \psi'^{\bar{\alpha}} \bar{p} & \rightarrow (S_L)^{\bar{\alpha}} \bar{\beta} \psi'^{\bar{\beta}} \bar{p}
\end{align*}
\]

- All the barred indices are now to be rotated. Consistent with Hassan
- The R-R sector can be also mapped to $O(D, D)$ spinors.

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach
If and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D, D)$ rotation flips the chirality of the theory, since
$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L}) S_{\bar{L}} \bar{\gamma}^{(D+1)}.$$ Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $\mathbf{O}(D, D)$ T-duality.

However, since $\bar{L}$ explicitly depends on the parametrization of $V_{A\dot{P}}$ and $\bar{V}_{A\dot{P}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $\mathbf{O}(D, D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.
If and only if $\det(\bar{L}) = -1$, the modified $O(D, D)$ rotation flips the chirality of the theory, since

$$\tilde{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L}) S_{\bar{L}} \tilde{\gamma}^{(D+1)}.$$  

Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $O(D, D)$ T-duality.

However, since $\bar{L}$ explicitly depends on the parametrization of $V_{A\dot{p}}$ and $\bar{V}_{A\dot{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $O(D, D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.
If and only if $\det(\bar{L}) = -1$, the modified $O(D,D)$ rotation flips the chirality of the theory, since

$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L}) S_{\bar{L}} \bar{\gamma}^{(D+1)}.$$ 

Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $O(D,D)$ T-duality.

However, since $\bar{L}$ explicitly depends on the parametrization of $V_{Ap}$ and $\bar{V}_{A\bar{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $O(D,D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.
With the semi-covariant derivative, we may construct YM-DFT:

\[ F_{AB} := \nabla_A \nabla_B - \nabla_B \nabla_A - i [\nabla_A, \nabla_B], \quad \nabla_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu \nu} \phi^\nu \end{pmatrix}, \]

\[ S_{YM} = \int_{\Sigma_D} e^{-2d} \text{Tr} \left( P^{AB} \tilde{P}^{CD} F_{AC} F_{BD} \right) \]
\[ \equiv \int d^D x \sqrt{-g} e^{-2\phi} \text{Tr} \left( f_{\mu \nu} f^{\mu \nu} + 2D_\mu \phi_\nu D^\mu \phi^\nu + 2D_\mu \phi_\nu D^\nu \phi^\mu + 2i f_{\mu \nu} [\phi^\mu, \phi^\nu] 
- [\phi_\mu, \phi_\nu] [\phi^\mu, \phi^\nu] + 2 (f^{\mu \nu} + i [\phi^\mu, \phi^\nu]) H_{\mu \nu \sigma} \phi^\sigma + H_{\mu \nu \sigma} H^{\mu \nu \tau} \phi^\sigma \phi^\tau \right). \]

Similar to topologically twisted Yang-Mills, but differs in detail.

Curved D-branes are known to convert adjoint scalars into one-form,
\[ \phi^a \rightarrow \phi^\mu, \quad \text{Bershadsky} \]

Action for ‘double’ D-brane Hull; Albertsson, Dai, Kao, Lin
With the semi-covariant derivative, we may construct YM-DFT:

\[ F_{AB} := \nabla_A \nabla_B - \nabla_B \nabla_A - i [\nabla_A, \nabla_B], \quad \nabla_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu\nu} \phi^\nu \end{pmatrix}, \]

\[ S_{YM} = \int \Sigma_D e^{-2d} Tr \left( P^{AB} \bar{P}^{CD} F_{AC} F_{BD} \right) \]

\[ \equiv \int dx^D \sqrt{-g} e^{-2\phi} Tr \left( f_{\mu\nu} f^{\mu\nu} + 2D_\mu \phi_\nu D^\mu \phi^\nu + 2D_\mu \phi_\nu D^\nu \phi^\mu + 2i f_{\mu\nu} [\phi^\mu, \phi^\nu] \right. \]

\[ \left. - [\phi_\mu, \phi_\nu] [\phi^\mu, \phi^\nu] + 2 (f^{\mu\nu} + i [\phi^\mu, \phi^\nu]) H_{\mu\nu\sigma} \phi^\sigma + H_{\mu\nu\sigma} H^{\mu\nu\tau} \phi^\sigma \phi^\tau \right). \]

Similar to topologically twisted Yang-Mills, but differs in detail.

Curved D-branes are known to convert adjoint scalars into one-form,

\[ \phi^a \rightarrow \phi_\mu, \quad \text{Bershadsky} \]

Action for ‘double’ D-brane  
Hull; Albertsson, Dai, Kao, Lin
With DFT-vielbein, it is possible to construct a rank-four tensor which is covariant with respect to $O(D,D)$ and ‘diagonal’ local Lorentz symmetry.

Gauge fixing the two vielbeins equal to each other, $e_{\mu m} = \bar{e}_{\bar{\mu} \bar{m}}$, gives

$$R_{mnpq} + D(p H_q)_{mn} - \frac{1}{4} H_{mn^r} H_{pqr} - \frac{3}{4} H_{m[n^r} H_{pqr]} .$$

This may provide a useful tool to organize the higher order derivative corrections to the effective action.

c.f. Hohm, Siegel, Zwiebach; Godazgar\textsuperscript{2}
With DFT-vielbein, it is possible to construct a rank-four tensor which is covariant with respect to $O(D, D)$ and ‘diagonal’ local Lorentz symmetry.

Gauge fixing the two vielbeins equal to each other, $e_{\mu m} = \bar{e}_{\mu \bar{m}}$, gives

$$R_{mnpq} + D_{(p} H_{q)m} - \frac{1}{4} H_{mn}{}^{r} H_{pq}{}_{r} - \frac{3}{4} H_{m[n} H_{pq]r}.$$ 

This may provide a useful tool to organize the higher order derivative corrections to the effective action. 

c.f. Hohm, Siegel, Zwiebach; Godazgar$^2$
With DFT-vielbein, it is possible to construct a rank-four tensor which is covariant with respect to $O(D, D)$ and ‘diagonal’ local Lorentz symmetry.

Gauge fixing the two vielbeins equal to each other, $e_{\mu m} = \bar{e}_{\mu \bar{m}}$, gives

$$R_{mnpq} + D_{(p} H_{q)m} - \frac{1}{4} H_{mn}{}^{r} H_{pqr} - \frac{3}{4} H_{m[n} H_{pq]}{}^{r}.$$  

This may provide a useful tool to organize the higher order derivative corrections to the effective action.

c.f. Hohm, Siegel, Zwiebach; Godazgar$^2$
The section condition is equivalent to the ‘coordinate gauge symmetry’,

\[ x^M \sim x^M + \varphi \partial^M \varphi'. \]

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in coordinate space.

The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

Hence, the finite transformation rules are not unique.

For example, the exponentiation of the generalized Lie derivative and a simple ansatz proposed by Hohm-Zwiebach. These two are fully equivalent up to the coordinate gauge symmetry.

\[ \text{see also Berman-Cederwall-Perry} \]
The section condition is equivalent to the ‘coordinate gauge symmetry’,

\[ x^M \sim x^M + \varphi \partial^M \varphi'. \]

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in coordinate space.

The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

Hence, the finite transformation rules are not unique.

For example, the exponentiation of the generalized Lie derivative and a simple ansatz proposed by Hohm-Zwiebach. These two are fully equivalent up to the coordinate gauge symmetry.
The section condition is equivalent to the ‘coordinate gauge symmetry’,
\[ x^M \sim x^M + \varphi \partial^M \varphi'. \]

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in coordinate space.

The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the ‘gauge orbits’.

Hence, the finite transformation rules are not unique.

For example, the exponentiation of the generalized Lie derivative and a simple ansatz proposed by Hohm-Zwiebach. These two are fully equivalent up to the coordinate gauge symmetry.

1304.5946 see also Berman-Cederwall-Perry
String propagates in doubled yet gauged spacetime, \( 1307.8377 \)

\[
S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} \sqrt{-h} h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M A_{jM},
\]

where

\[
D_i X^M = \partial_i X^M - A^M_i, \quad A^M_i \partial_M \equiv 0.
\]

The Lagrangian is symmetric with respect to the string worldsheet diffeomorphisms, Weyl symmetry, \( O(D, D) \) T-duality, target spacetime generalized diffeomorphisms and the coordinate gauge symmetry, thanks to the auxiliary gauge field, \( A^M_i \).

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

Further, after parametrization and integrating out \( A^M_i \), it can produce either the standard string action for the ‘non-degenerate’ Riemannian case,

\[
\frac{1}{4\pi \alpha'} \mathcal{L} \equiv \frac{1}{2\pi \alpha'} \left[ -\frac{1}{2} \sqrt{-h} h^{ij} \partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu \right],
\]

or novel chiral actions for ‘degenerate’ non-Riemannian cases, e.g. for \( \mathcal{H}_{AB} = \mathcal{J}_{AB} \),

\[
\frac{1}{4\pi \alpha'} \mathcal{L} \equiv \frac{1}{4\pi \alpha'} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu, \quad \partial_i Y^\mu + \frac{1}{\sqrt{-h}} \epsilon^i_j \partial_j Y^\mu = 0.
\]
String propagates in doubled yet gauged spacetime, $1307.8377$

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} \sqrt{-h} h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M A_{jM},$$

where

$$D_i X^M = \partial_i X^M - A_i^M, \quad A_i^M \partial_M \equiv 0.$$  

The Lagrangian is symmetric with respect to the string worldsheet diffeomorphisms, Weyl symmetry, $O(D, D)$ T-duality, target spacetime generalized diffeomorphisms and the coordinate gauge symmetry, thanks to the auxiliary gauge field, $A_i^M$. 

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

Further, after parametrization and integrating out $A_i^M$, it can produce either the standard string action for the ‘non-degenerate’ Riemannian case,

$$\frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{2\pi\alpha'} \left[ -\frac{1}{2} \sqrt{-h} h^{ij} \partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu \right],$$

or novel chiral actions for ‘degenerate’ non-Riemannian cases, e.g. for $\mathcal{H}_{AB} = J_{AB}$,

$$\frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{4\pi\alpha'} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu, \quad \partial_i Y^\mu + \frac{1}{\sqrt{-h}} \epsilon^i_j \partial_j Y^\mu = 0.$$
String propagates in doubled yet gauged spacetime, 1307.8377

\[ S = \frac{1}{4\pi\alpha'} \int d^2 \sigma \, \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} \sqrt{-h} h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M A_{jM}, \]

where

\[ D_i X^M = \partial_i X^M - A^M_i, \quad A^M_i \partial_M \equiv 0. \]

The Lagrangian is symmetric with respect to the string worldsheet diffeomorphisms, Weyl symmetry, \( O(D, D) \) T-duality, target spacetime generalized diffeomorphisms and the coordinate gauge symmetry, thanks to the auxiliary gauge field, \( A^M_i \).

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

Further, after parametrization and integrating out \( A^M_i \), it can produce either the standard string action for the ‘non-degenerate’ Riemannian case,

\[ \frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{2\pi\alpha'} \left[ -\frac{1}{2} \sqrt{-h} h^{ij} \partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu \right], \]

or novel chiral actions for ‘degenerate’ non-Riemannian cases, e.g. for \( \mathcal{H}_{AB} = \mathcal{J}_{AB} \),

\[ \frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{4\pi\alpha'} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu, \quad \partial_i Y^\mu + \frac{1}{\sqrt{-h}} \epsilon^i_j \partial_j Y^\mu = 0. \]
Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N} = 2 \ D = 10$ SDFT are, besides the fermions, the DFT-dilaton, $d$, DFT-vielbeins, $V_A^p$, $\bar{V}_{\bar{A}}^{-\bar{p}}$, and the R-R potential, $C^{\alpha \bar{\alpha}}$.

- Novel differential geometric ingredients:
  - projectors, $P_{AB} = V_A^p V_B^p$, $\bar{P}_{AB} = \bar{V}_{\bar{A}}^{-\bar{p}} \bar{V}_{\bar{B}}^{-\bar{p}}$, and semi-covariant derivative.
  - Spacetime being doubled yet gauged (section condition).

- $\mathcal{N} = 2 \ D = 10$ SDFT manifests simultaneously the symmetric structures:
  - $O(10, 10)$ T-duality
  - $DFT$-diffeomorphism (generalized Lie derivative)
  - A pair of local Lorentz symmetries, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R$
Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N} = 2 \ D = 10$ SDFT are, besides the fermions, the DFT-dilaton, $d$, DFT-vielbeins, $V_{Ap}$, $\bar{V}_{\bar{A}\bar{p}}$, and the R-R potential, $C^{\alpha \bar{\alpha}}$.

- Novel differential geometric ingredients:
  - projectors, $P_{AB} = V_{Ap} V^{p}_{B}$, $\bar{P}_{AB} = \bar{V}_{\bar{A}\bar{p}} \bar{V}^{\bar{p}}_{B}$, and semi-covariant derivative.
  - Spacetime being doubled yet gauged (section condition).

- $\mathcal{N} = 2 \ D = 10$ SDFT manifests simultaneously the symmetric structures:
  - $O(10, 10)$ T-duality
  - DFT-diffeomorphism (generalized Lie derivative)
  - A pair of local Lorentz symmetries, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R$
Conclusion

Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N} = 2$ $D = 10$ SDFT are, besides the fermions, the DFT-dilaton, $d$, DFT-vielbeins, $V_A p$, $\bar{V}_{A \bar{p}}$, and the R-R potential, $C^\alpha \bar{\alpha}$.

- Novel differential geometric ingredients:
  - projectors, $P_{AB} = V_A p V_B p$, $\bar{P}_{AB} = \bar{V}_{A \bar{p}} \bar{V}_{B \bar{p}}$, and semi-covariant derivative.
  - Spacetime being doubled yet gauged (section condition).

- $\mathcal{N} = 2$ $D = 10$ SDFT manifests simultaneously the symmetric structures:
  - $O(10, 10)$ T-duality
  - DFT-diffeomorphism (generalized Lie derivative)
  - A pair of local Lorentz symmetries, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R$
Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N} = 2 \ D = 10$ SDFT are, besides the fermions, the DFT-dilaton, $d$, DFT-vielbeins, $V_{Ap}$, $\bar{V}_{A\bar{p}}$, and the R-R potential, $C^{\alpha \bar{\alpha}}$.

- Novel differential geometric ingredients:
  - projectors, $P_{AB} = V_{Ap} V_{B\bar{p}}$, $\bar{P}_{AB} = \bar{V}_{A\bar{p}} \bar{V}_{B\bar{p}}$, and semi-covariant derivative.
  - Spacetime being doubled yet gauged (section condition).

- $\mathcal{N} = 2 \ D = 10$ SDFT manifests simultaneously the symmetric structures:
  - $O(10, 10)$ T-duality
  - DFT-diffeomorphism (generalized Lie derivative)
  - A pair of local Lorentz symmetries, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R$
Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N} = 2 \ D = 10$ SDFT are, besides the fermions, the DFT-dilaton, $d$, DFT-vielbeins, $V_{Ap}$, $\bar{V}_{A\bar{p}}$, and the R-R potential, $C^\alpha \bar{\alpha}$.

- Novel differential geometric ingredients:
  - projectors, $P_{AB} = V_{Ap} V_{Bp}$, $\bar{P}_{AB} = \bar{V}_{A\bar{p}} \bar{V}_{B\bar{p}}$, and semi-covariant derivative.
  - Spacetime being doubled yet gauged (section condition).

- $\mathcal{N} = 2 \ D = 10$ SDFT manifests simultaneously the symmetric structures:
  - $O(10, 10)$ T-duality
  - DFT-diffeomorphism (generalized Lie derivative)
  - A pair of local Lorentz symmetries, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R$
\( \mathcal{N} = 2 \ D = 10 \) SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian ‘metric-less’ backgrounds. For example,

\[
P_{MN} - \bar{P}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \quad \mathcal{N}^2 = 1.
\]

While the theory is unique, the Riemannian solutions are twofold.

\[ \implies \text{Unification of IIA and IIB.} \]

After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, \( \text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R \rightarrow \text{Spin}(1, 9)_D \), SDFT reduces to SUGRA.

\[ \text{A priori, in the covariant formalism, the R-R sector and the fermions are } \mathbb{O}(D, D) \text{ singlet.} \]
Yet, the diagonal gauge fixing, \( e_{\mu}^\rho \equiv \bar{e}_{\mu}^{\bar{\rho}} \), modifies the \( \mathbb{O}(D, D) \) transformation rule to call for a compensating \( \text{Pin}(D-1, 1)_R \) rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.
$N = 2\ D = 10$ SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian ‘metric-less’ backgrounds. For example,

$$\mathcal{P}_{MN} - \bar{\mathcal{P}}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \quad N^2 = 1.$$  

While the theory is unique, the Riemannian solutions are twofold.

$\implies$ Unification of IIA and IIB.

After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, $\text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R \rightarrow \text{Spin}(1, 9)_D$, SDFT reduces to SUGRA.

A priori, in the covariant formalism, the R-R sector and the fermions are $O(D, D)$ singlet. Yet, the diagonal gauge fixing, $e_\mu^p \equiv \bar{e}_\mu^\bar{p}$, modifies the $O(D, D)$ transformation rule to call for a compensating $\text{Pin}(D-1, 1)_R$ rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.
\( \mathcal{N} = 2 \ D = 10 \) SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian ‘metric-less’ backgrounds. For example,

\[
P_{MN} - \bar{P}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \quad \mathcal{N}^2 = 1.
\]

While the theory is unique, the Riemannian solutions are twofold.

\[ \implies \text{Unification of IIA and IIB.} \]

After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, \( \text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R \rightarrow \text{Spin}(1, 9)_D \), SDFT reduces to SUGRA.

\( A \text{ priori} \), in the covariant formalism, the R-R sector and the fermions are \( O(D, D) \) singlet. Yet, the diagonal gauge fixing, \( e_\mu^\rho \equiv \bar{e}_\mu^\rho \), modifies the \( O(D, D) \) transformation rule to call for a compensating \( \text{Pin}(D-1, 1)_R \) rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.
\( \mathcal{N} = 2 \; D = 10 \) SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian ‘metric-less’ backgrounds. For example,

\[
P_{MN} - \bar{P}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \quad \mathcal{N}^2 = 1.
\]

While the theory is unique, the Riemannian solutions are twofold.

\[ \implies \text{Unification of IIA and IIB.} \]

After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, \( \text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R \rightarrow \text{Spin}(1, 9)_D \), SDFT reduces to SUGRA.

\( A \text{ priori} \), in the covariant formalism, the R-R sector and the fermions are \( \mathcal{O}(D, D) \) singlet. Yet, the diagonal gauge fixing, \( e_\mu^\rho \equiv \bar{e}_\mu^\bar{\rho} \), modifies the \( \mathcal{O}(D, D) \) transformation rule to call for a compensating \( \text{Pin}(D-1, 1)_R \) rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.
\( \mathcal{N} = 2 \ D = 10 \) SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian ‘metric-less’ backgrounds. For example,

\[
P_{MN} - \bar{P}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \quad \mathcal{N}^2 = 1.
\]

While the theory is unique, the Riemannian solutions are twofold.

\( \implies \) Unification of IIA and IIB.

After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, \( \text{Spin}(1, 9)_L \times \text{Spin}(9, 1)_R \to \text{Spin}(1, 9)_D \), SDFT reduces to SUGRA.

\( \text{A priori} \), in the covariant formalism, the R-R sector and the fermions are \( \mathcal{O}(D, D) \) singlet. Yet, the diagonal gauge fixing, \( e_\mu^p \equiv \bar{e}_\mu^\bar{p} \), modifies the \( \mathcal{O}(D, D) \) transformation rule to call for a compensating \( \text{Pin}(D-1, 1)_R \) rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- $O(10,10)$ covariant Killing spinor equation: SUSY and T-duality are compatible.
- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity
- The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
- U-duality manifest $\mathcal{M}$-theory effective actions: Berman-Perry; Thompson, Godazgar; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- $O(10, 10)$ covariant Killing spinor equation: SUSY and T-duality are compatible.
- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity
- The uplift of type II SDFT to $M$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
- U-duality manifest $M$-theory effective actions: Berman-Perry; Thompson, Godazgar$^2$; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.

- $O(10, 10)$ covariant Killing spinor equation: SUSY and T-duality are compatible.

- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity

- The uplift of type II SDFT to $M$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal

- U-duality manifest $M$-theory effective actions: Berman-Perry; Thompson, Godazgar; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- \( O(10, 10) \) covariant Killing spinor equation: SUSY and T-duality are compatible.

- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity

- The uplift of type II SDFT to \( \mathcal{M} \)-theory, or the extension of T-duality to U-duality: West (\( E_{11} \)); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
- U-duality manifest \( \mathcal{M} \)-theory effective actions: Berman-Perry; Thompson, Godazgar\(^2\); JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- $O(10, 10)$ covariant Killing spinor equation: SUSY and T-duality are compatible.
- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity
- The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
- U-duality manifest $\mathcal{M}$-theory effective actions: Berman-Perry; Thompson, Godazgar; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

**Outlook**

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.

**O(10, 10)** covariant Killing spinor equation: SUSY and T-duality are compatible.

- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity

- The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal

- U-duality manifest $\mathcal{M}$-theory effective actions: Berman-Perry; Thompson, Godazgar$^2$; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

Outlook

- Further study and classification of the non-Riemannian ‘metric-less’ backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- $O(10, 10)$ covariant Killing spinor equation: SUSY and T-duality are compatible.
- The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity
- The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
- U-duality manifest $\mathcal{M}$-theory effective actions: Berman-Perry; Thompson, Godazgar; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben
Conclusion

**Outlook**

- Further study and classification of the non-Riemannian, metric-less backgrounds.
- Quantization of the string action on doubled yet gauged spacetime.
- $\mathbf{O(10,10)}$ covariant Killing spinor equation: SUSY and T-duality are compatible.

The relaxation of the section condition: Geissbuhler; Graña, Marqués, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee for Scherk-Schwarz and Blumenhagen, Fuchs, Lust, Sun for non-associativity.

The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West ($E_{11}$); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal.

U-duality manifest $\mathcal{M}$-theory effective actions: Berman-Perry; Thompson, Godazgar; JHP-Suh (U-geometry); Cederwall, Edlund, Karlsson; Musaev; Hohm-Samtleben.

Thank you.